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# Dynamische Systeme

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Abstract. This workshop, organized by Hakan Eliasson (Paris), Helmut Hofer (Princeton) and Jean-Christophe Yoccoz (Paris), continued the biannual series at Oberwolfach on Dynamical Systems that started as the "Moser–Zehnder meeting" in 1981. The workshop was attended by more than 50 participants from 12 countries.

The main theme of the workshop were the new results and developments in the area of classical dynamical systems, in particular in celestial mechanics and Hamiltonian systems. Among the main topics were KAM theory in finite and infinite dimensions, and new developments in Floer homology (Rabinowitz-Floer homology).

*Mathematics Subject Classification (2000):* 37XX.

## Introduction by the Organisers

The theme of this workshop were the new developments in classical dynamical systems. Specific topics covered KAM theory and integrable systems (Jacques Féjoz, Misha Bialy, Vadim Kaloshin, Serge Tabachnikov, Eugene Wayne), celestial mechanics (Sergey Bolotin), billiards (Karl Friedrich Siburg, Paul Wright) and horocycle flows (Giovanni Forni), maps of the interval and piecewise linear maps (Kristian Bjerklöv, Svetlana Katok, Tomasz Nowicki), hyperbolic dynamics (Pierre Berger, Carlos Matheus), and the use of $J$-holomorphic curves and Floer homology (Alberto Abbondandolo, Peter Albers, Samuel Lisi, Felix Schlenk). Particularly impressive was Artur Avila's lecture on Stratified analycity of the Lyapunov exponent and the global theory of one-frequency Schrödinger operators.

The meeting was held in a very informal and stimulating atmosphere. The weather was very rainy, so that we stayed most of the time inside and had much time for discussions. On the traditional Wednesday walk, only three of us were courageous enough to risk the walk through the stormy forest, among which, of course, Paul Rabinowitz.

**Workshop: Dynamische Systeme**

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# Abstracts

## Estimates and computations in Rabinowitz-Floer homology

Alberto Abbondandolo
(joint work with Matthias Schwarz)

Let $(W, \lambda)$ be a Liouville domain, that is a compact connected $2n$-dimensional manifold with boundary, equipped with a one-form $\lambda$ such that $\omega := d\lambda$ is a symplectic form on $W$, and that the Liouville vector field $Y$ defined by $\iota_Y \omega = \lambda$ is transverse to $\partial W$ and points in the outward direction. Recently, K. Cieliebak and U. Frauenfelder [5] have associated an algebraic invariant to such a structure, which they call Rabinowitz-Floer homology. This is the Floer homology of the free period Hamiltonian action functional

$$\mathbb{A}(x, \eta) := \int_{\mathbb{T}} x^* \lambda - \eta \int_{\mathbb{T}} H(x(t))\, dt,$$

which was used by P. Rabinowitz [8] in his proof of the Weinstein conjecture for star-shaped domains of $\mathbb{R}^{2n}$. Here $x$ is a loop of period 1 in $\hat{W} := W \cup_{\partial W} (\partial W \times [0, +\infty[)$, the completion of $W$, $\eta$ is a real number, and $H$ is a smooth Hamiltonian on $\hat{W}$ whose set of zeroes is $\partial W$ and whose associated Hamiltonian vector field restricts to the Reeb vector field of $(\partial W, \lambda|_{\partial W})$. The critical set of $\mathbb{A}$ consists of $\partial W \times \{0\}$, the space of constant loops on $\partial W$, and of two critical points $(y(\eta \cdot), \eta)$, $(y(-\eta \cdot), -\eta)$ for each closed Reeb orbit $y$ on $\partial W$ of (not necessarily minimal) period $\eta > 0$. The functional $\mathbb{A}$ is generically Morse-Bott and the Rabinowitz-Floer complex $RF(W, \lambda)$ is its Floer complex with cascades: It is based on counting solutions of the negative $L^2$-gradient equation for $\mathbb{A}$, that is the system of PDEs'

$$\begin{aligned} &\frac{\partial u}{\partial s}(s,t) + J_t(u(s,t)) \left( \frac{\partial u}{\partial t}(s,t) - \eta(s) X_H(u(s,t)) \right) = 0, \\ &\frac{d\eta}{ds}(s) = \int_{\mathbb{T}} H(u(s,t))\, dt, \end{aligned} \tag{1}$$

where $u$ is a map from the cylinder $\mathbb{R} \times \mathbb{T}$, endowed with coordinates $(s, t)$, to $\hat{W}$, $\eta$ is a real function on $\mathbb{R}$, $J_t$ is a loop of $\omega$-compatible almost complex structures on $\hat{W}$, and $X_H$ denotes the Hamiltonian vector field on $(\hat{W}, \omega)$ induced by $H$.

The Rabinowitz-Floer homology $HRF(W, \lambda)$ is the homology of the chain complex $RF(W, \lambda)$. It has the following important vanishing property: it vanishes whenever there is an embedding $\varphi : W \hookrightarrow W'$ into the interior part of another Liouville domain $(W', \lambda')$, such that $\varphi^* \lambda' - \lambda$ is exact and $\varphi(W)$ is displaceable within $W'$ by a Hamiltonian isotopy. The fact that $HRF(W, \lambda)$ vanishes implies the existence of closed Reeb orbits on $\partial W$, because otherwise $HRF(W, \lambda)$ would be isomorphic to the singular homology of $\partial W$.

Rabinowitz-Floer homology has already found quite a number of applications. It provides obstructions for the existence of contact embeddings of unit cotangent sphere bundles (see [5] and [6]), it allows to prove existence and multiplicity results for leaf-wise intersections points (see [3], [4], and P. Albers' abstract in this Report), and it plays a relevant role in the study of the dynamics and the symplectic topology of energy hypersurfaces of magnetic flows on cotangent bundles (see [7], where Rabinowitz-Floer homology is extended to domains whose boundary need not be of contact type, but satisfies substantially weaker assumptions).

An important ingredient in all these applications is to determine the Rabinowitz-Floer homology of $(D^*M, \lambda)$, the unit cotangent disk bundle of a closed Riemannian manifold $(M, g)$, equipped with the restriction of the canonical Liouville one-form of the cotangent bundle $T^*M$. In this case, the completion of $D^*M$ is $T^*M$, its boundary is $S^*M$, the unit cotangent sphere bundle, and the Reeb flow on $S^*M$ coincides with the geodesic flow. The Rabinowitz-Floer homology of $(D^*M, \lambda)$ has been recently computed by K. Cieliebak, U. Frauenfelder, and A. Oancea [6], by constructing an exact sequence of the form

$$\cdots \to H_k(\Lambda M) \to H_k RF(D^*M, \lambda) \to H^{1-k}(\Lambda M) \to H_{k-1}(\Lambda M) \to \dots, \tag{2}$$

where $\Lambda M$ is the free loop space of $M$. This exact sequence is found as a particular case of a more general exact sequence relating Rabinowitz-Floer homology to symplectic homology and cohomology.

The aim of this talk is to explain how (2) can be seen as the long exact sequence associated to a short exact sequence of chain complexes

$$0 \to C_k(\Lambda M) \to RF_k(D^*M, \lambda) \to C^{1-k}(\Lambda M) \to 0, \tag{3}$$

where $C_*(\Lambda M)$ and $C^*(\Lambda M)$ are the chain complex and the differential complex associated to a suitable cellular filtration of $\Lambda M$.

By standard infinite dimensional Morse theory, $C_*(\Lambda M)$ and $C^*(\Lambda M)$ can be identified with the Morse chain complex $M_*(\mathbb{E})$ and the Morse differerential complex $M^*(\mathbb{E})$ associated to the geodesic energy functional

$$\mathbb{E}(\gamma) = \int_0^1 g(\gamma'(t), \gamma'(t))\, dt$$

on the Hilbert manifold $W^{1,2}(\mathbb{T}, M)$ of absolutely continuous closed loops in $M$ with square integrable velocity. Then the constructions of the two chain maps in (3) is based on coupling the Rabinowitz-Floer complex produced by the Hamiltonian

$$H(q,p) = \frac{1}{2}(g_q^*(p,p) - 1), \quad \forall q \in M,\ p \in T_q^*M,$$

where $g^*$ is the inner product on $T^*M$ dual to $g$, with the Morse chain complex and the Morse differential complex of $\mathbb{E}$, and is similar to the construction introduced in [2]. The definition of the chain map

$$\Phi : M_*(\mathbb{E}) \to RF_*(D^*M, \lambda)$$

is based on counting solutions of the following mixed problem: Given $\gamma$ a critical point of $\mathbb{E}$ and $z$ a critical point of $\mathbb{A}$, we consider the space of pairs

$$u : [0, +\infty[ \times \mathbb{T} \to T^*M, \quad \eta : [0, +\infty[ \to \mathbb{R},$$

which solve the Rabinowitz-Floer equation (1), converge to $z$ for $s \to +\infty$, and satisfy the boundary conditions

$$\pi \circ u(0,\cdot) \in W^u(\gamma; -\nabla \mathbb{E}), \quad \eta(0) = \sqrt{\mathbb{E}(\pi \circ u(0,\cdot))}, \tag{4}$$

where $\pi : T^*M \to M$ is the projection and $W^u(\gamma; -\nabla\mathbb{E}) \subset W^{1,2}(\mathbb{T}, M)$ denotes the unstable manifold of $\gamma$ with respect to the $W^{1,2}$-negative gradient flow of $\mathbb{E}$. Actually, the Morse-Bott situation require us to consider solutions with cascades of the above problem. In order to keep the presentation simpler, we systematically ignore this point within this abstract. The second coupling condition in (4) is suggested by the inequality

$$\mathbb{A}(x, \sqrt{\mathbb{E}(\pi \circ x)}) \leq \sqrt{\mathbb{E}(\pi \circ x)}, \quad \forall x \in \Lambda T^*M,$$

and by the fact that the equality holds when $(x, \sqrt{\mathbb{E}(\pi \circ x)})$ is a critical point of $\mathbb{A}$. Indeed, the latter inequality allows us to prove the necessary compactness for the above mixed problem, and to construct a left inverse $\hat{\Phi}$ for $\Phi$, which is then injective. An important issue in the proof of compactness is to get $L^\infty$ estimates for solutions of the Rabinowitz-Floer equation (1) with uniformly bounded energy, in the case where the Hamiltonian $H$ has quadratic growth at infinity. These estimates are obtained by means of the A. D. Aleksandrov's integral version of the maximum principle, and by a version of this maximum principle with Neumann conditions on part of the boundary.

The definition of the chain map

$$\Psi : RF_*(D^*M, \lambda) \to M^{1-*}(\mathbb{E}),$$

and the proof that it admits a right inverse $\hat{\Psi}$ is similar. The composition $\Psi \circ \Phi$ might not be zero, but we can show that it is chain homotopic to zero, that is

$$\Psi \circ \Phi = P\partial + \delta P,$$

where the chain homotopy

$$P : M_*(\mathbb{E}) \to M^{-*}(\mathbb{E})$$

is defined by counting finite length cylinders over minimal closed geodesics, that is solutions

$$u : [-S, S] \times \mathbb{T} \to T^*M, \quad \eta : [-S, S] \to \mathbb{R},$$

of the Rabinowitz-Floer equation (1) for some $S > 0$, which satisfy the boundary conditions

$$\pi \circ u(-S, \cdot) = \gamma^-(\cdot),\ \eta(-S) = \sqrt{\mathbb{E}(\gamma^-)},\ \pi \circ u(S, \cdot) = \gamma^+(-\cdot),\ \eta(S) = -\sqrt{\mathbb{E}(\gamma^+)},$$

where $\gamma^-$ and $\gamma^+$ are closed geodesics of Morse index zero. The main point in the construction of $P$ and in the proof of the fact that it is a chain homotopy between $\Psi \circ \Phi$ and zero is to show that the length $2S$ of the cylinders in the above problem - and in the analogous problem in which the sum of the indices of the closed geodesics $\gamma^-$ and $\gamma^+$ is one - is bounded away from zero. Then $\Phi$ can be modified within its chain homotopy class, obtaining the chain map

$$\Theta := \Phi - \hat{\Psi} P \partial - \partial \hat{\Psi} P,$$

and our main result is the following:

**Theorem** [A. Abbondandolo and M. Schwarz, [1]] *The short sequence of chain maps*

$$0 \to M_*(\mathbb{E}) \xrightarrow{\Theta} RF_*(D^*M, \lambda) \xrightarrow{\Psi} M^{1-*}(\mathbb{E}) \to 0$$

*is exact and splits. After identifying Morse (co)homology with singular (co)homology by the isomorphisms*

$$H_k M(\mathbb{E}) \cong H_k(\Lambda M), \quad H^k M(\mathbb{E}) \cong H^k(\Lambda M),$$

*the associated long exact sequence takes the form*

$$\cdots \to H_k(\Lambda M) \xrightarrow{\Theta_* = \Phi_*} H_k RF(D^*M, \lambda) \xrightarrow{\Psi_*} H^{1-k}(\Lambda M) \xrightarrow{\Delta_*} H_{k-1}(\Lambda M) \to \dots$$

*where the connecting homomorphism $\Delta_*$ can be non-zero only in degree zero, and where*

$$\Delta_* : H^0(\Lambda M) \to H_0(\Lambda M)$$

*vanishes on the components of $\Lambda M$ containing non-contractible loops, and is the multiplication by the Euler number of $T^*M$ on the component of $\Lambda M$ consisting of contractible loops.*

The above result can be generalized to a fiber-wise uniformly convex compact domain $W$ in $T^*M$ whose interior contains a Lagrangian graph. Equivalently, $W$ is the set of points in $T^*M$ of energy $H$ not exceeding $c$, where $H$ is the Hamiltonian which is Fenchel-dual to a fiber-wise uniformly convex and superlinear Lagrangian function $L$ on $TM$, and $c$ is larger than the strict Mañé critical value of $L$.

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## Rabinowitz-Floer homology and leaf-wise intersection points

Peter Albers
(joint work with Urs Frauenfelder)

We address the leaf-wise intersection problem for hypersurfaces $\Sigma$ of restricted contact type in exact symplectic manifolds $M$ which are convex at infinity. This problem originates from work of Jürgen Moser in 1978. A leaf-wise intersection with respect to a Hamiltonian diffeomorphism $\phi$ is a point $x \in \Sigma$ such that $\phi(x)$ lies on the (not necessarily closed) Reeb orbit of $x$.

We prove that critical points of a perturbed Rabinowitz action functional give rise to leaf-wise intersections. After constructing Floer homology for this perturbed Rabinowitz action functional we derive several existence results for Hofer-small Hamiltonian diffeomorphism. For instance

**Theorem 1.** *If the Hofer norm of $\phi$ is smaller than the period of any Reeb orbit on $\Sigma$ which in contractible in $M$, then there exists a leaf-wise intersection with respect to $\phi$.*

Remark: Examples show that the bound on the Hofer norm is sharp.

**Theorem 2.** *If in addition $\phi$ is generic, then the number of leaf-wise intersections is at least as big as the sum of the Betti numbers of $\Sigma$.*

**Theorem 3.** *If $\Sigma$ is the unit cotangent bundle in $M = T^*B$ with respect to a Riemannian metric on the closed, simply connected manifold $B$, then for generic $g$ and $\phi$ the number of leaf-wise intersections is infinite.*

## Abundance of one dimensional non uniformly hyperbolic attractors for surface endomorphisms

Pierre Berger

We are interested in the following family of endomorphisms:

$$f_{a,B}: \ (x, y) \mapsto (x^2 + a + y, 0) + B(x, y)$$

where $a$ is a real number and $B$ is any fixed $C^2$ small function.

The motivation goes back to a question of Pesin-Yurchenko [4] in reaction-diffusion PDEs. Here is the answer to the question:

**Theorem 4.** *For every $B$ of $C^2$-norm $b$ sufficiently small, there exists a subset $\Lambda_B \subset [-2, 0)$ with positive Lebesgue measure such that for every $a \in \Lambda_B$, the map $f_{a,B}$ preserves a physical, ergodic, hyperbolic SRB probability $\mu$.*

We recall that $\mu$ is *physical* if its basin has positive Lebesgue measure.

The measure $\mu$ is *SRB* if its conditional measure with respect to any unstable manifold is absolutely continuous with respect to its Lebesgue measure.

The family of maps $f_{a,B}$ looks familiar.

In the case $B = 0$, the map $f_{a,B}$ preserves the horizontal real line, its restriction is the quadratic map$f_a(x) = x^2 + a$. Restricted to this case the above theorem is Jackobson's one.

In the case $B = b \cdot (0, x)$, the map $f_{a,B}$ is the Hénon map, with Jacobian $-b$. Restricted to this case, the above theorem is Benedicks-Carleson's one for the parameters exclusion [2], and Benedicks-Young's one for the SRB existence [3].

What is new in the statement is that the dynamics might be non-invertible with nasty singularities, and only $C^2$ (instead of $C^3$ for all the previous techniques).

Also our proof is different from [2] since it is based on a generalization of the geometric and combinatorial formalism of Yoccoz puzzles [5]. However many analytical and probabilistic ideas are mostly included in the easy parts of [2] and [6]. We will focus our exposition on some of the new aspects of this proof: the geometric operations and the characterization of suitable dynamics.

Before going into the details, we shall remind that the quadratic map $f_{-2}$ is semi-conjugated to the map of the circle $\theta \mapsto 2\theta$ via the map $\theta \mapsto 2\cos\theta$. This gives us some hyperbolicity for free for perturbations of $f_{-2}$ and $f_{-2,0}$.

Also for $a$ close to $-2$, the quadratic map $f_a$ has two repulsive fixed points $\mathcal{A} \approx -1$ and $\mathcal{B} \approx 2$. So the map $f_{a,B}$ has also two hyperbolic fixed points $\mathcal{A} \approx (-1, 0)$ and $\mathcal{B} \approx (2, 0)$.

**Regular dynamics.** To construct an SRB in the one dimensional case, we construct a *Markov structure* $(\alpha)_{\alpha\in\mathcal{Y}}$ of a subinterval $S$ of $[-\mathcal{B}, \mathcal{B}]$.

The set $\mathcal{Y}$ is formed by almost disjoint segments $S_\alpha$ endowed with an integer $n_\alpha$ such that $f^{n_\alpha}$ sends bijectively $S_\alpha$ onto $S$ with some distortion bounds and uniform hyperbolicity (hyperbolic times).

The *Markov piece* $\{S_\alpha\}_\alpha$ cannot cover all of $S$, but has to cover almost it. Even more, we ask for the integrability of the function $\sum n_\alpha 1\!\!1_{S_\alpha}$. If such a Markov structure exists, then the quadratic map $f_a$ preserves an SRB measure.

For other perspectives, we can assume furthermore that $S$ equals to $[\mathcal{A}, -\mathcal{A}]$. Then, the Markov pieces of $\mathcal{Y}$ are *puzzle pieces* and the quadratic map is *regular*.

We generalize this concept to the two dimensional case, by taking instead of $[\mathcal{A}, -\mathcal{A}]$ a continuous family of *flat and stretched (FS)* curves $\Sigma = (S^t)_{t\in T}$ parameterized by a compact metric space $T$. By FS we mean that each of them is $C^2$-close to the horizontal segment $[-1, 1]$ with end points in a local stable manifold of $\mathcal{A}$ close to an arc of the parabola $x^2 + y + a = -1$.

Each of these curves $S$ is still endowed with a partition by *puzzle pieces* $\mathcal{Y}(S)$: these are segments $S_\alpha$ of $S$ sent by $f^{n_\alpha}$ onto FS curves $S^\alpha$. As before we suppose some uniform distortion bound and hyperbolicity. Also we ask $S^\alpha$ to belong to $\Sigma$ and the map $(t, \alpha) \in \prod_{t\in T} \mathcal{Y}(S^t) \mapsto t \cdot \alpha \in T$ to be surjective.

Eventually the function on $X := \prod_t S^t$ equal to $\sum_{\alpha\in\cup_{t\in T}\mathcal{Y}(S^t)} n_\alpha 1\!\!1_{\{t\}\times S^t_\alpha}$ is supposed to be (more than) measurable and, restricted to each $S^t$, integrable.

In such a case the map $f = f_{a,B}$ is *regular* and then preserves an ergodic, hyperbolic, SRB measure $\mu$. Actually, the union of unstable manifolds $\cup_{t\in T} S^t$ has positive $\mu$-measure.

**Strongly regular dynamics.** The main difficulty is to prove the abundance of regular maps. For this end, we define independently to the parameters selection the strongly regular maps. The definition is combinatorial and geometric. By developing a few classical and easy techniques of [2] and [6] in an independent last part, we show the regularity of strongly regular dynamics. The algebraization of the geometric operations is more original.

A basic operation is the $\star$-product. Let $\alpha = \{S_\alpha, n_\alpha\}$ be a puzzle piece of an FS curve $S$ and let $\beta = \{S^\alpha_\beta, n_\beta\}$ be a puzzle piece of the FS curve $S^\alpha = f^{n_\alpha}(S_\alpha)$. Then the pair $\alpha \star \beta = \{f^{-n_\alpha}_{|S_\alpha}(S^\alpha_\beta), n_\alpha + n_\beta\}$ is a puzzle piece of $S$.

Before defining the other operation, let us recall that from the Chebychev map $f_{-2}$, for $a$ close to $-2$ and then $b$ small enough, a lot of puzzle pieces $s^M_-, \dots, s^2_-, s^2_+, \dots, s^M_+$ match for any FS curves. Here $M$ is the smallest positive integer s.t. $f_a^{M+1}(0)$ belongs to $[\mathcal{A}, -\mathcal{A}]$. The puzzle pieces $(s^k_\pm)_k$ are all disjoint and their union covers all $S$ but a small segment $S_\square$ close to 0. The puzzle pieces $(s^k_\pm)_k$ are called *simple* and satisfy $n_{s^k_\pm} = k$.

To state the strong regularity, we must understand how to construct new puzzle pieces on $S_\square$. The segment $S_\square$ is sent by $f^{M+1}$ to a curves $S^\square$ which starts from a very small local stable manifold of $\mathcal{A}$, comes around $(f_a^{M+1}(0), 0)$ and comes back to the same local stable manifold. Let us suppose that $S^\square$ is tangent to a local stable manifold $W^s_c$ of a point $c \in S$. Suppose $c$ is the unique point of an intersection $\cap_{i=0}^\infty S_{c_i}$ of puzzle pieces $c_i = \alpha_1 \star \alpha_2 \star \cdots \star \alpha_i$, where $(\alpha_k)_{k=1}^i$ are mostly simple for every $i \geq 1$. Then $W^s_c$ is $C^1$-close to a local stable manifold of $f_{a,0}$: an arc of parabola with end points as far from $c$ as $\theta := -(\log b)^{-1}$. By mostly simple we mean for every $i$:

$$\sum_{\{k \leq i+1,\ \alpha_k \text{ not simple}\}} n_{\alpha_k} \leq e^{-\sqrt{M}} \sum_{k \leq i} n_{\alpha_k} = e^{-\sqrt{M}} n_{c_i} \tag{1}$$

One can prove that the end points of $S_{c_i}$ have also a long stable manifold close to an arc of a parabola. Let $Y_{c_i}$ be the box bounded by these two stable manifolds and the two horizontal lines $\{y = \pm\theta\}$.

From elementary topology the set $Y_{c_i} \setminus Y_{c_{i+1}}$ intersects the curve $S^\square$ at (0 or) two segments that are pulled back via $f^{M+1}|S_\square$ to segments $S_{\Delta_-}$ and $S_{\Delta_+}$. Let $\Delta_\pm := \{S_{\Delta_\pm}, n_\Delta := M + 1 + n_{c_i}\}$.

We note that $f^{n_\Delta}$ sends each segment $S_{\Delta_\pm}$ to a curve with an end point in a small local stable manifold $W^s(z)$ of an end point $z$ of $S^{c_i}_{\alpha_{i+1}}$. If the image by $f^{n_{c_i}}$ of the tangency point $C$ between $S^\square$ and $W^s(c)$ is not too close to $W^s(z)$, then one shows that the curves $f^{n_\Delta}(S_{\Delta_\pm})$ can be extended to $C^1$-FS curves by following the curvature of $S^{c_i}$ after $W^s(z)$. Let $S^{\Delta_\pm}$ be these extensions. We note that the images of $S_{\Delta_\pm}$ by $f^{n_\Delta}$ are strictly included in $S^{\Delta_\pm}$. Therefore $\Delta_\pm$ are not Puzzle pieces of $S$. However the pairs $\Delta_\pm$ satisfies hyperbolic times inequalities. Thus if $\beta$ is a puzzle piece of $S^{\Delta_\pm}$ included in $f^{n_\Delta}(S_{\Delta_\pm})$ then $\Delta \star \beta$ defines a *topological* puzzle piece of $S$ included in $S_\square$: the segment $S_{\Delta\star\beta}$ is sent to a FS curve $S^{\Delta\star\beta}$ with some hyperbolicity. For instance $\beta$ can be equal to a simple piece $s$ or even of the form $\Delta' \star s$ to make a (topological) puzzle piece $\Delta \star \Delta' \star s$.

Let us state the combinatorial condition implying that the tangency point $C$ is not close to $W^s(z)$, with $s_-^2$ the simple piece next to $\mathcal{A}$:

$$\alpha_i = \cdots = \alpha_{i+k} = s_-^2 \Rightarrow Mi \geq 20k \tag{2}$$

A *strongly regular* map is endowed with a set $\Sigma$ of $FS$ curves $S$ equipped with a set $\mathcal{Y}(S)$ of almost disjoint topological puzzle pieces $\alpha$ s.t. $S^\alpha$ belongs to $\Sigma$. Also $\mathcal{Y}$ consists of the pieces of the form $\Delta_1 \star \cdots \star \Delta_n \star s$, where $s$ is simple and $\Delta_n$ is given by the above algorithm from a requested tangency between$(S^{\Delta_1 \cdots \Delta_{n-1}})^\square$ and $W_c^s$, with $c$ a sequence of puzzle pieces in $\mathcal{Y}$ satisfying $(1-2)$. Then the topological puzzle pieces of $\mathcal{Y}$ satisfy automatically the distortion and hyperbolic bounds and so are real puzzle pieces.

We can truncate the definition by asking for geometrical conditions with respect to $Y_{c_k}$ instead of $W_c^s$. Then regarding only the $\Delta$-constructions made from $c_j$, $j \leq k$. This makes the concept of *$k$-strong regularity* which is useful to show the abundance of strongly regular maps.

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# Integrable geodesic flows on surfaces

Misha Bialy

## 1. Introduction

Let $\Sigma$ be a closed orientable surface of genus $p \geq 1$. Given a Riemannian metric $g$ on $\Sigma$, let $g^t \colon T_1\Sigma \to T_1\Sigma$ be the corresponding geodesic flow acting on the unit circle bundle of $\Sigma$. It is an important question for dynamics and geometry if there exists a smooth function $F : T_1\Sigma \to \mathbb{R}$ which is invariant under the flow, that is $F(g^t x) = F(x)$. In this case $F$ is classically called a (first) integral of the geodesic flow. In this case, the Liouville-Arnold theorem implies that any connected component $L$ of the level set $\{F = c\}$, which satisfies $DF|_L \neq 0$, is a 2-torus invariant under the flow. Moreover, the dynamics of geodesic flow on this torus is linearizable. We shall use the following:

**Definition 5.** *A torus $L$ lying in a level of $F$ will be called regular if $DF|_L \neq 0$, on the other hand $L$ is singular if there exists a point on $L$ where $DF$ vanishes.*

In the search of smooth integrals, one usually requires some extra condition on $F$ which prohibits $F$ from being essentially constant. For example, one usually requires the set of regular points of $F$ to be dense. In this case the geodesic flow is called integrable.

The following theorem was proved in 1979 by V.V. Kozlov.

**Theorem 6.** *Assume the genus $p > 1$. Then any smooth integral $F : T_1\Sigma \to \mathbb{R}$ satisfying the following two conditions must be a constant.*

(K1) *$F$ has finitely many critical values.*
(K2) *For a dense set of points $x \in \Sigma$ the intersection of the fibre $\pi^{-1}(x)$ with any critical level $\{F = c\}$ is at most finite or coincides with the whole fibre.*

As a corollary Kozlov concluded that any real-analytic integral of the geodesic flow on surfaces of higher genus must be constant. The question on topological obstructions to the integrability of geodesic flows has found considerable interest, in the works of Taimanov, Paternain, Butler where generalizations to higher dimensions with various stronger assumptions on non-wildness of $F$ were proposed. On the other hand, new remarkable examples of manifolds which cannot have analytically integrable geodesic flows, but do admit a $C^\infty$-integrable geodesic flows were recently discovered. However, it seems that the question if such examples exist on compact surfaces of genus greater than one is still open.

We approach this question by a geometric idea which enables to relax significantly the conditions of Kozlov's theorem. On the other hand, we apply this method to the case of the 2-torus. For the 2-torus there are two known classes of metrics with integrable geodesic flows. These are rotationally symmetric metrics and, the so-called, Liouville metrics. The question if there exist other examples of integrable geodesic flows is widely open. By our method we get nontrivial information on the phase portraits for integrable geodesic flows. We show that they are essentially standard outside what we call separatrix chains. Namely the complement to the union of these chains is $C^0$ foliated by invariant sections. So all dynamical complications could be located only inside the chains.

The first ingredient of the method is the theory of minimal geodesics and rays on surfaces. It was invented by M. Morse and G. Hedlund and further treated by Bialy-Polterovich and Bangert in connection with Aubry-Mather theory. The second ingredient uses the properties of the projections of Lagrangian torii developed in works of Bialy and Polterovich. For our approach we shall require the metric $g$ and the function $F$ to be of class $C^3$, at least. This is the minimal regularity needed in order to use the properties of projections for Lagrangian torii and also for the Morse-Sard theorem for $F$ which is used below.

In order to state the first result let me formulate the main condition.

**Definition 7.** *We shall say that $F : T_1\Sigma \to \mathbb{R}$ satisfies* condition $\aleph$, *if for a dense set of $x \in \Sigma$ the intersection of the fibre $\pi^{-1}(x)$ with any connected component of the critical level $\{F = c\}$ is at most* countable *or coincides with the whole fibre.*

**Example 1.** *Any function which is real-analytic with respect to momenta satisfies this condition. In particular polynomials with respect to momenta are of great interest, since in all known examples of integrable geodesic flows the integral appears to be polynomial in the momenta variables.*

**Theorem 8.** *Suppose genus $p > 1$. Then any integral $F$ of the geodesic flow satisfying condition $\aleph$ must be a constant.*

**Corollary 1.** *There are no real-analytic with respect to momenta integrals for geodesic flows on surface of higher genus other than constants.*

Let me mention that the first condition (K1) of Kozlov's theorem is not required in theorem 8.

Our next results apply to the case of the Riemannian 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

**Theorem 9.** *Let $\Sigma = \mathbb{T}^2$. Let $F$ be a* non-constant *integral of the geodesic flow satisfying condition $\aleph$. Then there exists an invariant 2-torus $L$ of the geodesic flow lying in the regular level of $F$ which is a smooth section of $T_1\mathbb{T}^2$.*

In fact we can specify in the following way:

**Corollary 2.** *The torus $L$ in the theorem can be chosen in such a way that all the orbits of the flow on $L$ project to minimal closed geodesics of the same homotopy type.*

One can deepen theorem 9 in the following way, which can be interpreted as non-existence of the "instability" zones:

**Theorem 10.** *Let $F$ be a non-constant integral of the geodesic flow satisfying condition $\aleph$. Let $N$ be a domain in $T_1\mathbb{T}^2$ bounded by two disjoint sections $L_1, L_2$ which are singular invariant torii of the geodesic flow. Then there exists a regular torus $L$ lying inside $N$ which is a section invariant under the flow.*

**Definition 11.** *By a separatrix chain we mean a closed subset $X \subset T_1\mathbb{T}^2$ bounded by two different singular torii $L_+, L_-$ such that both of them are Lipshits sections invariant under the geodesic flow such that the intersection $L_+ \cap L_-$ equals the union of all periodic minimizing trajectories of a common rational rotation number. No other invariant sections are allowed to pass inside $X$.*

**Theorem 12.** *Let $F$ be a non-constant integral of the geodesic flow of the 2-torus satisfying condition $\aleph$. There are at most countably many separatrix chains. Through any point in the complement of their union passes a unique invariant section which is either regular or singular. In case it is singular, it is a limit from both sides of regular invariant sections. The complement to the union of separatrix chains is $C^0$ foliated by invariant sections.*

For example, there are no separatrix chains at all if the metric is flat (and in fact only in this case, by a theorem of E. Hopf). There are precisely two chains for rotationally symmetric Riemannian metrics and four of them for Liouville metrics. The number of these separatrix chains (it is always even, due to the symmetry of the metric) corresponds to the number of non-smooth points for the ball of the stable norm on $H_1(\mathbb{T}^2;\mathbb{R})$. This connection with the stable norm follows from Bangert's result.

**Corollary 3.** *If the integral $F$ is assumed to be real-analytic in momenta then there are at most finitely many separatrix chains.*

## Quasi-periodic perturbation of unimodal maps

Kristian Bjerklöv

We report on new results concerning the dynamics of quasi-periodically perturbed unimodal maps. The study is divided into two different models.

In the first model we consider perturbations of the quadratic map

$$f(x) = \frac{3}{2}x(1-x).$$

The unperturbed map $f$ has an attracting fixed point $x = 1/3$, which attracts the whole interval $(0,1)$. We study the one-parameter family

$$\Phi_\alpha : \mathbb{T} \times [0,1] \to \mathbb{T} \times [0,1] : (\theta, x) \mapsto (\theta + \omega, c_\alpha(\theta)p(x))$$

where

$$c_\alpha(\theta) = \frac{3}{2} + \frac{5}{2}\left(\frac{1}{1+\lambda(\cos 2\pi(\theta - \alpha/2) - \cos \pi\alpha)^2}\right)$$

and

$$p(x) = x(1-x).$$

Moreover, we assume that the frequency $\omega$ satisfies the Diophantine condition

$$\inf_{p\in\mathbb{Z}} |q\omega - p| > \frac{\kappa}{|q|^\tau} \quad \text{for all } q \in \mathbb{Z}\setminus\{0\}$$

for some constants $\kappa, \tau > 0$. Note that the function $c_\alpha(\theta)$ is close to $3/2$ except for $\theta$ close to 0 or $\alpha$, provided that $\lambda$ is large.

We introduce the notation $(\theta_n, x_n) = \Phi_\alpha^n(\theta, x)$.

**Theorem 1.** [1] For all $\lambda > 0$ sufficiently large there is a parameter value $\alpha$ such that the following holds for the map $\Phi_\alpha$:

1) For almost all $\theta \in \mathbb{T}$ and all $x \in (0,1)$ we have $x_n > 0$ for all $n > 0$ and $\inf_n x_n = 0$.
2) There exists a measurable function $\psi : \mathbb{T} \to (0,1)$ with a $\Phi_\alpha$-invariant graph, i.e.,
$$\psi(\theta + \omega) = c_\alpha(\theta)p(\psi(\theta)) \quad \text{for a.e. } \theta \in \mathbb{T}.$$
3) $|x_n - \psi(\theta_n)| \to 0$ exponentially fast as $n \to \infty$ for almost all $\theta \in \mathbb{T}$ and all $x \in (0,1)$.

In other words, the graph of $\psi$ attracts almost all points in $\mathbb{T} \times (0, 1)$. Moreover, from condition 1) it immediately follows that $\psi$ cannot be continuous. In fact, the set $\{\theta \in \mathbb{T} : \psi(\theta) < \varepsilon\}$ is dense in $\mathbb{T}$ for all $\varepsilon > 0$. The graph of $\psi$ is an example of a so-called Strange Nonchaotic Attractor (SNA). Strange in the sense of the geometry, and non-chaotic because of the non-positive Lyapunov exponents.

In the second model we investigate perturbations of unimodal maps $h : [0, 1] \to [0, 1]$ exhibiting an attracting period-3 point. By the Sharkovskij theorem, we know that such a map $h$ must have periodic points of all orders. The conditions we need on $h$ are (roughly) i) $h(0) = h(1) = 0$, ii) the critical point is non-degenerate and iii) $|h'(x)|$ is sufficiently large outside the immediate basin of attraction of the 3-cycle.

We let

$$\Phi : \mathbb{T} \times [0, 1] \to \mathbb{T} \times [0, 1] : (\theta, x) \mapsto (\theta + \omega, c(\theta)h(x))$$

where

$$c(\theta) = 1 + \frac{\kappa - 1}{1 + \lambda \sin^2 \pi\theta}.$$

In this study the base frequency $\omega$ will act as the parameter.

We present the following preliminary result [2]: For an open set of $\kappa > 1$ and for all $\lambda > 0$ sufficiently large there is a set $\Omega \subset \mathbb{T}$ (with $\mathrm{Leb}(\mathbb{T} \setminus \Omega) \to 0$ as $\lambda \to \infty$) such that for all $\omega \in \Omega$ we have:

1) There exists a measurable function $\psi : \mathbb{T} \to (0, 1)$ with a $\Phi$-invariant graph, i.e.,

$$\psi(\theta + \omega) = c(\theta)h(\psi(\theta)) \quad \text{for a.e. } \theta \in \mathbb{T}.$$

2) $|x_n - \psi(\theta_n)| \to 0$ exponentially fast for a.e. $(\theta, x) \in \mathbb{T} \times (0, 1)$.
3) For all $\theta \in \mathbb{T}$ there is an $x \in (0, 1)$ such that

$$\left| \frac{\partial x_n}{\partial x} \right| > e^n \quad \text{for all } n \geq 0.$$

Note that although the unperturbed map $h$ has an attracting 3-cycle, most of the dynamics of the map $\Phi$ is attracted to a single graph. As in the previous model, the function $\psi$ is highly discontinuous.

Unfortunately our conditions on the unperturbed map $h$ are not satisfied by any quadratic map $x \mapsto ax(1 - x)$. It would be interesting to know if similar results hold in that case.

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**Hill's determinant of a periodic orbit**

Sergey Bolotin

In his study of lunar orbits, Hill [2] encountered the equation $\ddot{x} = a(t)x$, $t$ mod $2\pi$. The behavior of solutions is determined by the eigenvalues $\rho, \rho^{-1}$ of the monodromy matrix. Hill found that

$$\frac{\rho + \rho^{-1} - 2}{e^{2\pi} + e^{-2\pi} - 2} = \det H,$$

where

$$H = \left(\frac{k^2\delta_{jk} + a_{k-j}}{k^2+1}\right)_{j,k\in\mathbb{Z}}, \qquad a(t) = \sum_{k\in\mathbb{Z}} a_k\, e^{ikt}.$$

Hill computed $\rho$ approximately by replacing the infinite matrix $H$ by a $3\times 3$ minor. Poincaré [6] gave a rigorous proof of Hill's formula. Hill's result was almost forgotten by the dynamical systems community until analogs appeared for discrete Lagrangian systems [5, 7, 8] and for continuous Lagrangian systems [1].

Consider a time-periodic Lagrangian system with configuration space $M^m$ and Lagrangian $L(x, v, t)$ on $TM \times (\mathbb{R}/\tau\mathbb{Z})$, strictly convex in the velocity. Periodic orbits are critical points of the action functional

$$A(\gamma) = \int_0^\tau L(\gamma(t), \dot\gamma(t), t)\, dt$$

on the space of $\tau$-periodic curves $\gamma\colon \mathbb{R}/\tau\mathbb{Z} \to M$. The second variation of $A$ at $\gamma$ is a bilinear form on the set $X$ of $\tau$-periodic $W^{1,2}$ vector fields along $\gamma$:

$$h(\xi, \eta) = \int_0^\tau \big((D\xi(t), D\eta(t)) + (U(t)\xi(t), \eta(t))\big)\, dt.$$

Here $(\,\cdot\,,\,\cdot\,) = \langle B(t)\,\cdot\,,\,\cdot\,\rangle$, $B(t) = L_{vv}(\gamma(t), \dot\gamma(t), t)$, and $D$ is a covariant derivative, i.e. a first order linear differential operator such that

$$\frac{d}{dt}(\xi, \eta) = (D\xi, \eta) + (\xi, D\eta)$$

for vector fields $\xi(t), \eta(t)$. Define a scalar product on $X$ by

$$\langle\langle \xi, \eta \rangle\rangle = \int_0^\tau \big((D\xi, D\eta) + (\xi, \eta)\big)\, dt.$$

Then $h(\xi, \bar\eta) = \langle\langle H\xi, \bar\eta\rangle\rangle$. The Hessian operator $H : X \to X$ has the form $H = I + K$, where $K = (-D^2 + I)^{-1}(U - I)$ is compact with eigenvalues $\lambda_k = O(k^{-2})$. Hence Hill's determinant

$$\det H = \prod_{k=1}^{\infty}(1 + \lambda_k)$$

converges absolutely.

Let $P$ be the linear Poincaré map of $\gamma$, and $Q$ the operator of parallel transport around $\gamma$. Let $\sigma = \det Q = \pm 1$ depending on whether $\gamma$ preserves or reverses the orientation. The following generalization of Hill's formula was proved in [1]:

$$\det H = \sigma(-1)^m \frac{e^{m\tau} \det(I - P)}{\det^2(e^\tau I - Q)}.$$

Suppose $\gamma$ is nondegenerate and $\sigma(-1)^{m+\operatorname{ind}\gamma} < 0$. Then we obtain $\det(P-I) < 0$, so the characteristic polynomial $\det(P-\rho I)$ has a real root $\rho > 1$ and $\gamma$ is linearly unstable. If $\gamma$ is degenerate, it is always linearly unstable. If $L$ is defined by a Riemannian metric and $\gamma$ is a closed geodesic, then Hill's formula degenerates to

$$\det H^\perp = \sigma(-1)^{m-1} \frac{e^{(m-1)\tau} \det(I - P^\perp)}{\det^2(e^\tau I - Q^\perp)},$$

where $P^\perp$ is the reduced monodromy operator, and $Q^\perp$ the operator of parallel transport around $\gamma$ of vectors orthogonal to $\gamma$. Thus a nondegenerate closed geodesic has a real multiplier $\rho > 1$ if $\sigma(-1)^{m+\operatorname{ind}\gamma} > 0$ [1, 8]. In [3] this result was obtained using Maslov index theory [4].

Hill's formula follows from a more general result. Fix $\rho \in \mathbb{C}$, $|\rho| = 1$. Let $X_\rho$ be the space of complex $\rho$-quasiperiodic $W^{1,2}$ vector fields $\xi$ along $\gamma$: $\xi(t+\tau) = \rho\xi(t)$. Choose $\ln\rho$ so that $0 \le \operatorname{Im}\ln\rho < 2\pi$ and identify $X_\rho$ with the complexification $X^{\mathbb{C}}$:

$$\xi(t) \in X^{\mathbb{C}} \to e^{\mu t}\xi(t) \in X_\rho, \quad \mu = \tau^{-1}\ln\rho.$$

Then the second variation $h|_{X_\rho}$ defines a Hermitian form $h_\rho$ on $X^{\mathbb{C}}$:

$$h_\rho(\xi,\bar\eta) = h(e^{\mu t}\xi, \overline{e^{\mu t}\eta}) = \int_0^\tau \big(((D+\mu I)\xi, \overline{(D+\mu I)\eta}) + (U\xi,\bar\eta)\big)dt = \langle\langle H_\rho\xi,\bar\eta\rangle\rangle,$$

where

$$H_\rho = (-D^2 + I)^{-1}(-(D+\mu I)^2 + U).$$

Let $P_N$ be the orthogonal projection to the eigenspace of $D$ corresponding to the eigenvalues in $[-N, N]$. Define

$$\det H_\rho = \lim_{N\to\infty} \det P_N H_\rho P_N^*,$$

Then $\det H_\rho$ converges and

$$\det H_\rho = \sigma(-1)^m \frac{e^{m\tau}\det(\rho I - P)}{\rho^m \det^2(e^\tau I - Q)}.$$

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## Area preserving disc maps via pseudoholomorphic foliations

Barney Bramham

We present a new proof of the following:

**Theorem 13.** *Let $\varphi : D \to D$ be a $C^\infty$-smooth, area preserving, diffeomorphism of the closed 2-disc, such that on the boundary circle $\varphi$ coincides with a rotation by an irrational number. Then if $\varphi$ has 2 periodic orbits, it has infinitely many.*

This originally follows from a celebrated result of John Franks [5], who remarkably proved this for maps which are only measure preserving *homeomorphisms* on the *open* disc; in particular without any boundary conditions at all. This was a major breakthrough in low dimensional topology in the early 90's that lead to the final resolution of an old conjecture; namely that $S^2$ equipped with any smooth Riemannian metric has infinitely many relatively prime closed geodesics, see Bangert [4].

In [3] we introduced new tools for disc maps using the theory of pseudoholomorphic curves, inspired by the approach in [6]. One outcome was the construction of infinitely many distinct *transversal foliations* for mapping tori, of which an informal definition is given below; see definition 14 and Figure 1. The proof we outline now of Theorem 13 uses just a few intuitive properties of two such transversal foliations. The details are in [1]. The proof of the, admittedly stronger, result of Franks makes use of Thurston's deep classification theorem of surface homeomorphisms. Suppose that $\varphi$ is an area preserving disc map with the assumptions of Theorem 13 and assume for brevity that $\varphi$ and its iterates are non-degenerate[1]; this assumption is easily removed by an approximation argument. We will derive a contradiction by assuming that $\varphi$ has only a finite number of periodic orbits, yet more than just one.

Without loss of generality, replacing $\varphi$ with an iterate, we may assume that all periodic orbits are fixed points. Let $\beta \in (0, 1)$ be the irrational rotation number of $\varphi$ on the boundary of the disc, where 1 denotes a complete rotation. Fix $n \in \mathbb{N}$ sufficiently large that $0 < \lfloor n\beta \rfloor$ and $\lceil n\beta \rceil < n$. We will only use later from this that

$$\{\lfloor n\beta \rfloor, \lceil n\beta \rceil\} \cap n\mathbb{Z} = \emptyset. \tag{1}$$

[1] We say $\varphi^n$ is non-degenerate if its linearization at any fixed point does not have 1 as an eigenvalue.

We shift the picture to mapping tori. That is, we take a flow on the solid torus $R/\mathbb{Z}\times D$ having $\varphi$ as the first return map on a transversal disc slice, and such that the flow on the boundary of the torus winds around in the meridianal direction a total of $\beta$ during its first lap around the longitude (and not just $\beta$ modulo $\mathbb{Z}$). We refer to this as our mapping torus for $\varphi$. All our references here to "winding" quantities are informal, for a rigorous exposition see [1]. Now we take a lift of this flow to $\mathrm{T}_n := \mathbb{R}/n\mathbb{Z}\times D$, the first return map on $\mathrm{T}_n$ is automatically the $n$th iterate $\varphi^n$. The flow on the boundary of $\mathrm{T}_n$, in the time it takes to return once to the initial disc, winds around the meridian a total angle $n\beta$ (and not just $n\beta$ modulo $\mathbb{Z}$).

**Definition 14.** *A **transversal foliation** of a mapping torus $T$ is a finite collection of periodic orbits we will call the **binding orbits**, and a singular foliation $\mathcal{A}$ whose leaves are embedded annuli transversal to all trajectories that are not binding orbits. $\mathcal{A}$ partitions into $\mathcal{A}_B\cup\mathcal{A}_I$; those leaves which touch the boundary of the mapping torus, and those which remain in the interior. Each leaf in $\mathcal{A}_I$ has boundary on two distinct binding orbits, while a leaf in $\mathcal{A}_B$ has one boundary component on a binding orbit, the other being just a closed loop contained in the boundary of the mapping torus. See also Figure 1.*

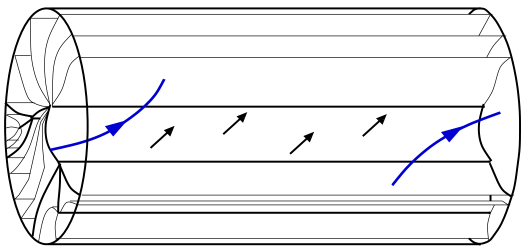

FIGURE 1. A hypothetical transversal foliation of a mapping torus. This example has 5 binding orbits. All other trajectories are strictly transverse to the leaves, as represented by the arrows on one leaf. In contrast to the illustration, the induced foliation of the boundary of the mapping torus will in general consist of closed loops which represent an homology class of the form [longitude] + $k$[meridian] where $k\in\mathbb{Z}$ is not necessarily zero. Indeed $k$ will be chosen to approximate the flow as close as possible.

The main result in [2], also in [3], is that any smooth, non-degenerate, area preserving diffeomorphism of the disc which on the boundary is exactly an irrational rotation, can be suitably represented by a mapping torus for which a pair of distinct *transversal foliations* exist which share a binding orbit, and which on the boundary of the torus follow the flow as closely as is possible from "either side". This is meant in the following weak sense: If the winding of the flow around the boundary of the torus in the time of first return is the non-integer $\alpha$, then the leaves of one, $\mathcal{A}_{slow}$ say, will wind around the meridian $\lfloor\alpha\rfloor$ many times with each lap of the longitude, while the leaves of the other, $\mathcal{A}_{fast}$ say, will wind around the meridian $\lceil\alpha\rceil$ many times with each lap of the longitude.

It turns out that we can arrange that our $n$-periodic mapping torus $\mathrm{T}_n$ chosen above is suitable for this construction, and so we obtain two transversal foliations of $\mathrm{T}_n$. If we denote these $\mathcal{A}_{slow}(\varphi^n)$ and $\mathcal{A}_{fast}(\varphi^n)$, then the leaves within these which touch the boundary of $\mathrm{T}_n$ do so in closed loops which wind around the meridian $\lfloor n\beta\rfloor$, respectively $\lceil n\beta\rceil$, many times with each lap of the longitude.

Away from the binding orbits, the flow is strictly transversal to the leaves, but in "opposite" directions. This gives us the key inequalities. Let $P^*$ denote the (it turns out unique) binding orbit contained simultaneously in both $\mathcal{A}_{slow}(\varphi^n)$ and $\mathcal{A}_{fast}(\varphi^n)$. Let $P$ be any second periodic orbit in $\mathrm{T}_n$ that is distinct from $P^*$ and which also closes up in the time it takes to go once around the longitude. (This is the point at which we use the assumption that $\varphi$ has more than one fixed point.) The pair of transversal foliations gives us a pair of inequalities on the integer $\mathrm{Tw}(P, P^*)$ which we use to denote the number of times $P$ and $P^*$ "twine" around one another in the time it takes to go once around the longitude. This can be made rigorous in the obvious way in terms of the homology class either orbit represents in the complement of the other.

It is easy to show that since the distinguished orbit $P^*$ is a binding orbit of $\mathcal{A}_{slow}(\varphi^n)$, any other orbit such as $P$ either hits a leaf in this transversal foliation in a particular direction or is itself a binding orbit. Either way, this implies

$$\mathrm{Tw}(P, P^*) \geq \lfloor n\beta\rfloor$$

because certain leaves in the foliation connect $P^*$ to the boundary of the mapping torus and "wind around" $P^*$ an amount $\lfloor n\beta\rfloor$ and do so "slower" than the flow.

Similarly, because $P^*$ is a binding orbit of $\mathcal{A}_{fast}(\varphi^n)$, whose leaves in some sense wind around $P^*$ faster than the flow, we are lead to the inequality

$$\mathrm{Tw}(P, P^*) \leq \lceil n\beta\rceil.$$

But we are assuming that the only periodic orbits of $\varphi$ are covers of fixed points, and the mapping torus $\mathrm{T}_n$ was chosen to be the $n$-fold lift of a mapping torus for $\varphi$. It follows that any pair of distinct periodic orbits in $\mathrm{T}_n$ which close up after traversing only once around the longitude, "twine" around one another some integer multiple of $n$. In particular,

$$\mathrm{Tw}(P, P^*) \in n\mathbb{Z}.$$

This contradicts the two inequalities above because we chose $n$ large enough that (1) holds. This completes the proof assuming non-degeneracy.

The extension to the non-degenerate case uses that the "twining number" of pairs of periodic orbits is purely topological. In particular, it does not require non-degeneracy to make sense, and is continuous under uniform convergence in $C^0$; unlike for example the Conley-Zehnder index.

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## Siegel disks with non locally connected boundary

Arnaud Chéritat

In [4], Handel constructed a $C^\infty$ area preserving diffeomorphism of the plane that has a minimal set that is a pseudo circle. His construction is in the spirit of the Anosov-Katok [1]. In [5], Herman adapted the construction to produce a $C^\infty$ diffeomorphism of the sphere that is conjugated to a rotation in the two complementary components of an invariant pseudo circle, and holomorphic in one of them. In the same spirit, in [6] Pérez Marco was able, using tube-log Riemann surfaces, to construct examples of injective holomorphic maps defined in a subset $U$ of $\mathbb{C}$ that have a Siegel disk compactly contained in $U$ whose boundary is a $C^\infty$ Jordan curve, which came as a surprise. Again the method is versatile and Kingshook Biswas used Pérez-Marco's construction to produce a set of interesting examples: [3, 2]. Here we add an ingredient to this construction and get:

**Theorem.** *There exists a holomorphic map $f$ defined in a simply connected open subset $U$ of $\mathbb{C}$ containing the origin, fixing $0$ and having at $0$ a Siegel disk $\Delta$ that is compactly contained in $U$ and whose boundary is a pseudo circle.*

**Remark.** We do not know whether the maps can be chosen entire (or entire meromorphic).

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## A simple proof of KAM

Jacques Féjoz

Let $K^o$ be a germ along $T_0 := \mathbf{T}^n \times \{0\}^n$ of a real analytic Hamiltonian in $\mathbf{T}^n \times \mathbf{R}^n = \{(\theta, r)\}$ for which $T_0$ is invariant and quasiperiodic with diophantine frequency vector $\alpha \in \mathbf{R}^n$. If $H$ is a small perturbation of $K^o$, there exists a Hamiltonian $K$ similar to $K^o$, i.e. of the form $K = c + \alpha \cdot r + O(r^2; \theta)$, an exact symplectomorphism $S$ and a vector $\beta \in \mathbf{R}^n$ such that

$$H = K \circ S + \beta \cdot r.$$

In the spirit of E. Zehnder's work [Z1, Z2], the existence of the Herman-Rüssmann normal form follows from an inverse function theorem between scales of Banach spaces. Yet the inverse function theorem which is needed here is simple [F].

If additionally

$$\int_{T_0} \frac{\partial^2 K^o}{\partial r^2}(\theta, 0)\, d\theta$$

is non degenerate, it follows from the existence of the normal form and from the differentiable dependence of $\beta$ with respect to the translations in the direction of the actions that $H$ has an $\alpha$-quasiperiodic invariant torus which is close to $T_0$ (Kolmogorov's theorem).

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## Limiting Distributions for Horocycle Flows

Giovanni Forni

(joint work with Alexander Bufetov)

We prove results on the existence and non-existence of limiting distributions for horocycle flows on compact hyperbolic surfaces and that all weak limits of probability distributions have non-trivial compact support. The correct normalization of ergodic integrals generally depends on the function, it is always given by a power for time, but it is not necessarily square-root. If the Laplace-Beltrami operator has "small" eigenvalues then for functions outside a subspace of finite codimension the limit exists, if the Laplace-Beltrami operator has no "small" eigenvalues the limit does not exist. The failure of existence of a limiting distribution is caused by (quasi)-periodic behavior in the space of limit distributions, hence it is possible to describe time sequences along which the limit exists. Our results imply in particular that the Central Limit Theorem (CLT) which holds for instance for the geodesic flow on hyperbolic surfaces (and for many other chaotic dynamical systems), does not hold for horocycle flows.

The unit tangent bundle of a compact hyperbolic surface can be identified with a quotient $M := \Gamma\backslash PSL(2,\mathbb{R})$ of the Lie group $PSL(2,\mathbb{R})$ with respect to a co-compact Fuchsian group $\Gamma < PSL(2,\mathbb{R})$.

The geodesic flow $\{\phi_t^X\}$ and the related stable horocycle flow $\{\phi_t^U\}$, unstable horocycle flows $\{\phi_t^V\}$ are given by the action on the right of the 1-parameter subgroup generated by the vector fields $X$, $U$, $V$, which form a basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ satisfying the commutation relations

$$[X,U] = U\,, \quad [X,V] = -V\,, \quad [U,V] = X\,.$$

Let $\mu > 0$ be an eigenvalue of the Laplace-Beltrami operator on the hyperbolic surface. By the theory of unitary representations for the group $PSL(2,\mathbb{R})$, there exists an irreducible component $H_\mu < L^2(M,\mathrm{vol})$ which contains the corresponding eigenfunction. We remark that irreducible components for $\mu \geq 1/4$ belong to the so-called *principal series*, while those for $0 < \mu < 1/4$ belong to the so-called *complementary series*. There is another series of irreducible representations, the so-called *discrete series*. Irreducible representations $H_n$ in the discrete series are not related to the spectrum of Laplacian but to spaces of holomorphic or anti-holomorphic sections of powers of the unit tangent bundle.

The space $L^2(M,\mathrm{vol})$ can be written as a direct orthogonal sum of such irreducible components. By the results of [1], the deviation of ergodic averages (from the mean) for functions which belong to the discrete series is of lower order and cannot be distinguished from the contributions of coboundaries. Thus we are able to analyze limiting distributions only for functions which have a non-zero component in the principal or complementary series. By the results of [1] there are functions which are not coboundaries which belong to the discrete series. For such functions we have no results.

For simplicity we will discuss only the case $\mu \neq 1/4$. Let $\nu := \sqrt{1-4\mu}$. We remark that $\nu \in (0,1)$ if $0 < \mu < 1/4$, while $\nu$ is purely imaginary if $\mu > 1/4$. It is precisely in the latter case that the quasi-periodic behavior of limiting distributions appears.

**Theorem 15.** *(*[1]*, Theorem 3.2) For any $\mu \in \mathbb{R}^+ \setminus \{1/4\}$, the sub-space of invariant distributions for the stable horocycle flow in $C^\infty(H_\mu)'$ has dimension 2; in fact, there exists a basis $\{\mathcal{D}_\mu^+, \mathcal{D}_\mu^-\}$ of distributional eigenvectors for the transfer operator $\mathcal{L}_X$ of the geodesic flow*

$$\mathcal{L}_X \mathcal{D}_\mu^\pm = -\frac{1 \pm \nu}{2}\,\mathcal{D}_\mu^\pm\,.$$

*For any $n \in \mathbb{N}$, the sub-space of invariant distributions for the stable horocycle flow in $C^\infty(H_n)'$ has dimension 1 and it is generated by a distributional eigenvector $\mathcal{D}_n$ of the transfer operator $\mathcal{L}_X$ of the geodesic flow:*

$$\mathcal{L}_X \mathcal{D}_n = -n\,\mathcal{D}_n\,.$$

For any $\mu \in \mathbb{R}^+ \setminus \{1/4\}$, the distributions $\mathcal{D}_\mu^\pm$ have Sobolev order equal to $(1 \pm \Re(\nu))/2$ ( while the distribution $\mathcal{D}_n$ has Sobolev order $n \in \mathbb{N}\setminus\{0\}$). It was proved in [1] that a smooth function is a coboundary for the horocycle flow iff it belongs to the kernel of all $U$-invariant distributions. Hence the horocycle flow is *smoothly stable* in the sense of A. Katok. This property, together with the fact that the horocycle flow is a 'fixed point' for a renormalization group (the geodesic flow), play a crucial role in the analysis of limiting distributions.

Let $s > 0$ be such that all invariant distributions for $\{\phi_t^U\}$ coming from the principal or complementary series have Sobolev order at most $s$ (it suffices to take any $s > 1$). Now take $T > 0$, $x \in \Gamma \setminus PSL(2, \mathbb{R})$ and consider the stable horocycle arc $[x, \phi_T^U x]$. Project the arc $[x, \phi_T^U x]$ onto the space of invariant distributions in the Sobolev space $W^{-s}(SM)$ and let $\alpha_\mu^\pm(x, T)$ be the coefficient at $\mathcal{D}_\mu^\pm$ in the projection. Now denote

$$\beta_\mu^\pm(x, T) = \lim_{t\to\infty} \frac{\alpha_\mu^\pm(g_{-t}x, T\exp(t))}{\exp(\frac{1\mp\nu}{2}t)}. \tag{1}$$

**Lemma 1.** *The limit in the right-hand side of (1) exists.*

For the complementary series ($0 < \mu < 1/4$) we have:

**Theorem 16.** *There exists $\varepsilon > 0$ such that the following holds. Let $f \in C^\infty(H_\mu)$ be any function such that $D_\mu^-(f) \neq 0$. Then*

$$\max_{T\in[0,1]} \Big| \frac{1}{D_\mu^-(f)\exp(\frac{1+\nu}{2}t)} \int_0^{T\exp(t)} f \circ \phi_\tau^U(x) d\tau - \beta_\mu^-(g_t x, T) \Big| = O(\exp(-\varepsilon t)).$$

*Let $f \in C^\infty(H_\mu)$ be any function such that $D_\mu^-(f) = 0$, but $D_\mu^+(f) \neq 0$. Then*

$$\max_{T\in[0,1]} \Big| \frac{1}{D_\mu^+(f)\exp(\frac{1-\nu}{2}t)} \int_0^{T\exp(t)} f \circ \phi_\tau^U(x) d\tau - \beta_\mu^+(g_t x, T) \Big| = O(\exp(-\varepsilon t)).$$

**Corollary 4.** *Let $X_t^\pm(T)$ be the families of random variables with values in $C[0,1]$, given by the formulas*

$$X_t^\pm(T) = \frac{1}{D_\mu^\pm(f)\exp(\frac{1\mp\nu}{2}t)} \int_0^{T\exp(t)} f \circ \phi_\tau^U(x) d\tau\,.$$

*If $f \in C^\infty(H_\mu)$ is any function such that $D_\mu^-(f) \neq 0$, the family $X_t^-$ converges in distribution as $t \to \infty$ to the random variable $\beta_\mu^-(\cdot, T)$. If $D_\mu^-(f) = 0$ but $D_\mu^+(f) \neq 0$, the family $X_t^+$ converges in distribution as $t \to \infty$ to the random variable $\beta_\mu^+(\cdot, T)$.*

For the principal series ($\mu > 1/4$) we have:

**Theorem 17.** *There exists $\varepsilon > 0$ such that the following holds. Let $f \in C^\infty(H_\mu)$ be any function. Then*

$$\begin{aligned}\max_{T\in[0,1]} \Big| \Big(\frac{1}{\exp(\frac{t}{2})} \int_0^{T\exp(t)} f \circ \phi_\tau^U(x) d\tau \\ - \sum_\pm \beta_\mu^\pm(g_t x, T) D_\mu^\pm(f) \exp(\pm\frac{\nu t}{2}) \Big| = O(\exp(-\varepsilon t)).\end{aligned} \tag{2}$$

We remark that since in this case $\nu$ is purely imaginary, by Theorem 17 the ergodic averages of any $f \in C^\infty(H_\mu)$, normalized by the square root of time, converge in distribution to a periodic motion with non-zero frequency $\Im(\nu)/4\pi$. In particular the limit does not exists.

The functions $\beta^\pm_\mu$ defined above are characterized by the following properties:

**Theorem 18.** *The following properties hold for any $\mu \in \mathbb{R} \setminus \{1/4\}$:*

(1) *(Cocycle property) For all $x \in SM$ and for all $S$, $T \in \mathbb{R}$:*

$$\beta^\pm_\mu(x, S+T) = \beta^\pm_\mu(x, S) + \beta^\pm_\mu(h_S x, T)\,;$$

(2) *(Hölder property) There exists a constant $C_\mu > 0$ such that for all $x \in SM$ and for all $T \in \mathbb{R}$,*

$$|\beta^\pm_\mu(x, T)| \le C_\mu T^{\frac{1 \mp \Re(\nu)}{2}}\,;$$

(3) *(Geodesic scaling) For all $x \in SM$, for all $t$, $T \in \mathbb{R}$,*

$$\beta^\pm_\mu(g_{-t}x, T\exp(t)) = \exp(\frac{1 \mp \nu}{2} t)\beta^\pm_\mu(x, T)\,.$$

As A. Avila has remarked at the end of the talk, the above Theorems 16 and 17 immediately imply that for all $\mu \in \mathbb{R}^+ \setminus \{1/4\}$ the functions $\beta^\pm_\mu(\cdot, T) \in H_\mu$. It follows that the the system of functions $\{\beta^\pm_\mu\}$ (defined on $M \times \mathbb{R}^+$) is linearly independent. As a consequence we have obtained a description of the limiting behavior of the distribution of ergodic averages for any sufficiently smooth function which has a non-zero projection on the infinitely dimensional subspace of $L^2(M, \mathrm{vol})$ orthogonal to all irreducible representations coming from the discrete series.

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## A transitive set of positive measure for a nearly integrable system

Vadim Kaloshin
(joint work with Ke Zhang and Yong Zheng)

The famous question called the quasi ergodic hypothesis suggested that for a typical Hamiltonian on a typical energy surface all but a set of zero measure of initial conditions have trajectories covering densely this energy surface itself. However, KAM theory showed that for nearly integrable systems there is a set of initial conditions of positive measure of almost periodic trajectories. This disproved the quasi ergodic hypothesis and forced to reconsider the problem. A weak quasi ergodic hypothesis asks if a typical Hamiltonian on a typical energy surface has a dense orbit. A definite answer whether this statement is true or not is still far out of reach of modern dynamics. There was an attempt to prove this statement by

E. Fermi, which failed. To simplify the weak quasi ergodic hypothesis, M. Herman formulated the following question:

Can one find an example of a $C^\infty$-Hamiltonian $H$ in a $C^r$-small neighborhood of $H_0(I) = \sum_i I_i^2/2$ such that on the unit energy surface there is a dense trajectory?

Many people, including Ehrenfest, Birkhoff, Arnold, believe that such examples do exist and are $C^r$-generic. In this paper we make a step in the direction of answering Herman's question.

For 3 degrees for freedom and any $r$ we construct a $C^\infty$-Hamiltonian, which is $C^r$-close to $H_0(I) = \sum_i I_i^2/2$, so that it has a trajectory dense in a set of positive measure. More exactly,

**Theorem.** Let $\theta \in \mathbb{T}^3$ and $I \in B^3$ in the unit ball $B^3 \subset \mathbb{R}^3$. For any $r \geq 3$ there is a small $C^r$–perturbation $\epsilon H_1(I, \theta, \epsilon)$ such that the Hamiltonian

$$H_\epsilon(I, \theta) = \sum_i I_i^2/2 + \epsilon H_1(I, \theta, \epsilon)$$

on a 5-dimensional energy surface $S = \{H_\epsilon = 1\}$ has an orbit dense in a set $F$ of positive 5-dimensional measure in $S$. Moreover, measure of $F$ tends to zero as $\epsilon \to 0$".

The above theorem improves a previous work of the author, joint with Mark Levi and Maria Saprykina, where a set $F$ has maximal Hausdorff dimension.

## Structure of attractors for $(a, b)$-continued fractions

Svetlana Katok

(joint work with Ilie Ugarcovici, DePaul University, Chicago, IL)

The standard generators $T(x) = x+1$, $S(x) = -1/x$ of the modular group $SL(2, \mathbb{Z})$ were used classically to define piecewise continuous maps acting on the extended real line $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ that lead to various continued fraction algorithms. In this talk we present a general method of constructing such maps, suggested for consideration by Don Zagier, and study their dynamical properties and associated generalized continued fractions.

Let $\Delta$ be the two-dimensional parameter set

$$\Delta = \{(a, b) \mid -1 \leq a \leq 0 \leq b \leq 1, b - a \geq 1\},$$

and consider the map $f_{a,b} : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ given by

$$f_{a,b}(x) = \begin{cases} x - 1 & \text{if } x \geq b \\ -\dfrac{1}{x} & \text{if } a \leq x < b \\ x + 1 & \text{if } x < a\,. \end{cases}$$

It defines what we call $(a,b)$-continued fractions using a generalized integral part function $\lfloor x \rceil_{a,b}$ :

$$\lfloor x \rceil_{a,b} = \begin{cases} \lceil x-b \rceil & \text{if } x \geq b \\ 0 & \text{if } a \leq x < b \\ \lfloor x-a \rfloor & \text{if } x < a\,, \end{cases}$$

where $\lfloor x \rfloor$ denotes the integer part of $x$ and $\lceil x \rceil = \lfloor x \rfloor + 1$.

We proved that any irrational $x$ can be expressed uniquely in the form

$$x = n_0 - \cfrac{1}{n_1 - \cfrac{1}{\ddots}} = \lfloor n_0, n_1, \cdots \rceil_{a,b},\ n_i \neq 0,$$

where $n_0 = \lfloor x \rceil_{a,b}$, $x_1 = -\frac{1}{x-n_0}$ and $n_{i+1} = \lfloor x_{i+1} \rceil_{a,b}$, $x_{i+1} = -\frac{1}{x_i - n_i}$, i.e. the sequence $r_k = \lfloor n_0, n_1, \ldots, n_k \rceil_{a,b} = \frac{p_k}{q_k}$ converges to $x$.

We mention three classical examples: the case $a = -1/2$, $b = 1/2$ gives "the closest-integer" continued fractions considered first by Hurwitz in [2], the case $a = -1$, $b = 0$ described in [6, 3] gives "minus" (backward) continued fractions, while the situation $a = -1$, $b = 1$ was presented in [5, 4] in connection with a method of coding symbolically the geodesic flow on the modular surface following Artin's pioneering work [1].

The main object of our study is the *natural extension map* of $f_{a,b}$,

$$F_{a,b} : \bar{\mathbb{R}}^2 \to \bar{\mathbb{R}}^2$$

(also called the *reduction map*) defined by

$$F_{a,b}(x,y) = \begin{cases} (x-1, y-1) & \text{if } y \geq b \\ (-\dfrac{1}{x}, -\dfrac{1}{y}) & \text{if } a \leq y < b \\ (x+1, y+1) & \text{if } y < a\,. \end{cases}$$

The *attractor* $D_{a,b}$ of the map $F_{a,b}$ is defined starting with a *trapping region*, i.e. a set $\Theta_{a,b} \subset \bar{\mathbb{R}}^2$ with the properties: (i) for every pair $(x,y) \in \bar{\mathbb{R}}^2$, there exists a positive integer $N$ such that $F^N(x,y) \in \Theta_{a,b}$; and (ii) $F(\Theta_{a,b}) \subset \Theta_{a,b}$, the existence of which we proved.

- The attractor $D_{a,b}$ of the map $F_{a,b}$, associated with the trapping region $\Theta_{a,b}$, is defined as $D_{a,b} = \bigcap_{n=0}^{\infty} D_n$, where $D_n = \bigcap_{i=0}^{n} F^i(\Theta_{a,b})$.

The structure of the attractor $D_{a,b}$ is actually "computed" from the data $(a,b)$ as follows. We associate to the points of discontinuity of the map $f_{a,b}$ $a$ and $b$ two forward orbits: to $a$, the *upper orbit* $\mathcal{O}_u(a)$ (i.e. the orbit of $Sa$) and the *lower orbit* $\mathcal{O}_\ell(a)$ (i.e. the orbit of $Ta$), and to $b$, the *upper orbit* $\mathcal{O}_u(b)$ (i.e. the orbit of $T^{-1}b$) and the *lower orbit* $\mathcal{O}_\ell(b)$ (i.e. the orbit of $Sb$).

- We say that $a$ (resp., $b$) has the *cycle property* if the upper and lower orbits meet forming a cycle.
- If the product over the cycle equals the identity matrix we say that the cycle property is *strong*.

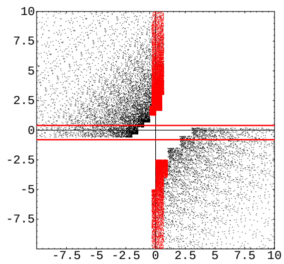


FIGURE 1. A typical attractor

We introduce the *truncated orbits* $\mathcal{L}_a$ and $\mathcal{U}_a$ by

$$\mathcal{L}_a = \begin{cases} \mathcal{O}_\ell(a) & \text{if } a \text{ has no cycle property} \\ \text{lower part of } a\text{-cycle} & \text{if } a \text{ has the cycle property} \end{cases}$$

$$\mathcal{U}_a = \begin{cases} \mathcal{O}_u(a) & \text{if } a \text{ has no cycle property} \\ \text{upper part of } a\text{-cycle} & \text{if } a \text{ has the cycle property} \end{cases}$$

and, similarly, $\mathcal{L}_b$ and $\mathcal{U}_b$.

- We say that $(a, b)$ satisfies the *finiteness condition* if the sets of values in the truncated orbits $\mathcal{L}_a$, $\mathcal{U}_a$, $\mathcal{L}_b$, and $\mathcal{U}_b$ are finite.

The main result of our work is the following theorem:

**Theorem 19.** *If $(a, b) \in \Delta$ satisfies the finiteness condition, the attractor $D_{a,b}$ of the map $F_{a,b}$ has finite rectangular structure, i.e. consists of two (or one, in degenerate cases) connected components bounded by non-decreasing step-functions with finitely many steps, and $F_{a,b} : D_{a,b} \to D_{a,b}$ is a bijection except for some images of the boundary of $D_{a,b}$.*

A typical attractor ($a = -0.8$, $b = 0.5$) is shown in Figure 1.

Notice that generically (almost surely) the finiteness condition comes from the strong cycle property, and in this case we have a stronger result that establishes the *Reduction Theory conjecture* proposed by Don Zagier:

**Theorem 20.** *If both $a$ and $b$ have the strong cycle property, then, in addition to the conclusions of Theorem 1, for every point $(x, y) \in \bar{\mathbb{R}}^2$ there exists an $N > 0$ such that $F^N(x, y) \in D_{a,b}$.*

This result can be used for "reduction" and, ultimately, coding of geodesics on the modular surface if the $(a, b)$-expansion has a so-called "dual". It was done for three classical cases in [4]. The general theory of coding via $(a, b)$-continued fractions is a subject of our paper in preparation.

The structure of the exceptional set $\mathcal{E} \in \Delta$ where the finiteness condition does not hold is also completely understood.

**Theorem 21.** *For any* $(a,b) \in \Delta$, $b \neq a+1$, *the finiteness condition holds. The set of exceptions to the finiteness condition* $\mathcal{E}$ *is an uncountable set of Lebesgue measure* 0 *on the diagonal boundary* $b = a+1$ *of* $\Delta$.

An example of $(a,b) \in \mathcal{E}$ is given by a non-periodic expansion of $b$,

$$b = \lim_{n\to\infty} (0, A_n)$$

where $(0, A_n)$ is a formal finite continued fraction and $A_n$ is a string of "digits" obtained by following "substitution" process for $m \geq 3$:

$$\{A_0 = -m, B_0 = -(m+1); A_{n+1} = A_n, A_n, B_n \text{ and } B_{n+1} = A_n, B_n\}.$$

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## Energy quantization for generalized pseudoholomorphic curves


Samuel Lisi

(joint work with Casim Abbas, Helmut Hofer)


We outline an approach for studying the dynamics of the Reeb vector field on a compact contact 3-manifold $M$. On a 3-manifold $M$, a contact form is a one-form $\lambda$ such that $\lambda \wedge d\lambda$ is a volume form. To each contact form $\lambda$, there is an associated Reeb vector field $R$, defined by the conditions $\lambda(R) = 1$ and $d\lambda(R, \cdot) = 0$. Up to a time reparametrization, the Reeb flow on $M$ is conjugate to the flow of an autonomous Hamiltonian having $M$ as a level set. To study the Reeb flow, we will construct a foliation by surfaces of section. These are obtained as solutions of an elliptic PDE that generalizes the pseudoholomorphic curve equation. These curves are deformations of a simple model foliation. The necessary deformation theory requires good compactness properties for solutions to the PDE. Our main result is an energy quantization that allows us to describe the non-compactness phenomena that occur.

Finite energy foliations have proven very effective in the study of Reeb dynamics [10, 9, 5]. A finite energy foliation is a foliation of $\mathbb{R} \times M$ by pseudoholomorphic curves, with the additional property that the projected curves in $M$ foliate the complement of a finite collection of periodic orbits and so that each curve projects to a surface of section.

The method to construct these foliations is by deforming a simple model foliation for a suitably chosen contact form, to obtain a foliation for the contact form of actual interest. The model foliation comes from modifying an open book decomposition supporting the contact structure, constructed by Giroux [7]. The key problem is that, for generic data, the deformation theory only allows the construction of a foliation by genus 0 curves. This follows from the relationship between the Fredholm index for pseudoholomorphic curves and a version of the adjunction formula [11]. By a result of Etnyre [6], however, certain contact manifolds cannot have a genus zero open book decomposition supporting the contact structure. For instance, any open book decomposition for the 3-torus with its standard contact structure must have pages with genus at least one.

Instead, following [8, 2], we consider a new elliptic problem that generalizes the standard pseudoholomorphic curve equation. Let $\lambda$ denote the contact form on $M$. Then, the Reeb vector field $R$ is defined by $\lambda(R) = 1$ and $d\lambda(R, \cdot) = 0$. Denote by $\pi : TM \to \ker\lambda$ the projection along $R$. Choose an almost complex structure $J$ on $\mathbb{R} \times M$, with $s$ denoting the coordinate on $\mathbb{R}$, by :

$$J\partial_s = R \quad \text{and} \quad J : \ker\lambda \to \ker\lambda$$

and by requiring that $J$ be invariant under translation in the $s$ coordinate, and oriented so that $ds^2+\lambda^2+d\lambda(\cdot, J\cdot)$ defines a Riemannian metric on $\mathbb{R}\times M$. Observe that such an almost complex structure splits the tangent space of $\mathbb{R} \times M$ into the sum of two complex line bundles $(\mathbb{R} \oplus \mathbb{R}R) \oplus \ker\lambda$. Let $\Sigma$ be a closed surface, and $\Gamma \subset \Sigma$ be a finite set of punctures. We now look for solutions $(\tilde{u}, j, \gamma)$, where $\tilde{u} = (a, u) : \Sigma \setminus \Gamma \to \mathbb{R} \times M$, $\gamma$ is a harmonic one-form on $\Sigma$ and $j$ is a complex structure on $\Sigma$, satisfying :

$$d\tilde{u} + J(u)d\tilde{u} \circ j + (\gamma \otimes \frac{\partial}{\partial s} + \gamma \circ j \otimes R) = 0,$$

which can also be rewritten as :

$$\begin{aligned} u^*\lambda \circ j &= da + \gamma \\ \pi du + J(u)\pi du \circ j &= 0. \end{aligned}$$

Note that if $\Sigma$ is a sphere, then $\gamma$ must vanish identically, and this equation specializes to the pseudoholomorphic curve equation. This equation has the correct Fredholm index, and formally satisfies all the requirements to provide a theory of finite energy foliations. Furthermore, by a result of Abbas, the pages of Giroux's open book can be deformed so that the pages lift to solutions to this generalized pseudoholomorphic curve equation [1].

The key difficulty in the study of these curves comes from the non-compactness of the space of harmonic forms. Indeed, if a family of solutions has uniformly bounded harmonic forms, the compactness properties are similar to those for standard pseudoholomorphic curves, as in [4]. In the case the harmonic forms are unbounded, these curves can become very wild : standard rescaling analysis gives convergence to a solution whose projection to $M$ has image in an arbitrary trajectory of the Reeb vector field. Indeed, at any given point, there are two possible scenarios :

(i) $||\gamma||_{L^\infty}$ blows up faster than $||\pi du||$

(ii) $||\gamma||_{L^\infty}$ and $||\pi du||$ both blow up, at comparable rates.

In case (i), the standard rescaling analysis gives us a pathological solution : a plane whose image lies in an arbitrary Reeb trajectory. In case (ii), rescaling gives a plane with $\int u^* d\lambda > 0$ and a constant harmonic form, which is then exact. There then exists a lift of the curve to $\mathbb{R} \times M$ that is pseudoholomorphic, but now may have infinite energy while having $\int_{\mathbb{C}} u^* d\lambda$ finite.

**Theorem 22** (Energy quantization [3])**.** *Let $M^3$ be a compact contact manifold with contact form $\lambda$ and adjusted almost complex structure $J$.*

*Suppose $\tilde{u} : \mathbb{C} \to \mathbb{R} \times M$ is a pseudoholomorphic plane with $0 < \int_{\mathbb{C}} u^* d\lambda < +\infty$. Then, there exists a periodic orbit of the Reeb vector field $R$ whose period $T$ satisfies*

$$T \leq \int_{\mathbb{C}} u^* d\lambda.$$

The key consequence of this theorem is that if $||\gamma||_{L^\infty}$ is unbounded, then case (i) above happens at all but finitely many points. In the case that these curves come from a finite energy foliation (i.e. corresponding to a foliation of $M$ by surfaces of section), an intersection-theoretic argument gives an *a priori* bound on $\gamma$, showing that the wildly non-compact behaviour described above is avoided.

The proof of Theorem 22 uses a construction of a *renormalization* of the potentially infinite $E$ energy plane. In the case the almost complex structure $J$ can be chosen to be flow invariant, a finite energy plane may be constructed from the original infinite energy plane by moving it in the Reeb flow. This case of a flow invariant almost complex structure is highly restrictive, however : it is prevented by the existence of hyperbolic orbits. The key construction in the proof of Theorem 22 is to consider a sequence of moving boundary value problems for which the Fredholm theory has automatic transversality and for which a compactness theory can be developed. Heuristically speaking, this shows a partial persistence of the features of the flow-invariant case.

The main step now missing from this outlined construction is a more refined intersection theory for generalized pseudoholomorphic curves. Some first steps in this direction are hinted at in [11]. The full details of this are a work in progress. Another area of future work is to understand the correct formulation of this generalized pseudoholomorphic curve equation in a compact symplectic manifold.

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## $C^1$-density of hyperbolicity for Benedicks-Carleson toy models

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(joint work with Carlos (Gugu) Moreira, Enrique Pujals)

Uniform hyperbolicity has been a long standing paradigm of complete dynamical description: any dynamical system such that the tangent bundle over its Limit set (the accumulation points of any orbit) splits into two complementary subbundles which are uniformly forward (respectively backward) contracted by the tangent map can be completely described from a geometrical and topological point of view. Nevertheless, uniform hyperbolicity is a property less universal than it was initially thought: *there are open sets in the space of dynamics containing only non-hyperbolic systems.* Actually, Newhouse showed that for smooth surface diffeomorphisms, the unfolding of a *homoclinic tangency* (a non transversal intersection of stable and unstable manifolds of a periodic point) generates open sets of diffeomorphisms such that their Limit set is non-hyperbolic (see [N1], [N2], [N3]).

To explain his construction, firstly we recall that the stable and unstable sets

$$W^s(p) = \{y \in M : \ dist(f^n(y), f^n(p)) \to 0 \text{ as } n \to \infty\},$$
$$W^u(p) = \{y \in M : \ dist(f^n(y), f^n(p)) \to 0 \text{ as } n \to -\infty\}$$

are $C^r$-injectively immersed submanifolds when $p$ is a hyperbolic periodic point of $f$. Secondly, let us recall that, for a given diffeomorphism $f : M \to M$, we say that $f$ exhibits a *homoclinic tangency* if there is a hyperbolic periodic point $p$ of $f$ such that the stable and unstable manifolds of $p$ have a non-transverse intersection.

It is important to say that a homoclinic tangency is (locally) easily destroyed by small perturbation of the invariant manifolds. To get open sets of diffeomorphisms with persistent homoclinic tangencies, Newhouse considers certain systems where the homoclinic tangency is associated to an invariant hyperbolic set with large fractal dimension. In particular, he studied the intersection of the local stable and unstable manifold of a hyperbolic set (for instance, a classical horseshoe), which, roughly speaking, can be visualized as a product of two Cantor sets whose thickness are large. Newhouse's construction depends on how this fractal invariant varies with perturbations of the dynamics, and actually this is the main reason that his construction works in the $C^2-$topology. In fact, Newhouse argument is based on the continuous dependence of the thickness with respect to $C^2$ perturbations. A similar construction in the $C^1-$topology leading to same phenomena is unknown (indeed, some results in the *opposite* direction can be found in [U]). In this setting, it was conjectured by Smale that

*Axiom A surface diffeomorphisms are open and dense in $Diff^1(M)$.*

In the present report, we'll discuss a recent progress on a special set of maps acting on a two dimensional rectangle. For this special type of systems, we show that, if one deals in $C^2-$topology, there are open sets of diffeomorphisms which are not hyperbolic, while in the $C^1-$topology, the Axiom A property is open and dense. A typical family where the Newhouse's phenomena hold is the so called Hénon maps. In fact, it was proved in [U2] that, for certain parameters of this family, the unfolding of a tangency leads to an open set of non-hyperbolic diffeomorphisms.

Numerical simulations indicate that the attractor of the Hénon map (i.e., the closure of the unstable manifold of its fixed saddle point) has the structure of the product of a line segment and a Cantor set with small dimension (when a certain parameter $b$ is close to zero). Although it is a great oversimplification (and many of the later difficulties on the analysis of Hénon attractors arise because of the roughness of such approximation), this idea gives a very good understanding of the geometry of the Hénon map. As a guide to what follows, it is worth to point out that Benedicks and Carleson [BC] have constructed a model where the point moves on a pure product space $(-1, 1) \times K$ where $K$ is the Cantor set obtained by repeated iteration of the division proportions $(b, 1-2b, b)$ and the dynamics on $(-1, 1)$ is given by a family of quadratic maps: in fact, the dynamical system over $(-1, 1)$ act as a movement on a fan of lines, where each line has its own $x$-evolution, while it is contracted in the $y$-direction.

More precisely, consider a one parameter family $\{f_y\}_{\{y\in[0,1]\}}$ such that

$$f_y : [-1, 1] \to [-1, 1]$$

is a $C^r-$unimodal map verifying that 0 is a critical point and $f_y(0)$ is the maximum value of $f_y$ for all $y \in [0, 1]$. We denote by $\mathcal{U}^r$ the set of families of $C^r-$unimodal maps satisfying the conditions stated above.

Let $k : [0, a] \cup [b, 1] \to [0, 1]$ be a $C^r$ function such that $k(0) = 0 = k(1)$, $k(a) = 1 = k(b)$ and $|k'| > \gamma > 1$. Put

$$K(x, y) = \begin{cases} K_1(y) & \text{if } x \geq 0, \\ K_2(y) & \text{if } x < 0, \end{cases}$$

where $K_1 = (k_{/[0,a]})^{-1}, K_2 = (k_{/[b,1]})^{-1}$.

The bulk of this article is the study of the dynamics of the maps $F : R = ([-1,1] \setminus \{0\} \times [0,1]) \to [-1,1] \times [0,1]$ given by

$$F(x,y) = (f(x,y), K(x,y)) = (f_y(x), K(x,y)). \tag{1}$$

We denote by $\mathcal{D}^r$ the set of such maps $F$ with the "usual" $C^r$-topology.

Now, let us recall that a set $\Lambda$ is called *hyperbolic* for a dynamical system $f$ if it is compact, $f$-invariant and the tangent bundle $T_\Lambda M$ admits a decomposition $T_\Lambda M = E^s \oplus E^u$ invariant under $Df$ and there exist $C > 0$, $0 < \lambda < 1$ such that

$$|Df^n_{/E^s(x)}| \leq C\lambda^n \text{ and } |Df^{-n}_{/E^u(x)}| \leq C\lambda^n \;\; \forall x \in \Lambda, \;\; n \in \mathbb{N}.$$

Moreover, a diffeomorphism is called *Axiom A* if the non-wandering set is hyperbolic and it is the closure of the periodic points. In the sequel, $\Omega(F)$ denotes the non-wandering set and $L(F)$ the limit set.

At this point, we are ready to state our main results:

**Theorem A.** *For $r \geq 2$, there exists an open set $\mathcal{U} \subset \mathcal{D}^r$ such that, for any $F \in \mathcal{U}$, the limit set $L(F)$ of $F$ is not a hyperbolic set.*

On the other hand, in the $C^1-$topology, the opposite statement holds:

**Theorem B.** *There exists an open and dense set $\mathcal{V} \subset \mathcal{D}^1$ such that $\Omega(F)$ is a hyperbolic set for any $F \in \mathcal{V}$.*

Concerning the proof of these results, a fundamental role will be played by certain points of the line $\{x = 0\}$: given $F \in \mathcal{D}^r$, consider $k : [0,a] \cup [b,1] \to [0,1]$ the Cantor map related to $F$ and denote by $K_0$ the Cantor set induced by $k$. For any $y \in K_0$, we call $c_y = (0,y)$ a *critical point* of $F$. In other words, the critical points are dynamical obstructions to hyperbolicity.

The relevance of this concept becomes clear from the following simple remark: it follows from the definition that if $c_y \in L(F)$, then $L(F)$ is not hyperbolic. Closing this report, we give a rough sketch of the proofs of these theorems and some possible generalizations:

*Sketch of proof of theorem A.* The proof of this results consists of a slight modification of Newhouse's argument [N1] (see also [PT]). More precisely, we take $0 < t < 1$, $t$ very close to 1 a real parameter and $a : [0,1] \to [\frac{7}{4}, 2)$ a $C^2$-map such that $|a'(y)| > 0$ for any $y \in [0, \frac{t}{2}] \cup [1 - \frac{t}{2}, 1]$. We define

$$F^t(x,y) = (1 - a(y)x^2, K^t(x,y)), \tag{2}$$

with

$$K^t(x,y) := \begin{cases} (k^t_{/[0,\frac{t}{2}]})^{-1}(y) & \text{if } x \geq 0, \\ (k^t_{/[1-\frac{t}{2},1]})^{-1}(y) & \text{if } x < 0, \end{cases}$$

where $k^t$ is the map

$$k^t(y) = \begin{cases} 2y/t & \text{if } 0 \leq y \leq t/2, \\ 2(1-y)/t & \text{if } 1 - \frac{t}{2} \leq y \leq 1. \end{cases}$$

Also, let $K_0 = K_0^t$ be the Cantor set induced by $k = k^t$.

Firstly, we prove a (simple) lemma saying that the maximal invariant set

$$\Lambda_{\frac{1}{3}} = \Lambda^t_{\frac{1}{3}} = \bigcap_{n\in\mathbb{Z}} (F^t)^n \left( \left\{ [-1, -\frac{1}{3}] \cup [\frac{1}{3}, 1] \right\} \times [0, 1] \right)$$

is hyperbolic. Indeed, it is easy to check that the horizontal direction $\mathbb{R} \cdot (1, 0)$ is the unstable direction and, using the standard cone field criterion, it is possible to construct also the complementary stable direction.

Next, we introduce the *tangency line*

$$L = F^2(\{0\} \times [0, 1]) = \{(1 - a(\frac{t}{2}y), \frac{t^2}{4}y)\}_{y\in[0,1]}.$$

At this stage, we conclude the argument by following Newhouse's scheme: using the geometric features of the $F = F^t \in \mathcal{D}^2$, namely, the fact that the projections of the horseshoe-like subset $\Lambda_{1/3}$ into the tangency line give rise to a pair of $C^{1+\alpha}$-dynamically defined *linked* Cantors set of large thickness, one can apply Newhouse's *gap lemma* to construct a *critical point* inside the limit set of $L(F)$, so that, by our previous discussion, it follows that $L(F)$ is *not* hyperbolic; furthermore, this construction is $C^2$-robust due to the continuity properties of the thickness, so that it suffices to take $\mathcal{U}$ a small neighborhood of $F^t$ in order to complete the proof of theorem A. □

*Sketch of proof of theorem B.* The proof of this theorem is inspired by the $C^1$-density of Axiom A among 1D unimodal maps of the interval. More precisely, we start by showing that the points of the non-wandering set staying away from the critical line $\{x = 0\}$ belong to a hyperbolic set; this is done by proving that any invariant compact set disjoint from the critical line exhibits a *dominated splitting* (via an argument involving the Pliss lemma [Pl]) and then we use theorem B in [PS1] to conclude hyperbolicity (here the simple geometry of the class $\mathcal{D}^1$ helps). Next, we exploit a recent theorem of Moreira [M] about the non-existence of $C^1$-stable intersections of Cantor sets plus the geometry of the maps $F \in \mathcal{D}^1$ to prove a dichotomy for the critical points of a generic $F$: either critical points fall into the basins of a finite number of periodic sinks or they return to some small neighborhood of the critical line. Finally, we prove that the critical points returning close enough to the critical line can be absorbed by the basins of a finite number of periodic sinks after a $C^1$-perturbation; thus, we conclude that the limit set of a generic $F \in \mathcal{D}^1$ is the union of a hyperbolic set with a finite number of periodic sinks, i.e., a generic $F \in \mathcal{D}^1$ is Axiom A. □

Ending the report, let's make a few comments on some possible extensions of theorem B. During the previous argument, we saw that an important role was played by the critical points. In particular, when dealing with more general situations (e.g., $C^1$-diffeomorphisms), it is relevant to have a good notion of *dynamical* critical points (because the usual notion of critical point doesn't make sense for $C^1$ diffeomorphisms). In this direction, a recent work of E. Pujals and F. Rodriguez-Hertz [PRH] proposes an intrinsic notion of dynamical critical points (for *dissipative systems*) such that the absence of such points completely *characterizes* the presence of dominated splittings (and, in view of the theorem B of [PS1], hyperbolicity). Therefore, we strongly believe (in fact this is a work in progress by the same authors of this report) that it is possible to use the previous scheme (combined with the Pujals-Rodriguez-Hertz notion of critical points) to get some further progress towards Smale's conjecture.

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## The structure of the invariant set for constant input error diffusion in acute simplices


Tomasz Nowicki

(joint work with Roy Adler, Grzegorz Świrszcz, Charles Tresser, Shmuel Winograd)


In [1] certain piecewise linear maps were defined in terms of a convex polytope. When the convex polytope is a simplex, the resulting map has a dual nature. On one hand, it is defined on $\mathbb{R}^N$ and acts as a piecewise translation. On the other hand, it can be viewed as a translation on the $N$-torus. What relates its two roles? A natural answer would be that there exists an invariant fundamental set into which all orbits under piecewise translation eventually enter. We prove this for $N = 1$, for acute and right triangles and in higher dimension for acute simplices and generic translations. We leave open the obtuse cases and unrestricted translations. Another open problem is the connectivity of the invariant fundamental set which arises and its shape – is it a (possibly non-convex) polytope?

This problem was motivated by investigation of greedy online algorithms for colour printing.

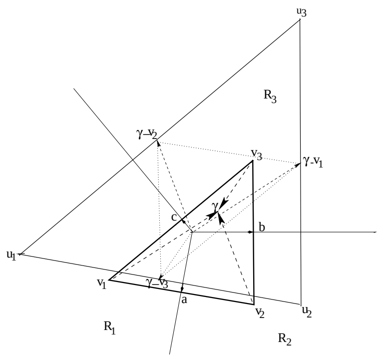


Figure 1. Original, inverted and big simplices. Note the Voronoi regions

Let $P$ be a non degenerate simplex with vertices $v_i$. We can define a partition of the space into Voronoi regions $V_i$ by collecting all the points which are closer (in Euclidean distance) to the vertex $v_i$ than to any other vertex. Some tie breaking rules need to be applied but the results are independent of this choice.

For $\gamma$ an internal point of the simplex define the piecewise translation map by $F_\gamma(x) = x + (\gamma - v_i)$ for $x \in V_i$. We will say that:

(1) the simplex is acute if any two co-dimension 1 faces meet at the acute angle (or that the scalar product of their external normal vectors is negative),
(2) the point is typical if its coordinates are rationally independent,
(3) the set is a fundamental set for a discrete lattice if its lattice translates are disjoint and cover the space,
(4) the set is absorbing if every trajectory eventually enters this set and never leaves.

It is worth observing that the map can be projected onto a torus (space divided by the simplex lattice) where it acts as rotation – translation by a single vector. The typicality condition is represented by ergodicity of this rotation.

**Theorem (Structure of the invariant set)** *For a typical point $\gamma$ in an acute simplex the minimal absorbing set for the trajectories of $F_\gamma$ is a fundamental set with respect to the lattice generated by the edges of the simplex.*

In the fundamental paper [1] it was proven that the trajectories of $F_\gamma$ are bounded for any convex polytope $P$ and the piecewise translations on Voronoi regions, even if we allow the internal point $\gamma$ to vary without restrictions at each iteration. The invariant limit set can be quite large depending on the geometry of the polytope [3]. The dynamics of $F_\gamma$ can be also viewed in a more general context of piecewise isometries. See for example [2], where a similar restriction of typicality was also imposed.

The proof of the Theorem is partially geometrical and partially ergodic. First one constructs a simplex similar to the original one but larger by a factor equal to the dimension of the space. Let $O$ be the common point of the closures of all

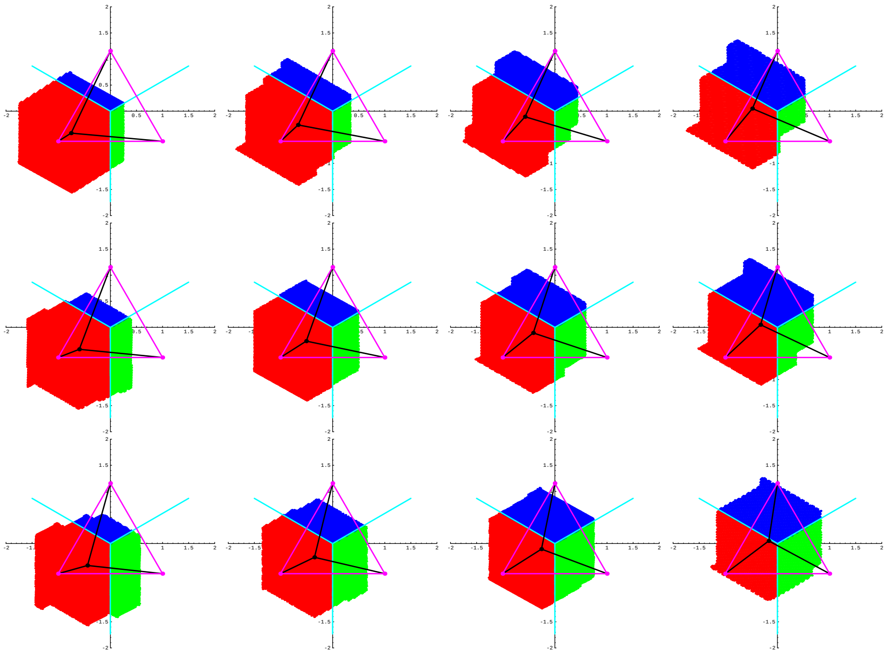

FIGURE 2. Examples of limit absorbing sets for different internal points of an equilateral triangle

Voronoi regions (equidistant to all the vertices). The inverted simplex $O+\gamma-P$ lies inside the large one and there it has no lattice equivalent points. Due to the acuteness condition the large simplex is invariant under $F_\gamma$. Because of ergodicity of the rotation every trajectory on the torus passes through the projection of the inverted simplex hence in the space every trajectory passes through the inverted simplex. Any two lattice equivalent points provide equivalent trajectories which enter the big simplex and collapse to one in the inverted simplex. On the other hand the absorbing set on the torus is the whole torus, so its lift to the space is included in the limit absorbing set, but by the previous reasoning without equivalent duplications.

There is more structure in the limit set.

**Theorem (Structure of the pieces)** *For any piecewise translation generated by an internal point $\gamma$ of a simplex (a partition maybe be different from the Voronoi one), if the limit set is a bounded fundamental set for the simplex lattice, then the intersection of the limit set with any nonempty union of partition elements produces a fundamental set for a lattice which can be explicitly expressed by the point $\gamma$ and a choice of vertices $v_i$.*

The proof is mostly algebraical, taking into account the fact that the map on the limit fundamental set is a bijective piecewise isometry, it uses the notion of return map. This Theorem suggests a possibility of a renormalization theory and some consequences into simultaneous approximations of irrational vectors by coding.

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## Symplectic embeddings of ellipsoids, Diophantine equations, and Fibonacci numbers


Felix Schlenk

(joint work with Dusa McDuff)


For $0 < a_1 < a_2$ consider the open ellipsoid

$$E(a_1, a_2) \;:=\; \left\{ (z_1, z_2) \in \mathbb{C}^2 = \mathbb{R}^4 \;\middle|\; \frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2} < 1 \right\}.$$

We are looking for the smallest ball $B^4(A) := E(A, A)$ into which $E(a_1, a_2)$ symplectically embeds. We can assume that $a_1 = 1$. We therefore would like to understand the function

$$c(a) \;:=\; \inf \left\{ A \mid E(1, a) \text{ symplectically embeds into } B^4(A) \right\}.$$

Since symplectic embeddings are measure preserving, an obvious lower bound for $c(a)$ is $\sqrt{a}$. Upper bounds for $c(a)$ have been found by L. Traynor, [6], and F. Schlenk, [5], via explicit embedding constructions. Recently, in [3], Dusa McDuff has determined $c(k)$ for $k \in \mathbb{N}$ by relating the above problem to the symplectic ball packing problem:

**Theorem $\mathbb{N}$ (McDuff).** *$E(1, k)$ symplectically embeds into $B^4(A)$ if and only if the disjoint union of $k$ balls $B^4(1) \cup \cdots \cup B^4(1)$ symplectically embeds into $B^4(A)$.*

One direction of this theorem is elementary, since it is not hard to symplectically embed $B^4(1) \cup \cdots \cup B^4(1)$ into $E(1, k)$.

The symplectic ball packing problem has a long history, starting with Gromov's work [2], followed by McDuff's and Polterovich's work [4], and culminating with Biran's work [1]. Set

$$c(w_1, \dots, w_k) \;=\; \inf \left\{ A \mid \amalg_{i=1}^k B^4(w_i) \text{ symplectically embeds into } B^4(A) \right\}.$$

It is shown in [1] that

$$c(w_1, \dots, w_k) \;=\; \max \left\{ \sqrt{\sum_i w_i^2}, \quad \sup \left\{ \frac{1}{d} \sum_i w_i m_i \right\} \right\}, \tag{1}$$

where the supremum is taken over all tuples $(d, m_1, \dots, m_k)$ of non-negative integers solving the system of Diophantine equations

$$\begin{cases} \sum_{i=1}^k m_i &= 3d-1, \\ \sum_{i=1}^k m_i^2 &= d^2+1. \end{cases}$$

The numbers $c(1,\dots,1)$ are readily computed, and so $c(k)$ is found.

Since the function $c(a)$ is continuous, it suffices to determine $c(a)$ for $a \in \mathbb{Q}$. There is a version of Theorem $\mathbb{N}$ for $a \in \mathbb{Q}$:

**Theorem $\mathbb{Q}$ (McDuff).** *To $a \in \mathbb{Q}$ one can associate a weight vector $w(a) = (w_1, \dots, w_k)$ such that $E(1,a)$ symplectically embeds into $B^4(A)$ if and only if the disjoint union of $k$ balls $B^4(w_1) \cup \dots \cup B^4(w_k)$ symplectically embeds into $B^4(A)$.*

Finding $c(a)$ now means computing (1). We have meanwhile succeeded in doing this. Let $\tau = \frac{1+\sqrt{5}}{2}$ be the golden ratio. Then the graph of $c(a)$ on $[1, \tau^4]$ is an infinite ladder determined by ratios of consecutive odd Fibonacci numbers: Denote by $g_n$, $n \geq 1$, the sequence of *odd* Fibonacci numbers. Thus the sequence $g_n$ starts with 1, 2, 5, 13, 34, .... Set $g_0 = 1$, and for each $n \geq 0$ define

$$a_n = \left(\frac{g_{n+1}}{g_n}\right)^2 \quad \text{and} \quad b_n = \frac{g_{n+2}}{g_n}.$$

Then $a_0 = 1 < b_0 = 2 < a_1 = 4 < b_1 = 5 < a_2 = \frac{25}{4} = 6\frac{1}{4} < b_2 = \frac{13}{2} = 6\frac{1}{2} < \dots$. More generally,

$$\dots < a_n < b_n < a_{n+1} < b_{n+1} < \dots, \quad \text{and } \lim a_n = \lim b_n = \tau^4.$$

Here is a simplified version of our main result.

**Theorem.**

(i) *For each $n \geq 0$, the function $c$ is constant with value $\sqrt{a_{n+1}}$ on the interval $[b_n, a_{n+1}]$.*

(ii) *For each $n \geq 0$, $c(a) = \frac{a}{\sqrt{a_n}}$ for $a \in [a_n, b_n]$.*

(ii) *For $a \in \left[\tau^4, 7\right]$ we have $c(a) = \frac{a+1}{3}$, and $c(a) = \frac{8}{3}$ on $[7, 7\frac{1}{9}]$.*

(iv) *There are finitely many intervals $I_j \subset [7\frac{1}{9}, 8\frac{1}{36}]$ such that $c(a) = \sqrt{a}$ for all $a \in \left[7\frac{1}{9}, 8\frac{1}{36}\right] \setminus \bigcup_j I_j$. Moreover, $c$ is piecewise linear in each $I_j$.*

(v) *$c(a) = \sqrt{a}$ for $a \geq 8\frac{1}{36}$.*

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**Billiards in ideal hyperbolic polygons**

Karl Friedrich Siburg
(joint work with Simon Castle and Norbert Peyerimhoff)

We present a general conjecture about billiards in ideal hyperbolic polygons, and prove it for several special cases. Our conjecture states that *regular* ideal $k$-gons are characterized by the property that their average length of all periodic trajectories with a fixed combinatorial structure is minimal amongst all ideal $k$-gons.

Consider a polygon $\Pi$ in the hyperbolic Poincaré disk $\mathbb{D}$. $\Pi$ is called *ideal* if all its vertices lie on the boundary $\partial\mathbb{D}$; an ideal $k$-gon $\Pi \subset \mathbb{D}$ is called *regular* if its symmetry group is the full dihedral group $D_k$. A billiard curve $c : \mathbb{R} \to \Pi$ is a piecewise smooth curve, parametrized by arc-length and consisting of geodesic arcs which are reflected at the sides of the polygon. To avoid technical difficulties, we do not consider billiard curves starting or ending in vertices of the polygon. Moreover, we identify billiard curves up to (orientation-preserving) reparametrizations, hence leading to the same trajectory.

A natural way to decode a billiard trajectory is to capture the order in which it hits the polygonal sides. By enumerating the sides of the polygon counter-clockwise from 1 to $k$, every billiard trajectory gives rise to a bi-infinite *billiard sequence* $(a_j)_{j\in\mathbb{Z}}$ with $a_j \in \{1,\dots,k\}$. Note that we identify sequences which are just shifts of each other and denote the set of all those (identified) billiard sequences by $\mathcal{S}(\Pi)$. One can show that, in contrast to Euclidean polygonal billiards, every billiard sequence uniquely determines the corresponding billiard trajectory in the hyperbolic polygon. The method of studying geodesics on hyperbolic surfaces with the help of symbolic dynamics has a long history and turned out to be very successful with beautiful connections to number theory; see, e.g., [6, 1, 9, 8].

Obviously, periodic billiard trajectories correspond to periodic billiard sequences $(a_j)$. A billiard sequence $(a_j)$ with period $n$ is also denoted by $\mathbf{a} = \overline{a_0, a_1, \dots, a_{n-1}}$. Let $\mathcal{S}_{per}(\Pi)$ be the set of all periodic sequences in $\mathcal{S}(\Pi)$. We can associate to $\mathbf{a} = \overline{a_0, \dots, a_{n-1}}$ a closed billiard trajectory which starts and ends at the side $a_0$ and hits the sides of the polygon $\Pi$ in the order $a_1, \dots, a_{n-1}$. The hyperbolic length of this closed piecewise geodesic curve is denoted by $L(\Pi, \mathbf{a})$. In this geometric interpretation, the shift $\overline{a_1, \dots, a_{n-1}, a_0}$ of $\mathbf{a}$ corresponds to the same closed billiard trajectory with a different start point but of the same finite length. As mentioned above, we identify all shifts in the set of all periodic billiard sequences $\mathcal{S}_{per}(\Pi)$. Note that $\mathbf{a}$ not only represents an element in $\mathcal{S}_{per}(\Pi)$, but also contains the information on its period. Thus, $\mathbf{a} = \overline{a_0, a_1, \dots, a_{n-1}}$ and $\mathbf{b} = \overline{a_0, \dots, a_{n-1}, a_0, \dots, a_{n-1}}$ both represent the same element in $\mathcal{S}_{per}(\Pi)$, but we have $L(\Pi, \mathbf{b}) = 2L(\Pi, \mathbf{a})$.

Next we introduce *cyclically related* closed billiard trajectories in a $k$-gon $\Pi$. $\mathbf{a} = \overline{a_0, a_1, \dots, a_{n-1}}$ and $\mathbf{b} = \overline{b_0, b_1, \dots, b_{n-1}}$ are called cyclically related if there is a fixed integer $s \in \mathbb{Z}$ such that

$$b_j \equiv a_j + s \mod k \quad \text{for all } j = 0, 1, \dots, n-1.$$

We write $\mathbf{a} \sim \mathbf{b}$, if $\mathbf{a}$ and $\mathbf{b}$ are cyclically related. Another more geometric way to view cyclically related billiard trajectories is to keep the symbolic encoding $\overline{a_0, \ldots, a_{n-1}}$, but to change the counter-clockwise enumeration of the sides of $\Pi$, i.e., to choose a different side with the label 1. This leads to different closed billiard trajectories in $\Pi$ which have, however, the same combinatorial structure. Note that "shifts" and "being cyclically related" are completely different concepts: for example, in a pentagon, $\overline{1524}$ and its shifts (e.g., $\overline{5241}$) represent the same periodic billiard sequence, whereas $\overline{1524}$ and its cyclically related sequences (e.g., $\overline{2135}$) are different elements in $\mathcal{S}_{per}(\Pi)$.

The *average length* $L_{av}(\Pi, \mathbf{a})$ is defined as the arithmetic mean of the lengths of all closed billiard trajectories $\mathbf{b}$ which are cyclically related to $\mathbf{a}$, i.e.,

$$L_{av}(\Pi, \mathbf{a}) = \frac{1}{k} \sum_{\mathbf{b} \sim \mathbf{a}} L(\Pi, \mathbf{b}).$$

If $\Pi$ is regular, we obviously have $L(\Pi, \mathbf{a}) = L(\Pi, \mathbf{b})$ for cyclically related closed trajectories and, therefore, also $L_{av}(\Pi, \mathbf{a}) = L(\Pi, \mathbf{a})$.

We believe that the following statement is true.

**Conjecture.** *Let $\Pi \subset \mathbb{D}$ be an ideal hyperbolic polygon with $k$ (counter-clockwise enumerated) sides and $\Pi_0$ be a regular ideal $k$-gon (also equipped with an counter-clockwise enumeration of its sides). Let $\mathbf{a} \in \mathcal{S}_{per}(\Pi)$. Then we have*

$$L_{av}(\Pi, \mathbf{a}) \geq L(\Pi_0, \mathbf{a}),$$

*with equality if and only if $\Pi$ is also a regular polygon.*

We confirm this conjecture for special billiard trajectories in quadrilaterals, pentagons and hexagons. Our methods of proof are elementary; we use only basic geometric and algebraic techniques in hyperbolic geometry, such as hyperbolic trigonometry and symmetry arguments. A general proof of our conjecture might be achieved using Teichmüller theory, but this is a project for future research.

It is natural to ask whether regular ideal polygons $\Pi \subset \mathbb{D}$ might have similar *quantum minimality properties*, i.e., whether the bottom $\lambda(\Pi)$ of the Dirichlet spectrum of ideal hyperbolic $k$-gons is minimal if and only if $\Pi$ is regular. For hyperbolic quadrilaterals within a *compact* geodesic ball $B \subset \mathbb{D}$, the minimality property of the regular quadrilateral was proved in [7]. In the case of ideal polygons with arbitrarily many sides, it might be possible to deduce this quantum minimality property from our conjecture via a Selberg-Gutzwiller trace formula argument.

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## The Mysterious Pentagram map

Serge Tabachnikov

This is a report on a joint work with V. Ovsienko and R. Schwartz [2, 3] building on the previous work by Schwartz [4, 5, 6].

Given an $n$-gon $P$ in the projective plane, draw its shortest diagonals and let $T(P)$ be the new $n$-gon whose vertices are the consecutive intersections of these diagonals. The map $T$ is the pentagram map. The pentagram map commutes with projective transformations. Let $C_n$ be the space of $n$-gons modulo projective transformations, and let $P_n$ be the space of twisted $n$-gons modulo projective equivalence. A twisted $n$-gon is a map $\phi : \mathbf{Z} \to \mathbf{RP}^2$ such that $\phi(n+k) = M \circ \phi(k)$ for all $k \in \mathbf{Z}$ and some fixed element $M \in PGL(3, \mathbf{R})$ called the *monodromy.*

Our main result is that the pentagram map $T : P_n \to P_n$ is completely integrable in the sense of Arnold–Liouville. Namely, there exists a $T$-invariant Poisson structure and a complete list of Poisson-commuting integrals for the map.

We associate to every vertex $v_i$ two numbers:

$$\begin{aligned} x_i &= [v_{i-2}, v_{i-1}, ((v_{i-2}, v_{i-1}) \cap (v_i, v_{i+1})), ((v_{i-2}, v_{i-1}) \cap (v_{i+1}, v_{i+2}))] \\ y_i &= [((v_{i-2}, v_{i-1}) \cap (v_{i+1}, v_{i+2})), ((v_{i-1}, v_i) \cap (v_{i+1}, v_{i+2})), v_{i+1}, v_{i+2}], \end{aligned}$$

see Figure 1, where the bracket is the cross ratio of 4 points in $\mathbf{RP}^1$ given by

$$[t_1, t_2, t_3, t_4] = \frac{(t_1 - t_2)(t_3 - t_4)}{(t_1 - t_3)(t_2 - t_4)}.$$

We call these coordinates the *corner invariants.*

The pentagram map is described in these coordinates as follows:

$$T^* x_i = x_i \frac{1 - x_{i-1} y_{i-1}}{1 - x_{i+1} y_{i+1}}, \qquad T^* y_i = y_{i+1} \frac{1 - x_{i+2} y_{i+2}}{1 - x_i y_i}.$$

Consider the *rescaling operation* given by

$$R_t : \quad (x_1, y_1, ..., x_n, y_n) \to (t x_1, t^{-1} y_1, ..., t x_n, t^{-1} y_n).$$

It follows that the pentagram map commutes with the rescaling operation.

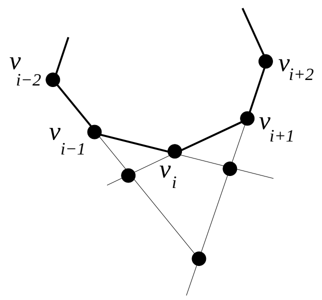


FIGURE 1. Definition of the corner invariants

The Poisson bracket on $P_n$ is as follows:

$$\{x_i, x_{i\pm1}\} = \mp x_i\, x_{i+1}, \qquad \{y_i, y_{i\pm1}\} = \pm y_i\, y_{i+1},$$

and all other brackets of coordinates functions vanish.

There is another coordinate system, assuming that $n \neq 0 \mod 3$. Lift the vertices of a polygon $v_i$ from $\mathbf{RP}^2$ to $\mathbf{R}^3$ so that $\det(V_i, V_{i+1}, V_{i+2}) = 1$. Then

$$V_{i+3} = a_i\, V_{i+2} + b_i\, V_{i+1} + V_i$$

where $a_1, b_1, \ldots, a_n, b_n$ are the new coordinates. Then the monodromy $M$ automatically lies in $SL(3, \mathbf{R})$. The relations between coordinates is as follows:

$$x_i = \frac{a_{i-2}}{b_{i-2}\, b_{i-1}}, \quad y_i = -\frac{b_{i-1}}{a_{i-2}\, a_{i-1}},$$

and the scaling acts as follows: $a_i \mapsto t a_i,\ b_i \mapsto t^{-1} b_i$.

Let

$$N_j = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & b_j \\ 0 & 1 & a_j \end{pmatrix}.$$

Then the monodromy $M = N_0 N_1 \ldots N_{n-1}$. The homogeneous components of Tr $M_n$ are the integrals $I_k$; likewise for $J_k$, applying the involution

$$a_i \mapsto -b_{-i}, \quad b_i \mapsto -a_{-i}.$$

Let $k = [n/2]$; the integrals $I_0, \ldots, I_k, J_0, \ldots, J_k$ are algebraically independent.

The main results concerning the Poisson bracket are:

(1) The Poisson bracket is invariant with respect to the Pentagram map.
(2) The integrals Poisson commute.
(3) The integrals $I_k, J_k$ and, in the even case, $I_0, J_0$, are Casimir functions.
(4) The Poisson bracket has corank 2 if $n$ if odd and corank 4 if $n$ is even.

These results imply that $P_n$ is foliated by symplectic leaves which carry a leafwise $T$-invariant Lagrangian foliation. The leaves of a Lagrangian foliation carry a canonical affine structure, in which $T$ is a parallel translation, cf. [1]. This is the Arnold–Liouville integrability.

To understand the dynamics on $C_n$, one needs to characterize $C_n$ as a subset of $P_n$. Since $M = \mathrm{Id}$ on closed polygons, one has "easy" relations: $\sum I_k = \sum J_k = 3$. Here is more:

$$\sum w(j) I_j = \sum w(j) J_j = 0, \ \sum w(j)^2 (I_j - J_j) = 0$$

where the weights of the integrals are $w(j) = 3j - k$ if $n = 2k$, and $w(j) = 3j - k + 1$ if $n = 2k + 1$. Conjecturally, there are no other univesal relations between the integrals on $C_n$.

We also consider (twisted) polygons inscribed into conics. Computer experiments suggest that, for inscribed polygons, one has $I_k = J_k$ for all $k$; this is an open conjecture.

The subspace of inscribed polygons is a *co-isotropic* submanifold in $P_n$. This implies that the following map of the space of twisted $n$-gons on $\mathbf{RP^1}$ is Poisson: take an inscribed polygon, read off its corner $x$- and $y$-coordinates, and construct a new polygon for which $x$ is the "old" $y$. We make the following conjecture: *this map is completely integrable.* This is true in the simplest case of $n = 3$: one has two independent integrals, one of which is Casimir.

Finally, let $P$ be an inscribed closed heptagon. Then $T(P)$ is projectively self-dual, and $T^2(P)$ is projectively dual to $P$:

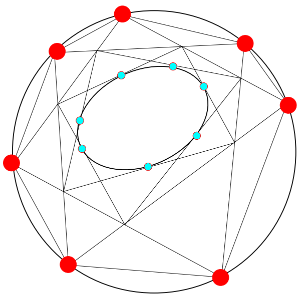

FIGURE 2. Heptagrammum Mysticum

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**Counter-propagating two solitons in the Fermi-Pasta-Ulam model**

C. Eugene Wayne
(joint work with Aaron Hoffman)

The Fermi-Pasta-Ulam (FPU) model is a model of coupled, nonlinear Hamiltonian oscillators. It was studied numerically by Fermi, Pasta and Ulam in an attempt to confirm the equipartition assumption that underlies statistical mechanics. To their surprise they found that the model did not satisfy equipartition and since that time the FPU model has continued to be a source of numerous interesting insights into the behavior of nonlinear dynamical systems. For instance, it was in a continuum approximation to the FPU model that the soliton solutions of the Korteweg-de Vries (KdV) equation were numerically discovered by Kruskal and Zabrusky in 1965.

In collaboration with Aaron Hoffman of Boston University I have recently studied the existence of counter-propagating two soliton solutions for the FPU model with a general potential. Even questions of the existence of (single) solitary waves are in general much more difficult to answer for lattice systems like the FPU model than they are for partial differential equations like the KdV equation since the equation for the traveling wave profile is not an ordinary differential equation like it is for the KdV equation but rather a differential-difference equation. However, recently, through the work of Friesecke and Pego ([1], [2], [3], [4]), Mizumachi ([9]), Mizumachi and Pego ([8]) and the authors, ([7]), the understanding of the existence and stability of solitary waves in the small amplitude, long-wavelength regime (which is essentially the regime that was originally studied numerically by Zabusky and Kruskal) has been worked out in detail. In particular, it has been shown that solitary waves exist and are both orbitally stable in $\ell^2$ as well as asymptotically stable in exponentially weighted norms in which the weight function translates with the solitary wave.

Once the properties of solitary waves are understood, it is natural to ask how two (or more) of these waves interact when they collide. This question is particularly natural in the present context since it was the numerical observation by Zabusky and Kruskal ([11]) that solitary waves in the KdV equation appeared to pass through each other and emerge with no change of form that sparked the interest in completely integrable Hamiltonian partial differential equations. While the FPU model is not believed to be completely integrable, except for very special cases like the Toda model, in certain parameter regimes it is well approximated by completely integrable models like the KdV or Toda models and one can ask to what extent the phenomena that exist in these completely integrable systems, like multi-soliton solutions, persist in small perturbations of these systems. In this sense the question is similar to that asked by the KAM theory though technically the methods used to answer the questions are quite different, largely due to the presence of dispersion in these systems, whereas the partial differential equations to which the KAM theory has been applied have been defined on bounded spatial regions and hence do not exhibit dispersive phenomena.

We prove that counter-propagating solitary waves in the FPU model behave very much like two-solitons in the Toda lattice in that they can be approximated by two solitary waves, with speeds, $c_+(t)$ and $c_-(t)$ for all times $-\infty < t < \infty$, and that as time goes to either plus or minus infinity, the speeds $c_\pm(t)$ approach asymptotic limits. We give precise estimates on the difference between the true solution of the FPU model and the pair of approximating solitary waves and show that this difference is much smaller than the solitary waves themselves.

The method of proof consists of dividing the evolution up into three parts:

- The *pre-interaction regime*: $-\infty < t < -T_0$. In this case the solution is well approximated by a pair of counter-propagating solitary waves. Because the waves are far apart they interact relatively little and this part is easy to control.
- The *interaction regime*: $-T_0 < t < T_0$. In this regime we build on prior work of G. Schneider and myself ([10]) which showed that over finite intervals of time solutions of FPU can be approximated by solutions of a pair of KdV equations and *gives detailed information about the error made in this approximation.*
- The *post-interaction regime*: $T_0 < t < \infty$. In this case the solution is approximated by a pair of solitary waves, perturbed by the additional terms generated in the interaction region.

The last regime is the most difficult to treat. It is in some sense a stability result – we wish to show that the two counter-propagating solitary waves are stable with respect to the perturbation generated by the collision. However, in contrast to a traditional stability result in which one states that a solitary wave is stable with respect to *sufficiently small* perturbations, here one does not have the luxury of making the perturbation small – one is forced to deal with whatever sort of perturbation is generated by the collision. This requires a careful study of the size of the basin of stability of the solitary wave. Furthermore, it turns out that not just the size of the perturbation is important, but also its spatial localization properties and thus we also have to study where (relative to the centers of the solitary waves) the perturbation terms are generated. In the end, we find open sets of initial conditions for which two-soliton like solutions exist. However, in contrast to the two-soliton solutions in the Toda lattice, in these solutions there may be small dispersive "tails" present, in addition to the solitary waves, as time goes to plus and minus infinity. In [6] we have refined our estimates to show that there are special sets of initial data for which the dispersive tails are absent as time goes to minus infinity, but so far it remains an open question as to whether or not, for general FPU potentials, there are solutions in which the dispersive tails are absent for time tending both to plus and minus infinity.

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## Billiards with small holes

Paul Wright
(joint work with Mark Demers and Lai-Sang Young)

Introducing a small hole into the phase space of an ergodic dynamical system causes almost every trajectory to eventually escape. Despite this, such *leaky dynamical systems*, or *dynamical systems with holes*, can have rich dynamics.

We consider the dynamical system arising from a point particle moving at unit speed on a billiard table defined by removing finitely many convex scatterers from $\mathbb{T}^2$. The particle interacts with the boundaries of the scatters via elastic collisions. Next, we introduce a hole $H$ on the table, i.e. a subset through which orbits escape. This leads to the following natural questions:

A. If a particle starts on the table, will it eventually escape?
B. If it does escape, how long will it take to do so?

The answer to Question A is affirmative (for almost every initial condition), because the dynamical system in question is known to be chaotic and ergodic [S]. However, this also means that the system has a sensitive dependence on initial conditions, and so Question B is essentially unanswerable when the initial conditions are fixed.

This motivates us to ask the following questions:

C. If infinitely many particles start on the table according to some distribution $\mu_0$, then at any given time, how much of the initial mass remains?
D. How does $\mu_n$, the distribution obtained by evolving $\mu_0$ forward to time $n$, behave?

E. How do the answers to the above questions change if we vary $H$ or $\mu_0$?

These are the questions we address in our work. As is customary, we reduce the dynamical system under consideration to the billiard map $T : M \circlearrowleft$ by taking $M$ as the cross-section to the flow corresponding to collisions with the boundaries of the scatterers. This is often referred to as the billiard map for a 2-dimensional dispersing periodic Lorentz gas, and we impose the further condition that the table has a finite horizon. The hole $H \subset M$ arises either from the deletion of a segment of the boundary of a scatter or from the deletion of a convex subset on the table away from the scatters. We let $\mathring{T}$ denote the map where orbits disappear when they enter $H$, which must be sufficiently small.

Let $\mu_{\rm phs}$ be the physical (SRB) invariant measure for $T$. $\mu_{\rm phs}$ is the projection of Liouville measure for the flow to the cross-section and has a density with respect to Lebesgue measure. The most basic result is the following.

**Theorem 23** (Escape rate [DWY1])**.** *$\exists \lambda \in (0,1)$ such that*

$$\mu_{phs}\big\{x : T^i x \notin H \text{ for } 0 \le i \le n\big\} \approx \lambda^n$$

*i.e.*

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_{phs}\big\{x : T^i x \notin H \text{ for } 0 \le i \le n\big\} = \log \lambda.$$

$-\log\lambda \in (0,\infty)$ is called the *(exponential) escape rate.*

Beyond the existence of an escape rate, we can in fact describe precisely how $\mathring{T}^n_* \mu_{\rm phs}$, the result of evolving $\mu_{\rm phs}$ forward to time $n$, behaves asymptotically. Clearly, $\mathring{T}^n_* \mu_{\rm phs}$ tends to zero for large $n$, because mass is escaping through the hole. Thus it is necessary to renormalize in order to obtain a nonzero limit.

**Theorem 24** (Limiting distribution [DWY1])**.** *$\mathring{T}^n_*\mu_{phs}/|\mathring{T}^n_*\mu_{phs}|$ converges as $n \to \infty$ to a limiting distribution $\mu_\infty$. $\mu_\infty$ shares the escape rate $-\log\lambda$ and is singular w.r.t. $\mu_{phs}$.*

Observes that $\mu_\infty$ is not an invariant measure for $T$. Rather, it is a *conditionally invariant measure*, which means that it is invariant after normalization, i.e. $\mathring{T}_*\mu_\infty/|\mathring{T}_*\mu_\infty| = \mu_\infty$. Furthermore, our proof shows that in fact $\mu_\infty$ is an *absolutely continuous conditionally invariant measure* (*a.c.c.i.m.*), because, conditioned on unstable leaves, it does have a density with respect to Lebesgue measure. It is natural to compare *a.c.c.i.m.*'s for systems with holes to SRB measures for systems without holes. However, one has to be careful, because given any escape rate, $\mathring{T}$ in fact has unaccountably infinitely many *a.c.c.i.m.*'s with that escape rate and overlapping supports [DY]. As a result, one must inquire if a physically meaningful *a.c.c.i.m.* exists. Our results show that $\mu_\infty$ is such a physically meaningful *a.c.c.i.m.*, because it reflects the asymptotic properties of $\mu_{\rm phs}$.

One may ask if the escape rate and the limiting distribution depend fundamentally on the initial distribution. In fact, our proofs show that they do not: One obtains the same escape rate and limiting distribution for all measures in a large class. This class contains the measures that are uniformly equivalent to $\mu_{\rm phs}$.

Furthermore, we can address the question of what happens when the hole size shrinks to zero.

**Theorem 25** (Dependence on a parameter [DWY1])**.** *Let $H_\varepsilon$, $\varepsilon \geq 0$, be a contracting sequence of holes. Assume*

- $d(H_\varepsilon, H_{\varepsilon'}) \leq |\varepsilon - \varepsilon'|$,
- *Lebesgue*$(H_\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

*Let* $-\log \lambda(\varepsilon)$ *and* $\mu_\infty(\varepsilon)$ *be the escape rate and limiting* a.c.c.i.m. *given by the theorems above. Then*

- $-\log \lambda(\varepsilon)$ *varys (Hölder) continuously with* $\varepsilon$.
- *As* $\varepsilon \to 0$, $\mu_\infty(\varepsilon)$ *converges weakly to* $\mu_{phs}$.

More recently, we have shown that the escape rate can also be determined by a variational principle involving *invariant* measures for $\mathring{T}$. These measures necessarily live on the *survivor set*, the zero-Lebesgue measure set of points whose trajectories never enter $H$.

**Theorem 26** (Variational principle for the escape rate [DWY2])**.** *Let $\mathcal{E}$ be the set of ergodic, invariant measures for $\mathring{T}$ that decay well near the singularities of $\mathring{T}$. Then*

$$\log \lambda = \sup_{\mu \in \mathcal{E}} \left( h_\mu(T) - \Lambda_\mu^+ \right),$$

*where $h_\mu(T)$ is the metric entropy and $\Lambda_\mu^+$ is the positive Lyapunov exponent for $\mu$.*

Our proof involves constructing a Markov extension (Young tower) over the billiard map $T$. Young towers were first constructed for dispersing billiards in [Y]. Our towers have one crucial additional property: $H$ lifts to a countable collection of Markov states on the tower, i.e. the Young tower *respects the hole*. This Markov structure is considerably easier to work with, and indeed results for the existence of an escape rate and a limiting *a.c.c.i.m.* have already been proved for expanding Young towers [BDM] with holes. Once we have constructed our towers to respect $H$, our proof consists of generalizing the results of [BDM] to a hyperbolic setting, and then projecting them back down to $T$. In principle, such tower constructions for dynamical systems with holes can be used to prove similar results for a wide variety of other maps, including the Hénon family, logistic maps, and other dispersing billiards in 2 or more dimensions.

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## Ergodicity of Integrable Hamiltonian Chains

Lai-Sang Young

In this talk we consider nonequilibrium steady states of a class of Hamiltonian chains with $N$ pinned particles and with moving particles running in the gaps between adjacent pinned particles. Energy is exchanged when the particles collide, and the two ends of the chain are connected to (unequal) heat baths. Ergodicity is claimed, and a few issues related to its proof are discussed. A numerical finding to the effect that when forced out of equilibrium, this model violates the principle of local thermal equalilibrium was also reported.

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