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Enveloping Algebras and Geometric Representation Theory

Organised by
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March 8th – March 14th, 2009

Abstract. The meeting brought together experts investigating Lie theory from the geometric, algebraic and combinatorial points of view to discuss recent progress and bring forward the research in this area by fostering scientific interaction.

Mathematics Subject Classification (2000): 17Bxx, 20Gxx, 14Lxx.

Introduction by the Organisers

The workshop *Enveloping Algebras and Geometric Representation theory*, organized by Shrawan Kumar (Chapel Hill), Peter Littelmann (Köln) and Wolfgang Soergel (Freiburg) was held March 8th–March 14th, 2009. It continues a series of conferences on enveloping algebras, with the extension of the title indicating a direction the whole subject has taken by including with great success more and more geometric methods.

The meeting was attended by over 50 participants from all over the world, including quite a few younger researchers. The lectures covered a broad range of topics from algebraic Lie theory, with strongly interrelated focal points in the study from a geometrical and cohomological point of algebraic varieties arising in Lie theory on the one hand and the study of related combinatorial structures on the other.

We had reserved tuesday and thursday afternoon for four shorter talks each by younger participants and had one "open problem session" on thursday evening, which also was quite a success. Apart from that we had usually two talks in the morning and two in the afternoon, with the reglementary excursion on wednesday afternoon, leaving ample time for discussion among the participants.

Particularly exciting seemed to us the new results on decompositions of tensor products in the case of quantum affine algebras and its relation to cluster algebras; Bruhat graphs in representation theory and geometry; differential operators and rational Cherednik algebras; construction of semisimple tensor categories; quiver varieties and branching; GIT cones and applications; and the brand new solution of Luna's longstanding conjecture on the classification of wonderful spherical varieties.

Workshop: Enveloping Algebras and Geometric Representation Theory

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Abstracts

Hodge polynomials of complete intersections in homogeneous spaces

Michel Brion

To each complex algebraic variety X, one associates its *Hodge polynomial* e_X, a polynomial in two variables u, v, uniquely determined by the following properties:
(i) $e_X(u,v) = \sum_{p,q} \dim H^p(X, \Omega_X^q)\, u^p\, v^q$ if X is smooth and projective.
(ii) (additivity) $e_X = e_{X_1} + \cdots + e_{X_N}$ whenever X is the disjoint union of locally closed subvarieties $X_1, \ldots, X_N$.

The existence of e_X follows from mixed Hodge theory, see e.g. [5]. From (i) and (ii), we see e.g. that e_X is a symmetric polynomial in u and v with integer coefficients, and satisfies $e_X(-1,-1) = \chi(X)$ (the topological Euler characteristic).

We address the problem of determining the Hodge polynomials of complete intersections in homogeneous spaces. Specifically, let X be a homogeneous variety under a connected linear algebraic group G, and let $Y_1, \ldots, Y_m \subset X$ be smooth hypersurfaces in general position; then $Y := Y_1 \cap \cdots \cap Y_m$ is a smooth complete intersection in X. There are algorithms to compute e_Y in the cases that X is a projective space $\mathbb{P}^n$ (by work of Hirzebruch, see [4]) or a torus $(\mathbb{C}^*)^n$ (by work of Danilov and Khovanskiĭ, see [2]). We sketch how to treat the case that $X = G/H$, where $[P,P] \subset H \subset P$ for some parabolic subgroup P of G.

We begin with some reduction steps, already implicit in [4].
1) e_Y is uniquely determined by the data of e_X and of the specialization $e_Y(-1,v)$ (this follows from a generalization of the Lefschetz hyperplane theorem to “open” varieties, see [3]). Moreover, e_X has been determined in [1], e.g., if $X = G/H$ where G and H are connected, then

$$e_X(u,v) = (uv)^{\dim(U_G)-\dim(U_H)} \frac{\prod_{i=1}^r (uv)^{d_i} - 1}{\prod_{j=1}^s (uv)^{e_j} - 1}$$

where $U_G \subset G$ denotes a maximal unipotent subgroup, $d_1, \ldots, d_r$ denote the degrees of the fundamental invariants of the Weyl group of G, and likewise for H.
2) For any smooth projective variety Z of dimension n, we have

$$e_Z(-1,v) = \sum_q \chi(Z, \Omega_Z^q)\, v^q = (-1)^n \sum_q \chi(Z, \Omega_Z^{n-q})\, v^q$$

by Serre duality. More generally, if Z is smooth of dimension n, and $\bar{Z}$ is a compactification with boundary $D := \bar{Z} \setminus Z$ being a smooth normal crossing divisor, then

$$e_Z(-1,v) = (-1)^n \sum_q \chi\big(\bar{Z}, \Omega_{\bar{Z}}^{n-q}(\log D)\big)\, v^q.$$

3) If $\bar{X}$ is a compactification of X as above, such that the $\bar{Y}_i$ are still smooth and in general position, then $e_Y(-1,v)$ is uniquely determined by the values of

$\chi\big(\bar{X}, \mathcal{L} \otimes \Omega^{n-q}_{\bar{X}}(\log D)\big)$ where $\mathcal{L}$ is a monomial in the invertible sheaves $\mathcal{L}_i := \mathcal{O}_{\bar{X}}(\bar{Y}_i)$, $i = 1, \ldots, m$.

To give a closed formula for $e_Y(-1, v)$, it is convenient to set

$$P_{\bar{X},D,\mathcal{F}}(t) := \sum_q (-1)^q \, \chi\big(\bar{X}, \mathcal{F} \otimes \Omega^{n-q}_{\bar{X}}(\log D)\big)\, t^q$$

for any locally free sheaf $\mathcal{F}$ on X, so that

$$e_Y(-1, v) = P_{\bar{Y}, D\cap\bar{Y}, \mathcal{O}_{\bar{Y}}}(-v).$$

Also, note that $P_{\bar{X},D,\mathcal{F}}(t)$ only depends on the class of $\mathcal{F}$ in the Grothendieck group $K(X)$, and this defines $P_{\bar{X},D,\xi}(t)$ for any class $\xi \in K(X)$. We may now state the adjunction formula

$$P_{\bar{Y}, D\cap\bar{Y}, \mathcal{F}}(t) = P_{\bar{X},D,\mathcal{F}_t}(t)$$

where we put

$$\mathcal{F}_t := \mathcal{F} \prod_{i=1}^m \frac{1 - [\mathcal{L}_i]}{1 - t[\mathcal{L}_i]}$$

(this product is to be expanded into a power series in t; the resulting power series expansion of $P_{\bar{X},D,\mathcal{F}_t}(t)$ is in fact a polynomial).

We may now determine the Hodge polynomials of complete intersections in the full flag variety $X = G/B$, where G is a connected reductive group and B a Borel subgroup. Then X is the disjoint union of the Bruhat cells C_w, $w \in W$, so that $e_Y = \sum_{w\in W} e_{Y\cap C_w}$. Moreover, each C_w has a compactification with boundary being a smooth normal crossing divisor: the Bott-Samelson-Demazure-Hansen variety $Z_{\underline{w}}$ associated with a reduced decomposition $\underline{w}$ of w, and its boundary $D_{\underline{w}}$. Specifically, we have a morphism

$$\varphi_{\underline{w}} : Z_{\underline{w}} \to X$$

which restricts to an isomorphism

$$Z_{\underline{w}} \setminus D_{\underline{w}} \cong C_w.$$

By the above discussion, it suffices to determine the polynomials

$$P_{\underline{w},\lambda}(t) := P_{Z_{\underline{w}}, D_{\underline{w}}, \varphi^*_{\underline{w}}\mathcal{L}(\lambda)}$$

where $\mathcal{L}(\lambda)$ denotes the G-linearized invertible sheaf on G/B associated with an arbitrary character λ of B. More generally, we may consider the polynomials

$$P_{\underline{w},\xi}(t) := P_{Z_{\underline{w}}, D_{\underline{w}}, \varphi^*_{\underline{w}}\xi}$$

for any $\xi \in K(G/B)$.

These polynomials are determined recursively as follows. Write $\underline{w} = (\underline{v}, s)$ where $s \in W$ is a simple reflection. Then

$$P_{\underline{w},\xi}(t) = (t-1)\, P_{\underline{v},D_s(\xi)}(t) + P_{\underline{v},\sigma_s(\xi)}(t)$$

where $D_s : K(G/B) \to K(G/B)$ denotes the Demazure operator associated with s, and $\sigma_s : K(G/B) \to K(G/B)$ denotes the operator induced by s.

To prove that recursive formula, one uses the structure of $Z_{\underline{w}}$ as a projective line bundle over $Z_{\underline{v}}$, together with the following geometric fact:

Let X, Y be smooth projective varieties, and $f : X \to Y$ a $\mathbb{P}^1$-bundle having a section $s : Y \to X$. Let E be a smooth normal crossing divisor on Y, so that $D := s(Y) + f^*E$ is a smooth normal crossing divisor on X. Then

$$P_{X,D,\xi}(t) = (t-1)\, P_{Y,E,f_*(\xi)}(t) + P_{Y,E,\sigma(\xi)}(t)$$

for any $\xi \in K(X)$, where $f_* : K(X) \to K(Y)$ denotes the push-forward in K-theory, and $\sigma : K(X) \to K(Y)$ denotes the unique $K(Y)$-linear map such that $\sigma[\mathcal{O}_X(nY)] = [\mathcal{O}_Y]$ for any integer n.

The case that $X = G/P$ for an arbitrary parabolic subgroup $P \supset B$ reduces to the former case, by using the decomposition of G/P into Bruhat cells C_{wP} and the existence of cells $C_w \subset G/B$, isomorphic to C_{wP} via the projection $G/B \to G/P$.

Finally, if $X = G/H$ where $[P,P] \subset H \subset P$, then the projection $G/H \to G/P$ is a principal bundle under P/H, a torus acting on the right on G/H. Thus, X may be compactified by a toric bundle on G/P with fiber a smooth projective toric variety, so that the boundary is a simple normal crossing divisor. Then the following geometric fact allows one to reduce to G/P:

Let X, Y be smooth projective varieties, and $f : X \to Y$ a toric bundle with boundary ∂X. Let E be a smooth normal crossing divisor on Y, so that $D := \partial X + f^*E$ is a smooth normal crossing divisor on X. Then

$$P_{X,D,\xi}(t) = (t-1)^r\, P_{Y,E,f_*(\xi)}(t)$$

for any $\xi \in K(X)$, where $r := \dim(X) - \dim(Y)$.

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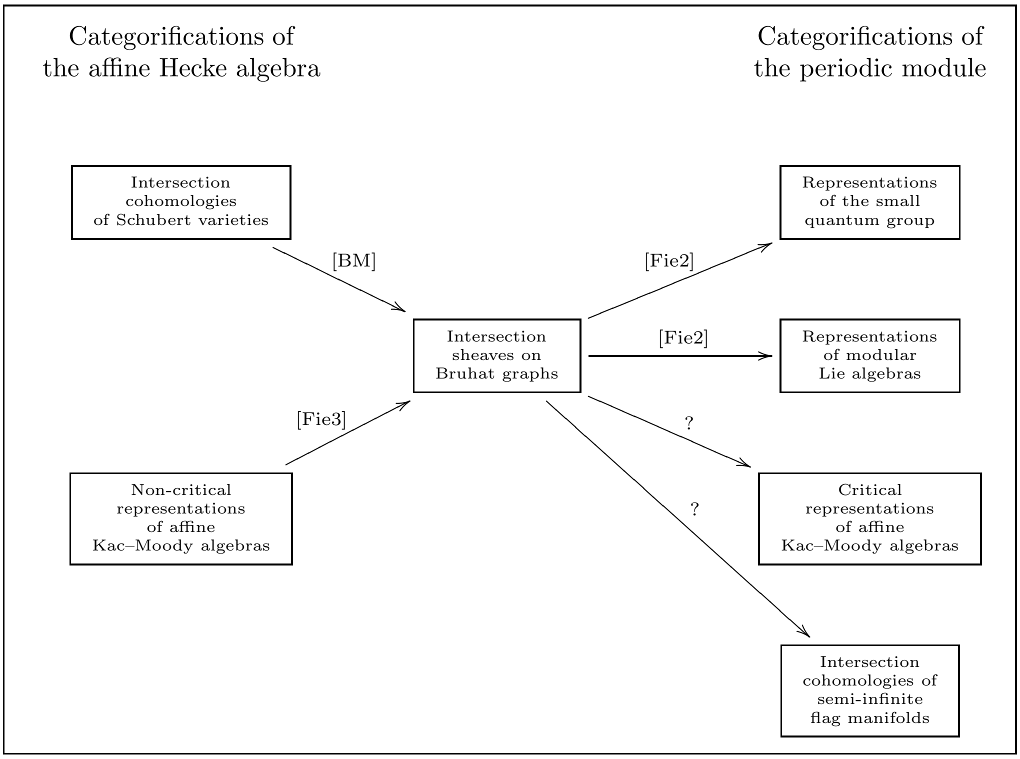
Moment graphs in representation theory and geometry

Peter Fiebig

1. Overview

To an abstract root system one can associate various objects of geometric or algebraic nature: several algebraic groups over a field k, their Lie algebras, some quantum groups, or various flag varieties. Each of these objects then gives rise to certain categories of representations or sheaves.

The main aim of my research is to show that some of these categories can be most conveniently described using the *Bruhat graph* associated to the root system or to its affinization. These descriptions then yield functors (and sometimes even equivalences) between the categories in question. As an application one can establish functors the existence of which is anticipated by the local geometric Langlands philosophy. The following table gives an overview over already established or still conjectural instances of the above approach.



2. Equivariant topology

The main ideas underlying the relations depicted above are the following. First we treat the geometric side. Let V be a complex projective algebraic variety acted upon by an algebraic torus T. Suppose that this action has only finitely many fixed points and finitely many one-dimensional orbits. Let F be a T-equivariant sheaf on V with complex coefficients (by this we mean an object in the T-equivariant derived category $D^+_T(V, \mathbb{C})$). We are interested in calculating the *equivariant cohomology* $\mathrm{H}^*_\mathrm{T}(V, F)$ of F.

For an embedding $i\colon W \to V$ of a subvariety we set $\mathrm{H}^*_\mathrm{T}(F)_W := \mathrm{H}^*_\mathrm{T}(W, i^*F)$. Let $V^T \subset V$ be the set of T-fixed points. A natural adjunction map yields the *localization map*

$$\mathrm{H}^*_\mathrm{T}(V, F) \to \bigoplus_{x \in V^T} \mathrm{H}^*_\mathrm{T}(F)_x.$$

The localization theorem (we mean the Goresky-Kottwitz-MacPherson version) states that in good situations the above map is injective and that its image is cut out by relations coming from the one-dimensional orbits in E.

Let E be a one-dimensional T-orbit. Its closure is then homeomorphic to $\mathbb{P}^1$ and picks up two T-fixed points, so $\overline{E} = E \cup \{x\} \cup \{y\}$. We get the following homomorphisms between the local equivariant cohomologies:

$$\mathrm{H}^*_\mathrm{T}(F)_x \xleftarrow{\sim} \mathrm{H}^*_\mathrm{T}(F)_{E \cup \{x\}} \to \mathrm{H}^*_\mathrm{T}(F)_E$$

(analogously for the fixed point y). The first homomorphism turns out to be an isomorphism, so by composing the inverse of the first with the second we get a map $\rho_{x,E}\colon\ \mathrm{H}^*_\mathrm{T}(F)_x \to \mathrm{H}^*_\mathrm{T}(F)_E$.

Theorem 2.1. [GKM] *Suppose that F is equivariantly formal. Then the localization map is injective and its image consists of all $(m_x) \in \bigoplus_{x \in V^T} \mathrm{H}^*_\mathrm{T}(F)_x$ with $\rho_{x,E}(m_x) = \rho_{y,E}(m_y)$ for all one-dimensional orbits E.*

An example of an equivariantly formal sheaf is the intersection cohomology sheaf on V.

3. Sheaves on moment graphs

Let us denote by $X = \mathrm{Hom}(T, \mathbb{C}^\times)$ the character lattice of the torus and by $S = S_\mathbb{C}(X \otimes_\mathbb{Z} \mathbb{C})$ the associated complex symmetric algebra. Then one has a natural action of S on $\mathrm{H}^*_\mathrm{T}(V, F)$, on $\mathrm{H}^*_\mathrm{T}(F)_x$ and on $\mathrm{H}^*_\mathrm{T}(F)_E$ for each fixed point x and each one-dimensional orbit E. All homomorphisms above respect this structure. Moreoever, if T rotates E according to the character $\lambda_E \in X$, then $\mathrm{H}^*_\mathrm{T}(F)_E$ is annihilated by λ_E. This leads to the following definitions.

To the variety V we associate the following *moment graph* $\mathcal{G}$. Its set of vertices is V^T and the edges are given by the one-dimensional orbits: We consider E as an edge connecting the two fixed points in its closure. Moreover, we remember the action of the torus by labelling E by the corresponding character $\lambda_E \in X$.

Now let k be an arbitrary field, and set $S_k := S(X \otimes_\mathbb{Z} k)$. We consider it as a graded algebra with $X \otimes_\mathbb{Z} k$ being the homogeneous component of degree 2. A *k-sheaf* $\mathbf{F}$ on $\mathcal{G}$ is given by an S_k-module $\mathbf{F}^x$ for any vertex x, an S_k-module $\mathbf{F}^E$ for any edge E with $\lambda_E \mathbf{F}^E = 0$ and a homomorphism $\rho_{x,E}\colon \mathbf{F}^x \to \mathbf{F}^E$ of S_k-modules for any vertex x lying on the edge E.

The *space of sections* $\Gamma(I, \mathbf{F})$ of a sheaf $\mathbf{F}$ on $I \subset V^T$ is defined as the set of (m_x) in $\bigoplus_{x\in I} \mathbf{F}^x$ such that $\rho_{x,E}(m_x) = \rho_{y,E}(m_y)$ for all edges E with $x, y \in I$. ¿From the constructions above we get a $\mathbb{C}$-sheaf $\mathbf{F}$ on $\mathcal{G}$ for any equivariant sheaf F on V. By the localization theorem, the equivariant cohomology of F is given by the global ($I = V^T$) sections of $\mathbf{F}$, provided F is equivariantly formal.

4. The Braden-MacPherson sheaf

Suppose now that V is endowed with a T-stable stratification. Under certain assumptions (being a Whitney stratification is one of them), Braden and MacPherson constructed in [BM] the sheaf $\mathbf{B}_{\mathbb{C}}$ on the moment graph that corresponds to the equivariant intersection cohomology complex on V (with complex coefficients). The algorithmic construction, however, makes sense for all fields k and yields a sheaf $\mathbf{B}_k$. Its stalks $\mathbf{B}_k^x$ turn out to be graded free S_k-modules of finite rank.

Let us denote by $\mathcal{G}_{\leq w}$ the graph associated to a finite or affine Schubert variety that corresponds to some w in the associated Weyl group. It can easily be constructed from the underlying finite or affine root system. Let $\mathbf{B}_k(w)$ be its Braden–MacPherson sheaf over the field k. Suppose that k is such that λ_E and $\lambda_{E'}$ are linearly independent in $X \otimes_{\mathbb{Z}} k$ for any two edges that share a common vertex (this is called the GKM-property). In this case we have the following conjecture.

Conjecture 4.1. *Let $h_{x,y}$ be the Kazhdan-Lusztig polynomial associated to the underlying Coxeter system. Then the rank of $\mathbf{B}_k(w)^x$ is $h_{x,w}(1)$.*

In [Fie3] it is shown that this conjecture is equivalent to the Kazhdan–Lusztig conjecture for complex simple Lie algebras or symmetrizable Kac–Moody algebras. In [Fie2] it is shown that the conjecture implies the conjecture of Lusztig on the irreducible characters of reductive algebraic groups for all relevant fields.

The conjecture is true if char $k = 0$ by [KL] and [BM], and for char k bigger than a certain explicit, but huge number, by [Fie4]. Moreover, the multiplicity one case holds in full generality by [Fie1]: If either the rank of $\mathbf{B}_k(w)^x$ or $h_{x,w}(1)$ is 1, then so is the other.

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Differential operators and Cherednik algebras

Iain G. Gordon
(joint work with V. Ginzburg, J.T. Stafford)

This is a report on joint work with V.Ginzburg and J.T.Stafford which will be published as [GGS].

1. **Notation.** Let $\mathfrak{S}_n$ denote the n^{th} symmetric group for some $n \geq 2$. For a parameter $c \in \mathbb{C}$ we write H_c for *the Cherednik algebra of type* $\mathfrak{S}_n$ with *spherical subalgebra* $\mathsf{U}_c = e\mathsf{H}_c e$, where $e = \frac{1}{n!}\sum_{w\in\mathfrak{S}_n} w \in \mathsf{H}_c$ is the trivial idempotent.

Let $\mathfrak{h} = \mathbb{C}^n$ denote the permutation representation of $\mathfrak{S}_n$ and write $\mathfrak{h}^{\text{reg}} = \mathfrak{h} \setminus \delta^{-1}(0)$ where $\delta = \prod_{i<j}(x_i - x_j) \in \mathbb{C}[\mathfrak{h}]$ is the discriminant; thus $\mathfrak{h}^{\text{reg}}$ is the subvariety of $\mathfrak{h}$ on which $\mathfrak{S}_n$ acts freely.

We identify U_c with a subalgebra of $\mathscr{D}(\mathfrak{h}^{\text{reg}}) \rtimes \mathfrak{S}_n$, the skew group ring of $\mathfrak{S}_n$ with coefficients in the ring of differential operators on $\mathfrak{h}^{\text{reg}}$: this follows from the Dunkl embedding of H_c into the same ring.

2. **Relation to Hilbert schemes.** For $a \in \mathbb{C}$, set

$${}_a\mathsf{P}_{a-1} = e\mathsf{H}_a\delta e \qquad \text{and} \qquad {}_{a-1}\mathsf{Q}_a = e\delta^{-1}\mathsf{H}_a e.$$

By induction, for $a \in b + \mathbb{Z}_{\geq 2}$, define

$${}_a\mathsf{P}_b = ({}_a\mathsf{P}_{a-1}) \cdot ({}_{a-1}\mathsf{P}_b) \quad \text{and} \quad {}_b\mathsf{Q}_a = ({}_b\mathsf{Q}_{b+1}) \cdot ({}_{b+1}\mathsf{Q}_a).$$

In these equations, the multiplication is taken inside $\mathscr{D}(\mathfrak{h}^{\text{reg}}) \rtimes \mathfrak{S}_n$ and this makes both ${}_a\mathsf{P}_b$ and ${}_a\mathsf{Q}_b$ into $(\mathsf{U}_a, \mathsf{U}_b)$-bimodules.

We can now construct a $\mathbb{Z}$*-algebra* $B_c = \bigoplus_{i\geq j\geq 0}\big({}_{c+i}\mathsf{P}_{c+j}\big)$ endowed with a natural matrix multiplication. The ring B_c has a natural filtration induced from the differential operator filtration on $\mathscr{D}(\mathfrak{h}^{\text{reg}}) \rtimes \mathfrak{S}_n$ and the main result [GS1] showed that for most c (and we know exactly which) the associated graded ring $\text{gr}B_c$ of B_c is the $\mathbb{Z}$-algebra that can be associated to the Hilbert scheme of n points on the plane, $\text{Hilb}^n\,\mathbb{C}^2$. This provides a bridge between Cherednik algebras and Hilbert schemes: H_c- or U_c-modules with good filtrations produce coherent sheaves on $\text{Hilb}^n\,\mathbb{C}^2$.

3. **Relation to $\mathscr{D}$-modules.** There is a second way of passing from H_c to a more geometric setting using hamiltonian reduction. Write $V = \mathbb{C}^n$, $\mathbb{P} = \mathbb{P}(V)$ and $\mathfrak{g} = \mathfrak{gl}(V)$; set $\mathfrak{G} = \mathfrak{g} \times V$ and $\mathfrak{X} = \mathfrak{g} \times \mathbb{P}$. There is an action of $GL(V)$ on both $\mathfrak{G}$ and $\mathfrak{X}$ which differentiates to produce a Lie algebra homomorphism $\mathfrak{g} \longrightarrow \mathsf{Vect}(\mathfrak{G})$ and thus an algebra mapping $\tau : U(\mathfrak{g}) \longrightarrow \mathscr{D}(\mathfrak{G})$. Now [GG] shows that

$$\mathsf{U}_c \cong \left(\frac{\mathscr{D}(\mathfrak{G})}{\mathscr{D}(\mathfrak{G}) \cdot \tau(I_{c+1})}\right)^{GL(V)}$$

where I_{c+1} is the ideal of $U(\mathfrak{g})$ generated by the elements $Y - (c+1)\text{tr}(Y)$ for all $Y \in \mathfrak{g}$. One then constructs the *functor of quantum hamiltonian reduction*

$$\mathbb{H}_c : (\mathscr{D}_{c+1}(\mathfrak{X}), SL(V))\text{-}\mathsf{mod} \longrightarrow \mathsf{U}_c\text{-}\mathsf{mod}, \qquad F \mapsto F^{SL(V)}$$

where $(\mathscr{D}_{c+1}(\mathfrak{X}), SL(V))$-$\mathsf{mod}$ denotes the category of $SL(V)$-equivariant modules for $(\mathscr{D}(\mathfrak{G})/\mathscr{D}(\mathfrak{G})\cdot\tau(\mathrm{Id}-n(c+1)))^{\mathbb{C}^\times}$ on which the $\mathfrak{sl}(V)$-action obtained by differentiating the $SL(V)$-action agrees with the action induced by τ. This approach connects the representation theory of H_c or U_c to differential operators on representation varieties.

4. **Main Theorem.** Our results compare these two approaches to the representation theory of H_c and U_c.

For $m \in \mathbb{Z}$, consider the space of semi-invariants

$$\mathfrak{D}_{c+1}^{\det^{-m}} = \{D \in \mathscr{D}(\mathfrak{G})/\mathscr{D}(\mathfrak{G})\tau(I_{c+1}) : g\cdot D = \det(g)^{-m}D \text{ for all } g \in GL(V)\}.$$

It is easy to check that $\mathfrak{D}_{c+1}^{\det^{-m}}$ is a $(\mathsf{U}_{c-m},\mathsf{U}_c)$-bimodule.

Theorem. *Fix $c\in\mathbb{C}$ and an integer $m \geq 1$ (with some explicit mild restriction on these parameters). Under the differential operator filtration on the two sides there is a filtered $(\mathsf{U}_{c-m},\mathsf{U}_c)$-bimodule isomorphism*

$$\Theta_{c,m} : \mathfrak{D}_{c+1}^{\det^{-m}} \xrightarrow{\sim} {}_{c-m}\mathsf{Q}_c.$$

There is also a description of ${}_c\mathsf{P}_{c-m}$ in terms of the $\mathfrak{D}_{d+1}^{\det^{-m}}$ for some d.

To prove this one first shows that, like ${}_{c-m}\mathsf{Q}_c$, the $(\mathsf{U}_{c-m},\mathsf{U}_c)$-bimodule $\mathfrak{D}_{c+1}^{\det^{-m}}$ is naturally embedded into $\mathsf{U}^{reg} = \mathsf{U}_c[\delta^{-2}]$. Now both of these bimodules are reflexive on at least one side (which requires the mild restriction on c and m), and the theorem is then proved by showing that such a bimodule is unique.

5. As a corollary of this theorem, we are able to give a direct and relatively short proof of one of the main results in [GS1] mentioned above in 2; previously this had previously relied on many key ingredients of Haiman's proof of the $n!$ theorem.

6. Characteristic cycles. A useful tool in the study of Cherednik algebras, just as for Lie algebras, is the concept of the characteristic cycle of a U_c-module. There are two completely different constructions of characteristic cycles of U_c-modules on $\mathrm{Hilb}^n\,\mathbb{C}^2$. The first, ch^{GS}, uses the $\mathbb{Z}$-algebra approach in 2: one starts with a U_c-module M with a good filtration and then produces a coherent sheaf on $\mathrm{Hilb}^n\,\mathbb{C}^2$. Although this sheaf depends on the choice of filtration on M, its cycle, $\mathsf{ch}^{GS}(M)$, does not. The second, ch^{GG}, is defined using the machinery of hamiltonian reduction. Starting again with a U_c-module M with a good filtration, one applies the adjoint functor to $\mathbb{H}_c$ to produce an $SL(V)$-equivariant $\mathscr{D}_{c+1}(\mathfrak{X})$-module with a good filtration. Taking the characteristic cycle of this module produces a cycle in $T^*\mathfrak{X}$ and applying classical hamiltonian reduction to this then produces $\mathsf{ch}^{GG}(M)$, a cycle in $\mathrm{Hilb}^n\,\mathbb{C}^2$.

We prove that these two constructions agree, thereby confirming a conjecture from [GG, (7.17)].

Theorem. *Assume that $n > 2$ and that $c \in \mathbb{C}\smallsetminus\mathbb{Q}_{<0}$. Then for any finitely generated U_c-module M one has an equality of algebraic cycles $\mathsf{ch}^{GS}(M) = \mathsf{ch}^{GG}(M)$.*

The key ingredient for the proof of this theorem is a comparison of the "shift functors" for rational Cherednik algebras and the effect on $\mathscr{D}$-modules of tensoring by the line bundles $\mathcal{O}_{\mathbb{P}}(nm)$ on $\mathfrak{X}$.

7. As a corollary of this theorem, we are able to give a complete description of the characteristic cycle of *any* object of the category $\mathcal{O}$ for H_c.

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Hilbert schemes of $\mathbb{C}^2$ and Elliptic Hall algebras

Olivier Schiffmann

(joint work with E. Vasserot)

We constuct a certain natural action of the spherical Double Affine Hecke algebra $S\ddot{H}_\infty$ of $\mathrm{GL}(\infty)$ on the equivariant K-theory of the Hilbert scheme of points in $\mathbb{C}^2$. This can be seen as a generalization, from homology to K-theory, of a classical result of Nakajima computing the homology of $\mathrm{Hilb}(\mathbb{C}^2)$ by means of a geometric action of a Heisenberg algebra. We relate the action of $S\ddot{H}_\infty$ on $K^T(\mathrm{Hilb}_n)$ with the action of the "virtual classes" of the natural correspondences $Z_{n,n\pm k} \subset \mathrm{Hilb}_n \times \mathrm{Hilb}_{n\pm k}$. Finally we explain an interpretation of our isomorphism in terms of the geometric Langlands conjecture for an elliptic curve, in the neighbourhood of the trivil local system.

Yangians and cohomology rings of instanton moduli spaces

Michael Finkelberg

(joint work with Boris Feigin, Andrei Negut, Leonid Rybnikov)

The moduli spaces $\mathfrak{Q}_{\underline{d}}$ were introduced by G. Laumon in [6] and [7]. They are certain partial compactifications of the moduli spaces of degree $\underline{d}$ based maps from $\mathbb{P}^1$ to the flag variety $\mathcal{B}_n$ of GL_n. In [2] we have studied the equivariant cohomology ring $H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}})$ where $\widetilde{T}$ is a Cartan torus of GL_n acting naturally on the target $\mathcal{B}_n$, and $\mathbb{C}^*$ acts as "loop rotations" on the source $\mathbb{P}^1$. The method of [2] was to introduce an action of $U(\mathfrak{gl}_n)$ on $V = \bigoplus_{\underline{d}} H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}}) \otimes_{H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt)} \mathrm{Frac}(H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(pt))$ by certain natural correspondences, and then to realize the cohomology ring $H^\bullet_{\widetilde{T}\times\mathbb{C}^*}(\mathfrak{Q}_{\underline{d}})$ as a certain quotient of the Gelfand-Tsetlin subalgebra $\mathfrak{A} \subset U(\mathfrak{gl}_n)$.

In this talk we adopt the following approach to the Gelfand-Tsetlin subalgebra going back to I. Cherednik. Namely, $\mathfrak{A}$ is the image of the maximal commutative subalgebra $\mathfrak{A}$ of the Yangian $Y(\mathfrak{gl}_n)$ (Gelfand-Tsetlin subalgebra) under the evaluation homomorphism to $U(\mathfrak{gl}_n)$ (see [8]). Composing the evaluation homomorphism $Y(\mathfrak{gl}_n)$ to $U(\mathfrak{gl}_n)$ with the action of $U(\mathfrak{gl}_n)$ on V we obtain an action of $Y(\mathfrak{gl}_n)$ on V. The main observation of this talk is that the "new Drinfeld generators" [1] of $Y(\mathfrak{sl}_n) \subset Y(\mathfrak{gl}_n)$ act on V by natural correspondences. In fact they are very similar to the correspondences used by M. Varagnolo [11] to construct the action of Yangians in the equivariant cohomology of quiver varieties.

There is an affine version of the Laumon spaces, namely the moduli spaces $\mathcal{P}_{\underline{d}}$ of parabolic sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$, see [3]. The similar correspondences give rise to the action of the affine Yangian $\widehat{Y}$ (two-parametric deformation of the universal enveloping algebra of the universal central extension of $\mathfrak{sl}_n[s^{\pm 1}, t]$, see [5]) on the localized equivariant cohomology $M = \bigoplus_{\underline{d}} H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{P}_{\underline{d}}) \otimes_{H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(pt)} \mathrm{Frac}(H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(pt))$ where the second copy of $\mathbb{C}^*$ acts by the loop rotation on the second copy of $\mathbb{P}^1$. We compute explicitly the action of Drinfeld generators of $\widehat{Y}$ in the fixed point basis of M.

Since the fixed point basis of V corresponds to the Gelfand-Tsetlin basis of the universal Verma module over $U(\mathfrak{gl}_n)$, we propose to call the fixed point basis of M the *affine Gelfand-Tsetlin basis.* In particular, we conjecture that M is isomorphic to the universal Verma module over $U(\widehat{\mathfrak{gl}}_n)$. Moreover, we expect that the specialization of the affine Gelfand-Tsetlin basis gives rise to a basis in the integrable $\widehat{\mathfrak{gl}}_n$-modules (which we also propose to call the affine Gelfand-Tsetlin basis). The set of affine Gelfand-Tsetlin patterns has a structure of $\widehat{\mathfrak{sl}}_n$-crystal of the integrable $\widehat{\mathfrak{gl}}_n$-module, equivalent to that of cylindric plane partitions [9]. We expect that the action of $\widehat{Y}$ on the integrable $\widehat{\mathfrak{gl}}_n$-modules coincides with D. Uglov's Yangian action [10].

We prove that the maximal commutative subalgebra of Cartan currents $\mathfrak{A}_{\mathrm{aff}} \subset \widehat{Y}$ (the affine Gelfand-Tsetlin algebra) surjects onto the cohomology ring of $\mathcal{P}_{\underline{d}}$. Furthermore, let $\mathfrak{M}_{n,d}$ denote the moduli space of torsion free sheaves of rank n and second Chern class d, trivialized at infinity. The equivariant cohomology ring $H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathfrak{M}_{n,d})$ is naturally a subring of $H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{P}_{d,\ldots,d})$. There is a natural embedding $Y(\mathfrak{gl}_n) \hookrightarrow \widehat{Y}$ which realizes the center $ZY(\mathfrak{gl}_n)$ as a subalgebra of $\mathfrak{A}_{\mathrm{aff}}$. This subalgebra surjects onto the cohomology ring $H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathfrak{M}_{n,d}) \subset H^\bullet_{\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*}(\mathcal{P}_{d,\ldots,d})$. In particular, the first Chern class of the determinant line bundle Δ_0 on $\mathfrak{M}_{n,d}$ is expressed as a certain noncommutative symmetric function (a power sum of the second kind, see [4]) $\Phi \in ZY(\mathfrak{gl}_n)$.

Our results are only proved when $n > 2$; however we expect them to hold for $n = 2$ as well, and it is instructive to compare them with the known results for $n = 1$. In this case $\mathfrak{M}_{n,d}$ is the Hilbert scheme $\mathrm{Hilb}^d(\mathbb{A}^2)$. The first Chern class of the determinant line bundle on $\mathrm{Hilb}^d(\mathbb{A}^2)$ was computed by M. Lehn as a certain infinite cubic expression (Calogero-Sutherland operator) of the generators of the

Heisenberg algebra acting by correspondences between Hilbert schemes. In our case the role of the Heisenberg algebra is played by $U(\widehat{\mathfrak{gl}}_n)$, and we were unable to express $c_1(\Delta_0)$ in terms of $U(\widehat{\mathfrak{gl}}_n)$, but there is an explicit formula for it in terms of $ZY(\mathfrak{gl}_n)$.

Finally, let us mention a trigonometric version of our note where the (affine) Yangian is replaced with the (toroidal) affine quantum group, and the equivariant cohomology is replaced with the equivariant K-theory. This is the subject of the preprint arXiv math/0903.0917 by A. Tsymbaliuk.

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Quantum cohomology of the Springer resolution

Alexander Braverman

We compute the equivariant quantum $\mathcal{D}$-module of the space $T^*\mathcal{B}$ where $\mathcal{B}$ is the flag variety of a semi-simple group G. The answer turns out to be related to the Calogero-Moser-Sutherland integrable system for the Langlands dual group $G^\vee$. We then recover the quantum $\mathcal{D}$-module of $\mathcal{B}$ itself by a limiting procedure (the quantum $\mathcal{D}$-module of $\mathcal{B}$ ist known to be related to the Toda integrable system, which is known to be a limit of the Calogero-Moser-Sutherland system).

On a construction of semisimple categories

Friedrich Knop

In [1], Deligne constructed a tensor category $\operatorname{Rep}(S_t)$ which depends on a free variable t which interpolates the representation categories of the symmetric groups S_n in the following sense:

Let $t = n \in \mathbb{N}$. Then $\operatorname{Rep}(S_n)$ is the quotient of $\operatorname{Rep}(S_t)|_{t=n}$ by its tensor radical.

Furthermore, for values of t which are not natural numbers, Deligne proved that $\operatorname{Rep}(S_t)$ itself is a semisimple tensor category. In this case, $\operatorname{Rep}(S_t)$ is an example of an abelian tensor category which is not Tannakian, i.e., does not come from any algebraic group.

In the talk, we presented a generalization of Deligne's construction. More precisely, in [2] we constructed many more non-Tannakian semisimple tensor categories, among which is one which interpolates $\operatorname{Rep}(\operatorname{GL}(n, \mathbb{F}_q))$, for fixed q and all $n \in \mathbb{N}$.

We start with a regular category $\mathcal{A}$. This means that $\mathcal{A}$ has all finite limits, every morphism has an image, and that images commute with pull-backs. Then one can define the category $\operatorname{Rel}(\mathcal{A})$ of *relations* in $\mathcal{A}$: the objects stay the same but a morphism from x to y is a subobject of $x \times y$. The composition of two relations $r \hookrightarrow x \times y$ and $s \hookrightarrow y \times z$ is defined as $r \circ s := \operatorname{image}(r \times_y s \to x \times z)$.

Now we modify this classical construction by using a degree function δ which assigns to every epimorphism e in $\mathcal{A}$ a number $\delta(e) \in \mathcal{C}$. This function should satisfy: the degree of an identity morphism is 1, the degree is multiplicative under composition, and the degree is invariant under pull-back. Then we define a new category $\mathcal{T}^0(\mathcal{A}, \delta)$ with the same objects as $\mathcal{A}$, the morphisms are $\mathcal{C}$-linear combinations of relations and the composition of two relations is modified to

$$r \cdot s := \delta(e)\; r \circ s$$

where e is the epimorphism $r \times_y s \to r \circ s$. Finally, $\mathcal{T}(\mathcal{A}, \delta)$ is obtained from $\mathcal{T}^0(\mathcal{A}, \delta)$ by first formally adjoining all direct sums and then adjoining all direct summands (the so-called pseudo-abelian closure).

Example: Let $\mathcal{A}$ be the category which is *opposite* to the category of finite sets. Then all degree functions are of the form $\delta(A \hookrightarrow B) = t^{|B \setminus A|}$ and $\mathcal{T}(\mathcal{A}, \delta)$ coincides with Deligne's category $\operatorname{Rep}(S_t)$.

Our main result is

Theorem 1. *Let $\mathcal{A}$ be a regular category and δ a degree function on $\mathcal{A}$. Then $\mathcal{T}^($$\mathcal{A}, \delta)$ is a semisimple (hence abelian) tensor category provided*

- *$\mathcal{A}$ is subobject finite, exact, and Mal'cev,*
- *$\omega_e \neq 0$ for all indecomposable epimorphisms e of $\mathcal{A}$.*

Explanation of terms: 1. The category $\mathcal{A}$ is subobject finite, if every object has only finitely many subobjects. This condition is necessary to make Hom-spaces finite dimensional. The category is exact if every equivalence relation $r \hookrightarrow x \times x$ has a quotient x/r. Moreover, $\mathcal{A}$ is Mal'cev if every reflexive relation is already an equivalence relation. Examples of such categories are the category $\{\text{finite sets}\}^{\text{op}}$, the category of finite dimensional $\mathbb{F}_q$-vector spaces, the category of finite groups and many more.

2. Let $e : x \to y$ be an epimorphism. The e is indecomposable if it is not the composition of two non-invertible epimorphism. Furthermore, ω_e is defined as

$$\omega_e := \sum_u \mu(u, x)\delta(e|_u) \in \mathcal{C}$$

where u runs through all subobjects of x such that $e|_u : u \to y$ is an epimorphism. Moreover, μ is the Möbius function of the poset of subobjects of x.

Example: Let $\mathcal{A} = \{\text{finite sets}\}^{\text{op}}$. Then the epimorphisms are the injective maps $e : A \hookrightarrow B$ between finite sets. The map e is indecomposable if $B = e(A) \cup \{b\}$ with $b \notin e(A)$. The subobject u corresponds to a surjective map $p : B \to C$. The condition on u means that $u|_{e(A)}$ is injective. Thus either $A \to C$ ($|A|$ cases) or $B \to C$ (1 case) is an isomorphism. We conclude $\omega_e = t - |A|$. Therefore, the second condition means $t \notin \mathbb{N}$ in accordance with Deligne's result.

The proof of Theorem 1 also yields a description of the simple objects:

Theorem 2. *Let $\mathcal{A}$, δ as in Theorem 1. Then the isoclasses of simple objects of $\mathcal{T}(\mathcal{A}, \delta)$ are classified by isoclasses of pairs (x, π) where x is an object of $\mathcal{A}$ and π is an irreducible representation of $\mathrm{Aut}_{\mathcal{A}}(x)$.*

Corollary. *There is an isomorphism*

$$K(\mathcal{T}(\mathcal{A}, \delta)) \cong \bigoplus_{x \in \mathrm{Ob}\mathcal{A}/\sim} \mathcal{R}(\mathrm{Aut}_{\mathcal{A}}(x)).$$

where $K(\cdot)$ denotes the Grothendieck group of an abelian category and $\mathcal{R}(\cdot)$ is the representation ring of a group.

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On a conjecture of Mirkovic and Vilonen

Daniel Juteau

1. The nilpotent cone of $\mathfrak{sl}_2$

First we explained the notions of constructible k-complexes, perverse sheaves and intersection cohomology complexes on a complex algebraic variety X, where k is any noetherian commutative ring of finite global dimension, as $\mathbb{C}$, $\mathbb{Z}$ or $\overline{\mathbb{F}}_p$.

Over $\mathbb{Z}$, there is an additional subtlety due to the presence of torsion, which does not behave well under duality: the derived dual of the torsion $\mathbb{Z}$-module $\mathbb{Z}/n$ is $\mathbb{Z}/n[-1]$. Consequently, we have two versions of perverse sheaves (which are the hearts of two t-structures) exchanged by the duality: the classical one, and a "+ version" where we use a truncation $\tau_{\leq -1_+}$ for which we keep the torsion part of the following cohomology degree 0 (this is explained at length in [Jut09]).

To illustrate these notions, we computed the intersection cohomology complex of the nilpotent cone $\mathcal{N}$ of $\mathfrak{g} = \mathfrak{sl}_2$, which has a simple surface singularity of type A_1. Indeed, we have

$$\mathfrak{g} = \mathfrak{sl}_2 \supset \mathcal{N} = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \middle| x^2 + yz = 0 \right\} = \mathcal{O}_{\mathrm{reg}} \cup \{0\} \simeq \mathbb{C}^2/\{\pm 1\}$$

where $\mathcal{O}_{\mathrm{reg}}$ denotes the regular nilpotent orbit. The stalks of $\mathcal{IC}(\mathcal{N}, \mathbb{Z})$, resp. $\mathcal{IC}^+(\mathcal{N}, \mathbb{Z})$, are given by

	-2	-1	0
$\mathcal{O}_{\mathrm{reg}}$	$\mathbb{Z}$	0	0
$\{0\}$	$\mathbb{Z}$	0	$(\mathbb{Z}/\mathbf{2})_+$

where $(\mathbb{Z}/2)_+$ means 0 for $\mathcal{IC}$, and $\mathbb{Z}/2$ for $\mathcal{IC}^+$. For k a field of characteristic p, we deduce that the stalks of $\mathcal{IC}(\mathcal{N}, k)$ are given by

	-2	-1	0
$\mathcal{O}_{\mathrm{reg}}$	k	0	0
$\{0\}$	k	$(k)_2$	0

where $(k)_2$ means k if $p = 2$, and 0 otherwise. This calculation has a representation theoretic interpretation: using the Fourier-Deligne transform version of Springer correspondence (both the classical one and a modular version), one can show that the decomposition matrix of a Weyl group (here $\mathfrak{S}_2$) is a submatrix of a decomposition matrix for equivariant perverse sheaves on the nilpotent cone [Jut07]. Here the decomposition matrices are

$$\begin{matrix} & (1^2) & (2) \\ (1^2) & \mathbf{1} & 0 \\ (2) & \mathbf{1} & 1 \end{matrix} \quad \text{for } \mathbf{Perv}_{GL_2}(\mathcal{N}), \quad \text{and} \quad \begin{matrix} & D_{(2)} \\ S_{(2)} & \mathbf{1} \\ S_{(1^2)} & \mathbf{1} \end{matrix} \quad \text{for } \mathfrak{S}_2.$$

2. The geometric Satake correspondence

Let $G \supset B \supset T$ be a split (simple, simply connected and) connected reductive group scheme over $\mathbb{Z}$, a split Borel subgroup and a split maximal torus, $X(T) \supset X(T)^+$ the weight lattice and the dominant weights, $\Phi \supset \Phi^+ \supset \Delta$ the root system, the positive roots and the simple roots. We have canonical morphisms

$\Delta(\lambda) \to \nabla(\lambda)$ from standard to costandard representations, for $\lambda \in X(T)^+$. For $k = \overline{\mathbb{F}}_p$, the simple G_k-modules are the $L(\lambda) = \mathrm{Im}(\Delta(\lambda)_k \to \nabla(\lambda)_k)$ and one wants to know the multiplicities $d^G_{\lambda,\mu} = [\Delta(\lambda)_k : L(\mu)]$. Lusztig gave a famous conjecture for p greater than the Coxeter number, which was proved by Andersen-Jantzen-Soergel for large p (unknown bound depending on the type), and by Fiebig using moment graphs (see his talk) with an explicit, but still very large bound. It is trivial that this multiplicity is 1 if $\lambda = \mu$, and 0 unless $\lambda \geq \mu$, that is, $\lambda - \mu \in \mathbb{N}\Delta$.

On the geometric side, we consider the complex affine Grassmannian $\mathcal{G}r := G^\vee(\mathcal{K})/G^\vee(\mathcal{O})$ of the dual Langlands group $G^\vee$, where $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. The $G^\vee(\mathcal{O})$-orbits are parametrized by $X(T)^+$. We denote them by $\mathcal{G}r_\lambda$, $\lambda \in X(T)^+$. The affine Grassmannian is an ind-scheme, direct limit of the finite dimensional projective varieties $\overline{\mathcal{G}r_\lambda}$. We consider the category $\mathbf{Perv}_{G^\vee(\mathcal{O})}(\mathcal{G}r, k)$ of $G^\vee(\mathcal{O})$-equivariant perverse sheaves with k coefficients on $\mathcal{G}r$. There are canonical morphisms $\mathcal{J}_!(\lambda, k) \to \mathcal{J}_*(\lambda, k)$ between standard and costandard objects and if k is a field, the simple objects are the $\mathcal{J}_{!*}(\lambda, k) = \mathrm{Im}(\mathcal{J}_!(\lambda, k) \to \mathcal{J}_*(\lambda, k))$.

Theorem 2.1 (Mirkovic-Vilonen). *We have a equivalence of tensor categories*

$$(G_k\text{-}mod, \otimes_k) \simeq (\mathbf{Perv}_{G^\vee(\mathcal{O})}(\mathcal{G}r, k), *)$$

where $*$ is a convolution product. Under this equivalence, the morphisms between standard and costandard objects correspond.

Besides, we have $\mathcal{J}_!(\lambda, \mathbb{Z}) \xrightarrow{\sim} \mathcal{J}_{!*}(\lambda, \mathbb{Z})$ and $\mathcal{J}_*(\lambda, \mathbb{Z}) \xrightarrow{\sim} \mathcal{J}^+_{!*}(\lambda, \mathbb{Z})$, and the standard and costandard objects over k are obtained from the standard and costandard objects over $\mathbb{Z}$ by applying the functor $k \otimes^L_{\mathbb{Z}} -$.

In particular, for $k = \overline{\mathbb{F}}_p$, we have $d^G_{\lambda\mu} = d^{\mathcal{G}r}_{\lambda\mu} := [\mathcal{J}_{!*}(\lambda, \mathbb{Z}) \otimes^L_{\mathbb{Z}} k : \mathcal{J}_{!*}(\mu, k)]$, so that the modular representation theory of G is encoded in the singularities of $\mathcal{G}r$. For example, a Levi lemma for decomposition numbers for reductive groups follows from an equivalence of singularities proved in [MOV05].

The IC stalks are a refinement of weight multiplicities. They are known for $k = \mathbb{C}$ (Kazhdan-Lusztig polynomials). If we knew them for $k = \overline{\mathbb{F}}_p$, then we could solve the big problem. All the information is contained in the stalks and costalks of $\mathcal{J}_!(\lambda, \mathbb{Z}) = \mathcal{J}_{!*}(\lambda, \mathbb{Z})$.

Conjecture 2.2 (Mirkovic-Vilonen). *The stalks of $\mathcal{J}_{!*}(\lambda, \mathbb{Z})$ are torsion-free.*

In type A, the singularities of $\mathcal{G}r$ are nilpotent singularities, and they checked that there is no torsion in IC stalks of nilpotent singularities up to $\mathfrak{sl}_6$. Bezrukavnikov noticed that this conjecture would imply a straightforward extension of his work on the unramified local geometric Langlands conjecture, from $\mathbb{C}$ to $\mathbb{Z}$ coefficients. However, we will see that **the conjecture is not true as stated** [Jut].

3. Minimal degenerations

It is hard to compute IC stalks in general, but the case of minimal degenerations is more tractable (it reduces to the case of an isolated singularity).

We say $\lambda > \mu$ is a minimal degeneration if there is no $\nu \in X(T)^+$ in between, and we write $\lambda \rightsquigarrow \mu$. We denote by $\overline{\alpha}$ the highest short root. By a theorem of Stembridge, if $\lambda \rightsquigarrow \mu$ then $\mathfrak{b} := \lambda - \mu \in \Phi^+$, and up to some Levi reduction, we have (with Bourbaki's numbering) either (1) $\mathfrak{b} \in \Delta$; (2) $\mathfrak{b} = \overline{\alpha}$ and $\mu = 0$, in any type; (3) $\mathfrak{b} = \overline{\alpha}$ and $\mu = \varpi_n$, in type B_n; (4) $\mathfrak{b} = \alpha_1 + \alpha_2$ and $\mu = 2\varpi_1$, in type G_2; (5) $\mathfrak{b} = \alpha_1 + \alpha_2$ and $\mu = \varpi_1$, in type G_2.

Theorem 3.1 (Malkin-Ostrik-Vybornov)**.** *The singularity* $\mathrm{Sing}(\overline{\mathcal{G}r_\lambda}, \mathcal{G}r_\mu)$ *is a simple surface singularity of type* $A_{\langle\lambda,\mathfrak{b}^\vee\rangle-1}$ *in case* (1)*, and a minimal nilpotent orbit closure singularity of the type of* $G^\vee$ *in case* (2)*, denoted by* $a_n, \ldots, g_2$*.*

The case of a singularity of type A_{m-1} is similar to the case of a singularity of type A_1, but with C_m instead of C_2. There is a torsion stalk $\mathbb{Z}/m$ for $\mathcal{IC}^+$, but for $\mathcal{IC}$ there is no torsion.

For minimal singularities, one has to compute the cohomology of the minimal nilpotent orbit of a simple Lie algebra [Jut08]. This can be expressed in terms of root combinatorics. The middle cohomology group allows to recover a known decomposition number, but most importantly there is torsion in other places (in all types but in type A), so that we have counter-examples to the conjecture of Mirkovic and Vilonen. However, there is torsion only for bad primes, so **one can still hope it is true if we replace $\mathbb{Z}$ coefficients by $\mathbb{Z}_p$ for p good**.

In the other cases, the singularity is called a quasi-minimal singularity and denoted by ac_n (resp. ag_2, cg_2), because it occurs in the affine Grassmannian of type C_n (resp. G_2), and has the same IC stalks as a_n (resp. a_2, c_2) over $\mathbb{Q}$.

Theorem 3.2. [Jut] *The following singularities are pairwise non-equivalent:* a_n *and* ac_n*;* a_2*,* ac_2 *and* ag_2*;* c_2 *and* cg_2*.*

Proof. In each case, there is some prime p for which the corresponding decomposition numbers for G are different, hence these singularities have different IC stalks with $\mathbb{F}_p$ coefficients. □

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Wonderful varieties and invariant Hilbert schemes

Stéphanie Cupit-Foutou

We show how to prove Luna's conjecture saying that wonderful varieties can be classified by combinatorial triples: the spherical systems. The proof we discuss relies on the use of some invariant Hilbert schemes.

Notation. Let G be a connected reductive algebraic group. Fix as usual a Borel subgroup B and a maximal torus $T \subset B$ of G.

1. Wonderful varieties

Definition 1. A wonderful G-variety of rank r is a smooth and projective algebraic variety endowed with an action of G such that

(1) X has an open G-orbit whose complement equals a finite union of smooth prime divisors D_i $(i = 1, \ldots, r)$ which intersect transversally;
(2) orbit closures are given by the partial intersections $\cap_{i \in I} D_i$, I being a subset of $\{1, \ldots, r\}$.

After Luna (*see* [8]), one can attach three invariants to any wonderful G-variety X: a set S_X^p of some simple roots of G, a set Σ_X of *spherical roots* and a Cartan pairing.

The subset S_X^p is defined as the subset of simple roots attached to the standard parabolic corresponding to the (unique) closed G-orbit of X. The set Σ_X is defined as the set of characters of the torus T of T_zX/T_zY, where T_zY stands for the tangent space at z of the closed G-orbit Y, the point z being the (unique) point of X fixed y the opposite Borel subgroup B^-. Finally, let Δ_X be the set of prime B-stable but not G-stable divisors of X. The set Δ_X forms a basis of the Picard group of X. The pairing $c : \Delta_X \times \Sigma_X \to \mathbb{Z}$ is defined by the identies $[D_\sigma] = \sum_{D \in \Delta_X} c(D, \sigma)[D]$ where D_σ is the prime divisor of X associated to the spherical root σ.

In case of group compactifications, the spherical roots are given as sum of simple roots $\alpha_i + \alpha_i'$ and the Cartan pairing corresponds to that of the Cartan matrix.

Luna proved that such triples enjoy nice properties: they are *spherical systems.* He thus conjectured that there corresponds a unique wonderful variety to any spherical system. Partially results were obtained previously by case-by-case considerations; *see* [8, 5, 2, 3]. The uniqueness part was obtained in full generality by Losev in [7] using Luna-Vust theory.

We shall follow another approach.

2. Invariant Hilbert schemes

We gather in this section notions and results of [1].

Given a finite set Γ of dominant weights $\lambda_1, \ldots, \lambda_s$, consider the finite dimensional G-module

$$V = V(\lambda_1) \oplus \ldots \oplus V(\lambda_s)$$

where $V(\lambda_i)$ is the irreducible G-module corresponding to λ_i.

Consider the functor which assigns to any scheme S (endowed with the trivial action of G) the following set of families $\pi : \mathcal{X} \to S$ such that

$$\pi_* \mathcal{O}_{\mathcal{X}} \cong \oplus_{\lambda \in \Gamma} \mathcal{F}_\lambda \otimes V(\lambda)^* \quad \text{as } \mathcal{O}_S - G\text{-modules}$$

where $\mathcal{F}_\lambda$ denotes an invertible sheaf.

This functor is representable by a quasiprojective scheme, the *invariant Hilbert scheme* Hilb^G_Γ.

Let X_0 be the G-orbit closure within V of $v_{\underline{\lambda}} = v_{\lambda_1} + \ldots + v_{\lambda_s}$ where v_{λ_i} denotes a highest weightvector of weight λ_i.

Then X_0 can be regarded as a closed point of Hilb^G_Γ. Furthermore, Hilb^G_Γ is endowed with an action of the adjoint torus T_{ad} of G; under this action it has finitely many orbits and X_0 is its unique fixed point.

To motivate the use of invariant Hilbert schemes to solve Luna's conjecture, let us recall the following result obtained in [6].

For an arbitrary wonderful variety X, one may consider its total coordinate ring

$$R(X) = \oplus_{(n_D) \in \mathbb{Z}^{\Delta_X}} H^0(X, \mathcal{O}_X(\sum_{D \in \Delta_X} n_D D)).$$

The quotient morphism $\mathrm{Spec}R(X) \to \mathrm{Spec}R(X)^G$ is a flat family. Moreover the (scheme-theoretic) fibers of q are normal varieties; the fiber $\tilde{X}_0$ over 0 is the unique horospherical variety and the fibers of q realize a deformation of $\tilde{X}_0$.

3. How to prove Luna's conjecture

Take a spherical system $\mathcal{S} = (S^p, \Sigma, \mathsf{A})$ of the group G. By [8], it is enough to consider a peculiar class of spherical systems, that are *cuspidal* and *primitive*. We claim that we can also reduce ourselves to the case where Σ does not contain any *loose* spherical root.

Let $\mathbb{G} = G \times T_{\mathsf{A}}$ where T_{A} denotes the torus whose charactergroup is spanned by the elements of A seen as characters of T. One can naturally associate to any spherical system a set of dominant weights, say $\lambda_1, \ldots, \lambda_s$. These dominant weights are in particular orthogonal to the given set S^p. Let V be the G-module whose highest weights are $\lambda_1, \ldots, \lambda_s$. We equip it naturally with a $\mathbb{G}$-module structure. We thus consider the invariant Hilbert scheme attached to such a $\mathbb{G}$ and V; we denote it for short $\mathrm{Hilb}(\mathcal{S})$.

Theorem 2. (i) *The tangent space at X_0 of* $\mathrm{Hilb}(\mathcal{S})$ *is multiplicity free as a T_{ad}-module and its set of T_{ad}-weights coincides with the given set Σ.*

(ii) *The invariant Hilbert scheme* $\mathrm{Hilb}(\mathcal{S})$ *is smooth.*

The first assertion of the above theorem is proved by means of a representation theoretical characterization of the tangent space given in [1]. To get the second assertion, we prove that the obstruction space of the invariant Hilbert functor is trivial which implies smoothness by Schlessinger's criterion. This is achieved by a nice characterization of this obstruction space which involves first cohomology groups of the isotropy Lie algebra of $v_{\underline{\lambda}}$.

As a consequence, we get that $\mathrm{Hilb}^\circ(\mathcal{S})$ is a toric T_{ad}-variety and it is in particular an affine space. Let $X_1 \in \mathrm{Hilb}^\circ(\mathcal{S})$ be such that its T_{ad}-orbit is dense within $\mathrm{Hilb}^\circ(\mathcal{S})$; regard it as a subvariety of V and consider its s-cone $\mathcal{C}(X_1)$.

Theorem 3. *Consider the closure within V of the s-cone $\mathcal{C}(X_1)$ and its quotient $X_{\mathcal{S}}$ by the algebraic torus $(\mathbb{C}^\star)^s$. Then the algebraic variety X is wonderful for the action of G and its spherical system is the given $\mathcal{S}$.*

The above theorem thus proves the existence part of Luna's conjecture; the uniqueness part may be obtained also by means of invariant Hilbert schemes.

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Fermionic formulas for eigenfunctions of the difference Toda Hamiltonian

Evgeny Feigin

(joint work with B. Feigin, M. Jimbo, T. Miwa, E. Mukhin)

1. **Central elements and Whittaker vectors.** The representation theory of quantum groups plays a very important role in the study of finite difference Toda Hamiltonian. In particular, one can construct eigenfunctions of H_{Toda} using Whittaker vectors in Verma modules (see [Br], [Sev], [Et]). We use pairing of Whittaker vectors with the dual ones.

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank l and let $U_v(\mathfrak{g})$ and $U_{v^{-1}}(\mathfrak{g})$ be two quantum groups with parameters v and v^{-1}. Let P, Q (resp. P_+, Q_+) be the weight and root lattices of $\mathfrak{g}$ (resp. their positive parts) and let $\mathcal{V}^\lambda = \sum_{\beta\in Q_+}(\mathcal{V}^\lambda)_\beta$ and $\overline{\mathcal{V}}^\lambda = \sum_{\beta\in Q_+}(\overline{\mathcal{V}}^\lambda)_\beta$ be Verma modules of $U_v(\mathfrak{g})$ and $U_{v^{-1}}(\mathfrak{g})$, respectively. In order to define a Whittaker vector θ^λ in the completion $\prod_{\beta\in Q_+}(\mathcal{V}^\lambda)_\beta$ of the

Verma module $\mathcal{V}^\lambda$ one fixes elements $\nu_i \in P$ and scalars c_i $(1 \le i \le l)$. Then the Whittaker vector, associated with these data, is defined by the condition

$$E_i K_{\nu_i}\ \theta^\lambda = \frac{c_i}{1 - v^2}\ \theta^\lambda \tag{0.1}$$

(for simplicity, we assume that $\mathfrak{g}$ is simply-laced). Here $E_i \in U_v(\mathfrak{g})$ are the Chevalley generators (which act as annihilating operators) and K_{ν_i} are certain elements from the Cartan subalgebra, associated with ν_i. Similarly, one defines the dual Whittaker vector $\bar\theta^\lambda$ in the completion of $\overline{\mathcal{V}}^\lambda$ by the formula

$$\bar E_i \bar K_{\nu_i}\ \bar\theta^\lambda = \frac{c_i^{-1}}{1 - v^{-2}}\ \bar\theta^\lambda \tag{0.2}$$

The main object for us is the following function

$$J^\lambda_\beta = v^{-(\beta,\beta)/2 + (\lambda,\beta)}\ (\theta^\lambda_\beta, \bar\theta^\lambda_\beta),$$

where $\theta^\lambda_\beta \in (\mathcal{V}^\lambda)_\beta$ is the weight $\lambda - \beta$ component of the Whittaker vector and $(\ ,\ )$ is the natural non-degenerate pairing between $\mathcal{V}^\lambda$ and $\overline{\mathcal{V}}^\lambda$. It can be shown that J^λ_β is independent of possible choices of ν_i and c_i.

Consider the generating function

$$F(q, z_1, \dots, z_l, y_1, \dots, y_l) = \sum_\beta J^\lambda_\beta \prod_{i=1}^l y_i^{(\beta,\omega_i)},$$

where $z_i = q^{-(\lambda,\alpha_i)}$, $q = v^2$ and ω_i (resp. α_i) are fundamental weights (resp. simple roots). Then F is known to be an eigenfunction of the quantum difference Toda operator ([Sev], [Et]). In order to prove this statement one uses central elements of the quantum group. Roughly, the procedure works as follows. If u is a central element, then the scalar product

$$(u\theta^\lambda_\beta, \bar\theta^\lambda_\beta) \tag{0.3}$$

can be written in two ways. On the one hand, one can compute the action of u on $\mathcal{V}^\lambda$ (the corresponding scalar). On the other hand, if a precise formula for u is known then one can compute (0.3) using the relation

$$(F_i w, \bar w) = (w, \bar E_i \bar w)$$

and formulas (0.1), (0.2).

The Toda Hamiltonian appears when one uses the central element written as the trace of products of R matrices in finite-dimensional $U_v(\mathfrak{g})$ modules. Our key observation is that if the Drinfeld Casimir element is used instead then one obtains a recursion relation for F which leads to the fermionic formulas. In the next subsection we describe those formulas in more details.

2. **Fermionic formulas.** Fermionic formulas appear in different problems of representation theory and mathematical physics (see for example [BM1], [FJMMT], [HKOTT], [SS]). Let us describe the class of formulas we treat in our paper.

Let $[r,s]=\{t\in\mathbb{Z}\mid r\le t\le s\}$ be a subset of $\mathbb{Z}$, where r,s are integers or $\pm\infty$. Let V be a vector space with a basis $e_{i,t}$ labeled by pairs $1\le i\le l$, $t\in[r,s]$. Let $\Gamma_+=\{\sum_{(i,t)}m_{i,t}e_{i,t}|m_{i,t}\in\mathbb{Z}_{\ge 0}\}$ be the positive part of the lattice generated by $\{e_{i,t}\}$. We fix a quadratic form $\langle\cdot,\cdot\rangle$ on V and a vector $\mu\in V$. Further, define maps w and d from V to the l-dimensional vector space with a basis $p_1,\dots,p_l$ via the formulas

$$w(\sum_{(i,t)}m_{i,t}e_{i,t})=\sum_{i=1}^{l}p_i\sum_{t\in[r,s]}m_{i,t},\qquad d(\sum_{(i,t)}m_{i,t}e_{i,t})=\sum_{i=1}^{l}p_i\sum_{t\in[r,s]}tm_{i,t}.$$

Define functions I_m depending on q, $z=(z_1,\dots,z_l)$ and $m=(m_1,\dots,m_l)$ as follows

$$I_m(q,z)=\sum_{w(\gamma)=m}z^{d(\gamma)}\frac{q^{\langle\gamma,\gamma\rangle+\langle\mu,\gamma\rangle}}{(q)_\gamma},\tag{0.4}$$

where the summands are labeled by

$$\gamma=\sum_{(i,t)}m_{i,t}e_{i,t}\in\Gamma_+\quad\text{and}\quad(q)_\gamma=\prod_{(i,t)}(q)_{m_{i,t}}\quad,\quad d(\gamma)=\prod_{i=1}^{l}z_i^{d(\gamma)_i}.$$

We call the right hand side of (0.4) a fermionic formula. The generating function $F(q,z,y)=F(q,z_1,\dots,z_l,y_1,\dots,y_l)$ is given by the formula

$$F(q,z,y)=\sum_m y^mI_m(q,z),\quad y^m=y_1^{m_1}\dots y_l^{m_l}.\tag{0.5}$$

Let the matrix of the quadratic form $\langle\cdot,\cdot\rangle$ be a tensor product $D=C\otimes G(r,s)$, where C is the Cartan matrix of $\mathfrak{g}$ (we assume here that C is symmetric) and $G=(G_{t,t'})_{i,j\in[r,s]}$, $G_{t,t'}=\min(t,t')$. Such matrices appear in [DS], [S] in the fermionic formulas for the Kostka polynomials. Let $[r,s]=[0,\infty)$. Then functions $I_m(q,z)$ satisfy the following recursion relation:

$$I_m(q,z)=\sum_{0\le a\le m}\frac{z^aq^{W(a)}}{(q)_{m-a}}I_a(q,z),\tag{0.6}$$

where $W(a)=\frac{1}{2}(Ca\cdot a-\mathrm{diag}C\cdot a)$, $\cdot$ denotes the standard scalar product and $0\le a\le m$ abbreviates the set of inequalities $0\le a_i\le m_i$. The relation (0.6) shows that $I_m(q,z)$ are determined by $I_0(q,z)$.

Recall the functions J^λ_β. Using the Drinfeld Casimir element and the procedure described in the end of subsection 1, we show that J^λ_β satisfy the relation

$$J^\lambda_\beta=\sum_{\beta'}\frac{1}{(q)_{\beta-\beta'}}q^{(\beta',\beta')/2-(\lambda+\rho,\beta')}J^\lambda_{\beta'}.$$

This leads to the following identification

$$J^\lambda_\beta = I_m(q,z),\ \beta = \sum_i m_i\alpha_i,\ z = q^{-(\lambda,\alpha_i)}.$$

In particular, this gives a fermionic formula for eigenfunctions of H_{Toda}.

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Diagonalizable and unitary representations of rational Cherednik algebras

Stephen Griffeth

(joint work with Charles Dunkl, Emanuel Stoica)

The goal of this talk is to describe some recent progress in the classification of unitary representations of rational Cherednik algebras. Part of the talk is based on the paper [3] by Etingof and Stoica, and its appendix by the author, and part is based on joint work in progress with Charles Dunkl and Emanuel Stoica.

Let n be a positive integer, let $\mathfrak{h}$ be an n-dimensional complex vector space, and let $W \subseteq \mathrm{GL}(\mathfrak{h})$ be a finite subgroup. Write $\mathfrak{h}^*$ for the dual space of $\mathfrak{h}$, let $T(\mathfrak{h}^* \oplus \mathfrak{h})$ be the tensor algebra, and let

$$T(\mathfrak{h}^* \oplus \mathfrak{h})\#W \tag{0.1}$$

be the *twisted group ring*, isomorphic to $T(\mathfrak{h}^* \oplus \mathfrak{h}) \otimes \mathbb{C}W$ as a $\mathbb{C}$-vector space and with multiplication

$$(f \otimes v)(g \otimes w) = f(v.g) \otimes vw \tag{0.2}$$

where for $g \in T(\mathfrak{h}^* \oplus \mathfrak{h})$ and $v \in W$, we write $v.g$ for the action of v on g.

Let

$$T = \{s \in W \mid \dim(\mathrm{fix}(s)) = n - 1\} \tag{0.3}$$

be the set of reflections in W, and let $t \in \mathbb{C}$ and $c_s \in \mathbb{C}$ be a collection of complex numbers such that $c_{wsw^{-1}} = c_s$ for all $s \in T$. The *rational Cherednik algebra* $\mathbb{H}_{t,c}$ corresponding to this data is the quotient of $T(\mathfrak{h}^* \oplus \mathfrak{h})\#W$ by the relations

$$x_1x_2 = x_2x_1, \text{ for } x_1, x_2 \in \mathfrak{h}^*, \quad y_1y_2 = y_2y_1, \text{ for } y_1, y_2 \in \mathfrak{h}, \tag{0.4}$$

and

$$yx = xy + t\langle x, y\rangle - \sum_{s \in T} c_s\langle \alpha_s, y\rangle\langle x, \alpha_s^\vee\rangle s \quad \text{for } x \in \mathfrak{h}^* \text{ and } y \in \mathfrak{h}, \tag{0.5}$$

where for each $s \in T$, the elements $\alpha_s \in \mathfrak{h}^*$ and $\alpha_s^\vee \in \mathfrak{h}$ are fixed subject to

$$s.x = x - \langle x, \alpha_s^\vee\rangle\alpha_s \quad \text{for } x \in \mathfrak{h}^*. \tag{0.6}$$

Motivated by the study of symplectic quotient singularities, in [2] Etingof and Ginzburg showed that multiplication in $\mathbb{H}_{t,c}$ gives a vector space isomorphism

$$S(\mathfrak{h}^*) \otimes \mathbb{C}W \otimes S(\mathfrak{h}) \cong \mathbb{H}_{t,c}, \tag{0.7}$$

where $S(\mathfrak{h}^*)$ and $S(\mathfrak{h})$ are the symmetric algebras. Thanks to this "triangular decomposition", many of the usual constructions of Lie theory can be carried out for the rational Cherednik algebra. The *Verma* (or "standard") module corresponding to an irreducible $\mathbb{C}W$-module S^λ is

$$M_{t,c}(\lambda) = \mathrm{Ind}_{S(\mathfrak{h})\otimes\mathbb{C}W}^{\mathbb{H}_{t,c}} S^\lambda, \tag{0.8}$$

and it carries a *contravariant form* $\langle\cdot,\cdot\rangle_{t,c}$, determined up to scalars as a Hermitian form satisfying

$$\langle w.f, w, g\rangle = \langle f, g\rangle \quad \text{for } f, g \in M_{t,c}(\lambda) \text{ and } w \in W, \tag{0.9}$$

$$\langle x.f, g\rangle = \langle f, x^*.g\rangle \quad \text{for } f, g \in M_{t,c}(\lambda) \text{ and } x \in \mathfrak{h}^*, \tag{0.10}$$

where $x \mapsto x^*$ is a W-equivariant conjugate linear isomorphism of $\mathfrak{h}^*$ onto $\mathfrak{h}$. The radical of the contravariant form is, in case $t = 1$, the radical of the module $M_{t,c}(\lambda)$. We write

$$L_{t,c}(\lambda) = M_{t,c}(\lambda)/\mathrm{Rad}(\langle\cdot,\cdot\rangle_{t,c}) \tag{0.11}$$

for the irreducible quotient of the Verma module by the radical of its bilinear form.

Cherednik posed the problem of determining those pairs (c, λ) for which the form $\langle\cdot,\cdot\rangle$ is positive definite on $L_{1,c}(\lambda)$. In [3], Etingof and Stoica initiated the study of this problem, and proved the following theorem for the symmetric group:

Theorem 1. For each partition λ of n not equal to (n) or (1^n), the set of c for which $L_{1,c}(\lambda)$ is unitary is contained in the union of the interval $[-\frac{1}{a(\lambda)}, \frac{1}{a(\lambda)}]$ with the finite set of isolated points $\frac{1}{k}$, for $b(\lambda^t) \leq k < a(\lambda)$ and $-a(\lambda) < k \leq -b(\lambda)$, where if λ has length l then $a(\lambda) = \lambda_1 + l - 1$ and $b(\lambda) = \lambda_1 + l - \lambda_l$.

In addition to obtaining an exact description of the set of c for which $L_{1,c}(\lambda)$ is unitary for those λ not covered by the above theorem, they conjectured that the set described in the theorem is exactly the set of pairs (c, λ) such that $L_{1,c}(\lambda)$ is unitary, and this conjecture was proved by the author in the appendix to [3] (completely solving the problem for the symmetric group). The key fact used to prove it is the classification, due to Cherednik [1] and Suzuki [4], of those pairs (c, λ) for which $L_{1,c}(\lambda)$ is diagonalizable with respect to the *Cherednik-Dunkl operators*

$$\epsilon_i^\vee = x_i y_i + c_0 \sum_{1 \leq i < j} s_{ij}. \tag{0.12}$$

Cherednik and Suzuki also describe the structure of the diagonalizable modules $L_{1,c}(\lambda)$ in terms of "periodic tableaux" on certain infinite skew diagrams. In joint work with Charles Dunkl and Emanuel Stoica we give a version of these results that applies to the infinite family $G(r, p, n)$ of complex reflection groups, and we hope to apply the resulting combinatorial description of the diagonalizable modules to obtain a classification of the unitary modules $L_{1,c}(\lambda)$. As a byproduct of our classification of diagonalizable irreducibles we obtain a number of new examples of finite dimensional rational Cherednik algebra modules, together with combinatorial formulas for their dimensions.

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Quantum affine algebras and cluster algebras

Bernard Leclerc
(joint work with David Hernandez)

My talk was a report on the recent preprint [12].

Let $\mathfrak{g}$ be a simple Lie algebra of type A_n, D_n or E_n, and let $U_q(\widehat{\mathfrak{g}})$ denote the corresponding quantum affine algebra, with parameter $q \in \mathbb{C}^*$ not a root of unity. The monoidal category $\mathcal{C}$ of finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules has been studied by many authors from different perspectives (see *e.g.* [1, 2, 6, 10, 18]). In particular its simple objects have been classified by Chari and Pressley, and Nakajima has calculated their character in terms of the cohomology of certain quiver varieties.

In spite of these remarkable results many basic questions remain open, and in particular little is known about the tensor structure of $\mathcal{C}$. When $\mathfrak{g} = \mathfrak{sl}_2$, Chari and Pressley [3] have shown that every simple object is isomorphic to a tensor product of simple objects of a special type called Kirillov-Reshetikhin modules. Conversely, they have shown that a tensor product $S_1 \otimes \cdots \otimes S_k$ of Kirillov-Reshetikhin modules is simple if and only if $S_i \otimes S_j$ is simple for every $i \neq j$. Moreover, $S_i \otimes S_j$ is simple if and only if S_i and S_j are "in general position" (a combinatorial condition on the roots of the Drinfeld polynomials of S_i and S_j). Hence, the Kirillov-Reshetikhin modules can be regarded as the *prime* simple objects of $\mathcal{C}$ [4], and one knows which products of primes are simple. For $\mathfrak{g} \neq \mathfrak{sl}_2$, the situation is far more complicated. Thus, already for $\mathfrak{g} = \mathfrak{sl}_3$, we do not know a general factorization theorem for simple objects, neither a tentative list of prime simple objects (see [4]).

Because of these difficulties, we decide to focus on some smaller subcategories. We introduce a sequence

$$\mathcal{C}_0 \subset \mathcal{C}_1 \subset \cdots \subset \mathcal{C}_\ell \subset \cdots, \qquad (\ell \in \mathbb{N}),$$

of full monoidal subcategories of $\mathcal{C}$, whose objects are characterized by certain strong restrictions on the roots of the Drinfeld polynomials of their composition factors. By construction, the Grothendieck ring R_ℓ of $\mathcal{C}_\ell$ is a polynomial ring in $n(\ell+1)$ variables, where n is the rank of $\mathfrak{g}$. Our starting point is that R_ℓ is naturally equipped with the structure of a cluster algebra.

Cluster algebras were introduced by Fomin and Zelevinsky [7] as a combinatorial device for studying canonical bases and total positivity. They found immediately lots of applications, including a proof of a conjecture of Zamolodchikov concerning certain discrete dynamical systems arising from the thermodynamic Bethe ansatz, called Y-systems [8]. As observed by Kuniba, Nakanishi and Suzuki [16], Y-systems are strongly related with the representation theory of $U_q(\widehat{\mathfrak{g}})$ via some other systems of functional relations called T-systems. It was conjectured in [16] that the characters of the Kirillov-Reshetikhin modules are solutions of a T-system, and this was later proved by Nakajima [19] in the simply-laced case, and by Hernandez in the general case [11]. Now it is easy to notice that in the simply-laced case the equations of a T-system are exactly of the same form as the exchange relations in a cluster algebra. This led us to introduce a cluster algebra structure on R_ℓ by using an initial seed consisting of a choice of $n(\ell+1)$ Kirillov-Reshetikhin modules in $\mathcal{C}_\ell$. The exchange matrix of this seed encodes $n\ell$ equations of the T-system satisfied by these Kirillov-Reshetikhin modules. (Note that the seed contains n frozen variables – or coefficients – in the sense of [7].) By definition of a cluster algebra, one can obtain new seeds by applying sequences of mutations to the initial seed. Then one of our main conjectures is that all the new cluster variables produced in this way are classes of simple objects of $\mathcal{C}_\ell$. In general, these simple objects are no longer Kirillov-Reshetikhin modules.

For $\ell = 0$, the cluster structure of R_0 is trivial: there is a unique cluster consisting entirely of frozen variables.

The case $\ell = 1$ is already very interesting, and most of the lecture was devoted to it. Fomin and Zelevinsky have classified the cluster algebras with finitely many cluster variables in terms of finite root systems [9]. It turns out that for every $\mathfrak{g}$ the ring R_1 has finitely many cluster variables, and that its cluster type coincides with the root system of $\mathfrak{g}$. Therefore, one may expect that the tensor structure of the simple objects of the category $\mathcal{C}_1$ can be described in "a finite way". In fact we conjecture that for every $\mathfrak{g}$ the category $\mathcal{C}_1$ behaves as nicely as the category $\mathcal{C}$ for $\mathfrak{sl}_2$, and we prove it for $\mathfrak{g}$ of type A_n and D_4.

More precisely, we single out a finite set of simple objects of $\mathcal{C}_1$ whose Drinfeld polynomials are naturally labeled by the set of almost positive roots of $\mathfrak{g}$ (*i.e.*, positive roots and negative simple roots). Recall that the almost positive roots are in one-to-one correspondence with the cluster variables [9], so we shall call these objects the *cluster simple objects*. To these objects we add n distinguished simple objects which we call *frozen simple objects*. Our first claim is that the classes of these objects in R_1 coincide with the cluster variables and frozen variables.

Recall also that the cluster variables are grouped into overlapping subsets of cardinality n called clusters [7]. The number of clusters is a generalized Catalan number, and they can be identified to the faces of the dual of a generalized associahedron [8]. Our second claim is that a tensor product of cluster simple objects is simple if and only if all the objects belong to a common cluster. Moreover, the tensor product of a frozen simple object with any simple object is again simple. It follows that every simple object of $\mathcal{C}_1$ is a tensor product of cluster simple objects and frozen simple objects. To prove this, we first show in a uniform way for all types that a tensor product $S_1 \otimes \cdots \otimes S_k$ of simple objects of $\mathcal{C}_1$ is simple if and only if $S_i \otimes S_j$ is simple for every $i \neq j$.

When $\ell > 1$ the ring R_ℓ has in general infinitely many cluster variables, grouped into infinitely many clusters. A notable exception is the case $\mathfrak{g} = \mathfrak{sl}_2$, for which R_ℓ is a cluster algebra of finite type A_ℓ in the classification of [9]. In this special case it follows from [3] that, again, the classes in R_ℓ of the simple objects of $\mathcal{C}_\ell$ are precisely the cluster monomials of R_ℓ. We conjecture that for arbitrary $\mathfrak{g}$ and ℓ, every cluster monomial of R_ℓ is the class of a simple object. We also conjecture that, conversely, the class of a simple object S in $\mathcal{C}_\ell$ is a cluster monomial if and only if $S \otimes S$ is simple. In this case, following [17], we call S a *real simple object.* We believe that real simple objects form an interesting class of irreducible $U_q(\widehat{\mathfrak{g}})$-modules, and the meaning of our partial results and conjectures is that their characters are governed by the combinatorics of cluster algebras.

Kedem [14] and Di Francesco [5] have studied another connection between quantum affine algebras and cluster algebras, based on other types of functional equations (Q-systems and generalized T-systems). Keller [15] has obtained a proof of the periodicity conjecture for Y-systems attached to pairs of simply-laced Dynkin diagrams using 2-Calabi-Yau categorifications of cluster algebras. More recently, Inoue, Iyama, Kuniba, Nakanishi and Suzuki [13] have also studied the connection between Y-systems, T-systems, Grothendieck rings of $U_q(\widehat{\mathfrak{g}})$ and cluster algebras, motivated by periodicity problems. These papers do not study the relations between cluster monomials and irreducible $U_q(\widehat{\mathfrak{g}})$-modules.

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Quiver varieties and Branching

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1. **Motivation.** Braverman-Finkelberg [1] recently proposed the geometric Satake correspondence for the affine Kac-Moody group G_{aff}. They conjecture that intersection cohomology sheaves on the Uhlenbeck compactification of the framed moduli space of G_{cpt}-instantons on $\mathbb{R}^4/\mathbb{Z}_r$ correspond to weight spaces of representations of the Langlands dual group $G_{\mathrm{aff}}^{\vee}$ at level r. When $G = \mathrm{SL}(l)$, the

Uhlenbeck compactification is the quiver variety of type $\mathfrak{sl}(r)_{\mathrm{aff}}$, and their conjecture follows from the author's earlier result [5] and I. Frenkel's level-rank duality [4]. They further introduce a convolution diagram which conjecturally gives the tensor product multiplicity [2]. Since the tensor product multiplicy corresponds to the branching multiplicity under the level-rank duality, the author develop the theory for the branching in quiver varieties and check this conjecture for $G = \mathrm{SL}(l)$ in the paper [6].

2. **Quiver varieties.** Suppose that a finite graph is given. Let I be the set of vertices and E the set of edges. Suppose that there are no edge loops. Let $\mathbf{C}$ be the Cartan matrix. Let $\mathfrak{g}$ be the corresponding (symmetric) Kac-Moody Lie algebra. Let H be the set of oriented edges (hence $\#H = 2\#E$), and we choose an orientation Ω of the graph (I, E).

Suppose that I-graded vector spaces V, W are given. Then we consider the vector space

$$\mathbf{M}(V, W) = \bigoplus_{h\in H} \mathrm{Hom}(V_{o(h)}, V_{i(h)}) \oplus \bigoplus_{i\in I} \mathrm{Hom}(W_i, V_i) \oplus \mathrm{Hom}(V_i, W_i),$$

where $o(h)$, $i(h)$ are the outgoing and incoming vertices of h. We denote the corresponding componets of the above decomposition by B_h, a_i, b_i. Let $G_V = \prod_{i\in I} \mathrm{GL}(V_i)$. It acts on $\mathbf{M}(V, W)$ by conjugation. The choice of the orientation gives us the symplectic form invariant under the G_V-action. Let $\mu\colon \mathbf{M}(V, W) \to (\mathrm{Lie}\, G_V)^*$ be the corresponding moment map vanishing at the origin. It is given by

$$\mu(B_h, a_i, b_i) = \sum_{h:i(h)=i} \varepsilon(h) B_h B_{\overline{h}} + a_i b_i$$

if we identify $(\mathrm{Lie}\, G_V)^*$ with $\mathrm{Lie}\, G_V$ by the trace. Here $\varepsilon(h)$ is 1 if $h \in \Omega$ and -1 otherwise, and $\overline{h}$ is the same edge with h but equipped with the opposite orientation.

We consider a quotient of $\mu^{-1}(0)$ by G_V in the sense of the geometric invariant theory. It depends on the choice, called the *stability parameter*. Let $\zeta = (\zeta_i) \in \mathbb{Z}^I$. We define the character χ_ζ of G_V given by $\chi_\zeta(g) = \prod_{i\in I}(\det g_i)^{-\zeta_i}$, and we consider the semi-invariants $A(\mu^{-1}(0))^{G,\chi_\zeta^n} = \{f \in A(\mu^{-1}(0) \mid f(gx) = \chi_\zeta(g)^n f(x)\}$. Then $\bigoplus_{n=0}^\infty A(\mu^{-1}(0))^{G,\chi_\zeta^n}$ is a graded ring, and we define the quiver variety by

$$\mathfrak{M}_\zeta(V, W) = \mathrm{Proj}(\bigoplus_{n=0}^{\infty} A(\mu^{-1}(0))^{G,\chi_\zeta^n}).$$

By a general result for the geometric invariant theory, $\mathfrak{M}_\zeta(V, W)$ is the set of ζ-semistable points modulo the so-called S-equivalences. (See [6] for the precise statement.) It contains the open subscheme $\mathfrak{M}_\zeta^{\mathrm{s}}(V, W)$ consisting of G_V-orbits of ζ-stable points. For example, if $\zeta = 0$, all points are ζ-semistable, and two points are S-equivalent if and only if their closure intersect. In this case, $\mathfrak{M}_0(V, W)$ is an affine algebraic variety given by $\mathrm{Spec}(A(\mu^{-1}(0))^{G_V})$.

The quiver variety depends on the choice of the stability parameter ζ, but its dependence is through the face F containing ζ. Here a face is given by the decomposition of the set $R_+(V)$ of positive roots with $\alpha = \sum m_i\alpha_i$ with $m_i \le \dim V_i$ into three parts $R_+(V) = R_+^+(V) \sqcup R_+^-(V) \sqcup R_+^0(V)$ as

$$F = \{\zeta \in \mathbb{Q}^I \mid \zeta\cdot\alpha > 0, < 0, = 0 \text{ for } \alpha \in R_+^+(V),\, \in R_+^-(V),\, R_+^0(V) \text{ respectively}\}.$$

We say a face F is a *chamber* if $R_+^0(V) = \emptyset$. For example, in [5] we use the parameter ζ^+ in the face given by $R_+^+(V) = R_+(V)$. If ζ is in a chamber, we have $\mathfrak{M}_\zeta(V,W) = \mathfrak{M}_\zeta^{\mathrm{s}}(V,W)$ and $\mathfrak{M}_\zeta(V,W)$ is nonsingular of dimension

$$\dim \mathfrak{M}_\zeta(V,W) = 2(\dim V, \dim W) - (\dim V, \mathbf{C}\dim V),$$

where $\dim V$, $\dim W$ are dimension vectors (in $\mathbb{Z}^I$) and $(\ ,\ )$ is the natural inner product on $\mathbb{Z}^I$.

If F' is in the closure of F, and if we take $\zeta' \in F'$, $\zeta \in F$, we have a projective morphism

$$\pi_{\zeta,\zeta'}\colon \mathfrak{M}_\zeta(V,W) \to \mathfrak{M}_{\zeta'}(V,W).$$

In particular, $\zeta' = 0$ is contained in the closure of any face, we always have $\mathfrak{M}_\zeta(V,W) \to \mathfrak{M}_0(V,W)$.

3. **Convolution algebra.** For the parameter $\zeta = 0$, we have a closed embedding $\mathfrak{M}_0(V,W) \subset \mathfrak{M}_0(V',W)$ for $V \subset V'$ by setting the data 0 on a subspace of V' complementary to V. We denote the direct limit by $\mathfrak{M}_0(W)$. If ζ is in a chamber, there is no obvious relation among different $\mathfrak{M}_\zeta(V,W)$'s, and we set $\mathfrak{M}_\zeta(W) = \bigsqcup_V \mathfrak{M}_\zeta(V,W)$ where V runs all isomorphism classes of I-graded vector spaces. For a general ζ, we have the closed embedding $\mathfrak{M}_\zeta(V,W) \subset \mathfrak{M}_\zeta(V',W)$ for $V \subset V'$, when the data $0 \in \mathbf{M}(V'/V,0)$ is ζ-semitable. We denote the inductive limit by $\mathfrak{M}_\zeta(W)$. We consider the fiber product

$$Z_{\zeta,\zeta'}(W) = \mathfrak{M}_\zeta(W) \times_{\mathfrak{M}_{\zeta'}(W)} \mathfrak{M}_\zeta(W),$$

when the faces F', F containing ζ', ζ satisfy $F' \subset \overline{F}$ for any choice of V. This is a union $\mathfrak{M}_\zeta(V^1,W) \times_{\mathfrak{M}_{\zeta'}(V,W)} \mathfrak{M}_\zeta(V^2,W)$ of various V^1, V^2 and a big vector space V containing both V^1 and V^2. Any irreducible component has at most $\dim \mathfrak{M}_\zeta(V^1,W) \times \mathfrak{M}_\zeta(V^2,W)/2$.

We assume ζ is in a chamber and consider

$$H_{\mathrm{top}}(Z_{\zeta,\zeta'}(W)),$$

where top means the degree $\dim \mathfrak{M}_\zeta(V^1,W) \times \mathfrak{M}_\zeta(V^2,W)$ for each summand $\mathfrak{M}_\zeta(V^1,W) \times_{\mathfrak{M}_{\zeta'}(V,W)} \mathfrak{M}_\zeta(V^2,W)$. This has a structure of the algebra given by the convolution product

$$c * c' = p_{13*}(p_{12}^*(c) \cap p_{23}^*(c')),$$

where p_{ab} is the projection from the triple fiber product to the fiber product of a^{th} and b^{th} factors.

In [5] the author constructed an algebra homomorphism

$$\mathbf{U}(\mathfrak{g}) \to H_{\mathrm{top}}(Z_{\zeta,0}(W)) \tag{0.1}$$

for $\zeta = \zeta^+$ as above. By the general theory of the convolution algebra (see [3]) the algebra $H_{\mathrm{top}}(\mathfrak{M}_\zeta(W))$ is the endomorphism algebra

$$\mathrm{End}_{\mathrm{Perv}(\mathfrak{M}_0(W))}(\pi_{\zeta,0*}(\mathbb{C}_{\mathfrak{M}_\zeta(W)}[\dim \mathfrak{M}_\zeta(W)])),$$

where the shift $\dim \mathfrak{M}_\zeta(W)$ means that we shift $\dim \mathfrak{M}_\zeta(V,W)$ for each component $\mathfrak{M}_\zeta(V,W)$. One can show that $\pi_{\zeta,0*}(\mathbb{C}_{\mathfrak{M}_\zeta(W)}[\dim \mathfrak{M}_\zeta(W)])$ is canonically isomorphic to each other independent of the choice of the chamber (containing ζ) by using a one parameter deformation of $\mathfrak{M}_0(W)$ and its similtaneous resolution. So we have a homomorphism (0.1) for any ζ.

Theorem 1. (1) Choose a subdiagram $I^\circ \subset I$. Take ζ' so that $\zeta'_i = 0$ for $i \in I^\circ$ and $\zeta'_i > 0$ for $i \notin I^\circ$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathbf{U}(\mathfrak{g}_{I^\circ}) & \longrightarrow & H_{\mathrm{top}}(Z_{\zeta,\zeta'}(W)) \\ \big\downarrow & & \big\downarrow \\ \mathbf{U}(\mathfrak{g}) & \longrightarrow & H_{\mathrm{top}}(Z_{\zeta,0}(W)), \end{array}$$

where $\mathfrak{g}_{I^\circ}$ is the Levi subalgebra of $\mathfrak{g}$ corresponding to I° and the bottom horizontal arrow is (0.1).

(2) Suppose that the graph (I,E) is affine. We choose a subdiagram $I_0^\circ \subset I_0$ of the corresponding finite type graph $I_0 = I \setminus \{0\}$. Take ζ' so that $\zeta'_i = 0$ for $i \in I_0^\circ$ and $\zeta'_i > 0$ for $i \in I_0 \setminus I^\circ$ and $\zeta' \cdot \delta = 0$ for the imaginary root δ. And take ζ from a chamber containing ζ' in its closure. Then we have a commutative diagram as above replacing $\mathbf{U}(\mathfrak{g}_{I^\circ})$ by $\mathbf{U}(\widehat{\mathfrak{g}}_{I_0^0})$ the enveloping algebra of the affine Lie algebra of the Levi subalgebra $\mathfrak{g}_{I_0^0}$ of the finite dimensional Lie algebra $\mathfrak{g}_{I_0}$.

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Dirac operators in the affine setting

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(joint work with Victor G. Kac, Pierluigi Mseneder Frajria)

1. Introduction

Let $\mathfrak{g} = \mathbb{k} \oplus \mathfrak{p}$ be an infinitesimal symmetric space. The adjoint representation gives a map $\mathbb{k} \to so(\mathfrak{p})$ and in turn we have a map between the corresponding affinizations $\widehat{\mathbb{k}} \to \widehat{so(\mathfrak{p})}$. Therefore, given a $\widehat{so(\mathfrak{p})}$-module, it makes sense to ask for its $\widehat{\mathbb{k}}$-decomposition. Kac and Peterson [6] discovered that this decomposition is finite for level 1 modules. Recall that a subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is called *quadratic* if the restriction to $\mathfrak{a}$ of a non-degenerate invariant form on $\mathfrak{g}$ is still non-degenerate. About 20 years ago there was much activity on the problem of classifying the quadratic subalgebras $\mathfrak{a}$ such that level 1 $\widehat{so(\mathfrak{p})}$-modules restrict finitely to $\widehat{\mathfrak{a}}$ and on that of finding actual decompositions. The first goal was achieved, and the above subalgebras might be split into three classes: certain equal rank subalgebras, a list of "exceptional" cases, and the symmetric subalgebras. Decompositions were known for the first two classes and in some instances of the third. Recently, we found a connection with the theory of abelian ideals in Borel subalgebras which allowed us to solve completely the problem (cf. [1]). It turns out that an affine analogue of Kostant's theory of multiplets [9] is the natural framework for a conceptual explanation of our formulas. This has been achieved by letting the Kac-Todorov field [7] play the role of Kostant cubic Dirac operator. We also found an analogue in affine setting of a theorem of Huang and Pandžić [2] which solves a conjecture of Vogan on Dirac cohomology. This result (see Theorem 3.2) allows us to prove a general multiplet theorem (see Theorem 3.1). We plan to investigate the applications of our methods in the context of finite and affine Lie superalgebras. Though this project is still at early stage of development, the construction of the Dirac field can be extended (with careful modifications) to the superalgebra case. We give a concise outline of this construction in Section 2 and in Section 4 we point out some of its consequences, notably a uniform proof of Freudenthal strange formula (4.1) for Lie superalgebras. The main results in the affine setting appear in Section 3.

2. The Dirac field

Let $\mathfrak{g}$ be a basic classical Lie superalgebra and σ an elliptic automorphism of $\mathfrak{g}$ (i.e., diagonalizable with modulus 1 eigenvalues). Let $(\cdot,\cdot)$ be a non-degenerate invariant supersymmetric form and assume that it is $\subset$ -invariant. Set $\bar{\mathfrak{g}} = P\mathfrak{g}$, where P is the parity reversing functor. Consider the conformal algebra $R = (\mathbb{C}[T] \otimes \mathfrak{g}) \oplus (\mathbb{C}[T] \otimes \bar{\mathfrak{g}}) \oplus \mathbb{C}K \oplus \mathbb{C}K'$ with λ-products

$$[a_\lambda b] = [a,b] + \lambda(a,b)K,\ [a_\lambda \bar{b}] = \overline{[a,b]},\ [\bar{a}_\lambda b] = p(b)\overline{[a,b]},\ [\bar{a}_\lambda \bar{b}] = (b,a)K',$$

K, K' being even central elements. Let $V(R)$ be the corresponding universal vertex algebra, and denote by $V^{k,1}(\mathfrak{g})$ its quotient by the ideal generated by $K - k|0\rangle$ and $K' - |0\rangle$. The relations are the same used in [4] for even variables.

Choose a homogeneous basis $\{x_i\}$ of $\mathfrak{g}$ and let $\{x^i\}$ be its dual basis. We assume that the Casimir operator of $\mathfrak{g}$ acts on $\mathfrak{g}$ as $2gI_{\mathfrak{g}}$. The element

$$G_{\mathfrak{g}} = \sum_i : x^i \overline{x}_i : -\frac{1}{3} \sum_{i,j} : \overline{[x^i, x_j]} \overline{x}^j \overline{x}_i : \in V^{k+g,1}(\mathfrak{g})$$

is called the Kac-Todorov operator. To enlighten how $G_{\mathfrak{g}}$ acts on representations, recall from [4] that the vertex algebra $V^{k+g,1}(\mathfrak{g})$ is isomorphic to $V^k(\mathfrak{g}) \otimes F(\overline{\mathfrak{g}})$, where the left factor is the universal affine vertex algebra of level k and the right factor is the universal fermionic vertex algebra. There is a natural notion of (σ-twisted) Spin-Weil module $SW^{\sigma}(\overline{\mathfrak{g}})$ for $F(\overline{\mathfrak{g}})$, hence given a σ-twisted module for $V^k(\mathfrak{g})$ (i.e., a representation M of the twisted affine superalgebra $\widehat{L}(\mathfrak{g}, \sigma)$), we may produce a $\sigma \otimes (-\sigma)$-twisted representation

$$X(M) = M \otimes SW^{-\sigma}(\overline{\mathfrak{g}})$$

of $V^{k+g,1}(\mathfrak{g})$. It turns out that $(\sigma \otimes (-\sigma))(G_{\mathfrak{g}}) = -G_{\mathfrak{g}}$, so that $Y^{X(M)}(G_{\mathfrak{g}}, z) = \sum_{n \in \mathbb{Z}} G_n^X z^{-n-\frac{3}{2}}$. Given a quadratic σ-stable subsuperalgebra $\mathfrak{a} \subset \mathfrak{g}$, we have an embedding $V^{k+1,g}(\mathfrak{a}) \subset V^{k+1,g}(\mathfrak{g})$, so that we may consider the field $G_{\mathfrak{g}} - G_{\mathfrak{a}}$, which turns out to act on $M \otimes SW^{-\sigma}(\overline{\mathfrak{p}})$ where $\mathfrak{p} = \mathfrak{a}^{\perp}$. We introduce the Kac-Todorov operator as

$$D_{\mathfrak{g},\mathfrak{a}} = (G_{\mathfrak{g}} - G_{\mathfrak{a}})_0^{M \otimes SW^{-\sigma}(\overline{\mathfrak{p}})}.$$

3. Main Theorems

Throughout this Section, $\mathfrak{g}$ is a Lie algebra, σ an elliptic automorphism of $\mathfrak{g}$ preserving the form and $\mathfrak{a}$ a quadratic subalgebra.

Write $\mathfrak{g} = \oplus_{\overline{j} \in \mathbb{R}/\mathbb{Z}} \mathfrak{g}^{\overline{j}}$, $\mathfrak{a} = \oplus_{\overline{j} \in \mathbb{R}/\mathbb{Z}} \mathfrak{a}^{\overline{j}}$, $\mathfrak{a}^{\overline{j}} = \mathfrak{a} \cap \mathfrak{g}^{\overline{j}}$.

Assumption. *We assume that there exists an elliptic automorphism of $\mathfrak{g}$ preserving the form, commuting with σ, and such that a Cartan subalgebra $\mathfrak{t}$ of the joint fixed points of σ and μ is a Cartan subalgebra of $\mathfrak{a}^{\overline{0}}$.*

Denote by $_0$ the Cartan subalgebra $Cent_{\mathfrak{g}^{\overline{0}}}(\mathfrak{t})$ of $\mathfrak{g}^{\overline{0}}$ and decompose it as $_0 = \mathfrak{t} \oplus {}_{\mathfrak{p}}$. Let $\widehat{W}_\sigma$ be the Weyl group of $\widehat{L}(\mathfrak{g}, \sigma)$ and $\widehat{\mathfrak{h}} = {}_0 \oplus \mathbb{C}K \oplus \mathbb{C}d$ the Cartan subalgebra. Set $\mathfrak{t}^{aff} = \mathfrak{t} \oplus \mathbb{C}K \oplus \mathbb{C}d$. We prove that the subgroup $\widehat{W}(\mu) = \{w \in \widehat{W}_\sigma \mid w\mu = \mu w\}$ is isomorphic to the group generated by the reflections s_β in the vectors $\beta = \alpha_{|\mathfrak{t}^{aff}}$ (where we stipulate that $s_\beta = Id$ if $\alpha_{|\mathfrak{t}^{aff}}$ is isotropic). We prove that the latter group is a Coxeter group which contains the Weyl group of $\widehat{L}(\mathfrak{a}, \sigma)$ as a reflection subgroup. Let $\widehat{W}'$ be the corresponding set of minimal right coset representatives. Let $\widehat{\rho}_\sigma$, $\widehat{\rho}_{\mathfrak{a},\sigma}$ be ρ-vectors for $\widehat{L}(\mathfrak{g}, \sigma)$, $\widehat{L}(\mathfrak{a}, \sigma)$ respectively.

Theorem 3.1. [5, Theorem 1.1] In the above setup, assume furthermore that $(\Lambda + \widehat{\rho}_\sigma)_{| \ \mathfrak{p}} = 0$. Then the following decomposition into a direct sum of irreducible $\widehat{L}(\mathfrak{a}, \sigma)$-modules holds:

$$Ker\,(D) = 2^{\lfloor \frac{\mathrm{rank}(\mathfrak{g}^{\overline{0}})) - \mathrm{rank}(\mathfrak{a}^{\overline{0}}) + 1}{2} \rfloor} \sum_{w \in \widehat{W}'} V(w(\Lambda + \widehat{\rho}_\sigma) - \widehat{\rho}_{\mathfrak{a}\,\sigma}).$$

By taking $\Lambda = 0$ and considering a symmetric subalgebra we recover via a multiplet approach the results obtained in previous papers for both the equal and non-equal rank cases. The proof proceeds along the lines of the finite-dimensional case, up to the fact that Parthasarathy's Dirac inequality is replaced by the following theorem, which can be viewed as an affine analogue of the "Vogan conjecture".

Theorem 3.2. [4, Theorem 8.1] Let $\mathfrak{g}$ be a semisimple Lie algebra, σ an elliptic automorphism of $\mathfrak{g}$ and $\mathfrak{a}$ a reductive quadratic subalgebra. Assume that the centralizer in $\mathfrak{g}^{\overline{0}}$ of Cartan subalgebra of $\mathfrak{a}$ is a Cartan subalgebra $_0$ of $\mathfrak{g}^{\overline{0}}$. Fix $\Lambda \in \widehat{\ }^*$ such that $\Lambda + \widehat{\rho}_\sigma$ is in the Tits cone of $\widehat{L}(\mathfrak{g}, \sigma)$ and let M be a highest weight module for $\widehat{L}(\mathfrak{g}, \sigma)$ with highest weight Λ. Let f be a holomorphic $\widehat{W_\sigma}$-invariant function on the Tits cone. Suppose that a twisted highest weight $\widehat{L}(\mathfrak{a}, \sigma)$-module of highest weight μ occurs in the Dirac cohomology of M. Then $f(\Lambda + \widehat{\rho}_\sigma) = f(\mu + \widehat{\rho}_{\mathfrak{a},\sigma})$.

4. Perspectives on the Lie superalgebra case

One of the key properties of the classical Dirac operator is the existence of a nice formula for its square. The replacement of the latter formula in our case is a nice expression for $[G_{\mathfrak{g}\,\lambda} G_{\mathfrak{g}}]$. Let now $\mathfrak{g}$ be a basic classical superalgebra, $\sigma = I_{\mathfrak{g}}$ and $M = L(\Lambda)$ be a highest module w.r.t. some positive system. If v is an highest weight vector in M, we compute that $G_0^X(v \otimes 1) = v \otimes (\overline{h}_{\overline{\Lambda}+\rho}) \cdot 1$ (here $\overline{\Lambda} = \Lambda_{|\ _0}$ and h_μ is defined by $\mu(h) = (h, h_\mu)$ for $\mu \in \ ^*_0$). By the above nice expression, $v \otimes 1$ is an eigenvector for $(G_0^X)^2$, so taking Λ such that $\overline{\Lambda} = -\rho$, we get the Freudenthal "strange" formula

$$(\rho, \rho) = \frac{g}{12} \mathrm{sdim} \mathfrak{g}. \tag{4.1}$$

For other (non-uniform) proofs of (4.1) see [8]. We also have a twisted version of this formula, which is an analogue of the "very strange formula". By applying the Zhu functor π_{Zhu} to our Dirac operator $D_{\mathfrak{g},\mathfrak{a}}$, we obtain a "finite-dimensional" Dirac operator in superalgebra setting which we are going to study in more detail. We have verified that $\pi_{Zhu}(D_{\mathfrak{g},\mathfrak{g}_0})$ is the Dirac operator defined in [3].

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Del Pezzo surfaces and homogeneous spaces

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(joint work with Alexei Skorobogatov)

Del Pezzo surfaces are smooth projective surfaces of degree d in $\mathbb{P}^d$. It is well known (see [3])that the Picard group of a del Pezzo surface X of degree $d\,(d \leq 6)$ contains a sublattice of roots of a simple Lie algebra of rank $r = 9 - d$ of Dynkin types $A_2 \times A_1$, A_4, D_5, E_6, E_7 and E_8 (this is the sublattice orthogonal to canonical class). Moreover, for $d > 1$, the number of exceptional divisors coincides with dimension of some minuscule representation of the corresponding algebraic group G. In 1990 Batyrev formulated the conjecture that X can be embedded into the quotient of the algebraic homogeneous space G/P (for a suitable parabolic P) by the action of the maximal torus H in G. This conjecture was proven by Popov for $d = 4$ and by Derenthal for $d = 3, 2$. We suggest another unified proof based on certain results about geometry of homogeneous spaces.

Let K be an algebraic closed field of characteristic zero. Let R be the root system of an algebraic group G from above list. Note that the previous Dynkin diagram of the list can be obtained from that of R by removing one simple root which we denote by α. Let P be the maximal parabolic subgroup corresponding to this root. The semisimple part of P is the previous algebraic subgroup G'. In all cases the nilpotent radical of the Lie algebra $\mathfrak{p}$ is abelian. The coroot $\check{\alpha}$ induces the $\mathbb{Z}$-grading on the Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

such that $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, $\mathfrak{g}_0$. Let V be the simple irreducible G-module with fundamental weight corresponding to α and v be a highest vector, so $Pv = Kv$. Note that in our case for $r \leq 7V$ is a minuscule representation. Consider the Z-grading

$$V = V_0 \oplus V_1 \oplus V_2 \oplus \dots$$

such that $V_0 = Kv$, $\mathfrak{g}_j V_i = V_{i-j}$. It is always true that $V_1 = \mathfrak{g}_{-1} V_0$. Moreover, V_1 is a simple G' module with highest vector $v' = r_\alpha(v)$. Recall that the orbit Gv is isomorphic to the affine cone $(G/P)_a$ of G/P. The following Lemma is true for an arbitrary G and maximal parabolic P.

Lemma 1. $(G/P)_a \cap V_1 = (G'/P')_a = G'v'$.

Let H be the maximal torus of G and T be the extension of H in $\mathrm{GL}(V)$ by scalar operators. Let $(G/P)_a^{sf}$ denote the set stable points in $(G/P)_a$ with trivial stabilizer in T.

Theorem 2. $Y = (G/P)_a^{sf}/T$ is a smooth quasi-projective variety and the torsor $f : (G/P)_a^{sf} \to Y$ is universal i.e. we have an isomorphism between the lattice of characters $\check{T}$ and $\mathrm{Pic}\, Y$.

The projection $V \to V_1$ induces the T-equivariant map $\pi : (G/P)_a \to V_1$, we denote by the same letter its projectivization $(G/P)^0 \to \mathbb{P}(V_1)$, where $(G/P)^0$ is obtained from G/P by removing the preimage of zero. Denote by S one dimensional subgroup in T

$$S = \left\{ g_t \colon g_{t|V_i} = t^{1-i}, t \in K^\times \right\}.$$

It is not difficult to see that for x\in $\mathbb{P}(V_1) \setminus (G'/P')$ the preimage $\pi^{-1}(x)$ is a free S-orbit. Moreover the following statement is true.

Theorem 3. The natural map $\bar{\pi} : (G/P)/S \to \mathbb{P}(V_1)$ is the blow up of G'/P' in $\mathbb{P}(V_1)$.

Now we formulate the main result.

Theorem 4. Let $\mathcal{T}$ be the universal torsor over del Pezzo surface X with torus T. There exists a T-equivariant embedding $\phi \colon \mathcal{T} \to (G/P)_a$ which induces an isomorhism $\check{T} \cong \operatorname{Pic} X \cong \operatorname{Pic} Y$. The restriction map $\operatorname{Pic} Y \to \operatorname{Pic} X$ maps T-invariant coordinate hyperplanes in V to exceptional divisors on X.

We prove the main theorem by induction starting with case $d = 5$ proven by Skorobogatov (see [6]). We may assume therefore that there exists an embedding $\mathcal{T}' \to (G'/P')_a \subset V_1$ satisfying all requirements of theorem. Note that X is obtained from X' by blowing up a point $x \in X'$ not lying on any of exceptional divisors in X'. Let Z denote the preimage of x in $\mathcal{T}'$. We were able to prove that there exists an element h in the centralizer of H in $\mathrm{GL}(V_1)$ such that $h(\mathcal{T}') \cap (G'/P') = Z$. In addition h should satisfy certain generality conditions which we do not discuss here (see for details [4]). Then we define $\mathcal{T}$ as the set of all stable point in Zariski closure of the preimage $\pi^{-1}(h\mathcal{T}')$.

One can also characterize $\mathcal{T}$ in the following way. Let F denote the centralizer of T in $\mathrm{GL}(V)$. Since V is minuscule, F itself is a torus which can be identified with open subset of V obtained by removing all coordinate hyperplanes.

Theorem 5. Let $k = r - 4$. There exists $h_1, \dots, h_k \in F$ such that

$$\mathcal{T} = h_1 (G/P)_a^{sf} \cap \dots \cap h_k (G/P)_a^{sf}.$$

In particular, it implies that the Cox ring of X is a quotient of a polynomial ring by ideal with quadratic generators (see also [7]).

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Exotic finite subgroups of E_8 and Springer's regular elements of the Weyl group

Bertram Kostant
(joint work with Nolan Wallach)

Distinguished elements of the Weyl group

1. Let $\mathfrak{g}$ be a complex simple Lie algebra and let G be a corresponding Lie group. Let $\ell = \operatorname{rank}\mathfrak{g}$ and let h be the Coxeter number of $\mathfrak{g}$. Let $\mathfrak{a} \subset \mathfrak{g}$ be a Cartan subalgebra and let $A \subset G$ be the corresponding group. Let $W = W(\mathfrak{a})$ be the Weyl group $\mathfrak{a}$ so that $W = \operatorname{Norm}\mathfrak{a}/A$. If $w \in W$, let $A_w \subset \operatorname{Norm}\mathfrak{a}$ be the coset of A defining w. Any element $a_w \in A_w$ will be called a lift of w. We will say that $w \in W$ is distinguished if $1 - w$ is invertible on $\mathfrak{a}$. (This terminology is motivated by a connection, established in [4], between such elements w and distinguished nilpotent elements in $\mathfrak{g}$.) If $w \in W$, then any two lifts of w are not, in general, conjugate in G. However, in fact one has

Proposition 1. *Let $w \in W$. Then a lift a_w of w is unique up to conjugacy in G if and only if w is distinguished.*

A Coxeter element $\sigma \in W$ is distinguished. We will write a_h for the unique (up to conjugacy) lift a_σ of σ. In the 1959 paper [5] Kostant proved

Theorem 2. *The element a_h is regular so there exists a unique Cartan subgroup H such that $a_h \in H$. Furthermore, if G is the adjoint group, then a_h has order h and a_h is the unique (up to conjugacy) regular element of minimal order.*

One knows $\dim\mathfrak{g} = \ell(h+1)$. The following is due to Victor Kac. See [3].

Theorem 3. (Kac) *Assume G is the adjoint group. Then there exists, up to conjugacy, a unique element $b_{h+1} \in G$ of order $h+1$ such that as an eigenvalue of the adjoint action of b_{h+1} every $h+1$ root of unity occurs with multiplicity ℓ.*

2. Let $\mathcal{F}_q$ be the finite field of q elements. We assume q is odd and write $q = 2\,d+1$. If $q \geq 5$, then the finite group $L_2(q)$ is simple and is of order $|L_2(q)| = 2\,(d+1)\,d\,(2d+1)$. Moreover $L_2(q)$ has an element b, referred to as elliptic, of order $d+1$, an element a, referred to as hyperbolic, of order d, and an abelian subgroup U_{2d+1} of order $q = 2d+1$ and normalized by the hyperbolic element a.

Some time ago Kostant conjectured that if $2h+1$ is a power of a prime, then for the adjoint group G, one has that $L_2(2h+1)$ embeds in G where, up to conjugacy, $a = a_h$ and $b = b_{h+1}$. This conjecture has been established. It is easy to prove this for a classical group. For the exceptional groups various people contributed to the proof.

$L_2(13)$ in G_2; $L_2(25)$ in F_4 and E_6; $L_2(37)$ in E_7; and the most subtle case (first proved by Cohen–Griess–Lisser in [1] and later, without the use of a computer by Serre in [6]) $L_2(61)$ in E_8. Subsequently Griess–Ryba proved, among other things, that $L_2(49)$ and $L_2(41)$ also embeds in E_8. See [2].

The main result of our paper here is to show, using results of T. Springer (see §9 in [7]), that there is a common pattern in the structure of the three subgroups $L_2(41)$, $L_2(49)$, $L_2(61)$ of E_8. To explain the pattern we first recall certain results in [5]. Assume $\mathfrak{g}$ is an arbitrary simple, complex Lie algebra and G is the adjoint group. There are two Cartan subalgebras associated to a Coxeter element, $\mathfrak{h}$ and $\mathfrak{a}_h$. The regular element a_h lies in a unique Cartan subgroup H and $\mathfrak{h} = \operatorname{Lie} H$. On the other hand $\mathfrak{a}_h$ is the Cartan subalgebra, stabilized by $\operatorname{Ad} a_h$, on which a_h induces a Coxeter element. One constructs $\mathfrak{a}_h$ from $\mathfrak{h}$ in a way, as follows, that exhibits a connection between the Coxeter element and the principal nilpotent in $\mathfrak{g}$. Let $\mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be a triangular decomposition of $\mathfrak{g}$. Let (x_h, e_h, f_h) be a principal $Sl(2)$ triple where $e_h \in \mathfrak{n}_+$ is principal nilpotent and $x_h \in \mathfrak{h}$. Then it is proved in [5] that z_h is a regular semisimple element where we put $z_h = e_h + e_{-\psi}$, where $e_{-\psi} \in \mathfrak{n}_-$ is the lowest root vector. Then $\mathfrak{a}_h = \mathfrak{g}^{z_h}$. Furthermore, if $a_h \in H$ is an element of order h in $\exp \mathbb{C} x_h$, then $\operatorname{Ad} a_h(z_h) = e^{2\pi i/h} z_h$ and $\operatorname{Ad} a_h|\mathfrak{a}_h \in W(\mathfrak{a}_h)$ is a Coxeter element. In addition, if $\mathcal{J}$ is a set of ℓ homogeneous generators of the ring of G-invariant polynomials on $\mathfrak{g}$, there exists a unique element $I_h \in \mathcal{J}$ of maximal degree h, and the conjugacy class (up to scalar multiplication) of z_h is determined by the fact that $I_h(z_h) \neq 0$, but $I(z_h) = 0$ for all other $I \in \mathcal{J}$.

Now assume $G = E_8$. To explain Springer's generalization of the results above and its connection with the three finite groups of Cohen–Griess–Lisser–Ryba, let $J = \{2, 8, 12, 14, 18, 20, 24, 30\}$ so that we can write $\mathcal{J} = \{I_j\}$, $j \in J$, where $\deg I_j = j$. Let $K = \{20, 24, 30\}$ so that K is the maximum 3-element subset of J. The finite groups then are $L_2(2k+1)$, $k \in K$. On the other hand, Springer's extension of the Coxeter case is to consider also the $SL(2)$ triples (x_k, e_k, f_k) where for $k = 24$, $e_k \in \mathfrak{n}_+$ is the subregular nilpotent, and for $k = 20$, $e_k \in \mathfrak{n}_+$ is the sub-subregular nilpotent element. He then proves, as in the regular case, $z_k = e_k + e_{-\psi}$ is regular semisimple so that for any $k \in K$, $\mathfrak{g}^{z_k} = \mathfrak{a}_k$ is a Cartan subalgebra. Also up to scalar multiplication the conjugacy class of z_k is characterized by the condition that $I_k(z_k) \neq 0$ but $I_j(z_k) = 0$ for $k \neq j \in J$. Next, if $a_k \in \exp \mathbb{C} x_k$ has order k, then $\operatorname{Ad} a_k(z_k) = e^{2\pi i/k} z_k$. Thus $\sigma_k \in W(\mathfrak{a}_k)$ is regular (Springer's definition) where $\sigma_k = \operatorname{Ad} a_k|\mathfrak{a}_k$. In addition, σ_k is distinguished and $\mathfrak{a}_k$ is its unique (up to conjugacy) lift. This establishes a connection between other nilpotent elements and certain regular, distinguished elements in the Weyl group. Moreover, as sort of a generalization of Theorem 2, we can characterize the conjugacy class of a_k in G in Theorem 4 below. Note that $240/k$ takes the values 8, 10, and 12 for $k = 30$, 24, and 20.

Theorem 4. *Let $a \in G$ have order k. Then $\dim \mathfrak{g}^a \geq 240/k$ where equality occurs if and only if a is conjugate to a_k.*

Now connecting with the finite groups $L_2(2k+1)$ of E_8, the following result relies almost exclusively on information from Alex Ryba.

Theorem 5. *The hyberbolic element in $L_2(2k+1) \subset E_8$ is conjugate in E_8 to a_k.*

The Euler number of k is 8 and in fact the characteristic polynomial of σ_k is the cyclotomic (degree 8) polynomial Φ_k. By factoring Φ_k over field $\mathbb{F}(d)$ when $d = 61$, 7, and 41 we can find the abelian subgroup, normalized by a_k, U_{2K+1} of $L_2(2k+1)$ in A_k where Lie $A_k = \mathfrak{a}_k$. With respect to the adjoint action of U_{2k+1} on Lie E_8 one has

Theorem 6. *The identity character of* U_{2k+1} *occurs* 8 *times. For the remaining* $2k$ *characters each occurs with multiplicity* $120/k$.

We deeply thank Alex Ryba for providing us with so much information about $L_2(2k+1)$.

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Weyl modules: A categorical approach

Ghislain Fourier
(joint work with Vyjayanthi Chari, Tanusree Pal)

Let $\mathfrak{g}$ be a simple complex Lie algebra and A a commutative, finitely generated algebra over $\mathbb{C}$ with a unit, then $\mathfrak{g} \otimes A$ can be equipped with a Lie structure by

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab.$$

Let $\mathcal{I}(A)$ be the category of $\mathfrak{g} \otimes A$ modules, that are locally finite as $\mathfrak{g}$ modules. For $A = \mathbb{C}[t]$, this category ihas been subject to a lot of research in the last decade, closely related also to finite dimensional modules for the quantum affine algebras. It is natural to extend the research to the general case, as it was done by [2] for example.
It is easy to define projective modules for $\mathcal{I}(A)$: Let V be a $\mathfrak{g}$ module, than $P(V) := \mathbb{U}(\mathfrak{g} \otimes A) \otimes_{\mathfrak{g}} V$ is a projective left $\mathfrak{g} \otimes A$ module. We want to restrict ourselves to a more suitable class of modules. If μ is a dominant integral weight, we define $P(V)^\mu$ to be the maximal quotient of $P(V)$, such that

$$\operatorname{Hom}_{\mathfrak{g}}(V(\tau), P(V)^\mu) \neq 0 \Rightarrow \tau \leq \mu$$

If λ is dominant integral, we denote by $V(\lambda)$ the irreducible $\mathfrak{g}$ module with highest weight λ and by $v_\lambda \in V(\lambda)$ a generator of the line of weight λ. So we can define $\mathbb{W}_A(\lambda) := P(V(\lambda))^\lambda$. Then $\mathbb{W}_A(\lambda)$ is a cyclic $\mathfrak{g} \otimes A$ module in the category $\mathcal{I}(A)$, generated by $1 \otimes v_\lambda$. $\mathbb{W}_A(\lambda)$ has a natural structure as a bi-module for $\mathfrak{h} \otimes A$, the Heisenberg algebra. Of course $\mathbb{W}_A(\lambda)$ is not cylcic for $\mathbb{U}(\mathfrak{h} \otimes A)$ but its finitely generated. Define

$$\mathbb{A}_\lambda := U(\mathfrak{h} \otimes A)/(\mathrm{Ann}_{\mathbb{U}(\mathfrak{h}\otimes A)}(1 \otimes v_\lambda)).$$

An important result is the following:

Irreducible $\mathfrak{g} \otimes A$ modules with highest weight λ are parametrized by points in MaxSpec($\mathbb{A}_\lambda$). These modules are finite dimensional, in fact they are tensor products of evaluation modules.

So we know the "smallest" finite dimensional modules with a given highest weight space. So obtain the "largest" we introduced the following functor $\mathbb{W}_A$ from the categroy of finite dimensional left $\mathbb{A}_\lambda$-modules to $\mathcal{I}(A)$

$$F \mapsto \mathbb{W}_A(\lambda) \otimes_{\mathfrak{h}\otimes A} F$$

we denote this module by $\mathbb{W}_A(F)$. It is easy to see, that $\mathbb{W}_A(F)$ is finite dimensional and its irreducible quotients are parametrized by the irreducible quotients of F as an $\mathbb{A}_\lambda$-module. We have the following theorem

The functor $\mathbb{W}_A$ is exact

So a Jordan-Hölder serie for F is sent to a filtration of $\mathbb{W}_A(F)$ by modules of the form $\mathbb{W}_A(\mathbb{C}_\xi)$, where $\mathbb{C}_\xi$ denotes the irreducible $\mathbb{A}_\lambda$-module corresponding to the point $\xi \in$ maxSpec($\mathbb{A}_\lambda$).
So to analyze the module $\mathbb{W}_A(F)$, for example getting knowledge about the dimension or $\mathfrak{g}$ structure, it is enough to analyze $\mathbb{W}_A(\mathbb{C}_\xi)$. We provide a description of $\mathbb{W}_A(\mathbb{C}_\xi)$ for an open dense subset in MaxSpec($\mathbb{A}_\lambda$) as a tensor product of the "smallest" Weyl modules, the modules with highest weight ω_i. So it remains to analyze $\mathbb{W}_A(\mathbb{C}_\xi)$, where $\xi \in$ MaxSpec(A_{ω_i}).
We provide a complete list for these in the case where $A = \mathbb{C}[t_1, \ldots, t_n]$ and $\mathfrak{g}$ is of type A, B, C, D.

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The index of centralizers of elements in a reductive Lie algebra

Anne Moreau
(joint work with Jean-Yves Charbonnel)

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over an algebraically closed field $\Bbbk$ of characteristic zero and consider the coadjoint representation $\mathrm{ad}(\mathfrak{g}^*)$. The *index* of $\mathfrak{g}$ is the minimal dimension of stabilizers $\mathfrak{g}^x$ of $x \in \mathfrak{g}^*$ (cf. [4]): $\mathrm{ind}\,\mathfrak{g} = \min_{x\in\mathfrak{g}^*} \dim \mathfrak{g}^x$. The notion of the index is very important in representation theory and also in invariant theory; by Rosenlicht's theorem, if $\mathfrak{g}$ is an algebraic Lie algebra, $\mathrm{ind}\,\mathfrak{g} = \deg\mathrm{tr}\,\Bbbk(\mathfrak{g}^*)^{\mathfrak{g}}$, where $\Bbbk(\mathfrak{g}^*)^{\mathfrak{g}}$ is the field of $\mathfrak{g}$-invariant rational functions over $\mathfrak{g}^*$. The index of a reductive algebra is equal to its rank. Computing the index of an arbitrary Lie algebra seems to be a wild problem. However, there is numbers of interesting results for several classes of nonreductive subalgebras of reductive Lie algebras. For example, the centralizers of elements form an interesting class of subalgebras (cf. [5], [7], [10]). This topic is closely related to the theory of integrable Hamiltonian systems [1]. Let us precise this link:

From now on, $\mathfrak{g}$ is supposed to be reductive and we denote by G the adjoint group of $\mathfrak{g}$. The symmetric algebra $\mathrm{S}(\mathfrak{g})$ carries a natural Poisson structure. A Poisson-commutative subalgebra $\mathcal{F}_x$ ($x \in \mathfrak{g}^*$) of $\mathrm{S}(\mathfrak{g}) = \Bbbk[\mathfrak{g}^*]$, called the *shift of argument subalgebra*, was defined in [6]. It is generated by the derivatives of all orders in the direction $x \in \mathfrak{g}^*$ of all elements of the algebra of $\mathfrak{g}$-invariants of $\mathrm{S}(\mathfrak{g})$. Moreover, if $G.x$ denotes the coadjoint orbit of $x \in \mathfrak{g}^*$:

Theorem 1 ([1], Theorem 2.1)**.** There is a Poisson-commutative family of polynomial functions on $\mathfrak{g}^*$, constructed by the shift of argument method, such that its restriction to $G.x$ contains $\frac{1}{2}\dim(G.x)$ algebraically independent functions if and only if $\mathrm{ind}\,\mathfrak{g}^x = \mathrm{ind}\,\mathfrak{g}$.

Motivated by the preceding result of Bolsinov, A.G. Elashvili formulated the conjecture:

Conjecture 2 (Elashvili)**.** Let $\mathfrak{g}$ be a reductive Lie algebra. Then $\mathrm{ind}\,\mathfrak{g}^x = \mathrm{rk}\,\mathfrak{g}$ for all $x \in \mathfrak{g}^*$, where $\mathrm{rk}\,\mathfrak{g}$ is the rank of $\mathfrak{g}$.

Elashvili's conjecture also appears in the following problem: Is the algebra $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ of invariants in $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ under the adjoint action a polynomial algebra? This question was formulated by A. Premet in [9, Conjecture 0.1]. Under certain hypothesis, and under the condition that Elashvili's conjecture holds, the algebra of invariants $\mathrm{S}(\mathfrak{g}^x)^{\mathfrak{g}^x}$ is polynomial in $\mathrm{rk}\,\mathfrak{g}$ variables, cf. [9, Theorem 0.3].

During the last decade, Elashvili's conjecture caught attention of many invariant theorists (e.g. [7], [2], [10], [3], [9]). The conjecture reduces to the case where $\mathfrak{g}$ is simple and where $x \in \mathfrak{g}^* \simeq \mathfrak{g}$ is a nilpotent element. Let us review what is known so far about Elashvili's conjecture. First, the conjecture is true for certain classes of nilpotent elements (e.g. regular, subregular, spherical,...); see [7], [8]. More recently, O. Yakimova proved the conjecture in the classical case [10]. To valid the conjecture in the exceptional type, W. Degraaf (and independenty J-Y. Charbonnel for the types E_7 and E_8) used the computer programme GAP (cf. [3]). Since there are many nilpotent orbits in the Lie algebras of exceptional type, it is difficult to present the results of such computations in a concise way. In 2004, Charbonnel published a case-free proof of Elashvili's conjecture applicable to all simple Lie algebras; see [2]. Unfortunately, the argument in [2] has a gap in the final part of the proof, which was pointed out by L. Rybnikov. To summarize, so far, there is no conceptual proof of that conjecture applicable to all finite-dimensional simple Lie algebras.

In a joint project with J.-Y. Charbonnel, we are currently trying to find such a conceptual proof. Our goal is almost reached. Our proof is unfortunately not for the moment totally GAP-free. Our approach is very different than those used before; we use Bolsinov's criterion to show that the conjecture reduces to the case of rigid nilpotent orbits. Our results can be summarized as follows:

- Conjecture 2 is true for all Richardson nilpotent elements, i.e. $\operatorname{ind}\mathfrak{g}^e = \operatorname{rk}\mathfrak{g}$ when $e \in \mathfrak{g}$ is a Richardson nilpotent element;
- If Conjecture 2 holds for all rigid nilpotent elements of any simple Lie algebra, then so does for all induced nilpotent elements of $\mathfrak{g}$;
- Conjecture 2 holds for "most" rigid nilpotent orbits. Namely, it always holds in classical type, in types G_2, F_4 and E_6 and for many rigid nilpotent orbits in types E_7 or E_8;
- It remains one rigid nilpotent orbit in type E_7 and 6 rigid nilpotent orbits in type E_8 to deal with. We handle these cases with the help of GAP.

The proofs of the first two points rely on Bolsinov's criterion. Our approach to deal with the rigid case is totally different; we use here properties of Slodowy slides.

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A combinatorial description of the $\widehat{\mathfrak{sl}}(n)_k$ fusion ring

Catharina Stroppel
(joint work with Christian Korff)

This talk presents a combinatorial construction of the fusion ring (or Verlinde algebra) associated with the $\widehat{\mathfrak{sl}}(n)_k$-Wess-Zumino-Novikov-Witten (WZNW) model in conformal field theory. The aim of the talk is to describe a precise relationship between this fusion ring and the (small) quantum cohomology of the Grassmannian. As a result we get explicit identities between the structure constants of the two rings. We describe the structure constants of the fusion ring in terms of cyclic non-commutative Schur functions acting on a space $\mathcal{H}_k$. This allows a simplified proof of associativity of the fusion product and a simple derivation of the Verlinde formula. Moreover, we explicitly construct a common eigenbasis for our Schur functions using the so-called Bethe Ansatz (a standard tool in quantum integrable models). The eigenvalues are then given by certain Weyl characters expressed in terms of commutative Schur functions, and one can deduce that the non-commutative symmetric functions share many nice properties with the usual commutative symmetric function. This construction directly relates to results of Rietsch ([12]) on the quantum cohomology side and is motivated by the work of Postnikov ([10]). Finally these eigenvalues will also be used to show (via the Verlinde formula) that our combinatorially defined ring is in fact the fusion ring, the eigenvalues from above turn up as entries in the modular S-matrix defining the structure constants of the fusion ring. Details and proofs will appear in [9].

Quantum Cohomology. Let $n, k \in \mathbb{Z}_+$ and denote by $\mathrm{Gr}(k, n+k)$ the Grassmannian of k-planes inside $\mathbb{C}^{n+k}$. Let $QH^\bullet(\mathrm{Gr}(k, n+k))$ be its small quantum cohomology ring. This is a $\mathbb{Z}[q]$-algebra which is isomorphic to $\mathbb{Z}[q] \otimes_\mathbb{Z} H^\bullet(\mathrm{Gr}(k, n+k))$ as a $\mathbb{Z}[q]$-module. In particular, the Schubert classes give a $\mathbb{Z}[q]$-basis $\{1 \otimes [\Omega_\lambda]\}$. Here λ runs through all partitions whose Young diagram fits into a box of size k times n. This module has a ring structure where the structure constants $C^\nu_{\lambda,\mu}(q) = \sum C^{\nu,d}_{\lambda,\mu} q^d$ are given by the so-called 3-point Gromov-Witten invariants $C^{d,\check{\nu}}_{\lambda,\mu}$ which count the number of rational curves of degree d passing through generic translates of Ω_λ, Ω_μ, Ω_ν. (In the cases $|\lambda| + |\mu| + |\nu| \neq kn + d(k+n)$, where the number of curves could be infinite, one just puts $C^{d,\nu}_{\lambda,\mu} = 0$.) Siebert and Tian ([14]) gave an explicit presentation of $QH^\bullet(\mathrm{Gr}(k, n+k))$ in terms of $\Lambda = \mathbb{Z}[e_1, e_2, \ldots]$, the ring of symmetric polynomials:

$$QH^*(\mathrm{Gr}(k, k+n) \cong \mathbb{Z}[q] \otimes \Lambda / \left\langle h_{n+1}, ..., h_{n+k-1}, h_{n+k} + (-1)^k q \right\rangle$$

where the e_i's are the elementary symmetric functions in k variables and the h_i's are the complete symmetric functions. Mapping a Schur polynomial s_λ to the Schur polynomial s_{λ^t} of the dual partition defines an isomorphism $QH^\bullet(\mathrm{Gr}(k, n+k)) \cong QH^\bullet(\mathrm{Gr}(n, n+k))$ which we call the *duality isomorphism*.

Fusion ring. Consider the non-twisted affine Lie algebra $\hat{\mathfrak{sl}}(n) = \mathfrak{sl}(n) \oplus \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\boldsymbol{c} \oplus \mathbb{C}\boldsymbol{d}$ obtained from the central extension of the loop algebra by adding a derivation $\boldsymbol{d}$. Consider the extended Cartan subalgebra $\hat{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}\mathbf{c} \oplus \mathbb{C}\mathbf{d}$ and let $\delta, \hat{\omega}_0 \in \hat{\mathfrak{h}}^*$ such that $\delta(\mathbf{d}) = 1$, $\delta(\mathbf{c}) = \delta(h) = 0$ and $\hat{\omega}_0(\mathbf{c}) = 1$, $\hat{\omega}_0(\mathbf{d}) = \hat{\omega}_0(h) = 0$, for $h \in \mathfrak{h}$. We denote by Γ the Dynkin diagram of $\widehat{\mathfrak{sl}}(n)$, which we view as a circle with n equidistant marked points. We name these points $0, 1, \ldots, n-1$ in clockwise direction. Let $\omega_i \in \mathfrak{h}^*$ be the fundamental weights for $\mathfrak{sl}_n$, considered as elements in $\hat{h}^*$ and set $\hat{\omega_i} = \omega_i + \hat{\omega}_0$. Now fix $k \in \mathbb{Z}_{>0}$ and $\gamma \in \mathbb{C}$ and consider

$$P_k^+ = \left\{ \hat{\lambda} = \sum_{i=0}^{n-1} m_i \hat{\omega}_i + \gamma\delta \;\middle|\; \sum_{i=1}^{n-1} m_i = k \right\}, \tag{0.1}$$

the set of *integral dominant weights of level k*. The m_i appearing here are often called Dynkin labels and will be denoted $m_i(\hat{\lambda})$ in the following. Since (up to a grading induced by the action of $\boldsymbol{d}$) the integrable highest weight modules are independent of the choice of γ, the particular choice will not be important for us and we therefore assume from now on $\gamma = 0$.

To get a connection with the quantum cohomology ring it is convenient to encode affine weights in terms of partitions (or equivalently their Young diagrams). The following is obvious

Lemma 1. With the notation from (0.1), there is a bijection of sets

$$\begin{array}{llll} \mathrm{P}: & P_k^+ & \longrightarrow & \{\text{Young diagrams with at most } n-1 \text{ rows and } k \text{ columns}\} \\ & \hat{\lambda} & \longmapsto & (\mu_1, \ldots, \mu_{n-1}, 0, \ldots) \text{ with } \mu_i - \mu_{i+1} = m_i. \end{array}$$

(The associated Young diagram has then exactly m_i columns of length i.)

Given Ω_λ, denote by λ' the preimage under P of the Young diagram obtained from the one for λ by removing all columns of length n.

The fusion ring $\mathcal{F}$ is defined as the free abelian group generated by the $\hat{\lambda} \in P_k^+$ equipped with the so-called fusion product

$$\hat{\lambda} \star \hat{\mu} = \sum_{\hat{\nu}} \mathcal{N}_{\hat{\lambda},\hat{\mu}}^{(k),\hat{\nu}} \hat{\nu}.$$

The structure constants are given by the so-called Verlinde formula

$$\mathcal{N}_{\hat{\lambda}\hat{\mu}}^{(k),\hat{\nu}} = \sum_{\hat{\sigma} \in P_+^k} \frac{\mathcal{S}_{\hat{\lambda}\hat{\sigma}} \mathcal{S}_{\hat{\mu}\hat{\sigma}} \bar{\mathcal{S}}_{\hat{\nu}\hat{\sigma}}}{\mathcal{S}_{0\hat{\sigma}}}, \tag{0.2}$$

where the S is the modular S matrix (see [1]). Its matrix entries are implicitly defined by $\mathfrak{sl}(n)$-characters or the Kac-Peterson formula ([15],[8]).

Our main result is the following

Theorem 1. (1) Sending $\hat{\lambda}$ to s_{λ^t} defines an isomorphism of rings

$$\mathcal{F} \cong \Lambda / \left\langle h_{n+1}, \ldots, h_{n+k-1}, h_{n+k} + (-1)^k e_k \right\rangle,$$

hence realizes the fusion ring $\mathcal{F}$ as a quotient of the quantum cohomology ring $QH^*(\mathrm{Gr}(k, k+n)$ by imposing the extra relation $e_k = q$.

(2) For classes $[\Omega_\lambda]$, $[\Omega_\mu]$, $[\Omega_\nu]$, we have $C^{\nu,d}_{\lambda,\mu} = \mathcal{N}^{(k),A^d(\nu')}_{\lambda',\mu'}$, where A is the Dynkin graph automorphism which sends vertex i to vertex $i+1$ (modulo n). The duality morphism translates into the level-rank duality.

(3) $\left(\mathcal{N}^{(k),\hat{\nu}}_{\hat{\lambda},\hat{\mu}}\right)_{\hat{\mu},\hat{\nu}} = \mathbf{s}_\lambda(a_1, a_2, \ldots, a_n)$, where $\mathbf{s}_\lambda(a_1, a_2, \ldots, a_n)$ denotes the Schur polynomial for λ in non-commutative variables acting on $\mathcal{H}_k[z]$ (see below).

For the case $\hat{\mathfrak{su}}(n)$, the connection between the fusion ring and the quantum cohomology is not new, it was already established by Gepner ([6], see also [16]). Presentations of $\mathcal{F}$ are also well-known. Our new input here is the construction of a combinatorial fusion ring from which all the results follow.

The combinatorial fusion ring. Let $\mathcal{H}_k = \mathbb{C}P^k_+$ be the vector space spanned by the set P^+_k (or alternatively by the set of partitions given by Lemma 0.2). Let $\mathcal{H} = \oplus_k \mathcal{H}_k$, and $\mathcal{H}[z] = \oplus_k \mathcal{H}_k[z]$ for a formal parameter z.

The *phase algebra* is the subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathcal{H})$ generated by the endomorphisms φ^*_i, φ_i and N_i, $1 \le i \le n$, where we have for $\lambda \in P^k_+$, $\varphi^*_i(\hat{\lambda}) = \hat{\lambda} + \hat{\omega}_i$, $\varphi_i(\hat{\lambda}) = \hat{\lambda} - \hat{\omega}_i$ if $m_i(\hat{\lambda}) > 0$, and $\varphi_i(\hat{\lambda}) = 0$ otherwise.

Lemma 2. The phase algebra acts faithfully on $\mathcal{H}$ and has a PBW-type basis

$$\{B_{\mathbf{b},\mathbf{a},\mathbf{c}} := {\varphi_1^*}^{b_1} {\varphi_2^*}^{b_2} \cdots {\varphi_n^*}^{b_n} \varphi_1^{a_1} \varphi_2^{a_2} \cdots \varphi_n^{a_n} N_1^{c_1} N_2^{c_2} \cdots N_n^{c_n}\}, \tag{0.3}$$

where $a_i, b_i, c_i \in \mathbb{Z}_{\ge 0}, a_i b_i c_i = 0$ for $1 \le i \le n$.

The *local affine plactic algebra* is the algebra A generated by $a_0, a_1, a_2 \ldots a_{n-1}$ modulo the relations

$$a_i a_j - a_j a_i = 0 \text{ if } |i-j| > 1 \mod n, \tag{0.4}$$

$$a_{i+1} a_i^2 = a_i a_{i+1} a_i \quad a_{i+1}^2 a_i = a_{i+1} a_i a_{i+1}, \tag{0.5}$$

where in (0.5) all variables are understood as elements in A by taking indices modulo n. This algebra generalizes the plactic algebra introduced by [7] and its local version from [4].

Proposition 1. There is an action of A on $\mathcal{H}_k[z]$ given by $a_j \mapsto \varphi_{j-1}\varphi^*_j$ for $j \neq 0$ and $a_n \mapsto z\varphi_{j-1}\varphi^*_j$ giving rise to a faithful representation of A on $\mathcal{H}[z]$.

Remark 1. There is a 2-parameter quantization $U^+_{\alpha,\beta}$ of the positive part of the universal enveloping algebra of $\hat{\mathfrak{sl}}(n)$ such that $U^+_{q,q} \cong U^+_q$ is Ringel's Hall algebra [13] and $U^+_{0,1} \cong A$ is the Hall algebra with generic extensions ([11], [3]). The action of A on $\mathcal{H}_k[z]$ gives rise to the crystal of the k-th exterior power of the vector representation.

For $0 \le r \le n-1$ define the r-th (noncommutative) elementary symmetric function

$$e_r(\mathcal{A}) = \sum_{|I|=r} \prod_{i \in I}^{\circlearrowleft} a_i \tag{0.6}$$

where the sum runs over all subsets of $\{1, 2, \dots n\}$ of order r and the variables in the monomials are ordered anti-clockwise, and set $a_n = z1$.

We use the phase algebra to define the *Yang-Baxter algebra.* The construction resembles the RTT-construction of Yangians, (but the matrix $L(0)$ is singular). For $i \in \{0, 1, 2, ..., n-1\}$, and $u \in \mathbb{C}^*$ the i-th *Lax matrix* $L_i = L_i(u)$ is the following endomorphism of $\mathbb{C}^2 \otimes \mathfrak{H}$

$$L_i(u) = \begin{pmatrix} 1 & u\varphi_i^* \\ \varphi_i & u1 \end{pmatrix} \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2 \otimes \mathfrak{H}) \,. \tag{0.7}$$

The complex variable $u \in \mathbb{C}$ is called the *spectral parameter.* The *monodromy matrix* is defined as

$$M(u) = L_{n-1}(u) \cdots L_1(u) L_0(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2 \otimes \mathfrak{H}). \tag{0.8}$$

Lemma 3 (cf. [2])**.** The monodromy matrix is a solution to the RTT-relation

$$R_{12}(u/v) M_1(u) M_2(v) = M_2(v) M_1(u) R_{12}(u/v), \qquad u, v \in \mathbb{C} \tag{0.9}$$

with

$$R(u) = \begin{pmatrix} \frac{u}{u-1} & 0 & 0 & 0 \\ 0 & 0 & \frac{u}{u-1} & 0 \\ 0 & \frac{1}{u-1} & 1 & 0 \\ 0 & 0 & 0 & \frac{u}{u-1} \end{pmatrix} \in \mathrm{End}\, \mathbb{C}^4 \cong \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2 \otimes \mathbb{C}^2) \,. \tag{0.10}$$

For generic u, the algebra generated by the coefficients in the power series A, B, C, D modulo the (0.9)-relations is called the Yang-Baxter algebra. The transfer matrix is the endomorphism $T(u) = A(u) + zD(u)$ of $\mathcal{H}_k[z]$. The RTT-relation directly implies $[T(u), T(v)] = 0$

Corollary 1 (T is generating function for the e's)**.**

(1) $T(u)_k = A(u)_k + zD(u)_k = \sum_{r=0}^{n} e_r(\mathfrak{A})_k u^r$

(2) The elementary symmetric polynomials e_r, $r = 1, \dots n$ (as endomorphisms of $\mathcal{H}[z]$) pairwise commute. In particular it makes sense to define the (non-commutative) Schur polynomials $\mathbf{s}_\lambda$ using the usual determinant formula.

The Bethe Ansatz gives eigenvectors for the action of the elementary symmetric functions. The Bethe Ansatz equations are equivalent to the equations $h_{n+1} = 0, ..., h_{n+k-1} = 0, h_{n+k} + (-1)^k e_k = 0$. Define the combinatorial fusion ring $\mathcal{F}_{\mathrm{comb}}$ to be the ring with basis $\hat{\lambda} \in P_k^+$ and multiplication $\hat{\lambda} \star' \hat{\mu} = \mathbf{s}_\lambda \hat{\mu}$.

Theorem 2. The two products ($\star$ and $\star'$) coincide, in particular $\mathcal{F} \cong \mathcal{F}_{comb}$.

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Equivariant Sheaves on Flag Varieties

Olaf M. Schnürer

The aim of our talk was to give an algebraic description of the Borel-equivariant derived category of sheaves on the flag variety of a connected reductive algebraic group.

Let G be a complex algebraic group acting on a complex variety X. We introduced the G-equivariant (bounded, constructible) derived category $\mathcal{D}^b_{G,c}(X)$ of sheaves of real or complex vector spaces on X (see [BL94]). It carries the perverse t-structure with heart the category of G-equivariant perverse sheaves. If G acts with finitely many orbits, there are only finitely many simple objects in this heart; we denote their direct sum by $\mathcal{IC}$. The extension algebra of this object is

$$\operatorname{Ext}(\mathcal{IC}) := \bigoplus_{n\in\mathbb{N}} \operatorname{Hom}(\mathcal{IC}, \mathcal{IC}[n]).$$

We view this graded algebra as a differential graded (dg) algebra with differential $d = 0$.

Let $\mathcal{A}$ be a dg algebra. We defined the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ (see e. g. [Kel98]). The perfect derived category $\operatorname{Perf}(\mathcal{A})$ of $\mathcal{A}$ is the thick subcategory of $\mathcal{D}(\mathcal{A})$ generated by $\mathcal{A}$ (i. e. the smallest full triangulated subcategory that contains $\mathcal{A}$ and is closed under taking direct summands). Its objects are precisely the compact objects in $\mathcal{D}(\mathcal{A})$.

The following conjecture of Soergel and Lunts (cf. [Lun95]) relates the geometric category $\mathcal{D}^b_{G,c}(X)$ and the algebraic category $\mathrm{Perf}(\mathrm{Ext}(\mathcal{IC}))$: If a complex reductive group G acts on a projective variety X with finitely many orbits, there is an equivalence of triangulated categories

$$\mathcal{D}^b_{G,c}(X) \cong \mathrm{Perf}(\mathrm{Ext}(\mathcal{IC})).$$

This conjecture (or a similar statement) is known to be true for a connected Lie group acting on a point ([BL94, 12.7.2]), for a torus acting on an affine or projective normal toric variety ([Lun95]), and for a complex semisimple adjoint group acting on a smooth complete symmetric variety (in the sense of de Concini and Procesi) ([Gui05]). We recently became aware of a related result for the loop rotation equivariant derived Satake category of the affine loop Grassmannian in [BF08]. Our main result is:

Theorem 1 ([Sch08])**.** Let G be a complex connected reductive affine algebraic group, $B \subset G$ a Borel subgroup, and $X = G/B$ the flag variety. Then there is an equivalence of triangulated categories

$$\mathcal{D}^b_{B,c}(X) \cong \mathrm{Perf}(\mathrm{Ext}(\mathcal{IC})).$$

We conclude with some remarks:

- Note that $\mathcal{D}^b_{B,c}(X)$ is equivalent to $\mathcal{D}^b_{G,c}(G \times_B X)$ or $\mathcal{D}^b_{G,c}(X \times X)$ by the induction equivalence. Hence our result fits into the setting of the conjecture.
- The perverse t-structure on $\mathcal{D}^b_{B,c}(X)$ corresponds to a t-structure on the perfect derived category $\mathrm{Perf}(\mathrm{Ext}(\mathcal{IC}))$ that can be described for a more general class of dg algebras (see [Sch08a]). This yields an algebraic description of the category of B-equivariant perverse sheaves on X.
- The algebra $\mathrm{Ext}(\mathcal{IC})$ is isomorphic to the endomorphism algebra of the B-equivariant hypercohomology of $\mathcal{IC}$ ([Soe01]); this hypercohomology can be described using Soergel's bimodules or the moment graph picture ([BM01]). In particular, the category $\mathcal{D}^b_{B,c}(X)$ depends only on the corresponding combinatorial data.
- The non-equivariant analog of this theorem is also true. In fact, we prove the theorem as a limit of equivalences that are similar to the non-equivariant analog. For the proof of the non-equivariant version we need the formality of a carefully constructed dg algebra; to obtain this formality we use mixed Hodge modules and purity results on intersection cohomology complexes.

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MV-polytopes/cycles and affine buildings

Michael Ehrig

1. Basic Notations and Definitions

We want to give a combinatorial construction of MV-polytopes. This is done by using the LS-gallery model by Gaussent and Littelmann [1], a discrete and building-theoretic version of Littelmann's path model. This gives a construction of MV-polytopes alternativ to the one given by Kamnitzer in [2] and [3] and independent of the type of the algebraic group. We start by fixing the basic notations.

Notation 1.1. By G we denote a complex, simply-connected, semi-simple algebraic group. We fix $B \subset G$ a Borel subgroup, $T \subset B$ a maximal torus, and denote by W its Weyl group. In addition we denote by B^- the Borel subgroup opposite to B, i.e., the Borel subgroup such that $B \cap B^- = T$, and by U^- its unipotent radical. Finally we denote by $\mathcal{O} = \mathbb{C}[[t]]$ the ring of formal power series and by $\mathcal{K} = \mathbb{C}((t))$ its field of fraction, the field of formal Laurent series.

Using these we have a number of associated objects.

Notation 1.2. Let us denote by $\mathcal{G} = G(\mathcal{K})/G(\mathcal{O})$ the affine Grassmannian, by $X^\vee$ the coweight lattice of G, and by $X^\vee_+$ the dominant coweights.

Let us now look at the basic geometric set-up:.

$$X^\vee \xhookrightarrow{i} \mathcal{G} \hookrightarrow \mathbb{P}(V) \xrightarrow{\boldsymbol{\mu}} X^\vee \otimes \mathbb{R}\,.$$

The inclusion i is an inclusion as T-fixed points and we denote the image of a coweight λ by t^λ, $\mathbb{P}(V)$ is a projective space over a suitable representation of the affine Kac-Moody group $\hat{\mathcal{L}}(G)$ corresponding to G, and the map $\boldsymbol{\mu}$ is its usual moment map.

Using the left multiplication of $G(\mathcal{K})$ on the affine Grassmannian the definition of MV-cycles, which first appear in [4], is then straight-forward.

Definition 1.3. Let $\lambda \in X^\vee_+$ and $\mu \in X^\vee$. An MV-cycle of coweight (λ, μ) is a non-empty $M \in \operatorname{Irr}(\overline{G(\mathcal{O}).t^\lambda \cap U^-(\mathcal{K}).t^\mu})$.

By a theorem of Mirković-Vilonen , MV-cycles for a fixed dominant coweight λ and arbitrary coweight μ form a natural basis for the irreducible heighest weight representation with heighest weight λ of $G^\vee$ the Langlands dual of G.

Since by definition an MV-cycle is a closed, T-stable subvariety of the affine Grassmannian, we can use results of Brion or Goresky and MacPherson to know that their image under the moment map $\boldsymbol{\mu}$ will be a convex polytope in $X^\vee \otimes \mathbb{R}$. This was first done by Anderson in [5] and later investigated further by Kamnitzer.

Definition 1.4. A polytope P in $X^\vee \otimes \mathbb{R}$ is called an MV-polytope (of coweight (λ, μ)) if there exists an MV-cycle (of coweight (λ, μ)), such that

$$P = \boldsymbol{\mu}(M).$$

2. MV-polytopes/cycles and the Bott-Samelson variety

To give a combinatorial construction of MV-polytopes, we first want to define a "good" dense subset of an MV-cycle that allows us to easily read off the fixed points needed for the corresponding moment polytope, the MV-polytope. For this we use a result of Gaussent and Littelmann from [1] as our starting point.

Theorem 2.1 (Gaussent, Littelmann)**.** Let $\lambda \in X^\vee_+$, $\mu \in X^\vee$, and γ_λ the type of a minimal gallery connecting 0 and λ in $X^\vee \otimes \mathbb{R}$. Then there exists a bijection

$$\{\text{LS-galleries of type } \gamma_\lambda \text{ ending in } \mu\} \leftrightarrow \{\text{MV-cycles of coweight } (\lambda, \mu)\}.$$

This is done by associating to each LS-gallery δ a subset of the affine Grassmannian D_δ whose closure is an MV-cycle M_δ. The problem is that this subset only contains a single fixed point of the cycle and thus is not very useful to construct the polytope. We want to look at this in a bit more detail.

$$\begin{array}{ccccc} \Gamma(\gamma_\lambda) & \xhookrightarrow{\;i\;} & \Sigma(\gamma_\lambda) & \xrightarrow{\;r_w\;} & \Gamma(\gamma_\lambda) \\ & & \big\downarrow{\scriptstyle \pi} & & \\ & & \overline{G(\mathcal{O}).t^\lambda} & & \end{array}$$

Here $\Gamma(\gamma_\lambda)$ denotes the set of all galleries in $X^\vee \otimes \mathbb{R}$ of type γ_λ, the map i is again an inclusion as T-fixed points, $\Sigma(\gamma_\lambda)$ is the Bott-Samelson variety for the given type, the map π is a resolution of singularities, and for each $w \in W$ the map r_w is called the retraction at infinity with direction w.

The dense subset of Gaussent and Littelmann is then defined as follows. Take δ an LS-gallery, then $D_\delta = \pi(C_\delta)$ with $C_\delta = r_e^{-1}(\delta)$, where e denotes the unit element of W.

To construct a "better" dense subset we combine the approach of Gaussent and Littelmann, with result of Kamnitzer to obtain the following.

Theorem 2.2 (E.). Let $M = M_\delta$ be an MV-cycle, then there exists a family of galleries $(\delta_w^M)_{w\in W}$ such that

(1) $M = \overline{\bigcap_{w\in W} wU^-(\mathcal{K})w^{-1}.t^{wt(\delta_w^M)}}$,
(2) $P = \boldsymbol{\mu}(M) = \mathrm{conv}(\{wt(\delta_w^M) \mid w \in W\})$,
(3) and for $x \in C_\delta$ generic, $r_w(x) = \delta_w^M$.

Remark 2.3. By definition $\delta_e^M = \delta$. Furthermore a simple calculation shows that $\delta_{w_0}^M$, with w_0 the longest element of W, is the unique minimal gallery of the same type as δ that lies in the anti-dominant chamber.

By this result we know which galleries will be needed to construct the polytope, but we still need a combinatorial way to construct them.

3. Combinatorial construction

For the combinatorial construction we use the fact that the set of LS-galleries is equipped with a crystal structure, especially that there exist raising operators e_α for each simple root α. We then define

$$\Xi_{s_\alpha}(\delta) = s_\alpha(e_\alpha^{\max}(\delta)),$$

where we first apply e_α to δ as often as it is defined and afterwards use the action of the Weyl group on the coweight lattice and apply the simple reflection s_α.

Remark 3.1. By definition of the crystal structure the gallery $\Xi_{s_\alpha}(\delta)$ is again an LS-gallery, but for the Borel wBw^{-1}, thus we can iterate the process and define our galleries inductively.

For $w \in W$ choose $w' \in W$ and α simple, such that $w = w's_\alpha$ and $l(w) > l(w')$ and assume that $\Xi_{w'}(\delta)$ is already defined, then we define

$$\Xi_w(\delta) = \Xi_{w's_\alpha w'^{-1}}(\Xi_{w'}(\delta)).$$

This is defined since $w'\alpha$ is a simple root for the positive roots of wBw^{-1} and by the above remark $\Xi_{w'}(\delta)$ is an LS-gallery. Of course one also needs to check that this is well-defined and independent of the chosen reduced decomposition of w. This leads to the following.

Theorem 3.2. For δ an LS-gallery and $M = M_\delta$, we have

$$\Xi_w(\delta) = \delta_w^{M_\delta}.$$

This theorem can be proved in a similar inductive way as the Ξ_w's are defined, but one needs to be careful since one makes iterative coordinate changes in the Bott-Samelson variety which have to be controlled quite strictly.

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Strange duality and the Hitchin Connection

Prakash Belkale

We will consider duality phenomena for non-abelian theta functions on a compact Riemann surface, in particular Beauville's symplectic strange duality conjecture. Recently this conjecture was proved for general curves by T. Abe. Using the Hitchin/WZW/KZB connection, and by studying its properties, we extend Abe's results from general curves to all curves.

Let $\mathcal{M}_G(X)$ denote the moduli-stack of principal G-bundles on X for a simple simply connected algebraic group G. Let $\mathcal{L}$ be a positive generator of the Picard group of $\mathcal{M}_G(X)$. We will first outline the construction of a connection over the moduli space of curves, on "the non-abelian G-theta" functions $H^0(\mathcal{M}_G(X), \mathcal{L}^k)$ (i.e. as X varies in a family). This connection can be obtained in many ways (through symplectic geometry of the moduli spaces and through conformal field theory) and interesting properties result from the different ways of looking at this connections. We will consider and give the answer (following the physicists) to the following question: If G is a subgroup of H, is the natural map from G-theta functions to H-theta functions flat for Hitchin's connection? This question casts a profound shadow on duality questions. The expectation of the physicists is that a duality phenomenon accompanies situations where the flatness statement is valid.

We will next move on to the case of symplectic duality. Consider the moduli stacks $\mathcal{M}_{\mathrm{Spin}(r)}(X)$ and $\mathcal{M}_{\mathrm{SO}(r)}(X)$ of principal Spin(r), and SO(r)-bundles, $r \geq 3$ on a smooth connected projective curve X of genus $g \geq 2$ over $\mathbb{C}$. Let $\mathcal{M}_{\mathrm{SO}(r)}(0)$ be the connected component of $\mathcal{M}_{\mathrm{SO}(r)}(X)$, which contains the trivial SO(r)-bundle. There is a natural map

$$p : \mathcal{M}_{\mathrm{Spin}(r)} \to \mathcal{M}_{\mathrm{SO}(r)}(0).$$

A line bundle κ on X is said to be a theta characteristic if $\kappa^{\otimes 2}$ is isomorphic to the canonical bundle K_X. The set of theta characteristics $\theta(X)$ forms a torsor for the 2-torsion $J_2(X)$ in the Jacobian of X, and hence $|\theta(X)| = 2^{2g}$. Recall that a theta characteristic κ is said to be even (resp. odd) if $h^0(\kappa)$ is even (resp. odd).

For each theta-characteristic κ on X there is a line bundle $\mathcal{P}_k$ on $\mathcal{M}_{\mathrm{SO}(r)}$ with a canonical section s_κ (constructed by Laszlo and Sorger). On $\mathcal{M}_{\mathrm{SO}(r)}(0)$, $s_\kappa = 0$ if and only if both κ and r are odd.

For theta characteristics κ and κ', the line bundle $p^*\mathcal{P}_\kappa$ is isomorphic to $p^*\mathcal{P}_{\kappa'}$. Set $\mathcal{P} = p^*\mathcal{P}_\kappa$ which is well defined upto isomorphism. The line bundle $\mathcal{P}$ is the positive generator of the Picard group of the stack $\mathcal{M}_{\mathrm{Spin}(r)}$. It is known that $\mathcal{P}$ does not descend to the moduli-space $M_{\mathrm{Spin}(r)}$, (similarly $\mathcal{P}_\kappa$ does not descend to the moduli-space $M_{\mathrm{SO}(r)}$). Clearly, $\mathcal{P}$ comes equipped with sections s_κ for each theta characteristic κ, coming from the identification $p^*\mathcal{P}_\kappa \xrightarrow{\sim} \mathcal{P}$ (s_κ are well defined up to scalars).

Let $\pi : \mathcal{X} \to S$ be a smooth projective relative curve of genus g. Assume by passing to an étale cover that the sheaf of theta-characteristics on the fibers of π is trivialized. For $s \in S$, let $X_s = \pi^{-1}(s)$. It is known that the spaces $H^0(\mathcal{M}_{\mathrm{Spin}(r)}(X_s), \mathcal{P})$ form the fibers of a vector bundle on S, which is equipped with a projectively flat connection (WZW or equivalently Hitchin's connection as described above).

Theorem 0.1. For even r, each section $s_\kappa \in H^0(\mathcal{M}_{\mathrm{Spin}(r)}(X_s), \mathcal{P})$, for $\kappa \in \theta(X_s)$ is projectively flat.

Theorem 0.2. For odd r, each section $s_\kappa \in H^0(\mathcal{M}_{\mathrm{Spin}(r)}(X_s), \mathcal{P})$, for even $\kappa \in \theta(X_s)$ is projectively flat.

It is known that the dimension of the space $H^0(\mathcal{M}_{\mathrm{Spin}(r)}(X_s), \mathcal{P})$ is equal to the number of theta characteristics (if r is odd, the number of even theta characteristics). It has been proved by Pauly and Ramanan that in Theorems 0.1 and 0.2, the sections are linearly independent, and hence form a basis. Our methods give a new proof of this result of Pauly and Ramanan.

We use Theorem 0.1 to show that the symplectic strange duality formulated by Beauville is, in a suitable sense, projectively flat for Hitchin's connection: Hence it is an isomorphism for all curves of a given genus if it is an isomorphism for some curve of that genus. Takeshi Abe [1, 2] has recently formulated a very interesting parabolic generalization of Beauville's conjecture, and has proved this conjecture for generic curves by using powerful degeneration arguments. His results imply Beauville's conjecture for generic curves. Therefore Abe's results (together with our work outliend above) imply that the symplectic strange duality conjecture of Beauville holds for all curves. It should be pointed out that Abe's parabolic symplectic duality conjecture has not yet been shown to hold for all curves.

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K-theory Schubert calculus of the affine Grassmannian

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(joint work with Thomas Lam, Anne Schilling)

1. Introduction

Let G be a simple simply-connected algebraic group over $\mathbb{C}$ and $T \subset G$ the maximal torus. Let Gr_G denote the affine Grassmannian of G. The T-equivariant K-cohomology $K^T(\mathrm{Gr}_G)$ and K-homology $K_T(\mathrm{Gr}_G)$ are Hopf-dual algebras over $K^T(\mathrm{pt})$ and are equipped with distinguished $K^T(\mathrm{pt})$-bases (denoted $\{[\mathcal{O}_{X_w^I}]\}$ and $\{\xi_w\}$), called Schubert bases. Our first main result is a description of the K-homology $K_T(\mathrm{Gr}_G)$ as a subalgebra $\mathbb{L}$ of the affine NilHecke algebra of Kostant and Kumar [KK90]. This generalizes work of Peterson [Pet] in homology. Our second main result is the identification, in the case $G = SL_n(\mathbb{C})$, of the Schubert bases of the non-equivariant K-(co)homology $K_*(\mathrm{Gr}_G)$ and $K^*(\mathrm{Gr}_G)$ with explicit symmetric functions called *K-k-Schur functions* and *affine stable Grothendieck polynomials* [Lam06]. This generalizes work of Lam [Lam08] in (co)homology.

The full paper containing these results is [LSS].

2. Kostant and Kumar's NilHecke ring

Let W_{af} be the affine Weyl group of G, with simple generators $\{r_i \mid i \in I\}$ where I is the affine Dynkin node set. The 0-Hecke algebra $\mathbb{K}_0$ is the ring generated by elements $\{T_i \mid i \in I\}$ and relations

$$\begin{aligned} T_i^2 &= -T_i \\ (T_iT_j)^{m_{ij}} &= (T_jT_i)^{m_{ij}} \qquad \text{whenever } i \neq j \text{ and } (r_ir_j)^{m_{ij}} = (r_jr_i)^{m_{ij}}. \end{aligned}$$

$\mathbb{K}_0$ has $\mathbb{Z}$-basis $T_w = T_{i_1}T_{i_2}\cdots T_{i_k}$ for $w \in W_{\mathrm{af}}$ where $w = r_{i_1}\cdots r_{i_k}$ is a reduced decomposition.

Extend the natural action of W on T to the level 0 action of $W_{\mathrm{af}} \cong W \ltimes Q^\vee$ on T by letting $Q^\vee$ act trivially. Then $\mathbb{K}_0$ acts on $R(T)$ by

$$T_i \cdot e^\lambda = (1 - e^{\alpha_i})^{-1}(e^{r_i(\lambda)} - e^\lambda).$$

The (small torus) affine nilHecke ring $\mathbb{K}$ is by definition the smash product of $\mathbb{K}_0$ and $R(T)$. Due to the relation

$$T_i\, q = (T_i \cdot q) + (r_i \cdot q)T_i \qquad \text{for } q \in R(T),$$

we have

$$\mathbb{K} = \bigoplus_{w \in W_{\mathrm{af}}} R(T)T_w.$$

The ring $\mathbb{K}$ is the algebraic model for the equivariant K-homology convolution ring $K_T(X_{\mathrm{ind}})$ where X_{ind} is the affine flag indvariety [Kum].

Note that if $Q(T)$ is the fraction field of $R(T)$ then $Q(T) \otimes_{R(T)}$ is isomorphic to the smash product $\mathbb{K}_Q$ of the group algebra $\mathbb{Q}[W_{\mathrm{af}}]$ and $Q(T)$.

3. Function realization of equivariant K-theory

The idea to realize equivariant K-theory by functions is due to Kostant and Kumar [KK90]. Let X be the "thick" affine flag manifold of [Kas], which has finite-codimensional Schubert varieties X_w for $w \in W_{\mathrm{af}}$. Let $K^T(X)$ be the Grothendieck group of T-equivariant coherent sheaves on X (see [KS1] for a precise definition). Then one has [KS1]

$$K^T(X) \cong \prod_{w \in W_{\mathrm{af}}} R(T)[\mathcal{O}_{X_w}].$$

For $w \in W_{\mathrm{af}}$ let i_w be the inclusion of a point into X with image w and let $i_w^* : K^T(X) \to K^T(\mathrm{pt}) \cong R(T)$ be the induced map. Let $\mathrm{Fun}(W_{\mathrm{af}}, R(T))$ be the $R(T)$-algebra of functions from W_{af} to $R(T)$ with pointwise multiplication. Then there is an injective map

$$\mathrm{res} : K^T(X) \to K^T(W_{\mathrm{af}}) \cong \mathrm{Fun}(W_{\mathrm{af}}, R(T))$$

which sends c to $(w \mapsto i_w^*(c))$. We regard $f \in \mathrm{Fun}(W_{\mathrm{af}}, R(T))$ as a function from $\mathbb{K}_Q \to Q(T)$ via $f(\sum_w a_w w) = \sum_w a_w f(w)$ for $a_w \in Q(T)$.

Theorem 1. [LSS] The image Ψ of res consists of the functions f such that:

$$f((1 - t_{\alpha^\vee})^d \, w) \in (1 - e^{\alpha})^d R(T) \tag{3.1}$$

and

$$f((1 - t_{\alpha^\vee})^{d-1}(1 - r_\alpha) w) \in (1 - e^{\alpha})^d R(T) \tag{3.2}$$

for all $d \in \mathbb{Z}_{>0}$, $w \in W_{\mathrm{af}}$, and affine real roots α.

This is the K-theoretic analogue of the cohomological result of Goresky, Kottwitz, and Macpherson [GKM04]. If one uses the maximal torus T_{af} in the affine Kac-Moody group then the condition is greatly simplified: one only needs (3.2) with $d = 1$ [HHH] [KK90].

4. Perfect pairing

There is an intersection pairing

$$K_T(X_{\mathrm{ind}}) \times K^T(X) \to R(T).$$

Our algebraic replacement is the $R(T)$-bilinear evaluating pairing

$$\begin{gathered} \mathbb{K} \times \Psi \to R(T) \\ \langle a \, , \, f \rangle = f(a). \end{gathered}$$

Letting $\psi^w \in \Psi$ be the image of $[\mathcal{O}_{X_w}]$ under res, we have the dual bases

$$\langle T_v \, , \, \psi^w \rangle = \delta_{vw} \qquad \text{for } v, w \in W_{\mathrm{af}}.$$

5. Affine Grassmannian

Let Gr_G be the thick affine Grassmannian (which is a natural quotient of the thick affine flag manifold X) and $\mathrm{Gr}_G^{\mathrm{ind}} \cong G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ the affine Grassmannian.

Proposition 2. [LSS] $K^T(\mathrm{Gr}_G)$ is isomorphic to the subring Ψ_{Gr} of Ψ defined by $f \in \Psi_{\mathrm{Gr}}$ if and only if

$$f(wv) = f(w) \qquad \text{for all } w \in W_{\mathrm{af}} \text{ and } v \in W. \tag{5.1}$$

Let W^0 be the set of minimum length coset representatives for W_{af}/W.
Our main theorem is the following.

Theorem 3. [LSS] The equivariant K-homology convolution ring $K_T(\mathrm{Gr}_G^{\mathrm{ind}})$ is isomorphic to the centralizer subalgebra $\mathbb{L} := Z_{R(T)}(\mathbb{K})$. The image $\xi_w \in \mathbb{L}$ of the Schubert class in $K_T(\mathrm{Gr}_G^{\mathrm{ind}})$ indexed by $w \in W^0$, is the unique element of

$$\mathbb{L} \cap \left(T_w + \bigoplus_{v \in W_{\mathrm{af}} \setminus W^0} R(T) T_v \right).$$

For $G = SL_n$ we use Theorem 3 to obtain explicit formulae for the nonequivariant analogues of algebra generators ξ_w of $K_T(\mathrm{Gr}_G^{\mathrm{ind}})$, and use these to realize $K_*(\mathrm{Gr}_G^{\mathrm{ind}})$ and $K^*(\mathrm{Gr}_G)$ and their dual Schubert bases, in terms of symmetric functions.

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$\mathfrak{g}$-commuting Dunkl operators and braided doubles

Arkady Berenstein
(joint work with Yuri Bazlov)

In my talk I shall introduce **q**-commuting analogues of Dunkl operatores that are acting on **q**-symmetric algebras. I shall explain the **q**-commutation phenomenon by constructing braided Cherednik algebras, for which the operators form a representation.

The classifications of braided Cherednik algebras is achieved in terms of braided doubles, that Yuri Bazlov an myself introduced earlier besides the ordinary rational Cherednik algebras and their braided tensor products. We obtained new algebras with triangular demcomposition attached to an infinite family of subgroups of even elements in complex refelction groups, so that the corresponding Dunkl operators pairwise anti-commute.

A Nonconventional Slice in the Coadjoint Space of the Borel and the Coxeter Element.

Anthony Joseph

Let $\mathfrak{g}$ be a simple Lie algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ and set $\mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h}$. Let P^+ denote the corresponding set of dominant weights. There is a canonically determined ideal $\mathfrak{b}_E$ of $\mathfrak{b}$ containing $\mathfrak{n}^+$ such that the algebra generated by semi-invariants on $\mathfrak{b}^*$ coincides with the algebra $Y(\mathfrak{b}_E)$ of invariants on $\mathfrak{b}_E^*$. Moreover the latter algebra is polynomial on rank $\mathfrak{g}$ generators [2]. By analogy with the semisimple case one can ask if there exists an affine slice $y + V$ of $\mathfrak{b}_E^*$ such that the restriction map induces an isomorphism of $Y(\mathfrak{b}_E)$ onto the algebra $R[y + V]$ of regular functions on $y + V$. This holds in type A and even extends to all biparabolic subalgebras [3]; but fails in general even with respect to the Borel.

Motivated by an attempt to construct $Y(\mathfrak{g})$ we consider the subalgebra

$$A(\mathfrak{g}) := \bigoplus_{\lambda \in P^+} Y(\mathfrak{n}^-)_{-\lambda} Y(\mathfrak{b}_E)_\lambda,$$

of $S(\mathfrak{g})$. Here the ultimate goal was to show that $((adU(\mathfrak{g}))A(\mathfrak{g}))^{\mathfrak{g}} = Y(\mathfrak{g})$. We remark that this is true in types A and C as shown in [1].

The main result we prove here is that there exists $y \in \mathfrak{g}$ and a subspace V of $\mathfrak{g}$ of dimension rank $\mathfrak{g}$ such that the restriction map induces an algebra isomorphism of $A(\mathfrak{g})$ onto $R[y + V]$. This may not give a slice in the conventional sense; but it still reflects the rather intricate structure of $A(\mathfrak{g})$ and consequently of $Y(\mathfrak{b}_E)$. A key point is that y is given by the "square root" of the unique longest element w_0 of the Weyl group acting on the standard regular ad-nilpotent element which is the sum of simple root vectors, rather than by an ad hoc procedure of writing down sums of root vectors. For example when the Coxeter number is divisible by 4, say equals $4n$, this element is the n^{th} power of a carefully chosen Coxeter element and has square equal to w_0. In principle this construction can give a more satisfactory understanding of slices of biparabolics in type A and in the appropriate cases for other types. (Such slices generally tend to exist when the Cartan subalgebra of the truncated biparabolic is sufficiently large). More generally it is hoped that it may provide nonconventional slices for truncated biparabolics at least when the invariant algebra is polynomial, which is frequently the case [1, 4].

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