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Strings, Fields and Topology

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Abstract. In recent years, the interplay between traditional geometric topology and theoretical physics, in particular quantum ?eld theory, has played a signi?cant role in the work of many researchers. The idea of this workshop was to bring these people together so that the ?elds will be able to grow together in the future. Most of the talks were related to various flavors of field theories and differential cohomology theories.

Mathematics Subject Classification (2000): 19xx, 55xx, 81xx.

Introduction by the Organisers

The goal of this workshop was similar to that organized by Stolz and Teichner in 2005, namely to bring together topologists and physicists interested in field theories. Various flavors of field theories were discussed in the talks, classical as well as quantum, topological as well as conformal field theories, with an emphasis on relations to topology and (higher) category category. Another focus were differential cohomology theories which are useful in certain physical models to describe charges, currents and fields. In the rest of this introduction, we will outline how some of the talks at the workshop fit into the themes described above.

A featured speaker was Kevin Costello with three lectures on *Factorization algebras in perturbative quantum field theory.* It it well-known that the observables in a classical mechanical system are the functions on the associated phase space which form a Poisson algebra, while the observables in a quantum mechanical system are operators which form an associative algebra. The goal of deformation quantization of classical systems is the classification of all associative algebras which modulo $\hbar$ reduce to the Poisson algebra of a given classical system. In his talks, Costello described an analog of deformation quantization for field theories. His first result (Theorem 1 in his abstract) describes the structure of observables of a classical field theory as a *factorization algebra* which is an algebra over a suitable operad P_0. Costello went on to define a *quantization* of such a structure as a factorization algebra over the Beilinson-Drinfeld operad, which modulo $\hbar$ agrees the original operad over P_0 (the Beilinson-Drinfeld operad has the same relationship to P_0 as the associative operad has to the Poisson operad).

Christoph Schweigert and Ingo Runkel showed in their two talks entitled *Conformal field theory and algebra in braided tensor categories* how to construct conformal field theories from 'special symmetric Frobenius algebras' in the modular tensor category provided by the representations of a suitable vertex operator algebra. This can be extended to make similar statements about open-closed theories ('D-branes') and theories involving defect lines. Concerning topological field theories, Kevin Walker talked about *Blob homology*, a new homological way to produce topological field theories, and Chris Schommer-Pries presented his work on the classification of 2 -dimensional *extended* field theories (which involve data associated to manifolds of dimension 0 in addition to those of dimension 1 and 2). Ralph Cohen made use of recent results of Hopkins-Lurie concerning the classification of extended field theories to relate string topology of a manifold to the sympletic field theory of its cotangent bundle.

Making use of differential real K -theory Dan Freed described the space of fields and the action of a 2 -dimensional string theory and its 10 -dimensional field theory approximation. An axiomatic approach to differential cohomology was presented by Thomas Schick and Ulrich Bunke described an explicit model for differential K -theory motived by index theory.

Not related to field theories (as far as we know today), but a spectacular result in topology is the recent solution (announced this spring) of an almost 50 year old question by Kervaire by Hill, Hopkins and Ravenel. Mike Hopkins described the fascinating history of the problem as well as an outline of their proof in his talk at this workshop as well as providing many details in a long evening session.

We would like to thank the Oberwolfach Institute for providing the ideal setting for this workshop and all participants for bringing lots of enthusiasm and energy, which showed e.g. in the (sometimes parallel) informal evening sessions initiated by the participants.

Workshop: Strings, Fields and Topology

Table of Contents

Abstracts

Conformal field theory and algebra in braided tensor categories

Christoph Schweigert
(joint work with Jens Fjelstad, Jürgen Fuchs, Ingo Runkel)

Let $\mathcal{V}$ be a conformal vertex algebra with the property that its representation category $\mathcal{C}$ carries the structure of a modular tensor category:

- The category $\mathcal{C}$ is an abelian semi-simple category enriched over the category of finite-dimensional complex vector spaces.
- It is noetherian, i.e. has finitely many isomorphism classes of simple objects for which we choose a set I of representatives. The tensor unit is assumed to be simple; I is chosen such that it contains the tensor unit.
- It is a ribbon category with a braiding that is non-degenerate in the sense that
$$\begin{array}{lcl} K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C} & \to & \mathrm{End}(\mathrm{id}_{\mathcal{C}}) \\ {[U]} & \mapsto & \alpha^{[U]} = (\alpha_V^{[U]})_{V \in \mathcal{C}} \end{array}$$
with $\alpha_V^{[U]} \in \mathrm{End}(V)$ given by the monodromy of U around V is an isomorphism of complex algebras.

It is well-known that any modular tensor category gives rise to a three-dimensional topological field theory, i.e. a symmetric tensor functor

$$\mathrm{tft}_{\mathcal{C}} : \mathrm{cobord}_{3,2}^{\mathcal{C}} \to \mathrm{vect}_{fd}(\mathbb{C})$$

from a category $\mathrm{cobord}_{3,2}^{\mathcal{C}}$ of (3,2)-cobordims decorated with objects and morphisms of the modular tensor category $\mathcal{C}$ with values in finite-dimensional complex vector spaces.

In this talk, we have introduced a category of oriented decorated surfaces X such that a double $\hat{X}$ of the surface can be seen as an object in $\mathrm{cobord}_{3,2}^{\mathcal{C}}$. The double $\hat{X}$ comes with an orientation reversing involution σ such that $\hat{X}/\langle\sigma\rangle \cong X$. The principle of holomorphic factorization states that the correlator $Cor(X)$ is an element in such a vector space $Cor(X) \in \mathrm{tft}_{\mathcal{C}}(X)$ that is required

(1) to be invariant under the action of the so-called relative mapping class group $\mathrm{Map}(X)^{\sigma} \equiv \{x \in \mathrm{Map} | x \circ \sigma = \sigma \circ x\} \cong \mathrm{Map}(X)$.
(2) to obey certain factorization constraints.

It can be shown that a solution to these constraints is given as follows [1]: the decoration data for X are given by the bicategory of special symmetric Frobenius algebras in $\mathcal{C}$. The Morita class of such a special symmetric Frobenius algebra labels a full local conformal field theory with chiral symmetries given by the vertex algebra $\mathcal{V}$. A-modules label boundary conditions, and A_1-A_2-bimodules label topological defect lines.

This solution is constructed in terms of a cobordism $\emptyset \xrightarrow{M_X} \hat{X}$ with embedded ribbon graph, where the ribbons are labelled by data from the bicategory such that

$$Cor(X) = \mathrm{tft}_{\mathcal{C}}(M_X)1 \in \mathrm{tft}_{\mathcal{C}}(\hat{X})\,.$$

Conversely, one can show [2] (restricting to the case of a single boundary condition and no defect lines in X) that any solution to the consistency constraints satisfying certain natural extra conditions (see Theorem 4.26 of [2]) is of this form.

References

[1] J. Fjelstad, J. Fuchs, I. Runkel and C. Schweigert, *TFT construction of RCFT correlators. V. Proof of modular invariance and factorisation*, Theor. Appl. Categor. **16** (2006) 342-433.
[2] J. Fjelstad, J. Fuchs, I. Runkel and C. Schweigert, *Uniqueness of open / closed rational CFT with given algebra of open states*, Adv. Theor. Math. Phys. **12** (2008), 1283-1375.

Conformal field theory and algebra in braided tensor categories II

Ingo Runkel
(joint work with Jürgen Fuchs, Liang Kong, Christoph Schweigert)

In part I of this series of two talks we have seen that the data decorating a conformal field theory world sheet is taken from the bicategory $\mathcal{F}rob(\mathcal{C})$ of special symmetric Frobenius algebras in the modular tensor category $\mathcal{C}$ provided by the representations of a suitable vertex operator algebra $\mathcal{V}$.

A Frobenius algebra $A \in \mathcal{F}rob(\mathcal{C})$ labels a full CFT with chiral symmetry $\mathcal{V}$. Part of the data of such a CFT is the space of bulk fields B. This is an object in the product category $\mathcal{C}_+ \boxtimes \mathcal{C}_-$, the $\mathbb{C}$-linear category whose objects are direct sums of pairs of objects, and whose morphism spaces are tensor products (over $\mathbb{C}$) of those in $\mathcal{C}$. The signs $\pm$ refer to the braiding and twist, $\mathcal{C}_+$ is just equal to $\mathcal{C}$, and $\mathcal{C}_-$ is $\mathcal{C}$ with inverse braiding and twist. The space of bulk fields B is associated to a marked point in the interior of the world sheet, and by evaluating correlators with several marked points on a sphere, the object B gets equipped with the structure of a commutative symmetric Frobenius algebra. We will now give a direct construction of B, starting from the modular tensor category $\mathcal{C}$ and a special symmetric Frobenius algebra $A \in \mathcal{C}$.

The starting point is a functor $R : \mathcal{C} \to \mathcal{C}_+ \boxtimes \mathcal{C}_-$ which is left and right adjoint to the tensor product functor T. The functor R acts on objects as

$$R(V) = \bigoplus_{i\in\mathcal{I}} (V \otimes U_i^\vee) \boxtimes U_i \ ,$$

where $\mathcal{I}$ is a set of labels for representatives U_i of the isomorphism classes of simple objects in $\mathcal{C}$ and $(\ )^\vee$ denotes the dual. One checks that R is naturally isomorphic to $R'(V) = \bigoplus_{i\in\mathcal{I}} U_i^\vee \boxtimes (V \otimes U_i)$. We will use R in what follows. It is in general not a tensor functor (it is so only if $\mathcal{C}$ is equivalent to the category of finite dimensional complex vector spaces), but it is a lax and colax tensor functor, and one can show that if A is a special symmetric Frobenius algebra, so is $R(A)$ [1, Prop. 2.25]. Typically, $R(A)$ is not commutative.

The algebra B we aim to describe is commutative, and turns out to be the centre of $R(A)$. To be more precise, in a braided tensor category there are two notions of a centre: The *left centre* $C_l(A)$ of an algebra A is the maximal subobject of A such that $m \circ c_{A,A} \circ (e \otimes id_A) = m \circ (e \otimes id_A)$, where m is the multiplication morphism of A, c is the braiding of $\mathcal{C}$, and e is the subobject embedding; the *right centre* $C_r(A)$ is the maximal subobject of A such that $m \circ c_{A,A} \circ (id_A \otimes e) = m \circ (id_A \otimes e)$. The left and right centre are in general not isomorphic, and possibly not even Morita equivalent. However, one can single out a preferred class of modules, the so-called local modules, and show that the categories of local $C_l(A)$-modules and local $C_r(A)$-modules are equivalent [2, Thm. 5.20].

The space of bulk fields of a CFT motivates a third notion of a centre.

Definition 1. [3, Def. 4.9] *The* full centre *of a special symmetric Frobenius algebra A in a modular tensor category $\mathcal{C}$ is $Z(A) = C_l(R(A)) \in \mathcal{C}_+ \boxtimes \mathcal{C}_-$.*

That C_l appears instead of C_r is linked to a choice made when defining the lax and colax tensor structure on R (cf. [1]). The full centre $Z(A)$ is a commutative special symmetric Frobenius algebra [3, Lem. 4.10], which contains $C_l(A) \boxtimes \mathbf{1}$ and $\mathbf{1} \boxtimes C_r(A)$ as subalgebras. It has another interesting property which neither the left nor right centre provide: $Z(A)$ separates Morita classes.

Theorem 1. [4, Thm. 1.1] *Let $\mathcal{C}$ be a modular tensor category and let A, B be simple special symmetric Frobenius algebras in $\mathcal{C}$. Then $Z(A) \cong Z(B)$ as algebras if and only if A and B are Morita equivalent.*

In CFT, the interpretation of $Z(A)$ is that as an object in $\mathcal{C}_+ \boxtimes \mathcal{C}_-$ it describes the space of bulk fields associated to a marked point in the interior of the world sheet, and its counit and multiplication encode the correlator of a sphere with three marked points. One can ask if every commutative symmetric Frobenius algebra $B \in \mathcal{C}_+ \boxtimes \mathcal{C}_-$ can be written in the form $Z(A)$ for some $A \in \mathcal{F}rob(\mathcal{C})$. This turns out to be true if we impose two more conditions: the algebra B must be simple and its quantum dimension must coincide with the global dimension of $\mathcal{C}$. In CFT these two conditions refer to the uniqueness of the bulk vacuum and modular invariance of correlators on the torus.

Theorem 2. [1, Thm. 3.4, 3.22] *Let $\mathcal{C}$ be a modular tensor category and let B be a simple commutative symmetric Frobenius algebra in $\mathcal{C}_+ \boxtimes \mathcal{C}_-$ with* $\dim(B) = \mathrm{Dim}(\mathcal{C})$. *Then there exists a simple special symmetric Frobenius algebra $A \in \mathcal{C}$ such that $B \cong Z(A)$ as Frobenius algebras.*

This means that every CFT which is defined on genus zero closed oriented surfaces, has the same rational vertex operator algebra $\mathcal{V}$ as left and right moving chiral symmetry, has a unique bulk vacuum, and is modular invariant on the torus, can be extended to a consistent set of correlators on open/closed world sheets.

If we describe a full CFT by an object $A \in \mathcal{F}rob(\mathcal{C})$, we have automatically also singled out a preferred boundary condition, namely the one labelled by A. Equivalent CFTs with different preferred boundary conditions correspond to Morita equivalent algebras in $\mathcal{C}$. In view of this it would be desirable to have a Morita invariant formulation of the datum defining a CFT. This is provided by the notion of module categories, which can be understood as the categorification of a module over a ring: A *right module category* over a tensor category $\mathcal{C}$ is a category $\mathcal{M}$ together with a bifunctor $\odot : \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ and associativity and unit isomorphism subject to coherence conditions, see, e.g., [5].

Given an algebra $A \in \mathcal{C}$, the category A-mod of left A-modules is a module category over $\mathcal{C}$ via $M \times V \mapsto M \otimes V$. Under suitable assumptions, in particular semi-simplicity and finiteness of the categories $\mathcal{C}$ and $\mathcal{M}$, there is a converse statement [5, Thm. 1]: The *internal end* of an object $M \in \mathcal{M}$ is the object $\underline{\mathrm{End}}(M) \in \mathcal{C}$ representing the functor $V \mapsto \mathrm{Hom}(M \odot V, M)$. It comes equipped with the structure of an algebra, and $\underline{\mathrm{End}}(M)$-mod is equivalent, as a module category, to $\mathcal{M}$.

The CFT interpretation of a module category $\mathcal{M}$ over the modular tensor category $\mathcal{C}$ is that the objects of $\mathcal{M}$ are the boundary conditions compatible with the chiral symmetry, and the internal end $\underline{\mathrm{End}}(M)$ for a given $M \in \mathcal{M}$ is the space of boundary fields assigned to a marked point on a boundary segment labelled by M. In fact, we can replace the bicategory $\mathcal{F}rob(\mathcal{C})$ by an equivalent bicategory whose objects are (suitable) module categories over $\mathcal{C}$, and whose morphisms from $\mathcal{M}$ to $\mathcal{N}$ are given by the category of module category functors $\mathcal{F}un(\mathcal{M}, \mathcal{N})$.

We will now see how to construct also the commutative Frobenius algebra B associated to marked points in the bulk directly from the module category. Given a module category $\mathcal{M}$ over $\mathcal{C}$ we define two functors α^+ and α^- from $\mathcal{C}$ to $\mathcal{F}un(\mathcal{M}, \mathcal{M})$, called *braided induction*. On objects they act in the same way, $\alpha^\pm_V$ is the functor $M \mapsto M \odot V$. One also has to provide an isomorphism $\alpha^\pm_V(M \odot U) \xrightarrow{\sim} \alpha^\pm_V(M) \odot U$. This is done with the braiding of $\mathcal{C}$, and the sign $\pm$ tells us to take either the braiding or its inverse.

We turn $\mathcal{F}un(\mathcal{M}, \mathcal{M})$ into a module category over $\mathcal{C} \boxtimes \mathcal{C}$ via a bifunctor $\circledast$: $\mathcal{F}un(\mathcal{M}, \mathcal{M}) \times \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{F}un(\mathcal{M}, \mathcal{M})$, given on objects by $F \circledast (U \boxtimes V) = \alpha^+_U \circ F \circ \alpha^-_V$. Let $Id_\mathcal{M}$ be the identity functor on $\mathcal{M}$ and define

$$Z_\mathcal{M} = \underline{\mathrm{End}}(Id_\mathcal{M}) \ \in \mathcal{C} \boxtimes \mathcal{C} \ .$$

Theorem 3. *Let $\mathcal{C}$ be a modular tensor category and let A be a special symmetric Frobenius algebra in $\mathcal{C}$. Then $Z_{A-\mathrm{mod}} \cong Z(A)$ as algebras.*

A corresponding statement holds for an algebra A over a field k: the endomorphisms of the identity functor on A-mod are isomorphic, as an algebra, to the centre of A.

In CFT terms, this theorem tells us that the internal end of the identity functor on the category of boundary conditions $\mathcal{M}$ provides us with the space of bulk fields. More generally, for $F \in \mathcal{F}un(\mathcal{M}, \mathcal{M})$, the internal end $\underline{\mathrm{End}}(F)$ describes the space of defect fields, i.e. the space of fields associated to a marked point on the world sheet that lies on a defect line labelled by F. The bulk fields occur as the special case of defect fields on the invisible defect, labelled by the identity functor.

References

[1] L. Kong and I. Runkel, *Cardy algebras and sewing constraints, I*, to appear in Commun. Math. Phys.

[2] J. Fröhlich, J. Fuchs, I. Runkel and C. Schweigert, *Correspondences of ribbon categories*, Adv. Math. **199** (2006) 192–329.

[3] J. Fjelstad, J. Fuchs, I. Runkel and C. Schweigert, *Uniqueness of open / closed rational CFT with given algebra of open states*, Adv. Theor. Math. Phys. **12** (2008), 1283-1375.

[4] L. Kong and I. Runkel, *Morita classes of algebras in modular tensor categories*, Adv. Math. **219** (2008) 1548–1576.

[5] V. Ostrik, *Module categories, weak Hopf algebras and modular invariants*, Transform. Groups **8** (2003) 177–206.

The geometry and topology of orientifolds

Daniel S. Freed

(joint work with Jacques Distler and Gregory W. Moore)

We study some *topology* in the perturbative *string* and its long-distance *field* theory approximation. Specifically, we investigate the orientifold construction in string theory. This has attracted much interest among string theorists as they use it to build models. We leave that aspect to the physicists; our concern here is the mathematical foundation. Specifically, our work gives a precise definition of the fields and action in both the perturbative 2-dimensional string theory and the 10-dimensional field approximation. We go on to prove two theorems of a purely topological nature about these theories. The first is the computation of the background Ramond-Ramond charge in KR-theory (with 2 inverted) in the 10-dimensional theory. The second is an anomaly cancellation on the worldsheet in the 2-dimensional theory.

This work has only recently been completed and we are beginning to write complete proofs as well as a more thorough exposition. A brief summary has appeared [1]. We make a few remarks here as introductory background to [1] and the subsequent papers.

Recall first how the low-dimensional fields emerge in string theory. One first investigates the string moving in 10-dimensional Minkowski spacetime M^{10}. Quantizing the theory on a 2-dimensional Lorentzian cylinder one obtains a Hilbert space with a representation of the 10-dimensional Poincaré group. In superstring theory there are two spin structures on the worldsheet, so on the cylinder four possibilities. They lead to a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-grading of the Hilbert space in which $\mathbb{Z}/2\mathbb{Z} = \{\text{Neveu-Schwarz}, \text{Ramond}\}$ is named after physicists who made important early contributions. The quotient $\mathbb{Z}/2\mathbb{Z}$-grading distinguishes bosons from fermions. There is an infinite set of particles—irreducible representations of Poincaré—of which one keeps only the *massless* ones in the low-energy or long-distance approximation. Then one writes a free field theory which reproduces this list of particles. Finally, one seeks to study the string on more general 10-manifolds (spacetimes) X^{10} and with more general surfaces (worldsheets) Σ^2 mapping in.

Fields which are differential forms on Minkowski spacetime are called *abelian gauge fields*. In quantum theories the associated charges are "quantized"—that is, constrained to live in a full lattice—according to Dirac. This is implemented by a cohomology theory; then the currents and fields are geometric objects in some generalized *differential* cohomology theory. In superstring theory the B-field and Ramond-Ramond field are of this type.

The orbifold construction in string theory is encoded by spacetimes which are orbifolds in the sense of Satake.

With this in mind we give the following.

Definition 1. *An* NSNS superstring background *consists of:*

1. *a 10-dimensional smooth orbifold X together with Riemannian metric and real-valued scalar (dilaton) field;*
2. *a double cover $\pi\colon X_w \to X$, the* orientifold double cover*;*
3. *a differential twisting $\check{\beta}$ of $KR(X)$, the* B-field*;*
4. *and a* twisted spin structure $\kappa\colon \Re(\beta) \to \tau^{KO}(TX-2)$.

The Ramond-Ramond field is self-dual, so is equipped with a quadratic refinement of the usual pairing between electric and magnetic currents (which are identified for self-dual fields). It is easiest to state on a 12-manifold, where it is integer valued. Recall that the representation ring of $\mathbb{Z}/2\mathbb{Z}$ is generated by the sign representation ϵ.

Definition 2. *Fix an NSNS superstring background as in Definition 1. Then*

1. *an* RR current *is an object $\check{j} \in \operatorname{ob} \check{KR}^{\check{\beta}}(X_w)$;*
2. *the required quadratic form on a 12-manifold M is the composition*

$$KR^{\beta}(M_w) \longrightarrow KO^{\Re(\beta)}_{\mathbb{Z}/2\mathbb{Z}}(M_w) \xrightarrow[\cong]{\kappa} KO^{\tau^{KO}(TM-4)}_{\mathbb{Z}/2\mathbb{Z}}(M_w) \longrightarrow KO^{-4}_{\mathbb{Z}/2\mathbb{Z}}(\mathrm{pt}) \longrightarrow \mathbb{Z}$$

$$j \longmapsto \overline{j}j \longmapsto \int_{M_w} \overline{j}j \longmapsto \epsilon\text{-component}$$

The data in these definitions are the bosonic fields in the 10-dimensional theory; they are fixed background data in the 2-dimensional theory. There are also fermionic fields, but we do not specify them here. Differential twistings of K-theory are classified by a cohomology theory with three nonzero homotopy groups in degrees 0,1,3. These correspond to degree shifts, double covers, and gerbes. It is important in our picture that all three are present, both in the 2-dimensional theory and the 10-dimensional theory. The twisted spin structure is an isomorphism of twistings of $KO(X)$. There is a Bott shift acting on the NSNS data, reflecting Bott periodicity, and theories related by a Bott shift are equivalent. The Ramond-Ramond current lives in twisted differential KR-theory.

We state the two topological theorems. Let $F \subset X_w$ be the fixed point set of the involution.

Theorem 3. *The center of symmetry of the quadratic form in Definition 2 is*

$$\mu = \frac{1}{2}\, i_* \left(\frac{\kappa^{-1}\,\Xi(F)}{\psi^{-1}\big(\kappa^{-1}\,\mathrm{Euler}(\nu)\big)} \right) \in KR[1/2]^{\beta}(X_w). \tag{4}$$

Here $\Xi(F)$ is a KO-theory analog of the Wu class; $\mathrm{Euler}(\nu)$ is the Euler class of the normal bundle, after localization; ψ is a twisted version of the Adams squaring operation, which is invertible after inverting 2; and i_* is pushforward in KR-theory by the inclusion of the fixed point set.

The background RR charge is $-\mu$.

The fields in the two-dimensional theory are enumerated in the following. We do not couple the superstring to the RR field.

Definition 5. *Fix an NSNS superstring background as in Definition 1. Then a* worldsheet *consists of*

(1) *a compact smooth 2-manifold Σ (possibly with boundary) with Riemannian structure;*
(2) *a spin structure α on the orientation double cover $\hat{\pi}\colon \widehat{\Sigma} \to \Sigma$ whose underlying orientation is that of $\widehat{\Sigma}$;*
(3) *a smooth map $\phi\colon \Sigma \to X$;*
(4) *an isomorphism $\phi^* w \to \hat{w}$, or equivalently a lift of ϕ to an equivariant map $\widehat{\Sigma} \to X_w$;*
(5) *a positive chirality spinor field ψ on $\widehat{\Sigma}$ with coefficients in $\hat{\pi}^*\phi^*(TX)$;*
(6) *and a negative chirality spinor field χ on $\widehat{\Sigma}$ with coefficients in $T^*\widehat{\Sigma}$ (the gravitino).*

Assume the boundary of Σ is empty. The exponentiated Euclidean action, after integrating out ψ and χ, has two factors on which we focus:

$$\exp\Big(2\pi i \int_{\Sigma} \check{\zeta} \cdot \phi^* \check{\beta}\Big) \cdot \mathrm{pfaff}\, D_{\widehat{\Sigma}}\big(\hat{\pi}^*\phi^*(TX) - 2\big), \tag{6}$$

the B-field amplitude and a pfaffian of a Dirac operator. Working over a parameter space S we find that each factor is *anomalous*, i.e., is a section of a hermitian line bundle $L_B \to S$ and $L_{\psi,\chi} \to S$, respectively. These bundles carry compatible covariant derivatives.

Theorem 7. *There is a trivialization of the tensor product $L_B \otimes L_{\psi,\chi} \to S$ determined by the twisted spin structure κ.*

We hope to get the more refined statement that this trivialization $\mathbf{1}$, which is geometric in the sense that $\|\mathbf{1}\| = 1$, $\nabla \mathbf{1} = 0$, is canonical. The trivialization $\mathbf{1}$ is part of the data which specifies the theory, the "setting of the quantum integrand".

Notice the quite different roles of the B-field and twisted spin structure in the theories. In the 10-dimensional theory they are used to define the RR current: the B-field twists differential KR-theory and the twisted spin structure is part of the quadratic form. In the 2-dimensional theory the B-field is integrated over the worldsheet and, in a novel twist, this integral is anomalous; the twisted spin structure is used to cancel the anomaly and set the quantum integrand.

References

[1] J. Distler, D. S. Freed, G. W. Moore, *Orientifold Précis*, arXiv:0906.0795

Factorization algebras in perturbative quantum field theory

Kevin Costello
(joint work with Owen Gwilliam)

In this series of lectures I described an analog, in quantum field theory, of the deformation quantization approach to quantum mechanics. In order to do this, I explained the following.

(1) The structure present on the collection of observables of a *classical* field theory. This structure is the analog, in the world of quantum field theory, of the Poisson algebra which appears in classical mechanics. This structure we called a *classical factorization algebra*.
(2) The structure present on the collection of observables of a *quantum* field theory to satisfy. This structure is that of a factorization algebra; this is a C^∞ analog of the notion of chiral algebra introduced by Beilinson and Drinfeld [1].
(3) A quantization theorem. In a wide range of situations, the classical factorization algebra associated to a classical field theory admits a quantization. Further, the set of quantizations accessed by this theorem is parameterized (order by order in $\hbar$) by the space of deformations of the Lagrangian describing the classical theory.

    This quantization theorem applies to examples of physical interest, including pure Yang-Mills theory.

Finally, I explained how (under certain restrictions) the factorization algebra associated to perturbative quantum field theory encodes the correlation functions of the theory.

0.1. The definition of a factorization algebra – which is the structure present on the set of observables of a quantum field theory – is rather straightforward to give.

Definition 1. *Let M be a manifold of dimension n. A factorization algebra $\mathcal{F}$ on M consists of the following data.*

(1) *For every connected open set $U \subset M$, a cochain complex of topological vector spaces, $\mathcal{F}(U)$.*
(2) *If $B_1, \dots, B_k$ are disjoint balls in M, all contained in a larger ball B_{k+1}, a continuous map*

$$\mathcal{F}(B_1) \otimes \cdots \otimes \mathcal{F}(B_k) \to \mathcal{F}(B_{k+1})$$

(where we use the completed projective tensor product).
(3) *These maps must satisfy an evident compatibility condition.*

Clearly, this definition is reminiscent of the definition of an E_n algebra. In fact, E_n algebras are equivalent to a special class of factorization algebras on $\mathbb{R}^n$. This definition is an analog of the definition of chiral algebra given by Beilinson and Drinfeld [1]. Chiral algebras are a geometric way of encoding the axioms of a vertex algebra.

The observables of a quantum field theory on M should form a factorization algebra. If $B \subset M$ is a ball, the space $\mathcal{F}(B)$ should be thought of as the space of measurements one can make in the region B of the space-time manifold M.

0.2. Factorization algebras form a symmetric monoidal category: if $\mathcal{F}, \mathcal{G}$ are factorization algebras, then we can define a factorization algebra $\mathcal{F} \otimes \mathcal{G}$ by the formula

$$(\mathcal{F} \otimes \mathcal{G})(B) = \mathcal{F}(B) \otimes \mathcal{G}(B)$$

for a ball $B \subset M$.

Definition 2. *A* commutative factorization algebra *is a commutative algebra in the category of factorization algebras.*

0.3. Suppose we have a classical field theory on M. We are working perturbatively, so we can assume that the space of fields is the space of sections of some graded vector bundle E on M. We will suppose that we are given a classical action, which is a local functional

$$S : \Gamma(M, E) \to \mathbb{R}.$$

satisfying some conditions detailed in [2].

The main object of interest in a classical field theory is the space of solutions to the Euler-Lagrange equation. If $U \subset M$ is an open set, let $\mathcal{EL}(U)$ be the formal space of sections of E on U which satisfy the Euler-Lagrange equations, and which are infinitesimally near the zero section. Sending $U \mapsto \mathcal{EL}(U)$ defines a sheaf of formal spaces on M.

This sheaf of solutions to the Euler-Lagrange equations can be encoded in the structure of a commutative factorization algebra. If $B \subset M$ is a ball, we will let $\mathcal{O}(\mathcal{EL}(B))$ denote the space of functions on $\mathcal{EL}(B)$.

Sending $B \mapsto \mathcal{O}(\mathcal{EL}(B))$ defines a commutative factorization algebra: given disjoint balls $B_1, \dots, B_n$ in a ball B_{n+1}, there is a restriction map

$$\mathcal{EL}(B_{n+1}) \to \mathcal{EL}(B_1) \times \cdots \times \mathcal{EL}(B_n).$$

Replacing the map of spaces by the corresponding map of algebras of functions yields the desired structure of commutative factorization algebra.

0.4. So far we have described the QFT analogs of commutative and associative algebras (namely, commutative factorization algebras and plain factorization algebras). It remains to describe the analog of Poisson algebras, and to state the quantization theorem.

Poisson algebras interpolate between commutative algebras and associative (or E_1) algebras. For us, the object describing the observables of a quantum field theory is not an E_1 in a symmetric monoidal category; instead, it is an E_0 algebra. An E_0 algebra in vector spaces is simply a vector space with an element; an E_0

algebra in any symmetric monoidal category is an object of this category with a map from the unit object. An E_0 algebra in the category of factorization algebras is simply a factorization algebra, as every factorization algebra has a unit.

Thus, the analog of the Poisson operad we are searching for is an operad that interpolates between the commutative operad and the E_0 operad. Such an operad was constructed by Beilinson and Drinfeld [1]; we will call it the BD operad.

Definition 3. *Let P_0 be the operad generated by a commutative and associatve product, $*$, and a Poisson bracket $\{-,-\}$.*

Let BD denote the differential graded operad over the ring $\mathbb{R}[[\hbar]]$ which, as a graded operad, is simply $P_0 \otimes \mathbb{R}[[\hbar]]$, but which is equipped with differential

$$\mathrm{d}* = \hbar\{-,-\}.$$

Modulo $\hbar$, the operad BD is the same as the operad P_0. However, if we invert $\hbar$, we find that

$$BD \otimes_{\mathbb{R}[[\hbar]]} \mathbb{R}((\hbar)) \simeq E_0 \otimes_{\mathbb{R}} \mathbb{R}((\hbar)).$$

Thus, we find that the operad P_0 bears the same relationship to the operad E_0 as the usual Poisson operad bears to the associative operad E_1.

Definition 4. *A* quantization *of a P_0 algebra V^{cl} (in some symmetric monoidal category) is a BD algebra V^q and an isomorphism of P_0 algebras $V^q/\hbar \simeq V^{cl}$.*

This definition is analogous to the definition of quantization of a Poisson algebra into an associative algebra.

0.5. The analog, in the world of QFT, of the structure of Poisson algebra, is a P_0 algebra in the category of factorization algebras on M (we will just call this a P_0 factorization algebra on M). Thus, we would expect that a classical field theory on M yields a P_0 factorization algebra on M.

One general source of P_0 algebras is the following lemma.

Lemma 1. *The algebra of functions on the derived critical locus of a function on a finite dimensional scheme (or stack) is a P_0 algebra.*

The space of solutions to the Euler-Lagrange equation is, of course, the critical locus of the action function on the infinite dimensional space of fields. This suggests that functions on the derived space of solutions to the Euler-Lagrange equation is a P_0 algebra. This is indeed the case:

Theorem 1. *Suppose we have a classical field theory on M. If $B \subset M$ is a ball, let $\mathcal{O}(\mathcal{EL}^h(B))$ denote the differential graded commutative algebra of functions on the derived space of solutions to the Euler-Lagrange equation. Then, $\mathcal{O}(\mathcal{EL}^h(B))$ has a canonical structure of P_0 algebra, compatible with the factorization structure.*

Thus, $\mathcal{O}(\mathcal{EL}^h(B))$ defines a P_0 algebra in the category of factorization algebras on M. We will denote this P_0 factorization algebra, associated to the classical field theory with action S, by $\mathcal{F}^{cl}_S$.

0.6. Now we have the definition of the algebraic structure associated to classical field theory, one can ask whether there is a quantization. The following theorem is proved using the effective field theory techniques of [2].

Theorem 2. *Let us fix a classical theory on M, with associated P_0 factorization algebra $\mathcal{F}^{cl}_S$.*

Let $\mathcal{Q}^{(n)}$ describe the simplicial set of possible quantizations of $\mathcal{F}^c l$ to a BD algebra in factorization algebras, up to order $\hbar^{n+1}$.

Then, there is a simplicial set $\mathcal{T}^{(n)}$ which fits into a commutative diagram

$$\begin{tikzcd} \mathcal{T}^{(n)} \arrow[r] \arrow[d] & \mathcal{Q}^{(n)} \arrow[d] \\ \mathcal{T}^{(n-1)} \arrow[r] & \mathcal{Q}^{(n-1)} \end{tikzcd}$$

such that $\mathcal{T}^{(n)}$ is a torsor over $\mathcal{T}^{(n-1)}$ for the simplicial abelian group of deformations of the local functional describing the classical theory.

This theorem allows us to quantize any classical theory using obstruction theory. Obstructions lie in H^1 of the cochain complex of deformations of the local functional describing the classical theory; deformations lie in H^0.

This theorem, together with results of [2], can be applied to produce a quantization of the classical factorization algebra associated to pure Yang-Mills theory on a flat manifold.

References

[1] A. Beilinson and V. Drinfeld, *Chiral algebras*, volume 51 of *American Mathematical Society Colloquium Publications*, American Mathematical Society, Providence, RI, 2004.
[2] K. Costello, *Renormalization and effective field theory*, (2009).

Differential cohomology theories

Thomas Schick
(joint work with Ulrich Bunke)

Given an ordinary cohomology theory E, a differential refinement $\hat{E}$ of E is a functor from the category of smooth manifolds to the category of graded abelian groups which includes information about $E(M)$ together with differential form information for de Rham representatives of the images of classes in $E(M)$ in $E(M)\otimes\mathbb{R}$. The classical example is Deligne cohomology (or equivalently Cheeger-Simons differential characters [5]) for ordinary integral cohomology.

This is motivated from considerations in string theory as presented by Dan Freed [6].

We give a precise axiomatic definition of such a smooth refinement, also including axioms for multiplicative version and for the appropriate notion of suspension in this context [2].

An abstract construction of such an extension (as functor with values in abelian groups) is given by Hopkins-Singer [7].

In the talk, we present geometric constructions of such differential refinements (with product and suspension) for K-theory [2], bordism theories and Landweber exact cohomology theories [4], and ordinary cohomology [1].

We present a uniqueness theorem for such refinements [3]; stating that any two such differential cohomology refinements are naturally isomorphic with a unique isomorphism, provided $E^k(pt) \otimes \mathbb{Q}$ is trivial for k odd, and provided they satisfy the suspension relation required above. The transformation is automatically a transformation of ring theories if one deals with multiplicative cohomology theories. On the other hand, we show that there are exotic abelian group structures (on smooth K-theory) without the suspension compatibility.

References

[1] Ulrich Bunke, Matthias Kreck and Thomas Schick. A geometric description of smooth cohomology arXiv:0903.5290

[2] Ulrich Bunke and Thomas Schick. Smooth K-theory. arXiv:0707.0046, 2007, to appear in Asterisque.

[3] Ulrich Bunke, Thomas Schick. Uniqueness of smooth extensions of generalized cohomology theories. arXiv:0901.4423, 2009.

[4] Ulrich Bunke, Thomas Schick, Ingo Schröder, and Moritz Wiethaup. Landweber exact formal group laws and smooth cohomology theories. arXiv:0711.1134, 2007, to appear in Algebraic and Geometric Topology.

[5] Jeff Cheeger and James Simons. Differential characters and geometric invariants. In *Notes of Stanford conference 1973*, volume 1167 of *Lecture Notes in Math.*, pages 50–49. Springer Verlag, New York, 1985.

[6] Daniel S. Freed. Dirac charge quantization and generalized differential cohomology. In *Surveys in differential geometry*, Surv. Differ. Geom., VII, pages 129–194. Int. Press, Somerville, MA, 2000.

[7] M. J. Hopkins and I. M. Singer. Quadratic functions in geometry, topology, and M-theory. *J. Differential Geom.*, 70(3):329–452, 2005.

Smooth K-theory

Ulrich Bunke

(joint work with Thomas Schick)

This talk was a report on the model of smooth K-theory [3] and related constructions [3]. In [3] smooth K-theory has been constructed in terms of cycles and relations. A cycle is a pair of a geometric family and a differential form. A cycle is trivial if the associated Dirac operator has an invertible perturbation so that the associated η-form is equal to the differential form of the cycle.

In the case of smooth K-theory the results of [2] apply so that our model gives the unique smooth extension of K-theory which in addition has a unique multiplicative structure. The geometric model furthermore allows the construction of an orientation structure which comprises the notion of a smooth K-orientation and the corresponding integration maps.

Riemann-Roch theorems describe the compatibility of integration maps with natural transformations. The prototypical example concerns the Chern character. Its smooth generalisation has been shown in [3].

Adams operations are operations on K-theory. In [1] was shown that they have unique refinements to smooth K-theory. The Riemann-Roch theorem for the Adams operations involves the cannibalistic class of the $Spin^c$-normal bundle. This K-theoretic characteristic class plays the same role as the $\hat{A}$-class in the classical index theorem. In [1], given a smooth K-orientation, a refinement of the cannibalistic class in smooth K-theory was constructed. The central result in that paper is that the smooth refinement of the Riemann-Roch theorem for the Adams operations holds true. This result has interesting applications e.g. to the structure of e-invariants for families of framed manifolds [1].

We refer to the literature cited below for a discussion of related work by other authors.

References

[1] U. Bunke, *Adams operations in smooth K-theory*, arXiv.org:0904.4355, 2009.

[2] U. Bunke and Th. Schick, *Uniqueness of smooth extensions of generalised cohomology theories*, arXiv.org:0901.4423, 2009.

[3] U. Bunke and Th. Schick *Smooth K-Theory*, arXiv.org:0707.0046., 2007.

Superconnections and Index Theory

Alexander Kahle

It is natural to consider studying the index theory of Dirac operators coupled to superconnections. In my talk I describe the results in [8], highlighting the main theorems and constructions, and sketching some proofs.

Superconnections were introduced to mathematics by Daniel Quillen [12] as a generalisation of the notion of a connection to the category of $\mathbb{Z}/2\mathbb{Z}$-graded vector bundles. A superconnection on a $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $V \to X$ is an odd derivation on $\Omega^\bullet(X; V)$. Concretely, every superconnection $\boldsymbol{\nabla}$ may be written

$$\boldsymbol{\nabla} = \nabla + \sum_i \omega_i, \tag{1}$$

where ∇ is a connection on V, and each ω_i is an odd element of $\Omega^i(X; \mathrm{End}(V))$.

One may form Dirac operators out of superconnections in a manner entirely analogous to the construction of Dirac operators from ordinary connections. The Dirac operator associated to a hermitian $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $V \to X$ with a superconnection $\boldsymbol{\nabla}$, where X is a spin and Riemannian manifold, is defined by the following sequence:

(2)

$$\mathcal{D}(\boldsymbol{\nabla}) : \Gamma\left(\mathbb{S}(X) \otimes V\right) \xrightarrow{(\nabla^{\mathbb{S}} \otimes \mathbf{1}) \oplus (\mathbf{1} \oplus \boldsymbol{\nabla})} \Omega^\bullet(X; \mathbb{S}(X) \otimes V) \xrightarrow{c(\cdot)} \Gamma\left(\mathbb{S}(X) \otimes V\right),$$

where here $\mathbb{S}(X)$ is the bundle of $\mathbb{Z}/2\mathbb{Z}$-graded spinors on X, $\nabla^{\mathbb{S}}$ is the Levi-Civita connection and $c(\cdot)$ denotes Clifford multiplication. We will take X to be compact and $\boldsymbol{\nabla}$ unitary in the appropriate sense. Under these conditions $\mathcal{D}(\boldsymbol{\nabla})$ is a formally self-adjoint first-order elliptic differential operator. In terms of the grading on $\Gamma(\mathbb{S} \otimes V)$, the Dirac operator decomposes as

$$\mathcal{D}(\boldsymbol{\nabla}) = \begin{pmatrix} & \not{D}(\boldsymbol{\nabla})^* \\ \not{D}(\boldsymbol{\nabla}) & \end{pmatrix}.$$

One should think of $\not{D}(\boldsymbol{\nabla})$ as being analogous to the classical Dirac operator.

The most basic question one may ask in studying the index theory of these operators is to compute their index. This is done easily as a corollary of the Atiyah-Singer index theorem [2]:

$$\operatorname{index} \not{D}(\boldsymbol{\nabla}) = (2\pi i)^{-\dim X/2} \int_X \hat{A}\left(\Omega^X\right) \operatorname{ch}(\boldsymbol{\nabla}).$$

This follows as the superconnection $\boldsymbol{\nabla}$ may be homotoped to a connection (by sending all the forms to zero), thus reducing to the usual Atiyah-Singer index theorem for Dirac operators coupled to connections. Any novelty, therefore, introduced by considering Dirac operators coupled to superconnections must lie at the level of geometry. In fact the key new features already appear when formulating a local index theorem in this setting. We recall that the original theorem [1, 7, 10] computes the $t \to 0$ limit of the supertrace heat kernel associated to a Dirac operator in terms of canonical differential forms associated to the geometry. This may lead one to expect a theorem for superconnections to read[1]

$$\lim_{t\to 0} \operatorname{Tr} p_t(x,x)\, \mathrm{d}x \overset{?}{=} (2\pi i)^{-\dim X/2} \left[\hat{A}(\Omega^X)\operatorname{ch}(\boldsymbol{\nabla})\right]_{(\dim X)}, \tag{3}$$

where here $p_t(x,y)$ is the integral kernel associated to the heat semigroup $e^{-t\mathcal{D}(\boldsymbol{\nabla})^2}$.

Unfortunately, the limit in Eq. 3 does not even converge! To understand where the problem arises, we recall that to prove the local index theorem for ordinary Dirac operators, one examines the behaviour of the heat kernel under dilation of X – superconnections however have terms in many cohomogical degrees, and Dirac operators coupled to them do not behave homogeneously under dilation. To take account of this inhomogeneity, we introduce an $\mathbb{R}^\times$-action on superconnections:

$$\boldsymbol{\nabla}^s = \nabla + \sum_i |s|^{(1-i)/2}\omega_i, \tag{4}$$

where $\omega_i \in \Omega^i(X; \operatorname{End}(V))$. The local index theorem then reads:

Theorem 1 (Getzler [6])**.** *Let X be a compact, spin and Riemannian manifold, with finite dimensional complex and hermitian $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $V \to X$ with unitary superconnection $\boldsymbol{\nabla}$. Then*

$$\lim_{t\to 0} \operatorname{Tr} p_{t,1/t}(x,x)\, \mathrm{d}x = (2\pi i)^{-\dim X/2} \left[\hat{A}(\Omega^X)\operatorname{ch}\boldsymbol{\nabla}\right]_{(\dim X)},$$

where $p_{t,s}(x,y)$ is the integral kernel associated to $\exp[-t\mathcal{D}(\boldsymbol{\nabla}^s)^2]$.

[1]In this note “Tr” always refers to the supertrace.

Getzler proved the theorem using stochastic techniques. In [8] we provide a new proof using only analytic methods.

One may enrich the local index theorem to a families index theorem. The good notion of "family" for Riemannian geometry is the so-called Riemannian submersion, given by a triple $(\pi : X \to Y, g^{X/Y}, P)$, where π is a smooth submersion, $g^{X/Y}$ is a metric on the vertical tangent bundle $T(X/Y)$, and $P : T(X) \to T(X/Y)$ is a projection. We require that the fibres be spin and compact. A hermitian $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle with unitary connection $(V, \boldsymbol{\nabla}) \to X$ then gives a family of Dirac operators parameterised by Y, constructed by applying the construction in Eq. 2 fibrewise. The families index theorem expresses the index of this family as an element of $K(Y)$. Bismut [3] constructs an explicit (infinite dimensional) $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle with superconnection representing this element for families of Dirac operators coupled to ordinary connections; we extend his construction to families of Dirac operators coupled to superconnections. In this case the Chern character of the pushed forward superconnection $\pi_! \boldsymbol{\nabla}$ is computed by

$$\lim_{t\to 0} \operatorname{ch} \pi_!^t \boldsymbol{\nabla} = (2\pi i)^{-\frac{1}{2}\dim X/Y} \pi_* \hat{A}(\Omega^\pi) \operatorname{ch}(\boldsymbol{\nabla}), \tag{5}$$

where Ω^π is the curvature of the Riemannian map (the family analogue of the Riemannian curvature) and π^t is the re-scaled Riemannian map $(\pi, |t|^{-1} g^{X/Y}, P)$. The $\mathbb{R}^\times$-action on superconnections is implicit in the rescaling of π: one computes that $\pi_!^t \boldsymbol{\nabla} = (\pi_! \boldsymbol{\nabla}^{1/t})^t$. Again the $\mathbb{R}^\times$-action on superconnections is crucial for the convergence of the limit.

Bismut and Freed [4, 5] (following work of Quillen [11]) showed how one may construct a geometric determinant line bundle (i.e. a hermitian line bundle with connection, with a section that represents the determinant) associated to a family of Dirac operators. We extend their construction to families of Dirac operators coupled to superconnections. Again, the $\mathbb{R}^\times$-action (Eq. 4) plays a crucial role: its presence in Th. 1 makes it essential to consider the entire family of Dirac operators $\mathcal{D}(\boldsymbol{\nabla}^s)$. These do not in general commute, and thus cannot be simultaneously diagonalised. One is thus lead to a dichotomy: one may either construct the determinant line bundle using the usual spectral definition and forsake geometric interpretations of its curvature and holonomy[2] or one must sacrifice a natural "spectral" definition interpretation of the determinant line bundle in favour of obtaining a geometric line bundle with curvature and holonomy given by natural index theoretic quantities. We choose the latter course, and construct a canonical geometric line bundle associated to a family of Dirac operators coupled to superconnections with curvature computed by

$$(2\pi i)^{-\frac{1}{2}\dim X/Y} \left[\pi_* \hat{A}(\Omega^\pi) \operatorname{ch}(\boldsymbol{\nabla})\right]_{(2)},$$

and holonomy computed by the appropriate η-invariant mod $\mathbb{Z}$, where the notation is as in Eq. 5.

[2]This construction is in fact done for a wider class of operators in [4, 5].

In order to construct the determinant line bundle we also briefly investigate the η-invariant for Dirac operators coupled to superconnections. Here again one encounters an essential dichotomy between emphasising the spectral or geometric nature of the original invariant. We choose to preserve the geometric aspects of the invariant (in this case, that it obey an APS-theorem).

To summarise: the essential novelty that pervades the index theory of Dirac operators coupled to superconnections is the presence of the $\mathbb{R}^{\times}$-action (Eq. 4). It presents numerous technical difficulties, both in the proof of the index theorems, and in the definition of secondary invariants and constructions associated to these operators. However, taken properly into account the basic theorems and formulas in the geometric index theory of Dirac operators coupled to superconnections end up very similar to those in classical geometric index theory.

References

[1] M. Atiyah, R. Bott and V. Patodi, *On the heat equation and the index theorem*, Invent. Math. **19** (1973), 279–330.
[2] M. Atiyah and I. Singer, *The Index of Elliptic Operators I*, Ann. Math. **87** (1968), 484-530.
[3] J.-M. Bismut, *The Atiyah-Singer index theorem for families of Dirac operators: two heat equation proofs*, Inv. Mathematicae **83** (1989), no.92, 91–151.
[4] J.-M. Bismut and D. Freed, *The analysis of elliptic families I: Metrics and connections*, Comm. Math. Phys. **106** (1986), 159–176.
[5] J.-M. Bismut and D. Freed, *The analysis of elliptic families II: Dirac operators, eta invariants, and the holonomy theorem of Witten*, Comm. Math. Phys. **107** (1986), 103–163.
[6] E. Getzler, *The Bargmann representation, generalized Dirac operators and the index of pseudodifferential operators on* $\mathbb{R}^n$, Symplectic Geometry and Quantization, Contemp. Math., vol. 179, Amer. Math. Soc., Providence RI (1994), pp. 61–81.
[7] P. Gilkey, *Curvature and the eigenvalues of the Dolbeault complex of Kaehler manifolds*, Advances in Math. 11 (1973), 311–325.
[8] A. Kahle, *Superconnections and Index Theory*, `arXiv:0810.0820`.
[9] H. McKean, and I. Singer, *Curvature and the eigenvalues of the laplacian*, J. Diff. Geom. **1** (1967), 43–69.
[10] V. Patodi, *Curvature and the eigenforms of the Laplace operator*, J. Diff. Geom. **5** (1971), 233–249.
[11] D. Quillen, *Determinants of Cauchy-Riemann operators over a Riemann surface*, Funct. Anal. Appl. **19** (1985), 31-34.
[12] D. Quillen, *Superconnections and the Chern Character*, Topology **24** (1985), 89–95.

∞-operads, $\mathcal{BV}_\infty$, and Hypercommutative$_\infty$

Gabriel C. Drummond-Cole
(joint work with Bruno Vallette)

Fix a ground field $\mathbf{k}$ of characteristic zero.

Theorem 1. [1, 6] *Let $(V, \cdot, \partial, \Delta)$ be a differential BV algebra which satisfies the "∂-$\bar{\partial}$" lemma. There exists the structure of a hypercommutative algebra on the homology $H(V, \partial)$.*

This theorem was generalized by Park [8], who weakened the "∂-$\bar{\partial}$" condition to a different one called "semiclassical". The "semiclassical" condition was then weakened further to "noncommutative Hodge to de Rham degeneration" by Terilla [9] (see also [5]). The construction was opaque and not clearly functorial. Our work illustrates an extension and illumination of the theorem above, and points the way to filling in the following chart, which schematically describes this passage:

Algebra	differential BV algebra	Hypercommutative algebra
Topology	Framed little disks (Version of genus zero moduli space $\mathcal{M}_{0,n}$)	Deligne-Mumford compactification of genus zero moduli space ($\overline{\mathcal{M}}_{0,n}$)
Geometry	?	Linear family of formal flat connections

There is a model category structure on operads which can be used to illuminate the homotopy theory of algebras over different operads, c.f. [7, 2]. Namely, the structure of an algebra over a cofibrant operad can be transferred across a quasiisomorphism; this is not true for operads in general. However, every operad $\mathcal{O}$ admits a cofibrant replacement $\mathcal{O}_\infty \xrightarrow{\sim} \mathcal{O}$, and so an $\mathcal{O}$-algebra structure can be transferred "up to homotopy" as a $\mathcal{O}_\infty$ structure. For instance, given an associative algebra V and a linear quasiisomorphism $f : V \to W$, it is not always possible to put an associative algebra structure on W which realizes f as a map of algebras. However, W can be given an A_∞ structure which realizes an extension of f as a quasiisomorphism of A_∞ algebras.

This principle is very useful, but also abstract. It would be nice to be able to realize this with a construction of a cofibrant replacement $\mathcal{O}_\infty \xrightarrow{\sim} \mathcal{O}$. One way to obtain such a replacement is through the composition of the bar and cobar functors between the categories of operads and cooperads:

$$\mathbf{B} : \mathbf{Op} \to \mathbf{coOp}$$

$$\mathbf{\Omega} : \mathbf{coOp} \to \mathbf{Op}_{\text{cofibrant}}$$

There is a canonical map

$$\mathbf{\Omega}\mathbf{B}\mathcal{O} \xrightarrow{\sim} \mathcal{O}$$

The "cobar-bar" resolution gives a particular realization of $\mathcal{O}_\infty$, but in practice this particular choice can be unwieldy. In order to find a smaller resolution, it would be nice to replace $\mathbf{B}\mathcal{O}$, which is an operad in chain complexes, with a smaller, quasiisomorphic complex, such as its homology. Then we could take

$$\mathcal{O}_\infty = \mathbf{\Omega} H\mathbf{B}\mathcal{O} \xrightarrow{\sim} \mathcal{O}$$

Unfortunately, in general, this runs into the same problem that exists with algebras: namely, the structure of a cooperad does not transfer nicely across quasiisomorphism. However, again like algebras we have a transfer theorem if we are willing to consider ∞-cooperads, that is, cooperads up to homotopy.

Theorem 2. *Let $\mathcal{C}$ be a coaugmented cooperad in the category of chain complexes.*

(1) *$H(\mathcal{C})$ carries the transferred structure of an ∞-cooperad which is quasiisomorphic as ∞-cooperads to the strict cooperad structure on $\mathcal{C}$.*
(2) *There exists a functor $\mathbf{\Omega}_\infty : \mathbf{coOp}_\infty \to \mathbf{Op}_{cofibrant}$ that respects quasiisomorphisms and that coincides up to quasiisomorphism with $\mathbf{\Omega}$ on strict cooperads.*

These theorems fit a familiar pattern but the explicit formulae are new and have been very useful in the work being reported.

Getzler [4] showed that the hypercommutative operad has the nice formality property, namely that $\mathbf{B}\mathcal{H}yc$ is quasiisomorphic as a strict cooperad to its homology, called $co\mathcal{G}rav$. So we get a small presentation of $\mathcal{H}yc_\infty$ as $\mathbf{\Omega} co\mathcal{G}rav$.

For the BV operad, $\mathbf{B}\mathcal{BV}$ is not quasiisomorphic as a strict operad to $H\mathbf{B}\mathcal{BV}$, but according to Theorem 2, we have a presentation of $\mathcal{BV}_\infty$ as $\mathbf{\Omega}_\infty H\mathbf{B}\mathcal{BV}$. Using recent work by Gálvez-Carillo, Tonks, and Vallette [3], we have proved:

Theorem 3. (1) *Viewed as a strict cooperad, $H(\mathbf{B}\mathcal{BV}) \cong \overline{\mathbf{k}[\delta]} \oplus co\mathcal{G}rav$.*
(2) *Viewed as ∞-cooperads, $H(\mathbf{B}\mathcal{BV})/\overline{\mathbf{k}[\delta]} \to co\mathcal{G}rav$ is an isomorphism.*

An analysis of the higher infinity cooperad structure on $H\mathbf{B}\mathcal{BV}$ relates the role that $\mathbf{k}[\delta]$ plays to the "noncommutative Hodge to de Rham degeneration" and Park's "semiclassical" condition and yields the following:

Theorem 4. (1) *Let $(V, \cdot, \partial, \Delta)$ be a differential BV algebra which satisfies the "noncommutative Hodge to de Rham degeneration" property. Then there exists the structure of a $\mathcal{H}yc_\infty$ algebra on the homology $H(V, \partial)$.*
(2) *If V is "semiclassical" then the structure is unique. In particular, there is a functor from the category of semiclassical differential BV algebras with "semiclassical $\mathcal{BV}_\infty$ morphisms" to the category of $\mathcal{H}yc_\infty$ algebras.*

Remark 1. *One can apply the homology functor to the $\mathcal{H}yc_\infty$ structure on $H(V, \partial)$ to recover the strict hypercommutative algebra of* [1].

By keeping the full $\mathcal{H}yc_\infty$ structure and not compressing to the strict algebra, the corollary yields higher level invariants, which correspond (in part) in the geometric world to extending a connection to a superconnection. This constitutes a hint as to what fills in the question mark in the chart above.

References

[1] S. Barannikov & M. Kontsevich, *Frobenius Manifolds and Formality of Lie Algebras of Polyvector Fields*, Internat. Math. Res. Notices (1998), 201–215
[2] C. Berger & I. Moerdijk, *Axiomatic homotopy theory for operads*, Comment. Math. Helv. **78** (2003), no. 4, 805–831
[3] I. Gálvez-Carillo, A. Tonks, & B. Vallette, *Homotopy Batalin-Vilkovisky Algebras*, preprint (2009), available at http://math.unice.fr/∼brunov/publications/HomotopyBValgebras.pdf
[4] E. Getzler, *Operads and Moduli Spaces of Genus 0 Riemann Surfaces* in *The Moduli Space of Curves (Texel Island, Netherlands, 1994)*, Progr. Math. **129**, Birkhäuser, Boston (1995), 199–230
[5] L. Katzarkov, M. Kontsevich, & T. Pantev, *Hodge theoretic aspects of mirror symmetry*, preprint (2008), available at http://arxiv.org/abs/0806.0107

[6] Y. Manin, *Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces*, Amer. Math. Soc., Providence, RI (1999)
[7] M. Markl, *Models for Operads*, Comm. Algebra **24** (1996), no. 4, 1471-1500
[8] J. Park, *Semi-Classical Quantum Fields Theories and Frobenius Manifolds*, Letters in Math. Physics **81** (2007), 41-59
[9] J. Terilla, *Smoothness Theorem for Differential BV Algebras*, Journal of Topology **1** (2008), 693–702.

Some algebra and applications related to mapping spaces

Scott O. Wilson
(joint work with Thomas Tradler, Mahmoud Zeinalian)

In this talk we begin with a useful language for some elementary concepts in algebraic topology, and then show how these can be used to define generalizations of Hochschild homology. We also give some applications to invariants and constructions that appear in settings such as Chern characters, Witten deformations, and PDE's related to fluids.

Recall that a differential graded algebra is precisely a strict monoidal functor from the category of finite sets (denoted $\mathcal{F}$) to the category of chain complexes (denoted Ch). We can generalize this definition by asking for a weak functor, i.e. one that is monoidal only up to a coherent natural transformation. We'll refer to these as *partial algebras*.

The meaning of such functors is illuminated by a theorem proved by the author: such partial algebras can be functorially rectified to E_∞-algebras. With this in mind, though, one may prefer to deal with the apparently small package of a partial algebra.

There are versions of this functor approach in many other settings: modules over algebras, algebras over any operad, and their modules, co-versions of all of these, etc.

Examples of these structures are abundant. The author has proved the following conjecture of J. McClure: the chains of a PL space form a partial co-algebra, where the structure maps are generalized diagonal maps and the natural transformation is given by the cartesian product of chains. An appropriate dual of this gives a partial algebra on cochains. By a theorem of Mandell, it's reasonable that this tidy package determines the integral homotopy type of a nilpotent space.

Now, for any partial algebra $A : \mathcal{F} \to Ch$, and any finite simplicial set $Y : \Delta \to \mathcal{F}$, we obtain by composition a simplicial object of chain complexes, whose total complex[1] we denote by $CH^Y(A)$. In fact, this forms a partial algebra itself. And there are module versions, etc.

This construction generalizes the Hochschild complex of an algebra and the higher Hochschild complexes of Pirashvilli [4], [2]. More recently Ginot, Tradler and Zeinalian have shown that for the algebra A of differential forms on X there

[1]This construction, defined for any simplicial set Y and partial algebra A, should also be related to K. Walker's *Blob Homology*, which is defined for (at least) any manifold M and category C. See his abstract in this report.

is, for any Y, an iterated integral map yielding a quasi-isomorphism from $CH^Y(A)$ to the forms on $Map(Y,X)$ (assuming certain connectivity hypothesese) [3]. The product in the domain is identified with a shuffle product and corresponds to the cup product on the mapping space.

For the case for $Y = S^1$, and (A,d) a strict dga, $CH^Y(A)$ is the usual Hochschild complex of A with differential D. The existence of the shuffle product, $*$, implies the exponential map is defined, and we can compute

$$e^{1\otimes x} = 1 + 1\otimes x + 1\otimes x\otimes x + 1\otimes x\otimes x\otimes x + \cdots$$

Furthermore,

$$De^{1\otimes x} = (1\otimes (dx + x^2)) * e^{1\otimes x}$$

This implies that Maurer-Cartan elements of A give cycles on $CH^{S^1}(A)$ and, if we imagine A as matrices of forms on M, it reminds us of the formula for curvature of a connection.

This analogy has been taken further in Getzler, Jones, and Petrack [1] by constructing, from a bundle with connection, a closed equivariant form in cyclic chains agreeing with Bismut's analytic construction, which has the property that, upon restriction to the constant loops $M \subset LM$, it gives the classical Chern character. We are working now to similarly construct a cycle in $CH^{S^1\times S^1}$ which restricts appropriately to the class above and satisfies an equivariance condition[2].

Another interesting example is given by the path space, i.e. $Y = I$ is the interval. For $A = \Omega(M)$ there is a differential D on $CH^I(A)$ induced by the standard action of A on itself (on the left and right).

For M Riemannian, the Hodge-star operator $\star$ induces a *dual module structure* given by $(x,y) \to \star^{-1}(x\wedge\star y)$, making (A,d^*) into a differential module over (A,d). Thus, on the same underlying vector space of $CH^I(A)$, we obtain a differential D^* corresponding to the usual right action and the dual left action. Clearly D^* is given by the transport of D by $id\otimes\cdots\otimes id\otimes\star$, so D^* is the formal adjoint of D. We call $\Delta = [D, D^*]$ the *Laplacian on the path space* and note that it has square root $D + D^*$.

For $x \in A$ of degree 1 and $s \in \mathbb{R}$ we compute

$$\Delta(e^{1\otimes s\cdot x\otimes 1}\cdot y) = e^{1\otimes s\cdot x\otimes 1}\cdot D^2_{x,s}(y) \tag{1}$$

where, letting L_x denote left multiplication by x, and $L_x^* = \star^{-1}L_x\star$ denote its adjoint, we have

$$D_{x,s} = d + d^* + sL_x + sL_x^*$$

This is the deformation of $d + d^*$ considered by Witten in [5], which can be used to prove the Poincaré-Hopf Index formula. It would be interesting to understand further properties of the operators $D + D^*$ and Δ on the path space, as well as their analogues defined on algebraic models of maps into a Riemannian manifold.

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[2] Some conference participants suggested that the data of gerbes with connections may be a more appropriate setting for this construction.

An interesting special case of (1) appears when we set $y = x$ and assume x is divergence free, $d^*x = 0$. Using $\star^{-1}(x \wedge \star x) = \|x\|^2$ we obtain

$$D^2_{x,s}(x) = d^*dx + s\left(\star^{-1}(x \wedge \star dx) + d\|x\|^2 + dx \wedge x\right) + s^2\|x\|^2 x$$

The *self-linking term* $dx \wedge x$ vanishes in dimension two and lower, though may be non-trivial in dimensions three and higher. The remaining terms are degree one, and modulo the s^2 term, can be seen in the Navier-Stokes equation for viscosity equal to one:

$$\dot{x} = \star^{-1}(x \wedge \star dx) + d\|x\|^2 + d^*dx + dp$$

Here x is now a time dependent 1-form (vector field) and the pressure p is determined uniquely (up to a constant) by the Hodge decomposition.

It may be fruitful to understand further connections between deformations of the Laplacian and non-linear PDE's such as this fluid equation.

References

[1] E. Getzler, J. Jones, Petrack, *Differential forms on loop spaces and the cyclic bar complex* Topology 30 (1991), no. 3, 339–371.

[2] Grégory Ginot, *Higher order Hochschild Cohomology*, C. R. Acad. Sci. Paris, Ser. I 340 (2005).

[3] G. Ginot, T. Tradler, M. Zeinalian, *A Chen model for mapping spaces and the surface product*, arXiv:0905.2231.

[4] Teimuraz Pirashvili, *Hodge Decomposition for higher order Hochschild Homology*, Ann. Scient. Éc. Norm. Sup., 4e série, t. 33 (2000) p. 151-179.

[5] E. Witten, *Supersymmetry and Morse theory*, J. Diff. Geom. 17 (1982), 661-692.

Topological Defects, *D*-Branes, and the Classification of Local TFTs in Low Dimensions

Christopher Schommer-Pries

Recently there have been many new and exciting developments relating higher categories and topology, particularly in the area of local (i.e. extended) topological field theory. In this talk we show how the higher-categorical point of view organizes and relates several existing notions in topological quantum field theory.

One particularly important recent development is the formulation and proof, due to M. Hopkins and J. Lurie [2], of the Baez-Dolan cobordism hypothesis [3]. Roughly speaking this characterizes the higher categorical bordism category in terms of an algebraic description. There are many variations of the bordism n-category and this algebraic description is the easiest to formulate in the case where bordism are equipped with framings. In this case the framed bordism n-category is the "free symmetric monoidal n-category generated by a single fully dualizable object".

As of the time of this talk, full details of the Hopkins and Lurie proof have not become publicly available. Instead we focus and an alternative concrete approach developed in the author's dissertation [1], and which is valid in low dimensions.

We discuss the results and techniques used to prove a generators and relations classification of local 2-dimensional topological field theories.

This explicit approach then leads naturally to a thorough understanding of natural transformations of field theories. Due to the duality present in the bordism bicategory, these natural transformations are automatically invertible. However, by altering the definition of natural transformation slightly (a notion we call *unnatural transformations*) we obtain a new concept which is similar to a natural transformation, but not necessarily invertible.

A careful and explicit examination shows that these *unnatural transformations* exactly reproduce the notions of topological defect encountered in the talks of C. Schweigert and I. Runkel. As a further specialization, an *unnatural transformation* between a theory and the trivial theory yields an open-closed field theory extending the original non-trivial one. This is also known as a topological D-brane. Thus the higher categorical setting provides a context in which all three of these notions can be seen as particular aspects of a unified concept.

References

[1] C. Schommer-Pries, *The Classification of Two-Dimensional Extended Topological Field Theories*, Ph. D. Dissertation (2009).

[2] J. Lurie, *On the Classification of Topological Field Theories*, preprint (2009).

[3] J. Baez and J. Dolan, *Higher-dimensional algebra and topological quantum field theory*, J. Math. Phys. 36 (1995), 6073-6105.

Invertible conformal nets

André Henriques

(joint work with A. Bartels, C. Douglas)

1. Conformal nets

We begin by quickly describing the symmetric monoidal 3-category $(\mathit{CN3}, \otimes)$ of conformal nets. The objects of $\mathit{CN3}$ are $\mathbb{Z}/2$-graded conformal nets. The arrows between two nets $\mathcal{A}$ and $\mathcal{B}$ are called $\mathcal{A}$-$\mathcal{B}$-defects. The 2-morphisms between $\mathcal{A}$-$\mathcal{B}$-defects D and E are called D-E-sectors. Finally, the 3-morphisms of $\mathit{CN3}$ are called homomorphisms of sectors. One of our main result is that $\mathit{CN3}$ is indeed a symmetric monoidal 3-category.

Before describing the objects of our 3-category $\mathit{CN3}$, we need a few facts about pin structures on one dimensional manifolds.

Definition 1.1. A *pin interval* is an interval equipped with a complex line bundle $S \to I$, and an isomorphism $S^{\otimes 2} \xrightarrow{\sim} T^*_{\mathbb{C}} I$ between the square of S and the complexified cotangent bundle of I.

An embedding between pin intervals (I', S') and (I, S) consists of an embedding $f : I' \hookrightarrow I$, along with an isomorphism $\beta : f^*S \to S'$. We allow β to be either $\mathbb{C}$-linear or $\mathbb{C}$-antilinear. If β is linear, then the first one of the following two diagrams

should commute. Otherwise, it is the second diagram that should commute:
(1)

$$\begin{tikzcd} f^*(S^{\otimes 2}) \arrow[r, "\sim"] \arrow[d, "\beta^{\otimes 2}"'] & f^*(T^*_{\mathbb{C}}I) \arrow[d, "T^*_{\mathbb{C}}f"] \\ S'^{\otimes 2} \arrow[r, "\sim"] & T^*_{\mathbb{C}}I' \end{tikzcd} \qquad \begin{tikzcd} f^*(S^{\otimes 2}) \arrow[r, "\sim"] \arrow[d, "\beta^{\otimes 2}"'] & f^*(T^*_{\mathbb{C}}I) = f^*(T^*I)\otimes\mathbb{C} \arrow[d, "T^*f\otimes(z\mapsto\bar z)"] \\ S'^{\otimes 2} \arrow[r, "\sim"] & T^*_{\mathbb{C}}I' = T^*I'\otimes\mathbb{C} \end{tikzcd}$$

We say an embedding (f, β) is $\mathbb{C}$-linear, respectively $\mathbb{C}$-antilinear, if β is so.

Let (I, S) be a pin interval. Its *pin involution* is the map γ given by the identity on I and negation on S. There are two other non-trivial involutions c_i and c_{-i} that restrict to the identity on I. We call them the *conjugating involutions.* In order to distinguish one from the other, we first introduce the notion of a *coorientation* of I. By this, we mean a coorientation of T^*I inside its complexification $T^*_{\mathbb{C}}I$. If the pin intervals (I, S), (I', S') are equipped with coorientations, we say that an embedding (f, β) preserves the coorientations if so does the right vertical arrow of the relevant diagram (1).

Let (I, S) be a pin interval equipped with a coorientation, and let v be a section of $T^*_{\mathbb{C}}I$ representing the coorientation. Let $\sqrt{v}$ be the section of S (defined up to sign) determined by the equation $\sqrt{v}\otimes\sqrt{v} = v$. For $j = i$ or $-i$, the conjugating involution c_j of (I, S) acts by j on $\mathrm{span}_{\mathbb{R}}\{\sqrt{v}\}$ and by $-j$ on $\mathrm{span}_{\mathbb{R}}\{\sqrt{-v}\}$.

Definition 1.2. We let INT_{Pin} be the topological category whose objects are pin intervals equipped with a coorientation. The morphisms are pin embeddings (either $\mathbb{C}$-linear or $\mathbb{C}$-antilinear) that do *not* need to preserve the coorientation.

Let VN denote the category whose objects are complex $\mathbb{Z}/2$-graded von Neumann algebras, and whose morphisms are given by

$$\hom_{\mathsf{VN}}(A, B) := \hom(A, B) \cup \hom(A, \bar{B}) \cup \hom(A, B^{op}) \cup \hom(A, \bar{B}^{op}).$$

The hom-sets are given the topology of pointwise convergence.

Definition 1.3. A $\mathbb{Z}/2$-graded conformal net is a continuous functor

(2) $$\mathcal{A} : \mathsf{INT}_{Pin} \to \mathsf{VN}.$$

To an embedding $f : J \hookrightarrow I$, it assigns a map $\mathcal{A}(f) : \mathcal{A}(J) \to \mathcal{A}(I)$ of the kind prescribed by the following table:

(3)

	f is $\mathbb{C}$-linear.	f is $\mathbb{C}$-antilinear.
f respects the coorientations.	$\mathcal{A}(f) \in \hom\big(\mathcal{A}(J), \mathcal{A}(I)\big)$	$\mathcal{A}(f) \in \hom\big(\mathcal{A}(J), \overline{\mathcal{A}(I)}\,\big)$
f does not respect the coorientations.	$\mathcal{A}(f) \in \hom\big(\mathcal{A}(J), \mathcal{A}(I)^{op}\big)$	$\mathcal{A}(f) \in \hom\big(\mathcal{A}(J), \overline{\mathcal{A}(I)}^{op}\big)$

Moreover, if γ is the pin involution of I, then $\mathcal{A}(\gamma)$ should be the grading involution of $\mathcal{A}(I)$, and if c_j is a conjugating involution, then $\mathcal{A}(c_j)$ should be the map $\#_j : \mathcal{A}(I) \to \overline{\mathcal{A}(I)}^{op}$. It is subject to the following axioms:
•***Isotony***: The image of an embedding $J \hookrightarrow I$ is an injective map $\mathcal{A}(J) \hookrightarrow \mathcal{A}(I)$.
•***Locality***: If $J \subset I$ and $K \subset I$ have disjoint interiors, then the images of $\mathcal{A}(J)$ and $\mathcal{A}(K)$ graded commute inside $\mathcal{A}(I)$.
•***Strong additivity***: If $I = J \cup K$, then $\mathcal{A}(J)$ and $\mathcal{A}(K)$ generate $\mathcal{A}(I)$.
•***Haag duality***: If $I = J \cup K$ and $J \cap K$ is a point, then the image of $\mathcal{A}(J)$ is the graded commutant of $\mathcal{A}(K)$ inside $\mathcal{A}(I)$.
•***Split property***: If J, K are disjoint subintervals of I and the inclusions are compatible with both orientations and coorientations, then the map from the algebraic tensor product $\mathcal{A}(J) \otimes_{alg} \mathcal{A}(K) \to \mathcal{A}(I)$ extends to the spacial tensor product

$$\mathcal{A}(J) \bar{\otimes} \mathcal{A}(K) \to \mathcal{A}(I).$$

•***Diff covariance***: If $\varphi : I \to I$ is a diffeomorphism that restricts to the identity in a neighborhood of ∂I, then $\mathcal{A}(\varphi)$ is an inner automorphism of $\mathcal{A}(I)$.
• ***Vacuum***: Let $S^1 \subset \mathbb{C}$ denote the unit circle. Every subinterval of S^1 acquires a pin structure from its embedding in $\mathbb{C}$. Let

$$I := \exp\big([0, \pi i]\big), \qquad I' := \exp\big([\pi i, 2\pi i]\big). \tag{4}$$

Equip I and I' with the inward coorientation. After upgrading the map $j : I' \to I$, $j(z) := \bar{z}$ to a pin isomorphism, it induces a homomorphism

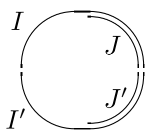
$$\mathcal{A}(j) : \mathcal{A}(I') \to \mathcal{A}(I)^{op}.$$

Let $H_0 := L^2(\mathcal{A}(I))$. We then have two left actions

$$\lambda : \mathcal{A}(I) \to \mathrm{B}(H_0), \qquad \rho : \mathcal{A}(I') \to \mathrm{B}(H_0),$$

given by the formulas $\lambda(a)(\xi) := a\,\xi$ and $\rho(b)(\xi) := (-1)^{|b||\xi|}\,\xi\,\mathcal{A}(j)(b)$.

Let $J \subset I$ be a subinterval such that $J \cap I' = \{1\}$ or $\{-1\}$, and let $J' := j(J)$.



Then the action of the algebraic tensor product

$$\lambda \otimes \rho \,:\, \mathcal{A}(J) \otimes_{alg} \mathcal{A}(J') \longrightarrow \mathrm{B}(H_0)$$

extends (uniquely) to an action of $\mathcal{A}(J \cup J')$.

2. $\mu = 1$ and invertibility

Let $\mathcal{A}$ be a conformal net. Its vacuum sector

$$H_0 := L^2\big(\mathcal{A}(\exp([0, \pi i]))\big)$$

is the identity on the identity defect of $\mathcal{A}$. It is an $\mathcal{A}(I)$-module for every $I \subset S^1$, where S^1 is endowed with the inward coorientation. We shall say that $\mathcal{A}$ is irreducible if its vacuum sector is irreducible. In other words, $\mathcal{A}$ is irreducible if $\mathcal{A} \neq 0$, and if

$$\bigvee_{I \subset S^1} \mathcal{A}(I) = \mathrm{B}(H_0).$$

This is equivalent to the algebras $\mathcal{A}(I)$ being factors i.e., having trivial center.

If ${}_AH_B$ is a bimodule between factors, then its statistical dimension $\dim({}_AH_B)$ is an invariant that lives in $\{0\} \cup [1, \infty]$. The statistical dimension is additive under direct sums, and multiplicative under tensor product and Connes fusion. Recall that if $I, J \subset S^1$ don't intersect, then $\mathcal{A}(J)\bar{\otimes}\mathcal{A}(I)$ acts on H_0 by the split property. Define intervals

$$I_1 := \exp\left([0, \tfrac{\pi i}{2}]\right), I_2 := \exp\left([\tfrac{\pi i}{2}, \pi i]\right), I_3 := \exp\left([\pi i, \tfrac{3\pi i}{2}]\right), I_4 := \exp\left([\tfrac{3\pi i}{2}, 2\pi i]\right).$$

Definition 2.1. The μ-index $\mu(\mathcal{A})$ of an irreducible conformal net $\mathcal{A}$ is the statistical dimension of the bimodule

$${}_{\mathcal{A}(I_1)\bar{\otimes}\mathcal{A}(I_3)} H_0 {}_{\mathcal{A}(I_2)^{op}\bar{\otimes}\mathcal{A}(I_4)^{op}}.$$

With the above definitions in place, we can now state our results:

Theorem 2.2. *A conformal net $\mathcal{A}$ is invertible in CN3 if and only if it is irreducible and $\mu(\mathcal{A}) = 1$.*

Theorem 2.3. *A conformal net is fully dualizable in CN3 if and only if it is a finite direct sum of irreducible nets $\mathcal{A}_i$, and all of those have $\mu(\mathcal{A}_i) < \infty$.*

Blob Homology

Kevin Walker

(joint work with Scott Morrison)

We define a chain complex $\mathcal{B}_*(M, C)$ (the "blob complex") associated to an n-category C and an n-manifold M. For $n = 1$, $\mathcal{B}_*(S^1, C)$ is quasi-isomorphic to the Hochschild complex of the 1-category C. So in some sense blob homology is a generalization of Hochschild homology to n-categories. The degree zero homology of $\mathcal{B}_*(M, C)$ is isomorphic to the dual of the Hilbert space associated to M by the TQFT corresponding to C. So in another sense the blob complex is the derived category version of a TQFT. A third specialization of the blob complex is when we take C to be trivial in dimensions less than n, so that C is essentially a commutative algebra thought of as an n-category. If we take this commutative algebra to be the polynomial algebra $\mathbb{C}[t]$, then $\mathcal{B}_*(M, C)$ is homotopy equivalent to $C_*(\Sigma^\infty(M))$, singular chains on the infinite symmetric power of M.

We hope to apply blob homology to tight contact structures on 3-manfolds ($n = 3$) and the extension of Khovanov homology to general 4-manifolds ($n = 4$). In both of these examples, exact triangles play an important role, and the derived category aspect of the blob complex allows this exactness to persist to a greater degree than it otherwise would.

$\mathcal{B}_0(M, C)$ is defined to be finite linear combinations of C-pictures on M. (A C-picture on M can be thought of as a pasting diagram for n-morphisms of C in the shape of M together with a choice of homeomorphism from this diagram to M.) There is an evaluation map from $\mathcal{B}_0(B^n, C)$ (C-pictures on the n-ball B^n) to the n-morphisms of C. Let U be the kernel of this map. Elements of U are called null fields. $\mathcal{B}_1(M, C)$ is defined to be finite linear combinations of triples (B, u, r) (called 1-blob diagrams), where $B \subset M$ is an embedded ball (or "blob"), $u \in U$ is a null field on B, and r is a C-picture on $M \setminus B$. Define the boundary map $\partial : \mathcal{B}_1(M, C) \to \mathcal{B}_0(M, C)$ by sending (B, u, r) to $u \bullet r$, the gluing of u and r. $\mathcal{B}_1(M, C)$ can be thought of as the space of relations we would naturally want to impose on $\mathcal{B}_0(M, C)$, and so $H_0(\mathcal{B}_*(M, C))$ is isomorphic to the generalized skein module (dual of TQFT Hilbert space) one would associate to M and C.

$\mathcal{B}_k(M, C)$ is defined to be finite linear combinations of k-blob diagrams. A k-blob diagram consists of k blobs (balls) $B_0, \ldots, B_{k-1}$ in M. Each pair B_i and B_j is required to be either disjoint or nested. Each innermost blob B_i is equipped with a null field $u_i \in U$. There is also a C-picture r on the complement of the innermost blobs. The boundary map $\partial : \mathcal{B}_k(M, C) \to \mathcal{B}_{k-1}(M, C)$ is defined to be the alternating sum of forgetting the i-th blob.

If M has boundary we always impose a boundary condition consisting of an $n{-}1$-morphism picture on ∂M. In this note we will suppress the boundary condition from the notation.

The blob complex has the following properties:

Functoriality. The blob complex is functorial with respect to diffeomorphisms. That is, fixing C, the association

$$M \mapsto \mathcal{B}_*(M, C)$$

is a functor from n-manifolds and diffeomorphisms between them to chain complexes and isomorphisms between them.

Contractibility for B^n. The blob complex of the n-ball, $\mathcal{B}_*(B^n, C)$, is quasi-isomorphic to the 1-step complex consisting of n-morphisms of C. (The domain and range of the n-morphisms correspond to the boundary conditions on B^n. Both are suppressed from the notation.) Thus $\mathcal{B}_*(B^n, C)$ can be thought of as a free resolution of C.

Disjoint union. There is a natural isomorphism

$$\mathcal{B}_*(M_1 \sqcup M_2, C) \cong \mathcal{B}_*(M_1, C) \otimes \mathcal{B}_*(M_2, C).$$

Gluing. Let M_1 and M_2 be n-manifolds, with Y a codimension-0 submanifold of ∂M_1 and $-Y$ a codimension-0 submanifold of ∂M_2. Then there is a chain map

$$\mathrm{gl}_Y : \mathcal{B}_*(M_1) \otimes \mathcal{B}_*(M_2) \to \mathcal{B}_*(M_1 \cup_Y M_2).$$

Relation with Hochschild homology. When C is a 1-category, $\mathcal{B}_*(S^1, C)$ is quasi-isomorphic to the Hochschild complex $\mathrm{Hoch}_*(C)$.

Relation with TQFTs and skein modules. $H_0(\mathcal{B}_*(M, C))$ is isomorphic to $A_C(M)$, the dual Hilbert space of the $n+1$-dimensional TQFT based on C.

Evaluation map. There is an 'evaluation' chain map

$$\mathrm{ev}_M : C_*(\mathrm{Diff}(M)) \otimes \mathcal{B}_*(M) \to \mathcal{B}_*(M).$$

(Here $C_*(\mathrm{Diff}(M))$ is the singular chain complex of the space of diffeomorphisms of M, fixed on ∂M.)

Restricted to $C_0(\mathrm{Diff}(M))$ this is just the action of diffeomorphisms described above. Further, for any codimension-1 submanifold $Y \subset M$ dividing M into $M_1 \cup_Y M_2$, the following diagram (using the gluing maps described above) commutes.

$$\begin{tikzcd}
C_*(\mathrm{Diff}(M)) \otimes \mathcal{B}_*(M) \arrow[r, "\mathrm{ev}_M"] & \mathcal{B}_*(M) \\
C_*(\mathrm{Diff}(M)) \otimes C_*(\mathrm{Diff}(M)) \otimes \mathcal{B}_*(M_1) \otimes \mathcal{B}_*(M_2) \arrow[u, "\mathrm{gl}_Y^{\mathrm{Diff}} \otimes \mathrm{gl}_Y"] \arrow[r, bend right, "\mathrm{ev}_{M_1} \otimes \mathrm{ev}_{M_2}"'] & \mathcal{B}_*(M_1) \otimes \mathcal{B}_*(M_2) \arrow[u, "\mathrm{gl}_Y"']
\end{tikzcd}$$

In fact, up to homotopy the evaluation maps are uniquely characterized by these two properties.

A_∞ categories for $n-1$-manifolds. For Y an $n-1$-manifold, the blob complex $\mathcal{B}_*(Y \times I, C)$ has the structure of an A_∞ category. The multiplication (m_2) is given my stacking copies of the cylinder $Y \times I$ together. The higher m_i's are obtained by applying the evaluation map to $i-2$-dimensional families of diffeomorphisms in $\mathrm{Diff}(I) \subset \mathrm{Diff}(Y \times I)$. Furthermore, $\mathcal{B}_*(M, C)$ affords a representation of the A_∞ category $\mathcal{B}_*(\partial M \times I, C)$.

Gluing formula. Let $Y \subset M$ divide M into manifolds M_1 and M_2. Let $A(Y)$ be the A_∞ category $\mathcal{B}_*(Y \times I, C)$. Then $\mathcal{B}_*(M_1, C)$ affords a right representation of $A(Y)$, $\mathcal{B}_*(M_2, C)$ affords a left representation of $A(Y)$, and $\mathcal{B}_*(M, C)$ is homotopy equivalent to $\mathcal{B}_*(M_1, C) \otimes_{A(Y)} \mathcal{B}_*(M_2, C)$.

Relation to mapping spaces. There is a version of the blob complex for C an A_∞ n-category instead of a garden variety n-category.

Let $\pi^\infty_{\leq n}(W)$ denote the A_∞ n-category based on maps $B^n \to W$. (The case $n = 1$ is the usual A_∞ category of paths in W.) Then $\mathcal{B}_*(M, \pi^\infty_{\leq n}(W))$ is homotopy equivalent to $C_*(\{\text{maps } M \to W\})$.

Product formula. Let $M^n = Y^{n-k} \times W^k$ and let C be an n-category. Let $A_*(Y)$ be the A_∞ k-category associated to Y via blob homology. Then

$$\mathcal{B}_*(Y^{n-k} \times W^k, C) \simeq \mathcal{B}_*(W, A_*(Y)).$$

There is a similar result for general fiber bundles.

Higher dimensional Deligne conjecture. The singular chains of the n-dimensional fat graph operad act on blob cochains.

The n-dimensional fat graph operad can be thought of as a sequence of general surgeries of n-manifolds $R_i \cup A_i \rightsquigarrow R_i \cup B_i$ together with mapping cylinders of diffeomorphisms $f_i : R_i \cup B_i \to R_{i+1} \cup A_{i+1}$. (Note that the suboperad where A_i, B_i and $R_i \cup A_i$ are all diffeomorphic to the n-ball is equivalent to the little $n+1$-disks operad.)

If A and B are n-manifolds sharing the same boundary, define the blob cochains $\mathcal{B}^*(A, B)$ (analogous to Hochschild cohomology) to be A_∞ maps from $\mathcal{B}_*(A)$ to $\mathcal{B}_*(B)$, where we think of both (collections of) complexes as modules over the A_∞ category associated to $\partial A = \partial B$. The "holes" in the above n-dimensional fat graph operad are labeled by $\mathcal{B}^*(A_i, B_i)$.

Background fields in twisted differential nonabelian cohomology

Urs Schreiber

(joint work with J. Baez, T. Nikolaus, H. Sati, Z. Škoda, J. Stasheff, D. Stevenson, K. Waldorf)

1. Motivation: background fields

An interesting supply of motivations for and applications of generalized notions of *cohomology* arises in formal higher energy physics in the context of theories that combine and generalize Maxwell's theory of the electromagentic field, Einstein's theory of the gravitational field and Yang-Mills' theory of general gauge fields.

In order to formalize and study certain phenomena exhibited by such higher *background fields* – such as what is called by the *Kalb-Ramond field in heterotic supergravity* or the *C-field in maximal supergravity* – Hopkins and Singer in their seminal work [1] developed the general theory of differential refinements of generalized Eilenberg-Steenrod cohomology. Based on this, Freed [2] explained certain subtle effects, previously observed semi-rigorously by physicists, systematically as phenomena exhibited by cocycles in differential generalized cohomology.

This involves notably various *twists* of one kind of cohomology by another. The most familiar example is twisted K-theory, the cohomology theory that in formal high energy physics describes the *Chan-Paton background field*.

But the physical applications indicate that this is only the simplest example in a more general theory of twisted generalized cohomology. The next example is the famous *Green-Schwarz mechanism* in *heterotic supergravity*, which, as Freed explained, amounts to asserting a kind of twist of a higher differential cohomology class. While this clarifies some of the structure, it remains noteworthy that the twist in the Green-Schwarz mechanism is fundamentally encoded not in *abelian* generalized coholomology, but by the class of a G-principal bundle for a nonabelian group G, hence by a coycle in *nonabelian cohomology*.

While abelian generalized cohomology has spectra as coefficients – maximally abelian spaces – degree n-nonabelian cohomology allows as coefficients arbitrary homotopy n-types. Such nonabelian cohomology is traditionally most familiar in the study of 1- and 2-gerbes [3] as well as in the higher Schreier theory [4] of nonabelian group extensions.

But it was already realized in [5] that generalized Eilenberg-Steenrod cohomology, sheaf cohomology as well as nonabelian cohomology all describe hom-sets in homotopy categories of (pre)sheaves with values in ∞-groupoids – called *simplicial presheaves*. This perspective was later refined by Joyal and Jardine's study [6] of the model category structure on simplicial presheaves. By the recent result of Lurie [7] we know that these constructions model precisely the theory of ∞-stacks [8].

This leads one to expect that a general theory of *smooth cohomology* that encompasses abelian as well as nonabelian phenomena concerns the $(\infty, 1)$-topos of ∞-stacks on a small version of the site Diff of smooth manifolds. This perspective on ∞-stacks as the truly general notion of cohomology is implied by Lurie's very notion of $(\infty, 1)$-*topos* as a context that "behaves like topological spaces". It can be found made explicit for instance in [9].

We therefore place ourselves in the context of the $(\infty, 1)$-topos $\mathbf{H}$ of ∞-stacks on Diff and discuss the following questions:

- What is the general notion of *differential cohomology* in $\mathbf{H}$?
- What is the general notion of *twisted cohomology* in $\mathbf{H}$?
- How does this describe phenomena exhibited by background fields in formal high energy physics?
- How does this induce the corresponding quantum field theories of objects charged under these background fields?

The last of these four questions is our main motivation. Here, however, we don't go into this last question except that it shall serve to motivate our answer to the first question:

2. Twisted differential nonabelian cohomology

2.1. Differential nonabelian cohomology. Recall that, as finally fully formalized in [10], an n-dimensional topological quantum field theory is an (∞, n)-functor $Z : \mathrm{Bord}_n \to \mathcal{V}$ on n-dimensional bordisms with values in an (∞, n)-category of something like n-vector spaces.

On the other hand, when we have on a target space object X a *background field* under which an n-dimensional object is charged, we expect a notion of parallel transport and holonomy encoded by an (∞, n)-functor $\exp(\int \nabla) : \mathrm{Bord}_n(X) \to \mathcal{V}$ from bordisms equipped with maps into X, that assigns to an n-dimensional bordism its parallel transport or holonomy, as a morphism in $\mathcal{V}$. The above suggests that those QFTs Z that arise as *σ-models* in that they are encoded by a background field ∇ are, in some sense, *extensions*

$$\begin{tikzcd} \mathrm{Bord}_n(X) \arrow[r, "\exp(\int \nabla)"] \arrow[d] & \mathcal{V} \\ \mathrm{Bord}_n \arrow[ur, dashed, "Z_\nabla"'] & \end{tikzcd} \qquad \begin{tikzcd} \text{field space} \arrow[r, "\text{background field}"] \arrow[d] & \mathcal{V} \\ \text{parameter space} \arrow[ur, dashed, "\text{quantum propagation}"] & \end{tikzcd}$$

along the obvious forgetful (∞, n)-functor $\mathrm{Bord}_n(X) \to \mathrm{Bord}(X)$. Whatever this extension procedure may be, a necessary prerequisite for studying it is a good grasp of how to encode differential cocycles on X in terms of functors on $\mathrm{Bord}_n(X)$.

In a series of articles [11, 12, 13, 14] it was shown that gerbes with connection and more generally higher principal bundles with connection are indeed encoded as morphisms in $\mathbf{H}$ of the form

$$\nabla : \mathcal{P}_n(X) \to \mathcal{V}$$

where $\mathcal{P}_n(X)$ is essentially the subobject of $\mathrm{Bord}_n(X)$ consisting only of topologically disk-shaped bordisms in X: the path n-groupoid of X.

Theorem 2.1. *Let G be a Lie group, $\mathbf{B}G$ the corresponding one-object smooth groupoid in $\mathbf{H}$ and $\mathbf{B}\mathrm{AUT}(G)$ the one-object 2-groupoid coming from the automorphism 2-group of G. Let X be manifold, then*

- $\mathbf{H}(\mathcal{P}_1(X), \mathbf{B}G) \simeq G\mathrm{Bund}_\nabla(X)$
  $= \{$*G-principal bundles with connection on X*$\}$
- $\mathbf{H}(\mathcal{P}_2(X), \mathbf{B}\mathrm{AUT}(H)) \simeq H\mathrm{Grb}_{\nabla_{\mathrm{ff}}}(X)$
  $= \{$*G-gerbes with fake-flat connection on X*$\}$.

In particular let $\mathbf{BB}U(1)$ be the smooth 2-groupoid with the Lie group $U(1)$ in degree 2, then

- $\mathbf{H}(\mathcal{P}_2(X), \mathbf{BB}U(1)) \simeq \mathrm{BdlGrb}_\nabla(X)$
  $= \{$*line bundle-gerbes with general connection on X*$\}$

In the abelian case a fake-flat connection is just a general connection, but in the nonabelian case, as well as in the abelian equivariant case, it is more restrictive than what one might expect. For full nonabelian differential cohomology the above is slightly too naive and replaced by the following.

Definition 2.2. *For every object $X \in \mathbf{X}$ there is an object $\Pi(X) = \varinjlim \mathcal{P}_n(X)$, the* fundamental ∞-groupoid *of X. For any coefficient object $A \in \vec{\mathbf{H}}$ we call $\mathbf{H}(\Pi(X), A)$ the* **flat differential A-cohomology of X**.

For A once deloopable there is a morphism $P : \mathbf{H}(X, A) \to \mathbf{H}(\Pi(X), \mathbf{B}A)$ whose image $P(c)$ of an A-cocycle c we call the **characteristic curvature class** *of c.*

Given a class $P \in \mathbf{H}(\Pi(X), \mathbf{B}A)$ we call the corresponding P-twisted flat differential A-cohomology $\mathbf{H}^P(\Pi(X), A)$ *the* **differential A cohomology with curvature P**.

The last clause uses the following definition of twisted cohomology.

2.2. **Twisted nonabelian cohomology.**

Definition 2.3. *Let $\hat{G} \to G \to A$ be a fibration sequence in $\mathbf{H}$. Notice that by the left-exactness of the Hom, for any object $X \in \mathbf{H}$ we obtain the fibration sequence* $\mathbf{H}(X, \hat{G}) \longrightarrow \mathbf{H}(X, G) \xrightarrow{\text{obstr}} \mathbf{H}(X, A)$ *in cohomology that characterizes $\hat{G}$-cocycles as those G-cocycles whose obstructing A-cocycle is trivializable. For $c \in \mathbf{H}(X, A)$ any possibly nontrivial A-cocycle on X, define the* **c-twisted $\hat{G}$-cohomology** $\mathbf{H}^c(X, A)$ *to be the homotopy pullback*

$$\begin{tikzcd} \mathbf{H}(X, \hat{G}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "* \mapsto c"] \\ \mathbf{H}(X, G) \arrow[r, "\text{obstr}"] & \mathbf{H}(X, A) \end{tikzcd} .$$

2.3. **Examples: background fields in twisted differential nonabelian cohomology.** These two definitions may be combined to yield a notion of *twisted differential nonabelian cohomology.* The local cocycle identities satisfied by the curvature characteristic forms of these twisted cocycles are the *twisted Bianchi identities* in the physics literature, as indicated in the following list of examples.

Claim 2.1. *fibration sequence:* $\mathbf{B}U(n) \to \mathbf{B}PU(n) \xrightarrow{c_1} \mathbf{B}^2U(1)$

- *twisting cocycle: lifting gerbe;*
- *twisted cocycle: twisted bundles / gerbe modules*
- *twisted Bianchi identity:* $dF_\nabla = H_3$
- *occurence: Freed-Witten anomaly cancellation on D-brane*

Claim 2.2. *fibration sequence:* $\mathbf{B}String(n) \to \mathbf{B}Spin(n) \xrightarrow{\frac{1}{2}p_1} \mathbf{B}^3U(1)$

- *twisting cocycle: Chern-Simons 2-gerbe;*
- *twisted cocycle: twisted nonabelian String-gerbe with conection*
- *twisted Bianchi identity:* $dH_3 \propto \langle F_\nabla \wedge F_\nabla \rangle$
- *occurence: Green-Schwarz anomaly cancellation*

Proof. [15]: use the BCSS model [16] of $String(n)$ with the construction from [17] of $\frac{1}{2}p_1$, then use [18, 20] to compute local differential form data. □

Claim 2.3. *fibration sequence:* $\mathbf{B}Fivebrane(n) \to \mathbf{B}String(n) \xrightarrow{\frac{1}{6}p_2} \mathbf{B}^7U(1)$

- *twisting cocycle: Chern-Simons 6-gerbe;*
- *twisted cocycle: twisted nonabelian Fivebrane-gerbe with connection*
- *occurence: dual Green-Schwarz anomaly cancellation for NS 5-brane magnetic dual to string*

Proof. Analogous to the above. And use [19, 20]. □

Claim 2.4. *fibration sequence:* $\mathbf{B}^2U(1) \to \mathbf{B}(U(1) \to \mathbb{Z}_2) \to \mathbf{B}\mathbb{Z}_2$

- *twisting cocycle: $\mathbb{Z}_2$-orbifold;*
- *twisted cocycle: orientifold gerbe / Jandl gerbe with connection*

Proof. Use [21] and the bosonic part of [22]. $\square$

References

[1] M. J. Hopkins, I.M. Singer, *Quadratic functions in geometry, topology, and M-theory*, J. Diff. Geom. **70** (2005) 329-452, [arXiv:math.AT/0211216].

[2] D. S. Freed, *Dirac charge quantization and generalized differential cohomology*, Surv. Differ. Geom., VII, 129–194, Int. Press, Somerville, MA, 2000, [arXiv:hep-th/0011220].

[3] Lawrence Breen, *Notes on 1- and 2-gerbes*, in J. Baez and P. May, eds., *Towards Higher Categories*, Institute for Mathematics and its Applications, [arXivmath.CT/0611317]

[4] Lawrence Breen, *Théorie de Schreier supérieure*, Ann. Sci. École Norm. Sup. (4) 25 (1992), no. 5, 465–514

[5] Kenneth S. Brown, *Abstract Homotopy Theory and Generalized Sheaf Cohomology*, Trans. of the Amer. Math. Soc., Vol. 186 (1973), 419-458

[6] J. F. Jardine, *Simplicial Presheaves*, Lectures at Fields Institute, January 2007, [http://www.math.uwo.ca/∼jardine/papers/Fields-01.pdf]

[7] Jacob Lurie, *Higher Topos Theory*, Princeton University Press, [http://arxiv.org/abs/math.CT/0608040]

[8] Bertrand Toën, *Higher and derived stacks: a global overview*, [arXiv:math/0604504]

[9] Daniel Dugger, *Sheaves and homotopy theory*, [http://www.uoregon.edu/∼ddugger/cech.html]

[10] Jacob Lurie, *On the Classification of Topological Field Theories* [arXiv:0905.0465]

[11] John Baez and U. S., *Higher Gauge Theory*, Categories in Algebra, Geometry and Mathematical Physics, Contemp. Math. 431, [arXiv:math/0511710]

[12] U. S., Konrad Waldorf, *Parallel Transport and Functors*, to appear in Journal of Homotopy and Related Structures, [arXiv:0705.0452]

[13] U. S., Konrad Waldorf, *Smooth Functors vs. Differential Forms*, [arXiv:0802.0663]

[14] U. S., Konrad Waldorf, *Connections on nonabelian gerbes and their holonomy*, [*arXiv:0808.1923*]

[15] Thomas Nikolaus, U. S., Zoran Škoda, Danny Stevenson, *Twisted differential nonabelian cohomology*, [http://ncatlab.org/schreiber/show/Differential+Nonabelian+Cohomology]

[16] John Baez, Alissa Crans, U. S., D. Stevenson, *From loop groups to 2-groups*, Homology, Homotopy and Applications, Vol. 9 (2007), No. 2, pp.101-135., [arXiv:math/0504123]

[17] Jean-Luc Brylinksi and Dennis McLaughlin, *Čech cocycles for characteristic classes*, Comm. Math. Phys. 178 (1996)

[18] Hisham Sati, U. S. and Jim Stasheff, *L_∞-connections and applications to String- and Chern-Simons n-transport*, in *Recent Developments in QFT*, eds. B. Fauser et al., Birkhäuser, Basel (2008), [arXiv:0801.3480]

[19] Hisham Sati, U. S. and Jim Stasheff, *Fivebrane Structures*, to appear in Rev. Math. Phys., [arXiv:0805.0564]

[20] Hisham Sati, U. S. and Jim Stasheff, *Twisted diufferential String- and Fivebrane-structures*, [http://www.math.uni-hamburg.de/home/schreiber/5twist.pdf]

[21] U. S., Christoph Schweigert and Konrad Waldorf, *Unoriented WZW Models and Holonomy of Bundle Gerbes*, Commun. Math. Phys., Vol. 274, No. 1, 2007, pp. 31-64, [arXiv:hep-th/0512283]

[22] Jacques Distler, Daniel Freed and Gregory Moore, *Orientifold Precis*, [arXiv:0906.0795]

String structures, 3-forms, and tmf classes

Corbett Redden

Consider $P \xrightarrow{\pi} M$, where P is a principal $Spin(k)$-bundle over a closed manifold M (compact without boundary). We then define a (topological) string structure on P to be a lift of the classifying map from $BSpin(k)$ to $BString(k)$. Here, $BString(k)$ is the homotopy fiber of the characteristic class $\frac{p_1}{2}$, as seen in the fibration sequence

$$BString(k) \to BSpin(k) \xrightarrow{\frac{p_1}{2}} K(\mathbb{Z}, 4).$$

While there are various descriptions of string structures, any construction will produce such a lift, and homotopy classes of lifts to $BString(k)$ have a convenient classification.

Definition. *A string class $\mathcal{S}$ is a cohomology class $\mathcal{S} \in H^3(P;\mathbb{Z})$ that restricts fiberwise to the stable generator of $H^3(Spin(k);\mathbb{Z})$.*

Proposition.

- *$\{String\ structures\}/(homotopy) \cong \{String\ classes\}$*
- *A string structure/class exists if and only if $\frac{p_1}{2}(P) = 0 \in H^4(M;\mathbb{Z})$.*
- *The set of string classes is a torsor for $H^3(M;\mathbb{Z})$ under the natural additive action of π^*.*

We now wish to describe the harmonic representative of a string class. A Riemannian metric on P determines the Hodge Laplacian Δ acting on differential forms. Hodge's Theorem implies that $\operatorname{Ker} \Delta^i$, the harmonic i-forms, is canonically isomorphic to $H^i(P;\mathbb{R})$.

Start with the data $(P \xrightarrow{\pi} M, g, A)$, where g is a Riemannian metric on M, and A is a connection on P. The connection A provides an orthogonal splitting of TP. Then, the choice of a bi-invariant metric g_{Spin} on $Spin(k)$ defines the 1-paremeter family of Riemannian metrics on P

$$g_\delta := \pi^* g \oplus \delta^2 g_{Spin}$$

for $\delta > 0$. Shrinking the fibers, or taking the limit as $\delta \to 0$, is known as the adiabatic limit. While the metric becomes singular at $\delta = 0$, work of Mazzeo–Melrose, Dai, and Forman [MM, Dai, For] show that the harmonic forms extend smoothly to $\delta = 0$. We denote this limit as

$$\mathcal{H}^i(P) := \lim_{\delta \to 0} \operatorname{Ker} \Delta^i_{g_\delta} \subset \Omega^i(P)$$

and note that $\mathcal{H}^i(P) \cong H^i(P;\mathbb{R})$.

Theorem. *Consider $(P \xrightarrow{\pi} M, g, A)$ with $\frac{p_1}{2}(P) = 0$. In the adiabatic limit, the harmonic representative of a string class $\mathcal{S}$ is of the form $CS_3(A) - \pi^* H_{\mathcal{S},g,A}$, where $CS_3(A)$ is the Chern–Simons 3-form, and $H_{\mathcal{S},g,A} \in \Omega^3(M)$; i.e.*

$$\begin{array}{ccc} H^3(P;\mathbb{Z}) \to H^3(P;\mathbb{R}) & \xrightarrow{\cong} & \mathcal{H}^3(P) \\ \mathcal{S} \longmapsto & & CS_3(A) - \pi^* H_{\mathcal{S},g,A}, \end{array}$$

(If one does not take the adiabatic limit, the difference between $CS_3(A)$ and the harmonic representative of $\mathcal{S}$ is *not* in general in $\pi^*\Omega^3(M)$.) This form $H_{\mathcal{S},g,A}$ satisfies two useful properties. First,

$$d^* H_{\mathcal{S},g,A} = 0 \in \Omega^2(M).$$

Secondly, the connection A determines a differential cohomology class $\frac{\check{p_1}}{2}(A)$ [CS], and $H_{\mathcal{S},g,A} = \frac{\check{p_1}}{2}(A)$ as differential classes. This is encoded in the following standard exact sequence:

$$\Omega^3_{\mathbb{Z}}(M) \to \Omega^3(M) \to \check{H}^4(M) \to H^4(M;\mathbb{Z}) \to 0$$
$$H_{\mathcal{S},g,A} \mapsto \frac{\check{p_1}}{2}(A) \mapsto \frac{p_1}{2}(P) = 0$$

In the language of differential characters, $\frac{\check{p_1}}{2}(A)$ is a homomorphism from 3-cycles to $\mathbb{R}/\mathbb{Z}$, and the form $H_{\mathcal{S},g,A}$ gives a specified lift of the homomorphism to $\mathbb{R}$. There is also the following equivariance: if one changes the string class by adding $\pi^*\psi \in \pi^* H^3(M;\mathbb{Z})$, then

$$H_{\mathcal{S}+\pi^*\psi,g,A} = H_{\mathcal{S},g,A} + H_{\psi,g}$$

where $H_{\psi,g}$ is the harmonic representative of ψ. This changes the lift of the character from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$ in the expected way. The above story can be duplicated with $Spin(k)$ replaced by any compact, simply-connected, semi-simple Lie group G, and $\frac{p_1}{2}$ replaced by a level $\lambda \in H^4(BG;\mathbb{Z})$.

Our motivation for dealing with string structures stems from

$$MString \xrightarrow{\sigma} tmf,$$

the $String$-orientation of the cohomology theory tmf or topological modular forms [Hop]. A spin manifold M^n with string class $\mathcal{S} \in H^3(Spin(TM);\mathbb{Z})$ naturally produces an element in string-bordism and a class $\sigma(M,\mathcal{S}) \in tmf^{-n}(pt)$ refining the Witten genus. The Witten genus is, heuristically, the S^1-equivariant index of $\not{D}_{LM}$, the Dirac operator on the free loop space LM. The string structure actually arises when constructing the mathematically rigorous spinor bundle on LM. It is hoped that the natural home for families index theorems on loop spaces will live in tmf, just as ordinary families index theorems live in K and KO-theory. The analogy between the Witten genus and the $\widehat{A}$-genus led Stolz to the following conjecture.

Conjecture (Stolz [Sto])**.** *Let M^n be a spin manifold with $\frac{p_1}{2}(M) = 0 \in H^4(M;\mathbb{Z})$. If M admits a metric of positive Ricci curvature, then the Witten genus of M is zero.*

One could also ask if something analogous to Hitchin's theorem might hold. Namely, if a string manifold M^n admits a positive Ricci curvature metric, then is $\sigma(M^n,\mathcal{S}) = 0 \in tmf^{-n}$? While there are no known counterexamples to Stolz' conjecture, the answer to the preceding question is most certainly no.

For example, consider $S^3 \cong SU(2)$. The Witten genus is 0 (it does not have dimension $4k$), yet the various framings produce different string structures which yield all elements in

$$MString^{-3}(pt) \cong tmf^{-3}(pt) \cong \pi_3^s \cong \mathbb{Z}/24.$$

Furthermore, the round metric on S^3 has positive Ricci curvature, and even positive sectional curvature. So, any attempt to generalize Hitchin's theorem must take into account both the geometry and the string structure. This leads to the following hypothesis, where $H_{\mathcal{S},g}$ is the 3-form constructed above with $P = Spin(TM)$ and A the Levi–Civita connection.

Hypothesis. *Let M^n be a spin manifold with $\frac{p_1}{2}(M) = 0 \in H^4(M;\mathbb{Z})$. If M admits a string class and metric $(\mathcal{S}, g)$ such that g has positive Ricci curvature and $H_{\mathcal{S},g} = 0$, then $\sigma(M, \mathcal{S}) = 0 \in tmf^{-n}(pt)$.*

The condition that $H_{\mathcal{S},g} = 0$ is quite strong as it implies that the differential class $\frac{\check{p}_1}{2}(g) = 0$. If we consider the situation of S^3, there is a useful 1-parameter family of left-invariant metrics, known as the Berger metrics, obtained by rescaling the fibers of the Hopf fibration. The above hypothesis holds true in this family of metrics, yet it would not if either condition were weakened. In particular, when g is the round metric and $\mathcal{S}$ is induced from D^4, the form $H = 0$ and the σ-invariant is 0. However, there is a metric for which the Ricci curvature is nonnegative but not positive; this metric and the right-invariant framing produce $H = 0$ and a generator of $tmf^{-3}(pt)$. There are also infinitely many other string classes and metrics which produce $H = 0$ but not positive Ricci curvature.

References

[CS] Jeff Cheeger and James Simons. Differential characters and geometric invariants. In *Geometry and topology (College Park, Md., 1983/84)*, volume 1167 of *Lecture Notes in Math.*, pages 50–80. Springer, Berlin, 1985.

[Dai] Xianzhe Dai. Adiabatic limits, nonmultiplicativity of signature, and Leray spectral sequence. *J. Amer. Math. Soc.*, 4(2):265–321, 1991.

[For] Robin Forman. Spectral sequences and adiabatic limits. *Comm. Math. Phys.*, 168(1):57–116, 1995.

[Hop] M. J. Hopkins. Algebraic topology and modular forms. In *Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002)*, pages 291–317, Beijing, 2002. Higher Ed. Press.

[MM] Rafe R. Mazzeo and Richard B. Melrose. The adiabatic limit, Hodge cohomology and Leray's spectral sequence for a fibration. *J. Differential Geom.*, 31(1):185–213, 1990.

[Red1] Corbett Redden. *Canonical metric connections associated to String structures*. PhD thesis, University of Notre Dame, 2006.

[Red2] Corbett Redden. Harmonic forms on principal bundles, 2008. [arXiv.dg:0810.4578].

[Sto] Stephan Stolz. A conjecture concerning positive Ricci curvature and the Witten genus. *Math. Ann.*, 304(4):785–800, 1996.

String Connections and Chern-Simons 2-Gerbes

Konrad Waldorf

String structures on a principal Spin(n)-bundle P over a smooth manifold M can be understood geometrically in two ways: (A) as lifts of the structure group of P from Spin(n) to a certain 3-connected cover, the *string group* [7], and (B) as lifts of the structure group of the looped bundle LP from LSpin(n) to its basic central extension [3]. I want to advertise a third way (C), which is equivalent to (A): string structures are trivializations of a certain geometrical object, namely a bundle 2-gerbe $\mathbb{CS}_P$ associated to P. In the following I want to outline the main results of my article [9] describing this approach.

The main advantage of my approach (C) is that the bundle 2-gerbe $\mathbb{CS}_P$ enjoys an explicit, smooth and finite-dimensional construction. This is in contrast to the approaches (A) and (B), which involve both non-smooth or infinite-dimensional smooth structures (the string group and loop spaces, respectively). I remark, however, that there is ongoing and promising research aiming at a finite-dimensional and smooth replacement for the string group in terms of certain generalized Lie 2-groups [6].

The bundle 2-gerbe $\mathbb{CS}_P$ is a certain *Chern-Simons bundle 2-gerbe* [1]. Let me give the idea of its construction. We start with a given principal Spin(n)-bundle P over M. The 2-fold fibre product $P^{[2]} := P \times_M P$ comes with a canonical map $g : P^{[2]} \to \mathrm{Spin}(n)$ which expresses the fact that P trivializes canonically when pulled back to its own total space. Over Spin(n) one finds the *basic bundle gerbe* $\mathcal{G}$, whose Dixmier-Douady class is the generator of $\mathrm{H}^3(\mathrm{Spin}(n), \mathbb{Z}) \cong \mathbb{Z}$. There exists a Lie-theoretic construction of $\mathcal{G}$ due to Gawędzki-Reis [2] and Meinrenken [4], finite-dimensional and smooth. The pullback of $\mathcal{G}$ along the map g is one part of the Chern-Simons 2-gerbe. The remaining ingredients are provided by a *multiplicative structure* on $\mathcal{G}$.

Like every bundle 2-gerbe, the Chern-Simons 2-gerbe has a characteristic class in $\mathrm{H}^4(M, \mathbb{Z})$. This class is

$$[\mathbb{CS}_P] = \tfrac{1}{2}p_1(P) \in \mathrm{H}^4(M, \mathbb{Z}),$$

the obstruction against string structures in the Stolz-Teichner approach (A). As a consequence, string structures on P exist if and only if $\mathbb{CS}_P$ admits trivializations. The situation is even better: there exists a canonical bijection between isomorphism classes of trivializations of $\mathbb{CS}_P$ and equivalence classes of string structures in the Stolz-Teichner approach (A). Summarizing, trivializations of the Chern-Simons 2-gerbe $\mathbb{CS}_P$ are a geometrical, smooth and finite-dimensional way to describe string structures.

One can now lift the whole construction to a setup with connections. This benefits particularly from the fact that we have only involved smooth, finite-dimensional manifolds. We assume that the principal Spin(n)-bundle P comes equipped with a connection A. One can show that this connection defines a canonical connection

∇_A on $\mathbb{CS}_P$. Let me just mention that part of this connection is a 3-form on P, namely the Chern-Simons 3-form $TP(A)$. Now we can look at trivializations of $\mathbb{CS}_P$ that respect the connection ∇_A in a certain way. This actually means to equip a trivialization with additional structure, that we call *string connection.* In my article [9] I show that

- To every string structure and every connection A on P there exists a string connection.
- The set of possible choices forms a contractible space.

The collection of a string structure and a string connection is called a *geometric string structure.* This notion of a geometric string structure has a number of interesting properties, which I want to outline in the following.

- Geometric string structures on (P, A) form a 2-groupoid, which is a module over the 2-groupoid of bundle gerbes with connection over M.
- On isomorphism classes, one obtains a free and transitive action of the differential cohomology $\hat{\mathrm{H}}^3(M, \mathbb{Z})$ on the set of isomorphism classes of geometric string structures is induced.
- Associated to every geometric string structure is a 3-form $H \in \Omega^3(M)$ whose pullback to P differs from the Chern-Simons 3-form $TP(A)$ by a closed 3-form with integral periods.

I remark that the notion of a geometric string structure in my approach (C) coincides with the original definition given by Stolz and Teichner [7] in the sense that both trivialize a certain Chern-Simons theory.

Another interesting link is to Redden's thesis [5], in which he constructs another 3-form $H_{g,A}$ associated to a string structure, a connection A on P, and a Riemannian metric g on M. One would like to have string connection associated to g and A, such that the two 3-forms coincide, $H = H_{g,A}$. During the workshop, Redden and I could at least show that such a string connection always exists.

Let me finally outline how my approach (C) to string structures relates to approach (B), namely to lifts of the structure group of LP from $L\mathrm{Spin}(n)$ to its basic central extension. For this purpose we look at the *transgression* of the Chern-Simons 2-gerbe $\mathbb{CS}_P$ to the free loop space LM. This is a bundle gerbe $\mathscr{T}_{\mathbb{CS}_P}$ over LM that one can explicitly construct from the given bundle 2-gerbe. On the level of characteristic classes, the construction covers the transgression homomorphism

$$\mathrm{H}^4(M, \mathbb{Z}) \to \mathrm{H}^3(LM, \mathbb{Z}).$$

What has this bundle gerbe $\mathscr{T}_{\mathbb{CS}_P}$ over LM to do with string structures? We use a result from [8] showing that the transgression of the basic bundle gerbe $\mathcal{G}$ defines a principal U(1)-bundle over LM, which underlies the basic central extension

$$1 \to \mathrm{U}(1) \to \widehat{L\mathrm{Spin}(n)} \to L\mathrm{Spin}(n) \to 1.$$

This fact makes the relation between $\mathbb{CS}_P$, which we have constructed using the basic bundle gerbe $\mathcal{G}$, and the string structures in the approach (B). More precisely, the bundle gerbe $\mathscr{T}_{\mathbb{CS}_P}$ *is* the lifting bundle gerbe associated to the problem of lifting the structure group of LP along the above central extension. As a consequence, string structures in the sense of trivializations of $\mathbb{CS}_P$ transgress to string structures in the sense of McLaughlin.

References

[1] A. L. Carey, S. Johnson, M. K. Murray, D. Stevenson, and B.-L. Wang, *Bundle gerbes for Chern-Simons and Wess-Zumino-Witten theories*, Commun. Math. Phys., 259(3):577–613, 2005.
[2] K. Gawędzki and N. Reis, *Basic gerbe over non simply connected compact groups*, J. Geom. Phys., 50(1–4):28–55, 2003.
[3] D. A. McLaughlin, *Orientation and string structures on loop space*, Pacific J. Math., 155(1):143–156, 1992.
[4] E. Meinrenken, *The basic gerbe over a compact simple lie group*, Enseign. Math., II. Sr., 49(3–4):307–333, 2002.
[5] D. C. Redden, *Canonical metric connections associated to string structures*, PhD thesis, University of Notre Dame, 2006.
[6] C. Schommer-Pries, *A finite-dimensional model for the string group*, in preparation.
[7] S. Stolz and P. Teichner, *What is an elliptic object?*, volume 308 of London Math. Soc. Lecture Note Ser., pages 247–343. Cambridge Univ. Press, 2004.
[8] K. Waldorf, *Multiplicative bundle gerbes with connection*, preprint, 2008.
[9] K. Waldorf, *String connections and chern-simons theory*, preprint, 2009.

String topology, field theories, and Fukaya categories

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(joint work with A. Blumberg and C. Teleman)

In this talk I will describe an ongoing project [6], in which we are attempting to understand the implications of recent work on "open-closed" topological field theories by Moore Segal [16], Costello [11], and Hopkins-Lurie [15], on the theory of "string topology", as introduced by Chas and Sullivan [7]. In particular we will attempt to use the classification of field theories contained in these works, to compare string topology to the symplectic field theory of the cotangent bundle.

In an open-closed topological field theory, one studies cobordisms beween compact one-dimensional manifolds, whose boundary components are labeled by an indexing set, $\mathcal{D}$. The cobordisms are those of manifolds with boundary, that preserve the labeling sets in a specific way. The set of labels $\mathcal{D}$ are referred to as "D-branes", and in the string theory literature these are boundary values of "open strings". An open-closed field theory is a monoidal functor from a category built out of such manifolds and cobordisms, that takes values in a linear category, such as vector spaces, chain complexes, or even the category of spectra. In this lecture we will restrict our attention to chain complex valued theories.

One common aspect of the work on open-closed theories in [16], [11], [15] is the attempt to understand the field theory $\mathcal{F}$ in terms of an associate (A_∞) category

$\mathcal{C}_{\mathcal{F}}$. The objects of $\mathcal{C}_{\mathcal{F}}$ are the set of D-branes, $\mathcal{D}$. The space of morphisms between λ_0 and $\lambda_1 \in \mathcal{D}$ is given by the value of the theory $\mathcal{F}$ on the object I_{λ_0,λ_1}, defined by the interval $[0,1]$ where the boundary component 0 is labeled by λ_0, and 1 is labeled by λ_1. The composition rules in this (A_∞) category are defined by the values of $\mathcal{F}$ on certain "open-closed" cobordisms.

In this talk I reported on a project whose goal is to understand how the "String Topology" theory of a manifold fits into this structure. This theory as originally introduced by Chas and Sullivan [7], is an intersection theory in the space of loops, or paths (with specified boundary conditions) of a closed, oriented n-dimensional manifold M. It was shown by Godin [14] that there is a (positive boundary) TCFT $\mathcal{S}_M$, which assigns to a circle the homology of the free loop space,

$$\mathcal{S}_M(S^1) = H_*(LM;k)$$

with field coefficients. This theory takes values in graded vector spaces (its a homological theory) rather than chain complexes. In this theory the set of D-branes $\mathcal{D}$ is the set of closed, oriented, connected submanifolds of M. The theory assigns to a compact one-manifold c with boundary levels, the homology of the mapping space,

$$\mathcal{S}_M(c) = H_*(Map(c,\partial;M)).$$

Here $Map(c,\partial;M)$ refers to the space of maps $c \to M$ that take the labeled boundary components to the submanifolds determined by the labeling. In particular, we write $\mathcal{P}_{N_0,N_1} = Map(I_{N_0,N_1},\partial;M)$ for the space of paths $\gamma : [0,1] \to M$ such that $\gamma(0) \in N_0$, and $\gamma(1) \in N_1$. An open-closed topological conformal field theory in the sense of Costello is a *chain complex* valued theory, and it is conjectured that the string topology theory has the structure of such a theory. The following theorem, which we report on in this paper, gives evidence for this conjecture.

Theorem 1. (1) *There is a category $\mathcal{S}_M$ enriched over chain complexes over a field k, whose objects are $\mathcal{D}_M$ = connected, oriented submanifolds of M, and whose space of morphisms $Mor_{\mathcal{S}_M}(N_1,N_2)$ is chain homotopy equivalent to the singular chains, $C_*(\mathcal{P}_{N_1,N_2})$. Furthermore the compositions in this category reflect the open-closed string topology operations on the level of homology.*

(2) *The Hochschild homology of this category $\mathcal{S}_M$ is the homology of the free loop space,*

$$HH_*(\mathcal{S}_M) \cong H_*(LM;k).$$

Note. In this theorem we produce a rigid category, not just an A_∞-category. The morphisms in this category are a particular model of the space of morphisms in the derived category of differential graded modules over the chains of the loop space, $C_*(\Omega M)$. Namely, Let $N \hookrightarrow M$ be a closed, oriented submanifold, defining an object of $\mathcal{S}_M$. Let $x_0 \in M$ be a fixed basepoint. Then the path space $\mathcal{P}_{N,x_0}$ is a model for the homotopy fiber of the inclusion, $N \hookrightarrow M$. Its chains (or more precisely the chains of an appropriate cofibrant replacement) can be viewed as a module over $C_*(\Omega M)$, and the morphisms in $\mathcal{S}_M$ between submanifolds N_1 and N_2, are defined to be

$$Mor_{\mathcal{S}_M}(N_1, N_2) = Rhom_{C_*(\Omega M)}(C_*(\mathcal{P}_{N_1,x_0}), C_*(\mathcal{P}_{N_2,x_0})).$$

Composition in this category is given by composition of morphisms. The key point in the above theorem is proving that this space of derived homomorphisms is chain equivalent to $C_*(\mathcal{P}_{N_1,N_2})$. This is done using a derived form of Poincare duality, where one studies coefficient systems of modules over the $C_*(\Omega M)$, rather than in the traditional setting of Poincare duality, where one studies modules over the group ring $k[\pi_1(M)]$, which can be viewed as the ring of path components of the DGA $C_*(\Omega M)$. This duality relies on the work of Dwyer-Greenlees-Iyengar [12]. Another aspect of this theorem is that showing that under this Poincare duality equivalence, composition of homomorphisms correspond to the Sullivan string product in homology. This uses the work on "umkehr maps" in [10].

Given any fixed submanifold N, the space of self-morphisms, $Mor_{\mathcal{S}_M}(N, N) \simeq C_*(\mathcal{P}_{N,N})$ is a differential graded algebra. Again, on the level of homology, this algebra structure is the string topology product introduced by Sullivan [18]. In this project we study the Hochschild cohomology, $HH^*(C_*(\mathcal{P}_{N,N}), C_*(\mathcal{P}_{N,N}))$, and prove that when M is simply connected, then for a large class of submanifolds, N,

$$HH^*(C_*(\mathcal{P}_{N,N}), C_*(\mathcal{P}_{N,N})) \cong H_*(LM)$$

as algebras. We show that the class of manifolds for which the above isomorphism holds includes the case when the inclusion $N \hookrightarrow M$ is null homotopic, (so for example all strict submanifolds of a sphere), as well as when $N \hookrightarrow M$ is the inclusion of the fiber of a fibration $p : M \to B$, or more generally, when $N \hookrightarrow M$ can be factored as a sequence of embeddings, $N = N_0 \hookrightarrow N_1 \hookrightarrow \cdots N_i \hookrightarrow N_{i+1} \cdots N_k = M$ where each $N_i \subset N_{i+1}$ is the inclusion of a fiber of a fibration $p_{i+1} : N_{i+1} \to B_{i+1}$. We also discuss the Morita theory of module categories over these DGA's.

The lecture ended with a discussion of the relationship of Fukaya categories of the cotangent bundle, T^*M with its canonical symplectic structure. This is the A_∞-category associated to the symplectic field theory of the cotangent bundle. It has been clear for some time that there is a close relationship between this field theory and the string topology field theory of the underlying manifold M. See for example, [1], [19], [8]. The objects of the Fukaya category $Fuk(T^*M)$ are exact, Lagrangian submanifolds $L \subset T^*M$. The morphisms are the "Lagrangian intersection Floer cochains", $CF^*(L_0, L_1)$. For a submanifold $N \hookrightarrow M$, let $\nu^*(N) \subset T^*M$ be the conormal bundle. These normal bundles are Lagrangian. Moreover Abbondandolo, A. Portaluri, and Schwarz [3] recently showed that the Floer cohomology, $HF^*(\nu^*(N_1), \nu^*(N_2))$ is isomorphic to the homology of the path space, $H_*(\mathcal{P}_{N_1,N_2})$. Using work of Fukaya-Seidel-Smith [13], Abouzaid [4], and Nadler [17], it is reasonable to conjecture that certain full subcategories of the Fukaya category $Fuk(T^*M)$ are isomorphic to corresponding full subcategories of the string topology category $\mathcal{S}_M$. Furthermore, with the classification of field theories due to Hopkins-Lurie [15], it was discussed how it is reasonable to believe that such isomorphisms of A_∞-categories should extend to imply isomorphisms of the corresponding open-closed field theories: the symplectic field theory of $T^*(M)$ on the one hand, and the string topology field theory $\mathcal{S}_M$ on the other hand. Evidence for this conjecture, and how it might be implied by the Hopkins-Lurie classification scheme was also discussed.

References

[1] A. Abbondandolo and M. Schwarz, *On the Floer homology of cotangent bundles*, Comm. Pure Appl. Math. **59**, (2006) , 254–316.

[2] A. Abbondandolo and M. Schwarz *Floer Homology of cotangent bundles and the loop poduct*, preprint MPI MIS no. 41, May 2008.

[3] A. Abbondandolo, A. Portaluri, and M. Schwarz, *The homology of path spaces and Floer homology with conormal boundary conditions* Journal of Fixed Point Theory and Applications 4 (2008), 263-293. math.SG. arXiv:0810.1977

[4] M. Abouzaid, *The wrapped Fukaya category and based loops*, preprint (2009).

[5] M. Abouzaid, *A topological model for the Fukaya categories of plumbings*, preprint (2009), arXiv:0904.1474.

[6] A. Blumberg, R.L. Cohen, and C. Teleman, *Open-closed field theories, string topology, and Hochschild homology*, preprint (2009) http://math.stanford.edu/∼ralph/arollabranes-submit.pdf

[7] M. Chas and D. Sullivan, *String Topology.* (preprint: math.GT/9911159).

[8] K. Cieliebak and J.Latschev, *The role of string topology in symplectic field theory*, preprint (2007), arXiv: mathSG/0706.3284

[9] R.L. Cohen and V. Godin, *A polarized view of string topology*, Topology, Geometry, and Quantum Field Theory, London Math. Soc. Lecture Notes **vol. 308**, (2004), 127-154.

[10] R.L. Cohen and J. Klein, *Umkehr maps*, Homology, Homotopy, and Applications, **vol. 11** (1), (2009), 17-33. preprint: arXiv:0711.0540

[11] K. Costello, *Topological conformal field theories and Calabi-Yau categories.* Adv. Math. 210 (2007), no. 1, 165–214.

[12] W. Dwyer, J. Greenlees, and S. Iyengar, *Duality in algebra and topology*, Adv. Math. 200(2006), no. 2, 357-402. preprint: math.AT/0510247

[13] K. Fukaya, P. Seidel, and I. Smith *The symplectic geometry of cotangent bundles from a categorical viewpoint* preprint (2007), arXiv:0705.3450.

[14] V. Godin, *Higher string topology operations*, preprint: math.AT/0711.4859

[15] J. Lurie, *On the classification of topological field theories*, preprint, (2009).

[16] G.W. Moore and G. Segal, *D-branes and K-theory in 2D topological field theory*, preprint: hep-th/0609042

[17] D. Nadler, *Microlocal branes are constructible sheaves* Preprint, math.SG/0612399.

[18] D. Sullivan, *Open and closed string field theory interpreted in classical algebraic topology*, Topology, geometry and quantum field theory, London Math. Soc. Lecture Notes, vol. 308, Cambridge Univ. Press, Cambridge, 2004, math.QA/0302332, pp. 344–357.

[19] C. Viterbo, *Functors and computations in Floer homology with applications, Part II*, Preprint, 1996.

The Kervaire invariant problem

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(joint work with Mike Hill and Doug Ravenel)

The authors recently proved that the elements θ_j do not exist for $j > 6$. Here θ_j is a hypothetical element of order 2 in the stable homotopy groups of spheres in dimension $2^{j+1} - 2$.
In 1960, Kervaire defined a $\mathbb{Z}/2$-valued invariant for closed, smooth manifolds with a stable framing. In geometric terms, the above result means that the only possible dimensions for such manifolds with nontrivial Kervaire invariant are

$$2, 6, 14, 30, 62, 126.$$

The first 5 dimensions were previously known to be realized, the first 3 by $S^j \times S^j$ for $j = 1, 3, 7$. The status of θ_6 (in dimension 126) remains open.
The theorem implies that the kernel and cokernel of the Kervaire-Milnor map (from the group of homotopy spheres to the homotopy group of spheres)

$$\Theta_n \to \pi_n^{st}/im(J)$$

are completely known finite abelian groups. Here Θ_n is the group of exotic smooth structures on S^n and the map associates to it the underlying framed manifold. The image of $J : KO_{n+1} \to \pi_n^{st}$ realizes the different choices of framings on such homotopy spheres.

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