Mathematisches Forschungsinstitut Oberwolfach

Report No. 35/2009

DOI: 10.4171/OWR/2009/35

Differentialgeometrie im Großen

Organised by
Olivier Biquard (Paris)
Bernhard Leeb (München)
Gang Tian (Princeton)

July 26th – August 1st, 2009

Abstract. The meeting continued the biannual conference series *Differentialgeometrie im Großen* at the MFO which was established in the 60's by Klingenberg and Chern. Global Riemannian geometry with its connections to geometric analysis, topology and geometric group theory is the guiding theme of the conference. Special emphasis was given to geometric structures on manifolds, the geometry of singular spaces, geometric evolution equations and the collapse of Riemannian manifolds.

Mathematics Subject Classification (2000): 53Cxx, 51K10, 58Jxx, 20Fxx, 32Qxx, 57Mxx.

Introduction by the Organisers

The meeting continued the biannual conference series *Differentialgeometrie im Großen* at the MFO which was established in the 60's by Klingenberg and Chern. Traditionally, the conference series covers a wide scope of different aspects of global differential geometry and its connections with geometric analysis, topology and geometric group theory. The Riemannian aspect is emphasized, but the interactions with the developments in complex geometry, symplectic/contact geometry/topology and physics play also an important role. Within this spectrum each particular conference gives special attention to two or three topics of particular current relevance.

Whereas in the recent previous conferences 22 (almost) one hour talks were delivered, the scientific program consisted this time of only 17 one hour talks allowing a less pressured schedule and leaving ample time for informal discussions. Apparently, not only the organizers but also many of the participants were quite satisfied with this modified approach.

A prominent theme of the workshop were *geometric structures and discrete subgroups of Lie groups*, represented by five talks concerned with hyperbolic and projective structures on manifolds and their deformations, representation varieties of surface groups, lattices and ergodic theory.

Another focus was the *geometry of singular spaces*, that is, metric spaces with upper or lower sectional curvature bounds (in the sense of Aleksandrov), including building theory from the perspective of comparison geometry, and Gromov-hyperbolic spaces with connections to geometric group theory.

An important role was again played by *geometric evolution equations* and, motivated by the desire to understand the degenerations of the Ricci flow, the study of the topology and geometry of collapsing Riemannian metrics in dimension three and higher subject to various kinds of curvature control.

Other talks presented results about conformal geometry, minimal surfaces and manifolds with positive sectional curvature.

There were 47 participants from 8 countries, more specifically, 19 participants from Germany, 16 from the United States of America, 5 from France, 5 from other European countries, and 2 from Canada. 13% of the participants (6) were women. 30% of the participants (14) were young researchers (less than 10 years after diploma or B.A.), both on doctoral and postdoctoral level.

The organizers would like to thank the institute staff for their great hospitality and support before and during the conference. The financial support by the new programme Oberwolfach Leibniz Graduate Students for young participants is gratefully acknowledged.

Workshop: Differentialgeometrie im Großen

Table of Contents

Abstracts

Fatness and beyond

Wolfgang Ziller

A Kaluza Klein metric, or connection metric, is a metric on a principal G-bundle $\pi\colon P \to B$ of the form

$$\langle U, V\rangle = g_B(\pi_*(U), \pi_*(V)) + Q(\theta(U), \theta(V))$$

where g_B is a metric on the base, θ a principle connection, and Q a fixed biinvariant metric on the Lie algebra of G. We will use also this terminology in the more general context of orbifold principle bundles, although we will assume that P is a manifold. This includes the case where G acts on P with finite isotropy groups. Such a connection metric makes the projection π into a Riemannian submersion with totally geodesic fibers.

Weinstein coined the concept of fatness for a general Riemannian submersion with totally geodesic fibers. A bundle is called *fat* if all vertizontal curvatures, i.e. all sectional curvature spanned by a horizontal and vertical vector are positive. In the case of a connection metric this turns out to be equivalent to a strong condition on the curvature Ω of θ:

$$\Omega_u = Q(\Omega, u) \text{ is non-degenerate on } \ker\theta \text{ for all } u \in \mathfrak{g}$$

which is a condition on the principle connection alone. Following an idea of Weinstein, we exhibit obstructions to the existence of a fat connection in terms of characteristic classes (joint work with Luis Florit). E.g., when $G = U(n)$ and $\dim B = 2m$, the existence of a fact connection implies the nonvanishing of the characteristic numbers

$$\sum_{n\geq\lambda_1\geq\cdots\geq\lambda_m\geq 0}\ \prod_{i=1}^{i=m} (n+m-i-\lambda_i)!\ \det\left(\sigma_{\lambda_i+j-i}(y)\right)_{1\leq i,j\leq m}\ \det\left(c_{\lambda_i+j-i}\right)_{1\leq i,j\leq m} \neq 0,$$

for all $0 \neq y = (y_1, \ldots, y_n)$, where $|\lambda| = m$, $\sigma_i(y) = \sigma_i(y_1, \ldots, y_n)$ is the elementary symmetric polynomial of degree i, and $c_i \in H^{2i}(B)$ are the Chern classes of the bundle.

We say that a connection is *hyperfat* if all sectional curvatures of the connection metric on P are positive, as long as the metric is scaled with a sufficiently small real number in the direction of the fibers. This condition is equivalent to:

$$\left(\nabla_x\Omega_u\right)(x, y)^2 < |i_x\Omega_u|^2 k_B(x, y),$$

for all orthonormal $x, y \in \ker\theta$ and $0 \neq u \in \mathfrak{g}$. In joint work with Karsten Grove and Luigi Verdiani we showed that:

There exists an orbifold bundle $SU(2) \to P^7 \to S^4$ with a hyperfat connection

In particular, P^7 is a new example with positive sectional curvature. We also observed that P is homeomorphic to the unit tangent bundle of S^4, although we do not know if it diffeonmorphic to it or not.

Equivariant classification of nonnegatively curved 4-manifolds with an isometric S^1-action

Burkhard Wilking

Hsiang-Kleiner and Kleiner showed that a nonnegatively curved 4-manifold which is compact, simply connected and has a nontrivial isometric S^1-action is homeomorphic to S^4, $\mathbb{C}P^2$, $S^2 \times S^2$ or $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$. Using purely topological methods Kim improved the conclusion to diffeomorphic.
On the other hand it is well known that each of these 4-manifolds admits lots of exotic S^1-action.
In joint work with K.Grove, I was able to show that in the above theorem the conclusion can be improved to equivariant diffeomorphic.
The basic problem arises if M^4/S^1 is a homotopy sphere containing a closed singular curve.
Combining the ellipticization conjecture with Alexandrov geometry we are able to show that such a curve cannot be knotted.

Locally collapsed 3-manifolds

John Lott
(joint work with Bruce Kleiner)

We give a proof of a result stated by Perelman, concerning closed 3-manifolds M that are locally volume-collapsed with respect to a local lower sectional curvature bound [3, Theorem 7.4].

Definition 0.1. Fix $\bar{w} \in (0, \frac{4\pi}{3})$. Given $p \in M$, the *$\bar{w}$-volume scale at p* is

$$r_p(\bar{w}) = \inf\{r > 0 \, : \, \text{vol}(B(p,r)) = \bar{w}\, r^3\}.$$

If there is no such r then we say that the $\bar{w}$-volume scale is infinite.

Definition 0.2. Given $p \in M$, the *curvature scale at p* is the (unique) number $r > 0$, if it exists, such that the infimum of the sectional curvatures on $B(p,r)$ equals $-r^{-2}$. Otherwise, we say that the curvature scale at p is infinite.

Theorem 0.3. *Let (M^α, g^α) be a sequence of closed connected Riemannian 3-manifolds. Let $K \geq 10$ be a fixed integer. Suppose that for each $w' \in (0, \frac{4\pi}{3})$,*

(1) *As $\alpha \to \infty$, the infimum of (the curvature scale divided by the w'-volume scale) tends to infinity.*

(2) *For each* $C < \infty$*, there is a number* $A(C, w') < \infty$ *with the following property. Given* $p \in M^\alpha$*, let* $r_p(w')$ *denote the* w'*-volume scale at* p*. Then for each integer* $k \in [0, K]$*,*

$$|\nabla^k \mathrm{Rm}| \leq A(C, w')\, r_p(w')^{-(k+2)}$$

on $B(p, Cr_p(w'))$*.*

Then for large α*,* M^α *is a graph manifold.*

There is also a version of Theorem 0.3 for manifolds-with-boundary.

Theorem 0.3 is used in Perelman's proof of the geometrization conjecture. It gives the nonhyperbolic pieces in the Thurston decomposition of the 3-manifold. Other proofs of Theorem 0.3 appear in [1, 2, 4].

In the talk we described three ideas in the proof :

(1) Stratification of the 3-manifold.
(2) Fake volume scale.
(3) Cloudy manifolds.

References

[1] L. Bessières, G. Besson, M. Boileau, S. Maillot and J. Porti, *Weak collapsing and geometrisation of aspherical 3-manifolds*, http://arxiv.org/abs/0706.2065

[2] J. Morgan and G. Tian, *Completion of the Proof of the Geometrization Conjecture*, http://arxiv.org/abs/0809.4040

[3] G. Perelman, *Ricci Flow with Surgery on Three-Manifolds*, http://arxiv.org/abs/math.DG/0303109

[4] T. Shioya and T. Yamaguchi, *Volume-Collapsed Three-Manifolds with a Lower Curvature Bound*, Math. Ann. **333** (2005), 131–155.

N^*-Bundles and Almost Ricci Flat Spaces

Aaron Naber

(joint work with Gamg Tian)

In this talk we study collapsing sequences $M_i \overset{GH}{\to} X$ of Riemannian manifolds with curvature bounded or curvature bounded away from a controlled subset. We introduce a structure over X which in an appropriate sense is dual to the N-structure of Cheeger, Dukaya and Gromov. As opposed to the N-structure, which lives over the M_i themselves and takes the form of a sheaf over vector fields, the N^*-bundle is an equivariant vector bundle $V^T \to X$ over the limit space. This point of view allows for a convenient notion of global convergence as well as the appropriate background structure for doing analysis on X.

Topologically the N^*-bundle $V^T \to X$ is defined by the property that if $f_i : M_i \to X$ are the equivariantly smooth Gromov-Hausdorff maps then, after passing to a subsequence, the pullback vector bundles f^*V^T over M_i are all isomorphic to the tangent bundles TM_i.This is particularly interesting because it is perfectly possible for no two of the M_i to even be homotopic, and yet their tangent bundles are all pullbacks of the same bundle.

As a more concrete application of this structure we give a generalization of Gromov's Almost Flat Theorem and prove new Ricci pinching theorems which extend those known in the noncollapsed setting. The key point of both theorems is the following lemma: Let $(M_i, g_i) \overset{GH}{\to} X$ with the M_i closed n-manifolds with $diam_i \leq 1$, $|sec_i| \leq 1$ and $|Rc_i| \to 0$. Then we have that X is a Ricci flat orbifold.

The assumptions of the theorem are sharp in that if any are dropped then we know very little about the limit space X. The lemma can be viewed as a generalization of a theorem of Fukaya, which makes the statement that X is a flat orbifold if further $|sec_i| \to 0$. Notice also that the conclusion is not just geometric in nature, but also restricts the type of topological singularities that can exist in X. As a consequence we prove the following Almost Ricci Flat Theorem: Let (M^n, g) be a closed n-manifold with $|sec| \leq K$. Then there exists a constant $\epsilon = \epsilon(n, K) > 0$ such that if $|Rc| \leq \epsilon$ then M is an orbifold bundle over a Rici flat orbifold X whose fibers are infranil. Further $|sec_X| \leq K$ and infranil fibers are almost totally geodesic.

References

[1] A. Naber; G. Tian, *Geometric Structures of Collapsing Riemannian Manifolds II: N^*-bundles and Almost Ricci Flat Spaces*, preprint

Space of Ricci Flows

Xiu-Xiong Chen

Inspired by the canonical neighborhood theorem of G. Perelman in 3 dimensional, we study the weak compactness of sequence of ricci flow with scalar curvature bound, Kappa non-collapsing and integral curvature bound.

All of these constraints are natural in the Kahler ricci flow in Fano surface. In particular, we prove the 2-dimensional Hamilton-Tian conjecture that, KRf sequentially converges to Kahler Ricci Soliton except a finite number of points. As an application, we give a ricci flow based proof to the Calabi conjecture in Fano surface.

Deforming projective structures on hyperbolic three-manifolds

Joan Porti
(joint work with Michael Heusener)

A hyperbolic manifold is equipped with a canonical projective structure, and we ask whether it can be deformed or not. The situation in dimension two is well understood independently by Choi-Goldman [1], Labourie [5] and Loftin [6]. In dimension three there are examples that can be deformed and examples that are locally rigid [2, 3].

The goal of this talk is to give a result that constructs infinitely many hyperbolic three manifolds whose projective structure is locally rigid.

Those manifolds are constructed by means of Dehn filling. Let M^3 be a orientable hyperbolic 3-manifold of finite volume and with one cusp. It has a compact core $\bar{M}^3$ with boundary a torus. A Dehn filling means gluing a solid torus $D^2 \times S^1$ to $\bar{M}^3$ along the boundary, in order to get a closed manifold. The homeomorphism type of the manifold only depends on the homotopy class in $\partial\bar{M}^3$ of the curve attached to the meridian $\partial D^2 \times \{*\}$. By Thurston's hyperbolic Dehn filling, all but finitely many Dehn fillings are hyperbolic manifolds.

Instead of local rigidity we shall prove infinitesimal rigidity, a stronger notion. For this we look at the variety of representations $Hom(\pi_1(M, PGL(4))$ to parameterize local deformations, by the theorem of Ereshman-Thurston.

The Zariski tangent space to $Hom(\pi_1(M, PGL(4))$ quotiented out by the tangent space of the orbit by conjugation is the first cohomology group:

$$H^1(M, \mathfrak{sl}(4))$$

where the coefficients are taken in the lie algebra twisted by the holonomy representation.

Definition. A closed hyperbolic three manifold N^3 is called *infinitesimally projectively rigid* if

$$H^1(N^3; \mathfrak{sl}(4)) = 0.$$

Infinitesimal rigidity implies local rigidity, but the converse is false, as the examples of [2, 3] show.

For cusped manifolds of finite volume M^3, we consider its compact core $\bar{M}^3$

Definition. A cusped hyperbolic manifold of finite volume M^3 is called *infinitesimally projectively rigid* if the natural map induces an injection

$$H^1(M^3; \mathfrak{sl}(4)) \hookrightarrow H^1(\partial\bar{M}^3; \mathfrak{sl}(4))$$

In this case, infinitesimal rigidity implies local rigidity relative to the boundary. We have checked that the figure eighth knot exterior an the Whitehead link exterior are both infinitesimally projectively rigid cusped manifolds.

Our main result is the following:

Theorem 0.1. *Let M^3 be a cusped hyperbolic manifold of finite volume. If M^3 is infinitesimally projectively rigid, then infinitely many Dehn fillings on M^3 are also infinitesimally projectively rigid.*

The proof uses the analycity of Thurston's Dehn filling deformation space. This space has a complex analytic structure and, in the one cusp case, it is equivalent to a neighborhood of the origin in the complex plane. The complete structure corresponds to the origin, and a countable number of points correspond to incomplete structures whose completion is the Dehn filling. This countable set accumulates at the origin. The idea of the proof is to provide conditions for these parameters that guarantee that the Dehn filling is infinitesimally rigid, and to prove that these conditions hold away from a proper real analytic subset. The structures corresponding to Dehn filling accumulate at the complete structures (the origin) in such way that they cannot lie all of them in a proper real analytic subset (cf. [4]).

References

[1] Suhyoung Choi and William M. Goldman. The classification of real projective structures on compact surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 34(2):161–171, 1997.
[2] D. Cooper, D. D. Long, and M. B. Thistlethwaite. Computing varieties of representations of hyperbolic 3-manifolds into $SL(4, \mathbb{R})$. *Experiment. Math.*, 15(3):291–305, 2006.
[3] D. Cooper, D. D. Long, and M. B. Thistlethwaite. Flexing closed hyperbolic manifolds. *Geom. Topol.*, 11:2413–2440, 2007.
[4] Michael Kapovich. Deformations of representations of discrete subgroups of $SO(3, 1)$. *Math. Ann.*, 299(2):341–354, 1994.
[5] François Labourie. Flat projective structures on surfaces and cubic holomorphic differentials. *Pure Appl. Math. Q.*, 3(4, part 1):1057–1099, 2007.
[6] John C. Loftin. Affine spheres and convex $\mathbb{RP}^n$-manifolds. *Amer. J. Math.*, 123(2):255–274, 2001.

Invariant subsets of finite volume homogeneous spaces

Yves Benoist

(joint work with Jean François Quint)

The main result of this talk is:

Theorem. *Let G be a simple real Lie group, Λ be a lattice of G and μ be a probability measure on G whose support is compact and generates a Zariski dense subgroup of G. Then every μ-ergodic μ-stationary probability measure on G/Λ either has finite support or is the Haar probability measure.*

Corollary. *Let G be a simple real Lie group, Λ a lattice of G, and Γ a Zariski dense semigroup of G. Every Γ-ergodic Γ-invariant probability measure on X either has finite support or is the Haar measure.*

a) Every Γ-invariant infinite closed subset of X is equal to X.

b) Every sequence of distinct finite Γ-orbit becomes equidistributed in X with respect to the Haar measure.

The simplest example to which this theorem and its corollary apply is $G = PSL(2, \mathbb{R})$, $\Lambda = PSL(2, \mathbb{Z})$, $\mu = \frac{1}{2}(\delta_{g_1} + \delta_{g_2})$ and Γ the semigroup generated by g_1 and g_2 as soon as it is non solvable.

These statements extend previous results of Eskin and Margulis, of Clozel, Oh and Ullmo and of Bourgain, Furman, Lindenstrauss and Mozes.

References

[1] Y. Benoist, J.-F. Quint *Mesures stationnaires et fermés invariants des espaces homogènes*, C. R. Math. Acad. Sci. Paris (2008) and preprint (2009).

On Tits' Center Conjecture

Carlos Ramos Cuevas

A sphere S with the round metric of radius 1 has only *small* proper convex subsets. Namely, a convex subset of S is contained in a closed ball of radius $\frac{\pi}{2}$, or it is the sphere itself.

A spherical building is a CAT(1)-space with the following rigidity property: any two points are contained in a top-dimensional convex subset isometric to a round sphere of radius 1. A natural question is whether the convex subsets of a spherical building are more *flexible* than those in the sphere. (cf. [2, Question 1.5]).

Question 0.1. *Let $C \subset B$ be a closed convex subset of a spherical building B. If C is not a subbuilding, is it true that $rad_C(C) \leq \frac{\pi}{2}$?*

If $dim(C) \leq 1$, then it is easy to see that the answer is yes.

A result in [1] says that in a CAT(1)-space X of finite dimension and radius $\leq \frac{\pi}{2}$ the convex set Z of circumcenters of X is non-empty and has radius $< \frac{\pi}{2}$. In particular, the group of isometries of X must fix the unique circumcenter of Z. Regarding isometric actions on spherical buildings and their fixed points, we have a weaker version of Question 0.1:

Question 0.2. *If C is not a subbuilding, is it true that the group of isometries of C has a fixed point?*

If $dim(C) \leq 2$, Question 0.2 has a positive answer. This was proven in [1].

A spherical building carries a natural structure of a polyhedral complex. If we restrict our attention to convex subsets that are also subcomplexes of the building, then a weaker version of Question 0.2 becomes Tits' Center Conjecture (compare [4] and [6, Conjecture 2.8]):

Conjecture 0.3 (Tits' Center Conjecture)**.** *Let $K \subset B$ be a convex subcomplex of a spherical building. Then K is a subbuilding or the group $Stab_{Aut(B)}(K)$ of building automorphisms of B preserving K has a fixed point in K.*

A building automorphism is an isometry preserving the polyhedral structure.

This conjecture has been proven for thick spherical buildings without factors of type F_4, E_6, E_7 or E_8 in [4]. The case of buildings of type F_4 was announced in a talk at Oberwolfach by Parker and Tent [5]. Both approaches use the incidence geometries associated with the different buildings.

Our approach to these questions is of differential-geometric nature, using methods of the theory of metric spaces with curvature bounded above. We are able to obtain the following results:

Theorem 0.4 (with B. Leeb [3])**.** *The Center Conjecture 0.3 holds for spherical buildings of type F_4 and E_6.*

Theorem 0.5. *The Center Conjecture 0.3 holds for spherical buildings of type E_7 and E_8*

This completes the proof of the Center Conjecture for arbitrary thick spherical buildings.

References

[1] A. Balser, A. Lytchak, *Centers of convex subsets of buildings*, Ann. Glob. Anal. Geom. **28**, No. 2, 201-209 (2005).
[2] B. Kleiner, B. Leeb, *Rigidity of invariant convex sets in symmetric spaces*, Inventiones Mathematicae **163**, No. 3, 657-676 (2006).
[3] B. Leeb, C. Ramos Cuevas, *The center conjecture for spherical buildings of types F_4 and E_6*, Preprint. arXiv:0905.0839v1.
[4] B. Mühlherr, J. Tits, *The center conjecture for non-exceptional buildings*, J. Algebra **300**, No. 2, 687-706 (2006).
[5] C. Parker, K. Tent, *Tits' Center Conjecture*, Oberwolfach report, http://www.mfo.de/programme/schedule/2008/04/OWR_2008_03.pdf, January 2008.
[6] J.-P. Serre, *Complète réductibilité*, Sém. Bourbaki, exp. 932, Astérisque **299** (2005).

Domains of Discontinuity for Surface Groups

Olivier Guichard
(joint work with Anna Wienhard)

In [4] F. Labourie introduced the notion of Anosov structures and their holonomy representations, so called Anosov representations, to study the Hitchin component for $\mathrm{SL}(n, \mathbf{R})$. Anosov representations are in some sense a dynamical analogue of holonomy representations of geometric structures (in the sense of Ehresmann), but the concept of Anosov representations is more flexible. Anosov representations have been proven to be a key tool in the study of higher Teichmüller spaces. In this note we show that Anosov representations of surface groups actually give rise to geometric structures on compact manifolds.

Theorem 0.2. *Let Σ be a closed connected orientable surface of negative Euler characteristic, and let G be a semisimple Lie group not locally isomorphic to $\mathrm{SL}(2, \mathbf{R})$.*

Suppose that $\rho : \pi_1(\Sigma) \to G$ is an Anosov representation, then there exist a parabolic subgroup $Q < G$ and a non-empty open set $\Omega \subset G/Q$ such that $\rho(\pi_1(\Sigma))$ preserves Ω and acts on it freely, properly discontinuously and with compact quotient.

Note that Anosov representations are easily seen to be faithful with discrete image [4, 2]. In particular, Anosov representations into $\mathrm{SL}(2, \mathbf{R})$ are exactly Fuchsian representations, thus their action on the projective line is minimal.

The proof of Theorem 0.2 is constructive, *i.e.* we construct an explicit $Q < G$ and a domain $\Omega \subset G/Q$ (see Section 3 for examples). The construction uses the equivariant curve $\xi : \partial\pi_1(\Sigma) \to G/P$ associated to an Anosov representation (see Proposition 1.2), and the parabolic group Q depends on P.

1. Anosov Representations

Let P_+, P_- be a pair of opposite parabolic subgroups of G and denote by $\mathcal{F}^{\pm} = G/P_{\pm}$ the flag variety associated to $P_{\pm}$. There is a unique open G-orbit $\mathcal{X} \subset$

$\mathcal{F}^+ \times \mathcal{F}^-$. We have $\mathcal{X} = G/(P_+ \cap P_-)$, and as an open subset of $\mathcal{F}^+ \times \mathcal{F}^-$ it inherits two foliations $\mathcal{E}_\pm$ whose corresponding distributions are denoted by $E_\pm$, $(E_\pm)_{(f_+,f_-)} \cong T_{f_\pm}\mathcal{F}^\pm$.

Given a representation $\rho : \pi_1(\Sigma) \to G$ we consider the corresponding flat G-bundle $\mathcal{P}$ over $T^1\Sigma$. Via the flat connection, the flow ϕ_t lifts to $\mathcal{P}$.

Definition 1.1 ([4]). *A representation $\rho : \pi_1(\Sigma) \to G$ is called a* P_+-Anosov representation *if the associated bundle $\mathcal{P} \times_G \mathcal{X}$*

(1) *admits a section σ that is flat along flow lines, and*
(2) *the action of the flow ϕ_t on $\sigma^* E_+$ (resp. $\sigma^* E_-$) is contracting (resp. dilating),* i.e. *there exist constants $A, a > 0$ such that for any e in $\sigma^*(E_\pm)_m$ and for any $t > 0$ one has*

$$\|\phi_{\pm t} e\|_{\phi_{\pm t} m} \leq A \exp(-at) \|e\|_m.$$

The set of P_+-Anosov representations is open in $\mathrm{Hom}(\pi_1(\Sigma), G)$ [4].

Proposition 1.2 ([4]). *Let Σ, G and P_+ be as above. Let ρ be a P_+-Anosov representation. Then*

(1) *there are two ρ-equivariant continuous maps $\xi^\pm : \partial\pi_1(\Sigma) \to \mathcal{F}^\pm$,*
(2) *for every $t_+ \neq t_- \in \partial\pi_1(\Sigma)$ we have $(\xi^+(t_+), \xi^-(t_-)) \in \mathcal{X}$,*
(3) *for every $\gamma \in \pi_1(\Sigma) - \{e\}$, the element $\rho(\gamma)$ is conjugate to an element in $P_+ \cap P_-$, having a unique attracting fix point in G/P_+ and a unique repelling fix point in G/P_-.*

2. A Special Case

Let V be a real vector space and F a non-degenerate bilinear form on V which we assume to be either skew-symmetric or symmetric indefinite of signature (p, q) (with $p \leq q$). Let $G_F = \{g \in \mathrm{GL}(V) \,|\, g^*F = F\}$, let $\mathcal{F}_0 = G_F/Q_0 = \{l \in \mathbb{P}(V) \,|\, F|_l = 0\}$ be the set of isotropic lines and $\mathcal{F}_1 = G_F/Q_1 = \{W \in \mathrm{Gr}_p(V) \,|\, F|_W = 0\}$ be the set of maximal isotropic suspaces ($p = \dim V/2$ when F is skew-symmetric). Let also $\mathcal{F}_{0,1} = \{(l, W) \in \mathcal{F}_0 \times \mathcal{F}_1 \,|\, l \subset W\}$ and $\pi_i : \mathcal{F}_{0,1} \to \mathcal{F}_i$, $i = 0, 1$, be the projections. Given a subset $A \subset \mathcal{F}_0$ we define the subset

$$K_A := \pi_1(\pi_0^{-1}(A)) \subset \mathcal{F}_1.$$

For an isotropic line $l \in \mathcal{F}_0$, $K_l \subset \mathcal{F}_1$ is the set of maximal isotropic subspaces containing l, and $K_A = \bigcup_{l \in A} K_l$. Similarily, given $B \subset \mathcal{F}_1$ we define $K_B \subset \mathcal{F}_0$.

Theorem 2.1. *Let Σ be as in Theorem 0.2 and let V, F and G_F as above with* $\dim V \geq 4$. *Suppose $\rho : \pi_1(\Sigma) \to G_F$ is a Q_i-Anosov representation, with $i = 0$ or 1, and let $\xi_i : \partial\pi_1(\Sigma) \to \mathcal{F}_i$ be the corresponding equivariant map. Define* $\Omega_\rho := \mathcal{F}_{1-i} - K_{\xi_i(\partial\pi_1(\Sigma))} \subset \mathcal{F}_{1-i}$.

Then Ω_ρ is non-empty, open and preserved by $\rho(\pi_1(\Sigma))$. Furthermore, the action of $\rho(\pi_1(\Sigma))$ on Ω_ρ is free, properly discontinuous and the quotient $\Omega_\rho/\rho(\pi_1(\Sigma))$ is compact.

The set $K_{\xi_i(\partial\pi_1(\Sigma))}$ is closed and (because $\dim V \geq 4$) of codimension at least 1 in $\mathcal{F}_{1-i}$; by ρ-equivariance of ξ_i it is preserved by $\rho(\pi_1(\Sigma))$, hence Ω_ρ is a

$\rho(\pi_1(\Sigma))$-invariant non-empty open subset of $\mathcal{F}_{1-i}$. That the action is free and properly discontinuous follows from the contraction estimates one can deduce from the representation ρ being Q_i-Anosov.

To prove compactness of the quotient $\Omega_\rho/\rho(\pi_1(\Sigma))$, we need to prove that $\mathrm{H}_n(\Omega_\rho/\rho(\pi_1(\Sigma)); \mathbf{F}_2)$ does not vanish. This is achieved by using algebraic topology tools (spectral sequence, Alexander duality, *etc.*).

The proof of the general situation (Theorem 0.2)

3. Examples

3.1. Maximal Representations into $\mathbf{Sp}(2n, \mathbf{R})$. Any maximal representation $\rho : \pi_1(\Sigma) \to \mathrm{Sp}(2n, \mathbf{R})$ is P-Anosov where P is the stabilizer of a Lagrangian subspace in $\mathbf{R}^{2n}$ (see [1] for definitions and proofs). Thus Theorem 2.1 applies and gives a domain of discontinuity $\Omega_\rho \subset \mathbf{R}\mathbb{P}^{2n-1}$.

3.2. Hitchin Representations into $\mathbf{SL}(n, \mathbf{R})$. Let $\rho : \pi_1(\Sigma) \to \mathrm{SL}(n, \mathbf{R})$ be a P_{min}-Anosov representation, and let $\xi = (\xi^1, \cdots, \xi^{n-1}) : \partial\pi_1(\Sigma) \to \mathcal{F}(\mathbf{R}^n)$ be the equivariant map into the flag variety. Examples of such representations are Hitchin representations [3, 4], but the construction applies also to other such representations.

The trace defines a non-degenerate bilinear form F on $V = \mathrm{End}(\mathbf{R}^n)$. Applying Theorem 2.1 to the Q_1-Anosov representation $\mathrm{Ad} \circ \rho : \pi_1(\Sigma) \to \mathrm{GL}(V)$ we obtain a domain of discontinuity $\Omega_{\mathrm{Ad}\circ\rho}$ in G_F/Q_0 which gives rise to a domain of discontinuity $\Omega_{\rho,\mathrm{Ad}} \subset \mathcal{F}_{1,n-1}(\mathbf{R}^n)$ in the space of partial flags consisting of a line and a hyperplane. $\Omega_{\rho,\mathrm{Ad}}$ is the complement of

$$\big\{(p, H) \in \mathcal{F}_{1,n-1}(\mathbf{R}^n) \,|\, \exists t \in \partial\pi_1(\Sigma),\, \exists\, 1 \leq k \leq n \text{ such that } p \subset \xi^k(t) \subset H\big\}.$$

3.3. Deformations of $\pi_1(\Sigma) \to \mathbf{SO}(2,1) \to \mathbf{SO}(n,1)$. Let $\rho : \pi_1(\Sigma) \to \mathrm{SO}(n,1)$, $n \geq 3$, be a (small enough) deformation of the embedding $\pi_1(\Sigma) \to \mathrm{SO}(2,1) \to \mathrm{SO}(n,1)$. Then the domain of discontinuity Ω_ρ constructed in Section 2 is the complement of the limit set of ρ in S^{n-1} and the quotient $\Omega_\rho/\rho(\pi_1(\Sigma)$ is homeomorphic to an S^{n-3}-bundle over Σ.

References

[1] M. Burger, A. Iozzi, F. Labourie and A. Wienhard, *Maximal representations of surface groups: Symplectic Anosov structures*, Pure and Applied Mathematics Quaterly. Special Issue: In Memory of Armand Borel. **1** (2005), no. 2, 555–601.

[2] O. Guichard, *Composantes de Hitchin et reprsentations hyperconvexes de groupes de surface*, J. Differential Geom. **80** (2008), no. 3, 391–431.

[3] N. Hitchin, *Lie groups and Teichmüller space*, Topology **31** (1992), no. 3, 449–473.

[4] F. Labourie, *Anosov flows, surface groups and curves in projective space*, Invent. Math. **165** (2006), no. 1, 51–114.

Surfaces, $SL(3)$ and an equation of Tzitzéica

John Loftin
(joint work with Ian McIntosh)

On a Riemann surface Σ equipped with a holomorphic cubic differential Q, we consider four equations of Tzitzéica type (depending on the choices of sign)

$$\Delta u \pm 4\|Q\|^2 e^{-2u} \pm 4e^u - 2\kappa = 0, \tag{0.1}$$

for Δ and κ respectively the Laplacian and curvature of a background metric. Each equation is an integrability condition of an immersed surface whose geometry is governed by a real form G of $SL(3,\mathbb{C})$. In each case, a solution to Tzitzéica's equation determines a representation of the surface group $\pi_1\Sigma$ into G. We recount the geometry of all four equations, three of the four already in the literature, and the last one being work in progress with Ian McIntosh.

Equation (0.1) with signs $(-,+)$ corresponds to minimal Lagrangian tori into $\mathbb{CP}^2$, with group $SU(3)$. This equation has been well studied using integrable systems techniques by Sharipov [9], Joyce [3], Haskins [2], Ma-Ma [7], McIntosh [8], Carberry-McIntosh [1], and others, with applications to special Lagrangian cones in $\mathbb{C}^3$. Solutions on tori which "close up" to tori in $\mathbb{CP}^2$ are studied by these authors.

Equation (0.1) with signs $(+,+)$ and $(+,-)$ correspond respectively to elliptic and hyperbolic affine spheres in $\mathbb{R}^3$, whose geometry is invariant under the action of $SL(3,\mathbb{R})$. These equations have been studied by Wang [10], Labourie [4], Loftin, and Loftin-Yau-Zaslow [6]. Applications are given to the Strominger-Yau-Zaslow conjecture in mirror symmetry and to the space $\mathrm{Hom}(\pi_1\Sigma, SL(3,\mathbb{R}))/SL(3,\mathbb{R})$.

Finally, in work in progress with McIntosh, we study solutions to (0.1) with signs $(-,-)$. Such solutions correspond to integrability conditions of minimal Lagrangian surfaces in the complex hyperbolic plane $\mathbb{CH}^2$. Solutions for cubic differential $Q = 0$ give rise to the standard Lagrangian $\mathbb{RH}^2 \subset \mathbb{CH}^2$. We are able to produce solutions for all Q which are small enough. We conjecture that these solutions parametrize a portion of the representation space $\mathrm{Hom}(\pi_1\Sigma, SU(2,1))/SU(2,1)$ similar to the subset of the representation space into $PSL(2,\mathbb{C})$ given by the quasi-Fuchsian representations.

References

[1] E. Carberry and I. McIntosh, *Minimal Lagrangian 2-tori in* $\mathbb{CP}^2$ *come in real families of every dimension*, J. London Math. Soc. **69** (2004), 531–544.

[2] M. Haskins, *Special Lagrangian cones*, Amer. J. Math. **126** (2004), 845–871.

[3] D. Joyce, *Special Lagrangian 3-folds and integrable systems.* Surveys on geometry and integrable systems, 189–233, Adv. Stud. Pure Math., **51**, Math. Soc. Japan, Tokyo, 2008.

[4] F. Labourie, *Flat projective structures on surfaces and cubic holomorphic differentials*, Pure Appl. Math. Q. 3 **4** (2007), no. 4, part 1, 1057–1099. 1057–1099.

[5] J. Loftin *Affine spheres and convex* $\mathbb{RP}^n$*-manifolds*, Amer. J. Math. **123** (2001), 255–274.

[6] J. Loftin, S.-T. Yau and E. Zaslow, *Affine manifolds, SYZ geometry and the "Y" vertex*, J. Diff. Geom. **71** (2005), 129–158, erratum at arXiv:math/0405061.

[7] H. Ma and Y. Ma, *Totally real minimal tori in* $\mathbb{C}\mathrm{P}^2$, Math. Z. **249** (2005), 241–267.

[8] I. McIntosh, *Special Lagrangian cones in* $\mathbb{C}^3$ *and primitive harmonic maps*, J. London Math. Soc. **67** (2003), 769–789.
[9] R. Sharipov, *Minimal tori in the five-dimensional sphere in* $\mathbb{C}^3$, Theoret. Math. Phys. **87** (1991), 363–369.
[10] C.-P. Wang, *Some Examples of Complete Hyperbolic Affine* 2*-Spheres in* $\mathbb{R}^3$, Global differential geometry and global analysis (Berlin, 1990), 271–280, Lecture Notes in Math., **1481**, Springer, Berlin, 1991.

Gromov hyperbolicity and CAT(−1)-spaces

Viktor Schroeder
(joint work with Thomas Foertsch)

In this talk we discussed the relation between Gromov hyperbolic spaces and CAT(−1)-spaces. A CAT(−1)-space is always Gromov hyperbolic and we are interested in the question to what extend the opposite holds.

A space X is called Gromov hyperbolic, if there exists a constant $\delta \geq 0$ such that for any quadruple of points $(x, y, z, w) \in X^4$ the two largest of the following three numbers

$$|xy| + |zw| \, , \; |xz| + |yw| \, , \; |xw| + |yz|$$

differ by at most 2δ. Here $|xy|, |zw|$ etc. denotes the distances. It is surprising that this simple property catches some important properties of the classical hyperbolic space $\mathbb{H}^n$ but is flexible enough in order to apply it for a broad class of spaces.

The classical hyperbolic space has an ideal boundary $\partial_\infty \mathbb{H}^n$ which is S^{n-1} in the unit ball and $\mathbb{R}^{n-1} \cup \{\infty\}$ in the upper half space model. In a similar way one can define an ideal boundary of a gromov hyperbolic space. One can also define a kind of metric structure on the boundary, which depends on a choosen basepoint $o \in X$ or a basepoint $\omega \in \partial_\infty X$. Actually for $o \in X$ respectively $\omega \in \partial_\infty X$ one can define quasi-metrics ρ_o, respectively ρ_ω on $Y = \partial_\infty X$.

In the case that the space is actually a CAT(−1)-space, the quasi-metrics ρ_o respectively ρ_ω are indeed metrics. This was proved by Bourdon [1], [2] respectively follows from result of Hamenstädt [4].

Our main observation is that the ideal boundary Y of a CAT(−1)-space X endowed with the Bourdon or Hamenstädt metric satisfies the so calle Ptolemy inequality,i.e. let $y_1, y_2, y_3, y_4 \in Y$, then

$$|y_1y_3||y_2y_4| \leq |y_1y_2||y_3y_4| + |y_2y_3||y_4y_1|.$$

Equality holds if and only if the convex hull of the four points is isometric to an ideal quadrilateral in the hyperbolic plane $\mathbb{H}^2$ such that the geodesics y_1y_3 and y_2y_4 are the diagonals.

Using the Ptolemy property we can give partial answers to the question, to what extend gromov hyperbolic spaces can be roughly isometrically embedded into CAT(−1)-spaces.

References

[1] M. Bourdon, *Structure conforme au bord et flot géodésique d'un* CAT(−1)*-espace*, Enseign. Math. (2) no. 41 (1995), 63–102.
[2] M. Bourdon, *Sur le birapport au bord des* CAT(−1)*-espaces* Inst. Hautes Études Sci. Publ. Math. No. 83 (1996), 95–104.
[3] Th .Foertsch, V. Schroeder, *Hyperbolicity,* CAT(−1)*-spaces and the Ptolemy Inequality*, Arxive math.MG/0605418.
[4] U. Hamenstädt, *A new description of the Bowen-Margulis measure*, Ergodic Theory Dynam. Systems 9 (1989), no. 3, 455–464.

Gradient flow in Alexandrov geometry

Anton Petrunin

I will give a quick introduction to the gradient flow technique and discuss at least the following applications:

1. Collapsing with lower curvature bound (an older work joint with Kapovitch and Tuschmann)
2. Compatibility of Alexandrov's definition with the definition of generalized lower curvature bound of Lott, Villani and Sturm
3. Existance of bi-Lipschitz distance embedding for Alexandrov space.
4. Maybe one more if time allows.

Lagrangian mean curvature flow for entire Lipchitz graphs

Jingyi Chen
(joint work with Albert Chau, Weiyong He, Chao Pang)

We consider the fully nonlinear parabolic equation $du/dt = \sum \arctan \lambda_j(u)$ where $\lambda_j(u)$ are the eigenvalues of the Hessian D^2u for $u : \mathbb{R}^n \to \mathbb{R}$. The graphs $(x, Du(x,t))$ are Lagrangian submanifolds in $\mathbb{R}^n \times \mathbb{R}^n$ and evolve by the standard mean curvature flow. We [1] show that if $D^2u \in L^\infty, -(1-\delta)I \leq D^2u \leq (1-\delta)I$, then there is a longtime smooth solution and $\sup |D^l u| < C(\delta, l)/t^{l-2}, l \geq 3$. The solution is unique and in fact for any C^0 initial data u_0 there is a unique C^0 viscosity solution [3]. Translating solitons, self-shrinkers, self-expanders are classified [2].

References

[1] A. Chau, J. Chen, W. He, *Lagrangian Mean Curvature flow for entire Lipschitz graphs*, arXiv:0902.3300.
[2] A. Chau, J. Chen, W. He, *Entire self-similar solutions to Lagrangian Mean curvature flow*, arXiv:0905.3869.
[3] J. Chen, C. Pang, *Uniqueness of viscosity solutions of a geometric fully nonlinear parabolic equation*, to appear in C.R. Math. Acad. Soc. Paris.

The deformation theory of hyperbolic cone-3-manifolds with cone-angles less than 2π

Hartmut Weiss

Let X be a hyperbolic cone-3-manifold with cone-angles less than 2π, i.e. contained in the interval $(0, 2\pi)$. Let $\Sigma \subset X$ denote the singular locus and $M = X \setminus \Sigma$ the smooth part of X. The singular locus Σ is an embedded geodesic graph. The smooth part M carries a smooth Riemannian metric of constant sectional curvature -1 which has conical singularities transverse to Σ. Let $\{e_1, \ldots, e_N\}$ denote the edges and $\{v_1, \ldots, v_k\}$ the vertices contained in Σ. We call the homeomorphism type of the pair (X, Σ) the topological type of X. Let $C_{-1}(X, \Sigma)$ denote the space of hyperbolic cone-manifold structures on X of fixed topological type (X, Σ). We are interested in properties of the map

$$\alpha = (\alpha_1, \ldots, \alpha_N) : C_{-1}(X, \Sigma) \to \mathbb{R}_+^N$$

which assigns the vector of cone-angles to a hyperbolic cone-manifold structure.

C.D. Hodgson and S.P. Kerckhoff showed in [1] that α is a local homeomorphism at the given structure if the cone-angles are $< 2\pi$ and Σ is not allowed to contain vertices. The author showed in [4] that the same conclusion holds true if the cone-angles are $\leq \pi$ without any restrictions on Σ. The current work [5] bridges the gap between these two results, namely we address the general case of cone-angles being less than 2π without restrictions on Σ.

As in [1] and [4], the main technical result is a vanishing theorem for (a substantial part of) the first L^2-cohomology group of M with values in some flat bundle $\mathcal{E}$. To be more precise, let $\mathrm{hol} : \pi_1 M \to \mathrm{SL}_2(\mathbb{C})$ denote the holonomy representation of the hyperbolic structure on M. Then the bundle $\mathcal{E} = \tilde{M} \times_{\mathrm{Ad} \circ \mathrm{hol}} \mathfrak{sl}_2(\mathbb{C})$ carries a flat connection $\nabla^{\mathcal{E}}$ and, since it it isomorphic as a vector bundle to $\mathfrak{so}(TM) \oplus TM$, a metric $h^{\mathcal{E}}$. Hence the L^2-cohomology group $H^1_{L^2}(M; \mathcal{E})$ is defined. Let N_j be the smooth part of the link of the vertex v_j. Since $\mathrm{hol}|_{\pi_1 N_j}$ fixes a point $p_j \in \mathbb{H}^3$, there is a splitting $\mathcal{E}|_{N_j} = \mathcal{E}^1_j \oplus \mathcal{E}^2_j$. The first factor corresponds to infinitesimal rotations and the second to infinitesimal translations at p_j.

Theorem 1. *Let $c \in H^1_{L^2}(M; \mathcal{E})$ be a class with the property that for all vertices $v_j \in \Sigma$ $(j = 1, \ldots, k)$ the following holds:*

$$c|_{H^1_{L^2}(N_j; \mathcal{E}^1_j)} = 0 \quad \text{or} \quad c|_{H^1_{L^2}(N_j; \mathcal{E}^2_j)} = 0.$$

Then $c = 0$.

The proof of Theorem 1 uses a Weitzenböck formula relating different Laplacians acting on differential forms with values in $\mathcal{E}$ and tools from analysis on singular spaces. In order to apply the Bochner method on the noncompact manifold M one has to ensure that one can integrate by parts. This is the main difficulty.

The deformation space of incomplete hyperbolic strucutures on M is locally homeomorphic to the space $X(\pi_1 M, \mathrm{SL}_2(\mathbb{C})) = R(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))/\,\mathrm{SL}_2(\mathbb{C})$, where $\mathrm{SL}_2(\mathbb{C})$ acts by conjugation. Equivalence classes of holonomy representations of hyperbolic cone-manifold structures of fixed topological type (X, Σ) lie in the subspace

$$X_0(\pi_1 M, \mathrm{SL}_2(\mathbb{C})) = \{[\rho] : \rho|_{\pi_1 N_j} \text{ fixes a point } p_j \in \mathbb{H}^3 \ \forall\, j = 1, \ldots, k\}.$$

Let $\chi_0 = [\mathrm{hol}]$. The tangent space $T_{\chi_0} X(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ may be identified with $H^1(\pi_1 M; \mathfrak{sl}_2(\mathbb{C})_{\mathrm{Ad} \circ \mathrm{hol}})$ via Weil's construction. This latter group in turn may be identified with $H^1(M; \mathcal{E})$. Hence Theorem 1 provides us with infinitesimal information about the spaces $X(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ and $X_0(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$. Using transversality arguments and Theorem 1 we obtain:

Theorem 2. $\dim_{\mathbb{R}} X_0(\pi_1 M, \mathrm{SL}_2(\mathbb{C})) = N$ *and the map*

$$t_\mu = (t_{\mu_1}, \ldots, t_{\mu_N}) : X_0(\pi_1 M, \mathrm{SL}_2(\mathbb{C})) \to \mathbb{R}^N$$

is a local diffeomorphism at χ_0.

From this one easily deduces the main result:

Theorem 3. *Let* X *be a hyperbolic cone-3-manifold with cone-angles less than* 2π. *Then the map*

$$\alpha = (\alpha_1, \ldots, \alpha_N) : C_{-1}(X, \Sigma) \to \mathbb{R}_+^N$$

is a local homeomorphism at the given structure.

It remains to establish a good local parametrization for $X(\pi_1 M, \mathrm{SL}_2(\mathbb{C}))$ and to identify the corresponding deformations geometrically. This is the subject of ongoing joint work with G. Montcouquiol.

An alternative approach to these questions – based on the deformation theory of Einstein metrics – has been developed by R. Mazzeo and G. Montcouquiol, cf. [2] and [3].

References

[1] C.D. Hodgson, S.P. Kerckhoff, *Rigidity of hyperbolic cone-manifolds and hyperbolic Dehn surgery*, J. Differential Geom. **48** (1998), no. 1, 1–59.

[2] R. Mazzeo, G. Montcouquiol, *Rigidity of 3-cone-manifolds and the Stoker problem*, in preparation.

[3] G. Montcouquiol, *Deformations of convex polyhedra and 3-cone-manifolds*, preprint 2009, arXiv:0903.4743.

[4] H. Weiss, *Local rigidity of 3-dimensional cone-manifolds*, J. Differential Geom. **71** (2005), no. 3, 437–506.

[5] H. Weiss, *The deformation theory of hyperbolic cone-3-manifolds with cone-angles less than* 2π, preprint 2009, arXiv:0904.4568.

On renormalized volume of conformally compact Einstein manifold

Alice Chang
(joint work with Chang-J. Qing-P. Yang; Chang-H. Fang and R. Graham)

Renormalized volume is a concept introduced in conformal field theory by Maldacena and well studied by mathematicians and people in mathematical physicists. In this talk I will discuss an integral formula of the renormalized volume in terms of curvature invariants on conformally compact Einstein manifolds with conformal infinity an odd dimensional compact manifold. The formula was known in the case of a four manifold by M. Anderson (2002), but here we will present a new proof (joint work with J.Qing and P. Yang) by exploring properties of the "Q-curvature" –a curvature of order the same as the dimension of the manifold and whose integral is a conformal invariant–of the formula, the proof has the advantage being generalized to arbitrary even dimensional conformally compact Einstein manifolds. In the second part of the talk, I will discuss a recent joint work with H. Fang and R. Graham of the precise integrand of the formula when the dimension is bigger than 5.

References

[1] A. Chang, J. Qing and P. Yang, *On the renormalized volumes for conformally compact Einstein manifolds*, (in Russian) Proceedings DFDE-2005, Contemporary Mathematics, Fundamental Directions.
[2] A. Chang and H. Fang, *On a new class of conformal variational functional*, to appear IMRN.
[3] R. Graham, *Extended obstruction tensors and renormalized volume coefficients*, to appear in Studia Math.

Geometric aspects of the Allen-Cahn equation

Frank Pacard
(joint work with M. del Pino, M. Kowalczyk and J. Wei)

We are interested in the understanding of entire solutions in $\mathbb{R}^N$, of the semilinear elliptic equation

$$\Delta u + (1 - u^2)\, u = 0\,, \tag{0.1}$$

known as the Allen-Cahn equation. This problem has its origin in the *gradient theory of phase transitions* [1] where one considers critical points of the energy

$$E_\varepsilon(u) := \frac{\varepsilon}{2}\int_M |\nabla^g u|_g^2 \, \mathrm{dvol_g} + \frac{1}{4\varepsilon}\int_M (1-u^2)^2 \, \mathrm{dvol_g},$$

where (M, g) is a N-dimensional compact Riemannian manifold. The critical points of E_ε satisfy the Euler-Lagrange equation

$$\varepsilon^2 \Delta_g u + (1 - u^2)\, u = 0\,, \tag{0.2}$$

in M. Working in local coordinates and changing x into x/ε, it is easy to see that (0.1) appears as the limit problem in the blow up analysis of (0.2) as ε tends to 0.

The relation between minimal interfaces and critical points of E_ε was first established by Modica in [12]. Let us briefly recall the main results in this direction. If u_ε is a family of *local minimizers* of E_ε for which $\sup_{\varepsilon>0} E_\varepsilon(u_\varepsilon) < +\infty$, then, up to a subsequence, u_ε converges in L^1 to $\mathbf{1}_\Lambda - \mathbf{1}_{\Lambda^c}$, as ε tends to 0, where $\partial\Lambda$ is a critical point of the volume functional. Here $\mathbf{1}_\Lambda$ (resp. $\mathbf{1}_{\Lambda^c}$) is the characteristic function of the set Λ (resp. $\Lambda^c = M - \Lambda$). Moreover, $E_\varepsilon(u_\varepsilon) \longrightarrow \frac{1}{\sqrt{2}}\,\mathcal{H}^{N-1}(\partial\Lambda)$. For *critical points* of E_ε with uniformly bounded energy, a related assertion is proven in [11]. In this case, the convergence of the interface holds with certain integer multiplicity to take into account the possibility of multiple transition layers converging to the same minimal hypersurface.

These results provide a link between solutions of (0.1) and the theory of minimal hypersurfaces which has been widely explored in the literature. Sequences of solutions concentrating along non-degenerate, minimal hypersurfaces of a compact manifold were found in [14]. More precisely, assume that $\Gamma \subset M$ is an oriented minimal hypersurface such that $M \setminus \Gamma = M_+ \cup M_-$ and n the unit normal vector field to Γ which is compatible with the orientation points towards M_+ and $-n$ points towards M_-. The Jacobi operator about Γ is given by

$$J_\Gamma := \Delta_{\mathring{g}} + \mathrm{Ric}_g\,(n,n) + |A_\Gamma|^2_{\mathring{g}}\,,$$

where $\Delta_{\mathring{g}}$ is the Laplacian on $(\Gamma, \bar{g})$ for $\mathring{g}$ the induced metric on Γ, Ric_g denotes the Ricci tensor on (M,g) and $|A_\Gamma|^2_{\mathring{g}}$ denotes the square of the norm of the shape operator. Assuming that Γ is non degenerate (namely, that J_Γ has no nontrivial kernel), it is proven in [14] that for all $\varepsilon > 0$ small enough, there exists u_ε, a critical point of E_ε, such that u_ε converges uniformly to 1 on compacts subsets of M^+ (resp. to -1 on compacts subsets of M^-) and $E_\varepsilon(u_\varepsilon) \longrightarrow \frac{1}{\sqrt{2}}\,\mathcal{H}^{N-1}(\Gamma)$, as ε tends to 0.

As far as multiple transition layers are concerned, given a non degenerate minimal hypersurface Γ such that $\mathrm{Ric}_g\,(n,n) + |A_\Gamma|^2_{\mathring{g}} > 0$ along Γ, it is proven in [8] that sequences of solutions of (0.2) with multiple transitions layers near Γ do exist.

In dimension $N = 1$, solutions of (0.1) which have finite energy are given by translations of the function H which is the unique solution of the problem

$$H'' + (1 - H^2)\,H = 0, \quad \text{with} \quad H(\pm\infty) = \pm 1 \quad \text{and} \quad H(0) = 0\,. \tag{0.3}$$

which is explicitly given by

$$H(y) = \tanh\left(\frac{y}{\sqrt{2}}\right)\,.$$

Then, for all $\mathbf{a} \in \mathbb{R}^N$ with $|\mathbf{a}| = 1$ and for all $b \in \mathbb{R}$, the function $u(\mathbf{x}) = H(\mathbf{a}\cdot\mathbf{x} + b)$ solves 0.1. A celebrated conjecture due to de Giorgi states that, in dimension $N \leq 8$, these solutions are the only ones which are bounded and monotone in one direction. In other words, if u is a (smooth) bounded solution of (0.1) and if $\partial_{x_1} u > 0$ then $u^{-1}(\lambda)$ is either a hyperplane or the empty set. In dimensions $N = 2, 3$, De Giorgi's conjecture has been proven in [10], [2] and (under some extra assumption) in the remaining dimensions in [16].

In view of de Giorgi's conjecture, it is natural to study the set of entire solutions of 0.1, namely, solutions which are defined in the entire $\mathbb{R}^N$. The functions $u(\mathbf{x}) = H(\mathbf{a}\cdot\mathbf{x}+b)$ are obvious solutions. In dimension $N = 2$, nontrivial examples (whose nodal set is the union of two perpendicular lines) were built in [5]. This construction can easily be generalised to obtain solutions with dihedral symmetry. X. Cabre and J. Terra [3] have obtained a higher dimensional version of this construction (using similar arguments) and they are able to find solutions in $\mathbb{R}^{2m}$ whose zero set is the minimal cone $\{(x,y)\in\mathbb{R}^{2m} : |x| = |y|\}$.

Recently, there has been some important progress on the existence of solutions of (0.1) which are defined in the entire space. All these new solutions are conterparts, in the noncompact setting, of the solutions obtained in [14] and rely on the knowledge of complete noncompact minimal hypersurfaces which are not invariant by dilations. Let us mention two deep results along these lines.

There is a rich family of minimal surfaces in $\mathbb{R}^3$ which are complete, embedded and have finite total curvature. Among these surfaces there is the catenoid, Costa's surface [4] and all k-ended surfaces and the embedded surfaces studied by J. Perez and A. Ros [15]. The main result in [9] asserts there exists solutions of (0.1) whose nodal set is close to a dilated version of any such minimal surface. More orecisely, if one consider the equation with scaling

$$\varepsilon^2\,\Delta u + u - u^3 = 0\,, \tag{0.4}$$

then, given Γ, a nondegenerate (i.e. all bounded Jacobi fields about Γ come from the action of rigid motions) complete, noncompact minismal surface with finite total curvature, for all $\varepsilon > 0$ small enough, there exists exists u_ε solution of (0.4), such that $u_\varepsilon^{-1}(0)$ converges uniformly on compacts to Γ.

Thanks to the result of Bombieri-de Giorgi-Giusti, it is known that there exists minimal graphs which are not hyperplanes in dimension $N \geq 9$. Following similar ideas, entire solutions of (0.1) which are monotone in one direction but whose level sets are not hyperplanes have been constructed in [7], provided the dimension of the ambient space is $N \geq 9$.

We assume from now on that the dimension is equal to $N = 2$. We say that u, solution of (0.1), has $2k$-*ends* if, away from a compact set, its nodal set is given by $2k$ connected curves which are asymptotic to $2k$ oriented half lines $\mathbf{a}_j\cdot\mathbf{x}+b_j = 0$, $j = 1,\ldots,2k$ (for some choice of $\mathbf{a}_j\in\mathbb{R}^2$, $|\mathbf{a}_j| = 1$ and $b_j\in\mathbb{R}$) and if, along these curves, the solution is asymptotic to either $H(\mathbf{a}_j\cdot\mathbf{x}+b_j)$ or $-H(\mathbf{a}_j\cdot\mathbf{x}+b_j)$.

Given any $k \geq 1$, we prove in [6] the existence of a wealth of $2k$-ended solutions of (0.1). To state our result in precise way, we assume that we are given $q_1,\ldots,q_k$ solutions of the *Toda system*

$$c_0\, q_j'' = e^{\sqrt{2}(q_{j-1}-q_j)} - e^{\sqrt{2}(q_j-q_{j+1})}\,, \tag{0.5}$$

for $j = 1,\ldots,k$, where $c_0 = \frac{\sqrt{2}}{24}$ and where we agree that $q_0 \equiv -\infty$ and $q_{k+1} \equiv +\infty$. The Toda system (0.5) is a classical example of integrable system which has been extensively studied [13]. It models the dynamics of finitely many mass points on the line under the influence of an exponential potential.

Given $\varepsilon > 0$, we define

$$q_{j,\varepsilon}(x) := q_j(\varepsilon\, x) - \sqrt{2}\left(j - \frac{k+1}{2}\right)\log \varepsilon\,. \tag{0.6}$$

It is easy to check that the $q_{j,\varepsilon}$ are again solutions of (0.5).

We agree that χ^+ (resp. χ^-) is a smooth cutoff function defined on $\mathbb{R}$ which is identically equal to 1 for $x > 1$ (resp. for $x < -1$) and identically equal to 0 for $x < -1$ (resp. for $x > 1$) and additionally $\chi^- + \chi^+ \equiv 1$. With these cutoff functions at hand, we define the 4 dimensional space

$$D := \text{Span}\,\{x \longmapsto \chi^{\pm}(x),\ x \longmapsto x\,\chi^{\pm}(x)\}\,, \tag{0.7}$$

and, for all $\mu \in (0,1)$ and all $\tau \in \mathbb{R}$, we define the space $\mathcal{C}^{2,\mu}_\tau(\mathbb{R})$ of $\mathcal{C}^{2,\mu}$ functions h which satisfy

$$\|h\|_{\mathcal{C}^{2,\mu}_\tau(\mathbb{R})} := \|(\cosh x)^\tau\, h\|_{\mathcal{C}^{2,\mu}(\mathbb{R})} < \infty\,.$$

Keeping in mind the above notations, we have the :

Theorem 0.1. [6] *For all $\varepsilon > 0$ sufficiently small, there exists an entire solution u_ε of (0.1) (here $N = 2$) whose nodal set is the union of k disjoint curves $\Gamma_{1,\varepsilon}, \ldots, \Gamma_{k,\varepsilon}$ which are the graphs of the functions*

$$x \longmapsto q_{j,\varepsilon}(x) + h_{j,\varepsilon}(\varepsilon\, x)\,,$$

for some functions $h_{j,\varepsilon} \in \mathcal{C}^{2,\mu}_\tau(\mathbb{R}) \oplus D$ satisfying

$$\|h_{j,\varepsilon}\|_{\mathcal{C}^{2,\mu}_\tau(\mathbb{R})\oplus D} \leq C\,\varepsilon^\alpha\,.$$

for some constants $C, \alpha, \tau, \mu > 0$ independent of $\varepsilon > 0$.

In other words, given a solution of the Toda system, we can find a one parameter family of $2k$-ended solutions of (0.1) which depend on a small parameter $\varepsilon > 0$. As ε tends to 0, the nodal sets of the solutions we construct become close to the graphs of the functions $q_{j,\varepsilon}$.

References

[1] S. Allen and J. W. Cahn, *A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening,* Acta. Metall. 27 (1979), 1084–1095.

[2] L. Ambrosio and X. Cabré, *Entire solutions of semilinear elliptic equations in* $\mathbb{R}$*s and a conjecture of de Giorgi,* Journal Amer. Math. Soc. 13 (2000), 725–739.

[3] X. Cabré and J. Terra, *Saddle shaped solutions of bistable diffusion equations in all* $\mathbb{R}^{2m}$, preprint (2008). arXiv:0801.3379.

[4] C.J. Costa *Example of a complete minimal immersion in* $\mathbb{R}^3$ *of genus one and three embedded ends*, Bol. Soc. Bras. Mat. 15 (2), (1984), 47-54.

[5] H. Dang, P.C. Fife and L.A. Peletier, *Saddle solutions of the bistable diffusion equation*, Z. Angew. Math. Phys. Vol. 43 (1992).

[6] M. del Pino, M. Kowalczyk and F. Pacard, *Multiple-end solutions to the Allen-Cahn equation in* $\mathbb{R}^2$, to appear in J. Funct. Analysis (2009).

[7] M. del Pino, M. Kowalczyk and J. Wei, *A conjecture by de Giorgi in large dimensions.* Preprint 2008.

[8] M. del Pino, M. Kowalczyk, J. Wei and J. Yang *Interface foliation of a minimal hypersurface in higher dimensional Riemannian manifolds.* Preprint (2009).

[9] M. del Pino, M. Kowalczyk and J. Wei *Entire solutions of the Allen-Cahn equation and complete embedded minimal surfaces of finite total curvature*Preprint (2008).
[10] N. Ghoussoub and C. Gui, *On a conjecture of de Giorgi and some related problems*, Math. Ann. Vol. 311 (1998), 481-491.
[11] J. E. Hutchinson and Y. Tonegawa, *Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory*, Calc. Var. 10 (2000), 49–84.
[12] L. Modica, *Convergence to minimal surfaces problem and global solutions of* $\Delta u = 2(u^3 - u)$. Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), pp. 223–244, Pitagora, Bologna, (1979).
[13] J. Moser, *Finitely many mass points on the line under the influence of an exponential potential–an integrable system.* Dynamical systems, theory and applications (Rencontres, Battelle Res. Inst., Seattle, Wash., 1974), pp. 467–497. Lecture Notes in Phys., Vol. 38, Springer, Berlin, 1975.
[14] F. Pacard and M. Ritoré, *From the constant mean curvature hypersurfaces to the gradient theory of phase transitions*, J. Differential Geom. 64 (2003), no. 3, 356–423.
[15] J. Perez and A. Ros, *The space of properly embedded minimal surfaces with finite total curvature*, Indiana Univ. Math. J. 45 (1996), no 1, 177-204.
[16] O. Savin, *Regularity of flat level sets in phase transitions.* To appear in Ann. of Math.

Recent Progress on Minimal Surfaces with Low Genus

Matthias Weber

Let X be a complete minimal surfaces of finite genus embedded in $\mathbb{R}^3$.

1. Examples with Genus 0

There are the catenoid, the helicoid, and a 1-parameter family of translation invariant minimal surfaces with infinitely many planar ends, discovered by Riemann [6]. He found them while classifying minimal surfaces that are foliated by circles in horizontal planes. Riemann's family limits on one side in the catenoid, and on the other in the helicoid.

2. Classification Genus 0

Meeks-Pérez-Ros [5], building on previous work by Meeks-Rosenberg and Colding-Minicozzi, prove:

Theorem 2.1. *A properly embedded, complete minimal surface of genus 0 in* $\mathbb{R}^3$ *is the catenoid, the helicoid, or one of Riemann's examples.*

Remark 2.2. *It is conjectured that the properness assumption can be removed. By Colding-Minicozzi, this is true if we assume finite topology (excluding the Riemann family)*

3. Examples Genus 1

3.1. **The Costa Surface.** Discovered by Costa, this surface has two catenoidal and one planar end. Hoffman-Meeks proved that it is embedded and were able to deform it so that the planar end becomes catenoidal.

3.2. **The Genus One Helicoid.** Hoffman-Karcher-Wei [2] proved the existence of a properly immersed minimal surface of genus 1 with one helicoidal end. Hoffman-Wei conjectured the existence of an embedded 1-parameter family of screw-motion invariant helicoids with handles, and of genus one in the quotient. Hoffman-Weber-Wolf [3] established this conjecture (see also Hoffman-White [4] for a different proof), thus proving that the genus one helicoid is embedded.

3.3. **The Costa-Riemann Surface.** Hauswirth-Pacard [1] succeeded in constructing a minimal surface by replacing a horizontal slab near a planar end of Riemann with a slightly tilted version of the Costa surface. The result is a 1-parameter family of properly embedded minimal surfaces of genus one with infinitely many ends.

3.4. **Riemann with a Handle.** Similarly, we conjecture that there is a 1-parameter family of Riemann-like minimal surfaces with just one handle added between two consecutive planar ends.

4. Classification of Genus 1

Conjecture 4.1. *A properly embedded minimal surface of genus 1 is one of the above examples.*

This conjecture is related to two other outstanding open problems:

Conjecture 4.2 (Genus One Helicoid Conjecture)**.** *A complete, embedded minimal surface of genus one with one end is the genus one helicoid.*

It is known by work of Bernstein-Breiner that the end of such a surface must be asymptotic to the helicoid and that the space of examples is compact.

Conjecture 4.3 (Hoffman-Meeks Conjecture)**.** *A complete, embedded minimal surface of finite total curvature and genus g has at most $g+2$ ends.*

If both these conjectures were true, one could prove the classification of embedded minimal surfaces of genus one in the finite genus case as follows:

By Colding-Minicozzi, any such surface is proper. By Collin, it has either one end, or finite total curvature. In the first case, the Genus One Helicoid conjecture applies. In the second case, the Hoffman-Meeks conjecture tells us that the surface has at most three ends. By Schoen, it cannot have less than three. By Costa, a 3-ended, complete, embedded minimal surface of finite total curvature, belongs to the Hoffman-Meeks family.

5. Higher Genus Examples

A classification of higher genus minimal surfaces appears very much out of reach at this point. In the finite total curvature point, the only known obstructions are the López-Ros theorem and the Hoffman-Meeks conjecture.

For the one-ended case, Meeks conjectures that there is precisely one genus g helicoid for any g.

For infinitely many ends, it is likely that any embedded finite total curvature example can be used for the Hauswirth-Pacard construction.

6. Motivation for the Conjectures

There is some good reason why the Hoffman-Meeks conjecture and the Genus-One Helicoid conjecture should be true. This reason comes from a construction method of Traizet [7], that describes the space of minimal surfaces near that part of its boundary where the surface limits in a noded surface all of whose components have genus 0. In this case, the period conditions can be expressed as a system of algebraic equations. At least for low genus, the solutions to these equations show that there are no counterexamples to the Hoffman-Meeks conjecture or the Helicoid conjecture, provided any such counterexample could be deformed to a noded limit.

References

[1] L. Hauswirth, F. Pacard, *Higher genus Riemann minimal surfaces*, Inventiones Mathematicae **169** (2007), 569–620.
[2] D. Hoffman, H. Karcher, F. Wei, *Adding handles to the helicoid*, Bulletin of the AMS, New Series **29** (1993), 77–84.
[3] D. Hoffman, M. Weber, M. Wolf, *An embedded genus-one helicoid*, Ann. of Math. **169** (2009), 347–448.
[4] D. Hoffman and B. White, *Genus-One Helicoids from a Variational Point of View*, preprint.
[5] W.H. Meeks III, J. Pérez, A. Ros, *Properly embedded minimal planar domains*, preprint.
[6] B. Riemann, *Über die Fläche vom kleinsten Inhalt bei gegebener Begrenzung*, Abh. Königl, d. Wiss. Göttingen, Mathem. Cl. **13** (1867), 3–52.
[7] M. Traizet, *An embedded minimal surface with no symmetries*, J. Differential Geometry **60** (2002), 103–153.

Reporter: Daniel Nicklas

Participants

Prof. Dr. Vestislav Apostolov
Dept. of Mathematics
University of Quebec/Montreal
C.P. 8888
Succ. Centre-Ville
Montreal , P. Q. H3C 3P8
CANADA

Prof. Dr. Christian Bär
Institut für Mathematik
Universität Potsdam
Am Neuen Palais 10
14469 Potsdam

Prof. Dr. Werner Ballmann
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn

Richard Bamler
Princeton University
Fine Hall
Washington Road
Princeton , NJ 08544-1000
USA

Prof. Dr. Yves Benoist
Laboratoire de Mathematiques
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Prof. Dr. Olivier Biquard
Inst. de Mathematiques de Jussieu
Universite Paris VI
175 rue du Chevaleret
F-75013 Paris

Dr. Christoph Böhm
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Dr. Maria-Cristina Caputo
Department of Mathematics
The University of Texas at Austin
1 University Station C1200
Austin , TX 78712-1082
USA

Prof. Dr. Sun-Yung Alice Chang
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton , NJ 08544-1000
USA

Prof. Dr. Jingyi Chen
Dept. of Mathematics
University of British Columbia
121-1984 Mathematics Road
Vancouver , BC V6T 1Z2
CANADA

Prof. Dr. Xiuxiong Chen
Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Drive
Madison , WI 53706-1388
USA

Prof. Dr. Vicente Cortes
Mathematik Department
Universität Hamburg
Schwerp.:Analysis&Diff.Geometrie
Bundesstr. 55
20146 Hamburg

Dr. Alessandro Ghigi
Dip. di Matematica e Applicazioni
Universita di Milano-Bicocca
Edificio U5
via Roberto Cozzi 53
I-20125 Milano

Prof. Dr. Sebastian Goette
Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg

Dr. Carlos Grossi-Ferreira
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn

Dr. Olivier Guichard
Laboratoire de Mathematiques
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Prof. Dr. Ursula Hamenstädt
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn

Hans-Joachim Hein
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton , NJ 08544-1000
USA

Sebastian Hensel
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn

Robert Kremser
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Prof. Dr. Urs Lang
Departement Mathematik
ETH Zürich
Rämistr. 101
CH-8092 Zürich

Dr. Nina Lebedeva
SFB 478
Geom. Strukturen in der Mathematik
Universität Münster
Hittorfstr. 27
48149 Münster

Prof. Dr. Bernhard Leeb
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Dr. Mario Listing
Mathematisches Institut
Abt. Reine Mathematik
Universität Freiburg
Eckerstr. 1
79104 Freiburg

Prof. Dr. John Loftin
Dept. of Mathematics & Computer Sc.
Rutgers University
101 Warren Street
Newark , NJ 07102
USA

Prof. Dr. Joachim Lohkamp
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. John Lott
Department of Mathematics
University of California
970 Evans Hall
Berkeley CA 94720-3840
USA

Ludovic Marquis
Laboratoire de Mathematiques
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Prof. Dr. Rafe Mazzeo
Department of Mathematics
Stanford University
Stanford , CA 94305-2125
USA

Dr. Aaron Naber
Department of Mathematics
Massachusetts Institute of
Technology
77 Massachusetts Avenue
Cambridge , MA 02139-4307
USA

Daniel Nicklas
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Prof. Dr. Frank Pacard
Centre de Mathematiques
Faculte de Sciences et Technologie
Universite Paris XII - Val de Marne
61 Avenue du General de Gaulle
F-94010 Creteil Cedex

Prof. Dr. Anton Petrunin
SFB 478
Hittorfstr. 27
48149 Münster

Prof. Dr. Joan Porti
Departamento de Matematicas
Universitat Autonoma de Barcelona
Campus Universitario
E-08193 Bellaterra

Carlos Ramos Cuevas
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Prof. Dr. Viktor Schroeder
Institut für Mathematik
Universität Zürich
Winterthurerstr. 190
CH-8057 Zürich

Prof. Dr. Natasa Sesum
Department of Mathematics
Columbia University
2990 Broadway
New York , NY 10027
USA

Prof. Dr. Iskander A. Taimanov
Sobolev Inst. of Mathematics
Siberian Branch of the Academy of
Sciences
pr. Koptjuga N4
630090 Novosibirsk
RUSSIA

Prof. Dr. Gang Tian
Department of Mathematics
Princeton University
609 Fine Hall
Washington Road
Princeton , NJ 08544
USA

Prof. Dr. Jeff Viaclovsky
Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Drive
Madison , WI 53706-1388
USA

Dr. Matthias Weber
Department of Mathematics
Indiana University at Bloomington
Rawles Hall
Bloomington , IN 47405-5701
USA

Dr. Hartmut Weiss
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Dr. Anna Wienhard
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton , NJ 08544-1000
USA

Prof. Dr. Burkhard Wilking
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Paul C. Yang
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton , NJ 08544-1000
USA

Prof. Dr. Wolfgang Ziller
Department of Mathematics
University of Pennsylvania
Philadelphia , PA 19104-6395
USA