Mathematisches Forschungsinstitut Oberwolfach

Report No. 30/2009

DOI: 10.4171/OWR/2009/30

Algebraische Zahlentheorie

Organised by
Guido Kings, Regensburg
Mark Kisin, Chicago
Otmar Venjakob, Heidelberg

June 21st – June 27th, 2009

Abstract. The workshop brought together researchers from Europe, the US and Japan, who reported on various recent developments in algebraic number theory and related fields. Dominant themes were p-adic methods, L-functions and automorphic forms but other topics covered a very wide range of algebraic number theory.

Mathematics Subject Classification (2000): 11R, 11S.

Introduction by the Organisers

The talks in this workshop gave a very broad perspective of recent developments in algebraic number theory. The topics treated can be grouped together in several dominating themes: Height pairings for cycles on Shimura varieties and derivatives of L-functions, p-adic methods (p-adic Galois representations, relative Fontaine theory and parallel transport for p-adic vector bundles), new results on Mordell-Weil groups for elliptic curves, Iwasawa theory and L-values, higher dimensional class field theory.

Three talks were related to the relation between height pairings of cohomologically trivial cycles and derivatives of L-functions. The talk by Bruinier reported on joint work with Yang about the Arakelov height pairing of cycles on the Shimura variety for the group $O(n, 2)$, where the cycles are defined by Shimura varieties for the group $O(n-1, 2)$. The talk by Zhang was devoted to his joint result with Yuan and Zhang on the relation of a height pairing of Gross-Schoen cycles on 3-fold products of Shimura curves to the derivative of the triple product L-function. Results in a similar direction were also presented by Howard (joint work with Yang). They show that the intersection numbers of Hirzebruch-Zagier cycles at finite places are encoded Fourier coefficients of the derivative of a non-holomorphic Eisenstein series.

Another group of three talks concerned Fontaine's theory of p-adic Galois representations of local fields. This theory is extremely active, in particular in connection with the p-adic Langlands program. Berger reported on the classification of potentially trianguline representations in dimension 2, a notion introduced by Colmez in connection with his work on the p-adic Langlands correspondence. Carouso reported on two results about the ramification of semi-stable Galois representations, treating the cases of tame and wild inertia actions. Fontaine's result concerned an elaboration of results by Kisin on finite group schemes.

Werner talked about joint work with Deninger on vector bundles on p-adic curves and parallel transport. In contrast to earlier result one can now also treat vector bundles which have strongly semi-stable reduction after pullback to a ramified covering.

The talk by Andreatta was about a relative version of Fontaine's theory and the application to Faltings' comparison result.

The generalization of the ∞-fern introduced by Gouvea, Mazur and Coleman for modular curves to the Galois representations of type $U(3)$ was presented by Chenevier.

Kerz presented a new approach to higher dimensional class field theory pioniered by Wiesend, which was refined and elaborated by him in joint work with Schmidt.

The talk by Geisser was somewhat related. He discussed Suslin homology and cohomology and especially its p-part. He formulates a generalization of a conjecture by Kato and explained the relation to higher dimensional class field theory.

Stix discussed non-abelian examples of the section conjecture. He showed that certain curves admit no sections by showing that the Brauer-Manin obstruction is the only obstruction to rational points.

A new approach to Ekedahl-Oort strata via level-1-truncations of loop groups was presented by Viehmann. In fact all known relations between these strata can be expressed in terms of group theoretical data of a loop group attached to the corresponding Shimura variety of PEL-type.

The talk by Jannsen was of a more algebraic geometric nature and presented a canonical resolution of singularities of 2-dimensional excellent schemes. This very strong result is needed in a lot of arithmetic applications.

Two talks presented new results on ranks of Mordell-Weil groups of elliptic curves. Dokchitser presented the result obtained with his brother about the parity of ranks of elliptic curves. They can show, that if the Shavarevich-Tate group is finite, then the parity of the Mordell-Weil rank is completely determined by the sign of the root number. The other result, by Mazur and Rubin, is that over each number field there are infinitely many elliptic curves of Mordell-Weil rank 0 and if the dimension of the 2-torsion of the Shavarevich-Tate group is even, then there are even infinitely many curves of rank 1.

There were two talks devoted to Iwasawa theory. Kakde talked about the results in his thesis about congruences in non-commutative Iwasawa theory for totally real fields. Building on work of Kato, he was able to prove the congruences necessary to show the Iwasawa main conjecture for some semi-direct products of abelian groups.

Van Order explained her results on the two variable main conjecture for elliptic curves over $\mathbb{Q}$ in the $\mathbb{Z}_p^2$-extension over an imaginary quadratic field. Here she obtains some divisibility results, building on work by Kato and Rohrlich.

Goncharov explained his construction of mixed motives via his theory of "Hodge correlators". The Hodge realization of these motives can be described in terms of Green functions and their derivatives. For modular curves one gets in particular the Beilinson-Kato elements.

Workshop: Algebraische Zahlentheorie

Table of Contents

Abstracts

Canonical embedded and non-embedded resolution of singularities for excellent two-dimensional schemes

Uwe Jannsen
(joint work with Vincent Cossart, Shuji Saito)

Mainly by work of Hironaka [7], there is a very strong form of resolution of singularities for schemes of characteristic zero. But there are only very few results on birational resolution for varieties over fields of positive characteristic, not to mention schemes of mixed characteristic. The talk presented the following results obtained in [4], which are valid for any excellent scheme X of dimension 2.

Theorem 1 (Canonical controlled resolution) There exists a canonical finite composition of morphisms

$$\pi : X' = X_n \to \ldots \to X_1 \to X_0 = X$$

such that X' is regular and, for each i, $X_{i+1} \to X_i$ is the blow-up of X_i in a permissible center $D_i \subset X_i$ which is contained in $(X_i)_{sing}$, the singular locus of X_i. In particular, π is an isomorphism over $X_{reg} = X - X_{sing}$. This sequence is functorial in the sense that it is compatible with automorphisms of X (every such automorphism extends to the sequence in a unique way) and with Zariski or étale localizations $U \to X$ (the pull-back to to U is the canonical resolution sequence for U after suppressing the morphisms which become isomorphisms over U).

Following Hironaka, a subscheme $D \subset X$ is called permissible, if D is regular and X is normally flat along D, i.e., all sheaves J^n/J^{n+1} are locally free $\mathcal{O}_X/J$-modules, where $J \subset \mathcal{O}_X$ is the ideal sheaf of D.

Theorem 2 (Canonical embedded resolution) Let $i : X \hookrightarrow Z$ be a closed immersion, with Z regular and excellent. There is a canonical commutative diagram

$$\begin{tikzcd} X' \arrow[r, "i'"] \arrow[d, "\pi"'] & Z' \arrow[d, "\pi_Z"] \\ X \arrow[r, "i"] & Z \end{tikzcd}$$

where X' and Z' are regular, i' is a closed immersion, and π and π_Z are proper and surjective morphisms inducing isomorphisms over $Z - X_{sing}$. The morphisms π and π_Z are compatible with automorphisms of (X, Z) and (Zariski or étale) localizations in Z.

In fact, starting from Theorem 1 one gets a sequence $Z' = Z_n \to \ldots Z_1 \to Z_0 = Z$ by blowing up "in the same centers" and identifying X_{i+1} with the strict transform of X_i in Z_{i+1}. Then all Z_i are regular since the blow-up of a regular scheme in a regular center is again regular. We obtain several refinements.

Theorem 3 (Canonical embedded resolution with boundary) Let $i : X \hookrightarrow Z$ be a closed immersion into a regular scheme Z, and let $B \subset Z$ be a simple normal crossings divisor such that no irreducible component of X is contained in B. Then there is a canonical commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{i'} & Z' & \supset & B' \\ {\scriptstyle \pi_X}\big\downarrow & & {\scriptstyle \pi_Z}\big\downarrow & & \\ X & \xrightarrow{i} & Z & \supset & B \end{array}$$

where i' is a closed immersion of regular schemes, $B' = \pi_Z^{-1}(B) \cup E$ (with E the exceptional locus of π_Z) is a strict normal crossings divisor on Z', and X' intersects B' transversally on Z'. Moreover, π_X and π_Z are projective, surjective, isomorphisms outside $X_{sing} \cup (X \cap B)$, and compatible with automorphisms of (Z, X, B) and with Zariski or étale localizations in Z.

In the paper [9], this theorem is applied to obtain finiteness results for certain motivic cohomology groups of varieties over finite fields. Another application is:

Corollary 1 Let Z be a regular excellent scheme (of any dimension), and let $X \subset Z$ be a reduced closed subscheme of dimension at most two. Then there exists a projective surjective morphism $\pi : Z' \longrightarrow Z$ which is an isomorphism over $Z - X$, such that $\pi^{-1}(X)$, with the reduced subscheme structure, is a strict normal crossings divisor on Z'.

In Theorem 3, π and π_Z are obtained by successive blow-ups in permissible centers D which are transversal with the respective normal crossing divisors, which in turn are obtained as the full transforms (including the exceptional divisors) of the previous normal crossing divisors. We also obtain a more general version, in which B can contain irreducible components of X. In addition, we get a variant for non-reduced schemes X, in which case $(X')_{red}$ is regular and normal crossing with B and X' is normally flat along $(X')_{red}$. Moreover, we obtain a variant, in which we only consider strict transforms for the normal crossings divisor, i.e., where we forget about the exceptional divisors. Theorem 1, i.e., the case where we do not assume any embedding for X, is also proved in a more general version, which allows a non-reduced scheme X as well as a so-called boundary on X, a notion which is newly introduced by us. Again this theorem comes in two versions, one with complete transforms and one with strict transforms. Our approach implies that these last results imply both Theorem 1 and Theorem 3. In particular, the canonical resolution sequence of Theorem 3 for $B = \emptyset$ and strict transforms coincides with the intrinsic sequence for X from Theorem 1.

To our knowledge, none of the three theorems is known, at least not in the stated generality. Even for $\dim(X) = 1$ we do not know a reference for these results, although they may be well-known. Zariski [12] proved Theorem 1 (without discussing canonicity or functoriality) for irreducible surfaces over algebraically closed fields of characteristic zero, and in [13] proved Corollary 1 for surfaces over fields of characteristic zero which are embedded in a non-singular threefold.

Abhyankar [2] extended the latter result to all algebraically closed fields (see also [5] for a shorter version). For schemes of characteristic zero and arbitrary dimension, Theorems 1, 2 and 3 were proved by Hironaka [7], but constructivity, canonicity or functoriality were only addressed in the later literature, see, e.g., [11], [3], and [6]. As for weaker versions of resolution, Abhyankar [1] showed how to resolve a surface over an algebraically closed field by so-called local uniformization, and Lipman [10] obtained resolution of singularities for arbitrary excellent two-dimensional schemes X, by a finite sequence $X_n \to \ldots X_1 \to X$ alternating normalization, and blow-ups in finitely many isolated singular points. But the processes of uniformization or normalization are not obtained by permissible blow-ups, and it is not known how to extend them to an ambient regular scheme Z like in Theorems 2 and 3, so these weaker results were not sufficient for the mentioned applications in [9].

Our approach is based on a method sketched by Hironaka (for hypersurfaces) in [8]. We use Hilbert-Samuel functions as invariants which measure the singularities and construct a sequence of blow-ups for which the invariants are non-increasing, and finally decreasing, so that in the end one reaches the regular situation. One blows up 'the worst locus', i.e., the stratum where the invariants are maximal, after possibly desingularising this stratum. The main point is to show that the invariants do finally decrease. In characteristic zero this is done by the method of maximal contact, but we show that maximal contact does not exist in positive characteristic, even for surfaces. Instead we use Hironaka's polyhedra.

References

[1] S. S. Abhyankar, *Local uniformization on algebraic surfaces over ground fields of characteristic $p \neq 0$*, Ann. of Math. (2) 63 (1956), 491–526.

[2] S. S. Abhyankar, *Resolution of singularities of embedded algebraic surfaces.* Pure and Applied Mathematics, Vol. 24 Academic Press, New York-London 1966 ix+291 pp.

[3] E. Bierstone, P. D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant.* Invent. Math. 128 (1997), no. 2, 207–302.

[4] V. Cossart, U. Jannsen, S. Saito, *Canonical embedded and non-embedded resolution of singularities for excellent two-dimensional schemes*, preprint math.AG.arXiv:0905.2191

[5] S. D. Cutkosky, *Resolution of singularities for 3-folds in positive charactersitic.* preprint.

[6] S. Encinas, H. Hauser, *Strong resolution of singularities in characteristic zero.* Comment. Math. Helv. 77 (2002), no. 4, 821–845.

[7] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero: I-II*, Ann. of Math. **79** (1964), 109–326.

[8] H. Hironaka, *Desingularization of excellent surfaces*, Advanced Science Seminar in Algebraic Geometry, (summer 1967 at Bowdoin College), Mimeographed notes by B. Bennet, Appendix to [CGO], 99–132.

[9] U. Jannsen, S. Saito, *Kato conjecture and motivic cohomology over finite fields*, preprint.

[10] J. Lipman, *Desingularization of two-dimensional schemes*, Ann. Math. (2) 107 (1978), no. 1, 151–207.

[11] O. Villamayor U., *Patching local uniformizations.* (English summary) Ann. Sci. École Norm. Sup. (4) 25 (1992), no. 6, 629–677.

[12] O. Zariski, *The reduction of the singularities of an algebraic surface*, Ann. of Math. (2) 40, (1939). 639–689.

[13] O. Zariski, *Reduction of the singularities of algebraic three dimensional varieties*, Ann. of Math. (2) 45, (1944). 472–542.

Parity of the rank of an elliptic curve

Vladimir Dokchitser
(joint work with Tim Dokchitser)

By the Mordell-Weil theorem, the group of K-rational points $E(K)$ of an elliptic curve E over a number field K is finitely generated. Its $\mathbb{Z}$-rank is the rank of the elliptic curve, $\mathrm{rk}_{E/K}$. I will discuss the parity of this rank. There is an obvious remark to make: if the rank is odd, then it is non-zero, and $E(K)$ must be infinite.

Root numbers. It should be directly pointed out that virtually nothing can be said concerning the parity of the Mordell-Weil rank without appealing to some conjectures. What will be discussed here is the *expected* behaviour of this parity.

Recall that the conjecture of Birch and Swinnerton–Dyer predicts that the rank of E should agree with its analytic rank, that is the order of vanishing at $s = 1$ of the L-function $L(E/K, s)$. Now $L(E/K, s)$ is expected to satisfy a functional equation of the form $L(E/K, s) \leftrightarrow \pm L(E/K, 2-s)$. Note that the sign determines the parity of the order of vanishing of the L-function at the central point $s = 1$. Part of the standard conjectural framework is a precise construction of the sign — it is given by the *global root number* $w(E/K)$. Thus we expect the parity formula

$$(-1)^{\mathrm{rk}_{E/K}} = w(E/K)\,.$$

By definition, the global root number is defined as the product of *local root numbers* $w(E/K_v) = \pm 1$:

$$w(E/K) = \prod\nolimits_v w(E/K_v)\,,$$

the product taken over all the places of K. The definition of the local root is rather elaborate and is also non-constructive (it is the same as the corresponding local epsilon-factor, except that it is scaled down to have absolute value 1; see [5, 6]). However, the crucial point is that it is a purely local invariant. In other words, the parity of the rank should be governed by purely local data.

Classification. To make the root numbers more concrete, here is a classification covering all cases except when E/K_v has additive reduction:

- If E/K_v has good reduction, then $w(E/K_v) = +1$.
- If $K_v = \mathbb{R}$ or $K_v = \mathbb{C}$, then $w(E/K_v) = -1$.
- If E/K_v has split multiplicative reduction, then $w(E/K_v) = -1$.
- If E/K_v has non-split multiplicative reduction, then $w(E/K_v) = +1$.

Most of the additive reduction cases can be found in [5] Thm 2. See also [3] Thms 1.3, 1.12 for a general, but slightly more cumbersome formula.

Examples. As explained above, the standard conjectures on L-functions and the Birch–Swinnerton-Dyer conjecture imply that the rank of E/K being even or odd is determined by whether the total product $\prod w(E/K_v)$ is $+1$ or -1. This very specific type of behaviour has strong arithmetic implications. Here are a few examples, whose proofs can be safely left as exercises. For a further discussion of examples 3, 4 and 5, see [4] and [2].

1. If $E/\mathbb{Q}$ is an elliptic curve and K an imaginary quadratic field in which all primes of bad reduction of E split (Heegner hypothesis), then $w(E/K) = -1$.

2. Every elliptic curve over $\mathbb{Q}$ should have even rank over $\mathbb{Q}(i, \sqrt{17})$. (The field has been chosen so that all rational primes split in it.)

3. The elliptic curve

$$y^2 + y = x^3 + x^2 + x$$

has rank 0 over $\mathbb{Q}$. It has split multiplicative reduction at 19 and good reduction at all other primes. Assuming the Birch–Swinnerton-Dyer conjecture, it must acquire a point of infinite order in the field $\mathbb{Q}(\sqrt[3]{m})$ for every cube-free $m \neq 0, 1$, as its rank over such a field must be odd.

4. The elliptic curve (of discriminant -11^4)

$$y^2 = x^3 + \tfrac{5}{4}x^2 - 2x - 7$$

has everywhere good reduction over the field $K = \mathbb{Q}(\sqrt[12]{-11})$. Its global root number is $+1$ over every finite extension of K, so its rank should be even over any number field containing $\sqrt[12]{-11}$. This elliptic curve does not have complex multiplication, which could in principle have accounted for this behaviour of the rank. Is it nevertheless possible to find some extra structure ("fake CM") on E that forces the rank to be always even?

5. The elliptic curve in the previous example already acquires everywhere good reduction over $F = \mathbb{Q}(\sqrt[6]{-11})$. As this field has an odd number of infinite places, the root number of E/F is -1. However it becomes $+1$ over every quadratic extension of F. It follows that for every $D \in F^\times/F^{\times 2}$ the quadratic twist of E/F by D should have positive (odd) rank. Thus Goldfeld's "$\frac{1}{2}$ average rank" conjecture for elliptic curves over $\mathbb{Q}$ fails over general number fields.

Arithmetic. The following is a recent result of T. Dokchitser and myself on the conjectural parity formula. See e.g. [1] §1 for a list of some other known results.

Theorem ([1] Thm 1.3, [3] Thm 1.2)**.** *Let E be an elliptic curve over a number field K, and set $F = K(E[2])$. If the Tate-Shafarevich group $Ш(E/F)$ is finite, then*

$$(-1)^{\mathrm{rk}_{E/K}} = w(E/K)\,.$$

Here is a sketch of the proof. I will derive a formula for the parity of the rank in terms of some local invariants, without taking the trouble to compare them to the local root numbers. At least morally, the latter is a purely local problem.

The crucial ingredient is Cassels' theorem, that the quantity

$$\frac{\mathrm{Reg}_{E/K} \cdot |Ш(E/K)| \cdot C_{E/K}}{\sqrt{\Delta_K} \cdot |E(K)_{\mathrm{tors}}|^2}$$

is the same for isogenous curves. Here $\mathrm{Reg}_{E/K}$ is the regulator, Δ_K is the discriminant of K, and $C_{E/K}$ is the product of the "local fudge factors" and periods of the curve that enters the Birch–Swinnerton-Dyer conjecture. (So for $K = \mathbb{Q}$, $C_{E/K} = \Omega_+ \prod_p c_p$, the real period multiplied by all the local Tamagawa numbers.)

Case 1: E admits a K-rational 2-isogeny $\phi : E \to E'$. Invoke Cassels' theorem, and look at the resulting expression *up to rational squares*. This eliminates the (difficult) contribution from Ш, and the resulting formula reads

$$\frac{\mathrm{Reg}_{E/K}}{\mathrm{Reg}_{E'/K}} = \frac{C_{E'/K}}{C_{E/K}} \cdot \square \,.$$

Using the fact that ϕ and its dual are adjoints with respect to the height pairing, one easily checks that the quotient of regulators is $2^{\mathrm{rk}_{E/K}} \cdot \square$. It follows that

$$\mathrm{rk}_{E/K} \equiv \mathrm{ord}_2 \frac{C_{E'/K}}{C_{E/K}} \mod 2\,,$$

a sum of local invariants.

Case 2: $\mathrm{Gal}(F/K) \simeq C_3$. Then $\mathrm{rk}_{E/K} \equiv \mathrm{rk}_{E/F} \mod 2$; apply Case 1 for E/F.

Case 3: $\mathrm{Gal}(F/K) \simeq S_3$. One checks that the two abelian varieties

$$E \times E \times \mathrm{Res}_{F/K} \qquad \text{and} \qquad \mathrm{Res}_{L/K}E \times \mathrm{Res}_{L/K}E \times \mathrm{Res}_{M/K}E$$

are isogenous; here L and $M = K(\sqrt{\Delta_E})$ are a cubic and a quadratic extension of K in F respectively, and Res denotes restriction of scalars. Invoking the analogue of Cassels' theorem for abelian varieties (Tate–Milne), looking modulo squares and making a regulator-computation leads to

$$\mathrm{rk}_{E/K} + \mathrm{rk}_{E/L} + \mathrm{rk}_{E/F} \equiv \mathrm{ord}_3 \frac{C_{E/F}C^2_{E/K}}{C_{E/M}C^2_{E/L}} \mod 2\,,$$

again a sum of local invariants. Case 1 expresses both $\mathrm{rk}_{E/L}$ and $\mathrm{rk}_{E/F}$ in terms of local data, so we deduce such an expression for $\mathrm{rk}_{E/K}$ as well.

Remark. The proof gives the following explicit formula for the parity of the rank, assuming $|Ш(E/F)| < \infty$. Write L/K for the smallest extension where E acquires a 2-torsion point, and E' for the corresponding isogenous curve. Then

$$\mathrm{rk}_{E/K} \quad \equiv \quad \begin{cases} \mathrm{ord}_2 \frac{C_{E/L}}{C_{E'/L}} & \text{if } [F{:}K]<6 \\ \mathrm{ord}_2 \frac{C_{E/L}C_{E/F}}{C_{E'/L}C_{E'/F}} + \mathrm{ord}_3 \frac{C_{E/F}C^2_{E/K}}{C_{E/K(\sqrt{\Delta_E})}C^2_{E/L}} & \text{if } [F{:}K]=6 \end{cases}$$

The terms on the right-hand-side can be computed in practice, see [3] footnote 1.

References

[1] T. Dokchitser, V. Dokchitser, *On the Birch–Swinnerton-Dyer quotients modulo squares*, 2006, arxiv: math.NT/0610290, to appear in Annals of Math.

[2] T. Dokchitser, V. Dokchitser, *Elliptic curves with all quadratic twists of positive rank*, Acta Arith. 137 (2009), 193–197.

[3] T. Dokchitser, V. Dokchitser, *Root numbers and parity of ranks of elliptic curves*, arxiv: 0906.1815.

[4] V. Dokchitser, *Root numbers of non-abelian twists of elliptic curves*, Proc. London Math. Soc. (3) 91 (2005), 300–324.

[5] D. Rohrlich, *Galois Theory, elliptic curves, and root numbers*, Compos. Math. 100 (1996), 311–349.

[6] J. Tate, *Number theoretic background*, in: Automorphic forms, representations and L-functions, Part 2 (ed. A. Borel and W. Casselman), Proc. Symp. in Pure Math. 33 (AMS, Providence, RI, 1979) 3-26.

Hodge correlators and generalized Rankin-Selberg integrals

Alexander Goncharov

1. A motivating example

Beilinson's conjectures on the special values of L-functions imply that special values of L-function of a motive are periods.

Periods are complex numbers which can be written as

$$\int_{\Delta_B} \Omega_A$$

where A, B are divisirs over $\mathbb{Q}$ in an n-dimensional smooth projective variety X over $\mathbb{Q}$, $\Omega_A \in \Omega^n_{\log}(X - A)$ is a form with logarithmic singularities along a divisor A, and Δ_B is an n-chain with boundary at $B(\mathbb{C})$, where B is a divisir in X.

Example. Let f be a weight 2 modular Hecke eigenform. Then

$$L(f, 2) = \int_0^\infty f(iy)ydy$$

The only way we know how to prove that this is a period is this. Let $Y(N)$ be the level N modular curve for sufficiently large N, so that $f(z)dz$ is a 1-form on $Y(N)(\mathbb{C})$. Let a be a degree zero divisor on $\overline{Y} - Y$. By Manin-Drinfeld Theorem, there exists a function $g_a \in \mathcal{O}(Y)^* \otimes \mathbb{Q}$ such that $\mathrm{div}(g_a) = (a)$. Then we have the following reamrkable facts:

A) The Rankin-Selberg method plus the work of Bloch and Beilinson tells that

$$\int_{Y(\mathbb{C})} \log|g_a| dlog|g_b| \wedge f(z)dz \sim L(f, 2)$$

where $\sim$ means equality up to certain explicitly known periods. (In particular, this implies that it is a period - we skip details here).

B) The above integral is the regulator of an element

$$\{g_a, g_b\} \in K_2(Y)$$

C) Finally, the elements $\{g_a, g_b\}$, suitably adjusted, form the Beilinson-Kato Euler system.

This leads to a natural

Question: Is there a general framework for this?

The Hodge correlators provide a general way to present periods of the motivic rational homotopy type of smooth varieties, togerther with their motivic avatars, Motivic correlators.

In the case when the variety is a modular curve, the simplest Hodge correlators deliver the Rankin-Selberg integrals, and their motivic avatars are the Beilinson's elements.

2. Hodge correlators

Let A be a graded algebra. Denote by $(\mathcal{C}_V, \delta)$ the cyclic Lie algebra complex. Namely,

$$\mathcal{C}_A := \oplus_{m=0}^{\infty}\Big(\otimes^m A[1]\Big)_{\mathbb{Z}/m\mathbb{Z}}$$

The differential δ on $\mathcal{C}_A$ is provided by products of neighbors in a cyclic word:

$$\delta(\overline{\alpha}_0 \otimes \ldots \otimes \overline{\alpha}_m)_{\mathcal{C}} = \mathrm{Cycle}_{m+1}(-1)^{|\alpha_m|}(\overline{\alpha}_0 \otimes \ldots \otimes \overline{\alpha}_{m-2} \otimes \overline{\alpha_{m-1} \cup \alpha_m})_{\mathcal{C}}.$$

Here Cycle_{m+1} means the sum of cyclic shifts, $\overline{\alpha} \in \mathbb{H}^*$ is the shifted by one element α and $|\overline{\alpha}|$ is its degree.

Now let X be a compact Kahler manifold of dimension n. Let us consider the following graded algebra without unit:

$$\mathbb{H}^* := \frac{H^*(X.\mathbb{C})}{(H^0(X.\mathbb{C}) \oplus H^{2n}(X.\mathbb{C})}$$

Set

$$\mathcal{H} := H_{2n}(X)[-2].$$

Theorem 2.1. *a) There is a canonical linear map, called the Hodge correlator map:*

$$\mathrm{Cor}^*_{\mathcal{H},a} : H^0_\delta\Big(\mathcal{C}_{\mathbb{H}^*} \otimes \mathcal{H}\Big) \longrightarrow \mathbb{C}. \tag{1}$$

b) It describes the real mixed Hodge structure on the rational homotopy type of X.

In particular, let X be now a smooth projective variety over $\mathbb{Q}$. Then $\mathbb{H}^* = \mathbb{H}^*_{\mathrm{DR}} \otimes \mathbb{C}$, where $\mathbb{H}^*_{\mathrm{DR}}$ is the reduced de Rham cohomology. Thanks to the part b), the image of the Hodge correlator map

$$\mathrm{Cor}^*_{\mathcal{H},a} : H^0_\delta\Big(\mathcal{C}_{\mathbb{H}^*_{\mathrm{DR}}} \otimes \mathcal{H}\Big) \longrightarrow \mathbb{C}$$

lies in the subring of periods.

The Hodge correlator map is defined as the correlator map assigned to a certain Feynman integral related to X.

It can be generalized to the case when X is open, e.g. an open curve. In the latter case we take

$$\mathbb{H}^* := \mathrm{gr}^W H^1(X)$$

In the case when the curve is the open modular curve, the Hodge correlator of the length three cyclic word

$$\mathcal{C}(\delta_a \otimes \delta_b \otimes [f(z)dz]) \tag{2}$$

is nothing esle as the Rankin-Selberg integral discussed above. Here δ_a is the class in $\mathrm{gr}^W H^1(X)$ assigned to the degree zero divisor a on $\overline{X} - X$.

3. Motivic correlators

Denote by $\mathrm{Colie}_{\mathrm{Mot}}$ the hypothetical Motivic Lie coalgebra of the categopry of all mixed motives over $\mathbb{Q}$. It is a Lie coalgebra in the category of all pure motives over $\mathbb{Q}$. Then, assuming the motivic formalism, there is a canonical map

$$\mathrm{Cor}^*_{\mathrm{Mot}} : H^0_\delta\Big(\mathcal{C}_{\mathbb{H}^*_{\mathrm{Mot}}} \otimes \mathcal{H}_{\mathrm{Mot}}\Big) \longrightarrow \mathrm{Colie}_{\mathrm{Mot}}$$

Here $\mathbb{H}^*_{\mathrm{Mot}}$ is a pure motive whose Betti realization is $\mathbb{H}^*$. Its composition with the natural period map

$$\mathrm{Colie}_{\mathrm{Mot}} \longrightarrow \mathbb{C}$$

is the Hodge correlator map.

In particular, the element (2) maps under the motivic correlator map to the Beilinson's element $\{g_a, g_b\}$ projected on the isotipical component corresponding the Hecke eigenform $f(z)dz$. For simplicity we assume it is defined over $\mathbb{Q}$.

The key point is that the map $\mathrm{Cor}^*_{\mathrm{Mot}}$ is a homomorphism of Lie coalgebras.

The details are available in [G1], [G2].

All known to me explicitly constructed elements in the motivic cohomology related to non-critial values of L-functions turnes out to be Motivic correlators, while the Hodge correlator delivers the corresponding Rankin-Selberg integral.

References

[G1] Goncharov A.B.: *Hodge correlators* arXiv:0803.0297.
[G2] Goncharov A.B.: *Hodge correlators II.* To appear in Moscow Math Journal, 2010. arXiv:0807.4855

Faltings heights of CM cycles and derivatives of L-functions

Jan Hendrik Bruinier
(joint work with Tonghai Yang)

Let E be an elliptic curve over $\mathbb{Q}$. Assume that its L-function $L(E, s)$ has an odd functional equation so that the central critical value $L(E, 1)$ vanishes. In this case the Birch and Swinnerton-Dyer conjecture predicts the existence of a rational point of infinite order on E. It is natural to ask if is possible to construct such a point explicitly. The work of Gross and Zagier [11] provides such a construction when $L'(E, 1) \neq 0$.

Let N be the conductor of E, and let $X_0(N)$ be the moduli space of cyclic isogenies of degree N of generalized elliptic curves. Let K be an imaginary quadratic field of discriminant D such that D is a square modulo $4N$. Gross and Zagier consider a divisor on $X_0(N)$ given by elliptic curves with complex multiplication by the maximal order of K. By the theory of complex multiplication, this divisor is defined over K. Taking the trace and using a modular parameterization $X_0(N) \to E$, one obtains a $\mathbb{Q}$-rational point $y^E(D)$ on E . The Gross-Zagier formula states that the canonical height of $y^E(D)$ is given by the derivative of the L-function of E over K at $s = 1$, more precisely

$$\langle y^E(D), y^E(D)\rangle_{NT} = C\sqrt{|D|}L'(E,1)L(E,\chi_D,1).$$

Here C is an explicit non-zero constant which is independent of K, and $L(E,\chi_D,s)$ denotes the quadratic twist of $L(E,s)$ by the quadratic Dirichlet character χ_D corresponding to $K/\mathbb{Q}$. It is always possible to choose K such that $L(E,\chi_D,1)$ is non-vanishing. So, in this case, $y^E(D)$ has infinite order if and only if $L'(E,1) \neq 0$.

The work of Gross and Zagier triggered a lot of further research on height pairings of algebraic cycles on Shimura varieties, see e.g. [9], [19], [20], [12], [15], [16]. In most of this work, the connection between a height pairing and the derivative of an automorphic L-function comes up in a rather indirect way.

In our joint work with T. Yang [7], we consider a different approach to obtain identities between certain height pairings on Shimura varieties of orthogonal type and derivatives of automorphic L-functions. It is based on the Borcherds lift [1] and its generalization in [4], [5]. We propose a conjecture for the Faltings height pairing of arithmetic special divisors and CM cycles. We compute the archimedean contribution to the height pairing. Using this result we prove the conjecture in certain low dimensional cases.

Let (V,Q) be a quadratic space over $\mathbb{Q}$ of signature $(n,2)$, and let $H = \mathrm{GSpin}(V)$. We realize the hermitian symmetric space corresponding to $H(\mathbb{R})$ as the Grassmannian $\mathbb{D}$ of oriented negative definite two-dimensional subspaces of $V(\mathbb{R})$. For a compact open subgroup $K \subset H(\mathbb{A}_f)$ we consider the Shimura variety

$$X_K = H(\mathbb{Q})\backslash\big(\mathbb{D} \times H(\mathbb{A}_f)/K\big).$$

It is a quasi-projective variety of dimension n, which is defined over $\mathbb{Q}$, see [13]. Note that for small n there are exceptional isomorphisms relating H to other classical groups. For instance $\mathrm{GSpin}(1,2) \cong \mathrm{GL}_2(\mathbb{R})$, so in the $n = 1$ case we are essentially looking at modular curves. Hilbert modular surfaces can be viewed as a particular $n = 2$ case and Siegel modular threefolds as a $n = 3$ case.

Let $L \subset V$ be an even lattice, and write L' for the dual of L. The discriminant group L'/L is finite. Throughout we assume that $K \subset H(\mathbb{A}_f)$ stabilizes $\hat{L} = L\otimes_{\mathbb{Z}}\hat{\mathbb{Z}}$ and that K acts trivially on L'/L. This is no loss of generality, since we can always fulfil this assumption by choosing K sufficiently small.

It is an important feature of such Shimura varieties that they come with natural families of algebraic cycles in all codimensions, see e.g. [13]. These special cycles arise from embeddings of rational quadratic subspaces $V' \subset V$ of signature $(n',2)$ with $0 \leq n' \leq n$. It is an interesting problem to consider height pairings of arithmetic versions of special cycles in complementary codimension, see [15]. In the present paper we study this problem for special divisors (where $n' = n-1$) and special 0-cycles (where $n' = 0$).

Let $U \subset V$ be a negative definite two-dimensional rational subspace of V. The Shimura variety corresponding to U is 0 dimensional and has a natural map to X_K. It defines a CM cycle $Z(U)$ on X_K, cf. [17]. Moreover, for any coset $\mu \in L'/L$ and any positive rational number m with $Q(\mu) \equiv m \mod 1$, we have a special divisor

$Z(m,\mu)$. It is given by Shimura subvarieties corresponding to rational quadratic subspaces $x^\perp$ for $x \in L+\mu$ with $Q(x)=m$.

An arithmetic divisor on X_K is a pair (x, g_x) consisting of a divisor x on X_K and a Green function g_x of logarithmic type for x. For the divisors $Z(m,\mu)$ we obtain such Green functions by means of the regularized theta lift of harmonic weak Maass forms. We consider the the subspace S_L of the space of Schwartz functions on $V(\mathbb{A}_f)$ generated by the characteristic functions $\phi_\mu = \mathrm{char}(\mu + \hat{L})$ of the cosets $\mu \in L'/L$. The metaplectic extension $\Gamma' = \mathrm{Mp}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{Z})$ has a Weil representation ρ_L on S_L, see e.g. [1].

Let $k \in \frac{1}{2}\mathbb{Z}$. We write $M^!_{k,\rho_L}$ for the space of S_L-valued weakly holomorphic modular forms of weight k for Γ' with representation ρ_L. Recall that weakly holomorphic modular forms are those meromorphic modular forms whose poles are supported at the cusps. The space of weakly holomorphic modular forms is contained in the space H_{k,ρ_L} of harmonic weak Maass forms of weight k for Γ' with representation ρ_L. Recall that harmonic weak Maass forms are real analytic modular forms which are annihilated by the weight k Laplacian and which may have poles at the cusps. An element $f \in H_{k,\rho_L}$ has a Fourier expansion of the form

$$f(\tau) = \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n \gg -\infty}} c^+(n,\mu) q^n \phi_\mu + \sum_{\mu \in L'/L} \sum_{\substack{n \in \mathbb{Q} \\ n<0}} c^-(n,\mu) \Gamma(1-k, 4\pi|n|v) q^n \phi_\mu,$$

where $\Gamma(a,t)$ denotes the incomplete Gamma function, and v is the imaginary part of $\tau \in \mathbb{H}$. Note that there are only finitely many $n<0$ for which $c^+(n,\mu)$ is non-zero. There is an antilinear differential operator $\xi : H_{k,\rho_L} \to S_{2-k,\bar{\rho}_L}$ to the space of cusp forms of weight $2-k$ with dual representation. It is surjective and its kernel is equal to $M^!_{k,\rho_L}$.

For $\tau \in \mathbb{H}$, $z \in \mathbb{D}$ and $h \in H(\mathbb{A}_f)$, let $\theta_L(\tau, z, h)$ be the Siegel theta function associated to the lattice L. For a harmonic weak Maass form $f \in H_{1-n/2,\bar{\rho}_L}$ of weight $1-n/2$, we consider the regularized theta integral

$$\Phi(z,h,f) = \int_{\mathcal{F}}^{reg} \langle f(\tau), \theta_L(\tau,z,h)\rangle \, d\mu(\tau),$$

see [4], [5]. It turns out that $\Phi(z,h,f)$ is a logarithmic Green function for the divisor

$$Z(f) = \sum_{\mu \in L'/L} \sum_{m>0} c^+(-m,\mu) Z(m,\mu)$$

in the sense of Arakelov geometry (see [18]). The pair $\hat{Z}(f) = (Z(f), \Phi(\cdot, f))$ defines an arithmetic divisor on X_K.

We aim to compute the Faltings height pairing of the arithmetic special divisor $\hat{Z}(f)$ and the CM cycle $Z(U)$. The pairing is a sum of an archimedean and a non-archimedean contribution. We begin by computing the archimedean part. It is given by the evaluation $\frac{1}{2}\Phi(Z(U), f)$ of the Green function of $\hat{Z}(f)$ at the cycle $Z(U)$.

By means of the splitting $V = U^\perp \oplus U$, we obtain definite lattices $N = L \cap U$ and $P = L \cap U^\perp$. Let

$$\theta_P(\tau) = \sum_{\lambda \in P'} q^{Q(\lambda)} \phi_\lambda = \sum_{\mu \in P'/P} \sum_{m \geq 0} r(m, \mu) q^m \phi_\mu$$

be the theta series in $M_{n/2,\rho_P}$ associated to the positive definite lattice P. The Fourier coefficients $r(m, \mu)$ are the representation numbers of m by the coset $\mu + P$. For to the negative definite 2-dimensional lattice N there is a similar theta series. The corresponding genus theta series is related to an incoherent Eisenstein series $E_N(\tau, s; 1)$ of weight 1 via the Siegel Weil formula. Its central derivative $\mathcal{E}_N(\tau) = E'_N(\tau, 0; 1)$ is a harmonic weak Maass form in H_{1,ρ_N}.

For a cusp form $g \in S_{1+n/2,\rho_L}$ with Fourier expansion

$$g = \sum_{\mu} \sum_{m>0} b(m, \mu) q^m \phi_\mu$$

we consider the Rankin type L-function

$$L(g, U, s) = (4\pi)^{-(s+n)/2} \Gamma\left(\tfrac{s+n}{2}\right) \sum_{m>0} \sum_{\mu \in P'/P} r(m, \mu) \overline{b(m, \mu)} m^{-(s+n)/2}.$$

Under mild assumptions on U, the completed L-function $L^*(g, U, s) := \Lambda(\chi_D, s+1) L(g, U, s)$ satisfies the functional equation $L^*(g, U, s) = -L^*(g, U, -s)$. Consequently, it vanishes at $s = 0$, the center of symmetry, and it is of interest to describe the derivative $L'(g, U, 0)$.

Theorem 0.1. *Let $f \in H_{1-n/2,\bar{\rho}_L}$, and assume that the constant term $c^+(0, 0)$ of f vanishes. We have*

$$\Phi(Z(U), f) = \deg(Z(U)) \cdot \left(\mathrm{CT}\left(\langle f^+, \theta_P \otimes \mathcal{E}_N^+ \rangle\right) + L'(\xi(f), U, 0)\right).$$

Here f^+ and $\mathcal{E}_N^+$ denote the holomorphic parts of the harmonic weak Maass forms f and $\mathcal{E}_N$. Moreover, $\mathrm{CT}(S)$ denotes the constant term of a q-series S.

The first summand on the right hand side is an explicit (rational) linear combination of the coefficients $\kappa(m, \mu)$ of $\mathcal{E}_N^+$. Each of these coefficients is a rational linear combination of $\log(p)$ for primes p, which can be computed explicitly.

The theorem can be proved by combining the approach of Kudla and Schofer to evaluate regularized theta integrals on special cycles (see [14], [17]) with results on harmonic weak Maass forms and automorphic Green functions obtained in [5].

When f is actually weakly holomorphic then $\xi(f) = 0$. So the second summand on the right hand side of the formula vanishes. Moreover, $\Phi(z, h, f) = -2 \log |\Psi(z, h, f)|^2$ where $\Psi(z, h, f)$ is a rational function on X_K, namely the Borcherds lift of f, see [1]. Hence Theorem 0.1 says that

$$\log |\Psi(Z(U), f)| = -\frac{\deg(Z(U))}{4} \mathrm{CT}\left(\langle f^+, \theta_P \otimes \mathcal{E}_N^+ \rangle\right).$$

One obtains an explicit formula for the prime factorization of $\Psi(Z(U), f)$, see [17]. It generalizes the formula of Gross and Zagier on singular moduli, that is, CM values of the j-function.

We now sketch a conjectural formula for the Faltings height pairing of arithmetic special divisors and CM cycles. Assume that there is a regular scheme $\mathcal{X}_K \to \operatorname{Spec}\mathbb{Z}$, projective and flat over $\mathbb{Z}$, whose associated complex variety is a smooth compactification of X_K. Let $\mathcal{Z}(f)$ and $\mathcal{Z}(U)$ be suitable extensions to $\mathcal{X}_K$ of the cycles $Z(f)$ and $Z(U)$, respectively. Such extensions can be found in many cases (when n is small) using a moduli interpretation of $\mathcal{X}_K$, see e.g. [15], [16], or by taking flat closures. Then the pair $\hat{\mathcal{Z}}(f) = (\mathcal{Z}(f), \Phi(\cdot, f))$ defines an arithmetic divisor.

Conjecture 0.2. *Let $f \in H_{1-n/2,\bar{\rho}_L}$, and assume that the constant term $c^+(0,0)$ of f vanishes. Then*

$$\langle \hat{\mathcal{Z}}(f), \mathcal{Z}(U)\rangle_{Fal} = \frac{\deg(Z(U))}{2} L'(\xi(f), U, 0).$$

In [7] we proved this conjecture in many cases of small dimension for $n = 0, 1, 2$. In particular, for $n = 1$ we obtained a new proof of the Gross-Zagier formula. For this we let V be the rational quadratic space of signature $(1, 2)$ given by the trace zero 2×2 matrices with the quadratic form $Q(x) = N \det(x)$, where N is a fixed positive integer. Then $H \cong \mathrm{GL}_2$. We chose the lattice $L \subset V$ and the compact open subgroup $K \subset H(\mathbb{A}_f)$ such that X_K is isomorphic to the modular curve $\Gamma_0(N)\backslash\mathbb{H}$. The special divisors $Z(m, \mu)$ and the CM cycles $Z(U)$ are both supported on CM points and therefore closely related.

The space $S_{3/2,\rho_L}$ can be identified with the space of Jacobi cusp forms of weight 2 and index N. Recall that there is a Shimura lifting from this space to cusp forms of weight 2 for $\Gamma_0(N)$, see [10]. Let G be a normalized newform of weight 2 for $\Gamma_0(N)$ whose Hecke L-function $L(G, s)$ satisfies an odd functional equation. There exists a newform $g \in S_{3/2,\rho_L}$ corresponding to G under the Shimura correspondence. It turns out that the L-function $L(g, U, s)$ is proportional to $L(G, s + 1)$.

We may choose $f \in H_{1/2,\bar{\rho}_L}$ with vanishing constant term such that $\xi(f) = \|g\|^{-2} g$ and such that the principal part of f has coefficients in the number field generated by the eigenvalues of G. Then $Z(f)$ defines an explicit point in the Jacobian of $X_0(N)$, which lies in the G isotypical component. In this case Conjecture 0.2 essentially reduces to the following Gross-Zagier type formula for the Neron-Tate height of $Z(f)$.

Theorem 0.3. *The Neron-Tate height of $Z(f)$ is given by*

$$\langle Z(f), Z(f)\rangle_{NT} = \frac{2\sqrt{N}}{\pi\|g\|^2} L'\big(G, 1).$$

The proof of this result which we give in [7] is quite different from the original proof of Gross and Zagier and uses *minimal* information on finite intersections between special divisors. Instead, we derive it from Theorem 0.1, modularity of the generating series of special divisors (Borcherds' approach to the Gross-Kohnen-Zagier theorem [2], [6]), and multiplicity one for the subspace of newforms in $S_{3/2,\rho_L}$. Another crucial ingredient is the non-vanishing result for coefficients of weight 2 Jacobi cusp forms by Bump, Friedberg, and Hoffstein [8]. Employing in addition the Waldspurger type formula for the coefficients of g [10], we also obtain the Gross-Zagier formula as stated at the beginning.

References

[1] *R. Borcherds*, Automorphic forms with singularities on Grassmannians, Inv. Math. **132** (1998), 491–562.

[2] *R. E. Borcherds*, The Gross-Kohnen-Zagier theorem in higher dimensions, Duke Math. J. **97** (1999), 219–233.

[3] *J.-B. Bost, H. Gillet, and C. Soulé*, Heights of projective varieties and positive Green forms. J. Amer. Math. Soc. **7** (1994), 903–1027.

[4] *J. H. Bruinier*, Borcherds products on $\mathrm{O}(2,l)$ and Chern classes of Heegner divisors, Springer Lecture Notes in Mathematics **1780**, Springer-Verlag (2002).

[5] *J. H. Bruinier and J. Funke*, On two geometric theta lifts, Duke Math. Journal. **125** (2004), 45–90.

[6] *J. H. Bruinier and K. Ono*, Heegner divisors, L-functions and harmonic weak Maass forms, Annals of Math., accepted for publication.

[7] *J. H. Bruinier and T. Yang*, Faltings heights of CM cycles and derivatives of L-functions, Invent. Math., accepted for publication.

[8] *D. Bump, S. Friedberg, and J. Hoffstein*, Nonvanishing theorems for L-functions of modular forms and their derivatives. Invent. Math. **102** (1990), 543–618.

[9] *B. Gross and K. Keating*, On the intersection of modular correspondences, Invent. Math. **112** (1993), 225–245.

[10] *B. Gross, W. Kohnen, and D. Zagier*, Heegner points and derivatives of L-series. II. Math. Ann. **278** (1987), 497–562.

[11] *B. Gross and D. Zagier*, Heegner points and derivatives of L-series, Invent. Math. **84** (1986), 225–320.

[12] *S. Kudla*, Central derivatives of Eisenstein series and height pairings. Ann. of Math. (2) **146** (1997), 545–646.

[13] *S. Kudla*, Algebraic cycles on Shimura varieties of orthogonal type. Duke Math. J. **86** (1997), no. 1, 39–78.

[14] *S. Kudla*, Integrals of Borcherds forms, Compositio Math. **137** (2003), 293–349.

[15] *S. Kudla*, Special cycles and derivatives of Eisenstein series, in *Heegner points and Rankin L-series*, Math. Sci. Res. Inst. Publ. **49**, Cambridge University Press, Cambridge (2004).

[16] *S. Kudla, M. Rapoport, and T.H. Yang*, Modular forms and special cycles on Shimura curves, Annals of Math. Studies series, vol 161, Princeton Univ. Publ., 2006.

[17] *J. Schofer*, Borcherds forms and generalizations of singular moduli, J. Reine Angew. Math., to appear.

[18] *C. Soulé, D. Abramovich, J.-F. Burnol, and J. Kramer*, Lectures on Arakelov Geometry, Cambridge Studies in Advanced Mathematics **33**, Cambridge University Press, Cambridge (1992).

[19] *S. Zhang*, Heights of Heegner cycles and derivatives of L-series, Invent. Math. **130** (1997), 99–152.

[20] *S. Zhang*, Heights of Heegner points on Shimura curves, Ann. of Math. **153** (2001), 27–147.

Twists of elliptic curves and Hilbert's Tenth Problem

Karl Rubin
(joint work with Barry Mazur)

This lecture is a report on investigations of the 2-Selmer rank in families of quadratic twists of elliptic curves over arbitrary number fields. For example, we show that under certain hypotheses an elliptic curve has many twists with trivial Mordell-Weil group, and (assuming the Shafarevich-Tate conjecture) many others with infinite cyclic Mordell-Weil group. Using work of Poonen and Shlapentokh, it follows from our results that if the Shafarevich-Tate conjecture holds, then Hilbert's Tenth Problem has a negative answer over the ring of integers of every number field. For details, see [5].

1. Results about ranks of twists

Let K be a number field. We will make use of the following weak version of the Shafarevich-Tate conjecture. Let $Ш(E/K)$ denote the Shafarevich-Tate group.

Conjecture $ШT_2(K)$**.** *For every elliptic curve E/K, $\dim_{\mathbb{F}_2} Ш(E/K)[2]$ is even.*

Theorem 1.

(1) *There are infinitely many elliptic curves E/K with $E(K) = 0$.*
(2) *If Conjecture $ШT_2(K)$ holds, then there are infinitely many elliptic curves E/K with $E(K) \cong \mathbb{Z}$.*

Fix an elliptic curve E defined over K. Let $\mathrm{Sel}_2(E)$ be the 2-Selmer group of E/K, and

$$d_2(E) := \dim_{\mathbb{F}_2} \mathrm{Sel}_2(E).$$

If F/K is a quadratic extension, let E^F denote the quadratic twist of E by F/K. Then Theorem 1 is a consequence of the following theorem.

Theorem 2. *Suppose $E(K)[2] = 0$, and suppose further that either K has a real embedding, or that E has multiplicative reduction at some prime of K. If $0 \le r \le \max\{d_2(E), 1\}$, then E has infinitely many twists with $d_2(E^F) = r$.*

When $K = \mathbb{Q}$, Chang [1, Theorem 1.1] proved a weaker version of Theorem 2, using similar methods to ours. Also when $K = \mathbb{Q}$, Ono and Skinner ([3, §1], [2, Corollary 3]) proved (by very different methods from ours) that, under the hypotheses of Theorem 2, E has infinitely many twists with $\mathrm{rank}(E(\mathbb{Q})) = 0$.

We also have the following, with stronger hypotheses and a stronger conclusion.

Theorem 3. *Suppose $\mathrm{Gal}(K(E[2])/K) \cong S_3$. Let Δ_E be the discriminant of some model of E, and suppose further that K has a place v satisfying one of the following conditions:*

- *v is real and $(\Delta_E)_v < 0$, or*
- *$v \nmid 2\infty$, E has multiplicative reduction at v, and $\mathrm{ord}_v(\Delta_E)$ is odd.*

Then for every $r \ge 0$, E has infinitely many twists with $d_2(E^F) = r$.

In both Theorems 2 and 3, we can replace "infinitely many" in the conclusion with a quantitative statement, namely that for $X \in \mathbb{R}^+$,

$$|\{\text{quadratic } F/K : d_2(E^F) = r \text{ and } \mathbf{N}_{K/\mathbb{Q}}\mathfrak{f}(F/K) < X\}| \gg X/(\log X)^{2/3}$$

where $\mathfrak{f}(F/K)$ denotes the finite part of the conductor of F/K.

2. Application to Hilbert's Tenth Problem

Theorem 4. *Suppose L/K is a cyclic extension of prime degree of number fields.*

(1) *There is an elliptic curve E over K with* $\operatorname{rank}(E(L)) = \operatorname{rank}(E(K))$.
(2) *If Conjecture* $\text{Ш}\mathrm{T}_2(K)$ *is true, then there is an elliptic curve E over K with* $\operatorname{rank}(E(L)) = \operatorname{rank}(E(K)) = 1$.

Theorem 4 has applications to Hilbert's Tenth Problem, thanks to the following result of Poonen. If K is a number field, $\mathcal{O}_K$ will denote its ring of integers.

Theorem 5 (Poonen, Theorem 1 of [4])**.** *Suppose $K \subset L$ are number fields, and E/K is an elliptic curve with* $\operatorname{rank}(E(K)) = \operatorname{rank}(E(L)) = 1$. *Then $\mathcal{O}_K$ is diophantine over $\mathcal{O}_L$.*

Using ideas of Poonen and Shlapentokh, Theorems 4 and 5 imply the following.

Theorem 6. *Suppose K is a number field, and L is the Galois closure of $K/\mathbb{Q}$. If Conjecture* $\text{Ш}\mathrm{T}_2(L)$ *holds, then Hilbert's Tenth Problem has a negative answer over $\mathcal{O}_K$.*

In particular if Conjecture $\text{Ш}\mathrm{T}_2(K)$ *holds for every number field K, then Hilbert's Tenth Problem has a negative answer over the ring of integers of every number field.*

3. Ideas of the proofs

Suppose E is an elliptic curve over K. For every place v of K, let $\mathcal{H}(E/K_v)$ denote the image of the Kummer map

$$E(K_v)/2E(K_v) \hookrightarrow H^1(K_v, E[2]).$$

The 2-Selmer group $\operatorname{Sel}_2(E) \subset H^1(K, E[2])$ is the (finite) $\mathbb{F}_2$-vector space defined by the exactness of the sequence

$$0 \longrightarrow \operatorname{Sel}_2(E) \longrightarrow H^1(K, E[2]) \longrightarrow \bigoplus_v H^1(K_v, E[2])/\mathcal{H}(E/K_v).$$

If E^F is a quadratic twist of E, then there is a natural identification of Galois modules $E[2] = E^F[2]$. This allows us to view $\operatorname{Sel}_2(E), \operatorname{Sel}_2(E^F) \subset H^1(K, E[2])$, defined by different sets of local subgroups $\mathcal{H}(E/K_v)$, $\mathcal{H}(E^F/K_v) \subset H^1(K_v, E[2])$. By choosing F carefully, and studying how the $\mathcal{H}(E/K_v)$ change, we will be able to compare $\operatorname{Sel}_2(E)$ and $\operatorname{Sel}_2(E^F)$.

Lemma 7. *If at least one of the following five conditions holds:*

(1) *v splits in F/K, or*
(2) *$v \nmid 2$ and $E(K_v)[2] = 0$, or*

(3) *E is multiplicative at v, F/K is unramified at v, and* $\mathrm{ord}_v(\Delta_E)$ *is odd, or*
(4) *v is real and $(\Delta_E)_v < 0$, or*
(5) *v is a prime where E has good reduction and v is unramified in F/K,*

then $\mathcal{H}(E/K_v) = \mathcal{H}(E^F/K_v)$.

Lemma 8. *If $v \nmid 2\infty$, E has good reduction at v, and v is ramified in F/K, then*

$$\mathcal{H}(E/K_v) \cap \mathcal{H}(E^F/K_v) = 0.$$

The next proposition follows from Lemmas 7, 8, and Poitou-Tate global duality.

Proposition 9. *Suppose F/K is a quadratic extension ramified at exactly one prime $\mathfrak{p}$, that E has good reduction at $\mathfrak{p}$, and that all of the following places split in F/K:*

- *all primes where E has additive reduction,*
- *all v of multiplicative reduction such that* $\mathrm{ord}_v(\Delta_E)$ *is even,*
- *all primes above 2,*
- *all real places v with $(\Delta_E)_v > 0$.*

Suppose further that the localization map

$$\mathrm{Sel}_2(E) \longrightarrow \mathcal{H}(E/K_\mathfrak{p})$$

is surjective. Then the kernel of this localization map is $\mathrm{Sel}_2(E^F)$*, and so*

$$d_2(E^F) = d_2(E) - \dim_{\mathbb{F}_2} \mathcal{H}(E/K_\mathfrak{p}).$$

The proof of Theorem 2 now proceeds as follows. If $E(K)[2] = 0$ and $d_2(E) > 1$, then (using the Cebotarev theorem) we can always find F satisfying the conditions of Proposition 9, and with $\dim_{\mathbb{F}_2} \mathcal{H}(E/K_\mathfrak{p}) = 2$. This allows us always to find a twist that reduces the 2-Selmer rank by 2. Under additional assumptions, we can find an F satisfying the conditions of Proposition 9, and with $\dim_{\mathbb{F}_2} \mathcal{H}(E/K_\mathfrak{p}) = 1$.

Once we find one twist of E with a given 2-Selmer rank, we can apply Proposition 9 again, with F such that $\mathcal{H}(E/K_\mathfrak{p}) = 0$, to find many other such twists.

References

[1] S. Chang, Quadratic twists of elliptic curves with small Selmer rank. Preprint available at `http://arxiv.org/abs/0809.5019`

[2] K. Ono, Nonvanishing of quadratic twists of modular L-functions and applications to elliptic curves. *J. Reine Angew. Math.* **533** (2001) 81–97.

[3] K. Ono, C. Skinner, Non-vanishing of quadratic twists of modular L-functions. *Invent. Math.* **134** (1998) 651–660.

[4] B. Poonen, Using elliptic curves of rank one towards the undecidability of Hilbert's tenth problem over rings of algebraic integers. In: Algorithmic Number Theory (Sydney, 2002), *Lecture Notes in Comput. Sci.* **2369**, Berlin: Springer-Verlag (2002) 33–42.

[5] B. Mazur and K. Rubin, Ranks of twists of elliptic curves and Hilbert's Tenth Problem. preprint available at `http://arxiv.org/abs/0904.3709`

Gross–Schoen cycles and triple product L-functions

Shou-Wu Zhang
(joint work with Xinyi Yuan and Wei Zhang)

Root numbers and local linear functionals. Let F be a number field with ring of adeles $\mathbb{A}$. Let $\sigma = \sigma_1 \otimes \sigma_2 \otimes \sigma_3$ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})^3$. In [2], Piatetski-Shapiro and Rallis defined triple product L-function $L(s, \sigma)$. Assume that the central character ω of σ is trivial when restricted to $\mathbb{A}^\times$

$$\omega|_{\mathbb{A}^\times} = 1.$$

Then the σ is self-dual and we have a functional equation for the Rankin L-series $L(s, \sigma)$

$$L(s, \sigma) = \epsilon(s, \sigma) L(1 - s, \sigma).$$

And the global root number $\epsilon(1/2, \sigma) = \pm 1$. For a fixed non-trivial additive character ψ of $F\backslash\mathbb{A}$, we have a decomposition

$$\epsilon(s, \sigma, \psi) = \prod \epsilon(s, \sigma_v, \psi_v).$$

The local root number $\epsilon(1/2, \sigma_v, \psi_v) = \pm 1$ does not depend on the choice of ψ_v. Thus we have a well-defined set of places of F:

$$\Sigma = \left\{ v : \qquad \epsilon(1/2, \sigma_v, \psi_v)\omega_{E_v/F_v}(-1) = -1. \right\}$$

These local sign can be also characterized by local linear functional:

$$v \in \Sigma \Longleftrightarrow \mathrm{Hom}_{\mathrm{GL}_2(F_v)}(\sigma_v, \mathbb{C}) \neq 0$$

where $\mathrm{GL}_{2,F}$ is embedded into $\mathrm{GL}_{2,E}$ induced by the embedding $F \subset E$. For each place v, let $\mathbb{H}_v$ denote a division quaternion algebra over F_v. Let π_v denote the Jacquet-Langlands correspondence of σ_v on $\mathbb{H}_{E_v}$(zero if σ_v is not discrete). Then the work of Prasad (non-archimedean) and Loke (archimedean) shows that

$$\dim \mathrm{Hom}_{\mathrm{GL}_2(F_v)}(\sigma_v, \mathbb{C}) + \dim \mathrm{Hom}_{\mathbb{H}_v^\times}(\pi_v, \mathbb{C}) = 1.$$

Let $\mathbb{B}$ be a quaternion algebra over $\mathbb{A}$ which is obtained from $M_2(\mathbb{A})$ with $M_2(F_v)$ replaced by $\mathbb{H}_v$ if $\epsilon(1/2, \sigma_v, \psi_v)\omega_{E_v/F_v}(-1) = -1$, and let π be the admissible representation of $\mathbb{B}_E^\times$ which is obtained from σ with σ_v replaced by π_v if $v \in \Sigma$. Then we have

$$\dim \mathrm{Hom}_{B_v^\times \times B_v^\times}(\pi_v \otimes \widetilde{\pi}_v, \mathbb{C}) = \dim \mathrm{Hom}_{B_v^\times}(\pi_v, \mathbb{C}) \otimes \mathrm{Hom}_{B_v^\times}(\widetilde{\pi}_v, \mathbb{C}) = 1$$

where $\widetilde{\pi}$ is the contragredient of π. An explicit element α in this space can defined by integration of matrix coefficients: for any $f_v \in \pi_v$ and $\widetilde{f}_v \in \widetilde{\pi}_v$, then we can form the integration of matrix coefficients:

$$\alpha_v(f_v, \widetilde{f}_v) := \frac{\zeta_{F_v}(2)}{\zeta_{E_v}(2)} \frac{L(1, \sigma_v, ad)}{L(1/2, \sigma_v)} \int_{F_v^\times \backslash B_v^\times} (\pi_v(b_v) f_v, \widetilde{f}_v) db_v^\times.$$

Here the Haar measure has been chosen for $B_v^\times$ such that it takes volume 1 on the maximal compact subgroup and the integral is normalized so it takes value 1 when everything is unramified.

Gross–Schoen cycles. Now we assume that the global root number

$$\epsilon(1/2, \sigma) = -1.$$

Then Σ is odd and the symmetry forces that the central value $L(\frac{1}{2}, \sigma) = 0$ and we are led to consider the first derivative $L'(\frac{1}{2}, \sigma)$. We assume further that F is a totally real field, and that for any $v|\infty$, all $\sigma_{i,v}$ are discrete of weight 2. It follows that the odd set Σ must contain all archimedean places.

For any open compact subgroup U of $\mathbb{B}_f^\times$, we have a Shimura curve X_U defined over F such that for any archimedean place τ, we have the usual uniformization as follows. Let $B = B(\tau)$ be a quaternion algebra over F with ramification set $\Sigma(\tau) := \Sigma \setminus \{\tau\}$ which acts on Poincaré double half plane $\mathcal{H}^\pm = \mathbb{C} \setminus \mathbb{R}$ by fixing an isomorphism $B \otimes_\tau \mathbb{R} = M_2(\mathbb{R})$. Then we have the following identification of analytic space at τ:

$$X_{U,\tau}^{\mathrm{an}} = B^\times \backslash \mathcal{H}^\pm \times \mathbb{B}_f^\times / U.$$

We also have a similar unformization as a rigid space at a finite place in Σ using Drinfeld's upper half plane.

Let $\Delta_{U,\xi}$ be the Gross–Schoen cycle on X_U^3 which is obtained form the diagonal cycle by some modification with respect to the Hodge class ξ (the unique class in $Pic^1(X)_\mathbb{Q} = \lim_U Pic^1(X_U)_\mathbb{Q}$ that is $\mathbb{B}_f^\times$-invariant) as constructed in [1] and [3]. It is shown in [1] that $\Delta_{U,\xi}$ is homologously trivial and the Beilinson-Bloch height pairing $\langle \Delta_{U,\xi}, \Delta_{U,\xi} \rangle$ is well defined unconditionally. More generally, one has a well-defined height pairing

$$\langle \Delta_{U,\xi}, \mathrm{T}(\phi) \Delta_{U,\xi} \rangle$$

for a Hecker operator defined by a function ϕ in the space $\mathcal{S}((\mathbb{B}_f^\times)^3)$ of locally constant with compact support on $(\mathbb{B}^\times)^3$ invariant under $U^3 \times U^3$. Here $\mathcal{S}(\mathbb{B}_f^\times)$ has two actions by $\mathbb{B}_f^\times$ from left and right translations. In fact, varying level structure U the Gross–Schoen cycle $\Delta_{U,\xi}$ forms a projective system but $\mathrm{T}(\phi)$ forms an inductive system. The projection formula ensures that the above paring does not depends on the choice of the open compact U.

Note that the Hodge class ξ is invariant (up to torsion) under $\mathbb{B}_f^\times$-translation. And the diagonal cycle and various partial diagonals are automatically invariant under the diagonal $\Delta(\mathbb{B}_f^\times) \subset (\mathbb{B}_f^\times)^3$. It follows from the projection formula that the linear form, denoted by γ_f, defined by $\phi \mapsto \langle \Delta_{U,\xi}, \mathrm{T}(\phi)\Delta_{U,\xi} \rangle$ is $\mathbb{B}_f^\times \times \mathbb{B}_f^\times$-invariant:

$$\gamma_f \in \mathrm{Hom}_{\mathbb{B}_f^\times \mathbb{B}_f^\times}(\mathcal{S}(\mathbb{B}_f^\times)^{\otimes 3}, \mathbb{C}).$$

Moreover, the height pairing depends only on the action of $\mathrm{T}(\phi_i)$ on the weight 2 forms ([1], Prop. 8.3). In other words, the linear form γ_f factors through the natural $(\mathbb{B}_f^\times \times \mathbb{B}_f^\times)^3$-equivariant projection

$$\mathcal{S}(\mathbb{B}_f^\times)^{\otimes 3} \longrightarrow \bigoplus_\pi \pi_f \otimes \widetilde{\pi}_f$$

where the sum is over the Jacquet-Langlands correspondences ρ on $\mathbb{B}^\times$ of all weight 2 cuspidal representation of $\mathrm{GL}_2(\mathbb{A})^3$. In particular, by restricting to the subspace

$\pi_f \otimes \widetilde{\pi}_f$ for one π, we have a well-defined height pairing:

$$\gamma \in \mathrm{Hom}_{\mathbb{B}_f^\times \times \mathbb{B}_f^\times}(\pi_f \otimes \widetilde{\pi}_f, \mathbb{C}). \tag{1}$$

It follows from the multiplicity one result that the two linear forms γ and α must differ by a constant. The main result of this paper is:

Theorem 1.

$$\gamma = \frac{\zeta_F(2)^2 L'(1/2,\sigma)}{2L(1,\sigma,ad)}\alpha, \tag{2}$$

Application to elliptic curves. Assume that $F = \mathbb{Q}$ and that π_i corresponds to elliptic curves over $\mathbb{Q}$ with same and square free conductor N. Then the central characters of π_i's are all trivial and the sign of triple product L-series is the product of the root numbers $w(E_i)$ of E_i. Assume this product is -1 and let M be the product of primes p such that the local product $\prod w_p(E_i) = -1$. Then M is the product of order number of primes. Thus there is indefinte quaternion algebra B over $\mathbb{Q}$ with discriminant M. Let X be the Shimura curves defined by B with minimal level structure. Then we parameterizations: $\pi_i : X \longrightarrow E_i$. For any subset I, let $X \longrightarrow E_1 \times E_2 \times E_3$ defined by π_i for $i \in I$ and zero map if $i \notin I$. Let Δ_X be the cycle on $E_1 \times E_2 \times E_3$ defined by the following formula:

$$\Delta_X := \sum_{\substack{I \subset \{1,2,3\} \\ I \neq \emptyset}} (-1)^{\#I-1} \pi_{I*} X.$$

Then Δ_X is homologously trivial thus a height of Δ_X can be defined by Arakelov theory. Our main theorem is the following conjecture of Gross–Kudla:

$$\langle \Delta_X, \Delta_X \rangle = c \cdot L'(2, H^1(E_1) \otimes H^1(E_2) \otimes H^1(E_3))$$

where c is an explicit positive constant. An interesting case is when $E_2 = E_3$ but not isogenous to E_1. Then the left hand side up to an explicit constant is equal to the Neron–Tate height of a rational point $x \in E_1$ defined as follows:

$$x = \sum_{E_1} \pi_{1*} \pi_2^*(O_{E_2})$$

where the right hand means the sum using group law on E_1 of a divisor making by pull-back and push forward of the origin O_{E_2} of E_2. In this way, we have further formula in terms of Neron–Tate height of a rational point:

$$\langle x, x \rangle_{NT} = c \cdot L(2, \mathrm{Sym}^2(E_2) \otimes E_1) \cdot L'(1, E_1).$$

References

[1] Gross, Benedict H.;Schoen, Chad. *The modified diagonal cycle on the triple product of a pointed curve.* Ann. Inst. Fourier (Grenoble) 45 (1995), no. 3, 649–679.

[2] Piatetski-Shapiro, I.; Rallis, Stephen. *Rankin triple L functions.* Compositio Math. 64 (1987), no. 1, 31–115.

[3] Zhang, Shou-Wu, *Gross–Schoen cycles and Dualising sheaves, Preprint http://www.math.columbia.edu/∼szhang/papers/gross-schoen.pdf*

Elliptic curves in dihedral towers and two-variable main conjectures of Iwasawa theory

Jeanine Van Order

1. Two-variable main conjectures

Let E be an elliptic curve of conductor N defined over $\mathbf{Q}$, parametrized by a cuspidal Hecke eigenform $f \in S_2(\Gamma_0(N))$. Let p be a rational prime of either good ordinary or multiplicative reduction for E. Fix an imaginary quadratic field k of discriminant prime to N. Let k_∞ denote the $\mathbf{Z}_p^2$-extension of k, with Galois group $G = \mathrm{Gal}(k_\infty/k)$. Let k^{cyc} denote the cyclotomic $\mathbf{Z}_p$-extension of k, and D_∞ the anticyclotomic $\mathbf{Z}_p$-extension of k. Let $\Gamma = \mathrm{Gal}(k^{\mathrm{cyc}})/k)$ and $H = \mathrm{Gal}(k_\infty/k^{\mathrm{cyc}})$. Given a profinite group $\mathcal{G}$, let $\Lambda(\mathcal{G})$ denote its Iwasawa algebra over $\mathbf{Z}_p$.

Theorem 1.1. *There exists a unique measure $L_p(f, k_\infty) \in \Lambda(G)$ whose specialization to any finite order character $\mathcal{W}$ of G satisfies*

$$\mathcal{W}\left(L_p(f, k_\infty)\right) = \eta \cdot \frac{L(f \otimes g_{\overline{\mathcal{W}}}, 1)}{8\pi^2 \langle f, f\rangle_N},$$

with $\eta = \eta(f, \mathcal{W})$ a product of algebraic constants, $L(f \otimes g_{\overline{\mathcal{W}}}, 1)$ the central value of the convolution L-function $L(f \otimes g_{\overline{\mathcal{W}}}, s)$, and $\langle f, f\rangle_N$ the Petersson inner product of f with itself.

The integrality of $L_p(f, k_\infty)$ can be deduced in two ways from the constructions given by Hida [3] and Perrin-Riou [6]. On the other hand, let L be an extension of k, and consider the short exact sequence

$$0 \longrightarrow E(L) \otimes \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow \mathrm{Sel}(E/L) \longrightarrow Ш(E/L)(p) \longrightarrow 0,$$

with $E(L)$ the Mordell-Weil group, $\mathrm{Sel}(E/L)$ the p^∞-Selmer group, and $Ш(E/L)(p)$ the p-primary part of the Tate-Shafarevich group of E/L. Let $X(E/L)$ denote the Pontryagin dual of $\mathrm{Sel}(E/L)$.

Theorem 1.2. (*Kato-Rohrlich) If E has good ordinary reduction at p, then $X(E/k^{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-torsion.*

The structure theory of $\Lambda(\Gamma)$-modules then gives a $\Lambda(\Gamma)$-module pseudoisomorphism

$$X(E/k^{\mathrm{cyc}}) \longrightarrow \left(\bigoplus_i \Lambda(\Gamma)/p^{m_i} \oplus \bigoplus_j \Lambda(\Gamma)/f_j^{n_j}\right), \tag{1}$$

with $m_i, n_j \in \mathbf{Z}$, and f_j monic irreducible distinguished polynomials (with respect to an isomorphism $\Lambda(\Gamma) \cong \mathbf{Z}_p[[T]]$). We may then define from right hand side of (1) the invariants

$$\mu_E(k) = \sum_i m_i, \quad \lambda_E(k) = \sum_j n_j \cdot \deg(f_j),$$

and a characteristic power series $\mathrm{char}_{\Lambda(\Gamma)} X(E/k^{\mathrm{cyc}}) = \prod_{i,j} p^{m_i} f^{n_j}$.

Proposition 1.3. *If E has good ordinary reduction at an odd prime p, then $X(E/k_\infty)$ is $\Lambda(G)$-torsion.*

A similar application of the structure theory then gives rise to a two-variable characteristic power series $\mathrm{char}_{\Lambda(G)}\, X(E/k_\infty)$ for $X(E/k_\infty)$.

Corollary 1.4. *If $\mu_E(k) = 0$, then*

(i) *The two-variable invariant $\mu_{\Lambda(G)} X(E/k_\infty)$ vanishes.*
(ii) *There exists a $\Lambda(H)$-module isomorphism $X(E/k_\infty) \cong \Lambda(H)^{\lambda_E(k)}$.*

Conjecture 1.5. *The dual Selmer group $X(E/k_\infty)$ is $\Lambda(G)$-torsion. Moreover, as ideals in $\Lambda(G)$, $(L_p(f, k_\infty)) = (\mathrm{char}_{\Lambda(G)}\, X(E/k_\infty))$.*

We remark that this conjecture is known for the special case of elliptic curves with complex multiplication over the imaginary quadratic fields by which they admit complex multiplication by works of Rubin (cf. eg. [8]) and Yager [10].

2. Dihedral main conjectures

We approach Conjecture 1.5 in the following way. Let K be any finite extension of k contained in k^{cyc}, viewed as a totally imaginary quadratic extension of its maximal totally real subfield F. Assume that the root number of the Hasse-Weil L-series $L(E/k, s)$ is $+1$. Assume also the following technical conditions:

(i) $p \geq 5$.
(ii) The Galois representation attached to the p-torsion $E[p]$ has image isomorphic to $GL_2(\mathbf{F}_p)$.
(iii) p does not divide the minimal degree of the modular parametrization $\varphi : X_0(N) \longrightarrow E$.
(iv) If $v^2 \mid N\mathcal{O}_F$ with $p \mid \mathbf{N}(v)+1$ for a prime $v \in F$, then $E[p]$ is an irreducible I_v-module, where I_v denotes the inertial subgroup of G_F at v.

Let $K[p^\infty]$ denote the p^∞-ring class tower over K, with $J_K(\infty) = \mathrm{Gal}(K[p^\infty]/K)$. Let $\mathcal{N}$ denote the integer defined by

$$\mathcal{N} = \begin{cases} pN & \text{if } E \text{ has good ordinary reduction at } p \\ N & \text{if } E \text{ has multiplicative reduction at } p. \end{cases}$$

Let $\mathfrak{f} \in S_2(\Gamma_0(\mathcal{N}))$ denote the eigenform of level $\mathcal{N}$ that arises from $f \in S_2(\Gamma_0(N))$. Let $\mathfrak{f}_F$ denote its basechange to F, and write ϕ_F for its Jacquet-Langlands lift, i.e. so that $\mathfrak{f}_F = JL(\phi_F)$.

Theorem 2.1. *There exists a unique measure $L_p(\phi_F, K[p^\infty]) \in \Lambda(J_K(\infty))$ whose specialization to any finite order character ρ_K of $J_K(\infty))$ satisfies*

$$\rho_K\left(L_p(\phi_F K[p^\infty])\right) = \kappa \cdot \frac{L(\phi_F \otimes g_{\rho_K^{-1}}, 1)}{\Omega_{\phi_F}},$$

with $\kappa = \kappa(\rho_K)$ some algebraic constant, $L(\phi_F \otimes g_{\rho_K}, 1)$ the central value of the convolution L-function $L(\phi_F \otimes g_{\rho_K}, s)$, and Ω_{ϕ_F} the Petersson inner product of ϕ_F with itself. Moreover, the μ-invariant attached to $L_p(\phi_F, K[p^\infty])$ is given by $2\nu_F$, with ν_F the largest integer such tht ϕ_F is congruent to a constant mod p^{ν_F}.

Using a generalization of the Euler system argument of Bertolini-Darmon [1], along with the nonvanishing theorem of Cornut-Vatsal [2], we obtain the following

Theorem 2.2. *The dual Selmer group $X(E/K[p^\infty])$ is $\Lambda(J_K(\infty))$-torsion. Moreover, as ideals in $\Lambda(J_K(\infty))$, $L_p(\phi_F, K[p^\infty]) \subseteq \mathrm{char}_{\Lambda(J_K(\infty))} X(E/K[p^\infty])$.*

We remark that Longo has obtained a similar result to Theorem 2.2 independently for a different but general case of the totally real base field F, using the theory of Hilbert modular forms. In any case, we obtain from Theorem 2.2 the following consequences for the setting of the two-variable main conjecture described above.

Corollary 2.3. *Assume that E has good ordinary reduction at p, with $\mu_E(k) = 0$. Then,* $\mathrm{corank}_{\Lambda(H)} X(E/k_\infty) = \lambda_E(k)$.

Example 2.4. Consider the elliptic curve $E = 19a1 : y^2 + y = x^3 + x^2 - 9x - 15$ at $p = 7$ over $k = \mathbf{Q}(\sqrt{-339})$. Computations of Pollack allow us to deduce that $Ш(E/k_\infty)(7)$ has $\Lambda(H)$-corank 4.

Let D_∞^K denote the compositum extension $D_\infty \cdot K$, with Galois group $\Omega_K = \mathrm{Gal}(D_\infty^K/K) \cong \mathbf{Z}_p$.

Corollary 2.5. *For any finite extension K of k contained in k^{cyc}, the dual Selmer group $X(E/D_\infty^K)$ is $\Lambda(\Omega_K)$-torsion. Moreover, as ideals in $\Lambda(\Omega_K)$, we have that* $(L_p(f, k_\infty)|_{\Omega_K}) \subseteq \left(\mathrm{char}_{\Lambda(\Omega_K)} X(E/k_\infty)\right)$.

While successive applications of this result do not a priori imply the desired divisibility $(L_p(f, k_\infty)) \subseteq \left(\mathrm{char}_{\Lambda(G)} X(E/k_\infty)\right)$, it seems that a modification of the inductive argument in [1] for instance might allow one to deduce this.

References

[1] M. Bertolini and H. Darmon, *Iwasawa's Main Conjecture for elliptic curves over anticyclotomic $\mathbf{Z}_p$-extensions*, Ann. of Math. **162** (2005), 1 - 64.

[2] C. Cornut and V. Vatsal, *Nontriviality of Rankin-Selberg L-functions and CM points*, L-functions and Galois Representations, Ed. Burns, Buzzard and Nekovár, Cambridge University Press (2007), 121 - 186.

[3] H. Hida, A p-adic measure attached to the zeta functions associated to two elliptic modular forms I, Invent. Math. **79**, 159-195 (1985).

[4] K. Kato, *p-adic Hodge theory and values of zeta functions of modular forms*, Cohomologies p-adique et applications arithmetiques III, Astérisque 295 (2004).

[5] M. Longo, *Anticyclotomic Iwasawa's Main Conjecture for Hilbert Modular Forms*, preprint (2009).

[6] B. Perrin-Riou, *Fonctions L p-adiques attachées á une courbe elliptique modulaire et á un corps quadratique imaginaire*, Jour. London Math Soc. **(2)** 38 (1988) 1-32.

[7] D. Rohrlich, *On L-funtions of elliptic curves and cyclotomic towers*, Invent. math. **75**, 409-423 (1984).

[8] K. Rubin, *The "main conjectures" of Iwasawa theory for imaginary quadratic fields*, Invent. math. **103** (1991), 25-68.

[9] J. Van Order, *Elliptic curves in dihedral towers and two-variable main conjectures of Iwasawa theory*, Ph.D. Thesis, University of Cambridge, 2009.

[10] R. Yager, *On two variable p-adic L-functions*, Ann. of Math. **115** (1982), 411-449.

Trianguline representations

Laurent Berger

Trianguline representations are a special class of p-adic representations. Let K be a finite extension of $\mathbf{Q}_p$ and let $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$. Fontaine has extensively studied p-adic representations (finite dimensional E-linear representations of G_K where E, the field of coefficients, is a finite extension of $\mathbf{Q}_p$). In particular, he has defined the important and useful notions of de Rham, semistable and crystalline representations. Trianguline representations have been defined by Colmez in the course of his work on the p-adic Langlands correspondence of Breuil. His definition is in terms of (φ,Γ)-modules over the Robba ring and we give it here in the case $K = \mathbf{Q}_p$ in order to simplify the notation.

Let $\mathcal{R} = \{f(X) = \sum_{n\in\mathbf{Z}} a_n X^n$ where $a_n \in E$ and there exists $\rho(f)$ such that $f(X)$ converges for $\rho(f) < |X|_p < 1\}$ be the Robba ring. The ring $\mathcal{E}^\dagger$ is the subring of $\mathcal{R}$ consisting of bounded power series and $\mathcal{O}_{\mathcal{E}}^\dagger$ is the set of $f(X) \in \mathcal{R}$ with $|a_n|_p \leq 1$ for all n. All of those rings are endowed with a frobenius φ given by $\varphi(f)(X) = f((1+X)^p - 1)$ and an action of the group $\Gamma \simeq \mathbf{Z}_p^\times$ given by $[a](f)(X) = f((1+X)^a - 1)$ where $[\cdot] : \mathbf{Z}_p^\times \to \Gamma$ denotes the isomorphism between $\mathbf{Z}_p^\times$ and Γ.

A (φ,Γ)-module is a free $\mathcal{R}$-module of finite rank d endowed with a semilinear frobenius φ such that $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\mathcal{R})$ and with a commuting semilinear continuous action of Γ. We say that such an object is étale if there exists a basis in which $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\mathcal{O}_{\mathcal{E}}^\dagger)$.

The main result relating (φ,Γ)-modules and p-adic Galois representations is the following (it combines theorems of Fontaine, Fontaine-Wintenberger, Cherbonnier-Colmez and Kedlaya) : if D is an étale (φ,Γ)-module, and if $\widetilde{\mathcal{R}}$ denotes one of Fontaine's rings, then $V = (\widetilde{\mathcal{R}} \otimes_{\mathcal{R}} \mathrm{D})^{\varphi=1}$ is a p-adic representation and the resulting functor gives rise to an equivalence of categories : $\{$étale (φ,Γ)-modules$\} \to \{p$-adic representations$\}$. In this way, one realizes the category of p-adic representations as a full subcategory of a larger one, the category of all (φ,Γ)-modules over $\mathcal{R}$.

We then say that a (φ,Γ)-module D is triangulable if it is an iterated extension of objects of rank 1, that is if we can write $0 = \mathrm{D}_0 \subset \mathrm{D}_1 \subset \cdots \subset \mathrm{D}_\ell = \mathrm{D}$ where each D_i is a (φ,Γ)-module and $\mathrm{D}_i/\mathrm{D}_{i-1}$ is of rank 1. If V is a p-adic representation, then we say that it is split-trianguline if the associated (φ,Γ)-module is triangulable, and we say it is trianguline if there exists some finite extension F/E such that $F \otimes_E V$ is split-trianguline.

Examples of trianguline representations include all semi-stable representations and also the representations associated to finite slope overconvergent modular forms (by a theorem of Kisin). In particular, trianguline representations are an important tool in the study of eigencurves and eigenvarieties, as in the work of Bellaïche and Chenevier. They are also used by Colmez (and were defined for that purpose) in his construction of the "unitary principal series of $\mathrm{GL}_2(\mathbf{Q}_p)$" which realizes Breuil's p-adic Langlands correspondence for trianguline representations.

In order to classify trianguline representations, one needs a classification of rank 1 (φ,Γ)-modules as well as the knowledge of the associated Ext^1 groups. If $\delta : \mathbf{Q}_p^\times \to E$ is a continuous character, one defines the (φ,Γ)-module $\mathcal{R}(\delta) = \mathcal{R}\cdot e_\delta$ where $\varphi(e_\delta) = \delta(p)e_\delta$ and $[a](e_\delta) = \delta(a)e_\delta$. It is then a result of Colmez that every (φ,Γ)-module of rank 1 is isomorphic to a $\mathcal{R}(\delta)$ for a well-defined δ. Note that one can define the slope of $\mathcal{R}(\delta)$ to be $\mathrm{val}_p(\delta(p))$ and the weight of $\mathcal{R}(\delta)$ to be $\lim_{a\to 1}\log_p \delta(a)/\log_p(a)$. In addition, although I have not defined (φ,Γ)-modules for $K \neq \mathbf{Q}_p$ they can also be defined and it is a result of Nakamura that there is a bijection between rank 1 (φ,Γ)-modules and continuous characters $\delta : K^\times \to E^\times$. Finally, th e Ext^1 groups were computed by Colmez (in most cases, and by Liu in the remaining cases); they are E-vector spaces of dimension 1 or 2, and in the latter case, the set of extensions is parameterized by a generalization of the $\mathcal{L}$-invariant.

We say that a p-adic representation is potentially trianguline if there exists a finite extension $K/\mathbf{Q}_p$ such that $V|_{G_K}$ is trianguline. Examples of such objects are given by de Rham representations and induced representations. Conversely, we have the following result : if V is a 2-dimensional potentially trianguline representation of $G_{\mathbf{Q}_p}$ then either (1) V is split trianguline, or (2) V is a direct sum of characters or an induced representation or (3) V is a twist of a de Rham representation (these three cases are of course not mutually exclusive). The proof of this result relies on the use of Galois descent : if a triangulation of the (φ,Γ)-module associated to such a representation does not descend, this imposes many conditions on the possible slopes and weights of the occuring rank 1 (φ,Γ)-modules, implying conditions either (2) or (3) (by using Fontaine's theory of $\mathbf{B}_{\mathrm{dR}}$-representations in the latter case).

It is an open problem to find an explicit example of a p-adic representation which is not potentially trianguline, although in recent joint work with Chenevier we show that they do exist.

References

[BlC09] J. Bellaïche et G. Chenevier – "Families of Galois representations and Selmer groups", *Astérisque* (2009), no. 324.

[BrC09] L. Berger et G. Chenevier – "Représentations non potentiellement triangulines", work in progress, 2009.

[Ber09a] L. Berger – "A p-adic family of dihedral (φ,Γ)-modules", preprint, 2009.

[Ber09b] ———, "Représentations potentiellement triangulines de dimension 2", preprint, 2009.

[Col08] P. Colmez – "Représentations triangulines de dimension 2", *Astérisque* (2008), no. 319, p. 213–258.

[Col09] ———, "La série principale unitaire de $\mathrm{GL}_2(\mathbf{Q}_p)$", *Astérisque*, to appear.

[Fon90] J.-M. Fontaine – "Représentations p-adiques des corps locaux. I", The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, p. 249–309.

[Fon94] ———, "Représentations p-adiques semi-stables", *Astérisque* (1994), no. 223, p. 113–184, Périodes p-adiques (Bures-sur-Yvette, 1988).

[Fon04] ———, "Arithmétique des représentations galoisiennes p-adiques", *Astérisque* (2004), no. 295, p. xi, 1–115, Cohomologies p-adiques et applications arithmétiques. III.

[Ked04] K. S. Kedlaya – “A p-adic local monodromy theorem”, *Ann. of Math. (2)* **160** (2004), no. 1, p. 93–184.
[Kis03] M. Kisin – “Overconvergent modular forms and the Fontaine-Mazur conjecture”, *Invent. Math.* **153** (2003), no. 2, p. 373–454.
[Liu08] R. Liu – “Cohomology and duality for (φ,Γ)-modules over the Robba ring”, *IMRN* (2008), no. 3.
[Nak09] K. Nakamura – “Classification of two dimensional split trianguline representations of p-adic fields”, Compositio Mathematica, to appear, 2009.

The infinite fern of Galois representations of type $\mathrm{U}(3)$

Gaëtan Chenevier

Let E be a number field, p a prime and let S be a finite set of places of E containing the primes above p and ∞. Consider the set of isomorphism classes of continuous semi-simple representations $\rho : G_{E,S} \to \mathrm{GL}_d(\overline{\mathbb{Q}}_p)$ of some fixed dimension d, where $G_{E,S}$ is the Galois group of a maximal algebraic extension of E unramified outside S. This is the set of $\overline{\mathbb{Q}}_p$-points of a natural rigid analytic space $\mathcal{X}$ over $\mathbb{Q}_p$, an interesting subset of which is the set $\mathcal{X}^g$ of the ρ which are geometric, in the sense that they occur as a subquotient of $H^i_{\mathrm{et}}(X_{\overline{E}},\overline{\mathbb{Q}}_p)(m)$ for some proper smooth variety X over E, some degree $i \geq 0$ and some Tate twist $m \in \mathbb{Z}$. Here are two basic, but presumably difficult, open questions about $\mathcal{X}^g$:

Does $\mathcal{X}^g$ have some specific structure ? Can we describe its Zariski-closure in $\mathcal{X}$?

A trivial observation is that $\mathcal{X}^g$ is countable, so it contains no subvariety of dimension > 0. When $d = 1$, class-field theory and the theory of complex multiplication describe $\mathcal{X}^g$ and $\mathcal{X}$, in particular $\mathcal{X}^g$ is Zariski-dense in $\mathcal{X}$ if Leopold's conjecture holds at p. When $d > 1$, the situation is actually much more interesting, and has been first studied by Hida, Mazur, Gouvêa and Coleman *when* $E = \mathbb{Q}$ *and* $d = 2$. A discovery of Gouvêa and Mazur is that in the most "regular" odd connected components of $\mathcal{X}$, which are open unit balls of dimension 3, then $\mathcal{X}^g$ *is still Zariski-dense.* Furthermore, it belongs to an intriguing subset of $\mathcal{X}$ they call the *infinite fern* [4], which is a kind of fractal 2-dimensional object in $\mathcal{X}$ built from Coleman's theory of finite slope families of modular eigenforms.

The aim of this talk is to present an extension of these results to the three-dimensional case $d = 3$, mostly by studying the contribution of $\mathcal{X}^g$ coming from the theory of Picard modular surfaces. From now on E is a quadratic imaginary field, p is an odd prime that splits in E, c is the non trivial element of $\mathrm{Gal}(E/\mathbb{Q})$ and the set S is stable by c. Let q be a power of p and fix a continuous absolutely irreducible Galois representation

$$\overline{\rho} : G_{E,S} \to \mathrm{GL}_3(\mathbb{F}_q)$$

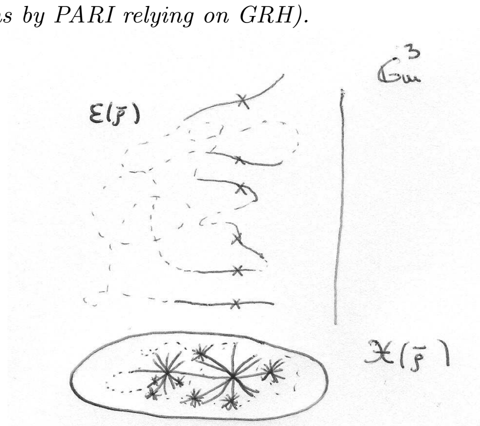
of type $\mathrm{U}(3)$, i.e. such that $\overline{\rho}^\vee \simeq \overline{\rho}^c$ (the latter being the outer conjugate by c). This last condition is equivalent to ask that $\overline{\rho}$ extends to a representation $\tilde{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_3(\mathbb{F}_q) \rtimes \mathrm{Gal}(E/\mathbb{Q})$ inducing the natural map $G_{\mathbb{Q},S} \to \mathrm{Gal}(E/\mathbb{Q})$ and where c acts on GL_3 via $g \mapsto {}^t g^{-1}$. Let us denote by $R(\overline{\rho})$ the universal $G_{E,S}$-deformation of type $\mathrm{U}(3)$ of $\overline{\rho}$ to the category of finite local $\mathbb{Z}_q = W(\mathbb{F}_q)$-algebras with residue field $\mathbb{F}_q$. This ring $R(\overline{\rho})$ might be extremely complicated in general, but we shall not be interested in these complications and rather assume that:

$$(H) \qquad H^2(G_{\mathbb{Q},S}, \mathrm{ad}(\tilde{\rho})) = 0.$$

In this case, one can show that $R(\overline{\rho})$ is formally smooth over $\mathbb{Z}_q$ of relative dimension 6. In particular, its analytic generic fiber $\mathcal{X}(\overline{\rho})$ in the sense of Berthelot is the open unit ball of dimension 6 over $\mathbb{Q}_q$. This space is actually a connected component of the locus of type $U(3)$ of $\mathcal{X}$. By definition its closed points x parameterize the lifts ρ_x of $\overline{\rho}$ such that $\rho_x^\vee \simeq \rho_x^c$. Such an x will be said *modular* if ρ_x is isomorphic to a p-adic Galois representation ρ_Π attached by Rogawski to some cohomological cuspidal automorphic representation Π of $\mathrm{GL}_3(\mathbb{A}_E)$ such that $\Pi^\vee \simeq \Pi^c$ and which is unramified outside S and at the two places above p. These Galois representations are cut out from the étale cohomology of (some sheaves over) the Picard modular surfaces of E. We say that $\overline{\rho}$ is modular if there is at least one modular point in $\mathcal{X}(\overline{\rho})$. It migh t well be the case that each $\overline{\rho}$ is modular (a variant of Serre's conjecture).

Theorem A: *Assume that $\overline{\rho}$ is modular and that (H) holds. Then the modular points are Zariski-dense*[1] *in $\mathcal{X}(\overline{\rho})$.*

Example: *If A is an elliptic curve over $\mathbb{Q}$, then $\overline{\rho} := (\mathrm{Symm}^2 A[p])(-1)$ is modular of type* U(3). *Assume that $E = \mathbb{Q}(i)$, $p = 5$ and let S be the set of primes dividing $10 \cdot \mathrm{condA} \cdot \infty$, then (H) holds whenever A is in the class labeled as $17A$, $21A$, $37B$, $39A$, $51A$, $53A$, $69A$, $73A$, $83A$, or $91B$ in Cremona's tables (this depends on some class number computations by PARI relying on GRH).*



A first important step in the proof of Theorem A is a result from the theory of p-adic families of automorphic forms for the definite unitary group U(3) ([2],[1]). Fix v a prime of E dividing p, so that $E_v = \mathbb{Q}_p$. Define a *refined modular point* as a pair $(\rho_\Pi, (\varphi_1/p^{k_1}, \varphi_2/p^{k_2}, \varphi_3/p^{k_3}))$ in $\mathcal{X}(\overline{\rho}) \times \mathbb{G}_m^3$ where ρ_Π is a modular Galois

[1] By Zariski-dense we simply mean here that if $t_1, t_2, \ldots, t_6$ are parameters of the ball $\mathcal{X}(\overline{\rho})$, then there is no nonzero power series in $\mathbb{C}_p[[t_1, \ldots, t_6]]$ converging on the whole of $\mathcal{X}(\overline{\rho})$ and that vanishes at all the modular points.

representation associated to Π, $k_1 < k_2 < k_3$ are the Hodge-Tate numbers of $\rho_{\Pi,v}$, and where $(\varphi_1, \varphi_2, \varphi_3)$ is an ordering of the eigenvalues of the crystalline Frobenius acting on $D_{\mathrm{cris}}(\rho_{\Pi,v})$ (recall that $\rho_{\Pi,v} := (\rho_\Pi)_{|G_{E_v}}$ is a crystalline representation of $G_{E_v} = G_{\mathbb{Q}_p}$). Define the *eigenvariety* $\mathcal{E}(\overline{\rho}) \subset \mathcal{X}(\overline{\rho}) \times \mathbb{G}_m^3$ as the Zariski-closure of the refined modular points. The main theorem from the theory of p-adic families of automorphic forms for $U(3)$ asserts that $\mathcal{E}(\overline{\rho})$ *has equi-dimension* 3. By construction the refined modular points are Zariski-dense in $\mathcal{E}(\overline{\rho})$, and even have some accumulation property. The complete *infinite fern of type* $U(3)$ is the set theoretic projection of $\mathcal{E}(\overline{\rho})$ in $\mathcal{X}(\overline{\rho})$. At a modular point in $\mathcal{X}(\overline{\rho})$ there are in general 6 branches of the fern passing through it, as there are in general six ways to refine a given modular point, hence 6 points in $\mathcal{E}(\overline{\rho})$ above it, so we get the above picture. (In any dimension d: $\dim \mathcal{X}(\overline{\rho}) = d(d+1)/2$, $\dim \mathcal{E}(\overline{\rho}) = d$ and there are up to $d!$ ways to refine a given modular point).

Theorem B: *There exist modular points $x \in \mathcal{X}(\overline{\rho})$ such that $\rho_{x|G_{E_v}}$ is irreducible and has $\neq$ crystalline Frobenius eigenvalues. If x is such a point, then*

$$\bigoplus_{y \mapsto x, y \in \mathcal{E}(\overline{\rho})} T_y(\mathcal{E}(\overline{\rho})) \longrightarrow T_x(\mathcal{X}(\overline{\rho}))$$

(the map induced on tangent space) is surjective.

Considering the Zariski-closure Z in $\mathcal{X}(\overline{\rho})$ of the modular points satisfying the first part of Theorem B, and applying Theorem B to a smooth such point of Z, we get Theorem A. The first part of Theorem B is a simple application of eigenvarieties, but its second part is rather deep. It relies on a detailled study of the properties at p of the family of Galois representations over $\mathcal{E}(\overline{\rho})$, especially around non-critical refined modular points, as previously studied in [1] (extending some works of Kisin and Colmez in dimension 2). There are several important ingredients in the proof but we end this short note by focusing on a crucial and purely local one.

Let L be a finite extension of $\mathbb{Q}_p$ and let V be a crystalline representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of any L-dimension d. Assume V is irreducible, with distinct Hodge-Tate numbers, and that the eigenvalues φ_i of the crystalline Frobenius on $D_{\mathrm{crys}}(V)$ belong to L and satisfy $\varphi_i\varphi_j^{-1} \neq 1, p$ for all $i \neq j$. Let $\mathcal{X}_V$ be the deformation functor of V to the category of local artinian L-algebras with residue field L. It is pro-representable and formally smooth of dimension d^2+1. For each ordering $\mathcal{F}$ of the φ_i (such an ordering is called a *refinement*), we defined in [1] the $\mathcal{F}$-trianguline deformation subfunctor $\mathcal{X}_{V,\mathcal{F}} \subset \mathcal{X}_V$, whose dimension is $d(d+1)/2+1$. Roughly, the choice of $\mathcal{F}$ corresponds to a choice of a triangulation of the (φ, Γ)-module of V over the Robba ring, and $\mathcal{X}_{V,\mathcal{F}}$ parameterizes the deforma tions such that this triangulation lifts. When the φ-stable complete flag of $D_{\mathrm{cris}}(V)$ defined by $\mathcal{F}$ is in general position compared to the Hodge filtration, we say that $\mathcal{F}$ is *non-critical.*

Theorem C: *Assume that d "well-chosen" refinements of V are non-critical (e.g. all of them), or that $d \leq 3$. Then on tangent spaces we have an equality*

$$\mathcal{X}_V(L[\varepsilon]) = \sum_{\mathcal{F}} \mathcal{X}_{V,\mathcal{F}}(L[\varepsilon]).$$

In other words "any first order deformation of a generic crystalline representation is a linear combination of trianguline deformations". See [3] for proofs of the results of this note.

References

[1] Bellaïche, J. & Chenevier, G. *Families of Galois representations and Selmer groups*, Astérisque 324 (2009).

[2] Chenevier, G. *Familles p-adiques de formes automorphes pour* $\mathrm{GL}(n)$, Journal für die reine und angewandte Mathematik 570, 143-217 (2004).

[3] Chenevier, G. *Variétés de Hecke des groupes unitaires*, Cours Peccot, Collège de France (2008), — `http://www.math.polytechnique.fr/~chenevier/coursPeccot.html` —.

[4] Mazur, B. *An infinite fern in the universal deformation space of Galois representations.* Collect. Math. 48, No.1-2, 155-193 (1997).

Higher dimensional global class field theory

Moritz Kerz

(joint work with Alexander Schmidt)

Let X be a regular, connected scheme which is flat separated and of finite type over $\mathbb{Z}$.

Problem: Describe Grothendieck's abelian fundamental group $\pi_1^{ab}(X)$.

In the one-dimensional case this problem is solved by global class field theory due to Hilbert, Takagi and Artin. A solution to the higher dimensional case of this problem was given in the work of Bloch, Parshin, Kato and Saito using Milnor K-theory, see [1] for the final result. Another more elementary approach has recently been given by Wiesend [4]. Wiesend's work has been completed and simplified in [2] and [3].

Question: How can we define an idele class group $C(X)$ generalizing the classical relative idele class group?

Wiesend's idea is to consider all curves $C \hookrightarrow X$, i.e. closed integral subschemes of X with $\dim(C) = 1$. He defines the idele group of X to be:

$$\mathrm{I}(X) = \bigoplus_{x \in |X|} \mathbb{Z} \oplus \bigoplus_{\substack{C \hookrightarrow X \\ v \in C_\infty}} k(C)_v^\times$$

where $|X|$ denotes the set of closed points of X and for a curve $C \hookrightarrow X$ we denote by C_∞ the set of places of the function field $k(C)$ which do not correspond to points of $\tilde{X}$ (the normalization of X). We endow the idele group $\mathrm{I}(X)$ with the direct sum topology.

The idele class group is now defined to be the quotient

$$\mathrm{C}(X) = \mathrm{coker}[\bigoplus_{C \hookrightarrow X} k(C)^\times \longrightarrow \mathrm{I}(X)]$$

endowed with the quotient topology.

Proposition 1. *Wiesend's class group satisfies the following basic properties:*

(1) $\mathrm{C}(-)$ *is, in a canonical way, covariant functorial.*
(2) *The intersection of all open subgroups of* $C(X)$ *is equal to the connected component* $\mathrm{D}(X)$ *of* 0 *in* $C(X)$.
(3) *There exists a continuous reciprocity homomorphism*

$$\rho : \mathrm{C}(X) \longrightarrow \pi_1^{ab}(X)$$

such that the composition

$$\mathbb{Z} \xrightarrow{x} \mathrm{C}(X) \xrightarrow{\rho} \pi_1^{ab}(X)$$

for a closed point $x \in X$ *sends* $1 \in \mathbb{Z}$ *to the Frobenius.*
(4) *The reciprocity map* ρ *is a natural transformation of functors.*

The fundamental theory of higher global class field theory in the sense of Wiesend says:

Theorem 2. *The sequence*

$$0 \longrightarrow \mathrm{D}(X) \longrightarrow \mathrm{C}(X) \xrightarrow{\rho} \pi_1^{ab}(X) \longrightarrow 0$$

is topologically exact.

The following famous corollary was first shown by Kato and Saito [1] using their version of higher dimensional class field theory. Nevertheless the proof via Wiesend's class field theory is considerably more elementary and does not use any K-theory.

Corollary 3. *The Chow group of zero cycles* $\mathrm{CH}_0(X)$ *is finite.*

References

[1] Kato, K.; Saito, S. *Global class field theory of arithmetic schemes.* (Boulder, Colo., 1983), 255–331, Contemp. Math., 55, Amer. Math. Soc., Providence, RI, 1986.

[2] Kerz, M. *Higher class field theory and the connected component.* preprint (2008). `http://arxiv.org/abs/0711.4485`

[3] Kerz, M., Schmidt, A. *Covering data and higher dimensional global class field theory.* preprint (2008), to appear in Journal of Number Theory. `http://arxiv.org/abs/0804.3419`

[4] Wiesend, G. *Class field theory for arithmetic schemes.* Math. Z. 256 (2007), no. 4, 717–729.

Truncations of level 1 of elements in the loop group of a reductive group

Eva Viehmann

Let k be an algebraically closed field of characteristic p. Let L be either $k((t))$ or $\mathrm{Quot}(W(k))$ where $W(k)$ is the ring of Witt vectors of k. Let $\mathcal{O}$ be the valuation ring of L. We denote by $\sigma : x \mapsto x^{p^r}$ the Frobenius of k over $\mathbb{F}_{p^r}$ for some fixed r and also the Frobenius of L over $F = \mathbb{F}_{p^r}((t))$ resp. $\mathbb{Q}_{p^r}$. Let $\mathcal{O}_F$ be the valuation ring of F. We denote the uniformizer t or p of $\mathcal{O}_F$ by ϵ.

Let G be a split connected reductive group over $\mathcal{O}_F$. Let B be a Borel subgroup of G and let A be a split maximal torus contained in B. Let $K = G(\mathcal{O})$ and let K_1 be the kernel of the projection $K \to G(k)$. Let W denote the Weyl group of A in G and $\widetilde{W} \cong W \ltimes X_*(A)$ the affine Weyl group. If M is a Levi subgroup of G containing A let W_M be the Weyl group of M and denote by ${}^M W$ the set of elements x of W that are shortest representatives of their coset $W_M x$. If $\mu \in X_*(A)$ we write ϵ^μ for the image of $\epsilon \in F^\times$ under $\mu : \mathbb{G}_m \to A$.

For $b \in G(L)$ we call $\{g^{-1} b \sigma(g) \mid g \in K\}$ the K-σ-conjugacy class of b, and $[b] = \{g^{-1} b \sigma(g) \mid g \in G(L)\}$ the σ-conjugacy class of b.

The goal of this talk is to describe the K-σ-conjugacy classes in $K_1 \backslash G(L)/K_1$.

Comparison with Ekedahl-Oort strata. Let X be a p-divisible group over an algebraically closed field k of characteristic p. Then the Dieudonné module of X is a pair $(\mathbf{M}, F)$ where $\mathbf{M}$ is a free $W(k)$-module of rank equal to the height h of X and where $F : \mathbf{M} \to \mathbf{M}$ is a σ-linear homomorphism. Choosing a basis for $\mathbf{M}$ we can write $F = b\sigma$ for some $b \in GL_h(W(k)[1/p])$. A change of the basis amounts to σ-conjugating b by an element of $GL_h(W(k)) = K$. As b is induced by the Dieudonné module $(\mathbf{M}, F)$, we have $b \in K p^\mu K$ for some minuscule $\mu \in X_*(A)$. Similarly, polarized p-divisible groups or abelian varieties lead to elements $b \in GSp_{2n}(W(k)[1/p])$ for n equal to the dimension of the p-divisible group.

In [O1] Oort shows that one obtains a discrete invariant of X (the so-called Ekedahl-Oort invariant) by considering the p-torsion points $X[p]$, or equivalently by studying the reduction of the module $\mathbf{M}$ together with the two maps $F : \mathbf{M} \to \mathbf{M}$ and $V = pF^{-1} : \mathbf{M} \to \mathbf{M}$ modulo p. In terms of the element b, this corresponds to considering the K_1-double coset. In other words, this situation is analogous to the above in the special case $G = GL_h$ or GSp_{2n} and μ minuscule for $\mathcal{O} = W(k)$. A classification of the Ekedahl-Oort invariant which is similar to our classification has been given by Moonen and Wedhorn in [MW].

To classify the K-σ-conjugacy classes in $K_1 \backslash G(L)/K_1$ in general let us first introduce some notation. For a dominant $\mu \in X_*(A)$ let M_μ be the centralizer of μ, let W_μ be the Weyl group of M_μ and let ${}^\mu W = {}^{M_\mu} W$. Let $x_\mu = w_0 w_{0,\mu}$ where w_0 denotes the longest element of W and where $w_{0,\mu}$ is the longest element of W_μ. Let $\tau_\mu = x_\mu \epsilon^\mu$. Then τ_μ is the shortest element of $W \epsilon^\mu W$.

Theorem 1. *Let $b \in G(L)$. Let $\mu \in X_*(A)$ be the unique dominant element with $b \in K \epsilon^\mu K$. There is a unique $w \in {}^\mu W$ such that the K-σ-conjugacy class of b contains an element of $K_1 w \tau_\mu K_1$.*

Definition 2. Let $b \in G(L)$. The pair (w, μ) as in Theorem 1 is called the *truncation of level* 1 of b.

In the case $L = k((t))$ we also consider the associated stratification of the loop group of G. In the Witt vector case one obtains analogous stratifications for example of the Siegel moduli space. Although it is not clear a priori whether these two cases have similar properties, there are comparison results (for example in [W]) which allow to translate our results for the loop group to the other situation.

Definition 3. Let $\mu \in X_*(A)$ be dominant, let $w \in {}^{\mu}W$ and assume that $\mathrm{char}(F) = p$. Then let $S_{w,\mu}$ be the reduced subscheme of the loop group of G such that $S_{w,\mu}(k)$ consists of those $g \in G(k((t)))$ whose truncation of level 1 is (w, μ).

The $S_{w,\mu}$ are bounded and admissible locally closed subschemes of the loop group and the closure of each stratum is a union of finitely many strata. The following criterion generalizes a corresponding result for Ekedahl-Oort strata by Wedhorn [W] to our situation.

Theorem 4. *$S_{w',\mu'} \subseteq \overline{S_{w,\mu}}$ if and only if there is a $\tilde{w} \in W$ with $\tilde{w}w'\tau_{\mu'}\tilde{w}^{-1} \leq w\tau_\mu$ with respect to the Bruhat order.*

Truncations of level 1 and σ-conjugacy classes. One interesting open question about Ekedahl-Oort strata is to determine which Newton polygons occur in a given Ekedahl-Oort stratum. Recently progress towards answering this question has been made in two ways. In a series of papers [Ha1], [Ha2], [Ha3] Harashita proves a conjecture of Oort ([O2], 6.9) giving a characterization of the generic Newton polygon in each Ekedahl-Oort stratum in the moduli space of principally polarized abelian varieties. Besides, Görtz, Haines, Kottwitz and Reuman [GHKR] study the intersections between Iwahori double cosets in the loop group of a reductive group and σ-conjugacy classes. We use results from [GHKR] to deduce similar conditions for the intersections between the truncation strata and σ-conjugacy classes. Especially, we obtain a generalization of Harashita's theorem to the loop group of any split connected reductive group.

Associated with each σ-conjugacy class $[b]$ there is a unique so-called minimal truncation type (w_b, μ_b). It satisfies $w_b\tau_{\mu_b} \in [b]$ and a technical property which ensures that $Iw_b\tau_{\mu_b}I$ is K-σ-conjugate to $w_b\tau_{\mu_b}$ (where I denotes the standard Iwahori subgroup associated to B). The element $w_b\tau_{\mu_b}$ is also called the minimal element in the given class. This is a generalization of the notion of minimal p-divisible groups (as in [O3]) to our context. The function field analog for general G and μ of Oort's conjecture is

Theorem 5. *Let $b \in G(L)$, and let (w, μ) be its truncation of level 1. Let $w_b\tau_{\mu_b}$ be the minimal element in the σ-conjugacy class of b. Then $w_b\tau_{\mu_b} \in \overline{S_{w,\mu}}$.*

From Theorem 5, one can easily deduce the following corollary which gives a characterization of the generic Newton polygon in a given stratum $S_{w,\mu}$.

Corollary 6. *Let $[b]$ be the generic σ-conjugacy class in $S_{w,\mu}$ for some $w \in {}^{\mu}W$. Then $[b]$ is the maximal element (with respect to the usual ordering on the associated Newton points) in the set of σ-conjugacy classes of minimal elements $w'\tau_{\mu'} \in \widetilde{W}$ such that $S_{w',\mu'} \subseteq \overline{S_{w,\mu}}$. This is also the same as the maximal class $[x]$ among all $x \in \widetilde{W}$ with $x \leq w\tau_\mu$.*

Here we use that $S_{w,\mu}$ is irreducible, so it contains a unique generic σ-conjugacy class.

There is a direct way to translate our Theorems back to the case of mixed characteristic. In particular, one obtains Oort's conjecture (as shown by Harashita) as well as an analog for non-polarized p-divisible groups.

References

[GHKR] U. Görtz, Th. J. Haines, R. E. Kottwitz, D. C. Reuman, *Affine Deligne-Lusztig varieties in affine flag varieties*, preprint, arXiv:0805.0045.

[Ha1] S. Harashita, *Ekedahl-Oort strata and the first Newton slope strata*, Journal of Algebraic Geometry **16** (2007), 171–199.

[Ha2] S. Harashita, *Configuration of the central streams in the moduli of abelian varieties*, to appear in the Asian Journal of Mathematics.

[Ha3] S. Harashita, *Generic Newton polygons of Ekedahl-Oort strata: Oort's conjecture*, http://www.ms.u-tokyo.ac.jp/∼harasita/gnp.ps

[He] X. He, *The G-stable pieces of the wonderful compactification*, Trans. AMS **359** (2007), 3005–3024.

[MW] B. Moonen, T. Wedhorn, *Discrete invariants of varieties in positive characteristic*, IMRN 2004, 3855–3903.

[O1] F. Oort, *A stratification of a moduli space of abelian varieties*, in: Moduli of abelian varieties (Texel Island, 1999), 345–416, Progr. Math., 195, Birkhäuser, Basel, 2001.

[O2] F. Oort, *Foliations in moduli spaces of abelian varieties*, J.A.M.S. **17** (2004), 267–296.

[O3] F. Oort, *Minimal p-divisible groups*, Ann. of Math. **161** (2005), 1021–1036.

[RR] M. Rapoport, M. Richartz, *On the classification and specialization of F-isocrystals with additional structure*, Compositio Math. **103** (1996), 153–181.

[W] T. Wedhorn, *Specialization of F-zips*, http://arxiv.org/abs/math/0507175.

Vector bundles on p-adic curves and parallel transport

Annette Werner

(joint work with Christopher Deninger)

Let $\overline{\mathbb{Q}}_p$ be the algebraic closure of $\mathbb{Q}_p$ and $\mathbb{C}_p$ its completion. By $\overline{\mathbb{Z}}_p$ and $\mathfrak{o}$, respectively, we denote their ring of integers. Both rings have the same residue field k, namely the algebraic closure of $\mathbb{F}_p$.

Let X be a smooth, projective and connected curve over $\overline{\mathbb{Q}}_p$ and let E be a vector bundle on $X_{\mathbb{C}_p} = X \otimes \mathbb{C}_p$.

Definition: *i) We say that E has strongly semistable reduction if there exists a proper, finitely presented, flat model $\mathcal{X}$ of X over $\overline{\mathbb{Z}}_p$ and a vector bundle $\mathcal{E}$ on $\mathcal{X} \otimes \mathfrak{o}$ extending E such that the special fibre $\mathcal{E}_k$ is strongly semistable on all normalized irreducible components of $\mathcal{X}_k$.*

ii) We say that E has potentially strongly semistable reduction if there exists a finite (possibly ramified) covering $\alpha : Y \to X$ of smooth, projective, connected *curves over* $\overline{\mathbb{Q}}_p$ *such that the vector bundle* α^*E *on* $Y_{\mathbb{C}_p}$ *has strongly semistable reduction.*

Recall that a strongly semistable vector bundle on a smooth, projective curve in postive characteristic is a semistable vector bundle such that all its pullbacks by powers of the absolute Frobenius remain semistable. One can show that a vector bundle with potentially strongly semistable reduction is itself semistable.

Now we want to define parallel transport for vector bundles with potentially strongly semistable reduction. For $x \in X(\mathbb{C}_p)$ the corresponding fibre functor F_x associates to every finite étale covering of X the set of points lying above x. The algebraic fundamental group $\pi(X, x)$ is given as the automorphism group of the fibre functor F_x.
Besides, if x and x' are two points in $X(\mathbb{C}_p)$, we call any isomorphism between the fibre functors F_x and $F_{x'}$ an étale path from x to x'.

Theorem: *Let* E *be a vector bundle of degree* 0 *and rank* r *on* $X_{\mathbb{C}_p}$ *with potentially strongly semistable reduction. For every étale path* γ *from* x *to* x' *there is an isomorphism*

$$\rho_E(\gamma) : E_x \to E_{x'}$$

of parallel transport which is functorial in γ. *The association* $E \mapsto \rho_E(\gamma)$ *is functorial, exact and compatible with tensor products and duals, Galois conjugation and pullbacks with respect to finite morphisms of* p*-adic curves. In particular, for every* $x \in X(\mathbb{C}_p)$ *we obtain a continuous representation*

$$\rho_{E,x} : \pi(X, x) \to GL(E_x).$$

This result can be regarded as a partial p-adic analogue of the result by Narasimhan and Seshadri who established an equivalence of categories between polystable vector bundles of degree zero on a compact Riemann surface and unitary representations of the fundamental group.

In [Fa], Faltings has even shown a p-adic analogue of Simpson's theory of Higgs bundles.

The previous theorem was proven in [DW1] for vector bundles which have strongly semistable reduction after pullback to an étale covering. The general case, allowing pullbacks to a ramified covering $\alpha : Y \to X$, is treated in [DW2]. If α^*E has strongly semistable reduction, it admits parallel transport along étale paths by [DW1]. It is easy to see that this parallel transport descends to a parallel transport for E along the étale paths in $U \subset X$, where U is the complement of the ramification points of α. In particular, for $x_0 \in U(\mathbb{C}_p)$ one obtains a representation $\rho : \pi_1(U, x_0) \to GL(E_{x_0})$. The main point is: This representation has no monodromy at the ramification points i.e. that it factors over $\pi_1(X, x_0)$. We were unable to prove this algebraically. Instead our proof uses Grothendieck's comparison theorem between algebraic and topological fundamental groups and so me considerations on Riemann surfaces.

References

[DW1] C. Deninger, A. Werner: *Vector bundles on p-adic curves and parallel transport.* Ann. Scient. Éc. Norm. Sup. **38** (2005), 553-597.

[DW2] C. Deninger, A. Werner: *Vector bundles on p-adic curves and parallel transport II.* Preprint 2009.

[Fa] G. Faltings: *A p-adic Simpson correspondence.* Adv. Math. **198** (2005), 847–862

Arithmetic Intersections on Shimura Surfaces

Benjamin V. Howard

(joint work with Tonghai Yang)

Fix a real quadratic field F with different $\mathfrak{D} \subset \mathcal{O}_F$ and let $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ be the moduli stack of triples (A, κ, λ) in which A is an abelian scheme of relative dimension two over an arbitrary base scheme, $\kappa : \mathcal{O}_F \to \mathrm{End}(A)$ is an action of $\mathcal{O}_F$ on A satisfying the Kottwitz determinant condition, and $\lambda : A \to A^\vee$ is an $\mathcal{O}_F$-linear polarization whose kernel is $A[\mathfrak{D}]$. Thus $\mathcal{X}$ is an integral model of the classical Hilbert modular surface $\mathcal{X}(\mathbb{C}) = \mathrm{SL}_2(\mathcal{O}_F)\backslash(\mathfrak{H} \times \mathfrak{H})$. For any such triple (A, κ, λ) define the space of *special endomorphisms*

$$L(A, \kappa, \lambda) = \{j \in \mathrm{End}(A) : j = j^* \text{ and } \kappa(t) \circ j = j \circ \kappa(t^\sigma)\ \forall t \in \mathcal{O}_F\}.$$

Here $j \mapsto j^*$ is the Rosati involution induced by λ and $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ is the nontrivial element. The space of special endomorphisms is a finite free $\mathbb{Z}$-module (on each connected component of the base) and comes equipped with the quadratic form $Q(j) = j \circ j^*$. For a positive integer m let $\mathcal{T}_m$ be the moduli stack of quadruples (A, κ, λ, j) in which (A, κ, λ) is as above, and $j \in L(A, \kappa, \lambda)$ is a special endomorphism satisfying $Q(j) = n$. Using the forgetful morphism $\mathcal{T}_m \to \mathcal{X}$ we view $\mathcal{T}_m$ as a cycle on $\mathcal{X}$. It can be shown that $\mathcal{T}_m$ is then of codimension one, and agrees with the *Hirzebruch-Zagier divisor* first defined in [1]. Now fix a totally complex quadratic extension E/F and consider the moduli stack $\mathcal{Y}_E$ of triples (A, κ, λ) exactly as in the definition of $\mathcal{X}$, except that now $\kappa : \mathcal{O}_E \to \mathrm{End}(A)$ is an action of $\mathcal{O}_E$ on A. Using the forgetful morphism $\mathcal{Y}_E \to \mathcal{X}$ we view $\mathcal{Y}_E$ as a codimension two cycle on $\mathcal{X}$, the *cycle of complex multiplication.* The problem, as per th e general program of Kudla [2], is to understand the intersection multiplicity $\mathcal{T}_m \cdot \mathcal{Y}_E$ and its relation to Fourier coefficients of automorphic forms.

Define a $\mathbb{Q}$-algebra $M = E \otimes_{\mathrm{id},F,\sigma} E$ (on the left E is an F-algebra via the inclusion $\mathrm{id} : F \to E$ and on the right E is an F-algebra by the conjugate embedding $\sigma : F \to E$) and let $E' \subset M$ be the subalgebra of elements fixed by the automorphism $a \otimes b \mapsto b \otimes a$. Then E' is either a quartic CM field or a product of two quadratic imaginary fields, and in either case we let $F' \subset E'$ be the maximal totally real subalgebra of E'. If (A,κ,λ) is an element of $\mathcal{Y}_E$ then the $\mathbb{Q}$-vector space

$$V(A,\kappa,\lambda) = L(A,\kappa,\lambda)\otimes_{\mathbb{Z}}\mathbb{Q}$$

admits a natural action of E', and there is a unique F'-quadratic form Q' on $V(A,\kappa,\lambda)$ with the property

$$Q(j) = \mathrm{Tr}_{F'/\mathbb{Q}}(Q'(j)).$$

For any nonzero $\alpha\in F'$ we now define $\mathcal{Y}_E(\alpha)$ to be the moduli stack of quadruples (A,κ,λ,j) in which (A,κ,λ) is an object of $\mathcal{Y}_E$ and $j\in L(A,\kappa,\lambda)$ satisfies $Q'(j) = \alpha$. By contemplation of the moduli problems there is a decomposition

$$\mathcal{T}_m \times_{\mathcal{X}} \mathcal{Y}_E = \bigsqcup_{\substack{\alpha\in F' \\ \mathrm{Tr}_{F'/\mathbb{Q}}(\alpha)=m}} \mathcal{Y}_E(\alpha).$$

Consider a triple (T,κ,λ) in which T is a free $\mathbb{Z}$-module, $\kappa:\mathcal{O}_E\to\mathrm{End}_{\mathbb{Z}}(T)$ is an action of $\mathcal{O}_E$ on T, and $\lambda: T\times T\to\mathfrak{D}^{-1}$ is a perfect $\mathcal{O}_F$-symplectic pairing. To such a triple one may attach the finite rank $\mathbb{Z}$-module

$$L(T,\kappa,\lambda) = \{j\in\mathrm{End}_{\mathbb{Z}}(T) : j = j^* \text{ and } \kappa(t)\circ j = j\circ\kappa(t^\sigma)\ \forall t\in\mathcal{O}_F\}$$

where $j\mapsto j^*$ is the adjoint with respect to the $\mathbb{Z}$-bilinear pairing $\mathrm{Tr}_{F/\mathbb{Q}}\circ\lambda$ on T. As above the $\mathbb{Q}$-vector space

$$V(T,\kappa,\lambda) = L(T,\kappa,\lambda)\otimes_{\mathbb{Z}}\mathbb{Q}$$

admits an action of E', which makes $V(T,\kappa,\lambda)$ into a free E'-module of rank one. The $\mathbb{Q}$-quadratic form $Q(j) = j\circ j^*$ on $V(T,\kappa,\lambda)$ has the form

$$Q(j) = \mathrm{Tr}_{F'/\mathbb{Q}}(Q'(j))$$

for a unique F'-quadratic form Q'. The F'-quadratic space $V(T,\kappa,\lambda)$ has signature $(2,0)$ at one archimedean place of F', and signature $(0,2)$ at the other, and by replacing Q' by $-Q'$ at the place of signature $(0,2)$ one obtains a quadratic space $V^*(T,\kappa,\lambda)$ over $F'_{\mathbb{A}}$ which is *incoherent* in the sense that it does not arise as the adelization of an F' quadratic space. By the theory of Siegel-Weil one can attach to this incoherent quadratic space an Eisenstein series which is a Hilbert modular form of weight one on a congruence subgroup of $\mathrm{GL}_2(\mathcal{O}_{F'})$. This Eisenstein series satisfies a functional equation which forces it to vanish at the point $s=0$. By summing over all isomorphism classes of triples (T,κ,λ) one obtains an Eisenstein series $E(\tau,s)$ which admits a Fourier expansion of the form

$$E(\tau,s) = \sum_{\alpha\in F'} c_\alpha(v,s)\cdot q^\alpha$$

where $v=\mathrm{Im}(\tau)$. Each Fourier coefficient $c_\alpha(v,s)$ vanishes at $s=0$.

The main result is as follows: suppose that $\alpha\in F'$ is totally positive. Then the stack $\mathcal{Y}_E(\alpha)$ is supported in a single nonzero characteristic, p. If p is unramified in E then (up to an explicit fudge factor)

$$c'_\alpha(v,0) = \sum_{x \in \mathcal{Y}_E(\alpha)(\overline{\mathbb{F}}_p)} \frac{1}{|\mathrm{Aut}(x)|} \mathrm{length}(\mathcal{O}^{\mathrm{sh}}_{\mathcal{Y}_E(\alpha),x})$$

where $\mathcal{O}^{\mathrm{sh}}_{\mathcal{Y}_E(\alpha),x}$ is the strictly Henselian local ring of $\mathcal{Y}_E(\alpha)$ at x. In particular the right hand side is finite and the left hand side is independent of v.

An earlier result of Yang [3] relates, under some very restrictive hypotheses, the intersection multiplicities $\mathcal{T}_m \cdot \mathcal{Y}_E$ to the derivative of the pullback of $E'(\tau, 0)$ via the diagonal embedding $\mathfrak{h} \to \mathfrak{h} \times \mathfrak{h}$. The above formula for $c'_\alpha(v,0)$ is a preliminary result which, when extended to include all characteristics p and to include archimedean intersections, will both refine and generalize the earlier result of Yang.

References

[1] F. Hirzebruch and D. Zagier, *Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus*, Invent. Math. **36** (1976), 57–1113.

[2] S. Kudla, *Special cycles and derivatives of Eisenstein series*, in Heegner Points and Rankin L-series, Math. Sci. Res. Inst. Publ. **49** (2004), 243–270.

[3] T. Yang, *Arithmetic Intersection on a Hilbert Modular surface and the Faltings height*, Preprint.

Bounding Galois action on semi-stable representations

Xavier Caruso

(joint work with T. Liu, D. Savitt)

Let p be a prime number and k be a perfect field of characteristic p. Consider $W = W(k)$ the ring of Witt vectors with coefficients in k and $K_0 = \mathrm{Frac}\, W$. Let K be a finite totally ramified extension of K_0 of degree e. It is a complete discrete valuation field with residue field k. Denote by $\bar{K}$ an algebraic closure of K, and by $G_K = \mathrm{Gal}(\bar{K}/K)$ the absolute Galois group of K.

1. Crystalline and semi-stable representations

In [10, 11], Fontaine has defined the notion of crystalline (resp. semi-stable) representation of G_K: they are representations of G_K in finite dimensional $\mathbb{Q}_p$-vector spaces that are "B_{cris}-admissible" (resp. "B_{st}-admissible"). We do not want to explain exactly what it means in this note, but rather first recall that all crystalline representations are semi-stable and then give two very important examples. The first one is obtained as follows. Let X be a proper smooth variety over K, and assume that X has a proper smooth (resp. proper semi-stable) model on the ring of integers $\mathcal{O}_K$. Then the étale cohomology group $H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$ is a crystalline (resp. semi-stable) representation. The second example is the Galois representation associated to a modular form of level prime to p, which is always crystalline (of dimension 2).

Recall also that a semi-stable representation V is in particular Hodge-Tate. If $\mathbb{C}_p$ denote the completion of $\bar{K}$, this means that there exists integers $h_1 \leq \cdots \leq h_d$ such that $V \otimes_{\mathbb{Q}_p} \mathbb{C}_p \simeq \bigoplus_{t=1}^d \mathbb{C}_p(h_t)$ where $\mathbb{C}_p(h)$ stands for Tate twist. Integers h_t are then uniquely determined and are called *Hodge-Tate weights* of V. We sometimes represent these numbers by joing points of coordinates $(t, h_1 + \cdots + h_t)$ in the plane. The obtained polygon is called the *Hodge polygon* of V.

One can (more or less) precise Hodge-Tate weights for examples given above. If V is a representation associated to a modular form of weight k, then its Hodge-Tate are 0 and $k-1$, whereas if $V = H^r_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$, we just know *a priori* that Hodge-Tate weights of V are in $\{-r, \ldots, 0\}$. In order to deal with non-negative integers, one sometimes prefer regarding the dual representation V^* instead of V.

2. Tame inertia action (joint work with D. Savitt)

Let V be a semi-stable representation of G_K and $R \subset V$ a G_K-stable $\mathbb{Z}_p$-lattice. Define $T = R/pR$. In the section, we are interested in the action of the tame inertia group on the semi-simplification of T. To this latter represention, one can attach, following [13] §1, some tame inertia weights. They are numbers $i_1 \leq \ldots \leq i_d$ belonging to $\{0, \ldots, p-1\}$.

If $K = K_0$ and V is crystalline with Hodge-Tate weights in $\{0, \ldots, p-2\}$, Fontaine-Laffaille's theory implies that $i_t = h_t$ for all t. Unfortunately, in [4], Breuil and Mézard computed explicit examples showing that equality between i_t and h_t no longer holds for semi-stable representations. Nevertheless, we have the following.

Theorem 1 (with D. Savitt [5])**.** Assume V is semi-stable with Hodge-Tate weights in $\{0, \ldots, r\}$ where r is an integer such that $er < p-1$. Then

$$e(h_1 + \cdots + h_t) \leq i_1 + \cdots + i_t$$

for all $t \in \{1, \ldots, d\}$. Furthermore, equality holds for $t = d$.

Theorem means that the Hodge polygon of V lies below its tame inetia polygon whose successive slopes are $\frac{i_1}{e}, \ldots, \frac{i_d}{e}$. Under the additional assumption $K = K_0$ and V is crystalline, Fontaine-Laffaille's result states the equility between these two previous polygons. In other words, inequality in Theorem 1 is indeed an equality for all t. At this level, one may wonder if V crystalline is enough to imply this. With Saviit, in [6], we gave a negative answer to this question by providing counter-examples (with two-dimensional representation) as soon as $K \neq K_0$. Fontaine-Laffaille's case appears then as very isolated.

3. Wild inertia action (joint work with T. Liu)

As before, pick V a semi-stable represention of G_K and $R \subset V$ a G_K-stable $\mathbb{Z}_p$-lattice. For all integer $n \geq 1$, set $T_n = R/p^n R$. In this section, we are interested in the action of the wild inertia subgroup on T_n. Let L_n be the finite extension of K corresponding to the finite index subgroup $\ker \rho_n \subset G_K$ where $\rho_n : G_K \to \mathrm{GL}(T_n)$ is the morphism that gives the action of G_K on T_n. We would like to bound ramification of L_n, and so we have first to consider a measure of this ramification. We will use the p-adic valuation of the different $\mathcal{D}_{L_n/K}$. To fix normalization, let v_K denote the valuation on $\bar{K}$ such that $v_K(K^\star) = \mathbb{Z}$.

Theorem 2 (with T. Liu [7])**.** Assume $p > 2$ and that Hodge-Tate weights of V are between 0 and r. Pick an integer n and write $\frac{nr}{p-1} = p^\alpha \beta$ with $\alpha \in \mathbb{N}$ and $\frac{1}{p} < \beta \leq 1$. Then

$$v_K(\mathcal{D}_{L_n/K}) \leq 1 + e(n + \alpha + \beta) - \frac{1}{p^{n+\alpha}}.$$

Before our work, some partial results were already known in this direction. First, in [8] and [9], Fontaine uses Fontaine-Laffaille's theory to get some bounds when $K = K_0$, $n = 1$, $r < p - 1$ and V is crystalline. In [1], Abrashkin follows Fontaine's general ideas to extend the result to arbitrary n (other restrictions remain the same). Later, with the extension by Breuil of Fontaine-Laffaille's theory to semi-stable case, it has been possible to achieve some cases where V is not crystalline. Precisely in [2][1], Breuil obtains bounds for semi-stable representations that satisfies Griffith transversality when $n = 1$ and $er < p - 1$. Very recently in [12], Hattori proved a bound for all semi-stable representations with $r < p - 1$ (e and n are arbitrary here). All these bounds have the same shape

$$e\left(n + \frac{r}{p-1}\right) + \text{cte} \tag{1}$$

with $0 \leq \text{cte} \leq 1$. Since r is always assumed to be $< p - 1$, one can see that these bounds are better than ours. However, the most important feature of Theorem 2 is to be applicable for any r! Furthermore, one remark that bounds of Theorem 2 have a logarithmic dependance in r, which may be quite surprising after (1) (where the dependance seems to be linear[2]). Actually, it is very plausible that, using analogous methods, one can improve Theorem 2 in order to fit with (1). Precisely, we conjecture the following.

Conjecture 3. Theorem 2 is true with α and β replaced by α' and β' defined by $\frac{r}{p-1} = p^{\alpha'}\beta'$, $\alpha' \in \mathbb{N}$ and $\frac{1}{p} < \beta' \leq 1$.

We finally wonder if better bounds exist when V is crystalline. It is actually the case when $e = 1$ and $r < p - 1$ by results of Fontaine and Abrashkin, but it is not clear to us how to extend this to a more general setting.

References

[1] V. Abrashkin, *Ramification in étale cohomology*, Invent. Math. **101** (1990), no. 3, 631–640
[2] C. Breuil, letter to Gross, December 3, 1998
[3] C. Breuil, W. Messing, *Torsion étale and crystalline cohomologies*, in *Cohomologies p-adiques et applications arithmétiques II*, Astérisque **279** (2002), 81–124

[1] See Proposition 9.2.2.2 of [3] for the statement

[2] Of course, it does not mean anything since these bounds are valid under the assumption for $r < p - 1$, and certainly not for r going to infinity.

[4] C. Breuil, A. Mézard, *Multiplicités modulaires et représentations de $GL_2(\mathbb{Z}_p)$ et de $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ en $\ell = p$*, Duke math. J. **115** (2002), 205–310
[5] X. Caruso, D. Savitt, *Polygones de Hodge, de Newton et de l'inertie modre des reprsentations semi-stables*, Math. Ann. **343** (2009), 773–789
[6] X. Caruso, D. Savitt, *Poids de l'inertie modre de certaines reprsentations cristallines*, to appear at J. Thor. Nombres Bordeaux
[7] X. Caruso, T. Liu, *Some bounds for ramification of p^n-torsion semi-stable representations*, preprint (2009)
[8] J.-M. Fontaine, *Il n'y a pas de variété abélienne sur $\mathbb{Z}$*, Invent. Math. **81** (1985), 515–538
[9] J.-M. Fontaine, *Schémas propres et lisses sur $\mathbb{Z}$*, in *Proceedings of the Indo-French Conference on Geometry (Bombay, 1989)*, 43–56, 1993
[10] J.-M. Fontaine, *Le corps des priodes p-adiques*, Astrisque **223**, Soc. math. France (1994), 59–111
[11] J.-M. Fontaine, *Reprsentations p-adiques semi-stables*, Astrisque **223**, Soc. math. France (1994), 113–184
[12] S. Hattori, *On a ramification bound of torsion semi-stable representations over a local field*, preprint
[13] J. P. Serre, *Proprits galoisiennes des points d'ordre fini des courbes elliptiques*, Invent. math. **15** (1972), 259–331

Remarks on Suslin's singular homology

Thomas Geisser

We discuss Suslin homology and cohomology, focusing on the p-part of Suslin homology in characteristic p and on finite base fields. We give a generalization of a conjecture of Kato in terms of Suslin homology, and discuss connections to class field theory. The contents are based on the article [4].

1. General fields

1.1. **Definition.** Let X be separated and of finite type over a perfect field k. We define C_*^X to be the complex of abelian group, which in degree $-i$ is the free abelian group generated by closed irreducible subschemes of $X \times \Delta^i$ which are finite and surjective over Δ^i. The differentials are given by alternating maps of pull-backs along face maps $\Delta^{i-1} \to \Delta^i$.

Suslin homology $H_i^S(X, A)$ of X with coefficients in the abelian group A is the homology of $C_*^X \otimes A$, and Suslin cohomology is by definition the dual of Suslin homology, i.e. for an abelian group A it is defined by $H^i_S(X, A) := \mathrm{Ext}^i_{\mathrm{Ab}}(C_*^X, A)$. Assuming resolution of singularities over k, Suslin homology has the following additional properties (and Suslin cohomology has the dual properties):

(1) If $i : Z \to X$ is a closed embeddding, $f : X' \to X$ is proper, and f induces an isomorphism $X' - X' \times_X Z \to X - Z$), then there is a long exact sequence

$$\cdots \to H^S_{i+1}(X, \mathbb{Z}) \to H^S_i(Z', \mathbb{Z}) \to H^S_i(X', \mathbb{Z}) \oplus H^S_i(Z, \mathbb{Z}) \to H^S_i(X, \mathbb{Z}) \to \cdots$$

(2) If X is proper, then motivic homology agrees with higher Chow groups, $H^S_i(X, \mathbb{Z}) \cong CH_0(X, i)$.

(3) If X is smooth of pure dimension d, then motivic homology agrees with motivic cohomology with compact support, $H_i^S(X,\mathbb{Z}) \cong H_c^{2d-i}(X,\mathbb{Z}(d))$.

In particular, if Z is a closed subscheme of a smooth scheme X of pure dimension d, then we have a long exact sequence

$$\cdots \to H_i(U,\mathbb{Z}) \to H_i(X,\mathbb{Z}) \to H_c^{2d-i}(Z,\mathbb{Z}(d)) \to \cdots .$$

The main theorem of Suslin and Voevodsky [5] states that if k is algebraically closed and m is invertible k, then Suslin cohomology $H^i_S(X,\mathbb{Z}/m)$ is isomorphic to etale cohomology $H^i_{\text{et}}(X,\mathbb{Z}/m)$.

1.2. **The mod p Suslin homology in characteristic p.** We are examining the p-part of Suslin homology in characteristic p. We assume that k is algebraically closed and resolution of singularities exists over k.

Theorem 1.1. *If X be separated and of finite type over k, then $H_i^S(X,\mathbb{Z}/p^r)$ are finite and vanish unless $0 \le i \le d$.*

The proof reduces to the smooth and projective case, in which case

$$H_i^S(X,\mathbb{Z}/p^r) \cong H_c^{2d-i}(X,\mathbb{Z}/p^r(d)) \cong H_c^{2d-i}(X,\nu_r^d(d)) \cong H_c^{2d-i}(X_{\text{et}},\nu_r^d(d)).$$

The latter group is known to be finite.

Together with the theorem of Suslin and Voevodsky, the thoerem shows that Suslin cohomology can be regarded as an improvement of etale cohomology: For m prime to the characteristic, it is usual etale cohomology, and for m a power of the characteristic, it is a finite group, dual to Suslin homology.

Proposition 1.2. *Assume X has a desingularization $p : X' \to X$ which is an isomorphism outside of the open set U. Then $H_i^S(U,\mathbb{Z}/p^r) \cong H_i^S(X,\mathbb{Z}/p^r)$. In particular, the p-part of Suslin homology is a birational invariant.*

The hypothesis is satisfied if X is smooth and U dense, or if U contains all singular points of X and a resolution of singularities exists which is an isomorphism outside of the singular points.

1.3. **Etale theory.** Let $\bar{k}$ be the algebraic closure of k with Galois group G_k, and let A be a continuous G_k-module A. Since Suslin homology does not have Galois descent, we redefine Suslin homology by imposing Galois-descent: We define Galois-Suslin homology $H_i^{GS}(X,A) = H^{-i}R\Gamma(G_k, C_*^X(\bar{k}) \otimes A)$ and Galois-Suslin cohomology to be $H^i_{GS}(X,A) = \operatorname{Ext}^i_{G_k}(C_*^X(\bar{k}),A)$. This agrees with the old definition if k is algebraically closed. The argument of Suslin-Voevodsky [5] shows:

Theorem 1.3. *If m is invertible in k and A finitely generated m-torsion G_k-module, then*

$$H^i_{GS}(X,A) \cong H^i_{\text{et}}(X,A).$$

Duality results for the Galois cohomology of a field k lead via theorem 1.3 to duality results between Galois-Suslin homology and cohomology over k. For example, if k be a finite field, A a finite G_k-module, and $A^\vee = \operatorname{Hom}(A,\mathbb{Q}/\mathbb{Z})$, then there is a perfect pairing of finite groups $H_{i-1}^{GS}(X,A) \times H^i_{\text{et}}(X,A^\vee) \to \mathbb{Q}/\mathbb{Z}$.

2. Finite base fields

We fix a finite field $\mathbb{F}_q$ with algebraic closure $\bar{\mathbb{F}}_q$ and consider the following conjecture, see [2]:

Conjecture P_0: *For all smooth and projective schemes X over the finite field $\mathbb{F}_q$, the groups $H_i^S(X,\mathbb{Q})$ vanish for $i \neq 0$.*

Conjecture P_0 is a consequence of Parshin's conjecture, or finite dimensionality in the sense of Kimura-O'Sullivan, or of Tate's and Beilinson's conjecture.

Assuming conjectureP_0, the groups $H_i^S(X,\mathbb{Q})$ are finite dimensional and vanish unless $0 \leq i \leq d$. If X is smooth, then they vanish for $i \neq 0$. One can also show that Conjecture P_0 holds if and only if the groups $H_i^S(X,\mathbb{Z})$ are finitely generated for all $X/\mathbb{F}_q$ if and only if the groups $H^i_S(X,\mathbb{Z})$ are finitely generated for all $X/\mathbb{F}_q$.

2.1. **Weil-Suslin theory.** Let X be separated and of finite type over $\mathbb{F}_q$, $\bar{X} = X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ and G be the Weil-group of $\mathbb{F}_q$. We define Weil-Suslin homology with coefficients in the G-module A, $H_i^W(X,A)$ to be the ith homology of the double complex

$$C_*^X(\bar{k}) \otimes A \xrightarrow{1-\varphi} C_*^X(\bar{k}) \otimes A,$$

where the Frobenius endomorphism φ acts diagonally. We obtain short exact sequences

$$0 \to H_i^S(\bar{X},A)_G \to H_i^W(X,A) \to H_{i-1}^S(\bar{X},A)^G \to 0.$$

If A is the restriction of a $\hat{G}$-module, then the results of [1] give a long exact sequence

$$\cdots \to H_i^{GS}(X,A) \to H_{i+1}^W(X,A) \to H_{i+1}^S(X,A\otimes\mathbb{Q}) \to H_{i-1}^{GS}(X,A) \to \cdots$$

Weil-Suslin cohomology of a G-module A is defined analogously, and the exact sequenes for Weil-Suslin homology have analog versions for cohomology. Analog to the result for Suslin homology we get that Conjecture P_0 holds if and only if the groups $H_i^W(X,\mathbb{Z})$ are finitely generated for all $X/\mathbb{F}_q$ if and only if the groups $H^i_W(X,\mathbb{Z})$ are finitely generated for all $X/\mathbb{F}_q$.

2.2. **A Kato type homology and class field theory.** Suslin-Kato-homology $H_i^{KS}(X,A)$ with coefficients in the G-module A, is the ith homology of the coinvariants of the complex considered above $(C_*^X(\bar{k})\otimes A)_\varphi$. It measures the difference between Suslin homology and Weil-Suslin homology: From the definition one deduces a long exact sequence

$$\cdots \to H_i^S(X,A) \to H_{i+1}^{WS}(X,A) \to H_{i+1}^{KS}(X,A) \to H_{i-1}^S(X,A) \to \cdots.$$

The cohomological theory can be defined analogously. The following is a generalization of a conjecture of Kato, see [3]

Conjecture 2.1. *If X is smooth and connected, then $H_i^{KS}(X,\mathbb{Z}) = 0$ for $i > 0$ and $H_0^{KS}(X,\mathbb{Z}) = \mathbb{Z}$.*

Using a theorem of Jannsen and Saito one can show that Conjecture 2.1 is equivalent to conjecture P_0. We have the following connection to class field theory:

Conjecture 2.2. *We have a canonical isomorphism*

$$H_1^{WS}(X,\mathbb{Z})^\wedge \cong \pi_1^t(X)^{ab}.$$

Under conjecture 2.1, $H_0^S(X,\mathbb{Z}) \cong H_1^{WS}(X,\mathbb{Z})$, and conjecture 2.2 is a theorem of Schmidt and Spieß.

References

[1] T.Geisser, Weil-étale cohomology over finite fields. Math. Ann. 330 (2004), no. 4, 665–692.

[2] T.Geisser, Parshin's conjecture revisited. In: K-theory and non-commutative geometry, EMS Series of Congress Reports (2008), 413–426.

[3] T.Geisser, Arithmetic homology and an integral version of Kato's conjecture, to appear in J. reine angew. Math.

[4] T.Geisser, Some remarks on Suslin homology, Preprint 2009.

[5] A.Suslin, V.Voevodsky, Singular homology of abstract algebraic varieties. Invent. Math. 123 (1996), no. 1, 61–94.

Congruences in non-commutative Iwasawa theory

Mahesh Kakde

In this talk we compute the K_1 groups of Iwasawa algebras of a special kind of a p-adic Lie group and certain localisations of it. As an application we will prove the main conjecture for certain type of extensions of totally real fields. This approach for attacking the main conjecture was suggested by Burns and Kato. In [1], Kato computed the K_1 group of the Iwasawa algebra of p-adic Heisenberg group and its localisation used in Iwasawa theory. He then used the strategy to prove the main conjecture. Our method is a generalisation of Kato's computations.

1. Computation of the K groups

Fix an odd prime p. Let $G = H \rtimes \Gamma$, where H is an abelian pro-p group and Γ is isomorphic to the additive group of p-adic integers. For a ring O (which for us will be a complete, unramified extension of $\mathbb{Z}_p$), we denote $\Lambda_O(G) = \varprojlim_U O[G/U]$, the Iwasawa algebra of G with coefficients in O. Let $G_i := H \rtimes \Gamma^{p^i}$, and $G_i^{ab} =: H_i \times \Gamma^{p^i}$. Here H_i is the appropriate quotient of H which gives the abelianisation of G_i. Then for each $i \geq 0$ we have the maps

$$K_1(\Lambda_O(G)) \longrightarrow K_1(\Lambda_O(G_i)) \longrightarrow K_1(\Lambda_O(G_i^{ab})) = \Lambda_O(G_i^{ab})^\times.$$

Here the first maps is the norm map and the second is the one induced by natural surjection of G_i onto G_i^{ab}. This gives a map

$$\theta : K_1(\Lambda_O(G)) \to \prod_{i=0}^{\infty} \Lambda_O(G_i^{ab})^\times.$$

Our goal is to show that θ is injective and describe its image.

Notation: Fix a topological generator γ of Γ. Then γ acts on H_i. We consider the map on $\Lambda_O(H_i \times \Gamma^{p^i})$ given by $x \mapsto \sum_{k=0}^{p^i-1} \gamma^k x \gamma^{-k}$. Let T_i be the image of this map. We also have the transfer homomorphism $ver : G_{i-1}^{ab} \to G_i^{ab}$ which induces the homomorphism (this induced homomorphism is chosen so that it acts as the absolute arithmetic Frobenius on the coefficients)

$$ver : \Lambda_O(G_{i-1}^{ab}) \to \Lambda_O(G_i^{ab}).$$

For $0 \leq j \leq i$, we have the maps

$$Nr : \Lambda_O(H_j \times \Gamma^{p^j})^\times \to \Lambda(H_j \times \Gamma^{p^i})^\times, \text{and}$$

$$\pi : \Lambda_O(H_i \times \Gamma^{p^i})^\times \to \Lambda_O(H_j \times \Gamma^{p^i})^\times.$$

Here again the first map is the norm map and the second is the one induced by surjection of H_i onto H_j.

Simplifying assumption: The p-power map $\varphi : G \to G$ $(g \mapsto g^p)$ induces a homomorphism $G_{i-1}^{ab} \to G_i^{ab}$. It can be checked that this homomorphism is the same as ver.

Let $S = \{f \in \Lambda_O(G) | \Lambda_O(G)/\Lambda_O(G)f \text{ is a finitely generated } \Lambda_O(H)-\text{module}\}$. It is proven in [2] that S is a left and right Ore set, multiplicatively closed and does not contain any zero divisors. Hence one may localise $\Lambda_O(G)$ at S. We have the *localised* analogue θ_S of θ.

$$\theta_S : K_1(\Lambda_O(G)_S) \to \prod_{i=0}^{\infty} \Lambda_O(G_i^{ab})_S^\times.$$

Theorem 1.1. *Let Φ (resp. Φ_S) be the the subgroup of tuples $(x_i)_{i=0}^\infty$ in $\prod_{i=0}^\infty \Lambda_O(G_i^{ab})^\times$ (resp. $\prod_{i=0}^\infty \Lambda_O(G_i^{ab})_S^\times$) such that (1) For all $0 \leq j \leq i$, $Nr(x_j) = \pi(x_i)$,*
(2) γ fixes x_i for all i.
(3) $ver(x_{i-1}) \equiv x_i (mod\ T_i)$ (resp. $(mod\ T_{S,i})$), for all $i \geq 1$. Then θ induces an isomorphism between $K_1(\Lambda_O(G))$ and Φ. Image of the homomorphism θ_S is contained in Φ_S. Hence $Im(\theta_S) \cap \prod_{i=0}^\infty \Lambda_O(G_i^{ab})^\times = Im(\theta)$.

2. Main conjecture

Let $\Lambda(G) = \Lambda_{\mathbb{Z}_p}(G)$. Let F be a totally real number field. Let F_∞ be a totally real p-adic Lie Galois extension of F in which only finitely many primes in F ramify. Assume that the cyclotomic $\mathbb{Z}_p$ extension of F, denoted by F^{cyc}, is contained in F_∞. Let Σ be a finite set of finite primes of F containing all primes which ramify in F_∞. Let $G = Gal(F_\infty/F)$. For any continuous homomorphism $\rho : G \to GL_n(O_L)$, where $L/\mathbb{Q}_p$ is a finite extension induces

$$K_1(\Lambda(G)_S) \to L \cup \{\infty\} \qquad (x \mapsto x(\rho))$$

$(*)$ Consider the complex $C^\cdot = RHom(R\Gamma_{\acute{e}t}(Spec(O_{F_\infty}[1/\Sigma]), \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p)$. Assume that the cohomology of $C^\cdot$ is S-torsion.

We also need the localisation sequence from K-theory

$$K_1(\Lambda(G)) \to K_1(\Lambda(G)_S) \xrightarrow{\partial} K_0(\Lambda(G), \Lambda(G)_S) \to 0.$$

Conjecture 2.1. *There is a unique* $\zeta(F_\infty/F) = \zeta \in K_1(\Lambda(G)_S)$ *such that*
(1) $\partial(\zeta) = -[C^\cdot]$, *and*
(2) For any Artin representation ρ *of* G *and any* $k > 0$, $k \equiv 0(mod\, p-1)$, *we have*

$$\zeta(\rho\kappa_F^k) = L_\sigma(\rho, 1-k),$$

where κ_F *is the cyclotomic character of* F *and* $L_\Sigma(\rho, 1-k)$ *is the value of the complex L-function associated to* ρ *with Euler factors at primes in* Σ *removed.*

Remark Under the assumption $(*)$ main conjecture is known to be true whenever G is abelian. This is the famous theorem of Wiles on Iwasawa main conjecture [3].

Now assume that $G = Gal(F_\infty/F) = H \rtimes \Gamma$, where $H = Gal(F_\infty/F^{cyc})$ and $\Gamma = Gal(F^{cyc}/F)$ and H is abelian pro-p as in the previous section. With the notations from the previous section consider the diagram

$$\begin{tikzcd}
K_1(\Lambda(G)) \arrow[r] \arrow[d, "\theta"'] & K_1(\Lambda(G)_S) \arrow[r] \arrow[d, "\theta_S"'] & K_0(\Lambda(G), \Lambda(G)_S) \arrow[r] \arrow[d] & 0 \\
\prod_0^\infty \Lambda(G_i^{ab})^\times \arrow[r] & \prod_0^\infty \Lambda(G_i^{ab})_S^\times \arrow[r] & \prod_0^\infty K_0(\Lambda(G_i^{ab}), \Lambda(G_i^{ab})_S) \arrow[r] & 0
\end{tikzcd}$$

Let F_i be the unique extension of F of degree p^i contained in F^{cyc}. Let K_i be the maximal abelian extension of F_i contained in F^{cyc}. Hence $G_i^{ab} = Gal(K_i/F_i)$.

Proposition 2.2. *(Burns, Kato) With the above assumptions on* F_∞/F, *the main conjecture for* F_∞/F *is true if and only if* $(\zeta(K_i/F_i))_0^\infty$ *belongs to* Φ_S.

Using the q-expansion principle of Deligne and Ribet [4] we prove the following theorem which proves the main conjecture in these cases.

Theorem 2.3. *With above assumptions on* F_∞/F, *the tuple* $(\zeta(K_i/F_i))$ *belongs to* Φ_S.

References

[1] K. Kato, *Iwasawa theory of totally real fields for Galois extensions of Heisenberg type*, Unpublished (2006).

[2] J. Coates, T. Fukaya, K. Kato, R. Sujatha, O. Venjakob *The GL_2 Main Conjecture for elliptic curves without complex multiplication*, Publ. Math. IHES (2005).

[3] A. Wiles *The Iwasawa conjecture for totally real fields*, Ann. of Math. *131* (1990), no.3, 493-540.

[4] P. Deligne, K. Ribet *Values of abelian L-functions at negative integers over totally real fields*, Inventiones Math. *59* (1980), 227-286.

Fontaine's theory in the relative setting and applications to comparison isomorphisms

Fabrizio Andreatta

(joint work with Olivier Brinon and Adrian Iovita)

Let $p > 0$ denote a prime integer and let k be a perfect field of characteristic p. Let K be a complete discrete valuation field with residue field k, fraction field

K and ring of integers $\mathcal{O}_K$. Denote by $\overline{K}$ a fixed algebraic closure of K and set $G_K := \mathrm{Gal}(\overline{K}/K)$. We write B_{cris} for the crystalline period ring defined by J.-M. Fontaine. Recall that B_{cris} is a topological ring, endowed with a continuous action of G_K, an exhaustive decreasing filtration $\mathrm{Fil}^\bullet B_{\mathrm{cris}}$ and a Frobenius operator.

Let $X \to \mathrm{Spec}(\mathcal{O}_K)$ be a smooth and proper morphism of relative dimension d with geometrically irreducible fibres and assume that X is defined over $\mathbb{W}(k)$. We consider the following *crystalline conjecture*:

Assume that there exist a $\mathbb{Q}_p$-adic étale sheaf $\mathcal{L}$ on X_K and a filtered-F-isocrystal $\mathcal{E}$ on X_K which are associated. Then, for every $i \geq 0$ there is a canonical and functorial isomorphism commuting with all the additional structures (namely, filtrations, G_K–actions and Frobenii)

$$\mathrm{H}^i(X^{\mathrm{et}}_{\overline{K}}, \mathcal{L}) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \cong \mathrm{H}^i_{\mathrm{cris}}(X_k/K, \mathcal{E}) \otimes_K B_{\mathrm{cris}}.$$

The clarification of what "being associated" means is part of the conjecture. The conjecture is now a theorem, proven by G. Faltings in [F1]. There are various approaches to the proof of the conjecture. One (and the first) is based on ideas of Fontaine and Messing using the syntomic cohomology on X; a full proof (for constant coefficients) using these methods was given by T. Tsuji. There is also an approach (for constant coefficients) based on a comparison isomorphism in K-theory which is due to W. Niziol. We follow the approach by Faltings, not in its original version but using a certain topology described in [F2]. The strategy consists in defining a new cohomology theory associated to X and proving that it computes both the left hand side (via the theory of almost étale extensions) and the right hand side of conjecture. The new inputs, compared to Faltings's original approach, are:

i) we systematically study the underlying sheaf theory of Faltings' topology;

ii) we introduce certain acyclic resolutions of sheaves of periods on Faltings' topology. This allows to avoid the need of providing a Poincaré duality in Faltings' setting compatible with Poincaré duality on crystalline cohomology and with Poincaré duality on étale cohomology.

Faltings' site has as objects the pairs $\big(U, W\big)$ where $U \to X$ is an étale morphism and $W \to U_{\overline{K}}$ is a finite and étale morphism. A family $\big(U_i, W_{ij}\big)_{i\in I, j\in J_i}$ is a covering family if $\amalg_{i\in I} U_i \to U$ is onto and if for every $i \in I$ the map $\amalg_{j\in J_i} W_{ij} \to W \times_U U_i$ is onto. One considers the topology defined by these covering families. As noticed by A. Abbes this definition does not coincide with Faltings' original definition, but it provides the right topos used also by Faltings. We let $\mathrm{Sh}(\mathfrak{X})$ be the category of sheaves of abelian groups for this topology. We have a morphism $w_*\colon \mathrm{Sh}(X^{\mathrm{et}}) \to \mathrm{Sh}(\mathfrak{X})$ given by $(U, W) \to U$. Given a quasi-coherent M sheaf on X^{et}, such as $\mathcal{O}_X$ or $\Omega^i_{X/\mathcal{O}_K}$, we write M for $w_*(M)$.

We denote by $\mathrm{Sh}(\mathfrak{X})^{\mathbb{N}}$ the category of inverse systems and by $\mathrm{Ind}\big(\mathrm{Sh}(\mathfrak{X})^{\mathbb{N}}\big)$ the category of inductive systems of inverse systems. The reason to introduce this category is that with Adrian Iovita we construct an object $\mathbb{B}_{\mathrm{cris}}$ of $\mathcal{O}_X \otimes B_{\mathrm{cris}}$-modules: working with inductive limits of inverse systems we manage to capture the fact that $\mathbb{B}_{\rm cris}$ is a "topological" sheaf. We also construct an integrable, quasi-nilpotent connection $\nabla\colon \mathbb{B}_{\rm cris} \longrightarrow \mathbb{B}_{\rm cris} \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathcal{O}_K}$ such that the complex

$$\mathbb{B}_{\rm cris} \xrightarrow{\nabla^1} \mathbb{B}_{\rm cris} \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathcal{O}_K} \xrightarrow{\nabla^2} \mathbb{B}_{\rm cris} \otimes_{\mathcal{O}_X} \Omega^2_{X/\mathcal{O}_K} \to \cdots \xrightarrow{\nabla^d} \mathbb{B}_{\rm cris} \otimes_{\mathcal{O}_X} \Omega^d_{X/\mathcal{O}_K} \to 0$$

is exact. Furthermore, $\mathbb{B}_{\rm cris}$ is endowed with a filtration satisfying Griffith's transversality i. e., such that $\nabla\left(\mathrm{Fil}^r\left(\mathbb{B}_{\rm cris}\right)\right) \subset \mathrm{Fil}^{r-1}\left(\mathbb{B}_{\rm cris}\right) \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathcal{O}_K}$ for every r. There is also a Frobenius structure. We let $\mathbb{B}^{\nabla}_{\rm cris}$ be the kernel of the connection. Consider $v_*\colon \mathrm{Sh}(\mathfrak{X}) \to \mathrm{Sh}(X^{\rm et})$ induced at the level of topologies by $X^{\rm et}U \mapsto (U, U_{\overline{K}})$. A key fact, based on joint work with O. Brinon, is that $\mathrm{R}^i v_* \mathbb{B}_{\rm cris} = 0$ for $i \geq 1$ and it coincides with $\mathcal{O}_X \otimes_{\mathcal{O}_K} B_{\rm cris}$ if $i = 0$.

Consider the functor $u_*\colon \mathrm{Sh}(X^{\rm et}_{\overline{K}}) \to \mathrm{Sh}(\mathfrak{X})$ induced at the level of topologies by $(U, W) \mapsto W$. Given a $\mathbb{Q}_p$-adic étale sheaf $\mathcal{L}$ we write $\mathcal{L}$ for $w_*\big(\mathcal{L}\big)$. We say that a $\mathbb{Q}_p$-adic étale sheaf $\mathcal{L}$ and a filtered-F-isocrystal $\mathcal{E}$ on X_K are *associated* if there exits an isomorphism of $\mathbb{B}_{\rm cris}$-modules $\mathcal{E} \otimes_K \mathbb{B}_{\rm cris} \cong \mathcal{L} \otimes_{\mathbb{Q}_p} \mathbb{B}_{\rm cris}$ compatible with all extra structures. This notion is equivalent to those of Faltings and of O. Brinon in [Br].

Suppose we have associated objects. Using the complex above we have that $\mathrm{H}^i\big(\mathfrak{X}, \mathcal{L} \otimes_{\mathbb{Q}_p} \mathbb{B}^{\nabla}_{\rm cris}\big)$ coincides with the hypercohomology of the complex $\mathbb{H}^i\big(\mathfrak{X}, \mathcal{E} \otimes_{\mathcal{O}_K} \Omega^\bullet_{X/\mathcal{O}_K} \otimes_{\mathcal{O}_K} \mathbb{B}_{\rm cris}\big)$. Due to the result on $\mathrm{R}^i v_* \mathbb{B}_{\rm cris}$, this coincides with the de Rham cohomology $\mathbb{H}^i_{\rm dR}\big(X, \mathcal{E} \otimes_{\mathcal{O}_K} \Omega^\bullet_{X/\mathcal{O}_K}\big) \otimes_{\mathcal{O}_K} B_{\rm cris}$ (compatibly with all extra structures). Note that $\mathbb{H}^i_{\rm dR}\big(X, \mathcal{E} \otimes_{\mathcal{O}_K} \Omega^\bullet_{X/\mathcal{O}_K}\big) \cong \mathrm{H}^i_{\rm cris}(X_k/K, \mathcal{E})$. On the other hand, a result of Faltings' states that for a $\mathbb{Q}_p$-adic étale sheaf $\mathcal{L}$ we have $\mathrm{R}^i u_*\big(\mathcal{L}\big) = 0$ for $i \geq 1$. Hence, $\mathrm{H}^i\big(\mathfrak{X}, \mathcal{L}\big) \cong \mathrm{H}^i\big(X^{\rm et}_{\overline{K}}, \mathcal{L}\big)$. One is then left to show that $\mathrm{H}^i\big(\mathfrak{X}, \mathcal{L} \otimes_{\mathbb{Q}_p} \mathbb{B}^{\nabla}_{\rm cris}\big) \cong \mathrm{H}^i\big(\mathfrak{X}, \mathcal{L}\big) \otimes_{\mathbb{Q}_p} B_{\rm cris}$ (compatibly with extra structures). Since $\mathcal{E}$ and $\mathcal{L}$ are associated, also the duals $\mathcal{E}^\vee$ and $\mathcal{L}^\vee$ are associated. We then prove by induction on i that $\mathbb{H}^i_{\rm dR}\big(X, \mathcal{E} \otimes_{\mathcal{O}_K} \Omega^\bullet_{X/\mathcal{O}_K}\big)$ (resp. $\mathbb{H}^{2d-i}_{\rm dR}\big(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_K} \Omega^\bullet_{X/\mathcal{O}_K}\big)$) is an admissible filtered F-module in the sense of Fontaine associated to the crystalline representation $\mathrm{H}^i\big(X^{\rm et}_{\overline{K}}, \mathcal{L}\big)$ (resp. $\mathrm{H}^{2d-i}\big(X^{\rm et}_{\overline{K}}, \mathcal{L}^\vee\big)$). This we do using Poincaré dualities on the étale side and de Rham side plus some fine criterion of admissibility of filtered F-modules proven by P. Colmez and J.-M. Fontaine.

References

[Br] O. Brinon: *Représentations p-adiques cristallines et de de Rham dans le cas relatif*, Mémoires de la SMF, **112**, pp. 158.

[F1] G. Faltings: *Crystalline cohomology and p-adic Galois representations*, In "Algebraic Analysis, Geomtery and Number Theory" (J.I.Igusa ed.), John Hopkins University Press, Baltimore, 25–80, (1998).

[F2] G. Faltings: *Almost étale extensions*, In "Cohomologies p-adiques et applications arithmétiques," vol. II. P. Berthelot, J.-M. Fontaine, L. Illusie, K. Kato, M. Rapoport eds. Astérisque **279** (2002), 185–270.

Nonabelian examples for the section conjecture in anabelian geometry

Jakob Stix

The present report continues [Sx09a], that dealt with p-adic local obstructions, by elaborating on the passage from local to global in the section conjecture. More details can be found in [Sx08] and the preprint [Sx09b].

1. The section conjecture of anabelian geometry

1.1. **The conjecture.** Let k be a field, k^{sep} a separable closure, and $\mathrm{Gal}_k = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ its absolute Galois group. The étale fundamental group $\pi_1(X, \bar{x})$ of a geometrically connected variety X/k with a geometric point $\bar{x} \in X$ above k^{sep}/k forms an extension

$$1 \to \pi_1(X \times_k k^{\mathrm{sep}}, \bar{x}) \to \pi_1(X, \bar{x}) \to \mathrm{Gal}_k \to 1, \tag{1}$$

which we abbreviate by $\pi_1(X/k)$ ignoring base points. A k-rational point $a \in X(k)$ yields by functoriality a section s_a of (1), which depends on the choice of an étale path from a to $\bar{x}$. Thus s_a is well defined only up to conjugation by elements from $\pi_1(X \times_k k^{\mathrm{sep}}, \bar{x})$. The section conjecture speculates the following, see Grothendieck [Gr83] for the case of a number field k.

Conjecture 1. *The map $a \mapsto s_a$ is a bijection of the set of rational points $X(k)$ with the set $\mathscr{S}_{\pi_1(X/k)}$ of conjugacy classes of sections of $\pi_1(X/k)$ if k is a number field or a finite extension of $\mathbb{Q}_p$ and X is a smooth, geometrically connected curve of genus g with boundary divisor D in its smooth projective completion, such that*

(i) *$2 - 2g - \deg(D)$ is negative, i.e., X is hyperbolic, and*
(ii) *D has no k-rational point.*

1.2. **Evidence.** It was known already to Grothendieck, that the map $a \mapsto s_a$ of Conjecture 1 is injective by an application of the weak Mordell-Weil theorem.

The real analogue of Conjecture 1 was proven by Mochizuki [Mz03] with alternative proofs later in [Sx08] Appendix A, and by Pal. Koenigsmann was able to prove a birational form over p-adic local fields [Ko05].

The note [Sx08] contains a p-adic local obstruction to the existence of sections and thus k-rational points that leads to a wealth of positive examples where Conjecture 1 holds, yet in the case that there are neither sections nor points. However, it is known that this ostensibly dull case of *empty curves* is crucial, see [Sx08] Appendix C. Shortly afterwards, in [HS08] Harari, Szamuely and Flynn gave examples for the section conjecture with still no points globally over $\mathbb{Q}$ but with local points everywhere.

Further evidence for the section conjecture can be found in the work of Ellenberg, Esnault–Hai, Esnault–Wittenberg, Wickelgren, Saïdi–Tamagawa, and Hoshi–Mochizuki.

Harari and Szamuely work with the abelianized extension $\pi_1^{(\mathrm{ab})}(X/k)$ obtained by pushing with the characteristic quotient $\pi_1 \twoheadrightarrow \pi_1^{\mathrm{ab}}$ of the maximal abelian quotient. The aim of this report is to discuss structural aspects of Conjecture 1 which go beyond the abelianized extension.

2. Adelic sections and Brauer–Manin obstructions

2.1. **Adelic sections.** An extension of algebraically closed base fields does not alter the fundamental group in characteristic 0. Hence for an extension K/k the extension $\pi_1(X \times_k K/K)$ is a pullback of $\pi_1(X/k)$ and we get a natural map

$$\mathscr{S}_{\pi_1(X/k)} \to \mathscr{S}_{\pi_1(X/k)}(K) := \mathscr{S}_{\pi_1(X\times_k K/K)} \quad s \mapsto s \otimes K.$$

Let k be a number field, k_v its completion at a place v and $\mathbb{A}_k \subset \prod_v k_v$ its ring of adels $\mathbb{A}_k \subset \prod_v k_v$. The space of adelic sections $\mathscr{S}_{\pi_1(X/k)}(\mathbb{A}_k) \subset \prod_v \mathscr{S}_{\pi_1(X/k)}(k_v)$ of $\pi_1(X/k)$ is the set of all tuples (s_v) such that for all quotients $\varphi : \pi_1(X) \twoheadrightarrow G$ with finite G all but finitely many of the $\varphi \circ s_v : \mathrm{Gal}_{k_v} \to G$ are unramified.

2.2. **Brauer–Manin obstruction for sections.** A class $\alpha \in \mathrm{H}^2(\pi_1(X), \mu_n)$ describes a function $\langle \alpha, - \rangle \ : \ \mathscr{S}_{\pi_1(X/k)}(\mathbb{A}_k) \to \mathbb{Q}/\mathbb{Z}$ on adelic sections of $\pi_1(X/k)$ by the formula $\langle \alpha, (s_v) \rangle = \sum_v \mathrm{inv}_v(s_v^*(\alpha))$, where the maps inv_v are the local invariant maps $\mathrm{H}^2(k_v, \mu_n) \subset \mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$.

Theorem 2. *The function $\langle \alpha, - \rangle$ is well defined because only finitely many summands in $\sum_v \mathrm{inv}_v(s_v^*(\alpha))$ do not vanish. The image of the global sections under the diagonal map $\mathscr{S}_{\pi_1(X/k)} \to \prod_v \mathscr{S}_{\pi_1(X/k)}(k_v)$ lands in the Brauer kernel*

$$\mathscr{S}_{\pi_1(X/k)}(\mathbb{A}_k)^{\mathrm{Br}} := \{(s_v) \in \mathscr{S}_{\pi_1(X/k)}(\mathbb{A}_k) \ ; \ \langle \alpha, (s_v) \rangle = 0 \textit{ for all } \alpha\}.$$

Proof: We only prove the second part which was independently also observed by O. Wittenberg. We compute

$$\langle \alpha, (s \otimes k_v) \rangle = \sum_v \mathrm{inv}_v\big((s \otimes k_v)^*(\alpha)\big) = \sum_v \mathrm{inv}_v\big(s^*(\alpha) \otimes_k k_v\big) = 0$$

by the Hasse–Brauer–Noether local global principle for the Brauer group. □

2.3. **Conditional results.** Because $\bigcup_n \mathrm{H}^2(\pi_1 X, \mu_n) \twoheadrightarrow \mathrm{H}^2(X, \mathbb{G}_m)$ is surjective for hyperbolic curves, the classical Brauer–Manin obstruction for rational points as in [Ma71] is subsumed under the corresponding obstruction for sections. We can therefore prove the following conditional result.

Theorem 3. *Let k be a number field such that Conjecture 1 holds for each completion k_v. If the Brauer–Manin obstruction against rational points is the only one for curves over k, then the section conjecture holds for hyperbolic curves over k.*

3. Beyond abelian sections

3.1. **The Reichardt–Lind curve.** We present an affine curve over $\mathbb{Q}$ that by an application of Section 2 can be shown not to admit sections. The corresponding empty example for the section conjecture over $\mathbb{Q}$ has adelic points but none that satisfies the Brauer–Manin obstructions, and moreover is not accounted for by the explicit examples of [HS08].

The affine Reichardt–Lind curve $U/\mathbb{Q}$ is defined by $2Y^2 = Z^4 - 17$ with $Y \neq 0$. Let $X/\mathbb{Q}$ be its smooth completion. The class $\alpha_U = \chi_Y \cup \chi_{17} \in \mathrm{H}^2(\pi_1(U), \mu_2)$, the cup product of the two characters defined via Kummer theory by Y and 17, lifts to $\alpha \in \mathrm{H}^2(X, \mu_2)$. The corresponding function $\langle \alpha, - \rangle$ takes the constant value $\frac{1}{2}$ on adelic sections subject to an extra condition.

Theorem 4. *The fundamental group extension $\pi_1(U/\mathbb{Q})$ for the affine Reichardt–Lind curve $U/\mathbb{Q}$ does not split. In particular, the section conjecture holds trivially for $U/\mathbb{Q}$ as there are neither rational points nor sections.*

More precisely, the maximal geometrically pro-2 quotient $\pi_1^{(2)}(X/\mathbb{Q})$ of $\pi_1(X/\mathbb{Q})$ for the projective Reichardt–Lind curve $X/\mathbb{Q}$ does not admit a section s that allows a lifting $\tilde{s}_p$

$$\begin{tikzcd} & \mathrm{Gal}_{\mathbb{Q}_p} \arrow[dl, dotted, "\tilde{s}_p"'] \arrow[d, "s\otimes\mathbb{Q}_p"] \\ \pi_1^{(2)}(U) \arrow[r, two heads] & \pi_1^{(2)}(X) \end{tikzcd}$$

locally at $p = 2$ and $p = 17$.

3.2. **Genus 2 curves.** Potentially, the Brauer–Manin obstruction against sections occurs only on a finer quotient than $\pi_1^{(\mathrm{ab})}(U/k)$, because it depends on H^2. This hope turns out to be illusory for the Reichardt–Lind curve. In order to have an explicit example X, where $\pi_1^{(\mathrm{ab})}(X/k)$ splits and yet there is no section, we resort to an argument of [Sx08] with some more care to prove the following.

Theorem 5. *Let $k/\mathbb{Q}_p$ be a finite extension for $p > 2$, and let X/k be a smooth projective curve of genus 2.*

(1) *If X has period 1, then $\pi_1^{\mathrm{ab}}(X/k)$ admits a section.*
(2) *If X has index 2, then the maximal geometrically metabelian quotient $\pi_1^{\mathrm{metab}}(X/k)$ does not split.*

Explicit examples for the setup of Theorem 5, even of curves over number fields that satisfy the conditions locally at some place, can be found in abundance.

References

[Gr83] A. Grothendieck, *Brief an Faltings (27/06/1983)*, in: Geometric Galois Action 1 (ed. L. Schneps, P. Lochak), LMS Lecture Notes **242** (1997), 49–58.

[HS08] D. Harari, T. Szamuely, *Galois sections for abelianized fundamental groups*, with an Appendix by E. V. Flynn, Math. Annalen **344**, No. 4 (2009), 779–800.

[Ko05] J. Koenigsmann, *On the 'section conjecture' in anabelian geometry*, J. Reine Angew. Math. **588** (2005), 221–235.

[Ma71] Y. I. Manin, *Le groupe de Brauer-Grothendieck en géométrie diophantienne*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, Paris, 1971, 401–411.

[Mz03] Sh. Mochizuki, *Topics surrounding the anabelian geometry of hyperbolic curves*, in Galois groups and fundamental groups, MSRI Publ. **41**, Cambridge Univ. Press (2003), 119–165.

[Sx08] J. Stix, *On the period-index problem in light of the section conjecture*, preprint, arXiv: 0802.4125v1[math.AG], Februar 2008, to appear in *Amer. J. Math.*

[Sx09a] J. Stix, *On the period-index problem in light of the section conjecture*, in: The Arithmetic of Fields, Oberwolfach report **6** no. 5 (2009).

[Sx09b] J. Stix, *Brauer–Manin obstructions for sections of the fundamental group*, preprint.

Finite group schemes and crystalline representations

Jean-Marc Fontaine
(joint work with Ariane Mézard)

This is a report on a part of a joint work in progress with Ariane Mézard whose aim is to understand what is behind the links between torsion (φ, Γ)-modules (resp. torsion φ-modules) and finite subquotients of crystalline representations studied by Wach [6] and Berger [1] (resp. by Breuil [2] and Kisin [5]).

1 – Finite p-groups in characteristic p.

Let k a perfect field of characteristic $p > 0$ and $W = W(k)$ the ring of Witt vectors with coefficients in k. We choose a formal power series $F \in W[[X]]$ such that $F(0) = 0$ and $F \equiv X^p (\bmod\ p)$ and let $S = W[[u]]$ the ring of formal power series in an indeterminate u with coefficients in W that we endow with the unique continuous endomorphism φ_S such that $\varphi_S u = F(u)$ and $\varphi_S s \equiv s^p (\bmod\ p)$ for all $s \in S$.

The ring $\mathcal{O}_{\mathcal{E}}$ which is the p-adic completion of $S[1/u]$ is a complete discrete valuation ring whose maximal ideal is generated by p. Its residue field $E = k((u))$ his itself the fraction field of the complete discrete valuation ring $\mathcal{O}_E = k[[u]] = S/pS$.

We consider the full sub-category $(\mathcal{O}_E)_{\text{flét}}$ of schemes over Spec $\mathcal{O}_E$ whose objects are those schemes such that the structural morphism $X \to \text{Spec } \mathcal{O}_E$ is flat with étale generic fiber. We put a Grothendieck topology on it by taking as covering finite surjective families of flat morphisms of the category.

Let $X = \text{Spec } A$ an object of $(\mathcal{O}_E)_{\text{flét}}$, set $X_\eta = \text{Spec } A_\eta$, with $A_\eta = E \otimes_{\mathcal{O}_E} A$ its generic fiber. For any $n \in \mathbb{N}$, we denote $\mathcal{O}_n(A_\eta)$ the unique étale $\mathcal{O}_{\mathcal{E}}/p^n$-algebra lifting A_η. There is a unique endomorphism φ of $\mathcal{O}_n(A_\eta)$ extending the Frobenius on $\mathcal{O}_{\mathcal{E}}/p^n$ and a unique homomorphism of rings $\mathcal{O}_n(A_\eta) \to W_n(A_\eta)$ (ring of Witt vectors of length n with coefficients in A_η) commuting with the Frobenius and inducing the identity by reduction mod p. It is injective and we use it to identify $\mathcal{O}_n(A_\eta)$ to a subring of $W_n(A_\eta)$. We set $\mathcal{O}_n(X) = \mathcal{O}_n(A) = W_n(A) \cap \mathcal{O}_n(A_\eta) (\subset W_n(A_\eta))$.

It is easy to see that $\mathcal{O}_n$ is a sheaf of S-algebras equipped with a Frobenius over $(\mathcal{O}_E)_{\text{flét}}$. Moreover $\mathcal{O}_1$ is the structural sheaf and, for $m, n \in \mathbb{N}$, we have a short exact sequence of abelian sheaves

$$0 \to \mathcal{O}_m \to \mathcal{O}_{m+n} \to \mathcal{O}_n \to 0$$

and we may consider $\mathcal{O}_\infty = \varinjlim_{m \in \mathbb{N}} \mathcal{O}_m$ as a p-divisible sheaf of torsion S-modules, equipped with a Frobenius.

A *φ-module over S* is an S-module M equipped with a φ_S-semi-linear map $\varphi_M : M \to M$, or, equivalently with a linear map $\Phi_M : \varphi_S^* M \to M$. With an obvious definiton of morphisms, they form an abelian category. We denote $\mathcal{M}^{tor}(S, \varphi)$ the full sub-category whose objects are the M's which are p-torsion

S-modules of finite type, without u-torsion and such that the S-module Coker Φ_M is of finite length.

Finally let $\mathcal{G}_p^{ffe}(\mathcal{O}_E)$ be the category of finite and flat commutative group schemes over $\mathcal{O}_E$ with étale generic fiber.

The following theorem is not hard to prove directly. It is also an easy consequence of the work of Zink [8].

Theorem 1.

i) *For any object J of $\mathcal{G}_p^{ffe}(\mathcal{O}_E)$, the φ-module $M(J) = \mathrm{Hom}_{\text{ab. sh.}}(J, \mathcal{O}_\infty)$ is an object of $\mathcal{M}^{tor}(S,\varphi)$.*

ii) *For any object M of $\mathcal{M}^{tor}(S,\varphi)$ and any object $X = \mathrm{Spec}\ A$ of $(\mathcal{O}_E)_{\text{flét}}$, set $J(M)(A) = \mathrm{Hom}_{\varphi-\mathrm{mod}}(M, \mathcal{O}_\infty(A))$. Then $J(M)$ is representable by an object of $\mathcal{G}_p^{ffe}(\mathcal{O}_E)$.*

iii) *The contravariant functor $J \mapsto M(J)$ is fully faithful and induces an anti-equivalence between $\mathcal{G}_p^{ffe}(\mathcal{O}_E)$ and $\mathcal{M}^{tor}(S,\varphi)$. The functor $M \mapsto J(M)$ is a quasi-inverse.*

Let $BT^e_{\mathbb{Q}_p}(\mathcal{O}_E)$ the category whose objects are Barsotti-Tate groups over $\mathcal{O}_E$ with étale generic fiber, and with $\mathrm{Hom}_{BT^e_{\mathbb{Q}_p}(\mathcal{O}_E)}(\Gamma_1,\Gamma_2) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{Hom}(\Gamma_1,\Gamma_2)$. By going to the limit, we may associate to any object Γ of $BT^e_{\mathbb{Q}_p}(\mathcal{O}_E)$ a free-$S[1/p]$-module M of finite rank equipped with an injective $S[1/p]$-linear map $\Phi_M : \varphi_S^* M \to M$ and the contravariant functor $J \mapsto (M, \Phi_M)$ is fully faithful. Moreover, for any Γ, the $S[1/p]$-module Coker Φ_M is a torsion module of finite type. As $S[1/p]$ is a principal domain, we may associate to Γ the invariant factors of Coker Φ_M. In particular, for any non zero $q \in S[1/p]$, we say that Γ *is q-finite* if Coker Φ_M is annihilated by a power of q. We say that Γ *is minimally q-finite* if it is q-finite and if ,for any subobject Γ' of Γ which is q-finite and such that $\Gamma' \to \Gamma$ induces an isomorphism on the general fiber, then $\Gamma' = \Gamma$.

2 – From characteristic 0 to characteristic p

Let's K be a finite totally ramified extension of the fraction field of W. Let $\overline{K}$ be an algebraic closure of K and $G = \mathrm{Gal}(\overline{K}/K)$. For any subfield L of $\overline{K}$, we call $\mathcal{O}_{\overline{L}}$ the intersection of L with the valuation ring of $\overline{K}$.

We choose a uniformizing parameter π_0 of K. We construct inductively a sequence $(\pi_n)_{n\in\mathbb{N}}$ of elements of $\overline{K}$ by requiring that ${}^{\sigma^{-n}}F(\pi_n) = \pi_{n-1}$ (where ${}^{\sigma^{-n}}F$ is the formal power series deduced from F by applying σ^{-n} to the coefficients). One sees easily that, for $n > 0$, $K_n = K[\pi_n]$ is a totaly ramified extension of K_{n-1} and that π_n is a uniformizing parameter of K_n (contrarly to the custom $K = K_0$ and K_0 may be different from $W[1/p]$).

There is a unique continuous homomorphism of W-algebras $\theta_0 : S \to \mathcal{O}_K$ sending u to π_0. Its kernel is a principal ideal and we choose a generator q of it.

In this context, the field E can be identified to the norm field of the extension K_∞/K [3,4,7]. Therefore (*loc. cit.*), to $\overline{K}$ corresponds a separable closure E^s of E together with an identification of $G_E = \mathrm{Gal}(E^s/E)$ to $\mathrm{Gal}(\overline{K}/K_\infty)$.

Let V be a p-adic continuous representation of G_K. We may view it as a Barsotti-Tate group over K up to isogeny. Similarly, when we restrict the action of G_K to G_E, we can view it as an etale Barsotti-Tate group over E up to isogeny.

Proposition 2. *Let V be a p-adic representation of G_K which is crystalline with non negative Hodge-Tate weights. There exists a unique object Γ_V of $BT^e_{\mathbb{Q}_p}(\mathcal{O}_E)$ which is minimally q-finite and such that its general fiber corresponds to V with the action of G_E.*

In the case where $F = X^p$, this is an easy consequence of a result of Kisin [5]. The extension to the general case is straightforward. It contains the cyclotomic case considered by Wach and Berger.

Conversely, one can construct an equivalence of categories between crystalline representations of G_K with non negative Hodge-Tate weights and pairs consisting of an object of $BT^e_{\mathbb{Q}_p}(\mathcal{O}_E)$ and a suitable data descent.

We may extend this notion of suitable data descent to the category $\mathcal{G}_p^{ffe}(\mathcal{O}_E)$. This give a way to define the notion of a p-torsion crystalline representation in full generality. Details will be given elsewhere.

References

[1] Berger, Laurent, *Représentations p-adiques et équations différentielles.* Invent. Math. 148 (2002), 219–284.

[2] Breuil, Christophe, *Schémas en groupes et corps des normes*, unpublished (1998) (http://www.ihes.fr/ breuil/publications.html).

[3] Fontaine, Jean-Marc et Wintenberger, Jean-Pierre, *Le ''corps des normes" de certaines extensions algébriques de corps locaux.* C. R. Acad. Sci. Paris 288 (1979), A367–A370.

[4] Fontaine, Jean-Marc et Wintenberger, Jean-Pierre, Extensions algébriques et corps des normes des extensions APF des corps locaux. C. R. Acad. Sci. Paris 288 (1979), A441–A444.

[5] Kisin, Mark, *Crystalline representations and F-crystals.* Algebraic geometry and number theory, 459496, Progr. Math., 253, Birkhaüser, Boston, MA, 2006.

[6] Wach, Nathalie, *Représentations cristallines de torsion.* Compositio Math. 108 (1997), 185–240.

[7] Wintenberger, Jean-Pierre, *Le corps des normes de certaines extensions infinies de corps locaux ; applications.* Ann. Sci. E.N.S. (1983), 59–89.

[8] Zink, Thomas, *The display of a formal p-divisible group.* Cohomologies p-adiques et applications arithmétiques, I. Astérisque No. 278 (2002), 127–248.

Reporter: Peter Barth

Participants

Prof. Dr. Fabrizio Andreatta
Dipartimento di Matematica
"Federigo Enriques"
Universita di Milano
Via Saldini 50
I-20133 Milano

Prof. Dr. Kenichi Bannai
Department of Mathematics
Faculty of Science and Technology
Keio University
3-14-1, Hiyoshi, Kohoku-ku
Yokohama 223-8522
JAPAN

Dipl.Math. Peter Barth
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Prof. Dr. Laurent Berger
Dept. de Mathematiques, U.M.P.A.
Ecole Normale Superieure de Lyon
46, Allee d'Italie
F-69364 Lyon Cedex 07

Prof. Dr. Gebhard Böckle
Fachbereich Mathematik
Universität Duisburg-Essen
45117 Essen

Dr. Thanasis Bouganis
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Prof. Dr. Jan Hendrik Bruinier
Fachbereich Mathematik
TU Darmstadt
Schloßgartenstr. 7
64289 Darmstadt

Dr. Oliver Bültel
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn

Dr. Xavier Caruso
U. F. R. Mathematiques
I. R. M. A. R.
Universite de Rennes I
Campus de Beaulieu
F-35042 Rennes Cedex

Prof. Dr. Gaetan Chenevier
Centre de Mathematiques
Ecole Polytechnique
Plateau de Palaiseau
F-91128 Palaiseau Cedex

Prof. Dr. Chandan Singh Dalawat
Harish-Chandra Institute
Chhatnag Road, Jhusi
Allahabad 211019
INDIA

Prof. Dr. Christopher Deninger
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Dr. Vladimir Dokchitser
Dept. of Pure Mathematics and
Mathematical Statistics
University of Cambridge
Wilberforce Road
GB-Cambridge CB3 0WB

Dr. Amir Dzambic
Institut für Mathematik
Universität Frankfurt
Robert-Mayer-Str. 6-10
60325 Frankfurt am Main

Sandra Eisenreich
Fakultät für Mathematik
Universität Regensburg
Universitätsstr. 31
93053 Regensburg

Dr. Tobias Finis
Mathematisches Institut
Heinrich-Heine-Universität
Gebäude 25.22
Universitätsstr. 1
40225 Düsseldorf

Prof. Dr. Matthias Flach
Department of Mathematics
California Institute of Technology
Pasadena , CA 91125
USA

Prof. Dr. Jean-Marc Fontaine
Laboratoire de Mathematiques
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Jochen Gärtner
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Prof. Dr. Thomas Geisser
Department of Mathematics
KAP 108
University of Southern California
3620 S. Vermont Avenue
Los Angeles CA 90089-2532
USA

Prof. Dr. Ulrich Görtz
Institut f. Experimentelle Mathematik
Universität Essen
Ellernstr. 29
45326 Essen

Prof. Dr. Alexander Goncharov
Department of Mathematics
Brown University
Box 1917
Providence , RI 02912
USA

Prof. Dr. Elmar Große-Klönne
Institut für Mathematik
Humboldt-Universität Berlin
Rudower Chaussee 25
12489 Berlin

Dr. Armin Holschbach
Fakultät für Mathematik
Universität Regensburg
Universitätsstr. 31
93053 Regensburg

Dr. Benjamin V. Howard
Department of Mathematics
301 Carney Hall
Boston College
Chestnut Hill , MA 02467-3806
USA

Prof. Dr. Annette Huber-Klawitter
Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg

Prof. Dr. Uwe Jannsen
Fakultät für Mathematik
Universität Regensburg
Universitätsstr. 31
93053 Regensburg

Dr. Mahesh Kakde
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton , NJ 08544
USA

Dr. Moritz Kerz
Naturwissenschaftliche Fakultät I
Mathematik
Universität Regensburg
93040 Regensburg

Prof. Dr. Guido Kings
NWF-I Mathematik
Universität Regensburg
93040 Regensburg

Prof. Dr. Klaus Künnemann
NWF-I Mathematik
Universität Regensburg
93040 Regensburg

Niko Naumann
Mathematisches Institut
Universität Bonn
Beringstr. 1
53115 Bonn

Prof. Dr. Wieslawa Niziol
Department of Mathematics
University of Utah
155 South 1400 East
Salt Lake City , UT 84112-0090
USA

Dr. Tadashi Ochiai
Dept. of Mathematics
Graduate School of Science
Osaka University
Machikaneyama 1-1, Toyonaka
Osaka 560-0043
JAPAN

Jeanine van Order
Dept. of Pure Mathematics and
Mathematical Statistics
University of Cambridge
Wilberforce Road
GB-Cambridge CB3 0WB

Dr. Sascha Orlik
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn

Prof. Dr. Florian Pop
Department of Mathematics
University of Pennsylvania
Philadelphia , PA 19104-6395
USA

Prof. Dr. Karl Rubin
Department of Mathematics
University of California, Irvine
Irvine , CA 92697-3875
USA

Prof. Dr. Alexander Schmidt
Naturwissenschaftliche Fakultät I
Mathematik
Universität Regensburg
93040 Regensburg

Prof. Dr. Peter Schneider
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Anthony J. Scholl
Dept. of Pure Mathematics and
Mathematical Statistics
University of Cambridge
Wilberforce Road
GB-Cambridge CB3 0WB

Prof. Dr. Romyar Sharifi
Department of Mathematics and Statistics
McMaster University
1280 Main Street West
Hamilton , Ont. L8S 4K1
CANADA

Dr. Jakob Stix
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Prof. Dr. Otmar Venjakob
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Dr. Eva Viehmann
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn

Prof. Dr. Torsten Wedhorn
Institut für Mathematik
Universität Paderborn
Warburger Str. 100
33098 Paderborn

Prof. Dr. Annette Werner
Institut für Mathematik
(Fach 187)
J. W. Goethe-Universität Frankfurt
60054 Frankfurt am Main

Prof. Dr. Kay Wingberg
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg

Prof. Dr. Jean-Pierre Wintenberger
Institut de Mathematiques
Universite de Strasbourg
7, rue Rene Descartes
F-67084 Strasbourg Cedex

Prof. Dr. Shouwu Zhang
Department of Mathematics
Columbia University
2990 Broadway
New York NY 10027
USA