Mathematisches Forschungsinstitut Oberwolfach

Report No. 36/2009

DOI: 10.4171/OWR/2009/36

Partielle Differentialgleichungen

Organised by
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August 2nd – August 8th, 2009

Abstract. The workshop dealt with partial differential equations in geometry and technical applications. The main topics were the combination of nonlinear partial differential equations and geometric problems, regularity of free boundaries, conformal invariance and the Willmore functional.

Mathematics Subject Classification (2000): 35 J 60, 35 J 35, 58 J 05, 53 A 30, 49 Q 15.

Introduction by the Organisers

The workshop *Partial differential equations*, organised by Tom Ilmanen (ETH Zürich), Reiner Schätzle (Universität Tübingen), Neil Trudinger (Australian National University Canberra) and Georg S. Weiss (University of Tokyo) was held August 2-8, 2009. This meeting was well attended by 46 participants, including 3 females, with broad geographic representation. The program consisted of 17 talks and 6 shorter contributions and left sufficient time for discussions.
New results combining partial differential equations and geometric problems were presented in the area of minimal surfaces, free boundaries and singular limits. Also there were several contributions to regularity of solutions of partial differential equations.
A major part of the leading experts of partial differential equations with conformal invariance attended the workshop. Here new results were presented in conformal geometry, for the Yamabe problem, Q-curvature and the Willmore functional.
The organisers and the participants are grateful to the Oberwolfach Institute for presenting the opportunity and the resources to arrange this interesting meeting.

Workshop: Partielle Differentialgleichungen

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Abstracts

Degenerate problems with irregular obstacles

Verena Bögelein
(joint work with F. Duzaar and G. Mingione)

In this talk we establish the natural Calderón & Zygmund theory for solutions of elliptic and parabolic obstacle problems involving possibly degenerate operators in divergence form, proving that the gradient of solutions is as integrable as that of the assigned obstacles. More precisely, we are interested in functions $u\colon \Omega_T \to \mathbb{R}$ belonging to the class

$$K = \Big\{ v \in C^0([0,T]; L^2(\Omega)) \cap L^p(0,T; W^{1,p}_0(\Omega)) : v \geq \psi \ \text{ a.e. on } \Omega_T \Big\}$$

and satisfying the variational inequality

$$(1) \qquad \int_0^T \langle \partial_t v, v-u \rangle \, dt + \int_{\Omega_T} a(Du) \cdot D(v-u) \, dz + \tfrac{1}{2} \| v(\cdot,0) - u_0 \|^2_{L^2(\Omega)} \geq 0 \, ,$$

for all $v \in K$ such that $\partial_t v \in L^{p'}(0,T; W^{-1,p'}(\Omega))$, where $p' = p/(p-1)$. Here, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W^{-1,p'}$ and $W^{1,p}$ and $\Omega_T := \Omega \times (0,T) \subset \mathbb{R}^{n+1}$ the parabolic cylinder over a domain $\Omega \subset \mathbb{R}^n$ and $\psi\colon \Omega_T \to \mathbb{R}$ is a given obstacle function with

$$(2) \qquad \psi \in C^0([0,T]; L^2(\Omega)) \cap L^p(0,T; W^{1,p}_0(\Omega)) \quad \text{and} \quad \partial_t \psi \in L^{p'}(\Omega_T).$$

For the initial values u_0 of u we shall assume that

$$(3) \qquad u(\cdot,0) = u_0 \in W^{1,p}_0(\Omega) \quad \text{and} \quad u_0 \geq \psi(\cdot,0).$$

The vector field $a\colon \mathbb{R}^n \to \mathbb{R}^n$ is supposed to be of class C^1, satisfying the following - possibly degenerate - growth and ellipticity assumptions:

$$(4) \qquad |a(w)| + \big(\mu^2 + |w|^2\big)^{\frac{1}{2}} |\partial_w a(w)| \leq L \left(\mu^2 + |w|^2\right)^{\frac{p-1}{2}} ,$$

$$(5) \qquad \partial_w a(w) \widetilde{w} \cdot \widetilde{w} \geq \nu \left(\mu^2 + |w|^2\right)^{\frac{p-2}{2}} |\widetilde{w}|^2 ,$$

for all $w, \widetilde{w} \in \mathbb{R}^n$ and some constants $0 < \nu \leq 1 \leq L$ and $\mu \in [0,1]$. The simplest model we have in mind and which is included by our assumptions is the variational inequality associated to the p-Laplacean operator, i.e. the case $a(Du) = |Du|^{p-2} Du$.

Remark 1. There is a general difference compared to the treatment of parabolic equations, since weak solutions belong to $W^{1,p'}(0,T; W^{-1,p'}(\Omega))$. But this is in general not known for parabolic variational inequalities. For that reason we have to formulate the problem in (1) in its *weak form* not involving the derivative in time $\partial_t u$ and which is obtained by an integration by parts. This is also the reason for the presence of the term involving the initial datum u_0 in (1). On the other hand, when the obstacle is more regular, then the solution indeed lies in $W^{1,p'}(0,T; W^{-1,p'}(\Omega))$ and the *strong form* makes sense.

Let us now state our main result.

Theorem 2. *Let $u \in K$ be a solution of the variational inequality (1) under the assumptions (2) - (5) with*

$$p > \frac{2n}{n+2}\,, \tag{6}$$

and suppose that

$$|D\psi|^p \in L^q_{\mathrm{loc}}(\Omega_T) \qquad \text{and} \qquad |\psi_t|^{\frac{p}{p-1}} \in L^q_{\mathrm{loc}}(\Omega_T)\,,$$

for some $q > 1$. Then $|Du|^p \in L^q_{\mathrm{loc}}(\Omega_T)$. Moreover, there exists a constant $c \equiv c(n, \nu, L, p, q)$ such that for any parabolic cylinder $Q_{z_0}(R) := B_{x_0}(R) \times (t_0 - R^2, t_0 + R^2)$ with $Q_{z_0}(2R) \subseteq \Omega_T$ there holds

$$\left[\fint_{Q_{z_0}(R)} |Du|^{pq}\, dz \right]^{\frac{1}{q}} \le c \left[\fint_{Q_{z_0}(2R)} |Du|^p\, dz + 1 + \left[\fint_{Q_{z_0}(2R)} \left(|D\psi|^{pq} + |\psi_t|^{\frac{pq}{p-1}} \right) dz \right]^{\frac{1}{q}} \right]^d , \tag{7}$$

where

$$d := \begin{cases} \frac{p}{2} & \text{if } p \ge 2 \\ \frac{2p}{p(n+2)-2n} & \text{if } p < 2\,. \end{cases}$$

Note that assumption (6) is unavoidable, since - even in the absence of obstacles - solutions cannot enjoy the regularity prescribed by Theorem 2. The a priori estimate (7) fails in being a reverse type homogeneous inequality in the case $p \neq 2$, by the presence of the natural exponent d, which has to be interpreted as *the scaling deficit of the problem.* This quantity already appears in the a priori estimates relative to the homogeneous parabolic p-Laplacean equation

$$\partial_t u - \operatorname{div}\left(|Du|^{p-2} Du \right) = 0\,.$$

The peculiarity of this equation is that the diffusive and the evolutionary parts possess a different scaling unless $p = 2$. Indeed, it is easily seen that multiplying a solution by a constant does not produce a solution of a similar equation and this clearly reflects in the fact that the a priori estimates available cannot be homogeneous, exactly as in (7). We also observe that when $p > 2$ the scaling deficit d is exactly given by the ratio between the scaling exponents of the evolutionary and the elliptic part, that is $p/2$. In the case $p < 2$, we have that $d \to \infty$ as p approaches the lower bound in (6) and this reflects the failure of estimates as (7) for $p \le 2n/(n+2)$. On the other hand, for elliptic variational inequalities, that is for the stationary case, the corresponding estimate we obtain indeed is homogeneous and takes the form

$$\left[\fint_{B_{x_0}(R)} |Du|^{pq}\, dx \right]^{\frac{1}{q}} \le c \fint_{B_{x_0}(2R)} (|Du| + \mu)^p\, dx + c \left[\fint_{B_{x_0}(2R)} |D\psi|^{pq}\, dx \right]^{\frac{1}{q}}.$$

Note that also the additive constant 1 is replaced by μ coming from the growth and ellipticity (4) and (5) of the vector field $a(\cdot)$.

Let us emphasize that we do not impose any growth assumption in time on the obstacle ψ. This is typically needed in the literature when dealing with irregular obstacles; for instance the obstacle is not allowed to increase in time. We also have proved an existence theorem for irregular obstacles including the ones considered above; particularly obstacles are not necessarily considered to be non-increasing in time.

The results presented are joint work with F. Duzaar and G. Mingione.

Inequalities on CR Manifolds and Nilpotent Groups

Sagun Chanillo

We first describe the results in [1] obtained jointly with J. Van Schaftingen and concerns sub-elliptic analogs of the Bourgain-Brezis inequalities. The Bourgain-Brezis inequalities we are concerned with are as follows.

Let Γ be a smooth, closed curve in $\mathbf{R}^n$. Let $\mathbf{T}$ denote the unit tangent vector to Γ. Then for any vector field $\mathbf{f}$ we have

$$|\int_\Gamma \mathbf{f}\cdot\mathbf{T}ds| \leq c(n)|\Gamma|\ ||\nabla\mathbf{f}||_n \tag{1}$$

(1) is actually an elementary consequence of another remarkable inequality obtained by Bourgain-Brezis. This inequality states:

Let $\mathbf{g}$ have vanishing distributional divergence. That is for all $\phi \in C_0^\infty(\mathbf{R}^n)$, assume,

$$\int_{\mathbf{R}^n} \mathbf{g}\cdot\nabla\phi = 0$$

Then,

$$|\int_{\mathbf{R}^n} \mathbf{f}\cdot\mathbf{g}| \leq c(n)||\mathbf{g}||_1\ ||\nabla\mathbf{f}||_n \tag{2}$$

A consequence of (2) is:

Let $\mathbf{g}$ have vanishing distributional divergence. Furthermore let,

$$\Delta\mathbf{u} = \mathbf{g}.$$

Then,

$$||\nabla\mathbf{u}||_{n/(n-1)} \leq c(n)||\mathbf{g}||_1.$$

We now describe the results in [1]. We shall consider a connected, nilpotent Lie group G with Lie algebra $\mathcal{G}$, satisfying the properties that,

(a) $\mathcal{G} = \oplus_{i=1}^p V_i$

(b) $[V_i, V_j] \subseteq V_{i+j},\ i+j \leq p,\ [V_i, V_j] = \{0\},\ i+j > p.$

(c) V_1 generates $\mathcal{G}$ via Lie brackets.

We also introduce the homogeneous dimension Q,

$$Q = \sum_{j=1}^{p} j \dim V_j .$$

Set $\dim V_1 = m$ and select a basis $\{Y_1, \dots, Y_m\}$ for V_1. Let $\mathbf{g} = (g_1, g_2, \dots, g_m)$ and $\mathbf{f} = (f_1, f_2, \dots, f_m)$. Assume furthermore that $\mathbf{f}$ has vanishing distributional divergence, that is

$$\int_G \sum_{i=1}^{m} f_i Y_i \phi = 0 \tag{3}$$

for all test functions ϕ on G. Then,

$$|\int_G \mathbf{f} \cdot \mathbf{g}| \leq c||\mathbf{f}||_1 \ || \sum_{i,j=1}^{p} |Y_i g_j| ||_Q \tag{4}$$

A consequence of (4) is as follows. Consider the Hörmander sums of squares operator Δ_b, where

$$\Delta_b = \sum_{i=1}^{m} Y_i^2 .$$

Then for any $\mathbf{f}$ satisfying the vanishing divergence condition (3), if

$$\Delta_b \mathbf{u} = \mathbf{f}$$

we have,

$$|| \sum_{i,j=1}^{m} |Y_i u_j| \ ||_{Q/(Q-1)} \leq c||\mathbf{f}||_1 .$$

We next describe joint work with Paul Yang [2]. We consider a smooth manifold M^3, equipped with a contact form θ, which satisfies the condition

$$\theta \wedge d\theta \neq 0.$$

We assume that the distribution $\Xi = \ker \theta$, which is the contact plane has an almost complex structure defined on it. We then have the Levi metric formed by using $d\theta$ and a connection introduced by Webster and Tanaka. These manifolds are called CR manifolds. Using this connection we have the notion of Webster curvature W. Our work [2] starts by studying the eqn. of geodesics introduced by Rumin. We first compute the Jacobi field eqns. associated with varying the geodesics. Then we can use our Jacobi fields to compare the volumes of balls in arbitrary manifolds with the volumes of balls on manifolds with constant Webster curvature. The constant curvature spaces being, $W = -1$ which is $SL(2, \mathbf{R})$, $W = 0$ which is the Heisenberg group and $W = 1$ which is the sphere S^3. As an application of our volume comparison results we have an isoperimetric inequality. To state our inequality we need some definitions. The volume element for our CR manifold is given by:

$$\theta \wedge d\theta$$

Next given a boundary $\partial\Omega$ of a domain Ω, we denote by e_1 the unit tangent vector to both $\partial\Omega$ and Ξ. Let e^1 denote the dual to e_1 in the duality bracket between tangent vectors and co-tangent vectors. Then the area element for $\partial\Omega$ is given by

$$\theta \wedge e^1.$$

Theorem 2: Let M^3 be a simply-connected, complete CR manifold with vanishing torsion tensor. Assume the Webster curvature is non-positive, i.e $W \leq 0$. Then for any domain $\Omega \subseteq M$, we have,

$$\mathrm{vol}(\Omega) \leq c|\partial\Omega|^{4/3}.$$

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On the stability of small crystals under exterior potentials

Alessio Figalli

The anisotropic isoperimetric inequality arises in connection with a natural generalization of the Euclidean notion of perimeter: given an open, bounded, convex set K of $\mathbb{R}^n$ containing the origin, define a "dual norm" on $\mathbb{R}^n$ by $\|\nu\|_* := \sup\{x \cdot \nu : x \in K\}$. Then, given a (smooth for simplicity) open set $E \subset \mathbb{R}^n$, its anisotropic perimeter is defined as

$$P_K(E) := \int_{\partial E} \|\nu_E(x)\|_* d\mathcal{H}^{n-1}(x).$$

Apart from its intrinsic geometric interest, the anisotropic perimeter P_K arises as a model for surface tension in the study of equilibrium configurations of solid crystals with sufficiently small grains, and constitutes the basic model for surface energies in phase transitions. In the former setting, one is naturally led to minimize $P_K(E)$ under a volume constraint. This is of course equivalent to study the isoperimetric problem (also called Wulff problem [3])

$$\inf\left\{\frac{P_K(E)}{|E|^{(n-1)/n}} : 0 < |E| < \infty\right\}. \tag{1}$$

Introduce the *isoperimetric deficit of E*

$$\delta_K(E) := \frac{P_K(E)}{n|K|^{1/n}|E|^{(n-1)/n}} - 1.$$

This functional measures, in terms of the relative size of the perimeter and of the measure of E, the deviation of E itself from being optimal in (1). The stability problem consists in quantitatively relating this deviation to a more direct notion of distance from the family of optimal sets, which are know to be translations and dilations of K and on which $\delta_K = 0$. In collaboration with F. Maggi and A. Pratelli, we proved the following sharp stability result [2]: if $|E| = |K|$, then

$$C(n)\delta_K(E) \geq \inf_{x\in\mathbb{R}^n}\left(\frac{|E\Delta(x+K)|}{|E|}\right)^2.$$

Here $C(n)$ is an explicit constant, with $C(n) \approx n^7$.

Then, in a joint work with F. Maggi [1] we apply the above improved version of the anisotropic isoperimetric inequality to understand properties of minima arising from the minimization problem

$$E \mapsto P_K(E) + \int_E g, \qquad |E| = m,$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is a given potential. The idea is that for small mass m the surface energy P_K dominates, and so one may apply the stability of the isoperimetric problem so say that a minimum F is (quantitatively) close in L^1 to a dilation of K. By combining this starting point with the minimality of F, we show that closedness holds in a quantitative way in some stronger norms: L^∞ for general K, and $C^{k,\alpha}$ if K is smooth and uniformly convex.

A further property that one would like to show is that, for small masses, all minima are actually convex (with no regularity assumptions on g). This last result is shown to be true in two dimension. Moreover, always in two dimensions we proved that if K is polyhedral, then for m small all minima are polyhedral, and their faces are parallels to the ones of K. This (quite surprising) stability property shows that two dimensional crystals are actually very rigid objects.

References

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Equilibrium configurations of epitaxially strained crystalline films

Nicola Fusco

We present some recent results on the equilibrium configurations of a variational model for the epitaxial growth of a thin film on a thick substrate introduced by Bonnetier–Chambolle in [1]. In the model only two dimensional morphologies are considered corresponding to three-dimensional configurations. The reference configuration of the film is

$$\Omega_h = \left\{z = (x,y) \in \mathbb{R}^2 : 0 < x < b,\, 0 < y < h(x)\right\},$$

where $h : [0,b] \to [0,\infty)$ and its graph Γ_h represents the free profile of the film. Denoting by $u : \Omega_h \to \mathbb{R}^2$ the planar displacement of the film with respect to the reference configuration, the strain is given by

$$E(u) = \frac{1}{2}(\nabla u + \nabla^T u)$$

and the energy associated to a smooth configuration (h, u) is

$$G(h,u) = \int_{\Omega_h} \Big[\mu|E(u)|^2 + \frac{\lambda}{2}(\mathrm{div} u)^2\Big]dz + \sigma\mathcal{H}^1(\Gamma_h)\,,$$

where μ and λ represent the *Lamé coefficients* of the film, σ is the surface tension on the profile (which up to to a rescaling we may assume to be equal to 1) and $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. One seeks to minimize G among all configurations (h, u) such that $h(0) = h(b)$, $u(x, 0) = e_0(x, 0)$, for $0 < x < b$, $e_0 > 0$, $u(b, y) = u(0, y) + e_0(b, 0)$ for $0 < y < b$, satisfying the volume constraint $|\Omega_h| = d > 0$.
However, smooth minimizing sequences may converge to irregular configurations, where the profile h is just a lower semicontinuous function of bounded variation. In particular, the extended graph of h may contain vertical segments and cuts. Let us denote by X the class of all reachable configurations (h, u), i.e. the class of all configurations such that $h : \mathbb{R} \to [0, \infty)$ is a b-periodic lower semicontinuous function of finite total variation in $(0, b)$ and $u \in H^1_{\mathrm{loc}}(\Omega_h; \mathbb{R}^2)$ satisfies the Dirichlet boundary condition $u(x, 0) = e_0(x, 0)$ and the periodicity assumption $u(b, y) = u(0, y) + e_0(b, 0)$. It has been proved in [1] (see also [2] for a variant of the model) that the relaxed energy associated to any pair $(h, u) \in X$ is given by

$$F(h,u) = \int_{\Omega_h} \Big[\mu|E(u)|^2 + \frac{\lambda}{2}(\mathrm{div} u)^2\Big]dz + \mathcal{H}^1(\Gamma_h) + 2\mathcal{H}^1(\Sigma_h)\,,$$

where

$$\Gamma_h = \{(x, y) : 0 \le x < b, h^-(x) \le y \le h^+(x)\},$$
$$\Sigma_h = \{(x, y) : 0 \le x < b,\ h(x) \le y < h^-(x)\}.$$

Here, $h^-(x) = \min\{h(x-), h(x+)\}$, $h^+(x) = \max\{h(x-), h(x+)\}$, and $h(x^\pm)$ denote the right and left limit at x. Notice that in the representation formula for F the *vertical cracks* (contained in Σ_h) are counted twice since they arise as limit of regular profiles. With this formula at hand one has (see [1]) the following existence result.

Theorem 1. *The minimum problem*

$$\min\big\{F(g, v) : (g, v) \in X, |\Omega_g| = d\big\} \tag{1}$$

has always a solution for any $d > 0$.

Let us now discuss the regularity properties of absolute (or local) minimizers for the constrained problem (1). First, we say that an admissible configuration $(h, u) \in X$ is a *local minimizer for F* if there exists $\delta > 0$ such that

$$F(h, u) < F(g, v)$$

for all pairs $(g, v) \in X$, with $|\Omega_g| = |\Omega_h|$, such that $0 < d_H(\Gamma_h \cup \Sigma_h, \Gamma_g \cup \Sigma_g) < \delta$. Here, for any two subsets A, B in $\mathbb{R}^2$, $d_H(A, B) = \inf\{\varepsilon > 0 : B \subset \mathcal{N}_\varepsilon(A)$ and $A \subset$

$\mathcal{N}_\varepsilon(B)\}$, where $\mathcal{N}_\varepsilon(A)$ denotes the ε-neighborhood of A. The use of d_H in measuring how far g is from h is due to the presence of the vertical cracks which are not seen by other kinds of possible distances such as the L^1 or the L^∞ one. However, if h is continuous, requiring that $d_H(\Gamma_h \cup \Sigma_h, \Gamma_g \cup \Sigma_g)$ is small is equivalent to requiring that $\sup\{|h(x) - g(x)| : 0 \leq x \leq b\}$ is small.

In order to state the regularity result proved in [2] we need another definition. We say that $(x, h^-(x))$, $x \in [0, b)$, is an *inward cusp point* if, either $g^-(x) = g^+(x)$ and $g'(x+) = -g'(x-) = +\infty$, or $g^-(x) < g^+(x)$ and $g^-(x) = g(x+)$, $g'(x+) = +\infty$, or $g^-(x) < g^+(x)$ and $g^-(x) = g(x-)$, $g'(x-) = -\infty$. The set of all cusp points in $[0, b)$ will be denoted by $\Sigma_{h,c}$.

Theorem 2. *Let $(h, u) \in X$ be a local minimizer for F. Then*

(i) *cusp points and vertical cracks are at most finite in $[0, b)$, i.e.,*
$$\text{card}\big(\{x \in [0, b) : (x, y) \in \Sigma_h \cup \Sigma_{h,c} \text{ for some } y \geq 0\}\big) < +\infty\,;$$
(ii) *the curve Γ_h is of class C^1 away from $\Sigma_h \cup \Sigma_{h,c}$;*
(iii) *$\Gamma_h \cap \{h > 0\}$ is of class $C^{1,\alpha}$ away from $\Sigma_h \cup \Sigma_{h,c}$ for all $\alpha \in (0, 1/2)$;*
(iv) *let $A := \{x \in \mathbb{R} : h(x) > 0$ and h is continuous at $x\}$. Then A is an open set of full measure in $\{h > 0\}$ and h is analytic in A.*

Notice that statement (ii) of Theorem 2 implies in particular the so-called *zero contact angle condition* (that is $h' = 0$) between the film and the substrate. We remark also that the regularity results in [2] refer to a slightly different model than the one considered here and to a slightly stronger notion of local minimality. However they apply also to the model under discussion.

We come now to the qualitative properties of solutions. The results presented here will appear in the forthcoming paper [3]. A first issue that will be discussed in the paper is to find sufficient conditions, based on a suitable notion of second variation for F, for an admissible configuration to be a local minimizer. To this aim, given a pair $(h, u) \in X$, with $h \in C^2([0, b])$, we say that (h, u) is a *critical point* for F if it satisfies the following set of Euler-Lagrange equation:

$$\begin{cases} \mu\Delta u + (\lambda + \mu)\nabla(\text{div}u) = 0 & \text{in } \Omega_h, \\ N(u)[\nu] = 0 & \text{on } \Gamma_h \cap \{y > 0\}, \\ N(u)(0, y)[\nu] = -N(u)(b, y)[\nu] & \text{for } 0 < y < h(0) = h(b), \\ k + \mu|E(u)|^2 + \frac{\lambda}{2}(\text{div}u)^2 = \text{const} & \text{on } \Gamma_h \cap \{y > 0\}\,, \end{cases} \tag{2}$$

where $N(u) = \mu\big(\nabla u + \nabla^T u\big) + \lambda \text{div} u$, ν is the exterior normal to Ω_h and k is the curvature of Γ_h. From the definition of F one has immediately that any sufficiently smooth local minimizer satisfies (2), hence is a critical point. Notice also that the *flat configuration* (h, u_0) *of volume* d, where

$$h \equiv \frac{d}{b}, \qquad u_0(x, y) = e_0\Big(x, \frac{-\lambda}{2\mu + \lambda} y\Big),$$

is always a critical point, i.e., satisfies (2). The first result proved in [3] deals with the local minimality of the flat configuration. In order to state it we need to introduce the *Grinfeld function* K defined (see [4]) for $y \geq 0$ as

$$K(y) = \max_{n\in\mathbb{N}} \frac{1}{n} J(ny)\,, \quad \text{where } J(y) := \frac{y + (3-4\nu_p)\sinh y \cosh y}{4(1-\nu_p)^2 + y^2 + (3-4\nu_p)\sinh^2 y}\,, \tag{3}$$

ν_p being the *Poisson modulus* of the elastic material, i.e., $\nu_p = \frac{\lambda}{2(\lambda+\mu)}$.

Theorem 3. *Let* $d_{\rm loc} : (0,+\infty) \to (0,+\infty]$ *be defined as* $d_{\rm loc}(b) := +\infty$*, if* $0 < b \leq \frac{\pi}{4}\frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}$*, and as the solution to*

$$K\Big(\frac{2\pi d_{\rm loc}(b)}{b^2}\Big) = \frac{\pi}{4}\frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}\frac{1}{b}\,, \tag{4}$$

otherwise. Then the flat configuration $(d/b, u_0)$ *is a local minimizer for* F *if* $0 < d < d_{\rm loc}(b)$*.*
The threshold $d_{\rm loc}$ *is critical: indeed, for* $d > d_{\rm loc}(b)$ *there exists* $(g,v) \in X$*, with* $|\Omega_g| = d$*, and* $d_H(\Gamma_{d/b}, \Gamma_g \cup \Sigma_g)$ *arbitrarily small such that* $F(g,v) < F(d/b, u_0)$*.*

In particular, if $0 < b \leq \frac{\pi}{4}\frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}$ then the flat configuration is always a local minimizer.
A crucial point in the proof of Theorem 3 is a local minimality criterion, based on the positive definiteness of a suitable notion of second variation of F. To define it, let us consider a critical point $(h,u) \in X$, with $h \in C^\infty([0,b])$, $h > 0$. Given a variation $\psi \in H^1(0,b)$, $\psi(0) = \psi(b)$, with $\int_0^b \psi\,dx = 0$, for $|t|$ small we set $h_t = h + t\psi$ and u_t the corresponding minimizer of the elastic energy in Ω_{h_t} under the usual Dirichlet and periodicity assumptions. Thus $(h_t, u_t) \in X$ and $|\Omega_h| = |\Omega_{h_t}|$. The second variation of F at (h,u) along the direction ψ is then defined as

$$\frac{d^2}{dt^2} F(h_t, u_t)_{|t=0}\,. \tag{5}$$

We say that the second variation at (h,u) is *positive definite* if (5) is positive for all $\psi \neq 0$.

Theorem 4. *Let* $(h,u) \in X$ *be a critical point for* F*, with* $h \in C^\infty([0,b])$ *and* $h > 0$*, and assume that the second variation of* F *at* (h,u) *is positive definite. Then* (h,u) *is a local minimizer.*

To the best of our knowledge, this result is the first example of a local minimality criterion based on the second variation in the framework of free boundary problems and we believe that many of the ideas introduced in [3] can be used in a large number of similar variational problems.
To conclude, we state a result dealing with the global minimality properties of the flat configuration. This theorem, as well as other qualitative properties of non-flat minimizers, is also contained in the forthcoming paper [3].

Theorem 5. *The following two statements hold.*

(i) *For every* $b > 0$*, there exists* $0 < d_{\rm glob}(b) \leq d_{\rm loc}(b)$ *(see Theorem 3) such that the flat configuration* $(d/b, u_0)$ *is a global minimizer if and only*

$0 < d \le d_{\rm glob}(b)$. Moreover, if $0 < d < d_{\rm glob}(b)$, then $(d/b, u_0)$ is the unique global minimizer.

(ii) *There exists $0 < b_{\rm crit} \le \frac{\pi}{4}\frac{2\mu+\lambda}{e_0^2\mu(\mu+\lambda)}$ such that $d_{\rm glob}(b) = +\infty$ if and only if $0 < b \le b_{\rm crit}$, i.e., the flat configuration $(d/b, u_0)$ is the unique global minimizer for all $d > 0$ if and only if $0 < b \le b_{\rm crit}$.*

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New results on elliptic and parabolic diagonal systems

Jens Frehse

We consider systems

$$(1) \qquad (u_t^\nu) - D_i(a_{ik}^\nu D_k u^\nu) + F^\nu(x,u,\nabla u)u^\nu = H^\nu(x,u,\nabla u) \quad \text{in } \Omega \subset \mathbb{R}^n$$

where $F^\nu \ge 0$, $|F^\nu| \le K + K|\nabla u|^2$, $(u \in L^\infty)$, and $|H^\nu| \le K|\nabla u||\nabla u^\nu| + \sum_{i=1}^{\nu} K|\nabla u^i|^2 + K$.

Boundary conditions and uniform ellipticity is assumed, further $a_{ik} \in H^{1,\infty}$.

Such systems arise from stachastic differential games with discount control. They are characterized by a controllable discount factor $e^{-tc(v)}$ in the costfunctional of the players and give rise to the term $F^\nu u^\nu$.

For $n > 3$ there is no existence theory (neither regularity) up to now; for $n = 2$ a paper of the author and Bensoussan (Stampacchia-memory-volume) is presented. We outline the prove of the theorem $u \in L^\infty \Rightarrow u \in C^\alpha \cap H^1$ under the conditions above for arbitrary u and indicate how this is used to reduce the system (1) to the case of a principle part not depending on ν (locally) (Up to now, the latter condition was assumed in the literature). We present an example of a non diagonal 2-d-system of type (1), $H^\nu = 0$, which has a bounded irregular solution. Up to now, there was a theorem of the author that there exists a smooth solution, but it was open whether every solution is regular (even under the condition $F_\nu = 0$).

Relative Minimizers of Energy are Relative Minimizers of Area

Stefan Hildebrandt
(joint work with Friedrich Sauvigny)

Let Γ be a closed, regular Jordan curve in $\mathbb{R}^3$ of class $C^{1,\mu}, 0 < \mu < 1$, and denote by $\mathcal{C}(\Gamma)$ the class of disk-type surfaces $X : B \to \mathbb{R}^3$ with $X \in H^{1,2}(B, \mathbb{R}^3) \cap C^0(\partial B, \mathbb{R}^3)$ and $B = \{w = (u, v) \in \mathbb{R}^2 : |w| < 1\}$ such that the "Sobolev trace" $X\big|_{\partial B}$ maps ∂B monotonically onto Γ. A mapping $X \in C^2(B, \mathbb{R}^3)$ is said to be a minimal surface if it satisfies the equations

$$\Delta X = 0 \tag{1.1}$$

and

$$|X_u|^2 = |X_v|^2, \ \ \langle X_u, X_v\rangle = 0 \ , \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^3$ with the norm $|\cdot|$.

Let $\mathcal{M}(\Gamma)$ be the set of minimal surfaces $X \in \mathcal{C}(\Gamma)$. It is well known that $\Gamma \in C^{1,\mu}$ implies

$$\mathcal{M}(\Gamma) \subset C^{1,\mu} \ . \tag{1.3}$$

Let $\mathcal{M}_{\mathrm{im}}(\Gamma)$ be the subclass of immersed minimal surfaces $X \in \mathcal{C}(\Gamma)$. For any such X there is a constant $\delta_0(X) > 0$ with

$$\Lambda(X) := |X_u|^2 = \frac{1}{2}|\nabla X|^2 \geq \delta_0(X) \ . \tag{1.4}$$

For $X \in H^{1,2}(B, \mathbb{R}^3)$ we define the area $A(X)$ and the Dirichlet integral $D(X)$ respectively by

$$A(X) := \int_B |X_u \wedge X_v| du\, dv$$

and

$$D(X) := \frac{1}{2}\int_B (|X_u|^2 + |X_v|^2) du\, dv \ .$$

We have

$$A(X) \leq D(X) \ \text{ for all } \ X \in H^{1,2}(B; \mathbb{R}^3) \ , \tag{1.5}$$

and the equality sign holds if and only if the equations (1.2) are satisfied a.e. on B. It is well known that

$$\inf_{\mathcal{C}(\Gamma)} A = \ \inf_{\mathcal{C}(\Gamma)} D \ . \tag{1.6}$$

A simple functional-analytic proof of this relation is given in [4]; cf. also [1], Section 4.10.

From (1.6) one infers:

Any minimizer of D in $\mathcal{C}(\Gamma)$ is a minimal surface that minimizes A in $\mathcal{C}(\Gamma)$, and conversely: any minimizer of A in $\mathcal{C}(\Gamma)$ satisfying (1.2) is a minimal surface which minimizes D in $\mathcal{C}(\Gamma)$.

This well-known fundamental result raises the question whether a similar result holds for relative minimizers of A and D. In fact, one direction is fairly obvious:

If $X \in \mathcal{M}(\Gamma)$ is a relative minimizer of A in $\mathcal{C}(\Gamma)$, i.e.

$$A(X) \le A(Y) \ \textit{for all}\ Y \in \mathcal{C}(\Gamma)\ \textit{with}\ ||X - Y|| < \epsilon \tag{1.7}$$

and for some $\epsilon > 0$, then

$$D(X) \le D(Y) \ \textit{for all}\ Y \in \mathcal{C}(\Gamma)\ \textit{with}\ ||X - Y|| < \epsilon \ .$$

The *proof* follows immediately from the relations

$$D(X) \overset{(1.2)}{=} A(X) \overset{(1.7)}{\le} A(Y) \overset{(1.5)}{\le} D(Y) \ \text{ for } \ ||X - Y|| < \epsilon \ .$$

Here we can choose any suitable norm for $|| \cdot ||$, say,

$$|| \cdot ||_{C^0} + \sqrt{D(\cdot)}, \ \ || \cdot ||_{C^1} \ , \ \text{ or } \ || \cdot ||_{C^{1,\nu}} \ \text{ with } \ 0 < \nu \le \mu \ ,$$

where C^0 stands for $C^0(\overline{B}, \mathbb{R}^3), C^{1,\nu}$ for $C^{1,\nu}(\overline{B}, \mathbb{R}^3)$, etc.

The converse is not at all clear; indeed, we do not know whether or not it is true in full generality. We can prove it only for *immersed* $X \in \mathcal{M}(\Gamma)$ which are relative minimizers of D in $\mathcal{C}(\Gamma)$, and for $|| \cdot ||$ we can only take the pair $\{|| \cdot ||_{C^1}, || \cdot ||_{C^{1,\mu}}\}$ as described in Theorem 1. Our proof does not work for the norms $|| \cdot ||_{C^{1,\mu}}$ or even $|| \cdot ||_{C^1}$ or $|| \cdot ||_{C^0} + \sqrt{D(\cdot)}$ alone, and it is an interesting question what is true in these cases.

Let us now state our main result:

Theorem 1. *Let $X \in \mathcal{M}_{\text{im}}(\Gamma)$ be a relative minimizer of D in the following sense: There is an $\epsilon > 0$ such that*

$$D(X) \le D(Z) \ \textit{for all}\ Z \in \mathcal{C}(\Gamma) \cap C^1(\overline{B}, \mathbb{R}^3) \ \textit{with}\ ||Z - X||_{C^1} < \epsilon \ . \tag{1.8}$$

Then there exists a $\delta(\epsilon) > 0$ such that

$$A(X) \le A(Y) \ \textit{for all}\ Y \in \mathcal{C}(\Gamma) \cap C^{1,\mu}(\overline{B}, \mathbb{R}^3) \quad \textit{with}\ ||Y - X||_{C^{1,\mu}} < \delta(\epsilon) \ , \tag{1.9}$$

i.e. X is a relative minimizer of A.

Remark 1. We can rephrase this result as follows: *A relative minimizer X of D with respect to the C^1-norm is a relative minimizer of A with respect to the $C^{1,\mu}$-norm.* Note that the C^1-minimum property is stronger than the $C^{1,\mu}$-minimum property since the C^1-norm is weaker than the $C^{1,\mu}$-norm.

Similarly one can prove

Theorem 2. *Suppose that the immersed minimal surface $X \in \mathcal{C}(\Gamma)$ is of the class $C^2(\overline{B}, \mathbb{R}^2)$, and assume that X is a relative minimizer of D with respect to the $C^{1,\mu}$-norm for some $\mu \in (0, 1)$. Then it is also a relative minimizer of A with respect to the C^2-norm.*

Remark 2. The second variations of A and D coincide even for possibly branched minimal surfaces in $\mathbb{R}^n, n \geq 3$ (cf. Section 1 in [7]).

This paper will be published in one of the forthcoming issues of Calc. Var.

On linear elliptic and parabolic equations with growing drift in Sobolev spaces without weights

Nikolai Krylov

In this talk we concentrate on problems in the whole space for uniformly elliptic and parabolic second-order equations with bounded leading and zeroth-order coefficients and possibly growing first-order coefficients. We look for solutions which are summable to the p-th power with respect to the usual Lebesgue measure along with their first- and second-order derivatives with respect to the spatial variables.

Here for brevity we only give the results for elliptic case.

Let $\mathbb{R}^d$ be a Euclidean space of points $x = (x^1, ..., x^d)$. We consider the following second-order operator L:

$$Lu(x) = a^{ij}(x)D_{ij}u(x) + b^i(x)D_iu(x) - c(x)u(t,x),$$

acting on functions defined on $\mathbb{R}^d$ (the summation convention is enforced throughout). Here

$$D_i = \frac{\partial}{\partial x^i}, \quad D_{ij} = D_iD_j.$$

We are dealing with the elliptic equation

$$Lu(x) = f(x), \quad x \in \mathbb{R}^d. \tag{1}$$

The solutions of (1) are sought in $W^2_p(\mathbb{R}^d)$, usual Sobolev space.

Our main result for elliptic case is Theorem 1 saying that under appropriate conditions the elliptic equation $Lu - \lambda u = f$ is uniquely solvable in $W^2_p(\mathbb{R}^d)$ if λ is large enough. Its proof is based on a corresponding result for parabolic equations. Interestingly enough, even if b is constant we do not know any other proof of Theorem 1 not using the parabolic theory.

For $p \in (1, \infty)$, $p \neq d$, define

$$q = d \vee p,$$

and if $p = d$ let q be a fixed number such that $q > d$.

Assumption 1. (i) The functions a^{ij}, b^i, c are measurable, $a^{ij} = a^{ji}$, $c \geq 0$.

(ii) There exist constants $K, \delta > 0$ such that for all values of arguments and $\xi \in \mathbb{R}^d$

$$\delta|\xi|^2 \leq a^{ij}\xi^i\xi^j \leq K|\xi|^2, \quad c \leq K.$$

(iii) The function $|b|^q$ is locally integrable on $\mathbb{R}^d$.

The following assumptions contain parameters $\gamma_a, \gamma_b \in (0, 1]$ whose value will be specified later. For $\alpha > 0$ we denote $B_\alpha = \{x \in \mathbb{R}^d : |x| < \alpha\}$.

Assumption 2 (γ_b)**.** There exists an $\alpha \in (0,1]$ such that on $\mathbb{R}^d$

$$\alpha^{-d} \int_{B_\alpha} \int_{B_\alpha} |b(x+y) - b(x+z)|^q \, dydz \le \gamma_b.$$

It is easy to check that Assumption 2 is satisfied with any $\gamma_b > 0$ if, for instance, b is such that $|b(x) - b(y)| \le K$ if $|x - y| \le 1$. We see that $|b(x)|$ can grow to infinity as $|x| \to \infty$.

Assumption 3 (γ_a)**.** There exists an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, $x \in \mathbb{R}^d$, and $i, j = 1, ..., d$ we have

$$\varepsilon^{-2d} \int_{B_\varepsilon} \int_{B_\varepsilon} |a^{ij}(x+y) - a^{ij}(x+z)| \, dydz \le \gamma_a. \tag{2}$$

Obviously, the left-hand side of (2) is less than

$$N(d) \sup_{|x-y| \le 2\varepsilon} |a^{ij}(x) - a^{ij}(y)|,$$

which implies that Assumption 3 is satisfied with any $\gamma_a > 0$ if, for instance, a is a uniformly continuous function. Recall that if Assumption 3 is satisfied with any $\gamma_a > 0$, then one says that a is in VMO.

Here is one of the main results.

Theorem 1. *There exist constants*

$$\gamma_a = \gamma_a(d, \delta, K, p) > 0, \quad \gamma_b = \gamma_{bb}(d, \delta, K, p, \varepsilon_0) > 0,$$

$$N = N(d, \delta, K, p, \varepsilon_0), \quad \lambda_0 = \lambda_0(d, \delta, K, p, \varepsilon_0, \alpha) \ge 0$$

such that, if the above assumptions are satisfied, then for any $u \in W_p^2(\mathbb{R}^d)$ and $\lambda \ge \lambda_0$ we have

$$\lambda \|u\|_{\mathcal{L}_p(\mathbb{R}^d)} + \|D^2 u\|_{\mathcal{L}_p(\mathbb{R}^d)} \le N \|Lu - \lambda u\|_{\mathcal{L}_p(\mathbb{R}^d)}. \tag{3}$$

Furthermore, for any $f \in \mathcal{L}_p(\mathbb{R}^d)$ and $\lambda \ge \lambda_0$ there is a unique $u \in W_p^2(\mathbb{R}^d)$ such that $Lu - \lambda u = f$.

One of surprising features of (3) is that N is independent of b if b is constant. Another one is that the set $(L - \lambda)W_p^2(\mathbb{R}^d)$ may not coincide with $\mathcal{L}_p(\mathbb{R}^d)$ if $|b|$ grows and yet it always contains $\mathcal{L}_p(\mathbb{R}^d)$. It is also worth noting that generally the constant λ_0 cannot be taken small.

The large genus limit of the infimum of the Willmore energy

Ernst Kuwert
(joint work with Yuxiang Li and Reiner Schätzle)

This is a report on joint work with Yuxiang Li (Tsinghua University, Beijing) and Reiner Schätzle (Universität Tübingen). The research was supported by the DFG research unit 469 and by the Humboldt fellowship of Yuxiang Li.

The Willmore energy of an immersed surface $\Sigma \hookrightarrow \mathbb{R}^n$ with mean curvature vector $\vec{H}$ and induced area measure μ is given by

$$\mathcal{W}(\Sigma) = \frac{1}{4}\int_\Sigma |\vec{H}|^2 \, d\mu.$$

Let $\mathcal{C}(n,p)$ be the class of oriented, closed (i.e. compact without boundary), smoothly immersed surfaces Σ with genus $(\Sigma) = p$, and put

$$\beta_p^n = \inf\{\mathcal{W}(\Sigma) | f \in \mathcal{C}(n,p)\}. \tag{1}$$

It is well-known that $\mathcal{W}(\Sigma) \geq 4\pi$ for any closed immersed surface, with equality only for round spheres [Wil82]. In [Sim93] L. Simon proved the existence of smooth minimizers in $\mathcal{C}(n,p)$ under the Douglas-type condition

$$\beta_p^n < 4\pi + \min\Big\{\sum_{i=1}^r (\beta_{p_i}^n - 4\pi) : 1 \leq p_i < p, \sum_{i=1}^r p_i = p\Big\} =: \tilde{\beta}_p^n. \tag{2}$$

In particular he obtained the existence for $p = 1$. The inequality (2) was proved later in [BaKu03], so β_p^n is attained for all n, p and $\beta_p^n > 4\pi$ for $p \geq 1$. By conformal invariance the area of a minimal surface in $\mathbb{S}^3$ equals the Willmore energy of the surface in $\mathbb{R}^3$ obtained by stereographic projection [Wei78], which leads to an upper bound for β_p^n. Namely, Pinkall [KP86] and independently Kusner [Kus87, Kus89] observed that the minimal surfaces $\xi_{p,1}$ in $\mathbb{S}^3$ described by Lawson in [Lw70] have area less than 8π. In summary we know that

$$4\pi < \beta_p^n < 8\pi \quad \text{for } p \geq 1. \tag{3}$$

An important consequence of the upper bound is that minimizers are automatically embedded, due to an inequality of Li and Yau [LY82]. It was conjectured that the β_p^n might be monotonically increasing in p, see [KP86, p. 446], and that the projected $\xi_{p,1}$ could in fact be minimizers for their genus [Kus89, p. 318 and p. 344]. For large p these surfaces look like two spheres connected by minimal handles, see [Wil93, p. 293] for $p = 5$, in particular their Willmore energy converges to 8π as $p \to \infty$ [Kus87]. Here we prove the following.

Theorem. *Let β_p^n be the infimum of the Willmore energy among oriented, closed surfaces of genus p immersed into $\mathbb{R}^n$. Then*

$$\lim_{p\to\infty} \beta_p^n = 8\pi. \tag{4}$$

We would like to thank Tom Ilmanen for asking the question addressed in this paper when one of us gave a talk in Zürich.

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A fully nonlinear version of the Yamabe problem on manifolds with umbilic boundary

Yanyan Li

In this talk I have recalled some results on the existence and compactness of solutions of the Yamabe problem, the Yamabe problem on manifolds with boundary, a fully nonlinear version of the Yamabe problem as well as the problem on manifolds with boundary. At the end of the talk, I have presented the following result with Luc Nguyen:

Theorem: Let (M^n, g) be a locally conformally flat Riemannian manifold with umbilic boundary of dimension $n \geq 3$. Assume that the Schouten tensor

$$A_g = \frac{1}{n-2}\Big(Ric_g - \frac{R_g}{2(n-1)}g\Big)$$

satisfies, for some $2 \leq k \leq n$,

$$\lambda(A_g) \in \Gamma_k, \qquad \text{on } \bar{M},$$

where

$$\Gamma_k = \{\lambda \in R^n : \sigma_l(\lambda) > 0, 1 \le l \le k\},$$
$$\sigma_l(\lambda) = \sum_{1\le i_1<\ldots<i_l\le n} \lambda_{i_1} \ldots \lambda_{i_l}$$

is the $l-$th elementary symmetric function, and $\lambda(A_g)$ denotes the eigenvalues of A_g with respect to the metric g. Assume also that h_g, the mean curvature of g on ∂M (with respect to the inner normal), is ≥ 0. Then for every real number c, there exists a conformal metric $\tilde{g} = u^{\frac{4}{n-2}} g$ such that

$$\sigma_k(\lambda(A_{\tilde{g}})) = 1, \quad \lambda(A_{\tilde{g}}) \in \Gamma_k, \qquad \text{on } \bar{M}$$

and

$$h_{\tilde{g}} = c \qquad \text{on } \partial M.$$

The $c = 0$ case was proved in [Chen, Szu-yu Sophie Boundary value problems for some fully nonlinear elliptic equations. (English summary) Calc. Var. Partial Differential Equations 30 (2007), no. 1, 1–15] and in [Jin, Qinian; Li, Aobing; Li, Yan Yan Estimates and existence results for a fully nonlinear Yamabe problem on manifolds with boundary. Calc. Var. Partial Differential Equations 28 (2007), no. 4, 509–543].

The $k = 1$ case corresponds to the Yamabe problem, which was solved for $c = 0$ in [Escobar, Jose F. Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary. Ann. of Math. (2) 136 (1992), no. 1, 1–50] and for $c \ne 0$ in [Han, Zheng-Chao; Li, Yanyan The Yamabe problem on manifolds with boundary: existence and compactness results. Duke Math. J. 99 (1999), no. 3, 489–542].

Regularity in Optimal Transportation

Jiakun Liu

(joint work with Neil S. Trudinger and Xu-Jia Wang)

1. Introduction

In this talk, we give some estimates for solutions to the Monge-Ampère equation arising in optimal transportation. The Monge-Ampère equation under consideration has the following type

$$\det\{D^2 u(x) - A(x, Du)\} = f(x) \quad \text{in } \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $A = \{A_{ij}\}$ is an $n \times n$ symmetric matrix defined in $\Omega \times \mathbb{R}^n$.

In optimal transportation, u is the potential function, the matrix A and the right hand side f are given by

$$A(x, Du) = D_x^2 c(x, T_u(x)), \tag{2}$$

$$f = |\det\{D_{xy}^2 c\}| \frac{\rho}{\rho^* \circ T_u}, \tag{3}$$

where $c(\cdot,\cdot)$ is the cost function, $T_u: x \to y$ is the optimal mapping determined by $Du(x) = D_x c(x,y)$, and ρ, ρ^* are mass distributions respectively in the initial domain Ω and the target domain Ω^*.

We assume that the cost function $c \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and satisfies the following conditions:

(**A1**) For any $x, p \in \mathbb{R}^n$, there is a unique $y \in \mathbb{R}^n$ such that $D_x c(x,y) = p$; and for any $y, q \in \mathbb{R}^n$, there is a unique $x \in \mathbb{R}^n$ such that $D_y c(x,y) = q$.
(**A2**) For any $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, $\det\{D^2_{xy} c(x,y)\} \neq 0$,
(**A3**) For any $x, p \in \mathbb{R}^n$, and any $\xi, \eta \in \mathbb{R}^n$ with $\xi \perp \eta$,

$$A_{ij,kl}(x,p)\xi_i\xi_j\eta_k\eta_l > c_0|\xi|^2|\eta|^2, \tag{4}$$

where $A_{ij,kl} = D^2_{p_k p_l} A_{ij}$ and A is given by (1.2).

Under above conditions on the cost function, the optimal mapping is uniquely determined by the corresponding potential function. Therefore, it suffices to study the regularity of potential functions, i.e. regularity of elliptic solutions of (1.1).

2. Regularity results

In the special case when the cost function is the Euclidean distance squared, the regularity of potential functions has been obtained by Caffarelli, Urbas and many other mathematicians. Our goal is to establish the corresponding regularity results for general cost functions satisfying conditions (A1)–(A3), assuming the mass distributions are merely measurable or Hölder continuous, [1, 2, 3].

The first one is the $C^{1,\alpha}$ regularity for potentials, [1]. The similar result was previously obtained by Loeper. We give a completely different proof and our exponent is optimal when the inhomogeneous term $f \in L^\infty$.

Theorem 1. *Let u be a potential function to the optimal transportation problem. Assume the cost function c satisfies conditions A1, A2, A3, Ω^* is c-convex with respect to Ω, and $f \geq 0$, $f \in L^p(\Omega)$ for some $p \in (\frac{n+1}{2}, +\infty]$. Then $u \in C^{1,\alpha}(\bar{\Omega})$, where $\alpha = \frac{\beta(n+1)}{2n^2+\beta(n-1)}$ and $\beta = 1 - \frac{n+1}{2p}$.*

Especially when $p = \infty$, our Hölder exponent $\alpha = \frac{1}{2n-1}$ is optimal.

The second result is the Hölder and more general continuity estimates for second derivatives, when the inhomogeneous term is Hölder and Dini continuous, together with corresponding regularity results for potentials, [2].

Theorem 2. *Assume the cost function c satisfies (A1)–(A3) and f satisfies $C_1 \leq f \leq C_2$ for some positive constants $C_1, C_2 > 0$. Let $u \in C^2(\Omega)$ be an elliptic solution of (1.1). Then for all $x, y \in \Omega_\delta$, we have the estimate*

$$|D^2u(x) - D^2u(y)| \leq C\left[d + \int_0^d \frac{\omega_f(r)}{r} + d\int_d^1 \frac{\omega_f(r)}{r^2}\right], \tag{5}$$

where $d = |x-y|$, $\Omega_\delta = \{x \in \Omega : \ dist(x, \partial\Omega) > \delta\}$, $C > 0$ depends only on $n, \delta, C_1, C_2, A, sup|Du|$, and the modulus of continuity of Du. It follows that:

(i) *If f is Dini continuous, then the modulus of continuity of D^2u can be estimated by* (5) *above;*

(ii) *If $f \in C^\alpha(\Omega)$ for some $\alpha \in (0,1)$, then*

$$\|u\|_{C^{2,\alpha}(\Omega_\delta)} \leq C\left[1 + \frac{\|f\|_{C^\alpha(\Omega)}}{\alpha(1-\alpha)}\right]; \tag{6}$$

(iii) *If $f \in C^{0,1}(\Omega)$, then*

$$|D^2u(x) - D^2u(y)| \leq Cd[1 + \|f\|_{C^{0,1}}|\log d|] \quad \forall x, y \in \Omega_\delta. \tag{7}$$

From the estimates in Theorem 2.2, we conclude the corresponding regularity results for potentials, which are semi-convex, almost everywhere elliptic solutions of equation (1.1).

Corollary 2.1. *Assume the cost function c satisfies (A1)–(A3) and ρ, ρ^* are Dini continuous, and uniformly bounded and positive, in Ω, Ω^* respectively. Then if the target domain Ω^* is c^*-convex, with respect to Ω, any potential function $u \in C^2(\Omega)$ and is an elliptic solution of (1.1), satisfying the estimates* (5), (6) *and* (7), *with f given by (1.3), where C depends on $n, \delta, C_1, C_2, \Omega, \Omega^*$ and c. Consequently if the densities ρ, ρ^* are Hölder continuous and Ω, Ω^* are c, c^*- convex with respect to each other, then the optimal mapping T_u is a $C^{1,\alpha}$ diffeomorphism from Ω to Ω^* for some $\alpha > 0$.*

The last result is the following $W^{2,p}$ estimate, which we have been working on most currently, [3].

Theorem 3. *Assume the cost function c satisfies (A1)–(A3). Let u be a (elliptic) weak solution of (1.1) on the domain $\Omega := S^0_{h,u}(x_0)$ defined by a sub-level set, and $u = \varphi + h$ on $\partial\Omega$, where φ is the c-support of u at x_0 (see the definitions in* [3]*).*

Then if f is continuous and pinched by two positive constants $C_1, C_2 > 0$ such that $C_1 \leq f \leq C_2$. We have $D^2u \in L^p(\Omega_{1/4})$ for any $1 \leq p < \infty$ and

$$\|u\|_{W^{2,p}(\Omega_{1/4})} \leq C(p, \sigma)$$

with σ the modulus of continuity of f and $\Omega_{1/4}$ is the 1/4-dilation of Ω with respect to its center of mass.

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Variational theory for an equation in self-dual gauge theory

Andrea Malchiodi
(joint work with David Ruiz)

We consider a compact orientable surface Σ with metric g and the equation

$$(1) \qquad -\Delta_g u = \rho \left(\frac{h(x)e^{2u}}{\int_\Sigma h(x)e^{2u}dV_g} - 1 \right) - 2\pi \sum_{i=1}^{m} \alpha_j \left(\delta_{p_j} - 1 \right).$$

Here ρ is a positive parameter, $h : \Sigma \to \mathbb{R}$ a smooth positive function, $\alpha_j \geq 0$ and $p_j \in \Sigma$. This equation arises from physical models such as the abelian Chern-Simons-Higgs theory and the Electroweak theory, see e.g. [1], [2] and the bibliographies therein for a recent and complete description of the subject.

Concerning (1), there is in literature some work concerning compactness and blow-up properties of solutions, while there are so far few existence results, which mostly rely on perturbative techniques. We mentions some program by C.S. Lin and some coauthors which aims to compute the Leray-Schauder degree of the equation.

Using some new improved version of the Moser-Trudinger inequality combined with global variational techniques, we obtain several existence results for the above equation for the case in which all the coefficient α_j belong to $(0, 1]$. To give an idea of our results we state this simple version only, when only one singularity is present: we call p the singular point, and α the corresponding coefficient. We notice that the standard Moser-Trudinger inequality gives immediately existence in the case $\rho < 4\pi$.

Theorem 1. *Suppose $\alpha \in (0, 1]$ and that $\rho \in (4\pi, 4\pi(1 + \alpha))$. Then, if Σ is not homeomorphic to the sphere,* (1) *has a solution. If $\alpha \in (0, 1)$ and if $\rho \in (4\pi(1 + \alpha), 8\pi)$, then* (1) *has solutions for every surface Σ.*

To prove the above theorem we use a min-max scheme which uses a topological set homeomorphic respectively to $\Sigma \setminus \{p\}$ and Σ respectively. To show existence, we need this set to be non contractible, and that is why we cannot cover the case of the sphere in the first situation. However, we point out that our assumptions are somehow natural. Indeed, in a work in progress by D.Bartolucci, C.S. Lin and G.Tarantello, it is shown that on the standard sphere, for $\rho \in (4\pi, 4\pi(1 + \alpha))$, (1) has no solution.

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An application of Q-curvature to an embedding of critical type

Luca Martinazzi

Let $\Omega \subset \mathbb{R}^{2m}$ be open, bounded and with smooth boundary, and let a sequence $\lambda_k \to 0^+$ be given. Consider a sequence $(u_k)_{k\in\mathbb{N}}$ of positive smooth solutions to

$$\begin{cases} (-\Delta)^m u_k = \lambda_k u_k e^{mu_k^2} & \text{in } \Omega \\ u_k = \partial_\nu u_k = \ldots = \partial_\nu^{m-1} u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Problem (1) arises from the Adams-Moser-Trudinger inequality [1, 10, 13]:

$$\sup_{u\in H_0^m(\Omega),\, \|u\|_{H_0^m}^2 \le \Lambda_1} \frac{1}{|\Omega|}\int_\Omega e^{mu^2}dx = c_0(m) < +\infty, \tag{2}$$

where $c_0(m)$ is a dimensional constant, $\Lambda_1 := (2m-1)!\mathrm{vol}(S^{2m})$, and $H_0^m(\Omega)$ is the Beppo-Levi space defined as the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_{H_0^m} := \|\Delta^{\frac{m}{2}} u\|_{L^2} = \left(\int_\Omega |\Delta^{\frac{m}{2}} u|^2 dx\right)^{\frac{1}{2}}, \tag{3}$$

where $\Delta^{\frac{m}{2}} u := \nabla \Delta^{\frac{m-1}{2}} u$ for m odd. In fact critical points of (2) under the constraint $\|u\|_{H_0^m}^2 = \Lambda_1$ solve (1). Then we have the following concentration-compactness result:

Theorem 1 ([9]). *Let (u_k) be a sequence of solutions to (1) such that*

$$\limsup_{k\to\infty} \|u_k\|_{H_0^m}^2 = \limsup_{k\to\infty} \int_\Omega \lambda_k u_k^2 e^{mu_k^2} dx = \Lambda < \infty. \tag{4}$$

Then up to a subsequence either

(i) $\Lambda = 0$ and $u_k \to 0$ in $C^{2m-1,\alpha}(\Omega)$, or

(ii) There exists a positive integer I such that $\Lambda \ge I\Lambda_1$, and there is a finite set $S = \{x^{(1)}, \ldots, x^{(I)}\}$ such that

$$u_k \to 0 \quad \text{in } C^{2m-1,\alpha}_{\mathrm{loc}}(\overline{\Omega}\backslash S),$$

and

$$\lambda_k u_k^2 e^{mu_k^2} \rightharpoonup \sum_{i=1}^I \alpha_i \delta_{x^{(i)}}, \quad \alpha_i \ge \Lambda_1,$$

weakly in the sense of measures.

Theorem 1 was proven by Adimurthi and M. Struwe [3] and Adimurthi and O. Druet [2] in the case $m = 1$, and by F. Robert and M. Struwe [11] for $m = 2$. Recently O. Druet [6] for the case $m = 1$, and M. Struwe [12] for $m = 2$ improved the previous results by showing that in case (ii) of Theorem 1 we have $\Lambda = L\Lambda_1$ for some positive $L \in \mathbb{N}$. Whether the same holds true for $m > 2$ is still an open question.

Part (ii) of the theorem shows an interesting threshold phenomenon: below the critical energy level Λ_1 we always have compactness. Moreover Λ_1 is the total Q-curvature of the sphere (see [8] for a short discussion of Q-curvature). We shall briefly explain how this remarkable connection with Riemannian geometry arises. It is easy to see that if we are not in case (i) of the theorem, then $\sup_\Omega u_k \to \infty$ as $k \to \infty$. Then one can blow up, i.e. define the scaled functions

$$\eta_k(x) := u_k(x_k)(u_k(x_k + r_k x) - u_k(x_k)) \quad \text{for } x \in r_k^{-1}\Omega - x_k,$$

where x_k is such that $u_k(x_k) = \max_\Omega u_k$ and $r_k \to 0$ is a suitably chosen scaling factor. Then it turns out that

$$\eta_k(x) \to \eta_0(x) \quad \text{in} \quad C^{2m-1}_{\text{loc}}(\mathbb{R}^{2m}), \quad \text{as } k \to \infty, \tag{5}$$

where η_0 is a solution of the Liouville-type equation

$$(-\Delta)^m \eta = (2m-1)! e^{2m\eta} \quad \text{on } \mathbb{R}^{2m}, \quad \int_{\mathbb{R}^{2m}} e^{2m\eta} dx < \infty. \tag{6}$$

We recall (see e.g. [8]) that if η solves $(-\Delta)^m \eta = V e^{2m\eta}$ on $\mathbb{R}^{2m}$, then the conformal metric $g_\eta := e^{2\eta}|dx|^2$ has Q-curvature V, where $|dx|^2$ denotes the Euclidean metric. Now the problem is to understand what is the solution η_0 or (equivalently) what is the conformal metric g_{η_0}.

A bunch of solution to (6) is given by the so-called *standard solutions*

$$\eta_{\lambda,x_0}(x) = \log \frac{2\lambda}{1 + \lambda^2 |x - x_0|^2}, \quad \lambda > 0, x_0 \in \mathbb{R}^{2m}.$$

These are "spherical" solutions, as the metric $e^{2\eta_{\lambda,x_0}}|dx|^2$ can be obtained by pulling-back the metric of the round sphere S^{2m} onto $\mathbb{R}^{2m}$ via the stereographic projection and a Möbius diffeomorphism.

While Chen and Li [5] proved that in the case $m = 1$ the only solutions to (6) are the standard solutions, Chang and Chen [4] showed that for $m > 1$ (6) possesses many other solutions. Therefore the problem of understanding η_0 starts to appear quite subtle, and the following classification result, due to the author [8], turns out to be crucial.

Theorem 2. *Let η be a solution to* (6) *and set*

$$v(x) := \frac{(2m-1)!}{\gamma_m} \int_{\mathbb{R}^{2m}} \log \left(\frac{|y|}{|x - y|} e^{2mu(y)} \right) dy,$$

where γ_m is such that $(-\Delta)^m \left[\frac{1}{\gamma_m} \log \frac{1}{|x|}\right] = \delta_0$. Then $\eta = v + p$, where p is a polynomial of degree at most $2m - 2$ and

$$\lim_{|x| \to \infty} \Delta^j v(x) = 0, \quad 1 \le j \le m - 1.$$

Moreover the following are equivalent:

(i) *η is a standard solution,*
(ii) *p is constant.*

Finally if η is not a standard solution there exist $1 \le j \le m-1$ and a constant $\alpha \neq 0$ such that

$$\lim_{|x|\to\infty} \Delta^j \eta(x) = \alpha. \tag{7}$$

Now the idea is to use Theorem 2 to prove the following proposition.

Proposition 3. *The function η_0 given by* (5) *is a standard solution to* (6)*.*

Proposition 3 yields

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega \lambda_k u_k^2 e^{mu_k^2}dx &\geq (2m-1)!\int_{\mathbb{R}^{2m}} e^{2m\eta_0}dx\\ &= (2m-1)!\int_{\mathbb{R}^{2m}} Q_{S^{2m}}\mathrm{dvol}_{g_{S^{2m}}} = \Lambda_1,\end{aligned}$$

This is the basic reason why $\alpha_i \geq \Lambda_1$ in case (ii) of Theorem 1.

In order to apply Theorem 2, one has to have a better understanding of the asymptotic behavior of the functions η_k and their derivatives. This is achieved in the following proposition, which is central to our argument.

Proposition 4. *For any $R > 0$, $1 \le \ell \le 2m-1$ there exists k_0 such that*

$$u_k(x_k)\int_{B_{Rr_k}(x_k)} |\nabla^\ell u_k| dx \le C(Rr_k)^{2m-\ell}, \quad \text{for all } k \geq k_0. \tag{8}$$

Equivalently

$$\int_{B_R(0)} |\nabla^\ell \eta_k| dx \le CR^{2m-\ell}, \quad \text{for all } k \geq k_0. \tag{9}$$

Observe that taking the limit in (9) one gets

$$\int_{B_R(0)} |\nabla^\ell \eta_0| dx \le CR^{2m-\ell}, \quad k \geq k_0(R), \tag{10}$$

and η_0 has to be a standard solution because (10) is not compatible with (7).

Finally, let us also comment on the proof of Proposition 4. The key idea is to prove that

$$\|\Delta^m(u_k^2)\|_{L^1(\Omega)} \le C. \tag{11}$$

This is an easy consequence of the following Lorentz-space estimate.

Proposition 5. *For every $1 \le \ell \le 2m-1$, $\nabla^\ell u_k$ belongs to the Lorentz space $L^{(2m/\ell,2)}(\Omega)$ and*

$$\|\nabla^\ell u_k\|_{(2m/\ell,2)} \le C.$$

This can be proven by interpolation observing that (4) implies that $\Delta^m u_k$ is bounded in the Zygmund space $L(\log L)^{\frac{1}{2}}$. Interestingly if we decide to be a bit sloppy and consider that (4) gives bounds for $\Delta^m u_k$ in $L^1(\Omega)$, then we get the bounds $\|\nabla^\ell u_k\|_{(2m/\ell,\infty)} \le C$ (here $L^{(p,\infty)}$ is the Marcinkievicz space). On the other hand these bounds are too weak to prove (11), hence Proposition 4. This also shows that (8), (9) and (10) are in some sense "sharp".

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Gradient estimates via non-linear potentials

Giuseppe Mingione

For the Poisson equation $-\triangle u = \mu$, here considered in the whole $\mathbb{R}^n$ and where μ is in the most general case a Radon measure with finite total mass, it is well-known that it is possible to get pointwise bounds for solutions via the use of Riesz potential

$$I_\beta(\mu)(x) := \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-\beta}}\,, \qquad \beta \in (0,n] \tag{1}$$

such as

$$|u(x)| \leq cI_2(|\mu|)(x)\,, \qquad \text{and} \qquad |Du(x)| \leq cI_1(|\mu|)(x)\,. \tag{2}$$

Similar local estimates ca be obtained using the localized version of the Riesz potential $I_\beta(\mu)(x)$ is given by the linear potential

$$\mathbf{I}^\mu_\beta(x_0,R) := \int_0^R \frac{\mu(B(x_0,\varrho))}{\varrho^{n-\beta}}\,\frac{d\varrho}{\varrho}\,, \qquad \beta \in (0,n] \tag{3}$$

with $B(x_0,\varrho)$ being the open ball centered at x_0, with radius ϱ. A question is now, *is it possible to give an analogue of estimates* (2) *in the case of general quasilinear* *equations such as for instance, the degenerate p-Laplacean equation with measure data*

$$-\text{div}\,(|Du|^{p-2}Du) = \mu\,? \tag{4}$$

A fundamental result is in the papers [10, 16], where - for suitably defined solutions to (4) - the authors give a first affirmative answer proving the following pointwise *zero order estimate* - i.e. for u - when $p \leq n$, via non-linear Wolff potentials:

$$|u(x_0)| \leq c \left(\fint_{B(x_0,R)} |u|^{p-1}\, dx \right)^{\frac{1}{p-1}} + c\mathbf{W}^{\mu}_{1,p}(x_0, 2R)\,, \tag{5}$$

where the constant c depends on the quantities n, p, and

$$\mathbf{W}^{\mu}_{\beta,p}(x_0, R) := \int_0^R \left(\frac{|\mu|(B(x_0,\varrho))}{\varrho^{n-\beta p}} \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \qquad \beta \in (0, n/p]\,, \tag{6}$$

is the non-linear Wolff potential of μ; here $\fint_B$ denotes the integral average over B. Estimate (5), which extends to a whole family of general quasi-linear equations, and which is commonly considered as a basic result in the theory of quasi-linear equations, is the natural non-linear analogue of the first linear estimate appearing in (2). *Here we present the non-linear analogue of the second estimate in* (2)*, thereby giving a pointwise gradient estimate via non-linear potentials which upgrades* (6) *up to the gradient/maximal level.*

To state the result in greater generality let us we shall consider $p \geq 2$, we shall therefore treat possibly degenerate elliptic equations when $p \neq 2$. Specifically, we shall consider general non-linear, possibly degenerate equations with p-growth of the type

$$-\text{div}\; a(x, Du) = \mu\,, \tag{7}$$

whenever μ is a Radon measure with finite total mass defined on Ω; eventually letting $\mu(\mathbb{R}^n \setminus \Omega) = 0$, without loss of generality we may assume that μ is defined on the whole $\mathbb{R}^n$. The continuous vector field $a\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be C^1-regular in the gradient variable z, with $a_z(\cdot)$ being Carathéodory regular and satisfying the following *growth, ellipticity and continuity assumptions*:

$$\begin{cases} |a(x,z)| + |a_z(x,z)|(|z|^2+s^2)^{\frac{1}{2}} \leq L(|z|^2+s^2)^{\frac{p-1}{2}} \\ \nu^{-1}(|z|^2+s^2)^{\frac{p-2}{2}}|\lambda|^2 \leq \langle a_z(x,z)\lambda, \lambda\rangle \\ |a(x,z) - a(x_0,z)| \leq L_1\omega(|x-x_0|)(|z|^2+s^2)^{\frac{p-1}{2}}\,, \end{cases} \tag{8}$$

whenever $x, x_0 \in \Omega$ and $z, \lambda \in \mathbb{R}^n$, where $0 < \nu \leq 1 \leq L$ and $s \geq 0, L_1 \geq 1$ are fixed parameters. Here $\omega\colon [0,\infty) \to [0,\infty)$ is a modulus of continuity i.e. a non-decreasing function such that $\omega(0) = 0$ and $\omega(\cdot) \leq 1$. On such a function we impose a natural decay property, which is essentially optimal for the result we are going to have, and prescribes a *Dini continuous dependence of the partial map*

$x \mapsto a(x,z)/(|z|+s)^{p-1}$:

$$\int_0^R [\omega(\varrho)]^{\frac{2}{p}} \frac{d\varrho}{\varrho} := d(R) < \infty\,, \tag{9}$$

for some $R > 0$. For the sake of simplicity we shall present our results in the form of a priori estimates - i.e. when solutions and data are taken to be more regular than needed, for instance $u \in C^1(\Omega)$ and $\mu \in L^1(\Omega)$ - but they actually hold, via a standard approximation argument, for general weak and very weak solutions - i.e. distributional solutions which are not in the natural space $W^{1,p}(\Omega)$ - to measure data problems such as $-\text{div}\; a(x, Du) = \mu$ where μ is a general Radon measure with finite total mass, defined on Ω. The reason for such a choice is that the approximation argument in question leads to different notions of solutions, according to the regularity/integrability properties of the right hand side μ. If the right hand side of (7) is integrable enough to deduce that $\mu \in W^{-1,p'}(\Omega)$, then our results apply to general weak energy solutions $u \in W^{1,p}(\Omega)$ to (7).

Theorem 1 ([6, 14])**.** *Let $u \in C^1(\Omega)$, be a weak solution to* (7) *with $\mu \in L^1(\Omega)$, under the assumptions* (8)*. Then there exists a constant $c \equiv c(n, p, L/\nu, L_1) > 1$, and a positive radius R_0 depending only on $n, p, L/\nu, L_1, \omega(\cdot)$, such that the pointwise estimate*

$$|Du(x_0)| \leq c \left(-\!\!\!\!\!\!\int_{B(x_0,R)} (|Du|+s)^{\frac{p}{2}}\, dx \right)^{\frac{2}{p}} + c\mathbf{W}^{\mu}_{1/p,p}(x_0, 2R) \tag{10}$$

holds whenever $B(x_0, 2R) \subseteq \Omega$, and $R \leq R_0$. Moreover, when the vector field $a(\cdot)$ is independent of x - and in particular for the p-Laplacean operator (4) *- estimate* (10) *holds with no restriction on R.*

Theorem 1 solves in the positive an old conjecture of Verbitsky. In the case $p = 2$ estimate (10) is exactly the local version of the second inequality in (2):

$$|Du(x_0)| \leq c -\!\!\!\!\!\!\int_{B(x_0,R)} (|Du|+s)\, dx + c\mathbf{I}_1^{|\mu|}(x_0, 2R)\,.$$

A non-local, level set version of (10) was previously obtained in [13]. Beside their intrinsic theoretical interest, the pointwise estimate (10) allows to unify and recast essentially all the known gradient estimates for quasilinear equations in rearrangement invariant function spaces. Indeed, by (10) it is clear that the behavior of Du can be controlled by that $\mathbf{W}^{\mu}_{1/p,p}$, which is in turn known via the behavior of Riesz potentials; this is a consequence of the pointwise bound of the Wolff potential via the Havin-Maz'ja non linear potential, that is

$$\mathbf{W}^{\mu}_{1/p,p}(x_0,\infty) = \int_0^\infty \left(\frac{|\mu|(B(x_0,\varrho))}{\varrho^{n-1}} \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \leq cI_{\frac{1}{p}} \left\{ \left[I_{\frac{1}{p}}(|\mu|) \right]^{\frac{1}{p-1}} \right\}(x_0)\,. \tag{11}$$

Ultimately, thanks to (11) and to the well-known properties of the Riesz potentials, we have $\mu \in L^q \Longrightarrow \mathbf{W}^{\mu}_{1/p,p} \in L^{\frac{nq(p-1)}{n-q}}$ for $q \in (1, n)$, while Marcikiewicz spaces must be introduced for the borderline case $q = 1$. Inequality (10) immediately allows to recast *for the model case equation* (4), all the classical gradient estimates for solutions such as those due to Boccardo & Gallöuet [3] - when q is "small" - and Iwaniec [12] and DiBenedetto & Manfredi [4] - when q is "large" - that is, for solutions to (4) it holds that $\mu \in L^q \Longrightarrow Du \in L^{\frac{nq(p-1)}{n-q}}$ where $q \in (1,n)$. Delicate borderline regularity cases as the Marcinkiewicz spaces estimates for $p = n$ - see [5] - also follows as a corollary (note that [4, 5] deal with the p-Laplacean system). Further results for parabolic equations can be found in [8, 10]. Moreover, since the operator $\mu \mapsto \mathbf{W}^{\mu}_{1/p,p}$ is obviously sub-linear, using the estimates related to (11) and classical interpolation theorems for sub-linear operators one immediately gets estimates in refined scales of spaces such Lorentz or Orlicz spaces, recovering some estimates of Talenti [15], but directly for the gradient of solutions, rather than for solutions themselves. As an example, an immediate corollary allows to establish a delicate borderline case in Lorentz spaces, i.e for $p < n$ we have that $\mu \in L(np/(np-n+p), q)$ implies $Du \in L(p, q(p-1))$. This solves an open problem stated several times in the literature ([1], [2], [9]).

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Singular Solutions to Fully Nonlinear Elliptic Equations

Nikolai Nadirashvili

We study a fully nonlinear second-order elliptic equations of the form (where $h \in \mathbf{R}$)

$$\mathbf{F}_h(D^2u) = \det(D^2u) - Tr(D^2u) + h\sigma_2(D^2u) - h = 0 \tag{1}$$

defined in a smooth-bordered domain of $\Omega \subset \mathbf{R}^3$, $\sigma_2(D^2u) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$ being the second symmetric function of the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of D^2u. Here D^2u denotes the Hessian of the function u. This equation is equivalent to the Special Lagrangian potential equation [HL1]:

$$SLE_\theta : \qquad Im\{e^{-i\theta}\det(I + iD^2u)\} = 0$$

for $h := -\tan(\theta)$ which can be re-written as

$$\mathbf{F}_\theta = \arctan\lambda_1 + \arctan\lambda_2 + \arctan\lambda_3 - \theta = 0.$$

The set

$$\{A \in Sym^2(\mathbf{R}^3) : \mathbf{F}_h(A) = 0\} \subset Sym^2(\mathbf{R}^3)$$

has three connected components, $C_i, i = 1, 2, 3$ which correspond to the values $\theta_1 = -\arctan(h) - \pi, \theta_2 = -\arctan(h), \theta_3 = -\arctan(h) + \pi$.

We study the Dirichlet problem

$$\begin{cases} \mathbf{F}_\theta(D^2u) = 0 & \text{in}\Omega \\ u = \varphi & \text{on}\partial\Omega \end{cases},$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ and φ is a continuous function on $\partial\Omega$.

For $\theta_1 = -\arctan(h) - \pi$ and $\theta_3 = -\arctan(h) + \pi$ the operator $\mathbf{F}_\theta$ is concave or convex, and the Dirichlet problem in these cases was treated in [CNS]; smooth solutions are established there for smooth boundary data on appropriately convex domains.

The middle branch $C_2, \theta_2 = -\arctan(h)$ is never convex (neither concave), and the classical solvability of the Dirichlet problem remained open.

In the case of uniformly elliptic equations a theory of weak (viscosity) solutions for the Dirichlet problem gives the uniqueness of such solutions, see [CIL], moreover these solutions lie in $C^{1,\varepsilon}$ by [CC],[T1],[T2]. However, the recent results [NV1],[NV2],[NV3] show that at least in 12 and more dimensions the viscosity solution of the Dirichlet problem for a uniformly elliptic equation can be singular, even in the case when the operator depends only on eigenvalues of the Hessian.

One can define viscosity solutions for non-uniformly elliptic equations (such as SLE_θ) as well, but in this case the uniqueness of viscosity solution is not known which makes the use of these solutions less convenient.

Recently a new very interesting approach to degenerate elliptic equations was suggested by Harvey and Lawson [HL2]. They introduced a new notion of a weak solution for the Dirichlet problem for such equations and proved the existence, the continuity and the uniqueness of these solutions.

We show that the classical solvability for Special Lagrangian Equations *does not* hold.

More precisely, we show the existence for any $\theta \in]-\pi/2, \pi/2[$ of a small ball $B \subset \mathbf{R}^3$ and of an analytic function ϕ on ∂B for which the unique Harvey-Lawson solution u_θ of the Dirichlet problem satisfies :

(i) $u_\theta \in C^{1,1/3}$;

(ii) $u_\theta \notin C^{1,\delta}$ for $\forall \delta > 1/3$.

Our construction use the Legendre transform for solutions of $\mathbf{F}_{\frac{1}{h}}(D^2u) = 0$ which gives solutions of $\mathbf{F}_h(D^2u) = 0$; in particular, for $h = 0$ it transforms solutions of $\sigma_2(D^2u) = 1$ into solutions of $\det(D^2u) = Tr(D^2u)$. This construction could be of interest by itself.

Finally, we think that the following conjecture is quite plausible:

Conjecture. *Any Harvey-Lawson solution of SLE_θ on a ball B lies in $C^1(B)$ (if φ is sufficiently smooth).*

In the case $\theta = 0$ these solutions lie in $C^{0,1}(B)$ by Corollary 1.2 in [T3].

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Joint minimization of Dirichlet and Willmore functionals

Pavel Igorevich Plotnikov

Let $\mathcal{S}_\infty(M)$ be a class of periodic surfaces $S \subset \mathbb{R}^3$ satisfying the following conditions. A surface $S \in \mathcal{S}_\infty$ admits a C^∞ parametrization $\mathbf{r} = \mathbf{u}(X)$, $X = (X^1, X^2)$ such that

$$\mathbf{u}(X + m\mathbf{l}_1 + n\mathbf{l}_2) = (m, n, 0) + \mathbf{u}(X) \quad \forall (m, n) \in \mathbb{Z}^2. \tag{1}$$

where the vectors $\mathbf{l}_1 = (T, 0)$ and $\mathbf{l}_2 = (-\alpha, T^{-1})$ generate the translation lattice Γ. Moreover, the inequality

$$\int\limits_{\mathbb{R}^2/\Gamma} \operatorname{Tr}\, \mathbf{a}(X)\, dX + \int\limits_{S/\mathbb{Z}^2} |\mathbf{A}|^2\, dS < M < \infty,$$

holds true for any $S \in \mathcal{S}_\infty(M)$. Here $\mathbf{a}$ and $\mathbf{A}$ are the first and second fundamental forms of S. We say that a surface $S = \{\mathbf{r} = \mathbf{u}(X)\}$ belongs to the class $\mathcal{S}(M)$ if there exists a sequence $S_k = \{\mathbf{r} = \mathbf{u}_k\} \in \mathcal{S}_\infty(M)$ such that

$$\mathbf{u}_k \to \mathbf{u} \text{ weakly in } H^{1,2}(D) \text{ for any disk } D \subset \mathbb{R}^2.$$

Denote by $\Omega \subset \mathbb{R}^3$ the domain below S and by $\Phi : \Omega \to \mathbb{R}$ a harmonic function satisfying the conditions

$$\partial_n(x^1 + \Phi) = 0 \text{ on } S, \quad \Phi\big(x + (m, n, 0)\big) = \Phi(x)\ \forall (m, n) \in \mathbb{Z}^2,$$
$$\lim_{x^3 \to -\infty} \Phi(x) = 0.$$

The function Φ is completely determined by S. Introduce a renormalized "kinetic energy"

$$\mathcal{D}(S) = \int\limits_{\Omega/\mathbb{Z}^2} |\nabla \Phi|^2\, dx - 2 \int\limits_{S/\mathbb{Z}^2} \Phi\, n_1\, dS,$$

and take the total energy associated with S in the form

$$\mathcal{E}(S, \mathbf{u}) = \int\limits_{\mathbb{R}^2/\Gamma} \operatorname{Tr}\, \mathbf{a}(X)\, dX + \int\limits_{S/\mathbb{Z}^2} H^2\, dS + \mathcal{D}(S) + \int\limits_{S/\mathbb{Z}^2} g(x^3) n_3\, dS. \tag{2}$$

Here g is an arbitrary smooth function and $\mathbf{n}$ is the outward normal vector to S. The aim of the work is to study qualitative properties of solutions to the variational problem

$$\mathcal{E}(S, \mathbf{u}) = \min_{\mathcal{S}(M)} \mathcal{E} \tag{3}$$

This problem has its origin in the theory of hydroelastic nonlinear waves on the surface of an ice ocean, see [1]. The first two integrals in the right hand side of (2) represent the elastic energy of the ice, $\mathcal{D}$ is the kinetic energy of a fluid, and the last integral in (2) is a potential energy of the fluid. In our framework the lattice Γ is unknown and can be regarded as an integral part of solution. It is worthy to note that for any solution to problem (3), the mapping $\mathbf{u} : \mathbb{R}^2/\Gamma \to S/\mathbb{Z}^2$ is conformal, and we can reformulate the problem as follows.

For any $S \in \mathcal{S}_\infty$, we denote by $\mathbf{r} = \mathbf{u}(X)$ a conformal parametrization of surface S satisfying periodicity conditions (1) and set

$$a_{11}(X) = a_{22}(X) = e^{2f(X)}, \quad \mathbf{e}_i = e^{-f}\partial_{X^i}\mathbf{u}, \quad \mathbf{n} = \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2. \tag{4}$$

For any $\delta > 0$, we denote by $\mathcal{M}_\infty(\delta)$ the set of all surfaces

$$S \in \bigcup_{M>0} \mathcal{S}^\infty(M)$$

such that

$$\int\limits_{S/\mathbb{Z}^2} |\mathbf{A}|^2\, dS < 16\pi/3 - \delta, \quad \|df\|_{L^2(\mathbb{R}^2/\Gamma)} \le \Big(8\pi/3\Big)^{1/2}.$$

We say that a surface S belongs to the class $\mathcal{M}(\delta)$ if there exists a sequence $S_k = \{\mathbf{r} = \mathbf{u}_k\} \in \mathcal{M}_\infty(M, \delta)$ such that $\mathbf{u}_k \to \mathbf{u}$ weakly in $H^{1,2}_{loc}(\mathbb{R}^2)$. The following lemma gives bounds for the conformal characteristics of elements of the class $\mathcal{M}(\delta)$.

Lemma 1. *The following inequalities hold true for any $S \in \mathcal{M}(\delta)$,*

$$\|df\|_{L^2(\mathbb{R}^2/\Gamma)} \le \Big(8\pi/3\Big)^{1/2} - \sqrt{\delta/2}, \tag{5}$$

$$\|d\mathbf{e}_1\|^2_{L^2(\mathbb{R}^2/\Gamma)} + \|d\mathbf{e}_2\|^2_{L^2(\mathbb{R}^2/\Gamma)} \le 16\pi \tag{6}$$

$$\|f\|_{C(\mathbb{R}^2)} \le C(\delta). \tag{7}$$

The proof of inequalities (5)-(6)is obtained by using the moving frame method, developed in [3], and the modified Wente inequalities. Inequality (7) follows from the Wente inequality and the simple observation that the surface $S/\mathbf{Z}^2$ is bounded in the horizontal direction.

The second lemma which is a consequence of the famous result by Li and Yau, see liYau, shows that for a suitable δ, elements of $\mathcal{M}(\delta)$ have no self-intersections and are physically admissible.

Lemma 2. *If $S \in \bigcup_{M>0} \mathcal{S}^\infty(M)$ and*

$$\frac{1}{4}\int\limits_{S/\mathbb{Z}^2} |\mathbf{A}|^2\, dS < 8\pi - 2\pi^2, \tag{8}$$

then S have no self-intersections

It follows from Lemmas 1,2 that the conformal characteristics of surfaces $S \in \mathcal{M}(\delta)$ are completely controlled by the Willmore energy. The following theorem is the main result of this work.

Theorem 3. *Let for some $\delta > 0$,*

$$\mathcal{E}(S, \mathbf{u}) = \inf_{\tilde{S} \in \mathcal{M}(\delta)} \mathcal{E}$$

and S satisfies (8). Then $S \in C^\infty$.

The proof is based on the Simon biharmonic approximation method, [2], and the following auxiliary lemma.

Lemma 4. *Let $\mathbf{r} = \mathbf{u}(X)$ be a smooth conformal embedding of the unit disk $D_1 = \{|X| \le 1\}$ into $\mathbb{R}^3$ and f, $\mathbf{e}_i$ are defined by (4). Furthermore assume that they satisfy the inequalities*

$$\|df\|_{L^2(D)} \le \varepsilon, \quad \|d\mathbf{e}_i\|_{L^2(D)} \le \varepsilon,$$
$$0 < \lambda^{-1} \le e^f \le \lambda < \infty, \quad 1 < \lambda < \infty.$$

Then there exist constants ε_0 and c, depending only on λ, such that for each $\varepsilon \in (0, \varepsilon_0)$, there are unit orthogonal vectors $(\mathbf{b}_i)_{i=1,2,3}$, a number $e^{f_0} \in (\lambda^{-1}, \lambda)$, and compact set $\mathcal{F} \subset D_1$ with the following properties. The one-dimensional Hausdorff measure of $\mathcal{F}$ is less than $c\varepsilon$ and

$$|\mathbf{b}_i - \mathbf{e}_i(X)| + |f(X) - f_0| \le \sqrt{\varepsilon} \quad \forall\, X \in D_1 \setminus \mathcal{F}.$$

Choose the Cartesian coordinates (x^i) in $\mathbb{R}^3$ such that the directions of the axes x^i coincide with directions of the vectors $\mathbf{b}_i$. Then there is a set $\mathcal{T} \subset \big((3\lambda)^{-1},\ (2\lambda)^{-1}\big)$ with meas $\mathcal{T} > 1/(9\lambda)$ such that for every $t \in \mathcal{T}$, the level set $\mathfrak{C}_t = \{X : (u^1)^2 + (u^2)^2 = t^2\}$ consists of finite number of smooth Jordan curves . Each connected component $\gamma \subset \mathfrak{C}_t$ admits the parametrization

$$X = \rho(\vartheta)(\cos\vartheta, \sin\vartheta), \quad \vartheta \in [0, 2\pi],$$

where

$$|\rho(\vartheta) - e^{-f_0} t| \le c\sqrt{\varepsilon}, \quad |\rho'(\vartheta)| \le C\sqrt{\varepsilon}.$$

The mapping $(x_1, x_2) = (u^1, u^2)(X)$ establishes the diffeomorphism between γ and the circumference $C_t = \{(x^1)^2 + (x_2)^2 = t^2\}$. Moreover, there is a neighborhood $\mathcal{O}$ of γ such that the surface $\{x = \mathbf{u}(X),\ X \in \mathcal{O}\}$ is a graph of a smooth function $x^3 = \eta(x^1, x^2)$ defined in some vicinity of C_t . The restrictions of η on C_t admit the estimates

$$|\eta| + |D\eta| \le c\sqrt{\varepsilon} \text{ on } C_t, \quad \|D^2\eta\|_{L^2(C_t)} \le c\|\mathbf{A}\|_{L^2(S_\lambda)}$$

where the annulus $S_\lambda = \mathbf{u}\left(D_1 \setminus D_{1/3\lambda}\right)$.

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Critical points of the Moser-Trudinger energy in the super-critical regime

Michael Struwe

On any bounded domain $\Omega \subset \mathbb{R}^2$ the energy functional

$$E(u) = \frac{1}{2}\int_\Omega (e^{u^2} - 1)\,dx$$

studied by Trudinger [13] and Moser [10] for any $\alpha \le 4\pi$ admits a maximizer in the space

$$M_\alpha = \{u \in H_0^1(\Omega);\, u \ge 0, ||\nabla u||^2_{L^2} = \alpha\}, \tag{1}$$

corresponding to a solution $0 < u \in M_\alpha$ of the equation

$$-\Delta u = \lambda u e^{u^2} \text{ in } \Omega \tag{2}$$

for some $\lambda > 0$; see [3] and [6]. For $\alpha > 4\pi$ the functional E is unbounded on M_α, but E still admits a relative maximizer in M_α when $\alpha > 4\pi$ is sufficiently small. One therefore may expect to see also critical points of saddle-type for such α; indeed, when Ω is a ball this conjecture is strongly supported by numerical evidence [9]. However, standard variational techniques fail in this "super-critical" range of energies and ad hoc methods devised to remedy the situation so far have only been partially succesful; compare [11].

In recent joint work [8] with Tobias Lamm and Frederic Robert we approach the problem by means of the following flow. Given a smooth function $u_0 \in M_\alpha$, we consider smooth solutions $u = u(t,x)$ to the equation

$$u_t e^{u^2} = \Delta u + \lambda u e^{u^2} \text{ in } [0,\infty[\times\Omega \tag{3}$$

with initial and boundary data

$$u(0) = u_0, \quad u = 0 \text{ on } [0,\infty[\times\partial\Omega, \tag{4}$$

and with a function $\lambda = \lambda(t) > 0$ determined so that the Dirichlet integral of u is preserved along the flow; that is, so that

$$\int_\Omega |\nabla u(t)|^2\,dx = \int_\Omega |\nabla u_0|^2\,dx = \alpha \text{ for all } t \ge 0\,. \tag{5}$$

Upon multiplying (3) by u_t and integrating, we then also obtain the relation

$$\int_\Omega u_t^2 e^{u^2} dx = \lambda \frac{d}{dt}E(u(t))\,; \tag{6}$$

that is, (3), (4), (5) is the L^2-gradient flow associated with E on M_α (with respect to the metric $g = e^{u^2} g_{\mathbb{R}^2}$).

Note that our equation (3) is similar to the equation for scalar curvature flow; in the case of 2 space dimensions the scalar curvature flow is the Ricci flow studied by Hamilton [7] and Chow [4].

Then we obtain the following result.

Theorem 1. *For any $\alpha > 0$ and any smooth $u_0 \in M_\alpha$ the evolution problem* (3), (4), (5) *admits a unique smooth solution $u > 0$ for small $t > 0$. The solution u may be continued smoothly for all $t > 0$, provided that $E(u(t))$ remains bounded. In this case, for a suitable sequence $t_k \to \infty$ the functions $u(t_k) \overset{w}{\rightharpoondown} u_\infty$ weakly in $H_0^1(\Omega)$, where $u_\infty \in H_0^1(\Omega)$ is a solution to the problem* (2) *for some constant $\lambda_\infty \geq 0$. Moreover, either i) $u(t_k) \to u_\infty$ strongly in $H_0^1(\Omega)$, $\lambda_\infty > 0$, and $0 < u_\infty$ is a critical point of E in M_α, or ii) there exist $i_* \in \mathbb{N}$ and points $x^{(i)} \in \overline{\Omega}$, $l_i \in \mathbb{N}$, $1 \leq i \leq i_*$, such that as $k \to \infty$ we have*

$$|\nabla u(t_k)|^2 dx \overset{w^*}{\rightharpoondown} |\nabla u_\infty|^2 dx + \sum_{i=1}^{i_*} 4\pi l_i \delta_{x^{(i)}}$$

weakly in the sense of measures. By (5) *then necessarily $4\pi \sum_{i=1}^{i_*} l_i \leq \alpha$.*

The above quantization result for (3), (4), (5) in the case of divergence is in complete analogy with the results of Adimurthi-Struwe [2] and Druet [5] for concentrating solutions of the corresponding elliptic equation (2), or the results of Struwe [12] for solutions $u_k \overset{w}{\rightharpoondown} 0$ in H^2 of the related fourth order equation $\Delta^2 u_k = \lambda_k u_k e^{2u_k^2}$ on a domain in $\mathbb{R}^4$. Our proof builds on these results.

Coupling Theorem 1 with a standard mountain-pass type construction we now are able to rigorously establish the existence of saddle-points of E in M_α for (sufficiently small) numbers $\alpha > 4\pi$.

Theorem 2. *There exists a number $\alpha_1 \in]4\pi, 8\pi]$ such that for any $4\pi < \alpha < \alpha_1$ there exists a pair of solutions $\underline{u}, \overline{u} \in M_\alpha$ of* (2) *with $0 < E(\underline{u}) < E(\overline{u})$.*

Theorem 1 completes Theorem 1.8 from [11], where the existence of a pair of solutions of (2) only was shown for almost every $4\pi < \alpha < \alpha_1$.

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Existence of mean curvature flow with non-smooth transport term

Yoshihiro Tonegawa

The study of time-global existence of mean curvature flow (MCF) was pioneered in the work of Brakke in 1978 using the theory of varifold. In the late 80's and early 90's various alternative existence theories were proposed such as the level set method, the phase field method and others. The level set method generally requires structures which allow the comparison theorem among solutions. Thus while having the nice uniqueness property, the applicability is limited within the scalar problems. The phase field method is more robust in general in that it can accommodate any number of unknown functions. In the talk I discuss the wide possibility of phase field approach as a tool to establish various time-global existence results of MCF-type in view of the recent rapid progress such as the resolution of 'Modified conjecture of De Giorgi' due to Matthias Röger and Reiner Schätzle. To illustrate the applicability we discuss the two-phase fluid flow problem coupled with the MCF equation.

Let $\Omega = (0, L)^d \subset \mathbf{R}^d$ and suppose that Ω at time t is separated into two disjoint sets $\Omega^{\pm}(t)$ which represent the domains of occupation of each phases, and suppose $\Gamma(t)$ is the $(d-1)$-dimensional interface separating $\Omega^{\pm}(t)$. Suppose that on each domain $\cup_{t\in(0,T)}\Omega^{\pm}(t) \times \{t\}$, v satisfies a non-Newtonian Navier-Stokes equation

$$v_t + v \cdot \nabla v = \operatorname{div}\left((1 + |e(v)|^2)^{\frac{p-2}{2}} e(v)\right) - \nabla P, \qquad \operatorname{div} v = 0$$

where $e(v) = \frac{\nabla v + \nabla v^t}{2}$ is the symmetric part of ∇v and P is the pressure. In case $p = 2$ this is the standard Navier-Stokes equation. On the interface $\Gamma(t)$ we assume that the surface tension force acts on the fluid so the balance of force materializes as

$$n \cdot \Big[(1 + |e(v)|^2)^{\frac{p-2}{2}} e(v) - pI\Big]_{\text{jump}} = \kappa_2 H$$

where n is the unit normal of $\Gamma(t)$, I is the $d \times d$ identity matrix, $\kappa_2 > 0$ is a positive constant and H is the mean curvature vector of $\Gamma(t)$. We also assume that the normal velocity V of the interface $\Gamma(t)$ is given by $V = (n \cdot v)n + \kappa_1 H$, thus the conventional kinematic condition ($\kappa_1 = 0$) is modified by the diffusive factor given by $\kappa_1 H$. The problem has the energy law:

$$\frac{d}{dt}\left(\frac{1}{2}\int |v|^2 + \kappa_2 \mathcal{H}^{d-1}(\Gamma(t))\right) = -\int ((1 + |e(v)|^2)^{\frac{p}{2}} - \kappa_1\kappa_2 \int_{\Gamma(t)} |H|^2.$$

We present the approximation scheme of this problem using the phase field method:

$$\begin{cases} v_t + v \cdot \nabla v = \operatorname{div}\left((1+|e(v)|^2)^{\frac{p-2}{2}} e(v)\right) - \nabla p - \frac{\varepsilon \kappa_2}{\sigma} \operatorname{div}(\nabla\phi \otimes \nabla\phi) * \eta, \\ \operatorname{div} v = 0, \\ \phi_t + (v * \eta) \cdot \nabla \phi = \kappa_1 \left(\Delta\phi - \frac{W'(\phi)}{\varepsilon^2} \right). \end{cases}$$

The suitable initial data are supplemented along with the periodic boundary conditions. Here σ is the normalizing constant determined by W. $*\eta$ is a convolution of mollifying function η which smooths out the interaction terms (particularly v) over the ball of radius ε^q, $0 < q < 1/d$. This smoothing of v is technically motivated, but it is interesting to note that such smoothing is regularly done in numerical simulations of turbulent flow in the Large Eddy Simulation. We chose q so that the resulting $|v * \eta|$ is bounded by $\varepsilon^{-\gamma}$, $\gamma < \frac{1}{2}$. Due to the energy law the solutions v_ε and ϕ_ε satisfy (where we now set $\kappa_1 = \kappa_2 = 1$ for simplicity)

$$\begin{aligned} &\sup_{[0,T]} \int_\Omega \left(\frac{\varepsilon|\nabla\phi_\varepsilon|^2}{2} + \frac{W(\phi_\varepsilon)}{\varepsilon} \right) + \int_{[0,T]\times\Omega} \frac{1}{\varepsilon} \left(\varepsilon\Delta\phi_\varepsilon - \frac{W'(\phi_\varepsilon)}{\varepsilon} \right)^2 \le C, \\ &\sup_{[0,T]} \int_\Omega \frac{|v_\varepsilon|^2}{2} + \int_{[0,T]\times\Omega} |\nabla v_\varepsilon|^p + \varepsilon^\gamma \|v_\varepsilon\|_{L^\infty([0,T]\times\Omega)} \le C \end{aligned}$$

where C is a constant depending only on the initial data but not on ε. From the property of the fluid equation we may also assume that v_ε converges to v strongly in $L^2([0,T]\times\Omega)$ as $\varepsilon \to 0$. Under those condition we prove the following.

Theorem (Chun Liu, Norifumi Sato, T. in preparation)
Let $d = 2, 3$ and $p > \frac{d+2}{2}$. There exists a subsequential limit of measures

$$d\mu_t^\varepsilon = \left(\frac{\varepsilon|\nabla\phi_\varepsilon|^2}{2} + \frac{W(\phi_\varepsilon)}{\varepsilon} \right)\Big|_t \, dx \to d\mu_t$$

for all $t \in [0,T]$ such that

(a) $\frac{1}{\sigma}\mu_t$ is integral,
(b) μ_t satisfies Brakke's inequality with transport: for $0 \le \forall t_1 < \forall t_2 \le T$ and $\forall \psi \in C_c^2(\mathbf{R}^d;\mathbf{R}^+)$,

$$\int \psi \, d\mu_{t_2} - \int \psi \, d\mu_{t_1} \le \int_{t_1}^{t_2} \int (-H\psi + \nabla\psi) \cdot (H + (T_x\mu_t)^\perp v) \, d\mu_t dt.$$

(c) The function v is defined as the trace in (b) and belongs to $L^2(dt \otimes d\mu_t)$.

Note that (b) is the integral form of Brakke's formulation. The construction of such weak solution would be extremely complex (and likely be intractable) if one tries to follow Brakke's original construction of MCF. I discuss the idea of the proof and future outlook of the problem in the talk.

Isoperimetric inequalities for linear and nonlinear eigenvalues

Cristina Trombetti
(joint work with Barbara Brandolini and Carlo Nitsch)

The Faber-Krahn inequality states that among all domains with given measure the first eigenvalue of $-\Delta$ is minimum on balls. It is also clear that it is not possible to bound it from above among all domains with given measure. In [1] we prove a reverse Faber-Krahn inequality for the eigenvalue of the Monge-Ampère operator. We show that, contrary to what happens for the operator $-\Delta$, this eigenvalue, among all convex sets with given measure, is bounded from above and it is maximum on the ellipsoids. In [2] we prove a kind of stability result for the first eigenvalue of $-\Delta_p$ (p-Laplace operator), $p > 1$. More precisely, if $\lambda_p(\Omega)$ denotes the first eigenvalue of $-\Delta_p$ in a bounded, convex domain $\Omega \subset \mathbb{R}^n$ and $\lambda_p(\Omega^\star)$ denotes the first eigenvalue of $-\Delta_p$ in the ball $\Omega^\star$ having the same perimeter as Ω, then we prove

$$0 \leq \frac{\lambda_p(\Omega) - \lambda_p(\Omega^\star)}{\lambda_p(\Omega)} \leq C(n,p)\delta(\Omega),$$

for some positive constant $C(n,p)$. Here $\delta(\Omega)$ is the isoperimetric deficit of Ω defined as

$$\delta(\Omega) = \frac{P(\Omega)^{\frac{n}{n-1}} - n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{P(\Omega)^{\frac{n}{n-1}}},$$

where $P(\Omega)$ and $|\Omega|$ are the perimeter and the measure of Ω, and ω_n is the measure of the unit ball in $\mathbb{R}^n$.

References

[1] B. Brandolini, C. Nitsch, and C. Trombetti, *New isoperimetric estimates for solutions to Monge-Ampère equations.* Ann. I. H. Poincaré - AN **26** (2009), 1265–1275.
[2] B. Brandolini, C. Nitsch, and C. Trombetti, *An upper bound for nonlinear eigenvalues by means of the isoperimetric deficit.* Preprint (2009).

Large amplitude nonlinear water waves with vorticity

Eugen Varvaruca
(joint work with Ovidiu Savin and Georg Weiss)

We study periodic travelling-wave solutions for the two-dimensional Euler equations describing the dynamics of an incompressible, inviscid, heavy fluid over a flat bottom and with a free surface. The corresponding mathematical problem is to find a domain Ω in the (X,Y)-plane, which lies above a horizontal line $\mathcal{B}_F := \{(X,F) : X \in \mathbb{R}\}$, where F is a constant, and below some a priori unknown curve $\mathcal{S} := \{(X, \eta(X)) : X \in \mathbb{R}\}$, where $\eta : \mathbb{R} \to \mathbb{R}$ is $2L$-periodic, and a function ψ in Ω which satisfies the following equations and boundary conditions:

$$\begin{aligned}
&\Delta\psi = -\gamma(\psi) \quad \text{in } \Omega,\\
&\psi = B \quad \text{on } \mathcal{B}_F,\\
&\psi = 0 \quad \text{on } \mathcal{S},\\
&|\nabla\psi|^2 + 2gY = Q \quad \text{on } \mathcal{S},\\
&\psi(X + 2L, Y) = \psi(X, Y) \quad \text{for all } (X, Y) \in \Omega,
\end{aligned}$$

where B, g, L are given positive constants, $\gamma \in C^{1,\alpha}([0, B])$ is a given *vorticity function* and Q, F are parameters. By a vertical translation, we may always assume either that $Q = 0$ or that $F = 0$.

In most of what follows we consider *solutions of type (SMG)* (symmetric monotone graphs), for which $\mathcal{S} := \{(X, \eta(X)) : X \in \mathbb{R}\}$, where η is even, $\eta'(X) < 0$ on $(0, L)$, and $\psi_Y < 0$ in Ω. We are interested in the existence and properties of *extreme waves*, which are waves with *stagnation points* ($\nabla\psi = (0, 0)$) on the free surface $\mathcal{S}$. At such points $\mathcal{S}$ need not be smooth, and we are interested in the shape of $\mathcal{S}$ close to such points. A famous conjecture of Stokes (1880) claims that (at least in the case of zero vorticity) the profile of any extreme wave has corners with included angle of 120° at stagnation points. Note however that for certain vorticity functions $\gamma : [0, B] \to \mathbb{R}$ there exist trivial extreme waves, whose free surface is a horizontal line all of whose points are stagnation points.

The first global theory of waves with general vorticity $\gamma : [0, B] \to \mathbb{R}$ was given by Constantin and Strauss (2004). Many authors have since then contributed to this theory. Constantin and Strauss (2004) proved, under very general assumptions on γ, the existence of *almost extreme waves*: a sequence of regular waves of type (SMG) $\{(\mathcal{S}_j, \mathcal{B}_0, \psi^j, Q_j)\}_{j\geq 1}$ for which

$$\max_{\overline{\Omega}_j} \psi^j_Y \to 0 \text{ as } j \to \infty.$$

(Being of type (SMG), they satisfy $\psi^j_Y < 0$ everywhere in Ω_j.) Numerical evidence suggests that this sequence converges either to an extreme wave which satisfies the Stokes conjecture, or to a smooth wave with a stagnation point on the bottom directly below the crest. Rigorous results on the existence of extreme waves and the Stokes conjecture had been known previously only in the case of zero vorticity.

The following theorem proves the existence of extreme waves for any nonpositive vorticity function.

Theorem 1 (Savin and Varvaruca (2009))**.** *Suppose that $\gamma(r) \leq 0$ for all $r \in [0, B]$. Let $\{(\mathcal{S}_j, \mathcal{B}_0, \psi^j, Q_j)\}_{j\geq 1}$ be a sequence of regular waves of type* (SMG) *such that*

$$\max_{\overline{\Omega}_j} \psi^j_Y \to 0 \textit{ as } j \to \infty.$$

Then $\{(\mathcal{S}_j, \mathcal{B}_0, \psi^j, Q_j)\}_{j\geq 1}$ *'converges' along a subsequence to an extreme wave* $(\widetilde{\mathcal{S}}, \mathcal{B}_0, \tilde{\psi}, \widetilde{Q})$ *with stagnation points at its crests, and for which the crests are the only stagnation points.*

The extreme wave obtained Theorem 1 is a weak solution of the problem, and the convergence obtained is in a weak sense, once new a priori estimates have been derived for the sequence of regular waves by using the maximum principle.

The following theorem proves a version of the Stokes conjecture which is in a certain sense optimal.

Theorem 2 (Varvaruca and Weiss (2009))**.** *Let* $(\mathcal{S}, \psi)$ *be an extreme wave, with* $Q = 0$*. Suppose that* $\mathcal{S} := \{(X, \eta(X)) : X \in \mathbb{R}\}$*, where* $\eta : \mathbb{R} \to \mathbb{R}$ *is continuous and locally of bounded variation. Suppose also that* $\psi_Y < 0$ *in* Ω*. Let* $(X_0, \eta(X_0))$ *be a stagnation point, i.e.* $\eta(X_0) = 0$*. Then*

$$\text{either} \lim_{X\to X_0\pm} \frac{\eta(X)}{X - X_0} = \mp\frac{1}{\sqrt{3}} \ \text{or} \lim_{X\to X_0\pm} \frac{\eta(X)}{X - X_0} = 0.$$

Moreover, if $\gamma(r) \geq 0$ *for all* $r \in [0, \delta]$*, for some* $\delta \in (0, B]$*, then*

$$\lim_{X\to X_0\pm} \frac{\eta(X)}{X - X_0} = \mp\frac{1}{\sqrt{3}}.$$

Note that, in the situation of the second part of Theorem 2, any stagnation point is *isolated* and hence there can be at most finitely many stagnation points on a period of the wave. This settles an open problem raised by Shargorodsky and Toland.

In the proof of Theorem 2, the behaviour close to any stagnation point of the curve $\mathcal{S}$ and the function ψ is studied by means of *blow-up sequences*. Suppose with no loss of generality that $X_0 = 0$. Let $\{\varepsilon_j\}_{j\geq 1}$ be a sequence such that $\varepsilon_j \searrow 0$ as $j \to \infty$, and let us consider the sequence $\{\psi^j\}_{j\geq 1}$ given by

$$\psi^j(X, Y) := \varepsilon_j^{-3/2}\psi(\varepsilon_j X, \varepsilon_j Y).$$

It is expected that any weak limit $\tilde{\psi}$ of $\{\psi_j\}_{j\geq 1}$ along a subsequence satisfies a *limiting problem*: find a locally rectifiable curve $\widetilde{\mathcal{S}}$ and a function $\tilde{\psi}$ in the unbounded domain $\widetilde{\Omega}$ below $\widetilde{\mathcal{S}}$, such that

$$\begin{aligned} &\Delta\tilde{\psi} = 0 \quad \text{in } \widetilde{\Omega}, \\ &\tilde{\psi} = 0 \quad \text{on } \widetilde{\mathcal{S}}, \\ &|\nabla\tilde{\psi}|^2 + 2gY = 0 \quad \mathcal{H}^1\text{-almost everywhere on } \widetilde{\mathcal{S}}. \end{aligned}$$

This has a trivial solution $(\widetilde{\mathcal{S}}_0, \tilde{\psi}_0)$, where $\widetilde{\mathcal{S}}_0 := \{(X, 0) : X \in \mathbb{R}\}$ and $\tilde{\psi}_0 \equiv 0$ in $\mathbb{R}^2_-$. Another solution, originally discovered by Stokes and nowadays called the *Stokes corner flow*, is the following: let $\widetilde{\mathcal{S}}^* := \{(X, \eta^*(X)) : X \in \mathbb{R}\}$, where

$$\eta^*(X) := -\frac{1}{\sqrt{3}}|X| \quad \text{for all } X \in \mathbb{R},$$

let $\widetilde{\Omega}^*$ be the domain below $\widetilde{\mathcal{S}}^*$, and let the function $\tilde{\psi}^*$ in $\widetilde{\Omega}^*$ be given, for all $(X,Y) \in \widetilde{\Omega}^*$, by

$$\tilde{\psi}^*(X,Y) := \frac{2}{3} g^{1/2} \operatorname{Im} \left(i(iZ)^{3/2} \right) \quad \text{where } Z = X + iY.$$

The key to the proof of Theorem 2 is the following uniqueness result.

Theorem 3 (Varvaruca and Weiss (2009))**.** *Any blow-up limit of a solution of the original problem is necessarily homogeneous of degree* 3/2*, and therefore is the Stokes corner flow.*

The proof uses a new ingredient, the *Monotonicity Formula*: the function

$$\Phi(r) := r^{-3} \int_{B_r(0)} \left(|\nabla\psi|^2 - 2\Gamma(\psi) - 2gY\chi_{\{\psi>0\}} \right) d\mathcal{L}^2$$
$$- \frac{3}{2} r^{-4} \int_{\partial B_r(0)} \psi^2 d\mathcal{H}^1 + \int_0^r s^{-4} \int_{B_s(0)} (2\Gamma(\psi) - 3\gamma(\psi)\psi)\, d\mathcal{L}^2\, ds$$

satisfies, for almost every r sufficiently small,

$$\frac{d}{dr}\Phi(r) = r^{-3} \int_{\partial B_r(0)} 2 \left(\nabla\psi \cdot \nu - \frac{3}{2}\frac{\psi}{r} \right)^2 d\mathcal{H}^1.$$

(The function $\Gamma : [0,B] \to \mathbb{R}$ is defined by $\Gamma(t) := \int_0^t \gamma(s)\, ds$ for all $t \in [0,B]$.) The proof of this formula is by direct verification, using a Pohozaev-type identity. Let $\{\psi^j\}_{j\geq 1}$ be the blow-up sequence

$$\psi^j(X,Y) := \varepsilon_j^{-3/2} \psi(\varepsilon_j X, \varepsilon_j Y).$$

It is immediate from the Monotonicity Formula that any weak limit $\tilde{\psi}$ of $\{\psi^j\}_{j\geq 1}$ along a subsequence is a function homogeneous of degree 3/2, and hence it is either identically 0 or coincides with the Stokes corner flow.

Landau damping

Cédric Villani

1. Introduction

A cornerstone of plasma physics, Landau damping predicts the stability of certain "stable" homogeneous equilibria of the Vlasov–Poisson equation. Well-understood at the linear level, it has remained elusive in the nonlinear regime for fifty years — except for some heuristic arguments in the quasilinear regime. One of the main difficulties, as outlined by Backus in 1960, is to check that the rapidly oscillating nonlinear terms, entailing a degradation of the smoothness as time goes by, will not destroy the linear damping. Another main challenge is to understand the robustness of the energy cascade from low to high velocity modes, under nonlinear perturbation.

In a recent work, Clément Mouhot and myself “solved” this problem; on that occasion we introduced several new tools (functional spaces, scattering control, functional inequalities) which may also prove useful in related issues. Here I shall only state the main result and refer to our preprint *Landau damping* (available on my Web page) for a more precise commented short presentation with comments; or to the full version of our article, *On Landau damping*.

2. Fourier transform

As in the classical theory of Landau damping, Fourier transform plays a crucial role in our study, It is important to distinguish between the Fourier transforms in the position and velocity variables. We consider periodic boundary conditions, which are simple and still retain the main physical issues.

For $x \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, $v \in \mathbb{R}^d$, let

$$\hat{f}(k,v) = \int e^{-2i\pi k\cdot x} f(x,v)\, dx, \qquad k \in \mathbb{Z}^d,$$

$$\tilde{f}(k,\eta) = \iint e^{-2i\pi k\cdot x}\, e^{-2i\pi \eta\cdot v}\, f(x,v)\, dx\, dv, \qquad k \in \mathbb{Z}^d,\ \eta \in \mathbb{R}^d.$$

Using the Fourier transform, we also introduce a norm measuring analyticity and decay (the latter only in velocity space):

$$\|f\|_{\lambda,\mu,\beta} = \sup_{k,\eta} \left|\tilde{f}(k,\eta)\right| e^{2\pi\lambda|\eta|}\, e^{2\pi\mu|k|} \;+ \iint |f(x,v)|\, e^{2\pi\beta|v|}\, dv\, dx.$$

3. Main result

Let $W : \mathbb{T}^d \to \mathbb{R}$ be an interaction potential, $f^0 = f^0(v) \geq 0$ be an analytic distribution (to be thought of as a stable equilibrium), and $f_i = f_i(x,v) \geq 0$ be another analytic distribution (to be thought of as a perturbation of f^0). For $\xi \in \mathbb{C}$, $k \in \mathbb{Z}^d$, let

$$\mathcal{L}(\xi,k) = -4\pi^2\, \widehat{W}(k) \int_0^\infty e^{2\pi|k|\xi^* t}\, |\tilde{f}^0(kt)|\, |k|^2\, t\, dt.$$

Assume that $\tilde{f}^0(\eta)$ decays exponentially fast as $|\eta| \to \infty$. Further assume that for some positive parameters λ, μ, β,

(i) $\inf_{k\in\mathbb{Z}^d} \inf_{0\leq \Re\xi<\lambda} \left|\mathcal{L}(\xi,k) - 1\right| > 0$ [linear stability of f^0]

(ii) $|\widehat{W}(k)| = O\left(\frac{1}{|k|^{1+\gamma}}\right)$, $\gamma > 1$ [interaction potential not too singular]

(iii) $\|f_i - f^0\|_{\lambda,\mu,\beta} \leq \varepsilon \ll 1$ [initial datum close to the stable equilibrium]

(Here ε depends on the other parameters in the problem.) Let $f = f(t,x,v)$ solve the nonlinear Vlasov–Poisson equation with initial datum f_i and interaction potential W:

$$\begin{cases} \dfrac{\partial f}{\partial t} + v \cdot \nabla_x f + F[f](t,x) \cdot \nabla_v f = 0 \\ f(0,\,\cdot\,) = f_i, \end{cases}$$

where

$$F[f](t,x) = -\iint \nabla W(x-y)\, f(t,y,w)\, dy\, dw$$

is the force induced by the distribution f. Then

(a) $F[f](t,\,\cdot\,) \longrightarrow 0$ exponentially fast as $t \to \pm\infty$ (in analytic norms);

(b) $f(t,\cdot,\cdot) \longrightarrow f_{\pm\infty}$ exponentially fast as $t \to \pm\infty$ (in the weak topology).

4. Coulomb or Newton potential

The physically most relevant case of Coulomb or Newton interactions are not covered by the previous theorem. However, in that situation we can adapt the proof to get damping over extremely large (exponential in $1/\varepsilon$) time scales. This can be considered as the reply to the objection formulated by Backus. We refer to the original works for more information.

Convex solutions to the mean curvature flow

Xu-jia Wang

We consider convex solutions u to the equation

$$(*) \qquad \sum_{i,j=1}^{n} (\delta_{ij} - \frac{u_i u_j}{\sigma + |Du|^2}) u_{ij} = 1,$$

where $\sigma \in [0,1]$ is a constant. If $\sigma = 0$, the level sets of u satisfy the mean curvature flow equation. When $\sigma = 1$, u is a translation solution to the mean curvature flow.

It was proved by Huisken-Sinestrari that if $\mathcal{M}$ is a mean convex flow, namely a mean curvature flow for hypersurfaces with positive mean convex, then the limit flow obtained by a proper blow-up near type II singular points is a convex translating solution. Separately White proved that a limit flow to the mean convex flow is an ancient convex solution. To classify the convex translating solutions of Huisken-Sinestrari and the convex solutions of White, one needs to classify convex solutions to the above equation (*) for $\sigma = 0$ or $\sigma = 1$. Our results can be summarized in the following theorems.

Theorem 1.1. *If $n = 2$, then any entire convex solution to (*) must be rotationally symmetric in an appropriate coordinate system.*

From Theorem 1.1 we obtain

Corollary 1.1. *A convex translating solution to the mean curvature flow must be rotationally symmetric if it is a limit flow to a mean convex flow in $\mathbf{R}^3$.*

Theorem 1.2. *For any dimension $n \geq 3$ and $1 \leq k \leq n$, there exist entire convex solutions to (*) which are not k-rotationally symmetric.*

Theorem 1.3. *Let u be an entire convex solution of (1.2). Let $u_h(x) = h^{-1}u\sqrt{h}\,x)$. Then there is an integer $2 \leq k \leq n$ such that after a rotation of the coordinate system for each h, u_h converges to*

$$\eta_k(x) = \frac{1}{2(k-1)} \sum_{i=1}^{k} x_i^2.$$

We say a solution to the mean curvature flow is *ancient* if it exists from time $-\infty$, and u is an *entire* solution if it is defined in the whole space $\mathbf{R}^n$. We say u is k*-rotationally symmetric* if there exists an integer $1 \leq k \leq n$ such that u is rotationally symmetric with respect to $x_1, \cdots, x_k$ and is independent of $x_{k+1}, \cdots, x_n$. The above results are proved in

Xu-Jia Wang, *Convex solutions to the mean curvature flow*, arXiv:math.DG/0404326..

Pseudo-hermitian geometry in 3-D

Paul Yang

In this talk, I summarize recent work on two topics in the study of CR invariants in dimension three.

A three manifold M with a contact 1-form θ satisfying $\theta \wedge d\theta \neq 0$, carries a distribution ξ given by the kernel of θ, and an almost complex structure J on ξ. There is a connection defined by Webster and Tanaka that solves the equivalence problem. The basic local invariants in this geometry are the torsion and the Webster scalar curvature R.

The first topic concerns the equation of mean curvature for a surface in this geometry. A particular example is the equation for a graph in the Heisenberg group,

$$H = \frac{(u_y + x)^2 u_{xx} - 2(u_y + x)(u_x - y)u_{xy} + (u_x - y)^2 u_{yy}}{D^3}$$

where D is the area element $D = \sqrt{(u_y + x)^2 + (u_x - y)^2}$. Intrisically, let e_1 denote the unit vector tangent to the surface as well as to the contact plane, the equation may be written as

$$\nabla_{e_1} e_1 = HJe_1 = He_2.$$

For a C^2 smooth surface, it is easy to construct a characteristic coordinate system $(s, t) \in \mathbb{R}^2$ so that $\partial s = fe_1$, $\partial t = gDJe_1$, then easy computation leads to the following analogue of the Codazzi equation:

$$D\text{"}D = 2(D' - 1)(D' - 2) + (H^2 + e_2(H))D^2.$$

In a series of joint work with Jih -Hsin Cheng, Jenn-Fann Hwang, and Andrea Malchiodi we formulate the notion of weak solution and proved this condition to be equivalent to the notion of minimizer.

Let Σ be a C^1 surface in M, a singular point is one where the tangent plane coincides with the contact point. For a C^1 weak solution of the mean curvature equation, we then derive enough regularity of the characteristic coordinates to recover the validity of the Codazzi equation. As a consequence, we can describe the structure of singular set in such a surface: it consists of isolated points and piecewise C^1 tree. We also determine the index of the line field e_1 near the singular points so that the Hopf index formula continues to hold.

The second topic is concerned with the analogue of the Yamabe equation to prescribe the Webster scalar curvature: To find a contact form $\theta' = u^2\theta$ for which the equation holds

$$Lu = -\Delta_b u + (1/4)Ru = (1/4)R'u^3,$$

where R' is a constant. Previously, Gamara and Yacoub showed the existence of solutions using the theory of critical points at infinity. It was not known whether these are minimizing solutions.

To find minimizing solutions, we introduce the notion of an asymptotically Heisenberg 3-manifold, and define a notion of mass for such a structure.

In a joint work in progress with Jih-Hsin Cheng, Hung-Lin Chiu, and Andrea Malchiodi, we provide a criteria for this mass to be positive under two positivity assumptions. The positivity of the operators L (already defined above) and that of the operator P which is the analogue of the Paneitz operator:

$$Pu = \Delta_b^2 u + T^2 u - 2i\{u_{11}A_{\bar{1}\bar{1}} - u_{\bar{1}\bar{1}} + u_1 A_{\bar{1}\bar{1},1} - u_{\bar{1}}A_{11,\bar{1}}\}.$$

It is necessary to remark that kernel of P contains the set of all plurisubharmonic functions, and hence is rather large.

This positive mass result allows the construction of test functions to provide for compactness in a minimizing sequence as in the case of the conformally invarint Yamabe problem.

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