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Algebraic K-Theory and Motivic Cohomology

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Abstract. Algebraic K-theory and the related motivic cohomology are a systematic way of producing invariants for algebraic or geometric structures. Its definition and methods are taken from algebraic topology, but it has also proved particularly fruitful for problems of algebraic geometry, number theory or quadratic forms. 19 one-hour talks presented a wide range of results on K-theory itself and applications. We had a lively evening session trading questions and discussing open problems.

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Introduction by the Organisers

The 2009 program on K-theory and motivic cohomology presented a varied series of lectures on the latest developments in the field. 19 one-hour talks were delivered. In addition, we had a lively evening session trading questions and discussing open problems. The participants represented all aspects of the community. The common bound was the interaction of K-theory and algebraic geometry in its many guises. We were particularly happy to see that a wide range of nationalities were present.

Algebraic K-theory is a systematic way of producing invariants for algebraic and geometric structures. Its definition and many methods are taken from algebraic topology, but it has proved particularly fruitful for problems of algebraic geometry, number theory or the theory of quadratic forms. Motivic cohomology is closely related: the two theories are related by a spectral sequence. Depending on the question, motivic cohomology is used to understand K-theory — or vice versa.

The field has matured and a wide range of methods (like topological cyclic homology or motivic homotopy theory) has been developed. At present these powerful tools are being used to get explicit results in examples and to settle some old problems. The project of extending Voevodsky's proof of the Milnor conjecture to the more general Bloch-Kato conjecture for other primes than 2 is ongoing and expected to be finished in the near future. A highlight of the conference was the announcement in Kerz's talk of the proof of Kato's conjecture on higher Hasse principles in case the coefficient characteristic is invertible.

We now want to describe in more detail the topics which were touched.

Computations in K-theory. Advances in understanding the relation of cyclic homology or topological cyclic homology with K-theory formed the basis for the lectures of Weibel, Gerhard and Walker, who showed how these foundational results have enabled a new series of computations. Cortiñas described new methods for proving homotopy invariance of K-groups of rings of continuous functions.

Special varieties/applications. Semenov used some of the methods that went into the proof of the Bloch-Kato conjecture to give an intrinsic characterization of those varieties that arise as "norm varieties" for an element of étale cohomology of the ground field. Using this result he gave a surprising application to a description of the finite subgroups of forms of E_8, completing a program of Serre's. Kahn described three methods, all relying on understanding forms of homogeneous spaces and group schemes, for detecting interesting elements in the K-theory of central simple algebras.

Categorical constructions for K-theory and motives. There were a number of lectures devoted to categorical aspects of the theory. Arapura showed how one could generalize Nori's Tannaka-like approach to give a theory of motivic sheaves over a base scheme. Ivorra constructed an A_∞-structure on the Rost complexes for cycle modules. Ayoub discussed the filtration by dimension in Voevodsky's triangulated category, giving rise to a theory of "motives mod algebraic equivalence". Schlichting showed us a new approach to the problem of localization for Grothendieck-Witt theory, outlining a proof that works in all characteristics.

$\mathbb{A}^1$-homotopy theory. Three talks discussed foundational aspects of $\mathbb{A}^1$-homotopy theory. Panin described the motivic version of the classical theorem of Conner and Floyd, the motivic version relating Voevodsky's algebraic cobordism with algebraic K-theory. Pelaez gave us his proof that the layers in Voevodsky's slice filtration in the motivic stable homotopy category are modules over motivic Eilenberg-Mac Lane spectrum (when the base is a perfect field) or over MGL (for general base). Wendt discussed unstable aspects of the theory, outlining a proof of the fundamental fact that the "naive" algebraic version of the classical singular complex construction does compute the $\mathbb{A}^1$-homotopy type for linear algebraic group schemes. This result is one ingredient in Morel's recently announced proof of the Friedlander-Milnor conjecture.

Motives/algebraic cycles. There were three lectures on classical Chow motives and algebraic cycles. Srinivas constructed examples of varieties with highly non-divisible Chow groups, relying on some classical geometry of Jacobian varieties. Kimura extended to arbitrary Chow motives Jannsen's result that "injectivity of the cycle map implies surjectivity", using what looks to be a very useful method of "idempotent correspondences with support". Vishik gave an overview of a series of results that describe how the motive of a quadric splits into Lefschetz motives over the algebraic closure of the ground field, and gave applications of these results to the theory of quadratic forms.

Arithmetic. Finally, we had three lectures on arithmetic aspect of the theory. Riou gave a report on a new proof of Gabber's theorem on absolute purity of étale cohomology, relying on Gabber's refinement of de Jong's theorem on modifications. Kerz described his proof with S. Saito of a conjecture of Kato, giving a generalization of some of the main results of class field theory. One important consequence of these results is the finiteness of certain mod ℓ Chow groups for a quasi-projective variety over a finite field. Lichtenbaum gave an overview of his ideas for constructing a cohomology theory for arithmetic varieties that would describe the order of vanishing and leading term of zeta functions.

Workshop: Algebraic K-Theory and Motivic Cohomology

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Abstracts

K-theory of cones of smooth varieties

Charles Weibel
(joint work with Guillermo Cortiñas, Christian Haesemeyer, Mark Walker)

Let R be the homogeneous coordinate ring of a smooth projective variety X over a field k of characteristic 0. We calculate the K-theory of R in terms of the geometry of the projective embedding of X. This answers an old question of Murthy, and clarifies several prior results. For example, we give the complete calculation for $R = k[x,y,z]/(xy = z^2)$: Murthy proved that $K_0(R) = \mathbb{Z}$; we show that $K_1(R) = K_1(k) \oplus k$ (the lower bound given by Srinivas) and more generally:

$$K_n(k[x,y,z]/(xy = z^2)) = K_n(k) \oplus \Omega_k^{n-1} \oplus \Omega_k^{n-3} \oplus \cdots .$$

By standard results, the Adams operations split $K_n(R)/K_n(k)$ into the direct sum of its eigenspaces $\tilde{K}_n^{(i)}(R)$, which have the additional structure of R-modules. The prototype is the Picard group $K_0^{(1)}(R)$, which is isomorphic to the R-module R^+/R, where R^+ is the seminormalization of R.

Under the philosophy of [1] and [2], the answer is expressed in terms of the cyclic homology over $\mathbb{Q}$ when $i < n$ and the *cdh*-cohomology of R over k when $i > n+1$. The *cdh*-cohomology is further interpreted as a graded module, indexed by twists $t > 0$, of Zariski cohomology groups of X. We have:

$$\tilde{K}_n^{(i)}(R) = \begin{cases} HC_{n-1}^{(i-1)}(R), & i < n; \\ \operatorname{tors}\Omega_R^{n-1}/d\operatorname{tors}\Omega_R^{n-2}, & i = n; \\ \left\{\oplus_t H^0(X, \Omega_X^n(t))\right\}/\Omega_R^n, & i = n+1; \\ \oplus_t H^{i-n-1}(X, \Omega_X^{i-1}(t)), & i \geq n+2. \end{cases}$$

In particular, $K_0(R) = \mathbb{Z} \oplus (R^+/R) \oplus \bigoplus_{j,t>0} H^j(X, \Omega_X^j(t))$. When X is a curve, the formulas simplify because we can ignore $H^j(X,-)$ for $j > 1$; for example $K_{-1}(R) = \oplus_t H^1(X, \mathcal{O}_X(t))$ and $K_0(R) = \mathbb{Z} \oplus (R^+/R) \oplus \bigoplus_t H^1(X, \Omega_X^1(t))$.

When X is defineable over a number field, the formulas simplify even more because $\Omega_X^* = \Omega_{X/k}^* \otimes \Omega_k^*$. When X is not defineable over a number field, the simplifications need to include the (arithmetic) twisted Gauss-Manin connection ∇. For example, we have the exact sequence

$$0 \to K_1^{(2)}(R) \to \frac{\oplus_t H^0(X, \Omega_{X/k}^1(t))}{\text{image } \Omega_{R/k}^1} \xrightarrow{\nabla} \Omega_k^1 \otimes \left(\oplus_t H^1(X, \mathcal{O}_X(t))\right) \to K_0^{(2)}(R) \to 0.$$

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Norm varieties and algebraic groups

Nikita Semenov

Let k be a field of characteristic 0, p a prime number, and n an integer. For simplicity we assume that k contains a primitive p-th root of unity.

Definition 1 ([2])**.** Let $0 \neq u \in H^n_{et}(k, \mu_p^{\otimes n})$. A smooth projective irreducible variety X over k is called a *norm variety for u* if

(1) $\dim X = p^{n-1} - 1 =: d$;
(2) $u_{k(X)} = 0 \in H^n_{et}(k(X), \mu_p^{\otimes n})$;
(3) $s_d(X) \neq 0 \mod p$, where s_d denotes the Milnor number of X.

Definition 2 ([3, Definition 5.1])**.** Let X be a smooth projective irreducible variety over k and $b = \dfrac{p^{n-1}-1}{p-1}$. Consider the complex

$$\mathrm{CH}^b(X) \xrightarrow{\pi_0^* - \pi_1^*} \mathrm{CH}^b(X \times X) \xrightarrow{\pi_0^* - \pi_1^* + \pi_2^*} \mathrm{CH}^b(X \times X \times X),$$

where π_i is the i-th projection in the diagram

$$X \leftleftarrows X \times X \;\overset{\leftarrow}{\leftleftarrows}\; X \times X \times X.$$

An element $\rho \in \mathrm{CH}^b(X \times X)$ is called a *special correspondence of type* (n,p) if

$$(\pi_0^* - \pi_1^* + \pi_2^*)(\rho) = 0$$

and $(\pi_0)_*(\rho^{p-1}) \neq 0 \mod p$ as an element in $\mathrm{CH}^0(X) = \mathbb{Z}$.

In [3, Proposition 7.14] Markus Rost showed the following theorem:

Theorem 3. *Let X be a smooth projective irreducible variety which possesses a special correspondence of type (n,p). Assume that $\deg(\mathrm{CH}_0(X)) \subset p\mathbb{Z}$. Then the Chow motive of X with $\mathbb{Z}_{(p)}$-coefficients has a direct summand R such that*

$$X \otimes R \simeq \bigoplus_{i=0}^{p-1} X\{bi\} \otimes \mathbb{Z}_{(p)}.$$

We show:

Theorem 4 ([4, Theorem 5.1])**.** *Let X be a smooth projective variety which possesses a special correspondence of type (n,p). Assume that $\deg(\mathrm{CH}_0(X)) \subset p\mathbb{Z}$. Then there exists $0 \neq u \in H^n_{et}(k, \mu_p^{\otimes(n-1)})$ such that X is a norm variety for u. For any field extension K/k we have $u_K = 0$ iff X has a zero-cycle of degree coprime to p.*

The proof of this theorem uses the technique of Voevodsky, the above result of Rost, and the Voevodsky conjecture about ν_n-varieties proved by Vishik.

Next we show:

Theorem 5 ([4, Theorem 8.8])**.** *Let G be a simple algebraic group over $\mathbb{Q}$ such that $G_{\mathbb{R}}$ is a compact Lie group of type* E_8. *Let K be a field of characteristic* 0. *If G_K splits, then $(-1)^5 \in H^5_{et}(K, \mathbb{Z}/2)$ is zero.*

This statement was conjectured by J.-P. Serre in 1999 in the context of the classification program of finite subgroups of Chevalley groups. The proof is based on the previous theorem and uses the J-invariant of algebraic groups introduced in my joint paper [1] with Petrov and Zainoulline.

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Algebraic K-theory of the dual numbers

Teena Gerhardt

(joint work with Vigleik Angeltveit, Lars Hesselholt)

It was proven by Soulé [5] that for a non-negative q, the relative algebraic K-theory group $K_q(\mathbb{Z}[x]/(x^2),(x))$ is a finitely generated abelian group of rank 1 if q is odd, and rank 0 if q is even. We improve upon Soulé's result with the following theorem:

Theorem 1. *For m a positive integer and i a non-negative integer,*

(1) $K_{2i+1}(\mathbb{Z}[x]/(x^m),(x))$ *is free of rank* $m-1$.
(2) $K_{2i}(\mathbb{Z}[x]/(x^m),(x))$ *is finite of order* $(mi)!(i!)^{m-2}$.

To prove this theorem we relate the algebraic K-theory groups in question to topological cyclic homology groups using the cyclotomic trace map of Bökstedt-Hsiang-Madsen [2]:

$$trc : K_q(\mathbb{Z}[x]/(x^m),(x)) \to \mathrm{TC}_q(\mathbb{Z}[x]/(x^m),(x)).$$

By a theorem of McCarthy [4], this map is an isomorphism after profinite completion. Thus, we aim to compute the relative topological cyclic homology groups $\mathrm{TC}_q(\mathbb{Z}[x]/(x^m),(x))$. A theorem of Hesselholt and Madsen [3] gives a formula for these groups in terms of certain equivariant homotopy groups of topological Hochschild homology. In particular, for each prime p we need to consider the equivariant homotopy groups

$$\mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p) = \pi_{q-\lambda}(T(\mathbb{Z})^{C_{p^{n-1}}}) = [S^q \wedge S^1/C_{p^{n-1}+}, S^\lambda \wedge T(\mathbb{Z})]_{S^1}$$

Here $C_{p^{n-1}} \subset S^1$ is the cyclic group of order p^{n-1}, $T(\mathbb{Z})$ denotes the topological Hochschild S^1-spectrum of $\mathbb{Z}$, and S^λ denotes the one-point compactification of the complex S^1-representation λ. These groups have several operators on them. Inclusion of fixed points induces a map

$$F : \mathrm{TR}^n_{q-\lambda}(\mathbb{Z};p) \to \mathrm{TR}^{n-1}_{q-\lambda}(\mathbb{Z};p)$$

called the Frobenius. This map has an associated transfer

$$V : \mathrm{TR}^{n-1}_{q-\lambda}(\mathbb{Z}; p) \to \mathrm{TR}^{n}_{q-\lambda}(\mathbb{Z}; p)$$

called the Verschiebung. Further, the topological Hochschild S^1-spectrum $T(\mathbb{Z})$ has the structure of a cyclotomic spectrum. From this cyclotomic structure we get a map

$$R : \mathrm{TR}^{n}_{q-\lambda}(\mathbb{Z}; p) \to \mathrm{TR}^{n-1}_{q-\lambda'}(\mathbb{Z}; p)$$

called the restriction. Here $\lambda' = \rho_p^*(\lambda^{C_p})$ where $\rho_p : S^1 \to S^1/C_p$ is the isomorphism given by the pth root. Evaluating the equivariant homotopy groups $\mathrm{TR}^{n}_{q-\lambda}(\mathbb{Z}; p)$ and the operators V and R is sufficient to evaluate the relative topological cyclic homology in question, using the formula of Hesselholt and Madsen.

We prove the following theorem, giving a partial computation of these groups:

Theorem 2. *Let n be a positive integer, and λ a finite dimensional complex S^1-representation. Then*

(1) $\mathrm{TR}^{n}_{2i-\lambda}(\mathbb{Z}; p)$ *is a free abelian group of rank equal to the number of integers* $0 \leq s < n$ *such that* $i = \dim_{\mathbb{C}}(\lambda^{C_{p^s}})$.

(2)

$$|\mathrm{TR}^{n}_{2i-1-\lambda}(\mathbb{Z}; p)| = \begin{cases} |\mathrm{TR}^{n-1}_{2i-1-\lambda'}(\mathbb{Z}; p)| p^{n-1}(i - \dim_{\mathbb{C}}(\lambda)) & \text{if } i > \dim_{\mathbb{C}}(\lambda) \\ |\mathrm{TR}^{n-1}_{2i-1-\lambda'}(\mathbb{Z}; p)| & \text{if } i \leq \dim_{\mathbb{C}}(\lambda) \end{cases}$$

(3) *For every integer q, the Verschiebung map*

$$V : \mathrm{TR}^{n-1}_{q-\lambda}(\mathbb{Z}; p) \to \mathrm{TR}^{n}_{q-\lambda}(\mathbb{Z}; p)$$

is injective. For q even, the cokernel is free.

The restriction maps in the formula of Hesselholt and Madsen can also be understood, so as described above we can recover the main theorem from Theorem 2. The proof of Theorem 2 relies on computations of the equivariant homotopy groups with $\mathbb{Z}/p\mathbb{Z}$-coefficients:

$$\mathrm{TR}^{n}_{q-\lambda}(\mathbb{Z}; p, \mathbb{Z}/p\mathbb{Z}) = [S^q \wedge S^1/C_{p^{n-1}+}, M_p \wedge S^\lambda \wedge T(\mathbb{Z})]_{S^1},$$

where M_p denotes the equivariant Moore spectrum given by the mapping cone of the multiplication by p map on the sphere S^1-spectrum. These groups were evaluated by Angeltveit and Gerhardt [1], and by Tsalidis [6].

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A category of motivic sheaves

Donu Arapura

The goal is to construct an abelian category of motivic "sheaves" over any quasi-projective base defined over a field of characteristic zero. The construction is based on Nori's method (c.f. [L]), and it almost certainly coincides with his category when $S = Spec\, k$.

Fix a field $k \subseteq \mathbb{C}$ and another field F. A more precise statement is as follows:

Theorem 1. *To every k-variety, there is an F-linear abelian category $\mathcal{M}(S;F)$ with an abelian full subcategory $\mathcal{M}_{tls}(S;F)$ of tame motivic "local systems" such that*

(1) *There are exact realization functors R_B, R_{et}, R_H where*

$$R_B : \mathcal{M}(S;F) \to Constr(S_{an}, F)$$

goes to the category of constructible sheaves of F-modules for the classical topology. The image $R_B(\mathcal{M}_{tls}(S;F))$ is contained in the subcategory of locally constant sheaves.

$$R_{et} : \mathcal{M}(S;F) \to Constr(S_{et}, F)$$

goes to the category of constructible sheaves of F-modules for the étale topology. In this case, F should be finite or $\mathbb{Q}_\ell$.

The image $R_{et}(\mathcal{M}_{tls}(S;F))$ is contained in the subcategory of locally constant sheaves.

$$R_H : \mathcal{M}_{tls}(S;\mathbb{Q}) \to VMHS(S_{an})$$

goes to the category of admissible variations of mixed Hodge structures.

(2) *To each good (e.g. projective) family $f : X \to S$, there exist motives $h^i_S(X)(n) \in \mathcal{M}(S;F)$ corresponding to $R^i f_* F(n)$ under realization. More generally, there is a motive $h^i_S(X,Y)$ for good pairs ($f : X \to S, Y \subseteq X$ closed), which roughly corresponds to the fiberwise cohomology of the pair.*

(3) *There are inverse images compatible with realizations.*

(4) *There are higher direct images (under some conditions) compatible with realizations.*

(5) *There are tensor products on $\mathcal{M}_{tls}(S;F)$ compatible with realizations, making this into a Tannakian category.*

(6) *The subcategory $\mathcal{M}_{pure}(S,\mathbb{Q}) \subset \mathcal{M}_{tls}(S,\mathbb{Q})$ generated by smooth projective families is a semisimple Tannakian category.*

In outline, $\mathcal{M}(S,F)$ is constructed as the universal theory for which:

(1) $\mathcal{M}(S;F)$ is an F-linear abelian category with a faithful exact functor R_B to the category of sheaves of F-modules on S with its classical topology.

(2) A morphism $X' \to X$ over S, taking Y' to Y would give rise to a morphism of $h^i_S(X,Y)(w) \to h^i_S(X',Y')(w)$ compatible with the usual map under R_B.
(3) Whenever $Z \subseteq Y \subseteq X$, there are connecting morphisms $h^i_S(X,Y)(w) \to h^{i+1}_S(Y,Z)(w)$ compatible with the usual maps.
(4) $h^{i+2}_S(X \times \mathbb{P}^1, X \times \{0\} \cup Y \times \mathbb{P}^1)(w) \cong h^i_S(X,Y)(w-1)$.
(5) Objects and morphisms of $\mathcal{M}(S;F)$ can be patched on a Zariski open cover[1].
(6) Objects and morphisms of $\mathcal{M}(S;F)$ can be patched on a Zariski cover.

Many of the properties can be deduced rather formally from the universality. However, the construction of direct images is technically the hardest part and is based on the method of the author given in [A] along with its refinement due to de Cataldo and Migliorini [CM].

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Purity and duality in étale cohomology (after Ofer Gabber)

Joël Riou

This talk was based on two preprints [3] and [4] which are texts written for a *groupe de travail* held at École Polytechnique during Spring 2006–2008. It was devoted to new results by Ofer Gabber, with applications to étale cohomology. We mainly focus on the new proof of Grothendieck's absolute purity conjecture in étale cohomology. The first proof given by Gabber, which used K-theory, was written by Fujiwara [1].

Theorem 1 (Gabber)**.** *Let ℓ be a prime number. We set $\Lambda = \mathbf{Z}/\ell^\nu\mathbf{Z}$ for some $\nu \geq 1$. Let $i\colon Y \to X$ be a closed embedding of codimension c between regular schemes on which ℓ is invertible. There is a canonical isomorphism $\mathrm{Cl}_i\colon \Lambda \xrightarrow{\sim} i^!\Lambda(c)[2c]$ in the derived category $D^+(Y,\Lambda)$ of étale sheaves of Λ-modules over Y.*

Corollary 2 (Gabber)**.** *Using the same notation as in theorem 1, there are canonical isomorphisms $H^{p-2c}_{\acute{e}t}(Y,\Lambda(q-c)) \xrightarrow{\sim} H^p_{\acute{e}t,Y}(X,\Lambda(q))$ for all $(p,q) \in \mathbf{Z}^2$ and long exact sequences for all $q \in \mathbf{Z}$:*

$$\cdots \to H^{p-2c}_{\acute{e}t}(Y,\Lambda(q-c)) \to H^p_{\acute{e}t}(X,\Lambda(q)) \to H^p_{\acute{e}t}(X-Y,\Lambda(q)) \to \ldots$$

We first construct these morphisms Cl_i and study their properties. They can be generalized for locally complete intersection morphisms (l.c.i.):

[1] This will probably be replaced with a stronger descent property in the next version of these notes.

Proposition 3 (Gabber). *We let Sch^{ic} be the category whose objects are $\mathbf{Z}[\frac{1}{\ell}]$-schemes admitting an ample bundle and whose morphisms are l.c.i. morphisms. For any morphism $f\colon Y \to X$ in Sch^{ic}, we let d_f be the virtual relative dimension of f and define a functor $f^? = f^!(-d_f)[-2d_f]\colon D^+(X,\Lambda) \to D^+(Y,\Lambda)$. We can define a morphism $\mathrm{Cl}_f\colon \Lambda \to f^?\Lambda$ in $D^+(Y,\Lambda)$ such that*

(i) *For any tuple of composable arrows $Z \xrightarrow{g} Y \xrightarrow{f} X$ in Sch^{ic}, the morphism $g^?(\mathrm{Cl}_g) \circ \mathrm{Cl}_g$ identifies to $\mathrm{Cl}_{f\circ g}$ in $D^+(Z,\Lambda)$;*
(ii) *If f is smooth, Cl_f is the "Poincaré duality isomorphism";*
(iii) *If $f\colon Y \to X$ is a regular closed embedding, Cl_f corresponds to the class in $H_Y^{-2d_f}(X,\Lambda(-d_f))$ defined in* [1].

The technical part of the proof lies in the verification of the fact that the classes Cl_f attached to regular embeddings are compatible with composition. The proof uses a geometric construction, called "modified blow-up": if $i\colon Y \to X$ is a closed embedding defined by a quasi-coherent sheaf of ideals $\mathcal{I}$ of finite type, for any epimorphism $\mathcal{E} \to \mathcal{I}/\mathcal{I}^2$ where $\mathcal{E}$ is a locally free sheaf on Y, we define a projective morphism $\pi\colon \mathrm{\acute{E}cl}_{Y,\mathcal{E}}(X) \to X$ which is an isomorphism over $X - Y$ and such that $\pi^{-1}(Y)$ identifies to the projective bundle $\mathbf{P}(\mathcal{E})$. To this datum is attached a class $\mathrm{Cl}_{i,\mathcal{E}} \in H_Y^{2r}(X,\Lambda(r))$ where r is the rank of $\mathcal{E}$. When i is a regular embedding and $\mathcal{E} \to \mathcal{I}/\mathcal{I}^2$ is the identity, $\mathrm{\acute{E}cl}_{Y,\mathcal{E}}(X)$ identifies to the blow-up X_Y; then, the class $\mathrm{Cl}_{i,\mathcal{I}/\mathcal{I}^2}$ is denoted Cl_i.

Remark 4. The construction of proposition 3 can be used to define pushforward maps $f\colon H^p(Y,\Lambda(q)) \to H^{p-2d_f}(X,\Lambda(q-d_f))$ in étale cohomology for projective morphisms f in Sch^{ic}.

The new proof of theorem 1 uses the same first reductions as the proof in [1] does: we have to prove theorem 1 for regular schemes of finite type over S where S is the spectrum of a discrete valuation ring. We may assume that S is of unequal characteristic. We use the notion of punctual purity so that it remains to prove that any regular scheme X of finite type over S is punctually pure. The case where X is smooth is easy. Then, we use [2] to get the case of schemes with semi-stable reduction. Then, we deduce from it the case of regular schemes which are the underlying scheme of a log-smooth log-scheme over S (where S is equipped with a chart $S \to \operatorname{Spec}\mathbf{Z}[\mathbf{N}]$ corresponding to a morphism of monoids $\mathbf{N} \to \mathcal{O}_S$ that sends 1 to a uniformizer of S). To obtain the general case, we may try to use an alteration $X' \to X$ where X' has semi-stable reduction (on some extension of S) and a finite group G acts on X' ($X' \to X$ should be generically a Galois covering of group G). If ℓ does not divide the order of G, a transfer argument shows that the punctual purity of X follows from that of X'. Otherwise, we consider an ℓ-Sylow H of G and use a desingularization of the quotient X'/H:

Theorem 5 (Gabber). *Let T be the spectrum of an excellent discrete valuation ring. Let Y be a log-smooth scheme over T equipped with a generically free and* *tame action of a finite group* H [1]. *Then, there exists a projective and birational morphism* $Y' \to Y/H$ *such that* Y' *is log-smooth over* T *and regular.*

Definition 6. Let X be an excellent scheme. A dimension function on X is a function $\delta\colon X \to \mathbf{Z}$ such that for any immediate specialisation of geometric points $\overline{x} \rightsquigarrow \overline{y}$ of X, there is an equality $\delta(y) = \delta(x) - 1$ where x and y are the points of X under $\overline{x}$ and $\overline{y}$ [2].

Dimension functions may or may not exist. Over excellent schemes, they exist at least locally for the Zariski topology. When the scheme is regular, $\delta = -\operatorname{codim}$ is a dimension function. For finite type schemes over a field, the usual notion of dimension gives a dimension function.

Theorem 7 (Gabber)**.** *Let* ℓ *be a prime number. We set* $\Lambda = \mathbf{Z}/\ell^\nu\mathbf{Z}$ *for some* $\nu \geq 1$. *Then, a noetherian excellent* $\mathbf{Z}[\frac{1}{\ell}]$*-scheme* X *has a dualising complex* K_X *(*i.e.*,* K_X *is an object in* $D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda)$ *such that the functor* $\mathbf{R}\operatorname{Hom}(-,K_X)$ *induces an involution on* $D^{\mathrm{b}}_{\mathrm{c}}(X,\Lambda)$*) if and only if* X *has a dimension function. Moreover, dualising complexes over noetherian excellent schemes have the following properties:*

(i) *If* X *is regular, the constant sheaf* Λ *is a dualising complex;*
(ii) *If* $f\colon Y \to X$ *is a separated morphism of finite type and* K_X *is a dualising complex on* X*, then* $f^!K_X$ *is a dualising complex on* Y*;*
(iii) *If* $f\colon Y \to X$ *is a regular morphism (*i.e.*,* f *is flat and has geometrically regular fibres* [3]*) and* K_X *is a dualising complex on* X*, then* $f^\star K_X$ *is a dualising complex on* Y*.*

The widest previous result regarding duality was due to Deligne in SGA $4\frac{1}{2}$ (case of schemes of finite type over regular schemes of dimension ≤ 1). The key-ingredient of the proof of theorem 7 is the notion of a candidate dualising complex associated to a dimension function δ on X, *i.e.*, an object $K \in D^+(X,\Lambda)$ equipped with pinnings at all points (identification of the cohomology of K with support on $\overline{x}$ with $\Lambda(\delta(x))[2\delta(x)]$) which satisfy some compatibilities with respect to Galois actions and immediate specialisations. We prove that these candidate dualising complexes enjoy properties (i)–(iii) (note that property (i) is highly related to cohomological purity), that they exist and are unique up to unique isomorphisms, and finally that they are dualising complexes.

[1]The action is tame if for any geometric point $\overline{x}$ or Y, the order of the stabilizer of $\overline{x}$ is invertible at $\overline{x}$.

[2]A specialisation $\overline{x} \rightsquigarrow \overline{y}$ is an X-morphism $X_{(\overline{x})} \to X_{(\overline{y})}$ between the corresponding strict henselisations of X. It is immediate when the closure of the image of the closed point $\overline{x}$ in $X_{(\overline{y})}$ is 1-dimensional.

[3]Regular morphisms may or may not be of finite type. For instance, if X is a local excellent scheme, the completion $\widehat{X} \to X$ is a regular morphism. However, if we request that f is of finite type, we precisely get the notion of smooth morphisms. A theorem of Popescu shows that regular morphisms (between affine schemes) are obtained as suitable projective limits of smooth morphisms.

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Linear algebraic groups in $\mathbb{A}^1$-homotopy theory

Matthias Wendt

$\mathbb{A}^1$-homotopy theory is one possible way of doing homotopy theory of algebraic varieties, using as basic object a model structure on the category of simplicial (pre)sheaves on the category Sm/S of smooth schemes over a finite type base scheme S. One can define $\mathbb{A}^1$-homotopy group (pre)sheaves $\pi_n^{\mathbb{A}^1}(X)$ for any S-scheme X, and it is an interesting matter to compute these. In the talk, I explained how to compute the $\mathbb{A}^1$-homotopy groups of linear algebraic groups. The main result is the following, cf. [3]:

Theorem 1. *Let Φ be a root system not equal to A_1, let S be the spectrum of an excellent Dedekind ring, and let $G(\Phi)$ be the simply-connected Chevalley S-group scheme associated to Φ. Then there are isomorphisms*

$$\pi_n^{\mathbb{A}^1}(G(\Phi), I)(U) \cong KV_{n+1}(\Phi, U)$$

for any smooth affine S-scheme U. The group schemes $G(\Phi)$ are pointed by the neutral element I, and the groups $KV_{n+1}(\Phi, U)$ are versions of the K-groups defined by Karoubi and Villamayor.

This result has been proven by F. Morel in the case of the root systems A_l, $l \geq 2$, cf. [2].

The main homotopical machinery to prove this type of result has been developed by Morel in [2]. Using this machinery, the result is a consequence of various factorizations in Chevalley groups, which generalize Abe's work [1] on Whitehead groups of Chevalley groups. More precisely, the main intermediate result, generalizing the work of Abe, is the following homotopy invariance of various K_1-functors associated to root systems.

Proposition 2. *Let Φ be a root system not equal to A_1, let R be an excellent Dedekind ring, and let B be a regular R-algebra which is smooth and essentially of finite type. Then there are isomorphisms*

$$K_1(\Phi, B[t_1, \ldots, t_n]) \cong K_1(\Phi, B),$$

where $K_1(\Phi, B) = G(\Phi, B)/E(\Phi, B)$ is the quotient of the B-points of the Chevalley group scheme $G(\Phi)$ modulo its elementary subgroup.

One can use Theorem 1 to provide descriptions of $\mathbb{A}^1$-homotopy groups of linear algebraic groups and rationally trivial homogeneous spaces. A sample application is the following description of $\mathbb{A}^1$-fundamental groups of simply-connected Chevalley groups over infinite fields.

Corollary 3. *Let k be an infinite field, and assume $\operatorname{rk}\Phi \geq 3$. If Φ is non-symplectic, there are isomorphisms*

$$\pi_1^{\mathbb{A}^1}(G(\Phi), I)(\operatorname{Spec} k) \cong K_2^M(k).$$

If Φ is symplectic and $\operatorname{char} k \neq 2$, there are isomorphisms

$$\pi_1^{\mathbb{A}^1}(G(\Phi), I)(\operatorname{Spec} k) \cong K_2^{MW}(k) \cong H_2(\mathrm{Sp}_\infty(k), \mathbb{Z}).$$

The assumptions $\operatorname{rk}\Phi \geq 3$ and $\operatorname{char} k \neq 2$ are probably not essential.

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Algebraic K-theory of toric varieties in characteristic p

Mark E. Walker
(joint work with Guillermo Cortiñas, Christian Haesemeyer, Chuck Weibel)

Let A be a finitely generated, cancellative, torsion-free, normal, abelian monoid, let k be a field, and let $k[A]$ denote the associated monoid-ring. Such an A is the monoid (under vector addition) of the integer lattice points contained in a strongly convex rational polyhedral cone in $\mathbb{R}^n$. The associated affine variety $U = \operatorname{Spec} k[A]$ is an affine toric variety. The variety U is normal, it contains the n-dimensional split torus T as an open subvariety, and the canonical action of T on itself extends to an action of T on all of U. Moreover, any affine variety with these properties is given as $\operatorname{Spec} k[A]$ for some such A. More generally, a toric variety is a normal k-variety X containing T as an open subvariety such that there is an action of T on X extending the canonical one of T on itself. Any toric variety is given locally by affine toric varieties associated to cones, and a general toric variety is built from a fan of cones in Euclidean space.

For any positive integer c, there is a natural endomorphism θ_c on $k[A]$ induced by the monoid endomorphism $a \mapsto a^c$ on A (where we write the product rule for the monoid A multiplicatively). We think of θ_c as a "Frobenius like" endomorphism. The endomorphism θ_c extends to any toric variety X. We write θ_c also for the endomorphism of the K-groups (Hochschild homology groups, etc.) of X induced from θ_c by functorality.

Conjecture 1 (Gubeladze)**.** Let $K_q(k[A])[\theta_c^{-1}]$ denote the result of formally inverting the action of θ_c on $K_q(k[A])$. If A has no non-trivial units, we have

$$K_q(k[A])[\theta_c^{-1}] \cong K_q(k).$$

More generally,

$$K_q(X)[\theta_c^{-1}] \cong KH_q(X)[\theta_c^{-1}]$$

for any toric variety X, where KH refers to Weibel's homotopy K-theory defined in [9].

Gubeladze proved this conjecture in the case char$(k) = 0$ in 2005 [6]. Cortiñas, Haesemeyer, Weibel and I gave a new proof of this case of Gubeladze's conjecture in 2008 [3]. Our new proof used the recent result due to Cortiñas, Haesemeyer, Schlichting, and Weibel [2] which asserts the existence of a homotopy cartesian square of spectra

$$\begin{array}{ccc} K(X) & \longrightarrow & KH(X) \\ \downarrow & & \downarrow \\ HN(X) & \longrightarrow & HN_{cdh}(X) \end{array} \tag{2}$$

for a variety X over a field of characteristic 0. Here, HN refers to negative cyclic homology (interpreted as a functor from varieties to Eilenberg-Mac Lane spectra) and HN_{cdh} its "*cdh*-sheafified" version. (Note that by Haesemeyer's Theorem [7], KH coincides with K_{cdh}, the "*cdh*-sheafified" version of K-theory, for varieties over a field of characteristic zero.) Using this square, Gubeladze's conjecture is seen to be equivalent to its analogue for Hochschild homology. We prove this analogue by relating the result of inverting θ_c on Hochschild homology with the Zariski cohomology of certain sheaves on toric varieties, $\tilde{\Omega}^q$, that were defined by Danilov [4].

In this talk, I describe a proof of Gubeladze's conjecture for any field k of characteristic $p > 0$:

Theorem 3. *Gubeladze's conjecture holds for a toric variety over a field of arbitrary characteristic.*

In characteristic $p > 0$, the analogue of the homotopy cartesian square (2) is given by the following theorem due to Geisser and Hesselholt [5].

Theorem 4 (Geisser-Hesselholt)**.** *For a variety defined over a field k of characteristic p over which "strong" resolution of singularities holds, there is a homotopy cartesian square*

$$\begin{array}{ccc} K(X) & \longrightarrow & KH(X) \\ \downarrow & & \downarrow \\ \{TC^n(X,p)\} & \longrightarrow & \{TC^n_{cdh}(X,p)\} \end{array}$$

of pro-spectra. Here, TC refers to topological cyclic homology (which is a pro-spectra) and TC_{cdh} refers to its "cdh-sheafified" version.

Recall that $TC^n(X,p)$ is the homotopy equalizer of the maps F, R defined on $TR^n(X,p)$. The spectrum $TR^n(X,p)$ is the C_{p^n}-fixed point spectra of the topological Hochschild homology spectrum $THH(X)$.

Our proof also relies on the following theorem of Hesselholt and Madsen [8]:

Theorem 5 (Hesselholt-Madsen)**.** *For a monoid A and field k of characteristic p, the map*

$$THH(k) \wedge N^{cy}(A) \to THH(k[A])$$

of S^1-spectra induces an equivalence on fixed-point spectra for all finite subgroups of S^1. Here, $N^{cy}(A)$ is the cyclic bar construction of the monoid A.

Using these theorems, we are able to reduce Gubeladze's conjecture to an assertion about the effect of inverting θ_c on $N^{cy}(A)$. Roughly, we prove that the functor from monoids to spaces

$$A \mapsto N^{cy}(A)[\theta_c^{-1}]$$

satisfies an appropriate analogue of *cdh*-descent. In more detail, using the above theorems due to Geisser-Hesselholt and Hesselholt-Madsen, Gubeladze's conjecture follows from the following two theorems of ours:

Theorem 6. *For a finitely generated, cancellative, torsion-free, normal monoid A, define the S^1-space*

$$\tilde{\Omega}_A = \coprod_{a \in A} N^{cy}(A[\tfrac{1}{a}])_a$$

where $A[\frac{1}{a}]$ refers to the monoid obtained from A by adjoining an inverse to a and $N^{cy}(A[\frac{1}{a}])_a$ is the connected component indexed by a. Then the map of S^1-spaces

$$N^{cy}(A)[\theta_c^{-1}] \to \tilde{\Omega}_A[\theta_c^{-1}]$$

is a homotopy equivalence on fixed-point subspaces for all finite subgroups of S^1.

Theorem 7. *The functor* $\mathrm{Spec}(k[A]) \mapsto THH(k) \wedge \tilde{\Omega}_A$ *(extended to a functor on toric varieties by imposing Zariski descent) satisfies descent for equivariant blow-ups of toric varieties.*

Finally, I comment on the fact that our proof of Gubeladze's conjecture is *not* dependent on assuming resolutions of singularities in characteristic p. This is because the proof of the theorem of Geisser-Hesselholt requires only that one work with a class of Cohen-Macaulay varieties for which singularities can be resolved by blowing up varieties X along centers C such that X is normally flat along C. It has been proven by Bierstone-Milman [1] that for toric varieties (which are automatically Cohen-Macaulay), singularities may be resolved in this manner.

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Algebraic K-theory of rings of continuous functions

Guillermo Cortiñas
(joint work with Andreas Thom)

Write Comp for the category of compact Hausdorff spaces and continuous maps. For $X \in$ Comp, consider the (C^*-) algebra $C(X)$ of continous functions $X \to \mathbb{C}$. The talk was about the algebraic K-theory of $C(X)$. The following results were presented.

Theorem 1. *For $n < 0$, the functor* Comp $\to \mathfrak{Ab}$*, $X \mapsto K_n(C(X))$ is homotopy invariant.*

Theorem 2. *$C(X)$ is K-regular.*

Theorem 3. *For $n \geq 1$, the group $K_n\mathbb{C}$ is equipped with a topology (which for $n = 1$ is the usual, euclidean one of $\mathbb{C}^*$). This topology is natural in the sense that the natural map*

$$K_n(C(X)) \mapsto \operatorname{map}(X, K_n\mathbb{C}), \qquad \xi \mapsto (x \mapsto ev_x(\xi))$$

factors through the set $C(X, K_n\mathbb{C})$ of continuous functions. Morevoer if X is contractible, then this map is an isomorphism

$$K_n(C(X)) \xrightarrow{\cong} C(X, K_n\mathbb{C}) \qquad (X \textit{ contractible.})$$

Remark 4. The first theorem was conjectured by Rosenberg [1, 3.7]. He showed (see [1, 3.8]) that his conjecture implies that $K_n(C(D)) = ku^n(D)$ for $n < 0$ and D a finite simplicial complex. The second theorem was first stated by Rosenberg; see [1, 3.5]. Unfortunately, the proof given in *loc. cit.* is incorrect.

References

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The motivic Conner-Floyd theorem

Ivan Panin
(joint work with Oliver Röndigs, Konstantin Pimenov)

Quillen's algebraic K-theory is reconstructed via Voevodsky's algebraic cobordism. More precisely, for a ground field k the algebraic cobordism $\mathbb{P}^1$-spectrum MGL of Voevodsky is considered as a commutative $\mathbb{P}^1$-ring spectrum. Setting

$$MGL^i = \bigoplus_{p-2q=i} MGL^{p,q},$$

we regard the bigraded theory $MGL^{p,q}$ as just a graded theory. There is a unique ring morphism $\phi : MGL^0(k) \to \mathbb{Z}$ which sends the class $[X]_{MGL}$ of a smooth projective variety X to the Euler characteristic $\chi(X, \mathcal{O}_X)$ of the structure sheaf $\mathcal{O}_X$. Our main result states that there is a canonical grade preserving isomorphism of ring cohomology theories

$$\phi : MGL^*(X, X - Z) \otimes_{MGL^0(k)} \mathbb{Z} \xrightarrow{\cong} K_{-*}(X \text{ on } Z) \cong K'_{-*}(Z),$$

on the category SmOp/k in the sense of [2], where $K_*(X \text{ on } Z)$ is Thomason-Trobaugh K-theory and K'_* is Quillen's K'-theory. In particular, the left-hand side is a ring cohomology theory. Moreover, both theories are oriented in the sense of [2] and ϕ respects orientations. The result is an algebraic version of a theorem due to Conner and Floyd. That theorem reconstructs complex K-theory via complex cobordism [1].

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On the surjectivity of the cycle map for motives

Shun-ichi Kimura

In [2, Theorem 3.6], U. Jannsen proved that when X is a smooth projective variety over a universal domain Ω, if the cycle map $cl : \mathrm{CH}^*(X)_\mathbb{Q} \to H^*(X, \mathbb{Q})$ is injective, then it is also surjective. This theorem was generalized to Deligne cohomology in [1] by H. Esnault and M. Levine. Also Murre and Srinivas proved that if $\mathrm{CH}^*(X_\Omega)_\mathbb{Q}$ is countable, then the motive of X is isomorphic to a direct sum of the twists of the Lefschetz motives (unpublished).

In this talk, we generalize Jannsen's theorem to Chow motives. The details are found in [3].

Theorem 1. *Let $M = (X, p, n)$ be a Chow motive over a universal domain Ω. If $\mathrm{CH}_*(X)_\mathbb{Q}$ is a finite dimensional vector space over $\mathbb{Q}$, then M is isomorphic to a direct sum of the twists of Lefschetz motives, $M \simeq \oplus \mathbb{L}^{d_i}$.*

Remark 2. As pointed out by J. Riou during the talk, as for the definition of the universal domain Ω, it is enough to assume that Ω is an algebraically closed field with infinite transcendental degree over the prime field.

The main tool is the supported correspondence.

Definition 3. A *correspondence* from X to Y is an element $\alpha \in \mathrm{CH}_*(X \times Y)$, and we denote it as $\alpha : X \vdash Y$.

When $S \subset X \times Y$ is a closed subscheme with the inclusion $i : S \to X \times Y$, we say that $\alpha : X \to Y$ is *supported* on S if $\alpha = i_*(\tilde{\alpha})$ for some $\tilde{\alpha} \in \mathrm{CH}_* S$. In this case, we say that α is *represented by* $\tilde{\alpha}$.

Lemma 4. *Assume that $\alpha : X \vdash Y$ and $\beta : Y \vdash Z$ are correspondences, supported on $S \times T$ and $U \times V$ respectively.*

(1) *$\beta \circ \alpha$ is canonically supported on $S \times V$.*
(2) *If $\dim S + \dim V < \dim(\beta \circ \alpha)$, then $\beta \circ \alpha = 0$.*
(3) *If $\dim T + \dim U < \dim Y$, then $\beta \circ \alpha = 0$.*

Definition 5. Let $\alpha : X \vdash Y$ be a correspondence, supported on $S \times T$, represented by $\tilde{\alpha}$. We call α a *supported idempotent* when $\alpha \circ \alpha$ is canonically represented by $\tilde{\alpha}$.

The main theorem easily follows from the Key Lemma.

Lemma 6 (Key Lemma)**.** *Let X/Ω be a d-dimensional smooth complete variety, and $0 \le m < d$ be an integer, $S_m \subset X$ a purely m-codimensional closed subscheme, and $\alpha_m : X \vdash X$ a supported idempotent correspondence, supported on $S_m \times X$. Assume that the image of $\alpha_{m*} : \mathrm{CH}_*(X)_{\mathbb{Q}} \to \mathrm{CH}_*(X)_{\mathbb{Q}}$ is a finitely generated vector space over $\mathbb{Q}$. Then there exists a purely 1-codimensional closed subscheme $S_{m+1} \subset S_m$, a purely m-dimensional closed subscheme $T_m \subset X$, a correspondence $\alpha_{m+1} : X \vdash X$ supported on $S_{m+1} \times X$, and a correspondence $\beta_m : X \vdash X$ supported on $S_m \times T_m$ such that*

(1) *$\alpha_{m+1} + \beta_m = \alpha_m$ as correspondences $X \vdash X$, supported on $S_m \times X$,*
(2) *α_{m+1} and β_m are supported idempotent, and*
(3) *α_{m+1} and β_m are supported orthogonal, namely $\alpha_{m+1} \circ \beta_m = 0$ and $\beta_m \circ \alpha_{m+1} = 0$ as supported correspondences.*

Bruno Kahn already generalized this result in the line of Murre-Srinivas.

Jannsen's theorem implies the Hodge conjecture for varieties for which his theorem applies. We hope that, by generalizing his theorem to motives (hence possibly applicable to all varieties), we can eventually prove the Hodge conjecture by this technique.

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The slice filtration for modules over ring spectra

Pablo Pelaez

Let X be a Noetherian separated scheme of finite Krull dimension, and $\mathcal{M}_X$ be the category of pointed simplicial presheaves on the smooth Nisnevich site Sm_X over X equipped with the Morel-Voevodsky motivic model structure [4]. We will denote by T the pointed simplicial presheaf represented by $S^1 \wedge \mathbb{G}_m$, where $\mathbb{G}_m$ is the multiplicative group over X pointed by 1; and by $Spt(\mathcal{M}_X)$ the category of symmetric T-spectra on $\mathcal{M}_X$ equipped with Jardine's motivic model structure [2]. The homotopy category of $Spt(\mathcal{M}_X)$ is a triangulated category which will be denoted by $\mathcal{SH}$.

Given an integer $q \in \mathbb{Z}$, we consider the following family of symmetric T-spectra

$$C^q_{eff} = \{F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \mid n, r, s \geq 0; s - n \geq q; U \in Sm_X\}$$

where F_n is the left adjoint to the n-evaluation functor

$$ev_n : Spt(\mathcal{M}_X) \to \mathcal{M}_X$$

Voevodsky [13] defines the slice filtration as the following family of triangulated subcategories of $\mathcal{SH}$

$$\cdots \subseteq \Sigma_T^{q+1}\mathcal{SH}^{eff} \subseteq \Sigma_T^{q}\mathcal{SH}^{eff} \subseteq \Sigma_T^{q-1}\mathcal{SH}^{eff} \subseteq \cdots$$

where $\Sigma_T^q\mathcal{SH}^{eff}$ is the smallest full triangulated subcategory of $\mathcal{SH}$ which contains C^q_{eff} and is closed under arbitrary coproducts.

It follows from the work of Neeman [5], [6] that the inclusion

$$i_q : \Sigma_T^q\mathcal{SH}^{eff} \to \mathcal{SH}$$

has a right adjoint $r_q : \mathcal{SH} \to \Sigma_T^q\mathcal{SH}^{eff}$, and that the following functors

$$f_q : \mathcal{SH} \to \mathcal{SH}$$

$$s_q : \mathcal{SH} \to \mathcal{SH}$$

are exact, where f_q is defined as the composition $i_q \circ r_q$, and s_q is characterized by the fact that for every $E \in Spt(\mathcal{M}_X)$, we have the following distinguished triangle in $\mathcal{SH}$

$$f_{q+1}E \xrightarrow{\rho_q^E} f_qE \xrightarrow{\pi_q^E} s_qE \longrightarrow \Sigma_T^{1,0} f_{q+1}E$$

We will refer to f_qE as the $(q-1)$-connective cover of E, and to s_qE as the q-slice of E.

Let A be a cofibrant ring spectrum with unit in $Spt(\mathcal{M}_X)$, and A-mod be the category of left A-modules in $Spt(\mathcal{M}_X)$. The work of Jardine [2, Proposition 4.19] and Hovey [1, Corollary 2.2] implies that the adjunction

$$(A \wedge -, U, \varphi) : Spt(\mathcal{M}_X) \to A\text{-mod}$$

induces a Quillen model structure $Spt^A(\mathcal{M}_X)$ in A-mod, this means that a map $f : M \to N$ in $Spt^A(\mathcal{M}_X)$ is a weak equivalence or a fibration if and only if Uf is a weak equivalence or a fibration in $Spt(\mathcal{M}_X)$.

It is easy to see that the homotopy category $\mathcal{SH}^A$ of $Spt^A(\mathcal{M}_X)$ is a triangulated category [8, Proposition 3.5.3].

Theorem 1. *Let A be an effective cofibrant ring spectrum with unit $u : \mathbf{1} \to A$ in $Spt(\mathcal{M}_X)$, i.e. A belongs to the triangulated subcategory $\Sigma_T^0 \mathcal{SH}^{eff}$ defined above. If $s_0(u)$ is an isomorphism in $\mathcal{SH}$, then for every $q \in \mathbb{Z}$ the functor*

$$s_q : \mathcal{SH} \to \mathcal{SH}$$

factors (up to a canonical isomorphism) through $\mathcal{SH}^A$

$$\begin{tikzcd} \mathcal{SH} \arrow[r, "s_q"] \arrow[dr, dashed, "\tilde{s}_q"'] & \mathcal{SH} \\ & \mathcal{SH}^A \arrow[u, "UR^A"'] \end{tikzcd}$$

where R^A denotes a fibrant replacement functor in $Spt^A(\mathcal{M}_X)$.

Proof. We refer the reader to [8, Theorem 3.6.20 and Lemma 3.6.21] or [7, Theorem 2.1(vi)]. □

This theorem has several interesting consequences.

Corollary 2. *Let $q \in \mathbb{Z}$ denote an arbitary integer and E denote an arbitrary symmetric T-spectrum in $Spt(\mathcal{M}_X)$.*

(1) *If the base scheme X is a perfect field k, then we have that the q-slice s_qE of E is equipped with a natural structure of $H\mathbb{Z}$-module in $Spt(\mathcal{M}_X)$, where $H\mathbb{Z}$ denotes Voevodsky's motivic Eilenberg-MacLane spectrum* [11]*. This proves a conjecture of Voevodsky* [13]*.*
(2) *If we restrict the base scheme X further, and assume that it is a field of characteristic zero; then we have that the q-slice s_qE of E is a big motive in the sense of Voevodsky, i.e. s_qE has transfers.*
(3) *For any Noetherian separated base scheme X of finite Krull dimension, we have that the q-slice s_qE of E is equipped with a natural structure of MGL-module in $Spt(\mathcal{M}_X)$, where MGL denotes Voevodsky's algebraic cobordism spectrum* [11]*. This implies that over any base scheme, the slices are always oriented cohomology theories.*

Proof. (1): This follows from the work of M. Levine [3] (over a perfect field) and Voevodsky [12] (over a field of characteristic zero), together with theorem 1. For the details we refer the reader to [8] or [7].

(2): This follows from the work of Röndigs and Østvær [9], together with what we have already proved in (1) above. For the details we refer the reader to [8] or [7].

(3): The work of Spitzweck [10, Corollaries 3.2 and 3.3] shows that MGL is effective and that the unit map $u : \mathbf{1} \to MGL$ induces an isomorphism $s_0(u)$ in $\mathcal{SH}$. Therefore the result is a direct consequence of theorem 1. □

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SK_1 and SK_2 of division algebras

Bruno Kahn

I presented 3 methods to produce Galois cohomology invariants of the form

$$SK_1(A) \to H^5_{\text{ét}}(F, \mathbb{Z}(3))/r[A] \cdot H^2_{\text{ét}}(F, \mathbb{Z}(2))$$

according to a conjecture of Suslin, and

$$SK_2(A) \to H^6_{\text{ét}}(F, \mathbb{Z}(4))/r[A] \cdot H^3_{\text{ét}}(F, \mathbb{Z}(3))$$

in case F contains an algebraically closed subfield.

Here F is a field, A is a division algebra of degree d prime to the characteristic of F, r is a divisor of d and $[A]$ is the class of A in the Brauer group $Br(F) \simeq H^3_{\text{ét}}(F, \mathbb{Z}(1))$.

The first method uses an étale Bloch-Lichtenbaum spectral sequence converging to a version of the étale K-theory of A (joint work with Marc Levine, [1, 6.9]). Here, $r = 1$. The second one uses the K-theory of the generalised Severi-Brauer variety $SB(r, A)$: here, $r = r$. The last one uses the formula

$$H^5_{\text{ét}}(\mathbf{SL}_1(A), \mathbb{Z}(3))/H^5_{\text{ét}}(F, \mathbb{Z}(3)) \simeq \mathbb{Z}$$

where $\mathbf{SL}_1(A)$ is the twisted form of $\mathbf{SL}_d$ associated to A. Here, $r = d$, that is, $r[A] = 0$.

An open problem is to compare these maps.

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The Mayer-Vietoris principle for Grothendieck-Witt groups

Marco Schlichting

This is a report on the author's preprint [5].

Definition 1 ([5, Definition 2.11])**.** Let $(\mathcal{E}, w, *, \eta)$ be an exact category with weak equivalences and duality [5]. Its *Grothendieck-Witt space*

$$GW(\mathcal{E}, w, *, \eta)$$

is the homotopy fibre of a map of topological spaces

$$|(wS^e_\bullet\mathcal{E})_h| \to |wS_\bullet\mathcal{E}|$$

where $S_\bullet\mathcal{E}$ is Waldhausen's $S_\bullet$-construction [9], $S^e_\bullet$ denotes its "edge-wise subdivision" [9, 1.9 Appendix] and, for a category with duality $(\mathbb{C}, *, \eta)$, the category of symmetric forms $\mathbb{C}_h$ has objects pairs (X, φ) where $\varphi : X \to X^*$ is a map in $\mathbb{C}$ satisfying $\varphi^*\eta_X = \varphi$ and maps $(X, \varphi) \to (Y, \psi)$ in $\mathbb{C}_h$ are maps $f : X \to Y$ in $\mathbb{C}$ such that $\varphi = f^*\psi f$.

Let X be a scheme, $Z \subset X$ be a closed subscheme, and L be a line-bundle on X. Set

$$\begin{aligned}
GW(X) &= GW(\mathrm{Vect}(X), iso, Hom(\ \ , O_X), \mathrm{can}) \\
GW^n(X, L) &= GW(\mathrm{Ch}^b\,\mathrm{Vect}(X), \mathrm{quis}, Hom(\ \ , O_X), \mathrm{can}) \\
GW^n(X \text{ on } Z, L) &= GW(\mathrm{Ch}^b_Z\,\mathrm{Vect}(X), \mathrm{quis}, Hom(\ \ , O_X), \mathrm{can})
\end{aligned}$$

where $\mathrm{Vect}(X)$ denotes the category of vector bundles on X, $\mathrm{Ch}^b\,\mathrm{Vect}(X)$ is the category of bounded chain complexes of vector bundles, $\mathrm{Ch}^b_Z\,\mathrm{Vect}(X)$ is the full subcategory of chain complexes which are acyclic outside Z, *iso* denotes the set of isomorphisms, quis denotes the set of quasi-isomorphisms, and can denotes the canonical isomorphism $E \to Hom(Hom(E, L), L)$ identifying a (complex of) vector bundle(s) with its double dual.

Proposition 2 ([5, Proposition 3.8])**.**

(1) $\pi_0 GW(X)$ *is Knebusch's Grothendieck-Witt group of* X *as defined in* [4].

(2) *For any exact category with weak equivalences and duality* $(\mathcal{E}, w, *, \eta)$*, the abelian group* $\pi_0 GW(\mathcal{E}, w, *, \eta)$ *is generated by isomorphism classes* $[X, \varphi]$ *of symmetric spaces* (X, φ) *in* $(\mathcal{E}, w, *, \eta)$ *subject to the following relations*

(a) $[X, \varphi] + [Y, \psi] = [X \oplus Y, \varphi \oplus \psi]$

(b) *if* $g : X \to Y$ *is a weak equivalence, then* $[Y, \psi] = [X, g^*\psi g]$*, and*
(c) *if* $(E_\bullet, \varphi_\bullet)$ *is a symmetric space in the category of exact sequences in* $\mathcal{E}$*, then*

$$[E_0, \varphi_0] = \left[E_{-1} \oplus E_1, \begin{pmatrix} 0 & \varphi_1 \\ \varphi_{-1} & 0 \end{pmatrix} \right].$$

Here "symmetric space" means a symmetric form (E, φ) *for which the map* $\varphi : E \to E^*$ *is a weak equivalence.*

Proposition 3 ([5, Proposition 6.5])**.** *Considering a vector bundle as a complex concentrated in degree* 0 *yields a homotopy equivalence*

$$GW(X) \xrightarrow{\simeq} GW^0(X, O_X).$$

Theorem 4 ([5, Theorem 9.2])**.** *Let* X *be a scheme with an ample family of line-bundles, let* $Z \subset X$ *be a closed subscheme with quasi-compact open complement* $j : U \subset X$*, and let* L *be a line bundle on* X*. Then for every* $n \in \mathbb{Z}$ *there is a homotopy fibration of Grothendieck-Witt spaces*

$$GW^n(X \text{ on } Z,\ L) \longrightarrow GW^n(X,\ L) \longrightarrow GW^n(U,\ j^*L).$$

Theorem 5 ([5, Theorem 9.3])**.** *Let* X *be a scheme with an ample family of line-bundles, let* $Z \subset X$ *be a closed subscheme with quasi-compact open complement, let* L *be a line bundle on* X*. Then for every* $n \in \mathbb{Z}$ *and every quasi-compact open subscheme* $j : V \subset X$ *containing* Z*, restriction of vector-bundles induces a homotopy equivalence*

$$GW^n(X \text{ on } Z,\ L) \xrightarrow{\sim} GW^n(V \text{ on } Z,\ j^*L).$$

Theorem 6 ([5, Corollary 10.13])**.** *Let* $X = U \cup V$ *be a scheme with an ample family of line-bundles which is covered by two open quasi-compact subschemes* $U, V \subset X$*. Then restriction of vector bundles induces a homotopy cartesian square*

$$\begin{tikzcd} GW(X) \arrow[r] \arrow[d] & GW(U) \arrow[d] \\ GW(V) \arrow[r] & GW(U \cap V). \end{tikzcd}$$

Remark 7. The theorems extend to negative degrees by introducing appropriate non-connective spectra; see [5, §10].

Remark 8.

(1) The K-theory analog of the theorems are due to Thomason [8].
(2) The Witt-theory analog of the theorems were proved by Balmer in [1] for regular noetherian separated schemes X with $\frac{1}{2} \in \Gamma(X, O_X)$. More generally, the following holds [7]. Let X be a scheme with an ample family of line-bundles, which is covered by two open quasi-compact subschemes $U, V \subset X$. Assume that $\frac{1}{2} \in \Gamma(X, O_X)$. Then the total homotopy cofibre

of the diagram of spectra representing Balmer's Witt-groups

$$\begin{tikzcd} W^*(X) \arrow[r] \arrow[d] & W^*(U) \arrow[d] \\ W^*(V) \arrow[r] & W^*(U\cap V) \end{tikzcd}$$

is the spectrum representing Tate cohomology

$$\hat{H}^*(\mathbb{Z}/2, \frac{K_0(U\cap V)}{Im(K_0(U)+K_0(V))}).$$

These cohomology groups are non-trivial, in general. This is the case, for instance, for the standard covering $\mathbb{P}^1_C = \mathbb{A}^1_C \cup \mathbb{A}^1_C$ of the projective line over a curve C with $K_{-1}(C) = \mathbb{Z}$ (e.g., the nodal curve).

(3) Theorem 6 was also proved by Hornbostel in [2] for regular noetherian separated schemes X with $\frac{1}{2} \in \Gamma(X, O_X)$ based on Balmer's results above and Karoubi's fundamental theorem [3].

Remark on the proof: Our proof follows [8]. For that we show Grothendieck-Witt theory analogs of Waldhausen's fibration and approximation theorems [9] and of Thomason's cofinality theorem [8]. The main difference to [8] is that contrary to K-theory the higher Grothendieck-Witt groups $\pi_i GW(\mathcal{E}, w, *, \eta)$ are not invariant under derived equivalences when $\frac{1}{2} \notin \mathcal{E}$, in general. In the talk we explained a counter example which will appear in [6]. The lack of "invariance under derived equivalences" prevents us from using perfect complexes and from extending our results to general quasi-compact and quasi-separated schemes as was done for K-theory in [8].

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Algebraic Cycles on Generic Abelian 3-Folds

Vasudevan Srinivas
(joint work with Andreas Rosenschon)

This is a report on joint work with Andreas Rosenschon. The details will appear in the forthcoming Proceedings of the Tata Institute Colloquium on *Cycles, Motives and Shimura Varieties*.

Let X be a smooth projective variety over an algebraically closed field k. Recall that $CH^i(X)$ denotes the Chow group of codimension i cycles modulo rational equivalence. If ℓ is a prime number, let

$$_{\ell}CH^i(X) = \ell\text{-torsion subgroup of } CH^i(X),$$

$$CH^i(X)/\ell = CH^i(X) \otimes_{\mathbb{Z}} \mathbb{Z}/\ell\mathbb{Z}.$$

We discuss finiteness statements for the groups $_{\ell}CH^i(X)$, $CH^i(X)/\ell$, focussing mainly on the case $k = \mathbb{C}$.

Some known results are as follows: It is "classical" that $_{\ell}CH^i(X)$, $CH^i(X)/\ell$ are both finite, for all ℓ, provided $i \leq 1$. Next, it is known that $_{\ell}CH^i(X)$, $CH^i(X)/\ell$ are finite, for all ℓ, for $i = \dim X$; the main content here is Roitman's theorem on torsion 0-cycles. It is known that $_{\ell}CH^2(X)$ is finite, for all ℓ; this is basically a consequence of the results of Merkurjev-Suslin relating K_2 of fields and division algebras, combined with the results of Bloch-Ogus.

In a related vein, the groups $_{\ell}CH^i(X)$, $CH^i(X)/\ell$ are "rigid": if $k \hookrightarrow K$ is an extension of algebraically closed fields, and ℓ is invertible in k, then Lecomte [1] showed that the natural base-change maps

$$_{\ell}CH^i(X) \to_{\ell} CH^i(X \times_k K), CH^i(X)/\ell \to CH^i(X \times_k K)/\ell$$

are *isomorphisms*. Though it is plausible they hold, the corresponding statements for ℓ equal to the characteristic of k do not seem to be present in the literature, as far as we know.

We show that these finiteness statements cannot be improved, in general, by showing that

- there exists a smooth projective variety X over $\mathbb{C}$ such that, for all but finitely many primes ℓ, the group $CH^i(X)/\ell$ is infinite for all $2 \leq i < \dim X$,
- there exists a smooth, projective variety X over $\mathbb{C}$ such that, for all but finitely many primes ℓ, the group $_{\ell}CH^i(X)$ is infinite, for all $3 \leq i < \dim X$.

In fact, by considering varieties of the form $Y \times \mathbb{P}^m$, we see that it suffices to construct (a) a smooth projective complex 3-fold X such that $CH^2(X)/\ell$ is infinite for all but finitely many ℓ, and (b) a smooth projective 4-fold X such that $_{\ell}CH^3(X)$ is infinite for all but finitely many ℓ. Now we may apply the following result of Schoen [3], together with Lecomte's rigidity result, to show that an example as in (b) can be constructed using an example as in (a):

Theorem 1. *Let $k \subsetneq K$ be an extension of algebraically closed fields. Let W be a smooth projective k-variety, and E be an elliptic curve over K with $j(E) \in K \setminus k$. Then the external product map*

$$CH^i(W) \otimes_{\mathbb{Z}} CH^1(E)_{\text{torsion}} \to CH^{i+1}(W \times_k E)_{\text{torsion}}$$

is injective, for all $i \geq 0$.

Thus, we reduce to proving our main result:

Theorem 2. *Let X be a generic abelian 3-fold over $\mathbb{C}$. Then $CH^2(X)/\ell$ is infinite for all but finitely many ℓ.*

We remark that, using an isogeny, it suffices to consider the case when X is principally polarized. Now, "generic" means the following: X determines a $\mathbb{C}$-valued point on the (coarse) moduli space of principally polarized abelian 3-folds, which is a geometrically integral $\mathbb{Q}$-scheme in a natural way; this $\mathbb{C}$-point should be a $\mathbb{Q}$-generic point of the moduli space.

We also note that there are earlier results of a similar kind due to Schoen (see [2], [4]):

Theorem 3. (1) *Let $E = \{x^3 + y^3 + z^3 = 0\} \subset \mathbb{P}^2_{\overline{\mathbb{Q}}}$ be the Fermat cubic, and let $X = E \times E \times E$. Then $CH^2(X)/\ell$ is infinite for all $\ell \equiv 1 (\mathrm{mod}\, 3)$.*

(2) *Let E be a generic elliptic curve over $\mathbb{C}$, and let $X = E \times E \times E$. Then $CH^2(X)/\ell$ is infinite for any $\ell \in \{5, 7, 11, 13, 17\}$.*

These results certainly suggest that our main result should hold!

Our technique of proof is by adapting an argument of Nori [6], as in the proof of Schoen [4]. This has two steps: first, one produces one example of a cycle which is homologically trivial in $CH^2(X)$, but whose class in $CH^2(X)/\ell$ is non-zero for all but finitely many ℓ. Next, one views the generic abelian 3-fold as the geometric generic member of a universal family over the moduli space, and uses the action of suitable modular correspondences to create infinitely many distinct cycles, by considering images of the non-trivial one.

To carry out the first step, one notes that the generic principally polarized abelian 3-fold is the Jacobian of the generic curve of genus 3, which is the generic fiber of a suitable universal family of curves over the moduli space of curves. The well-known Ceresa construction for cycles on a Jacobian produces a "universal Ceresa cycle", shown to be non-trivial by Hain in [7]; he shows that the union of the Ceresa cycles, in the total space of the universal family of Jacobians, has a non-torsion topolocigal cycle class in singular cohomology. By making a comparison with étale cohomology, and then applying a variant of the Bloch-Esnault method (see [5], [2], [4]), one deduces that the generic Ceresa cycle has non-zero image in $CH^2(X)/\ell$ for all but finitely many ℓ, where X is the geometric generic fiber of the universal family of Jacobians (or equivalently, of the universal family of principally polarized abelian 3-folds, say with level 3 structure).

Now, as in Nori's proof, we use that the Torelli map from the moduli of curves of genus 3 to the moduli of principally polarized abelian 3-folds (say, both with level 3 structure) is of degree 2, and is ramified along the hyperelliptic locus. Just as in Nori's argument, this allows one to use modular correspondences to generate infinitely many cycles mod ℓ, starting from the universal Ceresa cycle.

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n-Motivic Sheaves

Joseph Ayoub

This talk is based on our joint paper [1] with L. Barbieri-Viale. We fix a ground field k which we assume, for simplicity, to be of characteristic zero. Also for simplicity, we will work with rational coefficients. In the sequel, *motivic sheaf* is a shorthand for homotopy invariant sheaf with transfers [3], i.e., a motivic sheaf $\mathcal{F}$ is an additive contravariant functor from the category of smooth correspondences $\mathbf{Cor}(k)$ (see [3, Def. 1.5]) to the category of $\mathbb{Q}$-vector spaces such that:

(a) for every smooth k-scheme X, $\mathcal{F}(X) \to \mathcal{F}(\mathbb{A}^1_X)$ is invertible.
(b) the restriction of $\mathcal{F}$ to the category Sm/k of smooth k-schemes is a Nisnevich (or equivalently, an étale) sheaf with transfers.

If $\mathcal{F}$ satisfies (b) but not necessarily (a), we call it a sheaf with transfers. The category of sheaves with transfers will be denoted by $Str(k)$. We denote $\mathbf{HI}(k)$ its full subcategory of motivic sheaves. The obvious inclusion admits a left adjoint $h_0 : Str(k) \to \mathbf{HI}(k)$. It follows from [3, Th. 22.3] that h_0 is the given by the Nisnevich sheaf of the associated homotopy invariant presheaf with transfers. In particular, $\mathbf{HI}(k)$ is an abelian category and the inclusion $\mathbf{HI}(k) \hookrightarrow Str(k)$ is exact. In fact, there is a natural t-structure on Voevodsky's category $\mathbf{DM}_{\mathrm{eff}}(k)$ whose heart is canonically equivalent to $\mathbf{HI}(k)$. This gives a hint why motivic sheaves are important objects to study. Important examples include the following.

Example 1. Let X be a smooth k-scheme. We denote by $\widetilde{\mathrm{CH}}^p(X)$ the sheaf associated to the presheaf $U \rightsquigarrow \mathrm{CH}^p(U \times_k X)$. This is a motivic sheaf.

We recall the notion of an n-motivic sheaf from [1]. Fix an integer $n \in \mathbb{N}$ and let $\mathbf{Cor}(k_{\leq n}) \subset \mathbf{Cor}(k)$ be the full subcategory whose objects are the smooth k-schemes of dimension less than n. Let $Str(k_{\leq n})$ be the category of contravariant functors from $\mathbf{Cor}(k_{\leq n})$ to the category of $\mathbb{Q}$-vector spaces. There is an obvious restriction functor $\sigma_{n*} : Str(k) \to Str(k_{\leq n})$ which has a left adjoint σ_n^*.

Definition 2. An object $\mathcal{F} \in \mathbf{HI}(k)$ is an *n-motivic* sheaf if the obvious morphism

$$h_0\sigma_n^*\sigma_{n*}\mathcal{F} \to \mathcal{F}$$

is invertible. We denote by $\mathbf{HI}_{\leq n}(k) \subset \mathbf{HI}(k)$ the full subcategory of n-motivic sheaves.

It is formal to prove that $\mathbf{HI}_{\leq n}(k)$ is an abelian category. Given a morphism of n-motivic sheaves $a : \mathcal{F} \to \mathcal{G}$, $coker(a)$ is again an n-motivic sheaf and gives the cokernel of a in $\mathbf{HI}_{\leq n}(k)$. In other words, the inclusion $\mathbf{HI}_{\leq n}(k) \hookrightarrow \mathbf{HI}(k)$ is right exact. Unfortunately, it is an open problem whether or not this inclusion is left exact. In other words, we don't know that $ker(a)$ is n-motivic, and the kernel of a in $\mathbf{HI}(k)$ is a priori given by $h_0\sigma_n^*\sigma_{n*}ker(a)$. In fact, we conjecture much more than the left exactness of the inclusion $\mathbf{HI}_{\leq n}(k) \hookrightarrow \mathbf{HI}(k)$, namely:

Conjecture 3. There is a functor $(-)^{\leq n} : \mathbf{HI}(k) \to \mathbf{HI}_{\leq n}(k)$ which is a left adjoint to the obvious inclusion.

Unfortunately, the previous conjecture seems out of reach for $n \geq 2$. When $n = 0$ or $n = 1$, the situation is much easier and the functors $(-)^{\leq n}$ exist and are denoted respectively by π_0 and Alb. One can even write formulas:

$$\pi_0(\mathcal{F}) = \operatorname*{colim}_{X \to \mathcal{F}} \mathbb{Q}_{tr}(\pi_0(X)) \quad \text{and} \quad \mathrm{Alb}(\mathcal{F}) = \operatorname*{colim}_{X \to \mathcal{F}} \mathrm{Alb}(X)$$

where $\pi_0(X)$ is the étale k-scheme of connected components of X and $\mathrm{Alb}(X)$ is the Albanese scheme of X considered as a sheaf with transfers.

Example 4. Assume that k is algebraically closed and let X be a smooth k-scheme. Then one can prove that $\pi_0(\widetilde{\mathrm{CH}}^p(X))$ is the constant sheaf with value $\mathrm{NS}^p(X)$, the Neron-Severi group of codimension p-cycles up to algebraic equivalence.

We also address a (hopefully easy) conjecture.

Conjecture 5. Let X be a complex algebraic variety. Then $\mathrm{Alb}(\widetilde{\mathrm{CH}}^p(X))(\mathbb{C})$ is canonically isomorphic to the target of Walker's morphic Abel-Jacobi map (see [2]).

In fact, π_0 and Alb are defined on the whole category $Str(k)$ by the same formulas. An important issue is that these functors can be left derived, yielding two functors

$$\mathrm{L}\pi_0 : \mathbf{D}(Str(k)) \to \mathbf{D}(\mathbf{HI}_{\leq 0}(k)) \quad \text{and} \quad \mathrm{LAlb} : \mathbf{D}(Str(k)) \to \mathbf{D}(\mathbf{HI}_{\leq 1}(k)).$$

Moreover, these two functors pass to the $\mathbb{A}^1$-localization yielding two functors

$$\mathrm{L}\pi_0 : \mathbf{DM}_{\mathrm{eff}}(k) \to \mathbf{D}(\mathbf{HI}_{\leq 0}(k)) \quad \text{and} \quad \mathrm{LAlb} : \mathbf{DM}_{\mathrm{eff}}(k) \to \mathbf{D}(\mathbf{HI}_{\leq 1}(k))$$

which are left adjoint to the obvious inclusions.

We now give two applications. The first one gives an extension of the classical Neron-Severi groups to a bigraded cohomology theory.

Definition 6. Let X be a smooth k-scheme. We set

$$\mathrm{NS}^p(X, q) = \mathrm{L}_q\pi_0(\underline{\mathrm{Hom}}(X, \mathbb{Q}(p)[2p]))(k).$$

Then, $\mathrm{NS}^p(X, 0)$ is the classical Neron-Severi group $\mathrm{NS}^p(X)$ and we have a canonical morphism from Bloch's higher Chow groups:

$$\mathrm{CH}^p(X, q) \to \mathrm{NS}^p(X, q).$$

Except for $q = 0$, we do not expect this map to be surjective in general.

As a second application, we propose a definition of 2-motives.

Definition 7. A 2-*motive* is an object $M \in \mathbf{DM}_{\mathrm{eff}}(k)$ satisfying the following properties.

(a) $h_i(M) = 0$ *for* $i \notin \{0, -1, -2\}$.
(b) $h_0(M)$ *is a* 0-*motivic sheaf.*
(c) $h_{-1}(M)$ *is a* 1-*motivic sheaf.*
(d) $h_{-2}(M)$ *is a* 1-*connected* 2-*motivic sheaf.*
(e) $M[+1]$ *doesn't contains a non-zero direct summand which is a* 0-*motivic sheaf.*

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Cycle modules, Milnor K-theory and the intersection A_∞-algebra

Florian Ivorra

In this talk we give an overview of a *work in progress* on cycle modules and the *intersection theory* developed by M. Rost in [4]. The aim of this work is to show that for a cycle module $\mathscr{M}$ with a ring structure and a smooth separated scheme X of finite type over a field, the cycle complex $C^*(X, \mathscr{M})$ of X with coefficients in $\mathscr{M}$ defined by M. Rost has a *structure of an* A_∞-*algebra* and to develop some consequences of this fact.

The basic operation in Fulton's approach to intersection theory [2] is the *Gysin map* $f^* : \mathrm{CH}_p(X) \to \mathrm{CH}_{p-d}(Y)$ associated to a closed regular immersion $f : Y \hookrightarrow X$ of codimension d. This map is the composition of the specialization map $\mathrm{CH}_p(X) \to \mathrm{CH}_p(N_Y X)$ obtained via the *deformation to the normal cone* $N_Y X$

and the inverse of the pullback map $\mathrm{CH}_{p-d}(Y) \to \mathrm{CH}_p(N_Y X)$ which exists by homotopy invariance of Chow groups. For a smooth scheme X of pure dimension d the intersection product of two cycles $\alpha \in \mathrm{CH}_p(X)$ and $\beta \in \mathrm{CH}_q(X)$ is then given by $\alpha \cdot \beta := \Delta_X^*(\alpha \times \beta)$ where Δ_X is the diagonal immersion. Recall that the Chow group $\mathrm{CH}_p(X)$ is given as the cokernel of the divisor map

$$\mathrm{CH}_p(X) = \mathrm{coker}\left(\bigoplus_{x \in X_{(p+1)}} \kappa(x)^\times \xrightarrow{\mathrm{div}} \bigoplus_{x \in X_{(p)}} \mathbb{Z} \right)$$

and so is the 0-th homology group of the Gersten complex for Milnor K-theory

$$\cdots \to \bigoplus_{x \in X_{(p+r)}} K_r^M(\kappa(x)) \to \cdots \to \bigoplus_{x \in X_{(p+1)}} K_1^M(\kappa(x)) \xrightarrow{\mathrm{div}} \bigoplus_{x \in X_{(p)}} K_0^M(\kappa(x)) \to 0.$$

In [4] M. Rost generalizes the classical intersection theory in two directions, first by considering the whole *Gersten complex* and not only its 0-homology and secondly by allowing more general coefficients than Milnor K-groups: his so called *cycle modules*, which are essentially graded modules over Milnor K-theory endowed with a few extra maps. Given a cycle module $\mathscr{M}$, M. Rost builds his intersection theory on the associated cycle complex $C_*(X, \mathscr{M})$, a Gersten like complex with components given by

$$C_p(X, \mathscr{M}, n) := \bigoplus_{x \in X_{(p)}} \mathscr{M}_{n+p}(\kappa(x)),$$

entirely in terms of the deformation to the normal cone and four basic maps defined in a pointwise manner at the level of complexes: pullbacks, pushforwards, mutiplication by units and boundary maps. He constructs not only a Gysin map

$$f^* : \mathsf{H}_p(C_*(X, \mathscr{M}, n)) \to \mathsf{H}_{p-d}(C_*(Y, \mathscr{M}, n+d))$$

between the homology groups which coincides with Fulton's map for $n = -p$ and $\mathscr{M} = K_*^M$, but he also lifts this map to a *map of complexes*

$$I(f) : C_*(X, \mathscr{M}, n) \to C_{*-d}(Y, \mathscr{M}, n+d)$$

entirely defined in terms of the four basic maps. However, while the map f^* does not depend on any choices, the lifting $I(f)$ depends on the choice of a coordination of the normal bundle $N_Y X$, a variant of the usual notion of trivialization needed for technical reasons. M. Rost proves also some *weak functoriality.* More precisely, given a closed regular immersion $g : Z \hookrightarrow Y$, he shows that the maps $I(g) \circ I(f)$ and $I(f \circ g)$ are homotopic.

The key point in this ongoing work is the following observation: the *homotopy* constructed in [4] *looks very special.* It is reminiscent of the Gysin map by the very way it is defined in terms of the four basic maps and a space of double deformation to the normal cone. Given a cycle module $\mathscr{M}$ with a ring structure and a coordination of the tangent bundle of a smooth separated scheme X of finite type over a field, this remark suggests that the lack of associativity of the intersection product provided by the map $I(\Delta_X)$ is controlled by some *higher* *intersection products*. One is therefore lead to think that Rost's cycle complex $C^*(X, \mathscr{M})$ carries an A_∞*-algebra structure* which means that there should exist a family of bigraded maps of degree $(2-n, 0)$

$$m_n : C^*(X, \mathscr{M})^{\otimes n} \to C^*(X, \mathscr{M}) \tag{1}$$

with the following properties:

– m_1 is the differential of the cycle complex;
– $m_1 \circ m_2 = m_2 \circ (m_1 \otimes \mathbf{1} + \mathbf{1} \otimes m_1)$, and so m_2 commutes with the differentials;
– $m_2 \circ (\mathbf{1} \otimes m_2 - m_2 \otimes \mathbf{1}) = m_1 \circ m_3 + m_3 \circ (m_1 \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes m_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes m_1)$, and so the differential of m_3 is equal to the associator of m_2;
– more generally we have the relation

$$\sum_{n=r+s+t} (-1)^{r+st} m_{r+1+t} \circ \left(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}\right), \tag{2}$$

where the sum is taken over all decompositions $n = r + s + t$.

In the talk we explain how *generalized spaces of deformation* to the normal cone are to provide such an A_∞-algebra structure. To be more precise, for any integer $n \geq 2$, we describe the construction of a space of deformation $\mathscr{D}_{X,n}$ parametrized by the affine space $\mathbb{A}^{n-1}$ which allows to deform *simultaneously* all the diagonal immersions of X^{n-1} in X^n and coincides with the usual space of the deformation to the normal cone for $n = 2$. The *higher intersection maps* (1) are then obtained as a composition of the four basic maps via these higher deformation spaces $\mathscr{D}_{X,n}$. As explained by M. Rost in [4], a coordination of a vector bundle E over X defines a retraction of the pullback map $C^*(X, \mathscr{M}) \to C^*(E, \mathscr{M})$. The fibers over $\{0\}^{n-1}$ of the spaces $\mathscr{D}_{X,n}$ provide a bunch of vector bundles and the last step to establish the formula (2) is to check that all the associated retractions are compatible once a coordination of the tangent bundle of X is given.

Deformation to the normal cone spaces bearing some resemblance to the space $\mathscr{D}_{X,n}$ were also used within the context of the microlocal theory of sheaves [3] by J.-M. Delort in his work [1].

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On higher cohomological Hasse principles

Moritz Kerz
(joint work with Shuji Saito)

In joint work with Saito, relying on previous work of Jannsen and Saito, we prove Kato's conjectures on higher cohomological Hasse principles, formulated in [5], in case the coefficient characteristic is invertible on the scheme in question. Below we restrict for simplicity to the case of varieties over finite fields.

Let k be a finite field, l a prime number with $l \neq \mathrm{char}(k)$ and $\Lambda = \mathbb{Q}_l/\mathbb{Z}_l$. For a smooth projective algebraic curve X/k a classical result due to Hasse and Witt says that there is an exact sequence

$$0 \longrightarrow \mathrm{Br}(K) \longrightarrow \bigoplus_v \mathrm{Br}(K_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

of Brauer groups. The l-primary part of this sequence can be written cohomologically as

$$0 \longrightarrow H^2(k(X),\Lambda(1)) \longrightarrow \bigoplus_{x\in X_0} H^1(k(x),\Lambda) \longrightarrow \Lambda \longrightarrow 0.$$

Kato conjectued in [5] that for a smooth projective variety X/k of dimension d the analogous niveau complex

$$\begin{aligned} &0 \longrightarrow H^{d+1}(k(X),\Lambda(d)) \to \oplus_{x\in X_{d-1}} H^d(k(x),\Lambda(d-1)) \longrightarrow \cdots \\ &\longrightarrow \oplus_{x\in X_0} H^1(k(x),\Lambda) \longrightarrow \Lambda \longrightarrow 0 \end{aligned}$$

should be exact.

Indeed using the homological methods of Jannsen-Saito [4], a localization trick, an intersection theoretic pullback and Gabber's refinement of de Jong's theorem on alterations we can verify this conjecture:

Theorem 1. *Kato's niveau complex is exact.*

Remark 2. The exactness of Kato's complex had been known

- under the assumption of a strong form of resolution of singularities over finite fields (Jannsen [3], Jannsen-Saito [4]),
- up to degree 4 (Colliot-Thélène-Sansuc-Soulé [1], Colliot-Thélène [2], and Jannsen-Saito [4]).

Here the degree m part of Kato's complex is the part of the form $\oplus_{x\in X_m}\cdots$. An important corollary of Kato's conjecture is the finiteness of a certain higher Chow group. This is a special case of the so called Bass conjecture for higher Chow groups.

Corollary 3. *For all $q, m \geq 0$ the group $\mathrm{CH}^d(X,q)/l^m$ is finite.*

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Motives of quadrics

Alexander Vishik

The talk concerns the motivic structure of projective homogeneous varieties, namely, the largests class among them, the quadrics. These questions are empty over algebraically closed field, and give us an opportunity to study in a compressed form the effects which distinguish the algebraically closed field from the arbitrary one.

For the smooth projective quadric Q over the field k of characteristic not 2, let $M(Q)$ be its *motive* in the category $Chow(k)$ of Chow motives over k. Over the algebraic closure $\overline{k}$, our quadric becomes completely split, and so, cellular. This implies that $M(Q|_{\overline{k}})$ becomes isomorphic to a direct sum of Tate motives:

$$M(Q|_{\overline{k}}) \cong \oplus_{\lambda\in\Lambda(Q)}\mathbb{Z}(\lambda)[2\lambda].$$

Then the same happens to an arbitrary direct summand N of $M(Q)$, and we obtain a subset $\Lambda(N) \subset \Lambda(Q)$. This permits to define the following equivalence relation: for $\lambda, \mu \in \Lambda(Q)$ we say that λ and μ are *connected* in $\Lambda(Q)$, if, for any direct summand N of $M(Q)$, $\lambda \in \Lambda(N) \Leftrightarrow \mu \in \Lambda(N)$. The resulting decomposition of $\Lambda(Q)$ into *connected components* is called the *Motivic Decomposition Type.* This is an important discrete invariant of quadrics, and its interaction with such classical invariant as the *Splitting Pattern* gives many results about both. The natural question arises: What values of MDT are possible? This question for an arbitrary quadric can be easily reduced to that for an anisotropic quadric, and in the latter case, one shows that each connected component consists of an even number of elements. So, the minimal cardinality of such a component is two. As was shown by M.Rost ([3]), there is class of anisotropic quadrics where all the components are "binary" (consist of just two elements). These are, so-called, *excellent quadrics.* Examples include: Pfister quadrics, as well as anisotropic real quadrics of all dimensions. In general, in a given dimension, excellent quadrics are the best ones you could find. Our Main Theorem is the following:

Theorem 1. *All connections between Tate-motives which are present in the motives of an excellent quadric are also present in the motive of any anisotropic quadric of the same dimension.*

This result puts severe restrictions on the possible values of MDT as one can apply it not only to Q but to anisotropic parts of it over various field-extensions. As a corollary we get an estimate from below on the rank of an indecomposable direct summand in the motive of a quadric in terms of its dimension:

Theorem 2. *Let N be an indecomposable direct summand in the motive of some anisotropic quadric. Let $dim(N) + 1 = 2^{r_1} - 2^{r_2} + \ldots + (-1)^{s-1}2^{r_s}$, where $r_1 > r_2 > \ldots > r_{s-1} > r_s + 1 \geq 1$. Then:*

(1) *$rank(N) \geq 2s$;*
(2) *Moreover, we can say, which particular $2s$ Tate-motives are present in $N_{\overline{k}}$.*

As a corollary we get the Binary Motive Theorem:

Theorem 3. ([1, Theorem 6.1]) *Let N be a binary indecomposable direct summand in the motive of some quadric. Then $dim(N) = 2^r - 1$, for some r.*

and the theorem of N.Karpenko on $\mathfrak{i}_1(q)$:

Theorem 4. (N.Karpenko, [2]) *Let q be an anisotropic quadratic form of dimension m. Then $(\mathfrak{i}_1(q) - 1)$ is a remainder modulo 2^r of $(m - 1)$, for some $r < \log_2(n - 1)$.*

The bound from Theorem 2 is optimal:

Proposition 5. *In the notations of Theorem 2, for any given dimension, there exists an indecomposable direct summand N (in the motive of some anisotropic quadric), such that $dim(N)$ is as prescribed, and $rank(N)$ is exactly $2s$.*

In certain cases, it is also possible to produce the bound from above.

Theorem 6. *Let N be a direct summand of $M(Q)$, such that $dim(N) = dim(Q) = 2^r - 1$, for some r, and $\Lambda(N)$ does not contain $(2^{r-1} - 2^i)$, for $0 \leq i \leq n - 2$. Then N is binary.*

This result shows that instead of checking $dim(N)$ different Tate-motives you need to deal only with $\log_2(dim(N))$ of them. The methods of proof of this result are somewhat interesting and are similar to the proof of the, so-called, Main Tool Lemma ([4, Corollary 3.5]), the result concerning the field of definition of Chow group element, which has many applications and extensions in various directions. As such an extension, let me formulate the integral version, which is new:

Theorem 7. *Let Q be smooth projective quadric, and Y be smooth quasiprojective variety over the field k. Let $\overline{y} \in \mathrm{CH}^m(Y|_{\overline{k}})$. Suppose:*

- *$m < dim(Q)/2$;*
- *m is not divisible by 2^r - the smallest power of two $\geq \mathfrak{i}_1(q)$.*

Then:

$$\overline{y} \text{ is defined over } k(Q) \;\Leftrightarrow\; \overline{y} \text{ is defined over } k.$$

In the case of Pfister quadric, the second condition follows from the first one, and since the Pfister quadric is a Norm-variety for the pure symbol modulo 2, we get:

Theorem 8. *Let k be a field of characteristic not 2, and $r \in \mathbb{N}$. Then there exists a field extension E/k such that:*

1) $K_r^M(E)/2 = 0$;
2) *For any smooth quasiprojective variety Y/k, for any $m < 2^{r-1} - 1$, the respective restriction map on Chow groups is surjective:*

$$\mathrm{CH}^m(Y) \twoheadrightarrow \mathrm{CH}^m(Y_E).$$

Thus, the *mod* 2 and degree r cohomological invariants of algebraic varieties can not affect rationality of cycles of codimension up to $2^{r-1} - 2$ (and it is easy to see that this bound is sharp).

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Motivic cohomology and special values of Dedekind zeta-functions

Stephen Lichtenbaum

We explain a general philosophy of how special values of zeta-functions of number fields ought to be given by products of Euler characteristics of cohomology complexes.

Let X be an arithmetic scheme, i.e. a scheme of finite type over $\operatorname{Spec}\mathbb{Z}$. The scheme zeta-function $\zeta(X, s)$ of X is given by

$$\zeta(X, s) = \prod_{x \in |X|} (1 - N(x)^{-s})^{-1},$$

where $|X|$ is the set of closed points of X and $N(x)$ is the cardinality of the residue field $\kappa(x)$. In good cases, and possibly in all cases, we can extend $\zeta(X, s)$ to a meromorphic function in the plane.

We would like to give formulas for the order a_n of the zero and the leading term $\zeta^*(X, n)$ of $\zeta(X, s)$ at $s = n$, a non-negative integer.

If X is projective and smooth over a finite field, modulo generally accepted conjectures we can express $\zeta^*(X, n)$ as the Euler characteristic of the Weil-étale version of the motivic complex of sheaves $\mathbb{Z}(n)$ multiplied by an alternating product of Euler characteristics of coherent sheaves. The order of the zero is given by the generalized rank of the motivic complex.

Our goal is to find analogous formulas when X is $\operatorname{Spec} \mathcal{O}_F$, and $\mathcal{O}_F$ is the ring of integers in a number field F, and to discuss their relations with the usual functional equation, We close by pointing out that the coherent contribution to $\zeta^*(X,n)$ in the case of a projective smooth curve X over the finite field with q elements is given by

$$q^{n\chi(X,\mathcal{O}_X)-(n-1)\chi(X,\Omega)}.$$

This is replaced in the number field case by

$$(\chi(\mathcal{O}_F)^n)(\chi(D^{(-1)})^{-(n-1)}),$$

where D^{-1} is the inverse different. In the geometric case the coherent cohomology groups are finite and so do not contribute anything toward the order of the zero, but in the number field case these groups are only finitely generated, and so their ranks must be taken into account.

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