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Kommutative Algebra

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Abstract. The workshop brought together researchers on various areas of Commutative Algebra. New results in combinatorial commutative algebra, homological methods and invariants, characteristic p-methods, and in general commutative algebra and algebraic geometry were presented in the lectures of the workshop.

Mathematics Subject Classification (2000): 13xx, 14xx.

Introduction by the Organisers

The workshop on Commutative Algebra was very well attended by important senior researchers in the field and by many promising young mathematicians. Of particular help for the invitation of young participants was the support from the Leibniz Association within the grant "Oberwolfach Leibniz Graduate Students" (OWLG), from the National Science Foundation (NSF) and the Japan Association for Mathematical Sciences (JAMS).

The conference had a very lively atmosphere, made possible by the excellent facilities of the institute. There were 51 participants, and 24 talks. A considerable number of the lectures were given by young researchers. The program left plenty of time for cooperation and discussion among the participants. We highlight the areas in which new results were presented by the lecturers:

(a) *Combinatorial commutative algebra* The spectacular applications of commutative algebra to enumerative combinatorics two decades ago have developed into a subfield of commutative algebra that is very active now. This connection was one of the main topics of this conference. The basic objects are algebraic structures defined by monomials, in particular face rings of simplicial complexes and affine monoid algebras. The talks were devoted to depth properties and Hilbert functions. Also connections with game theory was one of the topics of the talks.

(b) *Homological methods and invariants* This area started from the fundamental theorems of Auslander, Buchsbaum and Serre on regular rings. Homological properties are used in the major classification of commutative rings and their modules. One the mysteries in this field is the structure of free resolutions of modules over local or graded rings. In this conference Eisenbud and Schreyer presented their proof of the Boij-Sderberg conjectures which is a major breakthrough in the field. Other major topics were the structure of Koszul algebras and resolutions over complete intersections and Gorenstein rings. Furthermore the derived category of coherent sheaves on algebraic varieties (in particular hypersurfaces) turned out to be an important object of study.

(c) *Characteristic p-methods* Characteristic p methods have had an extremely strong influence on the development of commutative algebra. They were crystalized by Hochster and Huneke in the notion of tight closure, and have led to remarkable results in ideal and module theory. Moreover there are strong connections to singularity theory and algebraic geometry. After the counterexample to the localization conjecture by Brenner and Monsky, Hochster is developing a variant of the notion of tight closure that satisfies localization. Of particular interest was also the action of Frobenius on (local) cohomology and applications thereof.

(e) *General commutative algebra and algebraic geometry* Commutative algebra is one of the main tools in algebraic geometry. Therefore the interactions of these two fields was also present in many lectures. Among the topics covered were log canonical threshholds which have a strong similarity to the jumping numbers in the theory of test ideals. Another interesting line of research was to find equations describing projective varieties.

Workshop: Kommutative Algebra

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Abstracts

Overview of Boij-Söderberg Theory, Part I

David Eisenbud
(joint work with Frank-Olaf Schreyer)

This talk and the one by Frank Schreyer presented a summary of recent developments stemming from a conjecture of Boij and Söderberg about free resolutions of graded modules over a polynomial ring, and now including information about cohomology tables of vector bundles and a sort of duality between the two subjects.

Since the fundamental papers of David Hilbert at the end of the 19th century, the Hilbert function of a graded module over a polynomial ring has played a fundamental role in commutative algebra and, more recently, in algebraic geometry. Hilbert himself calculated this invariant in terms of a finer invariant, the *graded Betti numbers* of the module. Determining which sets of graded Betti numbers actually come from modules is a problem that seems quite out of reach, but in late 2006 Mats Boij and Jonas Söderberg, motivated by the "multiplicity conjectures" of Herzog, Huneke and Srinivasan, made a remarkable conjecture specifying the possibilities up to a rational multiple [1]. In [6], Frank Schreyer and I showed that these conjectures were related to a group of (nearly) dual conjectures about vector bundles, and we proved these two conjectures together. A flurry of other papers and preprints including [2], [4], [7], [8], [9] and [10] have added to the basic picture and its applications. Schreyer and I were invited to present an overview of these developments in two lectures at this meeting, and this is a report of the first of these two lectures.

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k, graded with each x_i of degree 1, and let M be a finitely generated graded S-module. As usual, we write $S(-j)$ for the graded free module of rank 1 with generator in degree j. By the Hilbert Syzygy theorem there exist *finite free resolutions* of M; that is, sequences of graded free modules and degree 0 homomorphisms

$$\mathbf{F}: \quad F_0 \xleftarrow{\phi_1} \cdots \xleftarrow{\phi_m} F_m \longleftarrow 0$$

that are exact at F_i for $i > 0$, while coker $\phi_1 \cong M$. Such a resolution is said to be *minimal* if no proper summand of F_i maps onto the kernel of ϕ_{i-1}. Minimal resolutions are unique up to isomorphism, and have *length* $m \leq n$. In particular, if $\mathbf{F}$ is a minimal free resolution, and we write $F_i = \oplus_j S(-j)^{\beta_{i,j}(M)}$, then the *graded Betti numbers* $\beta_{i,j}(M) = dim((F_i \otimes_S k)_j)$ are invariants of M alone. We define the *Betti table* of M to be this collection of numbers $\{\beta_{i,j}(M)\}$.

We may regard the Betti table of M as an element of an (infinite-dimensional) rational vector space,

$$B := \bigoplus_{-\infty}^{\infty} \mathbb{Q}^{n+1}$$

with coordinates $\beta_{i,j}(M)$. Since $\beta_{i,j}(M \oplus N) = \beta_{i,j}(M) + \beta_{i,j}(N)$ the Betti tables of finitely generated modules form a sub-semigroup of this vector space. The following Theorem, conjectured by Boij and Söderberg, specifies the cone of positive rational linear combinations of Betti tables of finitely generated modules precisely.

One way of specifying a cone is to give it's extremal rays—the half-lines in the cone that are not in the convex hull of the remaining elements of the cone. In the case of the cone of Betti tables, the extremal rays will turn out to be the *pure* modules:

Definition 1. A finitely generated graded S-module M is called *pure* of type $d := (d_0, \dots, d_m)$ if

(1) In a minimal free resolution of M as above, the free module F_i generated by elements of degree d_i; that is, $\beta_{i,j} = 0$ when $j \neq d_i$.
(2) M is Cohen-Macaulay of codimension m; that is, $F_i = 0$ for $i > m$ and the annihilator of M is an ideal of codimension m.

It is easy to see that if there is a pure module of type d, then $d_0 < \cdots < d_m$. Much more is true: if M is a pure module, then a result of Herzog and Kühl [13] shows that the Betti table of M is determined by d up to a rational multiple: that is, there is a constant $r = r(M)$ depending on M such that

$$\beta_{i,d_i}(M) = \frac{r}{\prod_{t \neq i} |d_t - d_i|}.$$

Thus the pure modules of type d define a single ray in the cone of Betti tables.

One more preparation is necessary: we order the strictly increasing sequences d: we say that

$$d = (d_0, \dots, d_m) \leq d' = (d'_0, \dots, d'_{m'})$$

if $m \geq m'$ and $d_i \leq d'_i$ for $i = 1, \dots, m'$. (One can think of this as the termwise order if one simply extends each sequence $d = (d_0, \dots, d_m)$ to $d = (d_0, \dots, d_m, \infty, \infty, \dots)$.)

We can now state the main result of the theory concerning the cone of Betti tables:

Theorem 2. Let $S = k[x_1, \dots, x_n]$ be as above.

(1) For every strictly increasing sequence of integers $d = (d_0, \dots, d_m)$ with $m \leq n$, there exist pure finitely generated graded S-modules of type d.
(2) The Betti table of any finitely generated graded S-module may be written uniquely as a positive rational linear combination of the Betti tables of a set of pure finitely generated modules whose types form a totally ordered sequence.

The second statement of the Theorem has two nice interpretations, which may help clarify its meaning. First, geometrically, it really says that the cone of Betti tables is a *simplicial fan*, that is, it is the union of simplicial cones, meeting along facets, with each simplicial cone spanned by the rays corresponding to a set of pure Betti tables whose types form a totally ordered set. These simplices and cones are of course infinite dimensional; but one can easily reduce to the finite-dimensional case by specifying that one wants to work with resolutions where the free modules are generated in a given bounded range of degrees.

Second, algorithmically, the Theorem implies that there is a greedy algorithm that gives the decomposition. Rather than trying to specify this formally, we give an Example. For this purpose, we write the Betti table of a module M as an array whose entries in the i-th column are the $\beta_{i,j}$—that is, the i-th column corresponds to the free module F_i for reasons of efficiency and tradition, we put $\beta_{i,j}$ in the $(j-i)$-th row.

For our example we take $n = 3$, and let $M = S/(x^2, xy, xz^2)$. The minimal free resolution of M has the form

$$S \longleftarrow S(-2)^2 \oplus S(-3) \longleftarrow S(-3) \oplus S(-4)^2 \longleftarrow S(-5) \longleftarrow 0$$

and is represented by an array

$$\beta(M) = \begin{pmatrix} 1 & & & \\ & 2 & 1 & \\ & 1 & 2 & 1 \end{pmatrix}$$

where all the entries not shown are equal to zero.

To write this as a positive rational linear combination of pure diagrams, we first consider the "top row", corresponding to the generators of lowest degree in the free modules of the resolution. These are in positions

$$\begin{pmatrix} * & & & \\ & * & * & \\ & & & * \end{pmatrix}$$

corresponding to the degree sequence $(0, 2, 3, 5)$. There is in fact a pure module $M_1 = S/I_1$ with resolution

$$\beta(M_1) = \begin{pmatrix} 1 & & & \\ & 5 & 5 & \\ & & & 1 \end{pmatrix}.$$

The greedy algorithm now instructs us to subtract *the largest possible multiple* q_1 *of* $\beta(M_1)$ *that will leave the resulting table* $\beta(M) - q_1\beta(M_1)$ *having only non-negative terms.* We see at once that $q_1 = 1/5$.

We now repeat this process starting from $\beta(M) - q_1\beta(M_1)$; the Theorem guarantees that there will always be a pure resolution whose degree sequence matches the top row of the successive remainders. In this case we arrive at the expression $\beta(M) =$

$$\begin{pmatrix} 1 & & & \\ & 2 & 1 & \\ & 1 & 2 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & & & \\ & 5 & 5 & \\ & & & 1 \end{pmatrix} + \frac{1}{10}\begin{pmatrix} 3 & & & \\ & 10 & & \\ & & 15 & 8 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 1 & & & \\ & & & \\ & 4 & 3 & \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 & & & \\ & & & \\ & 1 & & \end{pmatrix}.$$

All the fractions and diagrams that occur are of course invariants—apparently new invariants—of M.

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On the lex-plus-powers conjecture

Jeff Mermin

Let $S = k[x_1, \dots, x_n]$, and let $F = (f_1, \dots, f_r) \subset S$ be a regular sequence of homogeneous elements in degrees $\deg f_i = e_i$ for an increasing sequence of degrees $2 \leq e_1 \leq \dots \leq e_r$. Put $P = (x_1^{e_1}, \dots, x_r^{e_r})$, the ideal of pure powers in the same degrees. In what follows, a *lex-plus-P* ideal means an ideal L of the form $L = \hat{L} + P$, where $\hat{L}$ is a lexicographic ideal.

The Lex-plus-powers conjecture makes two claims:

Conjecture 1. *Let I be any homogeneous ideal of S containing F. Then:*
(i) *There exists a unique lex-plus-P ideal L having the same Hilbert function as I.*
(ii) *The graded Betti numbers of L are greater than or equal to those of I, i.e., $b_{i,j}(S/L) \geq b_{i,j}(S/I)$ for all i, j.*

Part (i) of the conjecture is due to Eisenbud, Green, and Harris [4, 5], and part (ii) is generally credited to Graham Evans [6].

The special case in which $F = 0$ is essentially classical:

Theorem 2. *Let I be any homogeneous ideal of S.*

(i) (Macaulay [9]) *There exists a unique lex ideal L having the same Hilbert function as I.*
(ii) (Bigatti [1], Hulett [8], Pardue [14]) *The graded Betti numbers of L are greater than or equal to those of I.*

All known proofs of Theorem 2 begin by replacing the ideal I with an initial ideal. This results in an monomial ideal with the same Hilbert function as, and larger Betti numbers than, the original ideal I; combinatorial techniques can then be used to exploit the multigraded structure of the monomial ideal.

In general, however, taking an initial ideal does not preserve the property that I contains a regular sequence in the given degrees $(e_1, \dots, e_r)$. Thus, it is natural to consider the case that F is unchanged by passage to an initial ideal, i.e., $F = P$. In this case, part (ii) of the Lex-Plus-Powers conjecture was proved by the recent series of papers [11, 12, 10]:

Theorem 3. *Let I be a homogeneous ideal containing P.*
(i) (Clements, Lindström [3]) *There exists a unique lex-plus-P ideal L having the same Hilbert function as I.*
(ii) (–, Murai) *The graded Betti numbers of L are greater than or equal to those of I.*

Some new insight will likely be needed to establish the truth of the conjecture in general. The largest known case is due to Caviglia and Maclagan [2], who show that part (i) holds when the degrees of the regular sequence grow quickly enough:

Theorem 4 (Caviglia, Maclagan)**.** *Suppose that $e_i > \sum_{j=1}^{i-1} e_j$ for all i. If I is any homogeneous ideal containing F, then there is a lex-plus-P ideal L having the same Hilbert function as I.*

Several other, harder-to-state special cases are known as well; these are catalogued in [6].

In the meantime, it is interesting to consider what other properties of lex ideals continue to hold in the setting of ideals containing a regular sequence.

A common method of proving Theorem 2 involves showing that the Hilbert scheme parametrizing all ideals with a fixed Hilbert function is connected. (The Hilbert scheme is usually defined to parametrize the ideals with a fixed Hilbert *polynomial*, but it is possible, and more interesting in our setting, to define it for a fixed Hilbert function instead.)

These proofs usually establish, along the way, that, when L is the lex ideal with the same Hilbert function as I, the Betti numbers of L differ from those of I by *consecutive cancellations*: that is, there exist nonnegative integers $c_{i,j}$ such that $b_{i,j}(I) = b_{i,j}(L) - c_{i,j} - c_{i-1,j}$ for all i and j. This is not a corollary of the connectedness of Hilbert scheme, but a feature of the arguments used to study the lex ideal as a point inside the Hilbert scheme; see [15] for details.

Thus we have the following results in the case of theorem 2:

Theorem 5. *Let $\mathcal{H}$ be the Hilbert scheme parametrizing all ideals having the same Hilbert function as I, and let L be the lex ideal on H. Then:*

(i) (Hartshorne [7]) *$\mathcal{H}$ is connected.*
(ii) (Peeva [15]) *The Betti numbers of I differ from those of L by consecutive cancellations.*

If we restrict our attention to those ideals containing a regular sequence, we get new questions which seem related to the lex-plus-powers conjecture:

Question 6. *Let $I \supset F$ be given, let $\mathcal{H}$ be the Hilbert scheme parametrizing all ideals having the same Hilbert function as I, and let $\mathcal{G}$ be the subset of $\mathcal{H}$ parametrizing those ideals which contain a regular sequence in the degrees of F. Then:*
(i) *Is $\mathcal{G}$ connected?*
(ii) *If $\mathcal{G}$ contains a lex-plus-P ideal L, do the Betti numbers of I differ from those of L by consecutive cancellations?*

Question 6 has an affirmative answer if we restrict further to require that $F = P$:

Theorem 7. *Let $I \supset P$ be given, let $\mathcal{H}$ be the Hilbert scheme parametrizing all ideals having the same Hilbert function as I, and let $\mathcal{G}'$ be the subscheme of $\mathcal{H}$ parametrizing those ideals which contain P. Then:*
(i) (Murai, Peeva [13]) *If the field k has characteristic zero, then $\mathcal{G}'$ is connected.*
(ii) (–, Murai [10]) *The Betti numbers of I differ from those of the lex-plus-P ideal L by consecutive cancellations.*

The subscheme $\mathcal{G}'$ is naturally isomorphic to the Hilbert scheme parametrizing all ideals of the quotient ring S/P having the same Hilbert function as the image of I in this ring. Thus it is natural to ask whether the Betti numbers of I and L, viewed as ideals in the quotient ring, are comparable:

Question 8. *Let F, P, I, and L be as in Conjecture 1. Does one have $b_{i,j}^{S/P}(L) \geq b_{i,j}^{S/F}(I)$ for all i, j?*

Note that this is a statement about infinite resolutions. While Question 8 is open in general, there is again a positive answer in the case $F = P$.

Theorem 9 (Murai, Peeva [13])**.** *Suppose that I contains P and L is the lex ideal having the same Hilbert function. If the field k has characteristic zero, then $b_{i,j}^{S/P}(L) \geq b_{i,j}^{S/P}(I)$ for all i, j.*

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Exceptional sequences of invertible sheaves on rational surfaces

Markus Perling
(joint work with Lutz Hille)

In this talk I reported about joint work with Lutz Hille on the bounded derived category $D^b(X)$ of rational surfaces [6], [7]. In general, it is an open problem whether on an algebraic variety X so-called full exceptional, or strongly exceptional, sequences exist.

Definition: A coherent sheaf $\mathcal{E}$ on a smooth complete variety X is called *exceptional* if it simple and $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}) = 0$ for every $i \neq 0$. A sequence $\mathcal{E}_1, \dots, \mathcal{E}_n$ of exceptional sheaves is called an *exceptional* sequence if $\mathrm{Ext}^k_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E}_j) = 0$ for all k and for all $i > j$. If an exceptional sequence generates $D^b(X)$, then it is called *full*. A *strongly* exceptional sequence is an exceptional sequence such that $\mathrm{Ext}^k_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E}_j) = 0$ for all $k > 0$ and all i, j.

These definitions go back to work of Drezet and Le Potier [5] and the Seminaire Rudakov [8]. An important aspect is that strongly exceptional sequences provide a bridge to noncommutative geometry via tilting correspondence. Due to results of Bondal [3] (see also [1]), every full strongly exceptional sequences gives rise to an equivalence of categories

$$\mathbf{R}\mathrm{Hom}(\mathcal{T}, \, . \,) : D^b(X) \longrightarrow D^b(\mathrm{End}(\mathcal{T}) - \mathrm{mod}),$$

where $\mathcal{T} := \bigoplus_{i=1}^{n} \mathcal{E}_i$, which is sometimes called a tilting sheaf. This way the algebra $\mathrm{End}(\mathcal{T})$, at least in the derived sense, represents a non-commutative coordinization of X.

In our work we consider the special situation where X is a smooth complete rational surface and (strongly) exceptional sequences which consist only of invertible sheaves. Exceptional sequences of this type do exist in general, but in [6] we gave the first example of a toric surface which does not admit a strongly exceptional sequence of invertible sheaves. In [7] we managed to obtain an almost complete picture for all rational surfaces. The most important structural insight is the following result:

Theorem: Let X be a smooth complete rational surface, $\mathcal{O}_X(E_1), \dots, \mathcal{O}_X(E_n)$ be a full exceptional sequence of invertible sheaves on X, and set $E_{n+1} := E_1 - K_X$. Then to this sequence there is associated in a canonical way a smooth complete toric surface with torus invariant prime divisors $D_1, \dots, D_n$ such that $D_i^2 + 2 = \chi\big(\mathcal{O}_X(E_{i+1} - E_i)\big)$ for all $1 \leq i \leq n$.

Note that, if we denote Y the toric surface associated to an exceptional sequence of invertible sheaves, the theorem yields a canonical isometry $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ with respect to the intersection product.

From this we can derive various results concerning strongly exceptional sequences, among the most important ones are:

Theorem: Any smooth complete rational surface which can be obtained by blowing up a Hirzebruch surface two times (in possibly several points in each step) has a full strongly exceptional sequence of invertible sheaves.

Note that the case of $\mathbb{P}^2$ (and $\mathbb{P}^n$ in general) is well-known [2]. In the toric case, we can show that the converse is also true:

Theorem: Let $\mathbb{P}^2 \neq X$ be a smooth complete toric surface. Then there exists a full strongly exceptional sequence of invertible sheaves on X if and only if X can be obtained from a Hirzebruch surface in at most two steps by blowing up torus fixed points.

Conjecturally, the converse is also true for rational surfaces in general, though we were not yet able to prove this. It is well known that toric varieties have certain universal properties with respect to invertible sheaves (see [4]) and it is an interesting question whether analogous structural results as presented above also hold for the case of higher-dimensional rational varieties.

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Free resolutions over Koszul algebras

Luchezar L. Avramov
(joint work with Aldo Conca and Srikanth Iyengar)

Let K be a field and Q a commutative $\mathbb{N}$-graded K-algebra with $Q_0 = K$, Every graded Q-module M with $M_j = 0$ for $j \ll 0$ has a unique up to isomorphism minimal graded free resolution, F^M. The module F_i^M has a basis element in degree j if and only if $\operatorname{Tor}_i^Q(k, M)_j \neq 0$ holds, where $k = Q/Q_+$ for $Q_+ = \bigoplus_{j\geqslant 1} Q_j$, so important structural information on F^M is encoded in the sequence of numbers

$$t_i^Q(M) = \sup\{j \in \mathbb{Z} \mid \operatorname{Tor}_i^Q(k, M)_j \neq 0\}\,.$$

It is distilled through the notion of *Castelnuovo-Mumford regularity*, defined by

$$\operatorname{reg}_Q M = \sup_{i\geqslant 0}\{t_i^Q(M) - i\}\,.$$

One has $\operatorname{reg}_Q k \geq 0$, and equality means that Q is *Koszul*; see, for instance, [8].

When the K-algebra Q is finitely generated, $\operatorname{reg}_Q M < \infty$ holds for each finitely generated graded module M if and only if Q is a polynomial ring over some Koszul algebra, see [4]. As an alternative, we turn to the *slope* of M over Q, defined by

$$\operatorname{slope}_Q M = \sup_{i\geqslant 1}\left\{\frac{t_i^Q(M) - t_0^Q(M)}{i}\right\}\,.$$

We are interested in relations between the numbers $\operatorname{slope}_Q M$, $\operatorname{slope}_Q R$, and $\operatorname{slope}_R M$ for $R = Q/J$, when J is a *homogeneous* ideal and M a *non-zero* graded R-module. Initial information can be obtained by invoking only general techniques of homological algebra. Studying the convergence at the base and on the fiber of a classical change-of-rings spectral sequences of Cartan and Eilenberg [6], we get:

Theorem 1. *When $J \neq QJ_1$ holds there are inequalities*

$$\operatorname{slope}_R M \leq \max\left\{\operatorname{slope}_Q M\,, \sup_{i\geqslant 1}\left\{\frac{t_i^Q(R) - 1}{i}\right\}\right\} \leq \max\{\operatorname{slope}_Q M, \operatorname{slope}_Q R\}\,.$$

If the induced map $\mathrm{Tor}_i^Q(k,M) \to \mathrm{Tor}_i^R(k,M)$ *is injective for each* i*, then*

$$\operatorname{slope}_Q R \le 1 + \sup_{i\geqslant 2}\left\{\frac{t_i^R(M) - t_0^R(M) - 1}{i-1}\right\}.$$

When M is finitely generated and R is standard graded, a variant of the upper bound for $\operatorname{slope}_Q M$ was obtained by Aramova, Bărcănescu, and Herzog [1].

As an easy consequence of the theorem, one obtains:

Corollary 2. *If* R *is finitely generated over* K*, then for every finitely generated* R*-module* M *one has* $\operatorname{slope}_R M < \infty$.

Theorem 1 can be strengthened when R admits a simple resolution over Q:

Proposition 3. *If* $R = Q/(f)$ *for a non-zero divisor* $f \in Q_+$*, then one has:*

(1) $\operatorname{slope}_Q M \le \max\{\operatorname{slope}_R M, \deg(f)\}$ *with equality for* $f \notin (Q_+)^2$.

(2) $\operatorname{slope}_R M \le \max\{\operatorname{slope}_Q M, \deg(f)/2\}$ *with equality for* $f \in Q_+ \operatorname{Ann}_Q M$.

Following Backelin [5], we set $\operatorname{Rate} Q = \operatorname{slope}_Q Q_+$; thus, one has $\operatorname{Rate} Q \ge 1$, with equality if and only if Q is Koszul. The statement of the next result refers to the classical homological products in $\mathrm{Tor}^Q(k,R)$. Its proof depends on the use of DG algebra resolutions, and draws on results of [3].

Theorem 4. *Let* Q *be a standard graded* K*-algebra,* J *a non-zero homogeneous ideal such that* $\mathrm{Tor}_i^Q(k,k) \to \mathrm{Tor}_i^R(k,k)$ *is injective for each* i*, and set* $c = \operatorname{Rate} R$.

For every integer $i \ge 1$ *there are then inequalities*

$$t_i^Q(R) \le \operatorname{slope}_Q R \cdot i \le (c+1)\cdot i\,,$$

and the following conditions are equivalent:

(i) $t_i^Q(R) = (c+1)\cdot i$.

(ii) $t_h^Q(R) = (c+1)\cdot h$ *for* $1 \le h \le i$.

(iii) $t_1^Q(R) = c+1$ *and* $\mathrm{Tor}_i^Q(k,R)_{i(c+1)} = (\mathrm{Tor}_1^Q(k,R)_{c+1})^i \ne 0$.

The hypothesis on the induced map $\mathrm{Tor}_i^Q(k,k) \to \mathrm{Tor}_i^R(k,k)$ implies that J is contained in $(Q_+)^2$. When Q is Koszul the converse holds as well, so the first two conditions of the following theorem follow directly from Theorem 4. The other two involve, in addition, applications of results of Bruns and Wiebe.

Theorem 5. *If* Q *is a finitely generated Koszul* K*-algebra and* J *a homogeneous ideal with* $0 \ne J \subseteq (Q_+)^2$*, then for* $R = Q/J$ *and* $c = \operatorname{Rate} R$ *one has*

(1) $\max\{c,2\} \le \operatorname{slope}_Q R \le c+1$, *with* $c < \operatorname{slope}_Q R$ *when* $\operatorname{pd}_Q R$ *is finite.*

(2) $t_i^Q(R) = (c+1)\cdot i$ *for some* $i \ge 1$ *implies the following conditions:* $t_h^Q(R) = (c+1)\cdot h$ *for* $1 \le h \le i$ *and* $i \le \operatorname{rank}_k(J/Q_+J)_{c+1}$.

(3) $t_i^Q(R) < (c+1)\cdot i$ *holds for all* $i > \dim Q - \dim R$ *when* $\operatorname{pd}_Q R$ *is finite.*

(4) $\operatorname{reg}_Q R \le c \cdot \operatorname{pd}_Q R$*; when* Q *is a standard graded polynomial ring, equality holds if and only if* J *is generated by a* Q*-regular sequence of degree* $c+1$.

The result is new even in the case of a polynomial ring Q. Its conclusions were initially observed in case *LG-quadratic*, that is, $R \cong P/(I+L)$ for some polynomial ring P, ideal I with a quadratic Gröbner basis, and ideal L generated by linear forms that map to a regular sequence in P/I; see [7]. Such algebras are Koszul, and the work presented above was partly motivated by the following:

Question 6. Is every Koszul algebra LG-quadratic?

The *Betti numbers* $\beta_i^{\widetilde{R}}(R) = \sum_{j\in\mathbb{Z}} \operatorname{rank}_k \operatorname{Tor}_i^{\widetilde{R}}(k,R)_j$, where $\widetilde{R}$ denotes the symmetric algebra on R_1, might help separate the two notions. Indeed, when R is LG-quadratic one has $R \cong Q/L$ and $Q = \widetilde{Q}/I_Q$, where Q is a standard graded K-algebra, L is an ideal generated by a Q-regular sequence of linear forms, and the initial ideal $\operatorname{in}_\tau(I_Q)$ for some $\tau \in T(Q)$ is generated by quadrics. As a consequence, one has $\beta_1^{\widetilde{Q}}(Q) = \beta_1^{\widetilde{Q}}(\widetilde{Q}/\operatorname{in}_\tau(I_Q))$, so one gets

$$\beta_i^{\widetilde{R}}(R) \le \beta_i^{\widetilde{Q}}(\widetilde{Q}/\operatorname{in}_\tau(I_Q)) \le \binom{\beta_1^{\widetilde{Q}}(\widetilde{Q}/\operatorname{in}_\tau(I_Q))}{i} = \binom{\beta_1^{\widetilde{R}}(R)}{i},$$

from a standard deformation argument and the Taylor resolution. Thus, we ask:

Question 7. If R is a Koszul algebra, does $\beta_i^{\widetilde{R}}(R) \le \binom{\beta_1^{\widetilde{R}}(R)}{i}$ hold for every i?

The results reported here are obtained jointly with Aldo Conca and Srikanth Iyengar. Additional information and complete proofs may be found in [2].

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Frobenius maps on injective hulls and their applications

Mordechai Katzman

The talk focused on a technique which has been applied successfully to the solution of various characteristic p problems. I illustrate this technique by producing an explicit description of the parameter test ideal of Cohen-Macaulay rings suitable for algorithmic implementation. The full account of this construction can be found in [4].

Henceforth let S denote a commutative local ring of prime characteristic p. A p^e-weak parameter test element is an element $c \in S$ not in any minimal prime such that for all ideals $A \subseteq S$ generated by a system of parameters, s is in the tight-closure A^* of A in S if and only if $cs^{p^{e'}}$ is in the Frobenius power $A^{[p^{e'}]}$ for all $e' \geq e$. The above-mentioned p^e-weak parameter test ideal is defined to be the ideal generated by all p^e-weak parameter test elements. (See [1] or [2] for an introduction to tight closure and its properties.)

Given an S-module M, an eth *Frobenius map* on M is an additive function $\phi : M \to M$ with the property that $\phi(sm) = s^{p^e}\phi(m)$ for all $s \in S$ and $m \in M$. Consider the skew-polynomial ring $S[T; f^e] := \oplus_{i\geq 0} ST^i$ with multiplication $Ts = s^{p^e}T$ for $s \in S$. To give an e^{th} Frobenius map $\phi : M \to M$ amounts to giving M the structure of a left-$S[T; f^e]$ module with $Tm = \phi(m)$ for all $m \in M$. Consider $ST^e \subseteq S[T; f]$; it is an S-bimodule with $s(aT^e) = saT^e$ and $(aT^e)s = s^{p^e}aT^e$. We define the *Frobenius functor* on S-modules to be $F^e_S(M) = ST^e \otimes_S M$ where S acts on the left.

Henceforth, we fix (R, m) to be complete regular local ring over a field of characteristic $p > 0$, we fix an ideal $I \subset R$ and let $S = R/I$. We write E_R and E_S for the injective hulls of the residue fields of R and S, respectively. We denote the Matlis dual functor $(-)^\vee = \mathrm{Hom}(-, E_R)$. Given any $S[T; f]$-module M we can define an R-linear map $\alpha_M : F^e_R(M) \to M$ given by $\alpha(rT^e \otimes m) = rTm$ for all $r \in R$ and $m \in M$. Taking Matlis duals gives a map $\alpha^\vee_M : M^\vee \to F^e_R(M)^\vee$. If we further assume that M is Artinian, we have a functorial isomorphism $\gamma_M : F^e_R(M)^\vee \to F^e_R(M^\vee)$ and we obtain an R-linear map $M^\vee \xrightarrow{\gamma_M \circ \alpha^\vee_M} F^e_R(M^\vee)$. (Compare this to the construction of Gennady Lyubeznik's $\mathcal{H}$ functor in section 4 of [5].)

Let $\mathcal{C}^e$ be the category of Artinian $S[T; f^e]$-modules and let $\mathcal{D}^e$ be the category of R-linear maps $M \to F^e_R(M)$ where M is a finitely generated S-module. The construction above gives an exact contravariant functor $\Delta^e : \mathcal{C}^e \to \mathcal{D}^e$. We can also retrace our steps and obtain an exact contravariant functor $\Psi^e : \mathcal{D}^e \to \mathcal{C}^e$ such that $\Psi^e \circ \Delta^e(-)$ and $\Delta^e \circ \Psi^e(-)$ are canonically isomorphic to the identity functor (cf. [3] for details).

The technique mentioned in the first paragraph consists of translating a given problem into one involving the properties of a certain $S[T; f]$-module structure on E_S, and applying the functor Δ^e above to translate the problem into one phrased in terms of ideals of the power series ring R.

For example, the p^e-weak parameter test ideal of S has the following description in terms of the canonical $S[T; f]$-module structure of $\mathrm{H}^d_m(S)$.

Theorem 1. *Assume that S is Cohen-Macaulay and has a parameter-test-element. For all $e \geq 0$, the p^e-weak parameter test ideal of S is the image in S of*

$$\cap \left\{ (0 :_R ST^e M) \,|\, M \subseteq \mathrm{H}^d_m(S) \text{ is a } S[T;f]\text{-submodule}, \mathrm{ht}(0 :_R ST^e M)S > 0 \right\}.$$

When S has canonical module $\omega \subseteq S$ the short exact sequence of $S[T;f]$-modules $0 \to \omega \to S \to S/\omega$ yields a surjection of $S[T;f]$-modules

$$E_S = \mathrm{H}^{\dim S}_{\mathfrak{m}S}(\omega) \to \mathrm{H}^{\dim S}_{\mathfrak{m}S}(S).$$

Let the kernel of this surjection be $\mathrm{Ann}_{E_S} J$; J can be given explicitly as the image of the canonical module $\mathrm{Ext}^{\dim R - \dim S}_R(S, R)$ in $\mathrm{Ext}^{\dim R - \dim S}_R(\omega, R) \cong S$. An application of Δ^1 to E_S with the $S[T;f]$-module module structure induced from that of $\mathrm{H}^{\dim S}_m(S)$ yields a map $R/I \to R/I^{[p]}$ given by multiplication by some $u \in (I^{[p]} : I)$.

We can now lift the description of the parameter test ideal to one involving $S[T;f]$-submodules of E_S and we can apply then the functor Δ to obtain the following.

Theorem 2. *Assume that S is Cohen-Macaulay and that the image of $c \in R$ in S is a test element. For all $e \geq 0$, the p^e-weak parameter test ideal $\overline{\tau}_e$ of S is given by*

$$\left(\left((cJ + I)^{\star u} \right)^{[p^e]} :_R u^{1+p+\cdots+p^{e-1}} J \right) S$$

where for any ideal $A \subseteq R$, $(A)^{\star u}$ denotes the smallest ideal L containing A such that $uL \subseteq L^{[p]}$.

All the ingredients in the theorem above are readily computed in any given concrete problem and hence we obtain an algorithm for computing p^e-weak parameter test ideals.

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Boij-Söderberg Theory, Part II

Frank-Olaf Schreyer
(joint work with David Eisenbud)

This talk is a continuation of David Eisenbud talks on Boij-Söderberg theory about free resolutions of graded modules over a polynomial ring. We focus on the information coming from cohomology tables of vector bundles and a sort of duality between the two subjects.

In David's talk the cone of Betti tables was described in terms of its extremal rays. In this talk we focus on the facets of the cone. According to the simplicial structure of the cone [2], an outer facet corresponds to a sequence of three degree sequences which differ in at most two consecutive positions. For example the degree sequences to the following Betti tables form such a chain.

$$\begin{pmatrix} 3 & 8 & 6 & \\ & & & 1 \end{pmatrix} < \begin{pmatrix} 2 & 4 & & \\ & & 4 & 2 \end{pmatrix} < \begin{pmatrix} 1 & & & \\ & 6 & 8 & 3 \end{pmatrix}$$

The facet equation is defined by the vanishing on the smaller Betti table and all below and on the larger Betti table and all above. This allows to compute the coefficients of the facet equation recursively using zero coefficients on the support of the right hand table as start values.

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ 21 & -12 & 5 & 0 \\ 12 & -5 & 0 & 3 \\ 5 & 0 & -3 & 4 \\ \mathbf{0} & 3 & -4 & 3 \\ 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

What we have to prove is, that this linear form is non-negative on the Betti table of any minimal free resolution. Our key observation is that the numbers appearing are dimensions of cohomology groups of what we call supernatural vector bundles.

Definition 1. A vector bundle $\mathcal{E}$ on $\mathbb{P}^m$ has natural cohomology [9], if for each k at most one of the groups

$$H^i(\mathcal{E}(k)) \neq 0.$$

It has supernatural cohomology, if in addition the Hilbert polynomial

$$\chi(\mathcal{E}(k)) = \frac{\text{rank } \mathcal{E}}{m!} = \prod_{j=1}^{m} (k - z_j)$$

has m distinct integral roots $z_1 < z_2 < \ldots < z_m$.

For a coherent sheaf $\mathcal{E}$ on $\mathbb{P}^m$ we denote by

$$\gamma(\mathcal{E}) = (\gamma_{j,k}) \in \prod_{k=-\infty}^{\infty} \mathbb{Q}^{m+1} \text{ with } \gamma_{j,k} = h^j(\mathcal{E}(k))$$

its cohomology table. Analogous to the Theorem on free resolutions we have

Theorem 2. *The extremal rays of the rational cone of cohomology tables of vector bundles are generated by cohomology tables of supernatural vector bundles.*

The crucial new concept is the following pairing between Betti tables of modules and cohomology table of coherent sheaves. We define $\langle \beta, \gamma \rangle$ for a Betti table $\beta = (\beta_{i,k})$ and a cohomology table $\gamma = (\gamma_{j,k})$ by

$$\langle \beta, \gamma \rangle = \sum_{i \geq j} (-1)^{i-j} \sum_k \beta_{i,k} \gamma_{j,-k}$$

Theorem 3 (Positivity 1)**.** *For F any free resolution of a finitely generated graded $K[x_0, \ldots, x_m]$-module and $\mathcal{E}$ any coherent sheaf on $\mathbb{P}^m$ we have*

$$\langle \beta(F), \gamma(\mathcal{E}) \rangle \geq 0.$$

For example the facet equation above, is obtained from the vector bundles $\mathcal{E}$ on $\mathbb{P}^2$, which is the kernel of a general map $\mathcal{O}^5(-1) \to \mathcal{O}^3$. The coefficients of the functional $\langle -, \gamma(\mathcal{E}) \rangle$ are

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ 21 & -12 & 5 & 0 \\ 12 & -5 & 0 & 3 \\ 5 & 0 & -3 & 4 \\ 0 & 3 & -4 & 3 \\ 0 & 4 & -3 & 0 \\ 0 & 3 & 0 & -5 \\ 0 & 0 & 5 & -12 \\ 0 & 0 & 12 & -21 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

This is not quite what we wanted. We define truncate functionals $\langle -, \gamma \rangle_{\tau,c}$ by putting zero coefficients in the appropriated spots.

Theorem 4 (Positivity 2)**.** *For F any <u>minimal</u> free resolution of a finitely generated graded $K[x_0, \ldots, x_m]$-module and $\mathcal{E}$ any coherent sheaf on $\mathbb{P}^m$ we have*

$$\langle \beta(F), \gamma(\mathcal{E}) \rangle_{\tau,c} \geq 0.$$

To prove the positivity theorems we considering the tensor product of F with the Čech resolution

$$C : \ldots \to C^p(E) = \sum_{0 \leq i_0 < i_1 < \ldots < i_p \leq m} E[(x_{i_0} \cdot \ldots \cdot x_{i_p})^{-1}] \to \ldots$$

of $\mathcal{E}$, where E denotes any graded module whose associated sheaf is $\mathcal{E}$.

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
F_0 \otimes C^2 & \leftarrow & F_1 \otimes C^2 & \leftarrow & F_2 \otimes C^2 & \leftarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
F_0 \otimes C^1 & \leftarrow & F_1 \otimes C^1 & \leftarrow & F_2 \otimes C^1 & \leftarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
F_0 \otimes C^0 & \leftarrow & F_1 \otimes C^0 & \leftarrow & F_2 \otimes C^0 & \leftarrow & \dots
\end{array}$$

Since we want to prove a purely numerical statement, we can replace E with its translate under a general element of $PGL(m+1, K)$ to achieve that E and F are cohomologically transverse [11, 12]. The horizontal homology is then concentrated in the first column and the total complex has cohomology only in positive degrees. On the other hand the lower diagonal part of the vertical cohomology of internal degree zero is

$$\begin{array}{cccc}
 & & H^2(F_2 \otimes \mathcal{E}) & \dots \\
 & H^1(F_1 \otimes \mathcal{E}) & H^1(F_2 \otimes \mathcal{E}) & \dots \\
H^0(F_0 \otimes \mathcal{E}) & H^0(F_1 \otimes \mathcal{E}) & H^0(F_2 \otimes \mathcal{E}) & \dots
\end{array}$$

and the Euler characteristic of this diagram is the desired value $\langle \beta(F), \gamma(\mathcal{E}) \rangle$. We can split the spectral sequence which starts with the vertical cohomology and converge to the total cohomology as a sequence of K-vector spaces. The part displayed above has then no cohomology except the cokernel in total cohomological degree 0. So $\langle \beta(F), \gamma(\mathcal{E}) \rangle$ is the dimension of a vector space. Using the minimality one sees that the truncated functionals are even more positive.
The main remaining part of the proof of both Boij-Söderberg decompositions is now to establish the existence of supernatural vector bundles and pure resolution for arbitrary zero or degree sequences. There are two methods for both cases known. For equivariant resolution or homogeneous vector bundles one can use appropriate explicit Schur functors [4, 5, 8] in characteristic 0. For arbitrary fields, one can use a push down method [6]. For bundles this is a simple application of the Künneth formula applied to $\mathcal{E} = \pi_* \mathcal{O}(a_1, \dots, a_m)$, where π is a finite linear projection $\pi : \mathbb{P}^1 \times \dots \times \mathbb{P}^1 \to \mathbb{P}^m$ and $\mathcal{O}(a_1, \dots, a_m)$ is a suitable line bundle on the product. For resolutions this consists of an iteration of the Lascoux method [10] to get the Buchsbaum-Eisenbud family of complexes associated to generic matrices [1]: We start with $\mathcal{K}$, a Koszul complex on $\mathbb{P}^m \times \mathbb{P}^{m_1} \times \dots \times \mathbb{P}^{m_s}$ of $1 + m + \sum_{i=1}^{s} m_i$ forms of multidegree $(1, \dots, 1)$ tensored with $\mathcal{O}(d_0, a_1, \dots, a_s)$. Here s is the number of desired non linear maps and $m_j + 1$ their degrees. The spectral sequence for $R\pi_* \mathcal{K}$ of the projection $\pi : \mathbb{P}^m \times \mathbb{P}^{m_1} \times \dots \times \mathbb{P}^{m_s} \to \mathbb{P}^m$ give rise to the desired complex, if we choose $a_1, \dots, a_s$ suitably.

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Frobenius Actions

Manuel Blickle

In his groundbreaking work [Lyu97] Lyubeznik studies Frobenius actions on local cohomology modules of a regular ring of positive characteristic p to obtain strong finiteness results, extending earlier work of Huneke and Sharp [HS93]. The key idea is that the a priori infinitely generated R-modules $H^i_I(R)$ where I is an ideal in R, arise as direct limits of the type

$$M \xrightarrow{\gamma} F^*M \xrightarrow{F^*\gamma} F^{2*}M \longrightarrow \dots$$

where M is a *finitely generated* R-module and F is the Frobenius map. In this case we say that $H^i_I(R)$ is generated by M. Thus he relocates the study of local cohomology modules (or more generally: finitely generated unit $R[F]$-modules, i.e. R-modules $\mathcal{M}$ together with an isomorphism $F^*\mathcal{M} \longrightarrow \mathcal{M}$) to the study of *finitely generated* R-modules M with a map $M \xrightarrow{\gamma} F^*M$. The question I want to address is the following:

> Can one choose a canonical representative (M, γ) which generates $\mathcal{M}$?

This can be answered by introducing the concept of *minimality* in the category of γ-modules (i.e. coherent R modules with a map $\gamma : M \longrightarrow F^*M$). We call (M, γ) minimal, if M does not have nil-potent γ submodules nor nilpotent γ-quotient modules. Here we call a γ-module N nilpotent if $\gamma^i(N) = 0$ for some i, where $\gamma_i = F^*(\gamma^{i-1}) \circ \gamma$. Clearly, if N is nilpotent, then the above limit $\lim_i F^{i*}N = 0$.

This implies that two γ-modules N and M generate the same unit $R[F]$–module if N and M are nil-isomorphic, which roughly means that there is a map $N \longrightarrow M$ of γ-modules such that kernel and cokernel are nilpotent. We have the following statement.

Theorem 1 ([Bli08]). *Let R be regular and F-finite and $\mathcal{M}$ a finitely generated unit $R[F]$–module. Then there is a unique minimal γ-module M which generates $\mathcal{M}$.*

Already in [Lyu97] this was shown in the case that R is complete. In [Bli01] the local F-finite case was proven. Pick any M that generates $\mathcal{M}$. It is easy to see by passing from M to M/K where $K = \bigcup \ker \gamma^i$ that we may assume that γ is injective, such that M does not have nilpotent submodules (satisfies the first part in the definition of minimality). This property clearly passes to submodules, hence the goal is to find the *smallest* γ-submodule N such that the quotient M/N is nilpotent. The key point in showing that such smallest submodule exists, is an iterative procedure which leads to its construction. We define iteratively (with $M_0 = M$) the γ submodule M_{i+1} as the smallest γ-submodule of M_i such that $\gamma(M_i) \subseteq F^* M_{i+1}$. The key observations are

(1) $M_i = M_{i+1}$ if and only if M_i is minimal iff and only if $M_i = M_j$ for all $j > i$.
(2) The chain is functorial and commutes with localization.
(3) The chain stabilizes locally (known local case of the theorem)

Let U_i be the open subset of $x \in \operatorname{Spec} R$ such that the $(M_i)_x = (M_{i+1})_x$. By the functoriality, the open sets U_i are an increasing sequence whose union is all of $\operatorname{Spec} R$, by the local result. Since $\operatorname{Spec} R$ is noetherian, it is compact, hence $U_i = \operatorname{Spec} R$ for some i. But this implies that $M_i = M_{i+1}$ such that M_i is minimal.

This result allows one to pick for every finitely generated unit $R[F]$-module a unique minimal γ-module that generates it, i.e. the category of finitely generated unit $R[F]$-modules is equivalent to the category of minimal γ-modules.

Example 1. Let M be the γ module with structural map $\gamma : R \xrightarrow{r \mapsto f^a r^q} R \cong F^* R$ where $f \in R, a \in \mathbb{N}$, and $q = p^e$ are fixed. Then the unique minimal representative $M_{\min}$ of M is equal to the generalized test ideal $\tau(R, f^{\frac{a}{q-1} - \epsilon})$ for all small enough $\epsilon > 0$. In fact, the above construction of $M_{\min}$ shows also that the rational number $\frac{a}{q-1}$ is not an accumulation point of jumping number for the test ideals of (R, f). This yields an alternative proof of the discreteness and rationality result for F-thresholds in [BMS08].

Another application of these results, or rather a dual version of them, is to the cohomology of constructible sheaves of $\mathbb{F}_p$-vectorspace on the étale site. Let C be a smooth projective curve and N a constructible sheaf of $\mathbb{F}_p$-vectorspaces on C_{et}. The aim is to describe a bound for the Euler characteristic $\chi(C_{et}, N)$, similar to the Grothendieck-Ogg-Shafarevich formula in the characteristic zero case. An observation of Pink [Pin00] is that if N is the fixed points of the Frobenius acting on a locally free coherent $\mathcal{O}_C$-module M, then

$$\chi(C_{et}, N) \geq \chi(C, M)$$

where the right hand side is the coherent Euler characteristic which may be computed via Riemann-Roch to be $(1-g)\,\mathrm{rank}\, M + \deg M$. The question is now if one can choose M canonically? This is answered by Miller [Mil07] who observes that if $M^\vee$ is minimal (in the above sense), then $\deg M$ is maximal possible, hence the minimal $M^\vee$ yields to best possible such bound for the Euler characteristic. Interestingly, one can use this observation to attach new *coherent* invariants to constructible sheaves of $\mathbb{F}_p$ vectorspaces, by using the unique minimal γ-sheaf associated to them.

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Potential applications of commutative algebra to combinatorial game theory

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(joint work with Alan Guo and Michael Weimerskirch)

Finite combinatorial games involving two players taking turns on the same game board are much more complex when the last player to move loses (misère play, as in Dawson's chess) instead of winning (normal play, as in Nim). The goal of Combinatorial Game Theory, in this setting, is to describe—abstractly and algorithmically—the set of winning positions for any given game, or any given class of games.

Recent developments by Plambeck [5], and later also with Siegel [6], have introduced certain commutative monoids, called *misère quotients*, as contexts in which to classify winning positions in misère play; see [7] for a gentle introduction. Our aim is to develop *lattice games*, played on affine semigroups, to place arbitrary impartial combinatorial games—but particularly the historically popular notion of octal game—in a general context where commutative algebra might be brought to bear on periodicity questions. In this note, we state a precise conjecture to the effect that sets of winning positions in lattice games are finite unions of translates of affine semigroups, drawing analogies and connections to the combinatorics of monomial local cohomology and binomial primary decomposition.

In what follows, C is a fixed *affine semigroup* in $\mathbb{Z}^d$; thus C is a finitely generated submonoid of $\mathbb{Z}^d$. We additionally require that C be *pointed*: its identity is its only unit (i.e., invertible element). The games are basically played on C, with allowed lattice moves taken from a fixed set Γ.

Definition 1. A finite subset $\Gamma \subset \mathbb{Z}^d \smallsetminus \{0\}$ is a *rule set* if

1. $\mathbb{N}\Gamma$ is pointed, and
2. $\mathbb{N}\Gamma \supseteq C$.

A *game board* G is the complement in C of a finite Γ-order ideal in C called the set of *defeated positions*. Elements in G are called *positions*. A *move* proceeds from a position $p \in G$ to a point $p - \gamma$ for some $\gamma \in \Gamma$; the move is *legal* if $p - \gamma \in G$.

Normal play corresponds to the choice of $D = \varnothing$; in that case, the goal is to be the player whose move lands at the origin. *Misère play* corresponds to the choice of $D = \{0\}$; in that case, the goal is *not* to be the player whose move lands at the origin. In this sense, misère play is "normal play in which one tries to lose".

It is implicit in the definition that Γ induces a partial order on C; the elementary proof is omitted. Conjecture 3 will make sense with the above notion of rule set Γ, but one or more of the following stronger conditions on Γ might be required.

3. Γ is the minimal generating set for $\mathbb{N}\Gamma$.
4. For each ray ρ of C, there exists $\gamma_i \in \Gamma$ lying in the *negative tangent cone* $-T_\rho C = -\bigcap_{H \supset \rho} H_+$ of C along ρ, where $H_+ \supset C$ is the positive closed half space defined by a supporting hyperplane H for C.
5. Every $p \in C$ has a Γ-path to 0 contained in C; that is, given p, there exists a sequence $0 = p_0, p_1, \ldots, p_r = p$ in C with $p_k - p_{k-1} \in \Gamma$ for all $k > 0$.

Condition 4 is precisely what is necessary to guarantee that from every position there is a Γ-path ending in a neighborhood of the origin. Thus condition 5 implies condition 4, though we omit the proof.

Definition 2. Fix a game board G with rule set Γ. Then $W \subseteq G$ is the set of *winning positions*, and $L \subseteq G$ is the set of *losing positions*, if

1. W and L partition G,
2. $(W + \Gamma) \cap G = L$, and
3. $(W - \Gamma) \cap W = \varnothing$.

The last player to make a legal move wins; this holds both for normal play and misère play, as well as the generalizations for larger D. A position is winning if the player who just moved there can force a win. Condition 1 says that every position is either winning or losing. Condition 2 says that the losing positions are precisely those positions possessing legal moves to W: if your opponent lands on a losing position, then you can always move to W to force a win. Condition 3 says that it is impossible to move directly from one winning position to another.

Conjecture 3. *If W is the set of winning positions for a lattice game, then W is a finite union of translates of affine semigroups.*

The conjecture, if true, would furnish a finite data structure in which to encode the set of winning positions. This would be the first step toward effectively computing the winning positions. Note that the misère octal game *Dawson's chess* [1] remains open; that game initiated and still motivates much of the research on misère games, and one hope would be to use lattice games, along with whatever computational commutative algebra might arise, to crack it.

According to the theory invented by Sprague and Grundy in the 1930s [3, 8], all normal play impartial games are equivalent to the particularly simple game Nim under a certain equivalence relation. For misère games, the analogous equivalence relation is too weak (i.e., too many equivalence classes: not enough games are equivalent to one another). That is why Plambeck invented misère quotients [5]. In the setting of lattice games, the equivalence relation is as follows.

Definition 4. Fix a game board $G = C \smallsetminus D$ with winning positions W. Two lattice points p and $q \in C$ are *congruent* if $(p + C) \cap W = p - q + (q + C) \cap W$. The *misère quotient* of C is the set Q of congruence classes.

Thus p and q are congruent if the winning positions in the "cones" above them are translates of one another. It is elementary to verify that the quotient map $C \to Q$ is a morphism of monoids. Plambeck and Siegel have studied misère quotients in quite a bit of algebraic detail [6]; as this is the proceedings for a conference on commutative algebra, it is strongly recommended that the reader have a look at their work, as it is filled with commutative algebra of finitely generated monoids.

Theorem 5. *Conjecture 3 holds when the misère quotient is finite.*

Proof. Follows by properly interpreting the combinatorial description of primary decomposition [2] of the binomial presentation ideal of the semigroup ring for Q inside of the semigroup ring for C. □

From discussions with Plambeck, Siegel, and others, as well as from examples, it seems likely that binomial primary decomposition has a further role to play in open questions about misère quotients. Such questions include when finiteness occurs, and more complex "algebraic periodicity" questions, which have yet to be formulated precisely [7].

There is another analogy with commutative algebra that is worth bearing in mind. When I is a monomial ideal in an affine semigroup ring, and M is a finitely generated finely graded (i.e., $\mathbb{Z}^d$-graded) module, then the local cohomology $H^i_I(M)$ is supported on a finite union of translates of affine semigroups [4]. If Conjecture 3 is true, then perhaps one could develop a homological theory for winning positions in combinatorial games that explains why.

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Shokurov's ACC Conjecture for log canonical thresholds on smooth varieties

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(joint work with Tommaso de Fernex and Mircea Mustaţă)

Shokurov's ACC Conjecture [Sho] says that the set of all log canonical thresholds on complex varieties of any fixed dimension satisfies the ascending chain condition, that is, it contains no infinite strictly increasing sequences. This conjecture attracted considerable interest due to its implications to the Termination of Flips Conjecture (see [Bir] for a result in this direction). The first unconditional results on sequences of log canonical thresholds on smooth varieties of arbitrary dimension have been obtained in [dFM], and they were subsequently reproved and strengthened in [Kol2]. In a recent paper, we prove the conjecture is true for smooth varieties.

Theorem 0.1. For every n, the set

$$\mathcal{T}_n^{\mathrm{sm}} := \{\mathrm{lct}(\mathfrak{a}) \mid X \text{ is smooth, } \dim X = n,\ \mathfrak{a} \subsetneq \mathcal{O}_X\}$$

of log canonical thresholds on smooth varieties of dimension n satisfies the ascending chain condition.

In this talk we discuss various techniques that went into the proof of the above theorem.

Generic Limits

We review the construction from [Kol2], extending it from sequences of power series to sequences of ideals.

Let $R = k[\![x_1, \ldots, x_n]\!]$ be the ring of formal power series in n variables with coefficients in an algebraically closed field k, and let $\mathfrak{m}$ be its maximal ideal. If $k \subset L$ is a field extension, then we put $R_L := L[\![x_1, \ldots, x_n]\!]$ and $\mathfrak{m}_L := \mathfrak{m} \cdot R_L$.

For every $d \geq 1$, we consider the quotient homomorphism $R \to R/\mathfrak{m}^d$. We identify the ideals in $R/\mathfrak{m}^d$ with the ideals in R containing $\mathfrak{m}^d$. Let $\mathcal{H}_d$ be the Hilbert scheme parametrizing the ideals in $R/\mathfrak{m}^d$, with the reduced structure. Since $\dim_k(R/\mathfrak{m}^d) < \infty$, this is an algebraic variety. Mapping an ideal in $R/\mathfrak{m}^d$

to its image in $R/\mathfrak{m}^{d-1}$ gives a surjective map $t_d\colon \mathcal{H}_d \to \mathcal{H}_{d-1}$. This is not a morphism. However, by Generic Flatness we can cover $\mathcal{H}_d$ by disjoint locally closed subsets such that the restriction of t_d to each of these subsets is a morphism. In particular, for every irreducible closed subset $Z \subseteq \mathcal{H}_d$, the map t_d induces a rational map $Z \dashrightarrow \mathcal{H}_{d-1}$.

Suppose now that $(\mathfrak{a}_i)_{i\in I_0}$ is a sequence of ideals $\mathfrak{a}_i \subseteq R$ indexed by the set $I_0 = \mathbf{Z}_+$. We consider sequences of irreducible closed subsets $Z_d \subseteq \mathcal{H}_d$ for $d \geq 1$ such that

($\star$) For every $d \geq 1$, the projection t_{d+1} induces a dominant rational map $\phi_{d+1}\colon Z_{d+1} \dashrightarrow Z_d$.

($\star\star$) For every $d \geq 1$, there are infinitely many i with $\mathfrak{a}_i + \mathfrak{m}^d \in Z_d$, and the set of such $\mathfrak{a}_i + \mathfrak{m}^d$ is dense in Z_d.

Given such a sequence (Z_d), we define inductively nonempty open subsets $Z_d^\circ \subseteq Z_d$, and a nested sequence of infinite subsets

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots ,$$

as follows. We put $Z_1^\circ = Z_1$ and $I_1 = \{i \in I_0 \mid \mathfrak{a}_i + \mathfrak{m} \in Z_1\}$. For $d \geq 2$, let $Z_d^\circ = \phi_d^{-1}(Z_{d-1}^\circ) \subseteq \mathrm{Domain}(\phi_d)$ and $I_d = \{i \in I_0 \mid \mathfrak{a}_i + \mathfrak{m}^d \in Z_d^\circ\}$. It follows by induction on d that Z_d° is open in Z_d, and condition ($\star\star$) implies that each I_d is infinite. Furthermore, it is clear that $I_d \supseteq I_{d+1}$.

Sequences (Z_d) satisfying ($\star$) and ($\star\star$) can be constructed as follows. We first choose a minimal irreducible closed subset $Z_1 \subseteq \mathcal{H}_1$ with the property that contains $\mathfrak{a}_i + \mathfrak{m}$ for infinitely many indices $i \in I_0$. We set $J_1 = \{i \in I_0 \mid \mathfrak{a}_i + \mathfrak{m} \in Z_1\}$. By construction, J_1 is an infinite set and Z_1 is the closure of $\{\mathfrak{a}_i + \mathfrak{m} \mid i \in I_1\}$. Next, we choose a minimal closed subset $Z_2 \subseteq \mathcal{H}_2$ that contains $\mathfrak{a}_i + \mathfrak{m}^2$ for infinitely many i in J_1. As we have seen, t_2 induces a rational map $\phi_2\colon Z_2 \dashrightarrow Z_1$ and this has the property that the set $J_2 = \{i \in J_1 \mid \mathfrak{a}_i + \mathfrak{m}^2 \in Z_2\}$ is infinite, and that Z_2 is the closure of $\{\mathfrak{a}_i + \mathfrak{m}^2 \mid i \in J_2\}$. Note that by the minimality in the choice of Z_1, the rational map ϕ_2 is dominant. Repeating this process we select a sequence (Z_d) that satisfies ($\star$) and ($\star\star$) above.

Suppose now that we have a sequence (Z_d) with these two properties. The rational maps ϕ_d induce a nested sequence of function fields $k(Z_d)$. Let $K := \bigcup_{d\geq 1} k(Z_d)$. Each morphism $\mathrm{Spec}(K) \to Z_d \subseteq \mathcal{H}_d$ corresponds to an ideal $\mathfrak{a}'_d$ in $R_K/\mathfrak{m}_K^d$, and the compatibility between these morphisms implies that there is a (unique) ideal $\mathfrak{a}$ in R_K such that $\mathfrak{a}'_d = \mathfrak{a} + \mathfrak{m}_K^d$ for all d.

Definition 0.2. With the above notation, we say that the ideal $\mathfrak{a}$ is *a generic limit* of the sequence of ideals $(\mathfrak{a}_i)_{i\geq 1}$. More generally, for every field extension $L \supseteq K$, we say that $\mathfrak{a} \cdot R_L$ is *a generic limit* of the sequence $(\mathfrak{a}_i)_{i\geq 1}$.

Proposition 0.3. Let $\mathfrak{a} \subseteq R_K$ be a generic limit of a sequence $(\mathfrak{a}_i)_{i\geq 1}$ of ideals of R. Assume that $\mathfrak{a}_i \neq R$ for every i. For every d there is an infinite subset $I_d^\circ \subseteq I_d$ such that

$$\mathrm{lct}(\mathfrak{a} + \mathfrak{m}_K^d) = \mathrm{lct}(\mathfrak{a}_i + \mathfrak{m}^d) \quad \text{for every } i \in I_d^\circ.$$

Moreover, if E is a divisor over $\mathrm{Spec}(R_K)$ computing $\mathrm{lct}(\mathfrak{a}+\mathfrak{m}_K^d)$, then there is an integer d_E such that for every $d \geq d_E$ the following holds: there is an infinite subset $I_d^E \subseteq I_d^\circ$, and for every $i \in I_d^E$ a divisor E_i over $\mathrm{Spec}(R)$ computing $\mathrm{lct}(\mathfrak{a}_i + \mathfrak{m}^d)$, such that $\mathrm{ord}_E(\mathfrak{a} + \mathfrak{m}_K^d) = \mathrm{ord}_{E_i}(\mathfrak{a}_i + \mathfrak{m}^d)$.

Effective $\mathfrak{m}$-adic semicontinuity of log canonical thresholds

Another key technical tool to our proof is the following $\mathfrak{m}$-adic semicontinuity theorem for log canonical thresholds.

Theorem 0.4. Let X be a log canonical variety, and let $\mathfrak{a} \subsetneq \mathcal{O}_X$ be a proper ideal. Suppose that E is a prime divisor over X computing $\mathrm{lct}(\mathfrak{a})$, and consider the ideal sheaf $\mathfrak{q} := \{h \in \mathcal{O}_X \mid \mathrm{ord}_E(h) > \mathrm{ord}_E(\mathfrak{a})\}$. If $\mathfrak{b} \subseteq \mathcal{O}_X$ is an ideal such that $\mathfrak{b}+\mathfrak{q} = \mathfrak{a}+\mathfrak{q}$, then after possibly restricting to an open neighborhood of the center of E we have $\mathrm{lct}(\mathfrak{b}) = \mathrm{lct}(\mathfrak{a})$.

A version of this result was first proven by Kollár in [Kol2] in the setting of formal power series using deep results in the Minimal Model Program from [BCHM] and a theorem on Inversion of Adjunction due from [Kaw] (the original formulation for power series assumes that the ideals are principal and the center of E is the origin). By restricting to the algebraic setting, we give an elementary proof of this result which only uses the Connectedness Theorem of Shokurov and Kollár (see Theorem 7.4 in [Kol1]). It turns out that this geometric version of the result is enough for our purposes. On the other hand, our proof would extend to the setting of formal power series, provided the Connectedness Theorem could be extended to that setting.

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Rationality of F-jumping exponents on singular varieties

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(joint work with Manuel Bickle, Karl Schwede and Wenliang Zhang)

In this note, we will discuss the rationality of F-jumping exponents on singular varieties. First we recall the situation in characteristic zero.

Let X be a $\mathbb{Q}$-Gorenstein normal algebraic variety over a field of characteristic zero, $\mathfrak{a} \subseteq \mathcal{O}_X$ be an ideal sheaf of X. Let $\pi : Y \to X$ be a log resolution of the pair $(X, \mathfrak{a})$, that is, a proper birational morphism with $\widetilde{X}$ a nonsingular variety such that $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$ is an invertible sheaf and that $\mathrm{Exc}(\pi) \cup \mathrm{Supp}(F)$ is a simple normal crossing divisor. Fix a real number $t > 0$. Then the multiplier ideal $\mathcal{J}(\mathfrak{a}^t)$ of $\mathfrak{a}$ with exponent t is

$$\mathcal{J}(\mathfrak{a}^t) = \mathcal{J}(X, \mathfrak{a}^t) = \pi_*\mathcal{O}_Y(\lceil K_{Y/X} - tF \rceil) \subseteq \mathcal{O}_X,$$

where $K_{Y/X}$ is the relative canonical divisor of π. This definition is independent of the choice of the log resolution π.

By the definition of multiplier ideals, it is easy to see that the family of multiplier ideals $\mathcal{J}(\mathfrak{a}^t)$ of a fixed ideal $\mathfrak{a}$ is right continuous in t: for each $t > 0$, there exists $\epsilon > 0$ such that $\mathcal{J}(\mathfrak{a}^t) = \mathcal{J}(a^{t'})$ for all $t' \in [t, t+\epsilon)$.

Definition 1. A real number $t > 0$ is called a *jumping exponent* of $\mathfrak{a}$ if $\mathcal{J}(\mathfrak{a}^{t-\epsilon}) \supsetneq \mathcal{J}(\mathfrak{a}^t)$ for all $\epsilon > 0$.

Lemma 2. *All jumping exponents of* $\mathfrak{a}$ *form a discrete set of rational numbers.*

Proof. Write

$$F = \sum_{i=1}^{r} a_i E_i, \quad K_{Y/X} = \sum_{i=1}^{r} k_i E_i.$$

If t is a jumping exponent of $\mathfrak{a}$, then $k_i - ta_i$ should be an integer for some $1 \le i \le r$. Since a_i and k_i are rational numbers, t is also a rational number. □

Now we turn to the situation in positive characteristic.

Let R be a Noetherian reduced ring of characteristic $p > 0$. We denote by R° the set of elements of R which are not in any minimal prime ideal. Also, for each integer $e \ge 1$, denote by $F^e_* R$ the ring R viewed as an R-module via the e-times iterated Frobenius map $F^e : R \to R$ which sends x to x^{p^e}. The ring R is called *F-finite* if $F^1_* R$ is a finitely generated R-module.

Let $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a}\cap R^\circ \neq \emptyset$. Let $E := \bigoplus_{\mathfrak{m}} E_R(R/\mathfrak{m})$ be the direct sum of the injective hulls of the residue fields $R/\mathfrak{m}$ of R, where $\mathfrak{m}$ runs through all maximal ideals of R. Fix a real number $t > 0$. Then the $\mathfrak{a}^t$-tight closure $0_E^{*\mathfrak{a}^t}$ of the zero submodule in E is defined to be the submodule of E consisting of all elements $z \in E$ for which there exists $c \in R^\circ$ such that $c\mathfrak{a}^{\lceil tq \rceil} \otimes z = 0$ in $F_*^e R \otimes_R E$ for all large $q = p^e$. The generalized test ideal $\widetilde{\tau}(\mathfrak{a}^t)$ is

$$\widetilde{\tau}(\mathfrak{a}^t) = \widetilde{\tau}(R, \mathfrak{a}^t) = \mathrm{Ann}_R(0_E^{*\mathfrak{a}^t}) \subseteq R.$$

The reader is referred to [4] and [3] for basic properties of generalized test ideals.

Thanks to Hara and Yoshida's result, we can think of generalized test ideals as a characteristic p analogue of multiplier ideals.

Theorem 3 ([4])**.** *Let $(R, \mathfrak{m})$ be a $\mathbb{Q}$-Gorenstein normal local ring essentially of finite type over a perfect field of characteristic $p > 0$, and let $\mathfrak{a} \subseteq R$ be a nonzero ideal and $t > 0$ be a fixed real number. Assume that $(R, \mathfrak{a})$ is reduced from characteristic zero to characteristic $p \gg 0$, together with a log resolution $\pi : Y \to \mathrm{Spec}\ R$ of $(\mathrm{Spec}\ R, \mathfrak{a})$ giving the multiplier ideal $\mathcal{J}(\mathrm{Spec}\ R, \mathfrak{a}^t)$. Then*

$$\mathcal{J}(\mathrm{Spec}\ R, \mathfrak{a}^t) = \widetilde{\tau}(R, \mathfrak{a}^t).$$

Lemma 4. *If R is F-finite, then the following holds.*

(1) The formation of $\widetilde{\tau}(\mathfrak{a}^t)$ commutes with localization and completion.

(2) For each $t > 0$, there exists $\epsilon > 0$ such that $\widetilde{\tau}(\mathfrak{a}^t) = \widetilde{\tau}(\mathfrak{a}^{t'})$ for all $t' \in [t, t + \epsilon)$.

Definition 5. A real number $t > 0$ is called an *F-jumping exponent* of $\mathfrak{a}$ if $\widetilde{\tau}(\mathfrak{a}^{t-\epsilon}) \supsetneq \widetilde{\tau}(\mathfrak{a}^t)$ for all $\epsilon > 0$.

Blickle-Mustaţă-Smith proved that all F-jumping exponents are rational numbers if the ring is regular.

Theorem 6 ([1, 2])**.** *Let R be an F-finite regular ring of characteristic $p > 0$ and $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a} \cap R^\circ \neq \emptyset$. Suppose that one of the following conditions is satisfied:*

(1) $\mathfrak{a}$ is a principal ideal,

(2) R is essentially of finite type over a field.

Then all F-jumping exponents of $\mathfrak{a}$ form a discrete set of rational numbers.

Remark 7. Katzman–Lyubeznik–Zhang also proved a similar result in [5]: if R is an excellent (not necessarily F-finite) local ring of characteristic $p > 0$ and $\mathfrak{a}$ is a principal ideal, then all F-jumping exponents of $\mathfrak{a}$ form a discrete set of rational numbers.

We generalize Blickle-Mustaţă-Smith's result to the case of singular varieties.

Theorem 8. *Let R be an F-finite $\mathbb{Q}$-Gorenstein normal ring of characteristic $p > 0$ and $\mathfrak{a}$ be an ideal of R such that $\mathfrak{a} \cap R^\circ \neq \emptyset$. Assume that the order of the canonical module ω_R in the class group $\mathrm{Cl}(R)$ is not divisible by the characteristic p. In addition, suppose that one of the following conditions is satisfied:*

(1) $\mathfrak{a}$ is a principal ideal,
(2) R is essentially of finite type over a field.

Then all F-jumping exponents of $\mathfrak{a}$ form a discrete set of rational numbers.

Question 9. If the ring is not $\mathbb{Q}$-Gorenstein, does there exist a irrational F-jumping exponent?

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A tight closure theory in equal characteristic that commutes with localization

Melvin Hochster

(joint work with Neil A. Epstein)

We discuss recent joint work with Neil Epstein that provides a modification of tight closure theory, both in equal characteristic $p > 0$ and in equal characteristic 0, that commutes with localization.

It was recently shown by H. Brenner and P. Monsky that tight closure in characteristic p does not, in general, commute with localization, although many special cases of the localization question remain open.

We focus on the case of closure operations on ideals in a family of Λ-algebras, where Λ is some base ring. The most important choices for Λ are the integers, the rational numbers, and the integers modulo a prime p. The theory extends readily to modules for those closures, like tight closure, with the property that the question of whether $u \in M$ is in the closure of N is equivalent to the question of whether the image of u is in the closure of 0 in M/N. In such instances, M can be replaced by a free module G that maps onto it, N by its inverse image in G, and u by an element of G that maps to u.

Suppose that the closure of I is denoted I^*. We assume that the closures we are studying have the property that I^* is an ideal containing I such that $I^* = (I^*)^*$.

Suppose that we fix a family of systems Σ of polynomial equations over Λ: each such system Σ involves variables $X_1, \ldots, X_n$, Y, and $U_1, \ldots, U_h$ where n and h may vary with Σ. A closure is called *equational* over Λ if for $y \in R$ and I an ideal of R, y is in the closure of I precisely if at least one system Σ in the family has a solution in which the X_i are in I, $Y = y$, and the U_j are in R.

Equational closures are persistent, that is if u is in the closure of I in R and we have a homomorphism $R \to S$, then the image of u is in the closure of IS in S.

An equational closure is called *homogeneous* if for each element Σ of the family that determines it, the variables X_i, Y, U_j may be assigned positive integer degrees in such a way that every equation is (weighted) homogeneous with respect to this assignment of degrees.

Taking the radical is a homogeneous equational closure for the family in which a typical system is the single equation $Y^n - X = 0$ for some positive integer n. One may give X weight 1 and Y weight n. Integral closure, Frobenius closure, and plus closure of ideals all turn out to be homogeneous equational closures.

A persistent closure is equational if and only if all instances of the closure arise by base change from instances in a finitely generated Λ-algebra. That is, whenever u is in the closure of I in S, there is a finitely generated Λ-algebra R, an element u_0 in R, an ideal I_0 in R such that u_0 is in the closure of I_0, and a Λ-homomorphism $R \to S$ such that $u_0 \mapsto u$ and I_0S is contained in I.

A closure is homogeneous equational if and only if (†) the corresponding conditions are satisfied with the additional restrictions that R be a finitely generated nonnegatively graded Λ-algebra with $R_0 = \Lambda$, that I_0 be graded with generators of positive degree, and that u_0 be homogeneous of positive degree.

All homogeneous closures commute with localization because if homogeneous equations have a solution in the localized ring $W^{-1}R$, call it $(r_1/w, \ldots, r_n/w)$, where the variable in the i th spot has degree d_i, the equations also have a solution of the form $(w^{Nb_1-1}r_1, \ldots, w^{Nb_n-1}r_n)$ for large N: here it may be necessary to adjust the choice of the denominator w so that certain W-torsion is killed.

We now define homogeneous equational tight closure by allowing only those instances of tight closure that arise by base change from a graded instance as described in the condition labeled (†) two paragraphs above.

This gives new notions of tight closure both in prime characteristic $p > 0$ and in equal characteristic 0 that are persistent, captures colon (and, more generally, still give phantom homology for finite projective complexes when the original notion does), capture contracted expansions from integral extensions, yield a Briançon-Skoda theorem, have the property that every submodule of every module is closed over a regular ring, and may be tested modulo nilpotents, modulo minimal primes, or by maps to completions at maximal ideals, assuming that the ring is locally excellent. This closure may also be tested by all maps to complete local domains. It agrees with tight closure in a finitely generated graded algebra over any of the prime fields in the case of graded modules and their submodules. Beyond all that, the calculation of this closure commutes with localization.

One has for the new closure that in equal characteristic 0 it has the colon-capturing property and that, more generally, if one has a finite complex of modules each of which is locally free of constant rank that satisfies the standard conditons on rank and height (the height condition must hold modulo all minimal primes), then the cycles are in the homogeneous tight closure of the boundaries. Moreover, the tight closure of 0 in the augmentation module is the same as its homogeneous tight closure. The proofs depend on careful application of the Artin-Rotthaus theorem, properties of homogenization of affine algebras (corresponding to embedding in a weighted projective space), as well as proving new results in characteristic p that show that tight closure commutes with localization at an element t in certain cases (e.g., for parameter ideals) in a constructive sense: if I is a parameter ideal in R one can give an explicit bound $B > 0$ such that if $r \in R$ and $r/1 \in (IR_t)^*$ in R_t, then $t^B r \in I^*$.

In characteristic p, homogeneous tight closure lies between the plus closure and the usual tight closure, and while it might agree with the the former it is definitely, in general, smaller than the latter. It is particularly surprising that one gets a notion of tight closure in equal characteristic 0 that commutes with localization.

The new notion leads to a new class of rings: those for which every ideal is tightly closed in the sense of homogeneous tight closure. In characteristic p, this class contains all weakly F-regular rings and is closed under localization at any multiplicative system.

H-vectors of simplicial complexes and Serre's conditions

Satoshi Murai
(joint work with Naoki Terai)

The study of h-vectors of simplicial complexes is an interesting research area in combinatorics as well as in combinatorial commutative algebra. On h-vectors of simplicial complexes, one of fundamental problems is their non-negativity. For example, a classical result of Stanley guarantees that h-vectors of Cohen–Macaulay complexes are non-negative. We study the non-negativity of h-vectors in terms of Serre's condition (S_r).

Let $S = K[x_1, \ldots, x_n]$ be a standard graded polynomial ring over an infinite field K. Let $I \subset S$ be a graded ideal and $R = S/I$. The *Hilbert series* of R is the formal power series $F(R, \lambda) = \sum_{q=0}^{\infty} (\dim_K R_q)\lambda^q$, where R_q is the graded component of degree q of R. It is known that $F(R, \lambda)$ is a rational function of the form $(h_0 + h_1\lambda + \cdots + h_s\lambda^s)/(1-\lambda)^d$, where each h_i is an integer with $h_s \neq 0$ and where $d = \dim R$. The vector $(h_0(R), h_1(R), \ldots, h_s(R)) = (h_0, h_1, \ldots, h_s)$ is called the *h-vector* of R. We say that $R = S/I$ satisfies *Serre's condition* (S_r) if

$$\operatorname{depth} R_P \geq \min\{r, \dim R_P\}$$

for all graded prime ideals $P \supset I$ of S.

Let Δ be a simplicial complex on $[n] = \{1, 2, \ldots, n\}$. Thus Δ is a collection of subsets of $[n]$ satisfying that (i) $\{i\} \in \Delta$ for all $i \in [n]$ and (ii) if $F \in \Delta$ and $G \subset F$ then $G \in \Delta$. The squarefree monomial ideal $I_\Delta \subset S$ generated by all squarefree monomials $x_F = \prod_{i \in F} x_i \in S$ with $F \notin \Delta$ is called the *Stanley–Reisner ideal* of Δ. The ring $K[\Delta] = S/I_\Delta$ is the *Stanley–Reisner ring* of Δ. The vector $h(\Delta) = h(K[\Delta])$ is called the *h-vector* of Δ.

We say that Δ satisfies Serre's condition (S_r) if $K[\Delta]$ satisfies Serre's condition (S_r). It is not hard to see that Δ satisfies (S_r) if and only if, for every $F \in \Delta$, $\tilde{H}_i(\mathrm{lk}_\Delta(F); K) = 0$ for $i < \min\{r-1, \dim \mathrm{lk}_\Delta(F)\}$, where $\tilde{H}_i(\Delta; K)$ is the reduced homology groups of Δ over a field K and where $\mathrm{lk}_\Delta(F) = \{G \subset [n] \setminus F : G \cup F \in \Delta\}$ is the *link* of Δ with respect to a face $F \in \Delta$ (see [3, p. 454]). A homological characterization of (S_r) is also known. It is know that a $(d-1)$-dimensional simplicial complex Δ satisfies (S_r) with $r \geq 2$ if and only if $\dim(\mathrm{Ext}_S^{n-i}(K[\Delta], \omega_S)) \leq i - r$ for $i = 0, 1, \ldots, d-1$, where ω_S is the canonical module of S (see [1, Lemma 3.2.1]).

We remark some basic facts. Every simplicial complex satisfies (S_1). On the other hand, for $r \geq 2$, simplicial complexes satisfying (S_r) are pure and strongly connected. (S_2) states that Δ is pure and $\mathrm{lk}_\Delta(F)$ is connected for all faces $F \in \Delta$ with $|F| < \dim \Delta$. (S_d) is equivalent to the famous Cohen–Macaulay property of simplicial complexes.

A classical result of Stanley [2] guarantees that if Δ is Cohen–Macaulay (that is, if it satisfies (S_d)) then $h_k(\Delta)$ is non-negative for all k. We generalize this classical result in the following way.

Theorem 1. *If a simplicial complex Δ satisfies (S_r) then $h_k(\Delta) \geq 0$ for $k = 0, 1, \ldots, r$.*

We also study what happens if $h_k = 0$ for some $1 \leq k \leq r$. We get the next result.

Theorem 2. *Let Δ be a simplicial complex which satisfies (S_r). If $h_t(\Delta) = 0$ for some $1 \leq t \leq r$ then $h_k(\Delta) = 0$ for all $k \geq t$ and Δ is Cohen–Macaulay.*

It is known that, for all integers $2 \leq r < d$, there exists a $(d-1)$-dimensional simplicial complex Δ which satisfies Serre's condition (S_r) but $h_{r+1}(\Delta) < 0$ ([4, Example 3.5]). Thus we cannot expect that all the h_k are non-negative. However, we proved the following weak non-negative property for $h_k(\Delta)$ with $k \geq r$.

Theorem 3. *If a simplicial complex Δ satisfies (S_r) then $\sum_{k \geq r} h_k(\Delta) \geq 0$.*

To prove the above theorems, we prove the following algebraic result which might be itself of interest. For a finitely generated graded S-module M, let

$$\mathrm{reg}\, M = \max\{j : \mathrm{Tor}_i(M, K)_{i+j} \neq 0 \text{ for some } i\}$$

be the *(Castelnuovo-Mumford) regularity* of M.

Theorem 4. *Let $r \geq 1$ be an integer. Let $I \subset S$ be a graded ideal and d the Krull dimension of $R = S/I$. Suppose that $\mathrm{reg}(\mathrm{Ext}_S^{n-i}(R, \omega_S)) \leq i - r$ for $i =$*

$0, 1, \ldots, d-1$. *There exists a linear system of parameters* $\Theta = \theta_1, \ldots, \theta_d$ *of* R *such that*

$$h_k(R) = \dim_K(R/\Theta R)_k \quad \textit{for } k \leq r.$$

Theorem 1 follows from Theorem 4 as follows: If R is a Stanley–Reisner ring, then it is known that $\mathrm{Ext}_S^{n-i}(R, \omega_S)$ is a squarefree module (see [5]). The regularity of a squarefree module is always bounded by its dimension. Thus the homological characterization of (S_r) shows $\mathrm{reg}(\mathrm{Ext}_S^{n-i}(R, \omega_S)) \leq i - r$ for $i = 0, 1, \ldots, d-1$, where $d = \dim R$. Then apply Theorem 4.

It would be natural to ask whether Theorems 1, 2 and 3 hold for all graded ideals $I \subset S$. While we do not have a complete answer, we know that they are true for monomial ideals since polarization preserves Serre's conditions. We expect that there are other nice classes of graded ideals for which Theorems 1, 2 and 3 hold. Since a key point in the proof of Theorem 1 is the fact that $\mathrm{reg}(\mathrm{Ext}_S^{n-i}(K[\Delta], \omega_S)) \leq \dim(\mathrm{Ext}_S^{n-i}(K[\Delta], \omega_S))$, the following question might be of interest.

Question 5. *When* $\mathrm{reg}(\mathrm{Ext}_S^{n-i}(S/I, \omega_S)) \leq \dim(\mathrm{Ext}_S^{n-i}(S/I, \omega_S))$ *holds? Which graded ideals* I *satisfy* $\mathrm{reg}(\mathrm{Ext}_S^{n-i}(S/I, \omega_S)) \leq i - r$ *for* $i = 0, 1, \ldots, \dim S/I - 1$ *when* S/I *satisfies* (S_r)*?*

We do not know whether the above question is true even for monomial ideals (while we know that Theorems 1, 2 and 3 are true for monomial ideals).

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Torsion in the symmetric algebra and implicitization.

Marc Chardin

(joint work with Laurent Buse, Jean-Pierre Jouanolou and Aron Simis)

The problem we considered is to determine the image of a rational map from a projective space to another of dimension one more, in terms of the homogeneous polynomials (of same positive degree) defining the map. We only described precise results in the case where the map has finitely many base points and the image is a hypersurface.

There are two key ideas in this approach, both detailed in the work of Laurent Busé and Jean-Pierre Jouanolou. One is to remark that the equation of the image is the annihilator of the graded component of the Rees algebra of the ideal generated by the polynomials defining the map, in any non negative degree. (Recall that it is, almost by definition, the annihilator of the degree 0 component of the Rees algebra). A second key idea is to control the torsion in the symmetric algebra using the approximation complexes constructed by Juergen Herzog, Aron Simis and Wolmer Vasconcelos. In particular in cases where this torsion vanishes in high degrees, one derives an upper bound on the degree from which it is always zero, in terms of some basic invariants of the map.

In this talk, I reported on recent results and on work in progress concerning the understanding of the torsion in the symmetric algebra, and on its consequences for the implicitization problem. As mentioned above, only the case of maps with finitely many base points was treated. In a joint work with Laurent Buse and Jean-Pierre Jouanolou, we provided optimal bounds for the degree where the torsion of the symmetric algebra vanishes. We also analyzed the irreducible components of the symmetric algebra and their multiplicities, in order to describe the difference between the the cycles defined by the Rees algebra and the symmetric algebra in the biprojective space the naturally live in. This is of importance, since the algorithms developped for the implicitization problem actually computes the direct image of the cycle defined by the symmetric algebra, under the projection to the projective space corresponding to the target of the map.

Denoting by A and B the polynomial rings (over the same field) corresponding respectively to the origin and the target projective spaces, in respectively n and $n+1$ variables, by I the ideal generated by the $n+1$ forms in A of same degree $d>0$ defining the rational map λ, and by m the graded maximal ideal of A, the following statement contains several results mentioned above.

Theorem 1. *Let Z be the closed image of λ, $X := \mathrm{Proj}(A/I)$ be the base locus and $I_X := I :_A m^\infty$. If $X \subset \mathbf{P}^{n-1}$ is empty or zero dimensional and locally defined by at most n equations then,*

$$\mathrm{div}(\mathrm{Sym}_A(I)_\nu) = \deg(\lambda).[Z] + \sum_{x\in X}(e_x - d_x)[H_x],$$

if and only if $\nu \geq (n-1)(d-1) - \mathrm{indeg}(I_X)$; where the brackets denote the corresponding divisor, e_x is the Hilbert-Samuel multiplicity of the (not always reduced) point x, d_x its degree, and H_x a hyperplane defined by a linear form obtained by evaluation at x of any syzygy of the $n+1$ given generators of I.

For such a ν, this divisor can be computed as the determinant of the corresponding graded part of the approximation complex.

Recall that $e_x = d_x$ if and only if x is a locally complete intersection base point. The main consequences in applications to geometric modeling is the precise optimal value for ν where $\mathrm{Sym}_A(I)_\nu$ is torsion and gives the divisor associated to the direct image, and its computation as an alternated product of determinants.

Next I reported on joint work in progress with Laurent Busé and Aron Simis concerning the structure of the torsion of the symmetric algebra. It turns out that in a range of degrees that may be fairly big, the torsion is not zero but has a simple structure that can be completely described in terms of the syzygies of the polynomials defining the map. The knowledge of this simple structure can be used to compute the equation of the image of the map using determinants of much smaller sizes, which is of importance in practice. Particular cases of this method were studied first by David Cox, Ron Goldmann, and Zhang. In the case of a map given by "general polynomials" of some degree, this simple description is valid for half of the degrees where the torsion is not zero. We hope that the good understanding of the minimal free resolution of more graded components of the Rees algebra, given in our work, will have further applications.

With the notations as in Theorem 1, in the case where there is no base point, our result gives :

Theorem 2. *Let k be the base field and $J \subset A \otimes_k B$ be the kernel of the bigraded natural map $A \otimes_k B \to \mathrm{Rees}_A(I)$. Let $J[\ell]$ be the subideal of J generated by its elements of bidegree (a, b) with $b \leq \ell$ and $J[\ell]_\mu$ be the B-module of elements of bidegree $(\mu, *)$ in $J[\ell]$.*

If I is m-primary, then for all integer $\mu \geq \mathrm{reg}(I) - d$,

(i) $H^i_m(\mathrm{Sym}_R(I))_\mu = 0$ for $i > 0$, and

(ii) for all integers $\ell \geq 2$ the B-module $(J[\ell]/J[\ell-1])_\mu$ is free of rank $\dim_k(H_1)_{\mu+\ell d}$, where H_1 is the first Koszul homology module of the $n+1$ generators of I in A.

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The Classification of Graded Maximal Cohen-Macaulay Modules over a Graded Gorenstein Ring after Dmitri Orlov

Ragnar-Olaf Buchweitz

1. The aim of this talk was to present Dmitri Orlov's [5] fundamental results alluded to in the title. We emphasized in particular their significance in the classical context of Commutative Algebra and Algebraic Geometry.

2. Let $A = \oplus_{i \geq 0} A_i$ be a positively graded, not necessarily commutative, algebra over a field $K = A_0$, such that

(a) The ring A is noetherian on either side,
(b) As module on either side, A is of (the same) finite injective dimension d,
(c) For the (left or right) A–module $K = A/A_{>0}$, one has $\mathrm{Ext}^\bullet_A(K, A) \cong K(a)[-d]$ for some integer[1] a.

Such rings are called (*Artin-Schelter* or simply *AS*) *Gorenstein* of *virtual dimension* d with *Gorenstein invariant* a. If $a > 0$, the algebra is called *Fano*, if $a = 0$, it is *Calabi-Yau*, while for $a < 0$ it is of *general type*.

3. Just assuming (a) and (b) above, and A not necessarily graded, a finitely generated A–module M is *Maximal Cohen-Macaulay* (MCM) if $\mathrm{Ext}^i_A(M, A) = 0$ for $i \neq 0$. The *stable* category $\underline{\mathrm{MCM}}(A)$, of such modules modulo projective modules, is naturally triangulated, with Ω, the *syzygy functor*, as inverse of the translation functor. By [3], this category has the following equivalent incarnations:

(i) $(\underline{\mathrm{mod}} A)[\Omega^{-1}]$, the category obtained from the stable category of all finitely generated (graded) A–modules by inverting the syzyzgy functor. The equivalence assigns to (the stable class of a) finitely generated (graded) A–module N its *maximal Cohen-Macaulay approximation* $\mathbb{M}(N)$, see [2].
(ii) $\mathbb{K}^{\infty,\emptyset}(\mathrm{proj}\, A)$, the homotopy category of acyclic complexes of finite projective (graded) A–modules. The equivalence assigns to a maximal Cohen-Macaulay module M a (graded) *complete resolution* $\mathbb{CR}(M)$, and taking the zeroth syzygy module in such a complex provides the inverse.
(iii) $D^b(A)/D^b_{\mathrm{perf}}(A)$, the Verdier quotient of the bounded derived category of finitely generated (graded) A–modules modulo its thick subcategory of perfect complexes. The natural inclusion $\mathrm{mod}\, A \subseteq D^b(A)$ induces the equivalence from $(\underline{\mathrm{mod}} A)[\Omega^{-1}]$ in (i). Orlov sets $D_{\mathrm{sg}}(A) = D^b(A)/D^b_{\mathrm{perf}}(A)$ and calls it the "*triangulated category of singularities*" of A.
(iv) $\underline{\mathrm{MF}}(f)$, the homotopy category of (homogeneous) *matrix factorisations* of f, in case $A = K[\mathbf{x}]/(f)$ is the ring of a (homogeneous) hypersurface $f \in K[\mathbf{x}]$.

Either of the naturally present triangulated structures in (ii), (iii), or (iv) induces the mentioned triangulated structure on $\underline{\mathrm{MCM}}(A)$.

4. We now restrict to the graded context. The category $\mathsf{Modgr}\, A$ of all graded A–modules, with degree preserving homomorphisms $\mathrm{Homgr}_A(\ ,\ )$ as Hom–sets, contains $\mathsf{modgr}\, A$, its full subcategory of all finitely generated such modules, which in turn contains the full subcategory $\mathsf{projgr}\, A$ of finite projective graded modules. With $-(i)$, for $i \in \mathbb{Z}$, the degree-shift functors, and F a functor defined on graded modules, set $F_*(?) = \oplus_{i\in\mathbb{Z}} F(?(i))$ with homogeneous components F_i, so that $F_0 = F$. We write though Hom_A instead of $\mathrm{Homgr}_A(\ ,\ )_*$.

Given an integer c, notation such as $F_{\geq c}, F_{<c}$ should be self-explanatory, as should be $\mathsf{modgr}_{\geq c}, \mathsf{modgr}_{<c}$ etc. We denote further $\mathsf{projgr}^{<c} A$ the category of finite projectives that are *generated* in degrees less than c, while $\mathsf{projgr}^{\geq c} A = \mathsf{projgr}_{\geq c} A$ denotes those generated/concentrated in degrees at least c.

[1]Goto and Watanabe introduced this "a–invariant" in [6], however, with the opposite sign. We follow here Orlov's convention in (loc.cit.)

5. For a complex of graded finite projectives, its terms in $\mathsf{projgr}^{<c} A$ form a subcomplex, while the corresponding quotient complex has its terms in $\mathsf{projgr}^{\geq c} A$. Applied to complete resolutions, this provides for every integer c a functor

$$\mathbf{b}^{\geq c} : \underline{\mathrm{MCMgr}}(A) \cong \mathbb{K}^{\infty,\emptyset}(\mathsf{projgr}\, A) \to \mathbb{K}^{-,b}(\mathsf{projgr}^{\geq c} A) \cong D^b(\mathsf{modgr}_{\geq c} A)$$

This functor is fully faithful and left adjoint to the, necessarily dense, functor

$$\mathbb{M}_{\geq c} : D^b(\mathsf{modgr}_{\geq c} A) \subseteq D^b(\mathsf{modgr}\, A) \to \frac{D^b(\mathsf{modgr}\, A)}{D^b_{\mathrm{perf}}(\mathsf{modgr}\, A)} \cong \underline{\mathrm{MCMgr}}(A)$$

that assigns to a complex of graded A–modules, with finitely generated total homology concentrated in degrees not less than c, its (graded) maximal Cohen-Macaulay approximation.

6. On the geometric side, Serre's characterisation of (quasi-)coherent sheaves on projective schemes generalises by Artin-Zhang [1] to the situation here as follows.

With $\mathfrak{m} = A_{>0}$ the *irrelevant* maximal ideal of A, and M a (graded) A–module, the (graded) submodule of *local sections* of M at $\mathfrak{m}$ is given by

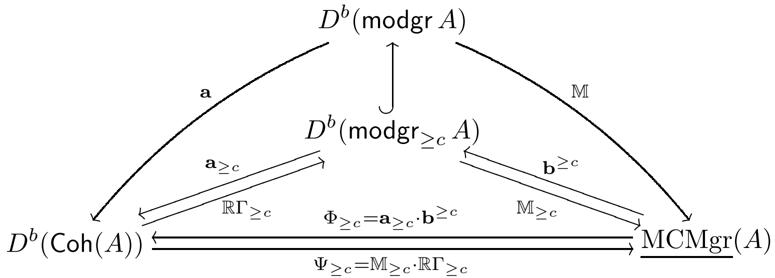
$$\Gamma_{\mathfrak{m}} M = \{m \in M \mid mA_{>i} = 0 \text{ for } i \gg 0\} \cong \varinjlim_i \mathrm{Hom}_A(A/A_{\geq i}, M) \subseteq M$$

The module is $(\mathfrak{m})$*–torsion* if $\Gamma_{\mathfrak{m}} M = M$. Torsion modules form a Serre subcategory $\mathsf{Tors}\, A$ of $\mathsf{Modgr}\, A$. The projection functor $\mathbf{a} : \mathsf{Modgr}\, A \to \mathsf{QCoh}(\mathrm{Proj}_K\, A) := \mathsf{Modgr}\, A/\,\mathsf{Tors}\, A$ serves as "*sheafification*", and the triple $(\mathsf{QCoh}(\mathrm{Proj}_K\, A), \mathcal{O}, -(1))$ represents the quasi-coherent sheaves on the "*projective scheme*" underlying A, with structure sheaf $\mathcal{O} = \mathbf{a}(A)$, and "*twists*" $\mathcal{M} = \mathbf{a}M \mapsto \mathcal{M}(i) = \mathbf{a}(M(i))$, $i \in \mathbb{Z}$. The exact sheafification functor $\mathbf{a}$ admits the fully faithful right adjoint

$$\Gamma_* : \mathsf{QCoh}(\mathrm{Proj}_K\, A) \to \mathsf{Modgr}\, A \quad , \quad \Gamma_*\mathcal{M} = \oplus_{i\in\mathbb{Z}} \varinjlim_j \mathrm{Homgr}(A_{\geq j}, M(i))$$

7. With $\mathsf{Coh}(A) := \mathbf{a}(\mathsf{modgr}\, A) \cong \mathsf{modgr}\, A/(\mathsf{tors}\, A = \mathsf{Tors}\, A \cap \mathsf{modgr}\, A)$ the full subcategory of "*coherent*" sheaves, fixing a "cut-off" $c \in \mathbb{Z}$, the functor $\Gamma_{\geq c}$ maps $\mathsf{Coh}(A)$ back into $\mathsf{modgr}_{\geq c} A$, and represents a right adjoint to the restriction $\mathbf{a}_{\geq c}$ of sheafification to $\mathsf{modgr}_{\geq c} A$. The exact functor $\mathbf{a}_{\geq c}$ passes trivially to the corresponding derived categories, while the right derived functor $\mathbb{R}\Gamma_{\geq c} = \oplus_{i \geq c} \varinjlim_j \mathbb{R}\,\mathrm{Homgr}_A(A_{\geq j}, M(i)) : D^b(\mathsf{Coh}(A)) \to D^b(\mathsf{modgr}_{\geq c} A)$ represents again a fully faithful right adjoint.

8**.** The situation so far is summarised in the following diagram of exact functors between triangulated categories, with left adjoints written above right adjoints,



Orlov's key result [5, Thm.2.5.] can then be stated thus:

9. **Theorem.** *The functor* $\Phi_{\geq c} = \mathbf{a}_{\geq c} \circ \mathbf{b}^{\geq c}$ *is* left adjoint *to* $\Psi_{\geq c} = \mathbb{M}_{\geq c} \circ \mathbb{R}\Gamma_{\geq c}$.

(1) *If* $a \geq 0$*, then* $\Phi_{\geq c}$ *is fully faithful and the thick subcategory* $\operatorname{Ker}\Psi_{\geq c}$ *is generated by the (strongly exceptional) sequence* $\mathcal{O}(-c-a+1), ..., \mathcal{O}(-c)$ *and equivalent to* $D^b(\mathrm{mod}(\oplus_{i,j=0}^{a-1} A_{j-i}))$*, the bounded derived category over an (upper) triangular "matrix" algebra of finite global dimension at most* a*.*

(2) *If* $a \leq 0$*, then* $\Psi_{\geq c}$ *is fully faithful and the thick subcategory* $\operatorname{Ker}\Phi_{\geq c}$ *is equivalent to* $D^b(\mathrm{mod}(\oplus_{i,j=a+1}^{0} A_{j-i}))$*, the bounded derived category over a (lower) triangular "matrix" algebra of finite global dimension at most* $-a$*.*

(3) *If* $a = 0$*, the functors* $\Phi_{\geq c}, \Psi_{\geq c}$ *are thus inverse equivalences.* □

10**.** This theorem has far-reaching consequences, certainly many yet to be explored. For instance, restricting to $a = 0$, it implies the remarkable fact that a complex C in $D^b(\mathsf{modgr}_{\geq c}\, A)$ is *saturated*, in that the natural morphism $C \to \mathbb{R}\Gamma_{\geq c}\mathbf{a}_{\geq c}C$ is an isomorphism, if and only if the natural morphism $\mathbf{b}_{\geq c}\mathbb{M}_{\geq c}C \to C$ is an isomorphism, a property that can be checked on the "graded Betti tables" of C and $\mathbb{R}\operatorname{Hom}_A(C, A)$. Moreover, compositions $\Phi_{\geq c}\Psi_{\geq c'}$, for different integers c, c', can produce nontrivial autoequivalences on $D^b(\mathsf{Coh}(A))$. We finally mention that the theorem has recently been greatly extended in [4] for *toric* Calabi-Yau algebras, producing equivalent triangulated categories parametrised by the *moment map*.

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Stratifying the derived category of a complete intersection

Srikanth B. Iyengar

Let A be a commutative noetherian ring and D the bounded derived category of finitely generated A-modules; its objects are complexes M of A-modules such that A-module $\mathrm{H}_i(M)$ is finitely generated for each i and zero when $|i| \gg 0$. There is a natural triangulated category structure on D, with exact triangles arising from mapping cone sequences of morphisms of complexes. A non-empty full subcategory of D is *thick* if it is a triangulated subcategory and closed under retracts; see [15].

An intersection of thick subcategories is again thick so each M in D is contained in a smallest, with respect to inclusion, thick subcategory, which I denote $\mathrm{thick}_A(M)$. The objects of $\mathrm{thick}_A(M)$ are exactly those complexes which can be built out of M using suspensions, finite direct sums, exact triangles, and retracts; in fact, the last two operations suffice. Thus, for example, a complex is in $\mathrm{thick}_A(A)$ if and only if it is *perfect*, i.e. isomorphic in $\mathsf{D}^{\mathsf{f}}(R)$ to a finite complex of finitely generated projective modules.

My talk was concerned with the following problem: *Classify the thick subcategories of* D. I started by trying to explain why thick subcategories of $\mathsf{D}^{\mathsf{f}}(A)$ are interesting from the point of view of homological algebra; this is discussed also in [11]. Such investigations concerning derived categories started with a remarkable result of Hopkins [10] and Neeman [13]:

If M, N are perfect complexes with $\mathrm{supp}_A M \subseteq \mathrm{supp}_A N$, *then* $M \in \mathrm{thick}_A(N)$.

Here $\mathrm{supp}_A M$ is the set $\{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathrm{H}(M)_{\mathfrak{p}} \neq 0\}$, the *support* of M. Various proofs of this theorem are discussed in [12]; for applications, see [8]. Given this theorem, it is easy to prove, see [13], that there is a bijection of sets:

$$\left\{\begin{matrix}\text{Thick subcategories}\\ \text{of } \mathrm{thick}_A(A)\end{matrix}\right\} \underset{\tau}{\overset{\sigma}{\rightleftarrows}} \left\{\begin{matrix}\text{Specialization closed}\\ \text{subsets of } \mathrm{Spec}\, A\end{matrix}\right\}$$

where a subset of $\mathrm{Spec}\, A$ is *specialization closed* if it is a (possibly infinite) union of closed subsets. The maps in question are

$$\sigma(\mathsf{C}) = \bigcup_{M \in \mathsf{C}} \mathrm{supp}_R M \quad \text{and} \quad \tau(\mathcal{V}) = \{M \mid \mathrm{supp}_R M \subseteq \mathcal{V}\}$$

This 'thick subcategory' theorem solves the classification problem stated when A is regular, for then $\mathrm{thick}_A(A) = \mathsf{D}$. Similar results have since been established for the derived category of perfect complexes of coherent sheaves on a noetherian scheme, by Thomason [14]; the stable module category of finite dimensional modules over the group algebra of a finite group, by Benson, Carlson, and Rickard [5]; and the category of perfect differential modules over a commutative noetherian ring, by Avramov, Buchweitz, Christensen, Piepmeyer and myself [2].

Let now A be a complete intersection; for simplicity assume $A = k[x_1, \dots, x_n]/I$, where k is a field, $x_1, \dots, x_n$ are indeterminates, and I is generated by a regular sequence. Set $c = n - \dim A$ and let $A[\chi_1, \dots, \chi_c]$ be the ring of cohomology operators constructed by Avramov and Sun [4]. Thus, $\chi_1, \dots, \chi_c$ are indeterminates over A of cohomological degree 2, and for each pair of complexes M, N of A-modules, $\mathrm{Ext}^*_A(M,N)$ is a graded R-module, which is finitely generated when M, N are in D; see [4, §§2,5], and also Gulliksen [9], for details. Set

$$\mathcal{V}_A(M) = \mathrm{supp}_R \, \mathrm{Ext}^*_A(M,M) \subseteq \mathrm{Spec}\, A[\chi_1, \dots, \chi_c]\,.$$

This construction is akin to the support variety of M in the sense of Avramov and Buchweitz [1]; only, it takes into account also the support of M as a complex of A-modules; see [7, §11]. A positive answer to the conjecture below takes us a long way towards a classification of thick subcategories of D for complete intersections.

Conjecture: For any M, N in D, if $\mathcal{V}_A(M) \subseteq \mathcal{V}_A(N)$, then $M \in \mathrm{thick}_A(N)$.

There are two points of view concerning homological algebra over complete intersections which lead one to such a statement: it is akin to that over regular rings, once we take into account the action of the cohomology operators; it is akin to that of group algebras of finite groups. Indeed, a result from [5] settles the conjecture above for the case when k is of positive characteristic p and $I = (x_1^p, \dots, x_n^p)$, for then A is the group algebra of $(\mathbb{Z}/p\mathbb{Z})^n$.

The simplest ring not covered by [5] is $A = k[x]/(x^d)$ with $d \geq 3$. The indecomposable A-modules are precisely $M_i = k[x]/(x^i)$, for $1 \leq i \leq d$. It is easy to verify that

$$\mathcal{V}_A(M_i) = \begin{cases} \{(x)\} & \text{for } i \neq d \\ \{(x), (x, \chi)\} & \text{for } i = d \end{cases}$$

Since $M_1 = k$ and $M_d = A$, the conjecture postulates that for $1 \leq i \leq d-1$ the subcategory $\mathrm{thick}_A(M_i)$ contains both A and k. In my talk, I demonstrated that this is indeed the case. This example is atypical for a general complete intersection is not of finite representation type, and one cannot expect to settle the conjecture with such direct computations.

Recently Benson, Krause, and I [6] gave a rather different proof of the result in [2]. It builds on the work in [3], which develops new tools for studying modules and complexes over complete intersections, and in [7], which develops a theory of local cohomology for the action of the ring of cohomology operators $A[\chi_1, \dots, \chi_c]$ on complexes of A-modules. The technique in [6] can be adapted to settle the conjecture above for all Artin complete intersection rings. The general case remains open, but I am optimistic that it will be settled in the near future.

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Bockstein homomorphisms in local cohomology

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(joint work with Uli Walther)

Let R be a polynomial ring in finitely many variables over the ring of integers. Let $\mathfrak{a}$ be an ideal of R, and let p be a prime integer. Taking local cohomology $H^\bullet_{\mathfrak{a}}(-)$, the exact sequence

$$0 \longrightarrow R \xrightarrow{\ p\ } R \longrightarrow R/pR \longrightarrow 0$$

induces an exact sequence

$$H^k_{\mathfrak{a}}(R/pR) \xrightarrow{\ \delta\ } H^{k+1}_{\mathfrak{a}}(R) \xrightarrow{\ p\ } H^{k+1}_{\mathfrak{a}}(R) \xrightarrow{\ \pi\ } H^{k+1}_{\mathfrak{a}}(R/pR)\,.$$

The *Bockstein homomorphism* β^k_p is the composition

$$\pi \circ \delta \colon H^k_{\mathfrak{a}}(R/pR) \longrightarrow H^{k+1}_{\mathfrak{a}}(R/pR)\,.$$

Fix $\mathfrak{a} \subseteq R$; we prove that for all but finitely many prime integers p, the Bockstein homomorphisms β^k_p are zero. More precisely:

Theorem 1. *Let R be a polynomial ring in finitely many variables over the ring of integers. Let $\mathfrak{a} = (f_1, \dots, f_t)$ be an ideal of R, and let p be a prime integer.*

If p is a nonzerodivisor on the Koszul cohomology module $H^{k+1}(\boldsymbol{f}; R)$, then the Bockstein homomorphism $\beta^k_p \colon H^k_{\mathfrak{a}}(R/pR) \longrightarrow H^{k+1}_{\mathfrak{a}}(R/pR)$ is zero.

This is motivated by Lyubeznik's conjecture [3, Remark 3.7] that for regular rings R, each local cohomology module $H^k_{\mathfrak{a}}(R)$ has finitely many associated prime ideals. This conjecture has been verified for regular rings of positive characteristic by Huneke and Sharp [2], and for regular local rings of characteristic zero as well as unramified regular local rings of mixed characteristic by Lyubeznik [3, 4]. It remains unresolved for polynomial rings over $\mathbb{Z}$, where it implies that for fixed $\mathfrak{a} \subseteq R$, the Bockstein homomorphisms β_p^k are zero for almost all prime integers p; the above theorem thus provides supporting evidence for Lyubeznik's conjecture.

The situation is different for hypersurfaces, as compared with regular rings:

Example 2. Consider the hypersurface

$$R = \mathbb{Z}[u, v, w, x, y, z]/(ux + vy + wz)$$

and ideal $\mathfrak{a} = (x, y, z)R$. A variation of the argument given in [5] shows that

$$\beta_p^2 \colon H^2_{\mathfrak{a}}(R/pR) \longrightarrow H^3_{\mathfrak{a}}(R/pR)$$

is nonzero for each prime integer p.

Huneke [1, Problem 4] asked whether local cohomology modules of Noetherian rings have finitely many associated prime ideals. The answer to this is negative since $H^3_{\mathfrak{a}}(R)$ in the hypersurface example has p-torsion elements for each prime integer p, and hence has infinitely many associated primes; see [5]. Indeed, the issue of p-torsion appears to be central in studying Lyubeznik's conjecture for finitely generated $\mathbb{Z}$-algebras.

We outline the proof of Theorem 1. One first verifies that if $\boldsymbol{f} = f_1, \ldots, f_t$ and $\boldsymbol{g} = g_1, \ldots, g_t$ are elements of R with $f_i \equiv g_i \mod p$ for each i, then there exists a commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^k_{(\boldsymbol{f})}(R/pR) & \longrightarrow & H^{k+1}_{(\boldsymbol{f})}(R/pR) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H^k_{(\boldsymbol{g})}(R/pR) & \longrightarrow & H^{k+1}_{(\boldsymbol{g})}(R/pR) & \longrightarrow & \cdots \end{array}$$

where the horizontal maps are the respective Bockstein homomorphisms, and the vertical maps are natural isomorphisms.

Another ingredient is the existence of endomorphisms of the polynomial ring $R = \mathbb{Z}[x_1, \ldots, x_n]$. For p a nonzerodivisor on $H^{k+1}(\boldsymbol{f}; R)$, consider the endomorphism φ of R with $\varphi(x_i) = x_i^p$ for each i. Since φ is flat, it follows that

$$H^{k+1}(\varphi^e(\boldsymbol{f}); R) \xrightarrow{\ p\ } H^{k+1}(\varphi^e(\boldsymbol{f}); R)$$

is injective for each $e \geq 0$. Thus, the Bockstein map on Koszul cohomology

$$H^k(\varphi^e(\boldsymbol{f}); R/pR) \longrightarrow H^{k+1}(\varphi^e(\boldsymbol{f}); R/pR)\,.$$

must be the zero map.

Suppose $\eta \in H^k_{\mathfrak{a}}(R/pR)$. Then η has a lift in $H^k(\varphi^e(\boldsymbol{f}); R/pR)$ for large e. But then the commutativity of the diagram

$$\begin{array}{ccc} H^k(\varphi^e(\boldsymbol{f}); R/pR) & \longrightarrow & H^{k+1}(\varphi^e(\boldsymbol{f}); R/pR) \\ \big\downarrow & & \big\downarrow \\ H^k_{(\varphi^e(\boldsymbol{f}))}(R/pR) & \longrightarrow & H^{k+1}_{(\varphi^e(\boldsymbol{f}))}(R/pR) \\ \big\downarrow & & \big\downarrow \\ H^k_{(\boldsymbol{f})}(R/pR) & \longrightarrow & H^{k+1}_{(\boldsymbol{f})}(R/pR)\,, \end{array}$$

where each horizontal map is a Bockstein homomorphism, implies that η maps to zero in $H^{k+1}_{\mathfrak{a}}(R/pR)$.

Stanley-Reisner ideals. For $\mathfrak{a}$ the Stanley-Reisner ideal of a simplicial complex, the following theorem connects Bockstein homomorphisms on reduced simplicial cohomology groups with those on local cohomology modules.

Let Δ be a simplicial complex, and τ a subset of its vertex set. The *link* of τ is

$$\operatorname{link}_\Delta(\tau) = \{\sigma \in \Delta \mid \sigma \cap \tau = \emptyset \ \text{ and } \ \sigma \cup \tau \in \Delta\}\,.$$

Theorem 3. *Let Δ be a simplicial complex with vertices $1, \dots, n$. Set R to be the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$, and let $\mathfrak{a} \subseteq R$ be the Stanley-Reisner ideal of Δ.*

For each prime integer p, the following are equivalent:

(1) the Bockstein homomorphism $H^k_{\mathfrak{a}}(R/pR) \longrightarrow H^{k+1}_{\mathfrak{a}}(R/pR)$ is zero;

(2) the Bockstein homomorphism

$$\widetilde{H}^{n-k-2-|\widetilde{\boldsymbol{u}}|}(\operatorname{link}_\Delta(\widetilde{\boldsymbol{u}}); \mathbb{Z}/p\mathbb{Z}) \longrightarrow \widetilde{H}^{n-k-1-|\widetilde{\boldsymbol{u}}|}(\operatorname{link}_\Delta(\widetilde{\boldsymbol{u}}); \mathbb{Z}/p\mathbb{Z})$$

is zero for each $\boldsymbol{u} \in \mathbb{Z}^n$ with $\boldsymbol{u} \leq \boldsymbol{0}$.

Example 4. Let Λ_m be the *m-fold dunce cap*, i.e., the quotient of the unit disk obtained by identifying each point on the boundary circle with its translates under rotation by $2\pi/m$; the 2-fold dunce cap Λ_2 is the real projective plane.

Suppose m is the product of distinct primes $p_1, \dots, p_r$. It is readily computed that the Bockstein homomorphisms

$$\widetilde{H}^1(\Lambda_m; \mathbb{Z}/p_i) \longrightarrow \widetilde{H}^2(\Lambda_m; \mathbb{Z}/p_i)$$

are nonzero. Let Δ be the simplicial complex corresponding to a triangulation of Λ_m, and let $\mathfrak{a}$ in $R = \mathbb{Z}[x_1, \dots, x_n]$ be the corresponding Stanley-Reisner ideal. The theorem then implies that the Bockstein homomorphisms

$$H^{n-3}_{\mathfrak{a}}(R/p_iR) \longrightarrow H^{n-2}_{\mathfrak{a}}(R/p_iR)$$

are nonzero for each p_i. It follows that the local cohomology module $H^{n-2}_{\mathfrak{a}}(R)$ has a p_i-torsion element for each $i = 1, \dots, r$.

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Syzygies of Veronese algebras

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(joint work with Winfried Bruns and Tim Römer)

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field and $R = S/I = \bigoplus_{i=0}^{\infty} R_i$ a graded quotient of it. Let $\mathfrak{m}_R$ denote the maximal homogeneous ideal of R. For every $c \in \mathbb{N}$ we consider the c-th Veronese algebra $R^{(c)}$ of R defined as $R^{(c)} = \bigoplus_{i=0}^{\infty} R_{ic}$. The question we want to address is how the degrees of the syzygies of $R^{(c)}$ vary with c. Normalizing the degrees, we consider $R^{(c)}$ as a standard graded algebra and have a surjective K-algebra map $T \to R^{(c)}$ where T is the symmetric algebra of the vector space R_c. So T is itself a polynomial ring over K whose Krull dimension equals the vector space dimension of R_c. We want to understand the degrees of the syzygies of $R^{(c)}$ as a T-module.

Several invariants can be used to measure the degrees of the syzygies. We recall those that play a role in our discussion. Given a graded and finitely generated R-module M we denote the (i,j)-th Betti number of M as an R-module by $\beta_{ij}^R(M)$. We set $t_i^R(M) = \sup\{j : \beta_{ij}^R(M) \neq 0\}$,

$$\operatorname{reg}_R(M) = \sup\{t_i^R(M) - i : i \geq 0\},$$

$$\operatorname{slope}_R(M) = \sup\{\frac{t_i^R(M) - t_0^R(M)}{i} : i > 0\}$$

and $\operatorname{Rate}(R) = \operatorname{slope}_R(\mathfrak{m}_R)$. While $\operatorname{reg}_R(M)$ can be infinite, $\operatorname{slope}_R(M)$ is finite for every finitely generated graded R-module M, see [2]. By definition, R is Koszul if and only if $\operatorname{Rate}(R) = 1$. The Castelnuovo-Mumford regularity $\operatorname{reg}(M)$ of M is, by definition, $\operatorname{reg}_S(M)$; it is finite and does not depend on S. The Green-Lazarsfeld index of R, denoted by $\operatorname{index}(R)$, is defined as:

$$\operatorname{index}(R) = \sup\{p : t_i^S(R) \leq i + 1 \text{ for every } i \leq p\}.$$

Only a few facts about the syzygies of the Veronese algebras are classical and well-known: $S^{(c)}$ is defined by quadrics, i.e. $\operatorname{index}(S^{(c)}) \geq 1$, and has Castelnuovo-Mumford regularity $\operatorname{reg}(S^{(c)})$ equal to $n - \lceil n/c \rceil$. Also, if R is defined by equations of degree a and smaller then $R^{(c)}$ is defined by equations of degree $\leq \max(2, \lceil a/c \rceil)$.

Let $K(\mathfrak{m}_R^c, R)$ denote the Koszul complex over R associated to the c-th power of $\mathfrak{m}_R$, by $H_\bullet(\mathfrak{m}_R^c, R)$ its homology, by $Z_\bullet(\mathfrak{m}_R^c, R)$ its cycles and by $B_\bullet(\mathfrak{m}_R^c, R)$ its boundaries. One notices that

$$\beta_{ij}^T(R^{(c)}) = \dim_K H_i(\mathfrak{m}_R^c, R)_{jc}.$$

Hence the study of the syzygies of $R^{(c)}$ is essentially equivalent to the study of the Koszul homology modules $H_i(\mathfrak{m}_R^c, R)$. Taking into account that $\mathfrak{m}_R^c H_i(\mathfrak{m}_R^c, R) = 0$ one can set up an inductive procedure leading to bounds for the regularity of $Z_\bullet(\mathfrak{m}_R^c, R)$. It follows that:

Theorem 1. *$H_i(\mathfrak{m}_R^c, R)_j = 0$ for $j \geq (i+1)c + \min(i\,\mathrm{Rate}(R), i + \mathrm{reg}(R))$. In particular, $\mathrm{index}(R^{(c)}) \geq c - \mathrm{reg}(R)$ and $\mathrm{index}(R^{(c)}) \geq c$ if R is Koszul.*

Theorem 1 has been proved by Green for Veronese subrings $S^{(c)}$ of polynomial rings S in characteristic 0, see [4]. For $n = 2$ or $c = 2$ the ring $S^{(c)}$ has a determinantal presentation and (at least in characteristic 0) the value of $\mathrm{index}(S^{(c)})$ can be deduced from the known resolutions of it. One has:

$$\mathrm{index}(S^{(c)}) = \begin{cases} \infty & \text{if } n = 2 \text{ or } (n = 3 \text{ and } c = 2) \\ 5 & \text{if } n > 3 \text{ and } c = 2. \end{cases}$$

Note however that Andersen [1] has showed that $\mathrm{index}(S^{(2)}) = 4$ in characteristic 5 if $n \geq 7$. In characteristic 0 and for $n > 2$ and $c > 2$ Ottaviani and Paoletti [5] have proved that

$$\mathrm{index}(S^{(c)}) \leq 3c - 3$$

with equality if $n = 3$. They conjectured that equality holds for every $n \geq 3$. We prove that the bound and the equality for $n = 3$ hold independently of the characteristic. In the proof an important role is played by the duality:

$$\dim_K H_i(\mathfrak{m}_S^c, S)_j = \dim_K H_{N-n-i}(\mathfrak{m}_S^c, S)_{Nc-n-j}$$

where $N = \dim S_c$ and the fact that for $n = 3$ the regularity of $S^{(c)}$ is ≤ 2. The duality above can be seen as a special instance of a duality of Avramov-Golod type, which is the algebraic counterpart of Serre duality. Our main contribution to the problem of finding $\mathrm{index}(S^{(c)})$ is the following improvement of Green's lower bound:

Theorem 2. *If K has characteristic 0 or $> c + 1$ then $\mathrm{index}(S^{(c)}) \geq c + 1$ for every n.*

For $c = 3$ and K of characteristic 0 Theorem 2 has been proved by Rubei in [6]. Set $Z_t = Z_t(\mathfrak{m}_S^c, S)$ and $B_t = B_t(\mathfrak{m}_S^c, S)$ and let Z_1^t denote the image of $\wedge^t Z_1$ in Z_t. The proof of Theorem 2 is based on three facts:

1) Z_t/Z_1^t is generated in degree $< (c+1)i$,

2) for every $a \in \mathbb{N}$ with $1 \geq a < c$, and for polynomials $f_1, \dots, f_{t+1} \in S_a$ and $g_1, \dots, g_{t+1} \in S_{c-a}$ one has

$$\sum_{\sigma \in \mathbb{S}_{t+1}} (-1)^\sigma f_{\sigma(t+1)} \otimes f_{\sigma(1)} g_1 \wedge f_{\sigma(2)} g_2 \wedge \cdots \wedge f_{\sigma(t)} g_t \in Z_t$$

where $\mathbb{S}_{t+1}$ is the symmetric group.

3) $(c+1)!\mathfrak{m}_S^{c-1}Z_1^c \subset B_c$

Indeed, the cycles in 2) are used together with a symmetrization argument to prove 3). Combining 1) and 3) one shows that $H_i(\mathfrak{m}_S^c)_{ic+j} = 0$ for $j \geq i+c-1$ and $i \geq c$ which, in turn, implies that $\operatorname{index}(S^{(c)}) \geq c+1$. The cycles described in 2) can be "explained" in terms of multilinear algebra and diagonal maps between symmetric and exterior powers of vector spaces. There are some indications that those cycles might generate $Z(\mathfrak{m}_S^c, S)$ as an S-algebra. For general R we prove the following:

Theorem 3. *One has* $\operatorname{index}(R^{(c)}) \geq \operatorname{index}(S^{(c)})$ *for every* $c \geq \operatorname{slope}_S(R)$.

As shown in [2], $\operatorname{slope}_S(R) = 2$ if R is Koszul. In particular, if R is Koszul then $\operatorname{index}(R^{(c)}) \geq \operatorname{index}(S^{(c)})$ for every $c \geq 2$.

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Nagata conjecture and symbolic Rees rings of space monomial curves

Kazuhiko Kurano

The aim of this note is to give a relation between finite generation of symbolic Rees rings of space monomial curves and Nagata conjecture.

1. Symbolic Rees rings of space monomial curves

Let k be a field. Assume that a, b, c are pairwise coprime positive integers, that is, $(a,b) = (b,c) = (c,a) = 1$. We denote the defining ideal of a space monomial curve (t^a, t^b, t^c) by $P(a,b,c)$, that is,

$$P(a,b,c) = \operatorname{Ker}(k[x,y,z] \xrightarrow{\phi} k[t]),$$

where ϕ is the k-algebra homomorphism defined by $\phi(x) = t^a$, $\phi(y) = t^b$, $\phi(z) = t^c$.

Herzog [3] proved that the ideal $P(a,b,c)$ was generated by at most 3 elements.

We set $R_s(P(a,b,c)) = \oplus_{m\geq 0} P(a,b,c)^{(m)}$.

If the ideal $P(a,b,c)$ is generated by two elements, then $R_s(P(a,b,c))$ coincides with the ordinary Rees ring of $P(a,b,c)$. In particular, it is Noetherian.

However, in many cases, $P(a,b,c)$ is not generated by two elements.

Many people study finite generation of $R_s(P(a,b,c))$ (e.g. Cutkosky, Goto, Herzog, Huneke, Nishida, Reed, Shimoda, Ulrich, Vasconcelos, Watanabe, etc.).

Even if $P(a,b,c)$ is not generated by two elements, there are many examples of Noetherian $R_s(P(a,b,c))$'s. For example, if $a+b+c > \sqrt{abc}$, then $R_s(P(a,b,c))$ is Noetherian by Cutkosky [1].

However, if $\mathrm{ch}(K) = 0$ and if $(a,b,c) = (25, 29, 72), \ldots$, then $R_s(P(a,b,c))$ is not Noetherian by Goto-Nishida-Watanabe [2].

Finite generation of such rings are deeply relatred to certain geometric phenomena on a rational surface.

Let $S = k[x,y,z]$ be the weighted polynomial ring with $\deg(x) = a$, $\deg(y) = b$ and $\deg(z) = c$. Let $\mathrm{Proj}(S)$ be the weighted projective space. Let $\pi : X \to \mathrm{Proj}(S)$ be the blow-up at a smooth point $V_+(P(a,b,c))$. Set $E = \pi^{-1}(P(a,b,c))$.

Since X is $\mathbb{Q}$-factorial, X has a $\mathbb{Q}$-valued intersection pairing.

Definition 1 We say that a curve C on X is a *negative curve* if $C \neq E$ and $C^2 < 0$.

Remark 2 (1) A negative curve exists if there exsit positive integers m_0 and d_0 such that $d_0/m_0 < \sqrt{abc}$ and $[P(a,b,c)^{(m_0)}]_{d_0} \neq 0$.

Existence of a negative curve depend on a, b, c and the characteristic of k.

(2) If a negative curve exists for (a,b,c) over some k with $\mathrm{ch}(k) > 0$, then $R_s(P(a,b,c))$ is Noetheran for (a,b,c) over this k (by Cutkosky [1]).

(3) If a negative curve exists for (a,b,c) over some k with $\mathrm{ch}(k) = 0$, then a negative curve exists for (a,b,c) over any field (by mod p reduction).

(4) We immediately obtain the following assertion by (2) and (3). If a negative curve exists for (a,b,c) over $\mathbb{C}$ (or $\mathbb{Q}$), then $R_s(P(a,b,c))$ is Noetheran for (a,b,c) over any field k of positive characteristic.

(5) Assume that $abc \geq 10$. If there is no negative curve for (a,b,c) over $\mathbb{C}$, *Nagata conjecture* is true for $r = abc$. We shall see this in the next section.

(6) We immediately obtain the following assertion by (4) and (5). If $R_s(P(a,b,c))$ is not Noetheran for (a,b,c) over some field k of positive characteristic, then Nagata conjecture is true for $r = abc$.

By (4) and (5) as above, existence of negative curves is very much important problem.

Remark 4 The cardinary of the set

$$A = \{(a,b,c) \in \mathbb{N}^3 \mid a \leq b \leq c \leq 300, \quad (a,b) = (b,c) = (c,a) = 1\}$$

is $1,291,739$. More than 90% of A satisfy the following condition:

(1) There exist m_0 and d_0 such that $\frac{m_0}{d_0} < \sqrt{abc}$ and $\dim_k S_{d_0} > \frac{m_0(m_0+1)}{2}$.

If $d > -a-b-c$, then we have

$$\dim_k [P(a,b,c)^{(m)}]_d = \dim_k S_d - \frac{m(m+1)}{2} + \dim_k H^2_{(x,y,z)}(P(a,b,c)^{(m)})_d.$$

Therefore, once the condition (1) is satisfied, then there exists a negative curve.

Unfortunately, there exist examples that do not satisfy the condition (1).

2. Rings of Nagata-type

For $s = (\alpha : \beta : \gamma) \in \mathbb{P}^2_{\mathbb{C}}$, we set

$$I_s = I_2 \begin{pmatrix} u & v & w \\ \alpha & \beta & \gamma \end{pmatrix} \subset B = \mathbb{C}[u, v, w].$$

For a finite set of closed points $H = \{s_1, \ldots, s_r\}$ in $\mathbb{P}^2_{\mathbb{C}}$, we set

$$R_H = \bigoplus_{m_1,\ldots,m_r \in \mathbb{Z}} [I_{s_1}^{m_1} \cap \cdots \cap I_{s_r}^{m_r}] \quad \text{and} \quad \Delta_H = \bigoplus_{m \geq 0} [I_{s_1}^{m} \cap \cdots \cap I_{s_r}^{m}].$$

Conjecture 5 (Nagata) Let $s_1, \ldots, s_r$ be general closed points in $\mathbb{P}^2_{\mathbb{C}}$. If $r \geq 10$ and $\frac{d}{m} \leq \sqrt{r}$, then $[I_{s_1}^m \cap \cdots \cap I_{s_r}^m]_d = 0$.

Remark 6 (1) If Nagata conjecture is true for r, then Δ_H is not Noetherian.
(2) If R_H is Noetherian, then so is Δ_H. However, the converse is not true. Totaro's example [6] satisfies that R_H is not Noetherian, but Δ_H is Noetherian.
(3) If $r = 4^2, 5^2, 6^2, \ldots$, then Nagata [5] solved the conjecture affirmatively. In the other cases, the conjecture is still open.
(4) The ring R_H is an invariant subring of a polynomial ring with a linear action. Nagata [5] obtained a counterexample to Hilbert's 14th problem by (1), (2), (3) and (4).

We define a ring homomorphism $S = \mathbb{C}[x, y, z] \longrightarrow B = \mathbb{C}[u, v, w]$ by $x \mapsto u^a$, $y \mapsto v^b$, $z \mapsto w^c$. We put $\zeta_n = e^{2\pi i/n}$ and

$$F_{a,b,c} = \left\{ (\zeta_a^{n_1} : \zeta_b^{n_2} : \zeta_c^{n_3}) \;\middle|\; \begin{array}{l} n_1 = 0, 1, \ldots, a-1 \;:\; n_2 = 0, 1, \ldots, b-1 \\ n_3 = 0, 1, \ldots, c-1 \end{array} \right\} \subset \mathbb{P}^2_{\mathbb{C}}.$$

Then it is easy to prove the following lemma:

Lemma 7 For each $m > 0$, $P(a, b, c)^{(m)} B = \bigcap_{s \in F_{a,b,c}} I_s^m$ holds.

It immediately follows from the above lemma that

$$R_s(P(a, b, c)) \otimes_S B = \Delta_{F_{a,b,c}}.$$

Therefore, $R_s(P(a, b, c))$ is Noetherian if and only if $\Delta_{F_{a,b,c}}$ is Noetherian.

In the rest of this note, we give an outline of a proof of Remark 2 (5).

Let a, b, c be pairwise coprime positive integers such that $abc \geq 10$.

Suppose that there exists a counterexample to Nagata conjecture for $r = abc$. That is, there exist positive integers m_0 and d_0 such that

$$\frac{d_0}{m_0} \leq \sqrt{abc} \quad \text{and} \quad \left[\bigcap_{s \in H} I_s^{m_0} \right]_{d_0} \neq 0,$$

where H is a set of general abc points.

By Remark 6 (3), we may assume that $\sqrt{abc}$ is irrational. Therefore, m_0 and d_0 satisfy

$$\frac{d_0}{m_0} < \sqrt{abc}.$$

Considering the specialization $H \rightsquigarrow F_{a,b,c}$, we know that $[P(a,b,c)^{(m_0)}]_{d_1} \neq 0$ for some $d_1 \leq d_0$. Therefore, there exists a negative curve in this case.

Acknowledgements In the conference, I said that if all of a, b and c are at most 300, then there exists a negative curve in characteristic 0, and therefore, the symbolic Rees ring is finitely generated in the case of positive charactieristic. However, there was a mistake in my proof. I would like to thank Prof. Kei-ichi Watanabe for pointing out the mistake.

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Interactive Visualization of Algebraic Surfaces — RealSurf

Peter Schenzel

1. Introduction

In most of the mathematical institutes of "traditional" german universities there is a collection of mathematical models. In recent times there is a strong effort in order to visualize mathematical models (e.g. (implicit) algebraic surfaces) on a computer by the aid of methods from computer graphics. The basic algorithm for implicitly given algebraic surfaces is the so-called ray tracing. One of the programs of this kind is Surf, developed by a group around Stephan Endraß, see [1].

Movies for the exploration of certain surfaces were built by a group of Herwig Hauser based on the free renderer POV-Ray, see Herwig Hauser's homepage [3]. During 2008, the year of Mathematics in Germany, the Oberwolfach Research Institute of Mathematics provides an exhibition Imaginary with a homepage [2]. For an interactive presentation the program Surfer (based on S. Endraß's program Surf) is used. It works in the background for a high-resolution image that occur after a certain rendering time depending on the complexity of the surface.

The aim of this talk is to present Christian Stussak's program RealSurf for an interactive visualization of algebraic surfaces in realtime, see [4]. The program is based on a recent technique of programming on the graphics programming unit

(GPU) with shader languages. It works well for computers with recent nVidia graphics cards.

2. Real Time and Singularities

A popular technique for the visualization of surfaces is based on polygonal meshes. It does not work correctly for singularities. Some times a singularity is not exhibited in a polygonal mesh, and hence does not occur for the visualization.

A typical example is the tangent surface F of the affine twisted cubic C given parametrically by $x = t, y = t^2, z = t^3$. It is easily seen that $F = 3x^2y^2 - 4x^3z - 4y^3 + 6xyz - z^2 = 0$. The curve C is a singular curve on its tangent variety F. There are numerical instabilities for drawing the singular locus correctly.

In Computer Graphics ray tracing is an appropriate technique in order to visualize scenes with complex details like singularities. It requires a ray for each pixel into a mathematically described scene and its interaction with further objects in order to compute the illumination. For obtaining fine details (like singularities) this iteration requires minutes resp. hours of computing time.

In recent times there is a new hardware development with computations and programming on the graphics processing unit (GPU). During the execution of the program the driver of the graphics card translates the program code into machine instructions for the graphics card. Because of the multiprocessor concepts of recent graphics cards this procedure ensures an essential increasing of the computing speed.

3. RealSurf

Christian Stussak's RealSurf is a program for the interactive visualization of (implicit) algebraic surfaces. It is based on the hardware development as mentioned above and uses the OpenGL Shading Language (GLSL). Figure 1 shows a screen shot of the program for the exploration of Barth's surface of degree 10.

Besides of the programming with the shader languages the implementation of the program requires several numerical considerations for the computation of zeros of algebraic equations. Because of numerical instabilities in the neighborhood of singularities Christian Stussak investigated methods in order to separate zeros of polynomial equations and implemented them.

The program RealSurf allows several features:

- Scaling, rotation and translation of the surface in real time is implemented by mouse actions.
- A list of classical surfaces is prepared, visible in the side bar. New surfaces can be added.
- The input of a surface is in the form $F \in \mathbb{R}[x, y, z]$. It works well for "sparse" F up to degree 13.
- The constants in F can be parameters. This allows deformations of given surfaces, also with several parameters.
- The change of light of the scene, the change of the material and the clipping against a cube or a sphere are available.

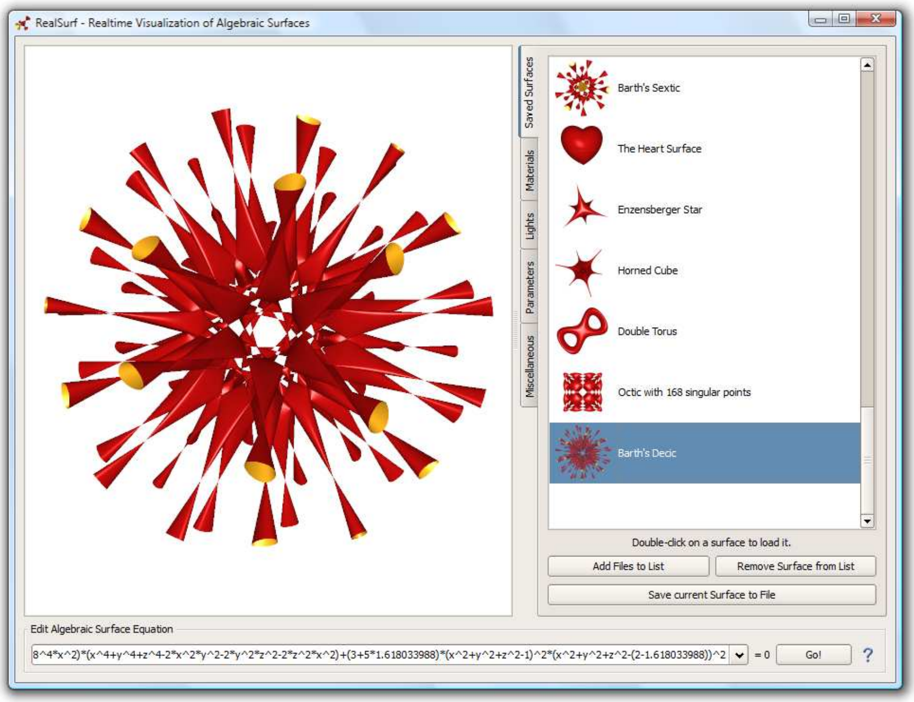

Figure 1. RealSurf

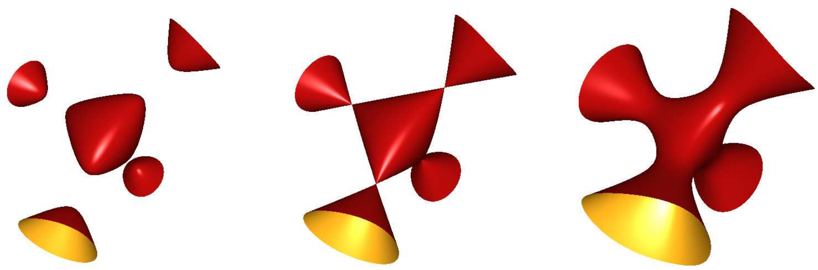

Figure 2. The cubic for $\alpha = 1.8, 2.0, 2.8$

- For further use a screen shot of a scene as a PNG file is possible. An appropriate background color might be chosen.

As an example we investigate the cubic surface

$$F(x, y, z) = x^2 + y^2 + z^2 - \alpha xyz - 1 = 0$$

with the parameter α, see Figure 2.

The program allows an interactive visualization of the surface by a continuous change of the parameter. In particular for $\alpha = 2$ it yields the Cayley cubic.

4. Concluding Remarks

The program RealSurf allows the interactive exploration of implicit given algebraic surfaces with singularities in real time. It is based on programming with the GLSL shading language. Presently it works well for surfaces up to degree 13 with "sparse" equations.

At the moment it is available for Windows XP and Windows Vista. It requires nVidia graphics hardware of the GeForce 7000 series or higher resp. corresponding Quadro cards. Upon request it is freely available via [4].

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Infinite matrices and the conjectural transcendence of some Hilbert-Kunz multiplicities

Paul Monsky

F is a finite field of characteristic p, and h is in $F[x_0, \ldots, x_r], h \neq 0, h(0) = 0$. $e_n(h)$ is the colength of the ideal generated by h and the qth powers of the variables where $q = p^n$. The "Hilbert-Kunz series", $\sum_h$, and the Hilbert-Kunz multiplicity, $\mu(h)$, are the formal power series $\sum e_n w^n$, and the limit as $n \to \infty$ of e_n/q^r. In all cases where $\sum$ and μ have so far been *provably* calculated, $\sum$ is in $Q(w)$ and μ is rational. My talk presented evidence that quite different things happen in general.

A result from [1] and [2] states that $\sum$ is in $Q(w)$ and μ is rational when h is a "disjoint" sum of 2-variable h_i (that is to say that the variables appearing in the various h_i are distinct). The proof involves the introduction of the space X of functions $[0,1] \cap Z\left[\frac{1}{p}\right] \to Q$, magnification operators $T_0 \ldots, T_{p-1}$ on X, and an element ϕ_h of X which encodes all the integers $e_n(h^k)$. Teixeira and I say that h is "strongly rational" when there is a finite dimensional subspace M of X stable under the T_i and containing ϕ_h. We have shown that 2-variable h are strongly rational, that the disjoint sum of strongly rationals is strongly rational, and that $\sum$ is in $Q(w)$ and μ in Q for strongly rational h; this last (easy) step involves the study of iterates of a linear operator on M.

However 3-variable h are unlikely to be strongly rational in general. In [3] I made a precise conjecture as to the value of $e_n(h^k)$ when $p = 2, r = 2$ and h defines a nodal cubic. The conjecture gives an infinite basis for the smallest subspace M of X that contains ϕ_h and is stable under T_0 and T_1, and describes the action of

T_0 and T_1 on the basis elements. Granting the conjecture, I showed in [3] that the $\sum$ attached to $f = uv + x^3 + y^3 + xyz$ lies in $Q(w, \sqrt{1-4w^2})$, and that $\mu(f) = \frac{4}{3} + \frac{5}{14\sqrt{7}}$.

In my talk I spoke about two further results that follow from the conjecture; as above we take $p = 2$.

I. If f is a disjoint sum of x^3+y^3+xyz and 2-variable h_i, then $\sum_f$ is algebraic over $Q(w)$ and $\mu(f)$ is algebraic over Q.

II. The Q-vector space spanned by the characteristic 2 Hilbert-Kunz multiplicities contains $\sum \binom{2n}{n}^2/(65,536)^n$. (This is interesting for the following reason. The sum is the quotient by π of a period of a Q-rational 1-form without zeros or poles on an elliptic curve defined over Q. Schneider long ago showed that such numbers are transcendental. So under my conjecture, transcendental Hilbert-Kunz multiplicities exist.)

The proof of I follows ideas from [1] and [2]. But now M is infinite dimensional, and to study the iterates of the linear operator, infinite matrices and walks on the half-line come into play. The proof of II starts with my calculation of the (conjectural) $\sum$ attached to $uv + x^3 + y^3 + xyz$ and uses the following surprising fact. Suppose h_i are disjoint polynomials in r_i variables. Set $\Phi_i = (1-2^{r_i+1}w)\cdot\sum_i$ where $\sum_i$ is the $\sum$ attached to $uv + h_i$. Then the Hilbert-Kunz multiplicity of $uv+\sum h_i$ is the Hadamard product of the Φ_i evaluated at $1/2^k$, where $k = \sum r_i+1$. This allows us to pass from the realm of algebraic power series to the realm of transcendental series via the Hadamard product.

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Generic bounds for tight closure

Helena Fischbacher-Weitz

(joint work with Holger Brenner)

Let $P = k[x_1, \ldots, x_d]$ be a standard-graded polynomial ring over a field k and let $a_1, \ldots, a_n$ be natural numbers. For a family $f_1, \ldots, f_n$ of homogeneous polynomials of degree $\deg(f_i) = a_i$ we look at the ideal $I = (f_1, \ldots, f_n)$. The Fröberg conjecture, which has been proved in dimension $d \leq 3$, claims that the Hilbert function

$$m \mapsto H(m) = \dim_k P_m/(f_1, \ldots, f_n)$$

has an easy description given by the coefficients of a certain power series defined by the degrees $a_1, \ldots, a_n$, provided that the f_i are choosen generically. In particular, this conjecture gives for $n \geq d$ an implicitly defined degree bound m_0 for ideal membership (depending only on the degrees $a_1, \ldots, a_n$), by which we mean

that $P_{\geq m_0} \subseteq (f_1, \dots, f_n)$. This degree bound is the smallest number where the predicted generic Hilbert function vanishes.
Now let R be any standard-graded k-algebra of dimension d. Does there exist a similar generic degree bound for ideal membership? Instead of asking,

When is $R_{m_0} \subseteq I$? **(1)**

we may replace the ideal by its tight closure:

When is $R_{m_0} \subseteq I^*$? **(2)**

The answer to (2) is nicer than the answer to (1). Indeed, as we can already see from the example of a parameter ideal in a hypersurface ring, the degree bound for (1) must depend on the structure of R. In contrast, there exists a generic degree bound for (2) which only depends on the dimension of R and the degrees a_i.

If m_0 is the generic degree bound in the polynomial ring, then $m_0 + d - 1$ is a generic tight closure bound for all standard-graded k-algebras of this dimension over a field of positive characteristic. This means that the containment in the tight closure behaves more uniformly than the containment in the ideal.

We recall the definitions of tight closure and Frobenius closure in positive characteristic.

Definition. Let R be a Noetherian ring containing a field of characteristic $p > 0$, and let $I = (f_1, \dots, f_n) \subseteq R$ be an ideal. Let

$$\begin{array}{ll} I^{[q]} & := (f_1^q, \dots, f_n^q) \subseteq R \text{ for } q = p^e,\ e \in \mathbb{N} \\ I^F & := \{x \in R : x^q \in I^{[q]} \text{ for some } q = p^e\} \\ I^* & := \{x \in R : \exists z \notin \text{min. prime} : zx^q \in I^{[q]} \text{ for almost all } q = p^e\} \end{array}$$

I^F is called the *Frobenius closure* of I and I^* is called the *tight closure* of I.

In a regular ring, such as a polynomial ring, every ideal is tightly closed (i.e. $I = I^*$), so both (1) and (2) generalize ideal membership in polynomial rings. It is an important feature of tight closure theory that we can often generalize statements about ideal membership in regular rings to non-regular rings if we replace the ideal by its tight closure, a typical example being the tight closure version of the Briançon-Skoda theorem. In our case, the general tight closure result follows from the regular ideal result by semicontinuity and by cohomological vanishing conditions.

We state our main result.

Theorem. Fix a degree type $(a_1, \dots, a_n)$, $n \geq d$. Suppose that there exist $g_1, \dots, g_n \in P = K[x_1, \dots, x_d]$, $\deg(g_i) = a_i$, such that $P_m \subseteq (g_1, \dots, g_n)$. Then for any d-dimensional standard-graded k-algebra R and n elements $f_1, \dots, f_n \in R$ of this degree type the containment

$$R_{m+d-1} \subseteq (f_1, \dots, f_n)^*$$

holds "generically" in an intersection of countably many open sets in the parameter space. In particular, this is true in the generic point of the parameter space, i.e.

for the "indetermined" ideal generators

$$G_i = \sum_{|\nu|=a_i} \alpha_\nu X^\nu \in R \otimes k[\alpha_\nu].$$

If R is normal, then we have

$$R_{m+d} \subseteq (f_1, \ldots, f_n)^F$$

for generic elements $f_1, \ldots, f_n$, i.e. in an open subset of the parameter space. If R is Cohen-Macaulay with a-invariant $a(R)$ and of dimension ≥ 2, then

$$R_{m+d+a(R)} \subseteq (f_1, \ldots, f_n)$$

for generic elements, i.e. in an open subset of the parameter space.

The "proof strategy" is as follows (for the tight closure result in dimension $d \geq 2$).

(1) We choose a homogeneous Noether normalization $P \subseteq R$ and choose $g_1, \ldots, g_n \in P$ for which the degree bound m_0 holds. This is directly related to the shape of a minimal resolution of the ideal $(g_1, \ldots, g_n)$ over P, and in particular to the Betti numbers in the last free module of this (finite) resolution.
(2) The pull-back of this resolution to R is still exact on the punctured spectrum and on $Y = \operatorname{Proj} R$. Under the condition that $m \geq m_0 + d - 1$, the twists in the last (splitting) syzygy bundle $\operatorname{Syz}_{d-1}(m)$ on Y are all non-negative.
(3) This means that a certain "tight closure"-cohomology condition is true for $H^{d-1}(Y, \operatorname{Syz}_{d-1}(m))$. This is an instance of the philosophy that top-dimensional cohomology classes of non-negative degree are "tightly zero". By "cohomology hopping" this implies the same cohomological property for $\operatorname{Syz}_1(m)$.
(4) In order to "deform away" from $g_1, \ldots, g_n \in P$ to more general elements $f_1, \ldots, f_n \in R$ (of the same degree type) we consider the whole situtation over a parametrizing space whose points determine the coefficients of the elements. One can apply semicontinuity for cohomology of a flat sheaf (namely the first syzygy sheaf) over a projective morphism to our cohomological property. Therefore this property holds for generic choice.
(5) Finally, the cohomological property implies that all elements of degree m belong to the tight closure.

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Determinantal Equations

Gregory G. Smith

(joint work with Jessica Sidman)

The qualitative study of systems of polynomial equations lies at the heart of commutative algebra. One important facet of this study relates geometric properties of a projective subscheme to structural features of its homogeneous ideal, such as being determinantal or having a free resolution with some simple form. For example, the homogeneous ideal of a rational normal curve, a Segre variety, or a quadratic Veronese embedding of projective space is given by the (2×2)-minors of a generic Hankel matrix, a generic matrix, or a generic symmetric matrix respectively. For these classic examples, the determinantal presentation leads to an explicit description of the minimal graded free resolution of the homogeneous ideal, and equations for their higher secant varieties. Mumford's "somewhat startling observation" [3] shows that a suitable multiple of every projective embedding is defined by the (2×2)-minors of a matrix of linear forms. Eisenbud [1, page 107] rephrases this observation as a "(vague) principle that embeddings of varieties by sufficiently positive bundles are often defined by ideals of (2×2)-minors". The aim of this work is to remove the ambiguity from this principle.

To achieve this, we need a source of appropriate matrices. Composition of linear series or equivalently multiplication in the Cox ring of the variety traditionally supply such matrices. To be more explicit, observe that, if $X \subset \mathbb{P}^r$ is a scheme embedded by the complete linear series $|L|$ for $L \in \operatorname{Pic}(X)$, then $H^0(X, L)$ is the space of linear forms on $\mathbb{P}^r$. Factoring L as $L = L_1 \otimes L_2$ for some $L_1, L_2 \in \operatorname{Pic}(X)$ yields a natural multiplication map

$$\mu \colon H^0(X, L_1) \otimes H^0(X, L_2) \to H^0(X, L_1 \otimes L_2) = H^0(X, L)\,.$$

By choosing ordered bases $y_1, \dots, y_m \in H^0(X, L_1)$ and $z_1, \dots, z_n \in H^0(X, L_2)$, one obtains the associated $(m \times n)$-matrix $A := [\mu(y_i \otimes z_j)]$ of linear forms on $\mathbb{P}^r$. Since the structure sheaf $\mathscr{O}_X$ is a sheaf of commutative rings, it follows that the (2×2)-minors of A vanish on X (see Proposition 6.10 in Eisenbud [1]). Moreover, we note that the homogeneous ideal $I_2(A)$ of (2×2)-minors of A is independent of the choice of bases for $H^0(X, L_1)$ and $H^0(X, L_2)$. Numerous classic examples of this construction, including the three given above, can be found in Room [4]. With this notation, the goal is to find conditions on the line bundles L_1 and L_2 to guarantee that the homogeneous ideal of X in $\mathbb{P}^r$ is $I_2(A)$.

We accomplish this goal by placing restrictions on certain modules arising from the line bundles L, L_1, and L_2. More precisely, if $L = L_1 \otimes L_2$ is very ample and the following three conditions hold:

- the module $\bigoplus_{d\geq 0} H^0(X, L\otimes L^d)$ has a linear presentation with respect to the polynomial ring $\mathrm{Sym}\big(H^0(X,L)\big)$,
- the module $\bigoplus_{d\geq 0} H^0(X, L_i\otimes L_j^d)$ has a linear presentation with respect to the polynomial ring $\mathrm{Sym}\big(H^0(X,L_j)\big)$ for $i\neq j$,
- the module $\bigoplus_{d\geq 0} H^0(X, L_1^2\otimes L_2^d)$ has a linear presentation with respect to the polynomial ring $\mathrm{Sym}\big(H^0(X,L_2)\big)$,

then the saturated homogeneous ideal of X in $\mathbb{P}^r$ is $I_2(A)$. By combining this with a cohomological criterion for a linear presentation and multigraded Castelnuovo-Mumford regularity, we establish the following principle: every projective embedding of a scheme determined by the complete linear series of a sufficiently ample line bundle is cut out by the (2×2)-minors of a matrix of linear forms. In other words, given a projective scheme X, there exists a line bundle L_0 on X such that, for all $L\in \mathrm{Pic}(X)$ for which $L\otimes L_0^{-1}$ is numerically effective (nef), the image of the map $\varphi_{|L|}\colon X\hookrightarrow \mathbb{P}\big(H^0(X,L)\big)$ corresponds to the ideal $I_2(A)$ for some matrix A of linear forms. Extending the work of Eisenbud-Koh-Stillman [2] for reduced irreducible curves, we also specify effective bounds for L_0 on products of projective spaces, Gorenstein toric varieties, and smooth n-folds. By considering more than one factorization of L or equivalently more than one matrix A of linear forms, we can weaken the explicit bounds.

Finally returning to our initial motivation, this work suggests that, for an embedding $X\subset \mathbb{P}^r$ given by the complete linear series of a sufficiently ample line bundle, the homogeneous ideal of the k^{th} secant variety $\mathrm{Sec}^k(X)$ is defined by the $(k+2)\times(k+2)$-minors of A. Assuming this is true, it would be interesting to have explicit bounds for "sufficiently ample" in this context.

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Toroidalization in Higher Dimensions

Steven Dale Cutkosky

Suppose that $\Phi : X \to Y$ is a dominant morphism of varieties over a field of characteristic zero. Φ is toroidal if there are simple normal crossing divisors D_X on X and D_Y on Y such that $\Phi^{-1}(D_Y) = D_X$ and for all $p\in X$, Φ is formally isomorphic to a morphism of toric varieties at p. In other words, Φ is locally a monomial mapping with respect to the divisors D_Y and D_X.

The problem of toroidalization is to construct a commutative diagram of morphisms

$$\begin{array}{ccc} X_1 & \stackrel{\Phi_1}{\to} & Y_1 \\ \downarrow & & \downarrow \\ X & \stackrel{\Phi}{\to} & Y \end{array}$$

such that the vertical arrows are products of monoidal transforms (blow ups of nonsingular subvarieties) and $\Phi_1 : X_1 \to Y_1$ is toroidal.

This problem is solved locally along a valuation in all dimensions in earlier work of the author ([C1], [C2]). The case when X has arbitrary dimension n and Y is a curve follows from the theorem of resolution of singularities for varieties of arbitrary dimension ([H] and more recent simplifications in the proof), and is in fact really equivalent to this theorem. There are several proof in the case when X and Y both are of dimension 2 (for instance [AKMW], [AK], [CP]). In earlier papers of the author, toroidalization is proved in the case when X has dimension 3 ([C3], [C4]).

An approach to proving toroidalization in all dimensions n of X is to make use of induction on the dimension of Y; as remarked earlier, the case when Y has dimension 1 follows from resolution of singularities.

We have recently proven that toroidalization can be proven in the case that X has arbitrary dimension n and Y has dimension 2. An interesting aspect of the proof is its relation with difficulties that come up in related problems such as resolution of vector fields and resolution of singularities.

The proof is by consideration of formal local representations of the map of the form

$$\begin{array}{rcl} u & = & (x_1^{a_1} \cdots x_t^{a_r})^n \\ v & = & P(x_1^{a_1} \cdots x_r^{a_r}) + x_1^{b_1} \cdots x_r^{b_r} F \end{array}$$

where $\gcd(a_1, \ldots, a_r) = 1$, P is a series, $x_1, \ldots, x_r$ do not divide F, and $x_1^{b_1} \cdots x_r^{b_r} F$ has no terms which are powers of $x_1^{a_1} \cdots x_t^{a_r}$. A main invariant is the order of F. This invariant is not so well behaved. In fact, it is not upper semicontinuous, and it can actually increase (but by at most 1) under blow ups. However, a related invariant which is the order of a log form associated to the map, is at least upper semicontinuous. Comparison of these two invariants under a series of algorithms allows us to construct a series of blow ups above X which lead to a reduction in order. This is the most difficult part of the proof. These algorithms require blow ups of general varieties through a point (or subvariety), and a lot of care is required in algebraizing and then patching the local algorithms together to get a global construction. We may also have to modify the divisor D_X. When this process is completed, we obtain a morphism $X_1 \to X$ such that the associated map $X_1 \to Y$ is “strongly prepared”. We finish the proof using the result of our joint paper with Olga Kascheyeva [CK], which shows that it is then possible to construct a sequence of blow ups above Y and X_1 to obtain a toroidal map.

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The a invariants of normal graded Gorenstein rings and varieties with even canonical class

Kei-ichi Watanabe

Let $R = \oplus_{n \geq 0} R_n$ be a Noetherian normal graded ring with $R_0 = k$ field and Proj $(R) = X$. We always assume that GCD of $\{n \mid R_n \neq 0\}$ is 1.

Such normal graded ring is described by so called DPD (Dolgachev- Pinkham-Demazure) construction (cf. [3] for the case $\dim R = 2$ and [2] for the general case). Namely, put $X = \text{Proj}\,(R)$ and fix a homogeneous element T of the quotient field of R. Then there exists unique $\mathbb{Q}$ divisor D on X such that ND is an ample Cartier divisor on X and

$$R = \oplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))T^n.$$

We denote this ring as $R = R(X, D)$.

Let K_R be a canonical module of R. The isomorphism class of canonical module is determined up to isomorphism as graded modules. We say that R is quasi-Gorenstein if K_R is a free R module, so that R is Gorenstein if and only if it is quasi-Gorenstein and Cohen-Macaulay. If this is the case, $K_R \cong R(a)$ for some $a \in \mathbb{Z}$ and we call this a the a-invariant of R (cf. [1]).

In the case R is generated by R_1 over R_0 (we say that R is a "standard graded ring"), if R is Gorenstein with $a(R) = \alpha$, then $\mathcal{O}_X(K_X) \cong \mathcal{O}_X(\alpha)$. Thus K_X should be $\mathbb{Q}$-cartier and either $K_X = 0$, or K_X or $-K_X$ is ample and $a(R)$ can take very limited values.

But if R is not standard, the situation is not so simple. For example if $X = \mathbb{P}^n$ with n odd, then for every integer $\alpha \neq 0$, there exists a normal Gorenstein ring with Proj $(R) = X$ and $a(R) = \alpha$. If $X = \mathbb{P}^n$ with n even, then for every *odd* integer α, there exists a normal Gorenstein ring with Proj $(R) = X$ and $a(R) = \alpha$. But $a(R)$ is never even.

Definition 1. Let X be a normal projective variety over k then we define

$$\mathcal{A}(X) = \{a(R) \mid R \text{ is quasi-Gorenstein and Proj } (R) = X\}.$$

We have the following facts.

Proposition 2. (1) For every normal projective variety X, $1 \in \mathcal{A}(X)$.

(2) If there is a normal Cohen-Macaulay graded ring R with Proj $(R) = X$, then there is a normal Gorenstein graded ring R with Proj $(R) = X$

Definition 3. Let X be a normal projective variety. We say X *has even canonical class* if $\mathrm{cl}(K_X) = 2\,\mathrm{cl}(D)$ for some divisor D in the divisor class group of X.

Theorem 4. If there exists a quasi-Gorenstein ring R with Proj $(R) = X$ and even $a(R)$, then X has even canonical class.

These statements follow from the characterization for Cohen-Macaulay and quasi-Gorenstein property for R (cf. [4]).

Example 5. Let X be a normal projective variety and H be an ample (integral $\mathbb{Q}$-Cartier Weil) divisor on X. Also, we assume that $\dim \mathrm{H}^0(X, \mathcal{O}_X(H)) \geq 2$ in (1) and (2) .

(1) If K_X is linearly equivalent to sH with some negative even integer s, then $\mathcal{A}(X) = \mathbb{Z} \setminus 0$.

(2) If X does not have even canonical class and K_X is linearly equivalent to sH with some negative odd integer $s \leq -3$, then $\mathcal{A}(X)$ is the set of all odd integers.

(3) If $K_X = 0$, then $\mathcal{A}(X) = \mathbb{Z}_{\geq 0}$, the set of all non-negative integers. (Conversely, if $0 \in \mathcal{A}(X)$, then $K_X = 0$.)

(4) If X has even canonical class and K_X is ample $\mathbb{Q}$-Cartier, then $\mathcal{A}(X) = \mathbb{Z}_{>}0$, the set of all positive integers.

(5) If X does not have even canonical class and K_X is ample $\mathbb{Q}$-Cartier, then $\mathcal{A}(X)$ is the set of all positive odd integers.

(6) If the divisor class group $\mathrm{Cl}(X) \cong \mathbb{Z}$ with the class of $-K_X$ as ample generator, then $\mathcal{A}(X)$ is the set of odd integers ≥ -1.

Question 6. (1) For every normal projective variety X, the set $\mathcal{A}(X)$ coincides with one of the sets appearing in (1)-(6) of Example 5 ?

(2) What is the condition for X so that $\mathcal{A}(X)$ contains a negative integer? It is easy to see that the condition "$-K_X$ is big" is a necessary condition. Is it also a sufficient condition?

(3) If X is a Fano variety with index 1, then is $\mathcal{A}(X)$ the set of odd integers ≥ 1 ?

Remark. In my talk at Oberwolfach, I overlooked the case (6) in Example 5.

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