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Singularities

Organised by
András Némethi, Budapest
Duco van Straten, Mainz
Victor A. Vassiliev, Moscow

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Abstract. Local/global Singularity Theory is concerned with the local/global structure of maps and spaces that occur in differential topology or theory of algebraic or analytic varieties. It uses methods from algebra, topology, algebraic geometry and multi-variable complex analysis.

Mathematics Subject Classification (2000): 14Bxx, 32Sxx, 58Kxx.

Introduction by the Organisers

The workshop *Singularity Theory* took place from September 20 to 26, 2009, and continued a long sequence of workshops *Singularitäten* that were organized regularly at Oberwolfach. It was attended by 46 participants with broad geographic representation. Funding from the Marie Curie Programme of EU provided complementary support for young researchers and PhD students.

The schedule of the meeting comprised 23 lectures of one hour each, presenting recent progress and interesting directions in singularity theory. Some of the talks gave an overview of the state of the art, open problems and new efforts and results in certain areas of the field. For example, B. Teissier reported about the Kyoto meeting on 'Resolution of Singularities' and about recent developments in the geometry of local uniformization. J. Schürmann presented the general picture of various generalizations of classical characteristic classes and the existence of functors connecting different geometrical levels. Strong applications of this for hypersurfaces was provided by L. Maxim. M. Kazarian reported on his new results and construction about the Thom polynomial of contact singularities; R. Rimányi used Thom polynomial theory to provide invariants for matroid varieties (e.g.

for line configurations in the plane) which answers some enumerative problems and explains certain deformation properties. M. Saito gave an overview of recent developments in the theory of jumping ideals and coefficients, spectra and b–functions, which has recently created a lot of activity and produces several new strong results. Sh. Ishii formulated several questions about the geometry of jet schemes.

Several connections with symplectic geometry were established and emphasized: N. A'Campo presented a new construction of 'vanishing spine' and (tête à tête) monodromies; Y. Namikawa about universal Poisson deformations of symplectic varieties; A. Takahashi spoke about the general program of homological mirror symmetry and exemplified it in the case of cusp singularities; M. Garay about the general KAM theorems.

Several talks targeted low–dimensional singularity theory: M. Borodzik's talk focused on the Tristam-Levine signature to understand the deformation of cuspidal plane singularities; P. Cadman characterized the δ–constant stratum; W. Ebeling presented the relation which connects the Poincaré series with the monodromy characteristic polynomial for some surface singularities; W. Veys provided a possible generalization of the 'Monodromy Conjecture' for normal surface singularities. J. F. de Bobadilla proved that the Nash Conjecture for normal surface singularities is topological (depends only on the resolution graph). The talk of I. Burban answered some classification questions about the structure of Cohen-Macaulay modules over non–isolated surface singularities.

The talks of C. Hertling, D. Mond and Ch. Sevenheck had their subject in the supplementary structures associated with universal unfoldings and free divisors. Mond provided several new constructions to produce free divisors. C. Sabbah overview his theory on 'Wild geometry' (of non–regular systems and singularities).

The meeting was closed by the talk of D. Siersma about Betti–number bounds of fibers of affine polynomial maps.

We think that the success of the meeting was also guaranteed by the fact that the younger participants also had the opportunity to present their work. Additionally, there was plenty of time for discussions, numerous collaborations started and continued.

Workshop: Singularities

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Abstracts

Spines and tête-à-tête monodromy

Norbert A'Campo

Introduction. Let (Σ, Γ) be a pair consisting of a a compact connected oriented surface Σ with non empty boundary $\partial\Sigma$ and a finite graph Γ that is embedded in the interior of Σ. We assume that the surface Σ is a regular neihborhood of the graph Γ and that the embedded graph has the tête-à-tête property. Moreover, we will construct for each pair (Σ, Γ) with the tête-à-tête property a relative mapping class T_Γ on $(\Sigma, \partial\Sigma)$. We call the mapping classes resulting from this construction tête-à-tête twists. The main result of asserts:

Theorem. *The geometric monodromy diffeomorphism of an isolated plane curve singularity is a tête-à-tête twist.*

Section 1. Tête-à-tête retractions to spines and tête-à-tête twists. Let Γ be a finite connected and metric graph with $e(\Gamma)$ edges and no vertices of valency 1. We assume, that the metric d_Γ on Γ is the path metric, that is given by parametrizations $E_e : [0, L_e] \longrightarrow \Gamma, L_e > 0, e = 1, \cdots, e(\Gamma)$ of the edges. We have $d_\Gamma(E_e(t), E_e(s)) = |t - s|, t, s \in [0, L_e]$.

Let Σ be a smooth, connected and oriented surface with non empty boundary $\partial\Sigma$. We say, that a map π of Γ into Σ is smooth if π is continuous, injective, $\pi(\Gamma) \cap \partial\Sigma = \emptyset$, the compositions $\pi \circ E_e, e = 1, \cdots, e(\Gamma)$, are smooth embeddings of intervals and moreover, at each vertex v of Γ all outgoing speed vectors of $\pi \circ E_e, v = E_e(0)$ or $v = E_e(L_e)$ are pairwise not proportional by a positive real number.

A safe walk along Γ is a continuous injective path $\gamma : [0, 2] \longrightarrow \Sigma$ with the following properties: $\gamma(t) \in \Gamma, t \in [0, 2]$; the speed, measured with the parametrization E_e at $t \in [0, 2]$ equals ± 1 if $\gamma(t)$ is in the interior of an edge e; if the path γ runs at $t \in (0, 2)$ into the vertex v, the path γ makes the sharpest possible right turn, i.e. the oriented angle at $v = \gamma(t) \in \Sigma$ inbetween the speed vectors $-\dot\gamma(t_-)$ and $\dot\gamma(t_+)$ is smallest possible.

It follows, that a safe walk γ is determined by its starting point $\gamma(0)$ and its starting speed vector $\dot\gamma(0)$. Futhermore, if the metric graph $\Gamma \subset \Sigma$ is without cycles of length less are equal 2, from each interior point of an edge two distinct safe walks start.

If we think of the graph as streets with intersections on the surface, we can imagine a safe walk as a walk staying always at the sidewalk of the street and making only right turns. So, in New York, a safe walk goes around the block by right turns only, and hence, in the same direction as the cars do. In Tokio, a safe walk is even safer, since it goes in opposite direction to the car traffic.

Definition: *Let (Σ, Γ) be the pair of a surface and regular embedded metric graph. We say that the tête-à-tête property holds for the the pair if: the graph Γ has no cycles of length ≤ 2; the graph Γ is a regular retract of the surface Σ; for* *each point* $p \in \Gamma$*,* p *not being a vertex, two distinct safe walks* $\gamma'_p, \gamma''_p : [0,2] \longrightarrow \Sigma$ *with* $p = \gamma'_p(0) = \gamma''_p(0)$ *exist and satisfy moreover* $\gamma'_p(2) = \gamma''_p(2)$*.*

Again thinking of the graph as streets, the tête-à-tête property of $\Gamma \subset \Sigma$ means, that two pedestrians being vis-à-vis with respect to the street will be again vis-à-vis after having done simultaneous safe walks over the distances of 2. It follows that the underlying metric graph of a pair (Σ, Γ) with tête-à-tête property is the union of its cycles of length 4.

We give basic examples of pairs (Σ, Γ) with tête-à-tête property: the surface is the cylinder $[-1,1] \times S^1$ and the graph Γ is the cycle $\{0\} \times S^1$ subdivided by 4 vertices in 4 edges of length 1. Here we think of $\{0\} \times S^1$ as a circle of length 4. The surface $\Sigma_{1,1}$ is of genus 1 with one boundary component and the metric graph $\Gamma \subset \Sigma$ is the biparted complet graph $K_{3,2}$ having edges of length 1. For $p, q \in \mathbf{N}, p > 0, q > 0$, the biparted complete graph $K_{p,q}$ is the spine of a surface $S_{g,r}, g = 1/2(p-1)(q-1), r = (p,q)$, such that the tête-à-tête property holds. For instance, let P and Q be two parallel lines in the plane and draw p points on P, q points on Q. We add pq edges and get a planar projection of the graph $K_{p,q}$. The surface $S_{g,r}$ is a regular thickening of the graph $K_{p,q}$, such that the given projection of $K_{p,q}$ into the plane extends to an immersion of $S_{g,r}$ into the plane. We give to all the edges of $K_{p,q}$ length 1.

Let (Σ, Γ) a pair of a surface and graph with tête-à-tête property. Our purpose is to construct for this pair a well defined element T_Γ in the relative mapping class group of the surface Σ. For each edge e of Γ we embed relatively a copy $(I_e, \partial I_e)$ of the interval $[-1,1]$ into $(\Sigma, \partial\Sigma)$ such that all copies are pairwise disjoint and such that each copy I_e intersects in its midpoint $0 \in I_e$ the graph Γ transversally in one point which is the midpoint of the edge e. We call I_e the dual arc of the edge e. Let Γ_e be the union of $\Gamma \cup I_e$. We consider Γ_e also as a metric graph. The graph Γ_e has 2 terminal vertices a, b.

Let $w_a, w_b : [-1,2] \longrightarrow \Gamma_e$ be the only safe walks along Γ_e with $w_a(-1) = a, w_b(-1) = b$. We displace by a small isotopy the walks w_a, w_b to smooth injective paths w'_a, w'_b, that keep the points $w_a(-1), w_b(-1)$ and $w_a(2), w_b(2)$ fixed, such that $w'_a(t) \notin \Gamma_e$ for $t \in (-1,2)$. The walks w_a, w_b meet each other in the midpoint of the edge e. Hence by the tête-à-tête property we have $w_a(2) = w_b(2)$. Let w_e be the juxtaposition of the pathes w'_a and $-w'_b$. We may assume that the path w_e is smooth and intersects Γ transversally. Let I'_e the image of the path w_e. We now claim that there exists up to relative isotopy a unique relative diffeomorphism ϕ_Γ of Σ with $\phi_\Gamma(I_e) = I'_e$. We define the tête-à-tête twist T_Γ as the class of ϕ_Γ.

For our first basic example $K_{2,2} \subset \Sigma_{1,2}$ we obtain back the classical right Dehn twist. The second example $K_{2,3} \subset \Sigma_{1,1}$ produces a tête-à-tête twist, which is the geometric monodromy of the plane curve singularity $x^3 - y^2$. The twists of the examples $(S_{g,r}, K_{p,q}), p, q \geq 2$, compute the geometric monodromy for the singularities $x^p - y^q$.

The family of Riemann surfaces $F_t := \{p \in \mathbf{C}^2 \mid x(p)^3 - y(p)^2 = t, \ ||p|| \leq R, \ t \in \mathbf{C}, t \neq 0\}$, can be obtained as follows. Let H_t be the interior of the real convex hull in $\mathbf{C}$ of $\{s \in \mathbf{C} \mid s^6 = t\}$. The surface $\bar{F}_t := (\mathbf{C} \setminus H_t) \cap D_{R'}$ has two boundary components, one component is a boundary with corners. Here we have denoted by $D_{R'}$ the disk of radious R' in $\mathbf{C}$. We subdivide the faces of that boundary by its midpoint obtaining 12 vertices and pieces. The Riemann surface F_t is obtained for some choice of R' by gluing orientation reversing two by two the 12 pieces. If, using the complex orientation of H_t, we enumerate the 12 vertices of H_t by $1, 2, \cdots 12$, the gluing is as follows. For i odd glue (cyclicly) the edge $[i, i+1]$ to $[i+8, i+7]$ and for i even glue the edge $[i, i+1]$ to $[i+7, i+6]$. We denote the gluing by N_t and write: $F_t = \bar{F}_t/N_t$. If t runs over a circle $|t| = r, 0 < r << R$, the hexagon H_t rotates by $\frac{2\pi}{6}$, hence the gluing scheme N_t is preserved. So, we obtain a monodromy diffeomorphism $\phi : F_t \longrightarrow F_t$.

Note that the gluing N_t converges in an appropriate microlocal topology, or arc space topology, for $t \longrightarrow 0$ to the normalization of $F_0 = \bar{F}_0/N_0, \bar{F}_0 := \mathbf{C}$. Conversely, the surface $\bar{F}_t$ can be obtained from $(S_{1,1}, K_{2,3})$ by cutting the surface $S_{1,1}$ along the graph $K_{2,3}$.

A similar description holds also for the singularities $x^p - y^q$: replace H_t by the convex hull of $\{s \in \mathbf{C} \mid s^{pq} = t\}$, which is a polygon with pq faces. We get $2pq$ vertices after midpoint subdivision. The gluing N_t is different. For the singularity E_8, given by $x^5 - y^3$, the gluing of the 30 pieces of ∂H_t is as follows. For i odd glue (cyclicly) the edge $[i, i+1]$ to $[i+11, i+10]$ and for i even glue (cyclicly) the edge $[i, i+1]$ to $[i+20, i+19]$.

In his seminal work on the ramification of integrals depending upon parameters Frédéric Pham has introduced the graphs $K_{p,q}$ as retracts of the local nearby fibers of the singularities $x^p - y^q$ [F].

Section 2. Relative tête-à-tête retracts and graphs. We prepare material, that will allow us to glue the previous examples. Let S be a connected compact surface with boundary ∂S. The boundary $\partial S = A \cup B$ is decomposed as a partition of boundary components of the surface S. We assume $A \neq \emptyset, B \neq \emptyset$.

Definition. *A relative tête-à-tête graph (S, A, Γ) in (S, A) is an embedded metric graph Γ in S with $A \subset \Gamma$. Moreover, the following properties hold: the graph Γ has no cycles of length ≤ 2; the graph Γ is a regular retract of the surface Σ; for each point $p \in \Gamma \setminus A$, p not being a vertex, the two distint safe walks $\gamma'_p, \gamma''_p : [0, 2] \longrightarrow \Sigma$ with $p = \gamma^+_p(0) = \gamma^-_p(0)$ satisfy to $\gamma^+_p(2) = \gamma^-_p(2)$; for each point $p \in A$, p not being a vertex, the only safe walk γ_p satisfies $\gamma_p(2) \in A$. The map $p \in A \mapsto w(p) := \gamma_p(2) \in A$ is called the boundary walk. The pair (A, w) is the boundary of the relative tête-à-tête graph (S, A, Γ).*

We now give a family of examples of relative tête-à-tête graphs. Consider the previous example $(S_{g,r}, K_{p,q}), g = 1/2(p-1)(q-1), r = (p, q)$. We blow up in the real oriented sense the p vertices of valency q, so we replace such a vertex $v_i, 1 \leq i \leq p$ by a circle A_i and attach the edges of $K_{p,q}$ that are incident with v_i to the circle in the cyclic order given by the embedding of $K_{p,q}$ in $S_{g,r}$. We get a surface $S_{g,r+p}$ and its boundary is partitioned in $A := \cup A_i$ and $B = \partial S_{g,r}$. The new graph is the union of A with the strict transform of $K_{p,q}$. So the new graph is in fact the total transform $K'_{p,q}$. We think of this graph as a metric graph. The metric will be such that all edges have positive length and that the tête-à-tête property remains for all points of $K'_{p,q} \setminus A$. We achieve this by giving the edges of A the length $2\epsilon, \epsilon > 0, \epsilon$ small and by giving the edges of $K'_{p,q} \setminus A$ the length $1-\epsilon$. The boundary walk is an interval exchange map from $w : A \longrightarrow A$. The boundary walk preserves length. We denote by the triple $(S_{g,r+p}, A, K'_{p,q})$ this relative tête-à-tête graph together with its boundary walk $w : A \longrightarrow A$.

Section 3. Gluing and closing of relative tête-à-tête graphs. First we describe the procedure of closing. We do it by an example. Consider $(S_{6,1+2}, A, K'_{2,13})$. We have two relative boundary components A_1 and A_2. In oder to close these components, we choose a piece-wise linear orientation reversing selfmap $s_1 : A_1 \longrightarrow A_1$ of order 2. The boundary component A_1 will be closed if we identify the pieces using the map s_1. In order to get the tête-à-tête property we do the same with the component A_2, but we have to take care: the involution $s_2 : A_2 \longrightarrow A_2$ is equivariant via the boundary walk $w : A_1 \longrightarrow A_2$ to the involution $s_1 : A_1 \longrightarrow A_1$. Hence we put $p \in A_2 \mapsto s_2(p) := w \circ s_1 \circ w^{-1}(p) \in A_2$. More concretely, we can choose for $s_1 : A_1 \longrightarrow A_1$ an involution that exchanges in an orientation reversing way the opposite edges of an hexagon. If we do so, we get a surface $S_{8,1}$ with tête-à-tête graph. The corresponding twist is the geometric monodromy of the singularity $(x^3 - y^2)^2 - x^5y$, see $[A'C]$. If we make our choices for the involution s_1 generically, the resulting graph Γ on $S_{8,1}$ will have 43 vertices, 58 edges, 13 vertices of valency 2, 30 vertices of valency 3. The length of the 26 edges that are incident with a vertex of valency 2 is $1-\epsilon$. The computation of the length of the remaining 32 edges is more difficult. The length of the boundary component A_1 is 26ϵ, hence the total length of the remaining edges is 26ϵ. A generic choice for the hexagon in the metric circle A_1 having opposite sides of equal length depends on 3 parameters. The following choice for s_1 is very special, but allows an easy description of the resulting metric graph. The involution s_1 is obtained by choosing the hexagon $H = [a, b, c, d, e, f]$ as follows: First take in the boundary component A_1 of $(S_{6,1+2}, A, K'_{2,13})$ two vertices, say vertex $a = 1$ and $c = 2$, where we label the thirteen vertices on A_1 cyclicly. Take for b the midpoint between a and c. Take d opposite to a, $e = 8$ opposite to b, and finally f opposite to c. So, d is the midpoint between the vertices 7 and 8 and f is the midpoint between vertices 8 and 9. The involution s_2 on component A_2 is deduced from s_1 by w-equivariance. The resulting graph on Γ on $S_{8,1}$ has 13 vertices of valency 2, 2 vertices of valency 6, 10 vertices of valency 4. Moreover, Γ has 6 edges of length ϵ, 10 of length 2ϵ and 26 of length $1-\epsilon$.

Now an example of gluing of relative tête-à-tête graphs. We glue in an walk equivariant way two copies, say $L =$ *left* and $R =$ *right* of $(S_{2,1}, A, K'_{2,5})$. So we glue the top boundary A_1^L orientation reversing, but isometrically to the top boundary A_1^R. We have a 1-dimensional family of gluings. We glue the bottom boundaries A_2^L and A_2^R w-equivariantly. We get a tête-à-tête graph on the surface $S_{5,2}$. The corresponding twist is the monodromy of the singularity $(x^3 - y^2)(x^2 - y^3)$, see $[A'C]$. We can glue in a special way, such that the 5 vertices on A_1^L match with the 5 vertices on A_1^R. We get a graph Γ on $S_{5,2}$ with 10 vertices of valency 4 and 10 of valency 2. The 10 edges connecting vertices of valency 4 have length 2ϵ, the

20 edges connecting vertices of valency 2 and 4 have length $1-\epsilon$. We can also glue by matching the vertices on A_1^L with midpoints in between vertices on A_1^R. We get a graph Γ on $S_{5,2}$ with 20 vertices of valency 3 and 10 of valency 2. The 20 edges connecting vertices of valency 3 have length ϵ, the 20 edges connecting vertices of valency 2 and 3 have length $1-\epsilon$.

Remark 1. The real analytic mapping $f : \mathbf{C}^2 \longrightarrow \mathbf{C}$ given by $f(x,y) = (x^3-y^2)^2 - x^4\bar{x}y$ has an isolated singularity at $0 \in \mathbf{C}^2$. This singularity is symplectic in the following sense: locally near the singularity at $0 \in \mathbf{C}^2$ the standard symplectic form $\omega := \frac{-1}{2i}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ on $\mathbf{C}^2$ restricts at each point p with $\mathrm{Rank}_{\mathbf{R}}(Df)_p = 2$ to a symplectic form on the smooth fiber of f through p. Moreover, the singularity at $0 \in \mathbf{C}^2$ of the map f admits a Milnor type fibration. The monodromy is the twist of the closure, as above, of the graph $(S_{6,1+2}, A, K'_{2,11})$. The corresponding knot is the obtained by cabling the trefoil with the $(2,11)$ torus knot. This cabling does not satisfy the Puiseux inequalities, so this knot is not the link of an isolated complex plane curve singularity. This knot is however the knot of a divide with 7 crossings, so still shares many properties with knots of isolated plane curve singularities.

Remark 2. Let $(\Sigma_{g,r}, \Gamma)$ be a tête-à-tête graph. For each oriented edge k of Γ, let $D_k \in H_1(\Sigma, \partial\Sigma, \mathbf{Z})$ be the relative cycle, that is represented by an relatively embedded copy of $[0,1]$ dual to the edge k. The cycle D_k is well defined and changes sign by changes the orientation of k.

The expression $\delta_k := D_k - T_\Gamma(D_k)$ is an absolut cycle in $H_1(\Sigma, \mathbf{Z})$. The map $D_k \mapsto \delta_k$ is a geometric model for the so called variation map $H_1(\Sigma, \partial\Sigma, \mathbf{Z}) \longrightarrow H_1(\Sigma, \mathbf{Z})$. We suspect that the cycles δ_k are indeed quadratic vanishing cycles: i.e cycles that vanish at a smooth point of the discriminant in the versal deformation. We enhance Γ by fixing an orientation for each of its edges. The map $k \in e(\Gamma) \mapsto \delta_k \in H_1(\Sigma, \mathbf{Z})$ induces a surjective linear map $\delta : \mathbf{Z}^{e(\Gamma)} \longrightarrow H_1(\Sigma, \mathbf{Z})$. For each vertex v of Γ, the relative cycle $R_v := \sum_{\{k \in e(\Gamma) | v \in k\}} \epsilon_{v,k} D_k$, where $\epsilon_{v,k} = \pm 1$ is the intersection number of D_k with the oriented edge k_v obtained from k by imposing upon k the orientation "*outgoing from*" v. We have $R_v = 0$ in $H_1(\Sigma, \partial\Sigma, \mathbf{Z})$, hence also $\rho_v := \sum_{\{k \in e(\Gamma) | v \in k\}} \epsilon_{v,k} \delta_k = 0$ in $H_1(\Sigma, \mathbf{Z})$. The fact $R_v = 0$ is very geometric, since the cycle $\sum_{\{k \in e(\Gamma) | v \in k\}} \epsilon_{v,k} D_k$ is the boundary of an relatively embedded disk in $(\Sigma, \partial\Sigma)$. The map $v \in v(\Gamma) \mapsto \rho_v$ induces a map $\kappa : \mathbf{Z}^{v(\Gamma)} \longrightarrow \mathbf{Z}^{e(\Gamma)}$ with $\kappa(v) = \sum_{\{k \in e(\Gamma) | v \in k\}} \epsilon_{v,k} k \in \mathbf{Z}^{e(\Gamma)}$. Let $\tau : \mathbf{Z} \longrightarrow \mathbf{Z}^{v(\Gamma)}$ be the linear map with $\tau(1) = \sum_{\{v \in v(\Gamma)\}} \kappa(v)$.

The tête-à-tête twist T_Γ acts on Γ by permutation of the sets $e(\Gamma)$ and $v(\Gamma)$. The action on $e(\Gamma)$ permutes the edges but does not necessary respect the choosen orientations of the edges. The action of T_Γ on $e(\Gamma)$ and an orientation of the edges leads to a signed permutation matrix. Moreover, the maps τ, κ and δ are T_Γ equivariant, so, since the sequence of maps τ, κ and δ is exact, we get a presentation by signed permutation matrices of semi-simple part of the action of T_Γ upon the homology $H_1(\Sigma, \mathbf{Z})$.

Remark 3. Let $(\Sigma_{g,r}, \Gamma)$ be a tête-à-tête graph describing the monodromy of a plane curve singularity. The alternating product of the characteristic polynomials of the action of T_Γ on $\mathbf{Z}, \mathbf{Z}^{v(\Gamma)}$ and $\mathbf{Z}^{e(\Gamma)}$ is the ζ-function of the monodromy of the singularity. For instance, we get $\zeta_{x^p+y^q}(t) = (1-t^{pq})(1-t^p)^{-1}(1-t^q)^{-1}(1-t)$.

This is work in progress. For isolated singularities of complex hypersurfaces $f : \mathbf{C}^{n+1} \longrightarrow \mathbf{C}$ we have a construction providing its Milnor fiber with a spine, that consists of lagrangian strata. Again the geometric monodromy is concentrated at the spine. The monodromy diffeomorphism is a generalized tête-à-tête twist. The link of the singularity is decomposed in pieces that are tangent sphere bundles over the langrangian strata. So, we have the possibility of describing combinatorially the contact structure of the link of the singularity.

References

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Nearby cycles and motivic characteristic classes

Jörg Schürmann

We are working in the complex algebraic context. Using Saito's deep theory of *algebraic mixed Hodge modules* [7, 8], we introduced in [3] the *motivic Chern class transformations* as natural transformations (commuting with proper push down) fitting into a commutative diagram:

$$\begin{array}{ccccc} G_0(X)[y] & \longrightarrow & G_0(X)[y,y^{-1}] & = & G_0(X)[y,y^{-1}] \\ {\scriptstyle mC_y}\uparrow & & {\scriptstyle mC_y}\uparrow & & \uparrow{\scriptstyle MHC_y} \\ K_0(var/X) & \longrightarrow & \mathcal{M}(var/X) & \xrightarrow{\chi_{Hdg}} & K_0(MHM(X))\,. \end{array}$$

Here $G_0(X)$ resp. $K_0(MHM(X))$ is the Grothendieck group of coherent sheaves resp. algebraic mixed Hodge modules on X, and $K_0(var/X)$ resp. $\mathcal{M}(var/X) := K_0(var/X)[\mathbb{L}^{-1}]$ is the (localization of the) relative Grothendieck group af complex algebraic varieties over X (with respect to the class of the affine line $\mathbb{L}$, compare e.g. [2, 5]). Finally $H_*(X)$ is either the Chow homology group $CH_*(X)$ or the Borel-Moore homology $H^{BM}_{2*}(X)$ of X (in even degrees).

The motivic Chern class transformations mC_y, MHC_y are a K-theoretical refinement of the *Hirzebruch class transformations* T_{y*}, MHT_{y*}, which can be defined by the (functorial) commutative diagram :

$$\begin{array}{ccccc} \mathcal{M}(var/X) & \xrightarrow{\chi_{Hdg}} & K_0(MHM(X)) & \xrightarrow{MHC_y} & G_0(X)[y,y^{-1}] \\ {\scriptstyle T_{y*}}\downarrow & & {\scriptstyle MHT_{y*}}\downarrow & & \downarrow{\scriptstyle td*} \\ H_*(X)\otimes\mathbb{Q}[y,y^{-1}] & \longrightarrow & H_*(X)\otimes\mathbb{Q}_{loc} & \xleftarrow{(1+y)^{-*}\cdot} & H_*(X)\otimes\mathbb{Q}[y,y^{-1}]\,, \end{array}$$

with $td_* : G_0(X) \longrightarrow H_*(X) \otimes \mathbb{Q}$ the *Todd class transformation* of Baum-Fulton-MacPherson [1, 4] and $(1+y)^{-*}\cdot$ the renormalization given in degree i by the multiplication

$$(1+y)^{-i}\cdot : H_i(-) \otimes \mathbb{Q}[y, y^{-1}] \longrightarrow H_i(-) \otimes \mathbb{Q}[y, y^{-1}, (1+y)^{-1}] =: H_*(-) \otimes \mathbb{Q}_{loc}\,.$$

These characteristic class transformations are motivic refinements of the (rationalization of the) *Chern class transformation*

$$c_* : F(X) \longrightarrow H_*(X)$$

of MacPherson [6], with $F(X)$ the abelian group of algebraically constructible functions. MHT_{y*} factorizes by [9] as

$$MHT_{y*} : K_0(MHM(X)) \longrightarrow H_*(X) \otimes \mathbb{Q}[y, y^{-1}] \subset H_*(X) \otimes \mathbb{Q}_{loc}\,,$$

fitting into a (functorial) commutative diagram

$$\begin{array}{ccccc} F(X) & \xleftarrow{\chi_{stalk}} & K_0(D^b_c(X(\mathbb{C}))) & \xleftarrow{rat} & K_0(MHM(X)) \\ {\scriptstyle c_*\otimes\mathbb{Q}}\downarrow & & {\scriptstyle c_*\otimes\mathbb{Q}}\downarrow & & \downarrow{\scriptstyle MHT_{y*}} \\ H_*(X)\otimes\mathbb{Q} & = & H_*(X)\otimes\mathbb{Q} & \xleftarrow{y=-1} & H_*(X)\otimes\mathbb{Q}[y,y^{-1}]\,. \end{array}$$

Here $D^b_c(X(\mathbb{C}))$ is the derived category of algebraically constructible sheaves on X (viewed as a complex analytic space), with rat associating to a (complex of) mixed Hodge module(s) the underlying perverse (constructible) sheaf complex, and χ_{stalk} is given by the Euler characteristic of the stalks. A famous result of Verdier states, that the MacPherson Chern class transformation c_* *commutes with specialization* [12]. Let $f : X \longrightarrow \mathbb{C}$ an algebraic function with $X_0 := \{f = 0\}$. Then Deligne's *nearby cycle functor*

$$\Psi_f : D^b_c(X) \longrightarrow D^b_c(X_0) \quad \text{induces} \quad \Psi_f : F(X) \longrightarrow F(X_0)$$

as a similar transformation for constructible functions. Assume now that X_0 is a hypersurface of codimension one, so that one also has a homological *Gysin homomorphism* for the inclusion $i : X_0 = \{f = 0\} \longrightarrow X$ ([12, 4]):

$$i^! : H_*(X) \longrightarrow H_{*-1}(X_0) \quad \text{and} \quad i^! : G_0(X) \longrightarrow G_0(X_0)\,.$$

Then Verdier's specialization result can be formulated as the equality of the following two transformations:

$$F(X) \xrightarrow[i^! c_*]{c_* \circ \Psi_f =} H_*(X_0)\,. \tag{1}$$

One can also consider the nearby cycle functor Ψ_f either on the motivic level of localized relative Grothendieck groups (see [2, 5]), or on the Hodge theoretical level of algebraic mixed Hodge modules ([7, 8]), "lifting" the corresponding functors on the level of algebraically constructible sheaves and functions, so that the following diagram commutes (with $\Psi_f'^H := \Psi_f^H[1]$ the shifted functor):

$$\begin{array}{ccc}
\mathcal{M}(var/X) & \xrightarrow{\Psi_f^m} & \mathcal{M}(var/X_0) \\
{\scriptstyle \chi_{Hdg}}\downarrow & & {\scriptstyle \chi_{Hdg}}\downarrow \\
K_0(MHM(X)) & \xrightarrow{\Psi_f'^H} & K_0(MHM(X_0)) \\
{\scriptstyle rat}\downarrow & & {\scriptstyle rat}\downarrow \\
K_0\left(D_c^b(X(\mathbb{C}))\right) & \xrightarrow{\Psi_f} & K_0\left(D_c^b(X_0(\mathbb{C}))\right) \\
{\scriptstyle \chi_{stalk}}\downarrow & & {\scriptstyle \chi_{stalk}}\downarrow \\
F(X) & \xrightarrow{\Psi_f} & F(X_0)\,.
\end{array}$$

Then we can prove the following counterpart of Verdier's specialization result [10]:

Theorem 1. *Assume that $X_0 = \{f = 0\}$ is a global hypersurface of codimension one. Then the motivic Hodge-Chern class transformation MHC_y commutes with specialization in the following sense:*

$$(1+y)\cdot MHC_y(\ \Psi_f'^H(-)\ ) = i^! MHC_y(-) \tag{2}$$

as transformations $K_0(MHM(X)) \longrightarrow G_0(X_0)[y, y^{-1}]$.

Another earlier result of Verdier [4, 11] states that the Todd class transformation td_* of Baum-Fulton-MacPherson *commutes* with the Gysin homomorphisms $i^!$ in these homology theories. Together with [9] one therefore gets the following commutative diagram of specialization results:

$$\begin{array}{ccc}
K_0(var(X)) & \xrightarrow[i^!\circ T_{y*}]{T_{y*}\circ\Psi_f^m =} & H_*(X_0)\otimes\mathbb{Q}[y] \\
{\scriptstyle \chi_{Hdg}}\downarrow & & \downarrow \\
K_0(MHM(X)) & \xrightarrow[i^!\circ MHT_{y*}]{MHT_{y*}\circ\Psi_f'^H =} & H_*(X_0)\otimes\mathbb{Q}[y,y^{-1}] \\
{\scriptstyle \chi_{stalk}\circ rat}\downarrow & & \downarrow{\scriptstyle y=-1} \\
F(X) & \xrightarrow[i^! c_*]{c_*\circ\Psi_f =} & H_*(X_0)\otimes\mathbb{Q}\,.
\end{array} \tag{3}$$

The motivic Chern class transformation MHC_y is defined in terms of the *filtered de Rham complex* of the filtered $\mathcal{D}$-module underlying a mixed Hodge module. In this context, our main result becomes a purely $\mathcal{D}$-module theoretic result about coherent $\mathcal{D}$-modules with a good filtration F, which are *strictly specializable* (in the sense of M.Saito [7]). Here one uses the $\mathcal{D}$-module description of nearby cycles in terms of the *V-filtration of Malgrange-Kashiwara*.

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Characteristic classes of complex hypersurfaces

Laurentiu Maxim
(joint work with Sylvain Cappell, Jörg Schürmann, Julius Shaneson)

An old problem in geometry and topology is the computation of topological and analytical invariants of complex hypersurfaces, such as Betti numbers, Euler characteristic, signature, Hodge numbers, etc. While the non-singular case is easier to deal with, the singular setting requires a subtle analysis of the relation between the local and global topological and/or analytical structure of singularities. For example, the Euler characteristic of a smooth projective hypersurface depends only of its degree and dimension. Similarly, the Hodge polynomial of a smooth hypersurface has a simple expression in terms of the degree and the cohomology class of a hyperplane section. However, in the singular context the invariants of a hypersurface inherit additional contributions from the singular locus. For instance, the Euler characteristic of a projective hypersurface with only isolated singularities differs (up to a sign) from that of a smooth hypersurface by the sum of Milnor numbers associated to the singular points. The purpose of this talk is to illustrate the interplay between local and global properties of singularities by calculating the motivic Hirzebruch classes [1] of singular hypersurfaces; see [2] for details.

Let $X \overset{i}{\hookrightarrow} M$ be the inclusion of an algebraic hypersurface X in a complex algebraic manifold M. If $N_X M$ denotes the normal bundle of X in M, then the *virtual tangent bundle* of X, that is,

$$T_{\mathrm{vir}}X := [i^*TM - N_X M] \in K^0(X), \tag{1}$$

is independent of the embedding in M, so it is a well-defined element in the Grothendieck group of vector bundles on X. Therefore, if cl^* denotes a cohomology characteristic class theory, then one can associate to the pair (M, X) an *intrinsic* (Borel-Moore) homology class defined as:

$$cl_*^{\mathrm{vir}}(X) := cl^*(T_{\mathrm{vir}}X) \cap [X]. \tag{2}$$

Assume, moreover, that there is a homology characteristic class theory $cl_*(-)$ for complex algebraic varieties, with good functorial properties, obeying the rule that for X smooth $cl_*(X)$ is the Poincaré dual of $cl^*(TX)$. If X is smooth, then clearly we have that $cl_*^{\mathrm{vir}}(X) = cl_*(X)$. However, if X is singular, the difference between the homology classes $cl_*^{\mathrm{vir}}(X)$ and $cl_*(X)$ depends in general on the singularities of X. The aim is to understand the difference class $cl_*^{\mathrm{vir}}(X) - cl_*(X)$ in terms of the geometry of singular locus of X.

For example, if $cl^* = L^*$ is the Hirzebruch L-polynomial and X is a compact complex hypersurface, the difference between the intrinsic homology class $L_*^{\mathrm{vir}}(X)$ and the Goresky-MacPherson L-class $L_*(X)$ was explicitly calculated by Cappell and Shaneson in terms of data of a fixed stratification of X as follows:

$$L_*^{\mathrm{vir}}(X) - L_*(X) = \sum_{V \in \mathcal{V}_0} \sigma(\mathrm{lk}(V)) \cdot L_*(\bar{V}), \tag{3}$$

where $\mathcal{V}_0$ is the collection of strata contained in the singular locus of X, all of which are assumed simply-connected, and $\sigma(\mathrm{lk}(V))$ is a signature invariant associated to the link pair of the stratum V in (M, X).

If $cl^* = c^*$ is the total Chern class in cohomology, the problem amounts to comparing the Fulton-Johnson class $c_*^{FJ}(X)$ with the homology Chern class $c_*(X)$ of MacPherson. The difference between these two is measured by the so-called *Milnor class*, a homology class supported on the singular locus of X. This was computed by Parusiński and Pragacz as a weighted sum in the Chern-MacPherson classes of closures of singular strata of X, the weights depending only on the normal information to the strata. For example, if X has only isolated singularieties, the Milnor class equals (up to a sign) the sum of the Milnor numbers attached to the singular points, which also explains the terminology.

Lastly, if $cl^* = td^*$ is the Todd polynomial, then the Verdier-Riemann-Roch theorem can be used to show that $td_*^{\mathrm{vir}}(X)$ equals in fact the Baum-Fulton-MacPherson Todd class $td_*(X)$ of X.

The main goal of this talk is to discuss the (unifying) case when $cl^* = T_y^*$ is the cohomology Hirzebruch class of the generalized Hirzebruch-Riemann-Roch theorem. The aim is to show that the results stated above are part of a more general philosophy, derived from comparing the intrinsic class

$${T_y}_*^{\mathrm{vir}}(X) := T_y^*(T_{\mathrm{vir}}X) \cap [X] \tag{4}$$

with the motivic Hirzebruch class $T_{y_*}(X)$ of [1]. This approach is motivated by the fact that the L-polynomial L^*, the Todd polynomial td^* and resp. the Chern class c^* are all suitable specializations of the Hirzebruch class T_y^*.

Assume in what follows that X is a complex algebraic variety, which is globally defined as the zero-set (of codimension one) of an algebraic function $f : M \longrightarrow \mathbb{C}$, for M a complex algebraic manifold. Next, recall that the motivic Hirzebruch class $T_{y_*}(X)$ is the value taken on the (class of the) constant Hodge sheaf $\mathbb{Q}^H_X$ by a natural transformation

$$MHT_{y_*} : K_0(\mathrm{MHM}(X)) \longrightarrow H^{BM}_{2*}(X) \otimes \mathbb{Q}[y, y^{-1}, (1+y)^{-1}] \tag{5}$$

defined on the Grothendieck group $K_0(\mathrm{MHM}(X))$ of algebraic mixed Hodge modules on X, with values in the even dimensional Borel-Moore homology and coefficients in the ring $\mathbb{Q}[y, y^{-1}, (1+y)^{-1}]$ (see [1]).

The main result of this talk is the following:

Theorem 2. *Let $\mathcal{V}$ be a fixed Whitney stratification of X, and denote by $\mathcal{V}_0$ the collection of all singular strata (i.e., strata of dimension strictly smaller than* $\dim X$*). Let F_v be the Milnor fiber of a point $v \in V$. Assume that all strata $V \in \mathcal{V}_0$ are simply-connected. Then:*

$$T^{\mathrm{vir}}_{y_*}(X) - T_{y_*}(X) = \sum_{V \in \mathcal{V}_0} \big(T_{y_*}(\bar{V}) - T_{y_*}(\bar{V} \setminus V)\big) \cdot \chi_y([\tilde{H}^*(F_v;\mathbb{Q})]) \,. \tag{6}$$

The requirement that all singular strata are simply-connected assures that all monodromy considerations become trivial to deal with. However, in some cases a lot of interesting information is readily available without any "monodromy" assumptions. For example, if X has only isolated singularities, the two classes $T^{\mathrm{vir}}_{y_*}(X)$ and resp. $T_{y_*}(X)$ coincide except in degree zero, where their difference is measured (up to a sign) by the sum of Hodge polynomials associated to the middle cohomology of the corresponding Milnor fibers attached to the singular points. These Hodge polynomials can in general be computed from the *Hodge spectrum* of singularities, and are just Hodge-theoretic versions of the Milnor numbers. For this reason, we regard the difference

$$\mathcal{M}T_{y_*}(X) := T^{\mathrm{vir}}_{y_*}(X) - T_{y_*}(X) \in H_*(X) \otimes \mathbb{Q}[y] \tag{7}$$

as a Hodge-theoretic Milnor class, and call it *the Milnor-Hirzebruch class* of the hypersurface X. In fact, it is always the case that by substituting $y = -1$ into $\mathcal{M}T_{y_*}(X)$ we obtain the (rationalized) Milnor class of X. Therefore, Theorem 2 specializes in this case to a computation of the (rationalized) Milnor class of X, and the resulted formula holds without any monodromy assumptions.

The key ingredient used in the proof of Theorem 2 is the specialization property for the motivic Hirzebruch class transformation MHT_{y_*} (see [3]). This is a generalization of Verdier's result on the specialization of the Chern-MacPherson classes, which was used for computing the Milnor class of X, and shows that the Milnor-Hirzebruch class $\mathcal{M}T_{y_*}(X)$ of $X = f^{-1}(0)$ is entirely determined by the vanishing cycles of the algebraic function $f : M \longrightarrow \mathbb{C}$. So the Milnor-Hirzebruch class is a measure of the complexity of singularities of X.

As an application of our result, we show that if X is a hypersurface with only isolated singularities which moreover is a rational homology manifold, then the Goresky-MacPherson L-class $L_*(X)$ can be deduced from the motivic Hirzebruch class $T_{y_*}(X)$ for the value $y = 1$ of the parameter. This confirms a conjecture of Brasselet-Schürmann-Yokura in this particular setting. When the hypersurface has only Du Bois singularities, we also obtain a characteristic class version of Steenbrink's cohomological insignificance.

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Nash problem for surface singularities is topological

Javier Fernández de Bobadilla

Nash problem [9] was formulated in the sixties (but published later) in the attempt to understand the relation between the structure of resolution of singularities of an algebraic variety X and the space of arcs (germs of algebroid curves) in the variety. He proved that the space of arcs centred at the singular locus (endowed with a infinite-dimensional algebraic variety structure) has finitely many irreducible components, and proposed to study the relation of these components with the essential irreducible components of the exceptional set a resolution of singularities. An irreducible component E of the exceptional divisor of a resolution of singularities

$$\pi : \tilde{X} \longrightarrow X$$

is called essential, if given any other resolution

$$\pi' : \tilde{X}' \longrightarrow X$$

the birational transform of E to $\tilde{X}'$ is an irreducible component of the exceptional divisor. Nash defined a mapping from the set of irreducible components of the space of arcs centred at the singular locus to the set of essential components of a resolution as follows: he assigns to each component Z of the space of arcs centred at the singular locus the unique component of the exceptional divisor which meets the lifting of a generic arc of Z to the resolution. Nash established the injectivity of this mapping and asked whether it is bijective. He viewed as a plausible fact that Nash mapping is bijective in the surface case, and also proposed to study the higher dimensional case.

Nash gave an affirmative answer to his problem in the case of A_k-singularities. Since then there has been much progress showing an affirmative answer to the problem for many classes of singularities: non-necessarily toric singularities of arbitrary dimension, quasi-ordinary singularities, certain infinite families of non-normal threefolds, minimal surface singularities, sandwiched surface singularities, and other classes of surface singularities defined in terms of the combinatorics of the minimal resolution (see [1],[2],[3],[4],[7],[8],[10],[11],[12],[14],[15]). However,

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Ishii and Kollár showed in [2] a 4-dimensional example with non-bijective nash mapping. Now the general problem has turned into characterising the class of singularities with bijective Nash mapping. Besides Nash problem, the study of arc spaces is interesting because it lays the foundations for motivic integration and because the study of its geometric properties reveals properties of the underlying varieties (see papers of Denef, Loeser, de Fernex, Ein, Ishii, Lazarsfeld, Mustata, Yasuda and others).

Nash problem seems different in nature in the surface case than in the higher dimensional case, since birational geometry in dimension 2 is much more simple than in higher dimension. For example the essential components are the irreducible components of the exceptional divisor of a minimal resolution of singularities. Although Nash problem in known for many classes of surfaces it is not yet known, for example, for the simple singularities E_6, E_7 and E_8. From now on we shall concentrate in the surface case, we let (X,O) be a normal surface singularity defined over a field of characteristic 0, and $\mathcal{X}_\infty$ denotes the space of arcs through the singular point.

Let us explain the approach to Nash problem based on wedges, due to M. Lejeune-Jalabert [5]. Let E_u be an essential component of a surface singularity (X,O). Denote by N_{E_u} the set of arcs whose lifting meets E_u. The space of arcs centred at the singular point splits as the union of the N_{E_u}'s. It is known (Remark 2.3 of [13]) that the N_{E_u}'s are constructible subsets of the space of arcs, and that E_v is *not* in the image of the Nash map if and only if N_{E_v} is in the Zariski closure N_{E_u} of a different essential component E_u. If Curve Selection Lemma were true in $\mathcal{X}_\infty$ then, for any arc

$$\gamma : Spec(\mathbb{K}[[t]]) \longrightarrow (X,O)$$

in N_{E_v} there should exists a curve in $\mathcal{X}_\infty$ with special point γ and generic point an arc on N_{E_u}. Giving a curve in $\mathcal{X}_\infty$ amounts to give a morphism

$$\alpha : Spec(\mathbb{K}[[t,s]]) \longrightarrow (X,O)$$

mapping $V(t)$ to O. Such a morphism is called a *wedge*. The lifting to $\tilde{X}$ of a generic arc in N_{E_v} is transversal to E_v, and if a wedge α has special arc equal to γ and generic arc in N_{E_u} it is clear that the rational lifting

$$\pi^{-1}\circ\alpha : \mathbb{K}[[t,s]]) \longrightarrow \tilde{X}$$

has an indetermination point at the origin, and hence there is no morphism lifting α to $\tilde{X}$. In [5], M. Lejeune-Jalabert proposes to attack Nash problem by studding the problem of lifting wedges whose special arc is a transversal arc through an essential component of (X,O).

Since Curve Selection Lemma in not known to hold in the space of arcs A. Reguera [13] introduced K-wedges, which are wedges

$$\alpha : Spec(K[[t,s]]) \longrightarrow (X,O)$$

defined over a field extension K of $\mathbb{K}$ and proved the following characterisation: an essential component E_v is in the image of the Nash map if and only if any wedge whose special arc is *the generic point* of N_{E_v} and whose generic arc lifts to E_v admits a lifting to the resolution. However the field of definition K of the involved wedges has infinite transcendence degree over K and, hence, it is not easy to work with them. Building on this result and assuming that $\mathbb{K}$ is uncountable, A. Reguera [13] and M. Lejeune-Jalabert (Proposition 2.9, [6]) proved a sufficient condition for a divisor E_v to be in the image of the Nash map based on wedges defined over the base field: it is enough to check that any $\mathbb{K}$-wedge whose special arc is transversal to E_i arc through a very dense collection of closed points of E_i lifts to $\tilde{X}$ (a very dense set is a set which intersects any countable intersection of dense open subsets). The results of [13] and [6] hold in any dimension.

Our first main result is a characterisation of all the possible adjacencies between essential components of the exceptional divisor of a resolution (a component E_u is adjacent to E_v if N_{E_v} is contained in the Zariski closure of N_{E_u}) in terms of wedges defined over the base field. We prove:

Theorem A. *Let (X, O) be a normal surface singularity defined over an algebraically closed field $\mathbb{K}$ of characteristic 0. Let E_u, E_v be different essential irreducible components of the exceptional divisor of a resolution. Equivalent are:*

(1) *the component E_u is adjacent to E_v.*
(2) *There exists a $\mathbb{K}$-wedge whose special arc has lifting transversal to E_v and with generic arc belonging to N_{E_u}.*

If the base field is $\mathbb{C}$ the following condition is also equivalent:

- *Given* any *convergent arc γ whose lifting is transversal to E_v there exists a convergent $\mathbb{C}$-wedge with special arc γ and generic arc belonging to N_{E_u}.*

An inmediate Corollary characterises the image of the Nash maps in terms of $\mathbb{K}$-wedges:

Corollary B. *Let (X, O) be a normal surface singularity defined over an algebraically closed field $\mathbb{K}$ of characteristic 0. Let E_v be an essential irreducible component of the exceptional divisor. Equivalent are:*

(1) *The component E_v is in the the image of the Nash map.*
(2) *There not exists a different component E_u and a $\mathbb{K}$-wedge whose special arc has lifting transversal to E_v and with generic arc belonging to N_{E_u}.*

If the base field is $\mathbb{C}$ the following condition is also equivalent:

- *There exists a convergent arc γ whose lifting is transversal to E_v such that there is no convergent $\mathbb{C}$-wedge with special arc γ and generic arc belonging to N_{E_u}, for a different component E_u of the exceptional divisor.*

Our result improves the result of [6] in the following sense: for proving that E_v is not in the image of the Nash map it is sufficient to exhibit a *single* wedge defined over the base field with the condition stated above. If $\mathbb{K} = \mathbb{C}$, in order to prove that E_v is in the image of the Nash it is sufficient to find a *single* convergent arc whose lifting is transversal to E_v such that any wedge having γ as special point has generic point in E_u. Our condition on the wedges is more precise that the liftability, since, when a component E_v is not in the image of the Nash map, we want to keep track of the responsible adjacencies. However we prove also an improvement of the result of [6] in terms of the original condition of lifting wedges of [5]:

Theorem C. *Let (X, O) be a normal surface singularity defined over $\mathbb{C}$. Let E_v be any essential irreducible component of the exceptional divisor of a resolution of singularities. If there exists a convergent arc γ whose lifting is transversal to E_v such that any $\mathbb{C}$-wedge having γ as special arc lifts to $\tilde{X}$, then the component E_v is in the image of the Nash map.*

The ideas of the proofs are as follows: if the Zariski closure of N_{E_u} contains N_{E_v} we use Corollary 4.8 of [13] to obtain a K-wedge (with K an infinite transcendence degree extension of $\mathbb{K}$) whose special arc is *the generic point* of N_{E_v} and whose generic arc lifts to E_u, after we follow a specialisation procedure to obtain a $\mathbb{K}$-wedge whose special arc has lifting transversal to E_v and with generic arc belonging to N_{E_u}.

In the other direction, given an arc γ whose lifting to $\tilde{X}$ is transversal to E_v, a $\mathbb{K}$-wedge whose special arc is γ and with generic arc belonging to N_{E_u} (with E_u another component of the exceptional divisor) will be called a wedge realising an adjacency from E_u to γ. We use Popescu's Approximation Theorem to replace such a $\mathbb{K}$-wedge by an algebraic one with the same property with respect to another transversal arc γ'. After, using Stein Factorisation, we "complete" the wedge and factorise it through a finite covering of normal surface singularities realising an adjacency from E_u to γ'. We prove that there exists a wedge realising an adjacency from E_u to γ' if and only if there exists a finite covering realising an adjacency from E_u to γ'. Using Lefschetz Principle we can reduce the existence of a finite covering realising an adjacency from E_u to a γ' to the analogue statement in the complex analytic case. After doing this, using a topological argument and a suitable change of complex structures we prove that given two convergent arcs γ and γ' on a complex analytic normal surface singularities there exists a finite covering realising an adjacency from E_u to a γ if and only if there exists a finite covering realising an adjacency from E_u to a γ'. This results allows to move wedges in a very flexible way, and is the key to the characterisation given above.

Using the same kind of technique we prove also the following surprising result:

Theorem D. *The set of adjacencies between exceptional divisors of a normal surface singularity is a combinatorial property of the singularity: it only depends on the dual weighted graph of the minimal good resolution. In the complex analytic case this means that the set of adjacencies only depends on the topological type of the singularity, and not on the complex structure.*

The last result allows us to play with combinatorial and topological arguments in order to study and compare the adjacency structure of different singularities. This is exploited in the last section: we prove reductions of Nash problem for singularities with symmetries in the dual weighted graph of the minimal good resolution. We prove a result showing that if there is an adjacency between two divisors, there should exists a certain path of rational components in the exceptional divisor. We prove results comparing the adjacency structure of different singularities. We also reduce the Nash problem in the following sense: we introduce *extremal graphs*, which is a subclass of the class of dual graphs with only rational vertices and no loops and *extremal rational homology spheres*, which are the plumbing 3-manifolds associated with extremal graphs.

Corollary E. *If the Nash mapping is bijective for singularities whose minimal good resolution graph is extremal then it is bijective in general. Equivalently, if the Nash mapping is bijective for all complex analytic normal surface singularities having extremal* $\mathbb{Q}$*-homology sphere links then it is bijective in general.*

The last Corollary improves Proposition 4.2 of [6], which reduces Nash problem for surfaces to the class of surfaces having only rational vertices in its resolution, and makes essential use of Theorem D.

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A viewpoint on local resolution of singularities

Bernard Teissier

In the talk I gave a brief report on the progress made towards resolution of singularities in positive characteristic as it was presented by various groups during the RIMS workshop of December 2008. (See the following homepage for details and documents: http://www.kurims.kyoto-u.ac.jp/ kenkyubu/proj08-mori/index.html)

Apart from the work of Cossart-Piltant proving resolution of singularities in dimension 3 in the equicharacteristic case (see [2], [3]), all approaches follow the approach of Hironaka's fundamental paper, as modified by Villamayor to put the idealistic exponents, or basic objects, at the center of the process, as the only objects which need to be resolved (see [4], [8]).

An idealistic exponent on a non singular space W is a pair (J, b) of a coherent Ideal J on W and an integer $b \geq 1$. Its singular locus is the set of points x of W where $\frac{\nu_x(J)}{b} > 1$. The order $\nu_x(J)$ is the largest integer n such that $J_x \subseteq m_x^n$.

One then defines a permissible center for (J, b) as a non singular subvariety Y of W which is contained in the singular locus of (J, b). The transform of (J, b) by the blowing-up $W' \longrightarrow W$ with center Y is then defined as $(J', b') = ((I_Y \mathcal{O}_{W'})^{-b} J\mathcal{O}_{W'}, b)$.

The goal is then essentially to prove the existence of a finite sequence of permissible blowing-ups such that the final singular locus is empty. In fact all groups try to prove the existence of a canonical process, and one has to use a richer definition of idealistic exponents and their transforms, taking into account at each stage the exceptionnal divisors created by the previous blowing-ups. In order to produce a canonical process one associates to an idealistic exponent an "invariant" at each point of W, with values in an ordered set and such that the set of points of W where the invariant is the worst (largest) is non singular, or at least has simple normal crossings, and that blowing it up (or blowing up its components in some order) will make the worst invariant decrease strictly.

The main problem in positive characteristic is the non-existence of "hypersurfaces of maximal contact" with (J, b) in W. In characteristic zero, one can define on such non singular hypersurfaces a "trace" of the idealistic exponent which retains enough information about the order of the ideal and its behavior under permissible blowing-up to permit a proof by induction on the dimension. All the attempts to prove resolution in positive characteristic replace the idealistic exponent by (different) graded algebras which are stable under derivation, finitely generated and in several cases integrally closed. The generators are expected to play the role of maximal contact by allowing an inductive process.

The generators of the graded algebras just mentioned are monomials of the form $x_i^{p^{e_i}}$ where p is the characteristic, so that comparison with monomial ideals plays a role in all programs.

In the last years I have been led to try to prove local uniformization (a very local version of resolution) by a completely different method, in which the basic idea is to compare a given singular germ *by deformation* with a space whose resolution is easy and blind to the characteristic. The spaces in question are affine toric varieties, which are defined by prime binomial ideals. I refer to [7] for their toric *embedded* resolution and to [1] for the proof of a canonical embedded resolution by composition of blowing-ups with equivariant non singular centers.

Given a base field k, which we assume to be algebraically closed, the algebra $k[t^\Gamma]$ of an affine toric variety over k is the semigroup algebra over k of a finitely generated semigroup Γ, for example a polynomial ring $k[u_1, \dots, u_N]$. A binomial ideal in this algebra is an ideal generated by differences of terms, where a term is the product of a monomial with an element of k^*. It turns out that there is a deep relation between the most important valuations from the viewpoint of local uniformization and affine toric varieties. See [9].

Let R be a noetherian excellent equicharacteristic local domain with an algebraically closed residue field $k = R/m$. Let ν be a valuation on R, corresponding to an inclusion $R \subset R_\nu$ of R in a valuation ring of its field of fractions. We may assume that R_ν dominates R in the sense that $m_\nu \cap R = m$ and that the residual injection $R/m \hookrightarrow R_\nu/m_\nu$ is an isomorphism. This corresponds to the fact that the point picked by ν in all schemes birationally dominating Spec$R$ by a proper map is a *closed* point.

In many important cases (see [9], [11]), one can check that there exists a formal embedding of (SpecR, m) in an affine space $(\mathbf{A}^N(k), 0)$ with the following properties: there is a system of coordinates such that the intersection of (SpecR, m) with the torus $T^N(k)$ consisting of the complement of the coordinate hyperplanes is dense, and a birational map of toric varieties $Z \longrightarrow \mathbf{A}^N(k)$ with Z regular, which is equivariant with respect to $T^N(k)$ and such that the strict transform of (SpecR, m) in Z is non singular and transversal to the non dense orbits of Z *at the point of this strict transform picked by the valuation* ν.
Such a result is a constructive form of local uniformization, at least if one can effectively construct the embedding.

In the case of plane branches (see [5]) and more generally of quasi-ordinary hypersurfaces (see [6]), the smallest embedding with this property can be explicitely constructed in characteristic zero from (generalized) Puiseux expansions.

Since it seems much easier to glue up the embeddings corresponding to various valuations (by compactness of the Zariski-Riemann manifold a finite number suffices) than to glue up *à la Zariski* various birational models, this led me to ask in [10] the following:

Question: *Given a noetherian excellent equicharacteristic local domain* R *with an algebraically closed residue field* $k = R/m$*, does there exist a formal embedding of* (SpecR, m) *with a toric birational map of toric varieties* $Z \longrightarrow \mathbf{A}^N(k)$ *in appropriate coordinates on* $\mathbf{A}^N(k)$ *such that the strict transform of* Spec$R$ *in* Z *is non singular and transversal to the non dense orbits at each point mapped to the closed point* m *of* Spec$R$.
One may ask that in addition the singular locus of Spec$R$ should be the union of intersections with Spec$R$ of sets of coordinate hyperplanes in this new embedding. The map from the strict transform to Spec$R$ is then an isomorphism outside of the singular locus. I think of this as a generalization of the condition of non-degeneracy with respect to a Newton polyhedron.

This would imply local resolution of singularities and the difficulty is moved from the study of the behaviour of the order of ideals under certain blowing ups to the search of functions in R having very special properties. The simplest non trivial example is the plane curve $(y^2-x^3)^2-x^5y=0$ which can be resolved by a single toric modification of the ambient space only after being embedded in $\mathbf{A}^3(k)$ by the functions x, y, y^2-x^3. See [5], [9] and [11].

After hearing me mention this last January at the Workshop on Toric Geometry (see [11]), Jenia Tevelev kindly sent me a proof of the following:
Theorem (Tevelev) Let k be an algebraically closed field of characteristic zero and let $X \subset \mathbf{P}^n(k)$ be a projective algebraic variety. Then, for a sufficiently high order Veronese reembedding $X \subset \mathbf{P}^N(k)$ one can choose projective coordinates $z_0 : \ldots : z_N$ such that if $T^N(k)$ is the torus $(k^*)^N$ consisting of the complement of the coordinate hyperplanes in $\mathbf{P}^N(k)$,

- The intersection of X with $T^N(k)$ is dense in X,
- There exists a nonsingular toric variety Z and an equivariant map $Z \longrightarrow \mathbf{P}^N(k)$ such that the strict transform of X is non singular and transversal to the non dense toric orbits in Z.

The proof uses resolution of singularities and answers the question in characteristic zero for algebraizable singularities while of course one would hope to prove local resolution in the manner I have described. Still, it is very encouraging.

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The universal unfolding is an atlas of Stokes data for the simple and the simple elliptic singularities

Claus Hertling

In 2007 Céline Roucairol and I did some joint work on the Stokes data of the simple singularities $A_\mu, D_\mu, E_6, E_7, E_8$ and the simple elliptic singularities $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$. The results are complete, but we still have to write them up (hopefully soon).

In the case of germs of functions with isolated singularities and especially in the case of quasihomogeneous functions, the Lefschetz thimbles correspond to vanishing cycles, and Stokes data are equivalent to distinguished bases of vanishing cycles. These have been studied by Looijenga [Lo] and Deligne [De] in the ADE case and later in general by A'Campo, Brieskorn, Ebeling, Gabrielov, Gusein-Zade, Kluitmann, Voigt and others. Most of this work has a topological and combinatorial flavour. But [Lo] and [De] together imply a beautiful global 1-1 correspondence in the ADE cases, which is formulated in [Mi, 39.] and [Yu, 4.6.3], but still not well known.

Roughly, the base $M \cong \mathbb{C}^\mu$ of a universal unfolding is an atlas of Stokes data. More precisely, after some choice, M obtains a complicated decomposition into Stokes walls (real hypersurfaces with boundaries) and simply connected Stokes regions. For example in the A_μ case there are $(\mu+1)^{\mu-1}$ Stokes regions. By [Lo] and [De] these Stokes regions are in 1-1 correspondence with the distinguished bases up to signs. And the combinatorial structure of the Stokes regions and Stokes walls reflects the braid group action on the distinguished bases. Here the surjectivity of the correspondence is the simpler part, the injectivity is more difficult and follows from equality of numbers. The number of Stokes regions is the degree of the Lyashko-Looijenga map. This degree is the result of a simple calculation in [Lo]. The number of distinguished bases is more difficult to determine, for A_μ it is in [Lo], for D_μ and E_μ it is one main point of [De].

There is a second coarser 1-1 correspondence between Stokes regions up to isomorphism and Coxeter-Dynkin diagrams up to signs. One obtains it from the 1-1 correspondence above by dividing out on both sides a finite group of automorphisms. In the A_μ case it is cyclic of order $\mu + 1$. It acts on M by rotation and on the distinguished bases by cyclic renumbering.

Roucairol and I have generalized both correspondences to the case of the simple elliptic singularities $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$. We started with the Legendre normal form with parameter space $M_0 = \mathbb{C} - \{0, 1\}$ and chose a global everywhere universal unfolding on $M \cong$ (a vector bundle or rank $\mu - 1$ on M_0). This global unfolding and M are not canonical, but the universal covering $M^{univ} \longrightarrow M$ and a certain quotient $M/\sim_M$ by an analytic equivalence relation with finite classes are canonical; then $M/\sim_M \cong M^{univ}/G$ where the group G is a finite extension of $PSL(2, \mathbb{Z})$. The finer 1-1 correspondence compares Stokes regions in M^{univ} with distinguished bases up to sign (the numbers of both are infinite). The coarser 1-1 correspondence compares Stokes regions in the quotient $M/\sim_M$ with Coxeter-Dynkin diagrams up to sign (the numbers of both are finite and equal).

In order to control the Lyashko-Looijenga maps from M and $M/\sim_M$, we needed partial compactifications with good behaviour with respect to the Lyashko-Looijenga maps. SINGULAR was useful for finding them. They show especially that the Lyashko-Looijenga map is covering from the complement of caustic and Maxwell stratum to $\mathbb{C}^\mu -$ *discriminant*, in coincidence with [Ja], but they also allow to determine its degree. For the finer 1-1 correspondence we could not simply compare numbers, as they are infinite. An argument from [He, ch. 13.2] on symmetries of singularities is useful. It applies also to ADE and is there more conceptual than the comparison of numbers. Our work completes also work of Kluitmann. He had calculated the number of Coxeter Dynkin diagrams in the cases $\widetilde{E}_6$ [Kl1][Kl2] and $\widetilde{E}_7$ [Kl2], but not in the case $\widetilde{E}_8$. We calculated all three by a completely different method, via the degree of the Lyashko-Looijenga map and the coarser 1-1 correspondence above (and obtained the same values for $\widetilde{E}_6$ and $\widetilde{E}_7$).

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Adding divisors to make them free

David Mond
(joint work with Mathias Schulze)

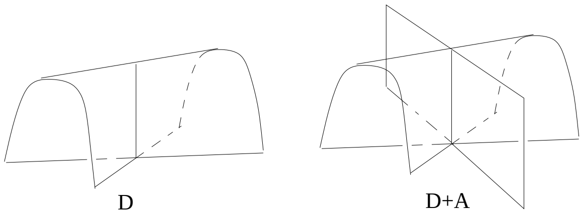
A hypersurface $D \subset \mathbb{C}^{n+1}$ is a *free divisor* if the $\mathcal{O}_{\mathbb{C}^{n+1}}$-module $\mathrm{Der}\,(-\log D)$ is locally free. We prove three theorems along similar lines.

1. Adding the Adjoint

Theorem 1.1. *Let* $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^{n+1}, 0)$ *be a stable map-germ of corank 1, let* D *be the image of* f*, and let* A *be an adjoint divisor. Then* $D + A$ *is a free divisor.*

Note that D itself is not free. Any stable map $\mathbb{C}^n \longrightarrow \mathbb{C}^p$ with $n < p$ normalises its image. By *adjoint divisor* we mean a divisor $A \subset \mathbb{C}^p$ such that $f^*(A)$ is the conductor of the normalisation. An adjoint divisor has the property that as sets, $D \cap A = D_{\text{sing}}$.

Example 1.2. (1) Consider the stable map-germ $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^3, 0)$ given by $f(u, x) = (u, x^2, ux)$, whose image is the Whitney umbrella. Denote the first coordinate in the target by U. Then one can take as the adjoint divisor the hyperplane $\{U = 0\}$.



(2) The image of a stable map of multiplicity k contains points at which it is locally a normal crossing divisor of multiplicity k, with equation $y_1 \cdots y_k = 0$ in suitable local coordinates. As adjoint at such a point one can take the divisor $\{y_2 \cdots y_k + y_1 y_3 \cdots y_k + \cdots + y_1 \cdots y_{k-1} = 0\}$.

Theorem 1.3. *Let $D \subset \mathbb{C}^\mu$ be the discriminant in the base-space of an $\mathcal{R}_e$-miniversal deformation of a weighted homogeneous function germ $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$. Let d be the weighted degree of f, and let $d_1, \ldots, d_\mu$ be the weighted degrees of the homogeneous members of a $\mathbb{C}$-basis of the Jacobian algebra of f, with d_1 the weight of the Hessian. Assume that $d - d_1 + 2d_i \neq 0$ for $i = 2, \ldots, \mu$. Let A be an adjoint divisor of D. Then $D + A$ is a free divisor.*

Here D itself is already a free divisor. The numerical condition holds for the simple singularities, since there $d_1 < d$. It is easy to check that it holds for irreducible functions of two variables, of the form $f(x, y) = x^p + y^q$. We do not know whether the statement holds if this numerical condition is not met, nor indeed whether it holds for function-germs which are not weighted homogeneous.

Let $D \subset V$ be as in Theorems 1.1 or 1.3, and let $\bar{D}$ be its normalisation (smooth in both cases). Let $\mathcal{F}_1$ be the first Fitting ideal of $\mathcal{O}_{\bar{D}}$ as $\mathcal{O}_V$- module.

Let h be an equation for the adjoint A. A key step in the proof of Theorems 1.1 and 1.3 involves showing that dh defines a surjective morphism

$$\text{Der}\,(-\log D) \longrightarrow \mathcal{F}_1. \tag{1}$$

That $dh(\text{Der}\,(-\log D)) \subset \mathcal{F}_1$ holds because $\mathcal{F}_1$ is intrinsically determined by D and logarithmic vector fields are infinitesimal automorphisms of D. It seems that the representation of the Lie algebra $\text{Der}\,(-\log D)$ on $\mathcal{F}_1$ contains a lot of information.

2. Pulling back a free divisor

A free divisor $D \subset \mathbb{C}^n$ is *linear* if there is a basis of $\mathrm{Der}\,(-\log D)$ consisting of vector fields of weight zero - i.e. whose coefficients are all linear forms. The simplest example is the normal crossing divisor $\{x_1 \cdots x_n = 0\} \subset \mathbb{C}^n$. More examples may be found in [1] and [2].

If D is a linear free divisor then the group $G_D \subset \mathrm{Gl}(\mathbb{C}^n)$ consisting of linear automorphisms preserving D has an open orbit in $\mathbb{C}^n$ whose complement is D itself, and moreover the Lie algebra of G_D is isomorphic, under the infinitesimal action of $\mathfrak{gl}_n$, to the Lie algebra of weight-zero members of $\mathrm{Der}\,(-\log D)$. If $D_1 \ldots, D_k$ are the irreducible components of D, and $f_1, \ldots, f_k$ are homogeneous equations for them, then by results of [5] it follows that there exist vector fields $\chi_j \in \mathrm{Der}\,(-\log D)$ such that $df_i(\chi_i) = f_i$ and $df_i(\chi_j) = 0$ if $i \neq j$.

Theorem 2.1. *Suppose that $D = \cup_{i=1}^k D_i \subset \mathbb{C}^n$ is a free divisor and for $i = 1, \ldots, k$ let f_i be a reduced equation for D_i. Suppose that for $j = 1, \ldots, k$, there exist vector fields χ_j such that $df_i(\chi_i) = f_i$ and $df_i(\chi_j) = 0$ if $i \neq j$. Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}^k$ be the map with components $f_1, \ldots, f_k$ (so D is the preimage of the normal crossing divisor $N := \{y_1 \cdots y_k = 0\} \subset \mathbb{C}^k$). Let $E \subset \mathbb{C}^k$ be a divisor such that $N + E$ is free. Then $D + f^{-1}(E)$ is free.*

The proof is elementary, and makes use of nothing more than Saito's criterion ([4]). The theorem can be applied in an obvious way to a linear free divisor with irreducible components $D_1, \ldots, D_k$. It may also be applied taking as the D_i the unions of disjoint collections of the irreducible components of D. For example, if $g \in \mathcal{O}_{\mathbb{C}^2,0}$ is any germ not divisible by either of the variables, then for any $n > 1$ and any k with $1 \leq k < n$, the divisor with equation

$$x_1 \cdots x_n \times g(x_1 \cdots x_k,\, x_{k+1} \cdots x_n) = 0$$

is free.

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Gauß-Manin systems and Frobenius manifolds for linear free divisors

Christian Sevenheck
(joint work with David Mond, Ignacio de Gregorio)

Linear free divisors have been recently introduced by Mond and Buchweitz ([1]) as special examples of free divisors. A reduced divisor $D = h^{-1}(0) \subset V := \mathbb{C}^n$ is called free if the coherent $\mathcal{O}_V$-module $Der(-\log D) := \{\vartheta \in Der_V \mid \vartheta(h) \subset (h)\}$ is $\mathcal{O}_V$-free of rank n. In that case, one can write any basis $\vartheta_1, \ldots, \vartheta_n$ of $Der(-\log D)$ in terms of the coordinate vector fields as $\vartheta_i = \sum_j a_{ij}\partial_{x_i}$, and we call D linear free if all a_{ij} are linear functions on V. The most simple example is the normal crossing divisor given by $h = x_1 \cdot \ldots \cdot x_n$, but many more examples are constructed as discriminants in quiver representation spaces. Linear free diviors are related to the classical theory of prehomogeneous vector spaces, that is, tuples (V, G) such that G acts linearly on V and has a Zariski open orbit. As have been shown in [3], given a linear free divisor D, the group $G_D := \{g \in GL(V) \mid g(D) = D\}$ makes V into a prehomogeneous vector space, where the open orbit is the complement $V\backslash D$. Of particular importance is the case where G_D is reductive, then we call D a reductive linear free divisor.

By work of Sabbah and Douai ([2]), it is known that a universal unfolding space of the Laurent polynomial $\widetilde{f} := x_1 + \ldots + x_{n-1} + \frac{1}{x_1 \cdot \ldots \cdot x_{n-1}}$ carries a Frobenius structure, which is known to be isomorphic to the quantum cohomology ring of $\mathbb{P}^{n-1}$. This Laurent polynomial can be seen as the restriction of the ordinary polynomial $f = x_1 + \ldots + x_n$ to the non-singular fibres of the equation $h = x_1 \cdot \ldots \cdot x_n$. Hence it seems natural to study linear functions on non-singular fibres and on the central fibre of a morphism given by an equation defining a linear free divisor. Let $f \in \mathbb{C}[V]$ be a linear function. We want to study deformations of f modulo coordinate changes preserving the morphism $h : V \to T = \operatorname{Spec}\mathbb{C}[t]$, this is referred to as $\mathcal{R}_h$-equivalence. The corresponding deformation (or Jacobi) algebra is given by

$$\mathcal{T}^1_{\mathcal{R}_h/T}(f) := \frac{\mathcal{O}_V}{df(Der(-\log h))},$$

where $Der(-\log h) := \{\vartheta \in Der_V \mid \vartheta(h) = 0\} \subset Der(-log\, D)$. Actually, we have the direct sum decomposition $Der(-log\, D) = \mathcal{O}_V E \oplus Der(-\log h)$, where $E = \sum_{i=1}^n x_i\partial_{x_i}$. The first result concerns the finiteness of the above family of Jacobian algebras.

Proposition 1 ([5]). *Let D be reductive. Let V^* be the dual space of V, equipped with dual action of G. Then (V^*, G) is again prehomogenous, and the complement D^* of the open orbit in V^* is again linear free. If $f \in V^*\backslash D^*$, then $h_*\mathcal{T}^1_{\mathcal{R}_h/T}(f)$ is $\mathcal{O}_T$-free of rank n. Moreover, the restriction of f to $D_t := h^{-1}(t)$, $t \neq 0$ has n non-degenerate critical points.*

In order to construct Frobenius structures associated to the restrictions $f_{|D_t}$ (and to $f_{|D}$), we study families of Brieskorn lattices. More precisely, define

$$G_0 := \frac{\Omega^{n-1}_{V/T}(\log D)[\theta]}{(\theta d - df\wedge)\Omega^{n-1}_{V/T}(\log D)[\theta]}$$

which is an $\mathcal{O}_T[\theta]$-module, equipped with connection operators $\theta^2\nabla_\theta$ and $\theta t\nabla_t$, i.e., with a connection

$$\nabla : G_0 \longrightarrow G_0 \otimes \theta^{-1}\Omega^1_{\mathbb{C}\times T}((\log \{0\} \times T) \cup (\mathbb{C} \times \{0\})).$$

Here

$$\Omega^\bullet_{V/T}(\log D) := \frac{\Omega^\bullet_V(\log D)}{h^*\Omega^1_T(\log\{0\}) \wedge \Omega^{\bullet-1}_V(\log D)}$$

is the relative logarithmic de Rham complex. One of the main results of [5] is the following.

Theorem 2. *Let $f \in V^*\backslash D^*$. Then*

1. *The restrictions $f_{|D_t}$ are cohomological tame functions in the sense of* [6].
2. *G_0 is $\mathcal{O}_T[\theta]$-free of rank n.*
3. *There is a basis $\underline{\omega}$ such that*

$$\nabla(\underline{\omega}) = \underline{\omega}\cdot\left[(A_0\frac{1}{z} + A_\infty)\frac{dz}{z} + (-A_0\frac{1}{z} + A'_\infty)\frac{dt}{nt}\right]$$

where A_0 and A_∞ are constant, $A'_\infty := \mathrm{diag}(0, 1, \ldots, n-1) - A_\infty$ and A_∞ is diagonal. These diagonal entries are not necessarily the spectral numbers of the tame functions $f_{|D_t}$ but can be turned into them after some base change which is meromorphic along $0 \in T$, i.e., there is some other basis $\underline{\omega}'$ of $G_0 \otimes \mathcal{O}_T[\theta, t^{-1}]$, in which the connection also takes the above form and where the diagonal entries of A_∞ are the correct spectral numbers.

As a consequence (by some more arguments concerning the duality theory for these families of Brieskorn lattices), one obtains the following results.

Theorem 3.

1. *The semi-universal unfoldings $(M_t, 0)$ of the tame functions $f_{|D_t}$ can be equipped with the structure of Frobenius manifolds, depending (among other things) on the choice of a primitive and homogenous section of G_0. Any element ω'_i of the above mentioned basis $\underline{\omega}'$ can be chosen as such a form.*
2. *The germs $(M_t, 0)$ of Frobenius manifolds glue to a germ $(T^* \times \mathbb{C}^{\mu-1}, T^* \times \{0\})$, where $T^* := T\backslash\{0\} = \mathbb{C}^*$.*
3. *Under some conjecture on the duality theory of G_0, there exists a "limit" Frobenius structure associated to $f_{|D}$, which is constant, i.e., its potential is a polynomial of degree three.*

In order to understand the properties of the duality theory of G_0, it is desirable to have some more concrete informations on the possible values that occur in the matrix A'_∞. It turns out that they are related to the roots of the Bernstein polynomial $b_h(s)$ of h. It has been shown in [4] that for a reductive linear free divisor, these roots (normalized so that they are in the intervall $(-\infty, 1)$) actually lie in $(-1, 1)$, are symmetric around 0 and that $\deg(b_h) = n$. The comparison result with the diagonal entries of A'_∞ can be stated as follows.

Theorem 4 ([7]). (1) *The roots of $b_h(s)$ are equal to the diagonal entries of A'_∞.*

(2) *Consider the restriction $G_0(h)/(h)$, which is a free $\mathbb{C}[\theta]$-module of rank n quipped with a connection operator $\theta^2\nabla_\theta$. This object does not depend on the choice of f, has a regular singularity at $\theta = 0$, and the eigenvalues of the residue of ∇_θ on its saturation are, up to multiplication by n, equal to the roots of $b_h(s)$.*

Notice that the second part of this result is an analogue of the classical theorem of Malgrange relating the roots of the Bernstein polynomial of an isolated hypersurface singularity to the residues eigenvalues of the saturation of the Brieskorn lattice.

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Universal Poisson deformations of affine symplectic varieties

Yoshinori Namikawa

A symplectic variety (X, ω) is a pair of a normal algebraic variety X and a holomorphic symplectic 2-form ω on the regular part X_{reg} of X such that ω extends to a (not necessarily non-degenerate) holomorphic 2-form on a resolution $\tilde{X}$ of X. Then X_{reg} admits a natural Poisson structure induced by ω. By the normality of X, this Poisson structure uniquely extends to a Poisson structure on X. In this lecture, I talked on the Poisson deformation of $(X, \{\ ,\ \})$ obtained from a symplectic variety (X, ω). One can define the Poisson deformation functor PD_X from the category of local Artin $\mathbf{C}$-algebras with residue field $\mathbf{C}$ to the category of sets. The first main theorem is:

Theorem 1. *Let (X, ω) be an affine symplectic variety. Then PD_X is unobstructed.*

Let (X, ω) be the same as in Theorem 1. By Birkar, Cascini, Hacon and McKernan, one can take a $\mathbf{Q}$-factorial terminalization $\pi : Y \longrightarrow X$. By definition, Y has only $\mathbf{Q}$-factorial terminal singularities and π is a birational, crepant, projective morphism. The symplectic 2-form ω is pulled-back to a symplectic 2-form on $\pi^{-1}(X_{reg})$. Note that $\pi^{-1}(X_{reg})$ is contained in the rular locus Y_{reg} of Y. Since π is semi-small, $\pi^*(\omega)$ further extends to a holomorphic symplectic 2-form ω' on Y_{reg} and (Y, ω') becomes a symplectic variety. Therefore, Y has a Poisson structure, and we get the Poisson deformation functor PD_Y. It is relatively easy to prove that PD_Y is unobstructed. Since X has rational singularities, there is a natural blowing-down map of functors $\pi_*; PD_Y \longrightarrow PD_X$. The map π_* is a finite Galois covering. Let R and S be the pro-representable hulls of PD_X and PD_Y respectively. Then there are formal universal Poisson deformations $\mathcal{X}_{formal}$ and $\mathcal{Y}_{formal}$ over the base spaces $Spec(R)$ and $Spec(S)$ respectively. The birational map π induces a birational map $\mathcal{Y}_{formal} \longrightarrow \mathcal{X}_{formal}$. It is not clear at all that these are algebraizable. So, we assume the following condition

(*): X has a $\mathbf{C}^*$-action with positive weights and ω is also positively weighted with respect to the action.

Then everything can be algebraized. As a corollary of this construction, we have the following remarkable result:

Theorem 2. *Under the assumption (*), the following are equivalent:*
(a): X has a crepant resolution.
(b): X has a smoothing by a Poisson deformation.

Wild geometry

Claude Sabbah

Linear differential equations of one variable in the complex domain lead to the Stokes phenomenon and generalized monodromy data. Recent results of T. Mochizuki [5, 6] and K. Kedlaya [3] on vector bundles with meromorphic connection having irregular singularities make it possible to develop the Stokes phenomenon in higher dimensions, using previous results of H. Majima [4] and the author [8]. After a short survey of these results we propose tentative results for the underlying geometry, called "wild geometry" in analogy with the wild ramification in arithmetic. Some examples of Stokes-perverse sheaves are given, which mix usual perverse sheaves in complex analytic geometry together with real constructible sheaves on the boundary of real blow-up spaces of a manifold along a divisor.We also give an example of computation of the Stokes filtration of a direct image $\mathcal{D}$-module.

In the usual complex algebraic geometry,

- the underlying spaces are complex algebraic varieties (or complex analytic spaces),
- The monodromy phenomenon is treated sheaf-theoretically with local systems,
- introducing singularities in these local systems leads to $\mathbb{C}$-constructible sheaves, and then to perverse sheaves,
- one can realize each perverse sheaf as the sheaf of solutions of a system of holonomic differential equations with regular singularities (the connection matrix can be reduced to a normal form with logarithmic poles along a normal crossing divisor),
- Hodge theory extends in this setting (pure or mixed Hodge $\mathcal{D}$-modules of M. Saito).
- Moreover (Griffiths-Schmid), Hodge theory implies tameness (the natural extension of a variation of Hodge structures defines a meromorphic connection with regular singularities).
- Usual systems of differential equations in algebraic geometry (Gauss-Manin systems) have regular singularities (i.e., are tame).

Wild geometry addresses the question of extending these properties to differential equations having possibly irregular singularities (the matrix of the connection cannot be reduced to a matrix having logarithmic poles). The word "wild" is given with analogy to "wild ramification" in arithmetic. What is the usefulness for algebraic geometry?

- The classical theory of oscillating integrals produces such wild objects. If $F : X \longrightarrow \mathbb{A}^1$ is a morphism from a smooth quasi-projective variety to the affine line, the function $I(\tau) = \int_X e^{-\tau F}\omega$, for some algebraic differential form of maximal degree on X, satisfies a differential equation which has an irregular singularity at infinity.
- Such irregular connections occur in the notion of non-commutative Hodge structure (and variations of such) introduced by Katzarkov, Kontsevich and Pantev [2] as a model for the quantum cohomology of the projective space. This is strongly related to the notion of TERP structure of Hertling [1].

1. Stokes filtration on local systems and the R-H correspondence

The main part of the talk introduces the tools to define the Stokes filtration in the following setting.

- X smooth complex projective variety.
- D is a divisor with simple normal crossings.
- $\widetilde{X}$ is the real blow-up of the components of D (local polar coordinates).

We define the Stokes filtration for a meromorphic bundle on X with flat connection, with poles on D at most. The associated local system $\mathcal{L}$ on $X \smallsetminus D$ is naturally extended on the boundary of $\widetilde{X}$. The Stokes filtration is defined by subsheaves of the pull-back of $\mathcal{L}$ on the étale space of the sheaf $\mathcal{O}_X(*D)/\mathcal{O}_X$ (or ramified variants).

2. Example of computation of a Stokes filtration by direct image

We use the following setting: (E, ∇) is free $\mathcal{O}_\Delta[x]$-module, equipped with a meromorphic connection on $\Delta \times \mathbb{A}^1$ (coordinates (z, x)) with logarithmic poles along the divisor $S \cup (\Delta \times \{\infty\})$ and $p : \Delta \times \mathbb{A}^1 \longrightarrow \Delta$ denotes the projection. The exponentially twisted Gauss-Manin connection has underlying bundle $N = \operatorname{coker}\big(E(*S) \xrightarrow{\nabla_{\partial_x}+\mathrm{id}} E(*S)\big)$, and has connection induced by ∇_{∂_z}. This corresponds to considering equations satisfied by integrals $\int_p f(z,x)e^x dx$ where f is a multivalued horizontal section of ∇ out of S.

Question. *To compute the formal normal form of N at $z = 0$ and the Stokes filtration in terms of the local system $\ker \nabla$ on $(\Delta \times \mathbb{A}^1) \smallsetminus S$.*

After ramification w.r.t. z, we can assume that the components S_i of S going through $(0, \infty)$ have equation $(1/x) = z^{q_i}u_i(z)$, $u_i(0) \neq 0$. Set $\varphi_i(z) = 1/[z^{q_i}u_i(z)]$.

Theorem (C. Roucairol [7])**.** *The formal normal form of N is (up to ramification) a direct sum of terms $\nabla_i + d\varphi_i$, where ∇_i has logarithmic poles. Each ∇_i acts on a vector bundle of rank equal to $\operatorname{rk} E$ and has monodromy whose characteristic polynomial equals that of the monodromy of the nearby cycles of (E, ∇) along S_i.*

Theorem (C.S.)**.** *The Stokes-filtered local system attached to N is obtained by direct image from the (2-dimensional) Stokes-filtered local system attached to $(E(*S), \nabla + dx)$.*

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Thom polynomials and non-associative Hilbert schemes

Maxim Kazarian

Isomorphism classes of local singularities determine loci in the source manifolds of holomorphic mappings. These loci represent cohomology classes that can be expressed as universal polynomials (known as *Thom polynomials*) in the relative Chern classes c_i of the mapping (see e.g. the review paper [1] and references therein). For a given singularity type α its Thom polynomial Tp_α depends neither on the particular (generic) mapping nor on the dimensions of the manifolds involved in the mapping provided that the relative dimension ℓ is fixed. It was shown previously by Rimányi and Fehér [3] that for any singularity type the dependence on the relative dimension can be expressed in an infinite series of the form

$$\mathrm{Tp}_\alpha = \sum_{(i_1,\dots,i_\mu)\in\mathbf{Z}^\mu} w_{i_1,\dots,i_\mu} c_{\ell+1+i_1}\dots c_{\ell+1+i_\mu}$$

with independent of ℓ integer coefficients $w_{i_1,\dots,i_\mu}$, where μ is the dimension of the local algebra of the singularity diminished by 1. It is convenient to encode such an infinite sum by a formal Laurent-type generating series

$$S_\alpha(t_1,\dots,t_\mu) = \sum_{(i_1,\dots,i_\mu)\in\mathbf{Z}^\mu} w_{i_1,\dots,i_\mu} t_1^{i_1}\dots t_\mu^{i_\mu}.$$

Theorem ([2])**.** *The generating series for coefficients of the Thom polynomial of any (contact) singularity type is rational. Moreover, it can be written explicitly in the following form*

$$S_\alpha = \frac{\prod_{i=1}^{\mu} t_i^{e_i} \prod_{1\le i<j\le\mu}(t_j-t_i)}{\prod_{\substack{1\le i\le j<k\le\mu\\ w_i+w_j\le w_k}}(t_k-t_i-t_j)} \quad P_\alpha(t_1,\dots,t_\mu)$$

where $e_1,\dots,e_\mu$ and $w_1,\dots,w_\mu$ are certain numerical invariants of the singularity and P_α is a polynomial.

The definition of P_α is not very explicit; this polynomial is known in some cases but in general its computation is a problem for future investigations.

The theorem implies that the whole infinite Thom series can be uniquely recovered from a finite number of its initial coefficients.

The proof of the theorem is based on a new realization of the local Hilbert scheme of finite-codimensional ideals in the ring of function germs. The smooth ambient space of this construction is called the *non-associative Hilbert scheme.* Its points parameterize certain finite-dimensional commutative nilpotent algebras which are not necessary associative. The (usual) Hilbert scheme can be identified with the sublocus determined by the associativity condition.

A partial case of the theorem for the case when α is a Morin singularity A_μ (i.e. when the local algebra is isomorphic to $\mathbb{C}[x]/(x^{\mu+1})$, $\mu \geq 1$) is a reformulation of a result by Bérczi and Szenes [5].

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Equivariant classes of matrix matroid varieties

Richárd Rimányi

Thom polynomials of (local) singularities express the cohomology class represented by singularity submanifolds of (global) maps between manifolds. A recently found property of Thom polynomials of contact singularities sheds light on the behavior of equivariant classes of matroid realization varieties.

In Section 1 we discuss the so-called "d-stability property" of Thom polynomials of contact singularities. In Section 2 we study classical geometry problems, under the name of "linear Gromov-Witten invariants for matroids", using Thom polynomial techniques.

1. Thom series of contact singularities of maps

The results of this section are joint with L. Fehér [FR]. They are also inspired by a recent work of G. Bérczi and A. Szenes [BS]; and strongly influenced by communication with M. Kazarian.

Let Q be a complex, commutative, finite dimensional, local algebra. For given positive integers $n < p$ let $\xi_Q^{n,p}$ be the contact singularity of germs $(\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^p, 0)$ with local algebra Q.

1.1. **Thom polynomials.** For a map $f : N^n \longrightarrow P^p$ between compact complex manifolds we define the singularity submanifold

$$\xi_Q^{n,p}(f) = \{x \in N : f \text{ has singularity } \xi_Q^{n,p} \text{ at } x\}.$$

Under certain transversality assumptions the cohomology class represented by the closure of $\xi_Q^{n,p}$ in N can be computed by substituting the characteristic classes $c_i(TN)$, $f^*c_i(TP)$ into a polynomial depending only on $\xi_Q^{n,p}$. This polynomial is called the Thom polynomial of the contact singularity $\xi_Q^{n,p}$.

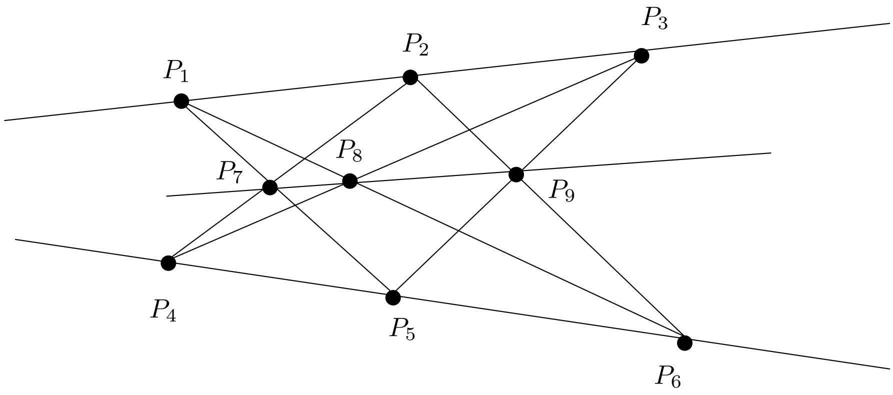

FIGURE 1. The Pappus configuration

1.2. **Structure of Thom polynomials.** It is known (Thom-Damon [D]) that the Thom polynomial of contact singularities is a polynomial of the quotient variables c, defined

$$1 + c_1t + c_2t^2 + \ldots = \frac{1 + b_1t + b_2t^2 + \ldots + b_pt^p}{1 + a_1t + a_2t^2 + \ldots + a_nt^n},$$

where a_i (resp. b_i) are the variables where $c_i(TN)$ (resp. $f^*(TP)$) are to be substituted.

Recently we showed [FR] how the Thom polynomial of $\xi_Q^{n,p+1}$ determines $\xi_Q^{n,p}$. As a consequence the Thom polynomials of $\xi_Q^{n,p}$ for different n and p can be organized into one formal power series, that we named Thom series, in infinitely many variables. Eg. the Thom series of $Q = \mathbb{C}[x]/(x^3)$ is

$$d_0^2 + d_{-1}d_1 + 2d_{-2}d_2 + 4d_{-3}d_3 + 8d_{-4}d_4 + \ldots,$$

where $c_i = d_{i+p-n+1}$. It also follows that the Thom polynomial, expressed in the quotient variables c_i (or d_i) has width (the number of factors in each term) at most the dimension of Q minus 1. We named this property "d-stability".

It is showed by Weber and Pragacz [PW] that in the so-called Schur basis, Thom polynomials have non-negative coefficients.

2. Linear Gromov-Witten invariants for matroids

The results of this section are joint with L. Fehér and A. Némethi [FNR]. We start with an example. Consider Figure 1, the Pappus configuration of nine points on the complex projective plane. Suppose $l_1, \ldots, l_8$ are straight lines, and Q is a point on the complex projective plane, in general position. One can ask how many Pappus configurations exist in the plane with $P_i \in l_i$ for $i = 1, \ldots, 8$, and $P_9 = Q$. (Answer: 5.)

The difficulty in this and other similar questions is that the variety of points representing given configurations (embedded in appropriate moduli spaces) is hopeless to describe with equations. These equations are not only the equations describing collinearities of the configuration, but also the geometric theorems valid for that configurations (in the case of Pappus: various Menelaus, Ceva, Pappus, and maybe other theorems). In fact, these varieties (named "matroid representation spaces") are universal objects in algebraic geometry, in the sense that any complication of varieties can be modeled on them (Mnëv's theorem). [Refreshing exceptions are the so-called Schubert varieties—corresponding to very special configurations—, whose singularities, hierarchy, and behavior in general, are well understood. The enumerative questions corresponding to Schubert varieties (Schubert calculus) are mostly solved at least as outcomes of algorithms. The study of enumerative questions of matroid representation spaces in general is outside the scope of Schubert calculus.]

Techniques inspired by Thom polynomial calculations in singularity theory provide the answer. In a joint work with L. Fehér and A. Némethi we study the following objects. To each subset I of $\{1,\dots,k\}$ associate an integer $r(I)$. Denote by X the collection of those $n \times k$ matrices for which the rank of a union of columns corresponding to a subset I is $r(I)$, for all I. The group $GL(n)$ times the group of diagonal matrices of size k, acting on $\mathbb{C}^{n\times k}$, leaves X invariant. The equivariant class ("Thom polynomial") represented by the closure of X encodes the answers of the enumerative questions discussed above. For example, one of the more than 10,000 terms of the Thom polynomial corresponding to the Pappus configuration is $5d_1d_2d_3d_4d_5d_6d_7d_8$. The coefficient 5 is the answer to the enumerative problem of the introduction.

The challenge is the calculation of the equivariant classes of matroid realization varieties. In general, one knows numerous conditions such a class must satisfy. One such condition is the so-called interpolation condition. According to this, the equivariant class must vanish under certain substitutions. These substitutions come from geometry, they express the restriction of the equivariant class to an orbit outside the closure of X. In other situations, where the underlying representation is "equivariantly perfect", these substitutions determine the equivariant class. However, the representation in question is not perfect.

An extra strong condition on these classes which makes them calculable in practice, is the analogue of the d-stability condition from Section 1, and its corollary, as follows. After substituting 0 into the Chern classes of the group of diagonal $k \times k$ matrices, the remaining polynomial (in the Chern classes of $GL(n)$) has width as most $k - r(\{1,\dots,k\})$.

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Jumping coefficients, spectra and *b*-functions of hyperplane arrangements

Morihiko Saito

Recent developments in the theories of jumping coefficients, (singularity) spectra, and b-functions of hyperplane arrangements are explained. Some part is done by joint work with A. Dimca and N. Budur.

Those invariants associated with hypersurface singularities are known by the difficulty of their explicit calculations. In the case of hyperplane arrangements, the first problem is whether they combinatorial invariants. We show that the jumping coefficients and the spectrum are determined by certain combinatorial data. In case the rank (i.e. the dimension of the ambient space) is small, we give explicit formulas in terms of relatively simple combinatorial invariants (like the numbers of edges with given multiplicities) [BS]. There are too ways for the proofs. One is by induction on the rank. The other uses the Hirzebruch-Riemann-Roch theorem together with the combinatorial description of the cohomology ring of a good compactification of the ambient space [DP]. Note that a formula for multiplier ideals was obtained by M. Mustaţă [Mu] (and has been improved in [Te]). However, his formula implies that we have to calculate the dimension of the parameter space of hypersurfaces of a given degree passing through certain singular points of the arrangement. So the problem is rather nontrivial. Note also that the above result does not imply an answer to the conjecture that the Milnor cohomology groups of hyperplane arrangements are combinatorially determined. This conjecture is recently studied with A. Dimca and N. Budur [BDS]. However, no known method seems to work, and the conjecture is very difficult.

As for the b-functions, the situation is not so good as in the case of the above invariants since the calculation is much more difficult. Recently Malgrange's formula for the b-function of an isolated hypersurface singularity is generalized to the non-isolated singularity case assuming that the support of the vanishing cycle sheaf with a given eigenvalue of the Milnor monodromy is isolated [Sa1]. Combining this with the solution of Aomoto's conjecture on the combinatorial description of the cohomology of certain local systems which is due to Esnault, Schechtman, Terao, Varchenko, and Viehweg ([ESV], [STV]), we can calculate the b-function of hyperplane arrangements in case the rank is 3 and the degree is small [Sa2]. Note that the b-function of a generic central hyperplane arrangement was determined by U. Walther [Wa] except for the multiplicity of the root -1 which is solved in [Sa1]. (It does not seem easy to determine the multiplicity of -1 without using the theory of weights on mixed perverse sheaves.) Using these theories, some attempts are now being made with N. Budur and S. Yuzvinsky [BSY] to solve the topological monodromy conjecture of J. Denef and F. Loeser concerning the poles of the local topological zeta function and the roots of the b-function. Note that the conjecture concerning the Milnor monodromy was settled by [BMT] in the hyperplane arrangement case. The conjecture for the b-function is now solved in the reduced case with rank 3. However, the higher rank case and the non-reduced case seem to be very difficult.

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General KAM theorems

Mauricio Garay

The aim of this work is to obtain a general analytic KAM theory, by including it into the general framework of group actions. Already in the simplest cases, the general KAM theorems show that the non-degeneracy conditions on the frequency are not necessary to ensure the existence of invariant tori, that families of isochronous tori may be preserved under perturbation, that other invariant lagrangian varieties than tori can be preserved under perturbations, that degeneration of lagrangian invariant manifolds may occur etc. But, let us first recall the basic setting of hamiltonian mechanics.

Let U be an open subset of $T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ with coordinates $q_1, \dots, q_n$, $p_1, \dots, p_n$. The symplectic form $\omega = \sum_{i=1}^n dq_i \wedge dp_i$ induces a Poisson bracket in $T^*\mathbb{R}^n$ defined by

$$\{f, g\}\omega^n = df \wedge dg \wedge \omega^{n-1},$$

An analytic function $H : U \longrightarrow \mathbb{R}$ defines an *hamiltonian dynamical system* given by *Hamilton's equations* :

$$\begin{cases} \dot{q}_i &= -\partial_{p_i} H &= \{H, q_i\}, \\ \dot{p}_i &= \partial_{q_i} H &= \{H, p_i\} \end{cases}$$

The function H is called the *hamiltonian function* of the dynamical system.

This dynamical system is called *Liouville-integrable* or simply *integrable* if there exists analytic functions $f_1, \dots, f_n$ with commuting hamiltonian vector fields which generate at each point the tangent spaces to the fibres of the *moment map* :

$$f = (f_1, \dots, f_n) : U \longrightarrow \mathbb{R}^n.$$

In such a case, the flows of the hamiltonian vector fields of the f_i''s define an affine structure on the fibres of f.

In general, the solution of Hamilton's equation might exhibit a complicated behaviour. The case of integrable systems is, as a general rule, simple. For instance, if the moment map is proper then Arnold-Liouville's theorem (also called Arnold-Liouville-Mineur's theorem) states that these fibres are tori and that the dynamic is linear in the local coordinates induced by the hamiltonian flows of the f_i's [1, 3, 4]. Such a motion is called *quasi-periodic*, it is fully determined by the vector giving the direction of the trajectories in these affine coordinates. This vector is called the *frequency vector*. If we multiply the frequency vector by a constant then the trajectory do not change, therefore as long as we are not interested in the parametrisation of the trajectories, the motion is determined by the point in projective space $\mathbb{R}P^{n-1}$ corresponding to the frequency vector. This point is called the *frequency* on the given torus.

Consider the following example, consisting of two harmonic oscillators with no interaction :

$$H = \frac{1}{2}(p_1^2 + q_1^2) + \frac{\sqrt{2}}{2}(p_2^2 + q_2^2), \;\; f_i = p_i^2 + q_i^2.$$

On every smooth fibre of $f = (f_1, f_2)$, the motion is quasi-periodic with frequency $(1 : \sqrt{2}) \in \mathbb{R}P^1$. All trajectories are dense in each torus. If we consider now the motion corresponding to the hamiltonian

$$H' = \frac{1}{2}(p_1^2 + q_1^2) + \frac{\sqrt{2}}{2}(p_2^2 + q_2^2) + \frac{1}{2}(p_1^2 + q_1^2)^2 + \frac{1}{2}(p_2^2 + q_2^2)^2$$

then the frequency changes accordingly with the torus. If the frequency defines a rational point in $\mathbb{R}P^1$ then each trajectories is periodic on the corresponding torus, otherwise each one is dense.

The dynamics of an integrable system defined by a proper moment mapping being governed by quasi-periodic motions, is something elementary. Poincaré observed that this is not a generic motion : if we perturb it slightly, then the dynamics usually becomes chaotic...

In 1954, Kolmogorov observed that although some quasi-periodic motions may disappear under perturbation, there are some tori on which the motion might be preserved. The proof of Kolmogorov's theorem was completed by Moser in

1962 and Arnold in 1963. The theorem was called by the acronym *KAM theorem* although Arnold always referred to it as Kolmogorov's theorem.

The KAM theorem applies only for motions on a torus satisfying two conditions. First, the frequency should be *diophantine*, which roughly means (for $n = 2$) that it is badly approximated by rational points. The second condition is that the map which assigns to each torus its frequency should be smooth. For instance, KAM theorems says nothing concerning our first example. Nevertheless using general KAM theorem one can easily prove that ... *any deformation H_t of H admits a family of invariant tori $L_t = \{f_t = 0\}$ provided that t is small enough.* The values of t for which the theorem holds *do depend* on the perturbation. A deeper study shows that the phase space is separated between zones of conflict, on one part the quiet KAM zone with his quasi-periodic motions, on the other side a turbulence zone, but let us return to invariant tori...

The KAM theorem is a theorem on group actions. In Kolmogorov's situation, one searches a symplectomorphism φ_t which sends the perturbed hamiltonian H_t to a hamiltonian H' whose restriction to the diophantine torus defines the same dynamics as the original one. This can be stated in algebraic terms. If I is the ideal of the lagrangian torus, we search a symplectomorphism φ_t such that $\varphi_t(H_t)$ is equal to H modulo I^2. In the space of hamiltonians, we are trying to prove that I^2 is a transversal to the action of the group of symplectomorphisms on functions.

Therefore, one has to work out a theory for group actions in infinite dimensional spaces, the spaces involved in this study being neither Banach nor Fréchet spaces.

This theory is build up from the classical notion of *Banach scale*. A Banach scale consists of an increasing family of Banach spaces $(E_s), s \in]0,1[$ such that the inclusions have norm at most one. The space of holomorphic function germs along a compact admits various Banach scales. A vector space on which we have fix such a scale is called a *scaled vector space*.

In [2], J. Féjoz and myself proved a theorem for group actions on scaled vector space. This result gives sufficient conditions to ensure that there is no difference between locally homogeneous spaces and infinitesimally locally homogeneous spaces. Although this theorem has a wide range of applications, it does not apply to the KAM situation. Therefore, one needs to develop a thorough, but elementary, deformation theory for scaled vector spaces.

In this way, one obtains a theorem for group actions in deformed scaled vector spaces which implies all sort of theorems in dynamical systems. The general KAM theorems give sufficient condition under which a deformation of a hamiltonian function belonging to a lagrangian ideal induces a lagrangian deformation of the ideal which contains the deformation, and eventually describe the dynamics of the deformed hamiltonian system.

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On deformations of plane curve singularities

Maciej Borodzik

Let us consider a germ (C, z_0) of a plane curve singularity. Let $(U, E) \longrightarrow (C, z_0)$ be its minimal embedded resolution, where E is the exceptional divisor. Let K be the canonical divisor on U and C' the strict transform. Let $D = C' + E_{red}$, where "red" means that we take reduced scheme structure, i.e. all coefficients are 1. Finally, let

$$K + D = H + N$$

be the Zariski–Fujita decomposition of $K + D$.

Definition 1. (see [BZ]) The $\overline{M}$ *number* of the singular point z_0 is $K(K + D)$. A *fine* M number is $K(K + D) + N^2$.

The $\overline{M}$ number is strongly related to the parametric form of the singularity. Namely, it is the codimension of the equisingularity stratum in a suitably defined parameter space. Therefore it is natural to ask the following question.

Question 2. Is the $\overline{M}$ number upper-semicontinuous in deformations of plane curve singularities?

The answer is obviously no, if we do not assume that the (local) geometric genus of members of deformation's family is fixed, or, equivalently, the deformation is not δ-constant.

Therefore we reformulate the above question adding an assumption

Question 3. Is the $\overline{M}$ number upper-semicontinuous in $\delta-$constant deformations of plane curve singularities?

We can approach to this problem via knot-theoretic invariants of links of singular points. We recall one definition.

Definition 4. Let L be a link in S^3 and S its Seifert matrix. The *Tristram–Levine signature* of L is a function $\sigma_L(\cdot)$ that associates with a complex number z of modulus 1 a signature of the Hermitian form

$$(1 - z)S + (1 - \bar{z})S^T.$$

The Tristram–Levine signature is computable for iterated torus knots (see [Bo2]). In particular we have

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Proposition 5 (see [Bo2]). *Let C be a germ of a plane curve singularity with one branch. Let L be the corresponding link of singularity, and μ, M respectively the Milnor number and the M number of the singular point. Then*

$$0 < -3\int_0^1 \sigma_L(e^{2\pi i x}) - \mu - M < \frac{2}{9}. \tag{1}$$

We want to apply the above result in studying deformations. So let us assume that we are given a family $\{C_s\}_{s\in D}$ of singular plane curves (D is a unit disk in the complex plane) such that there exists a small ball $B \in \mathbb{C}^2$ that $C_s \subset B$, $\partial C_s \subset \partial B$ and C_s is transverse to ∂B for all s.

Assume moreover that C_0 has one singular point z_0 and $\partial B \cap C_0$ is isotopic to the link L_0 of C_0 at z_0. Let us pick some $s \neq 0$ and look at singularities of C_s. Let $z_1, \dots, z_N$ be the singular points of C_s and $L_1, \dots, L_N$ corresponding links of singularities. The Tristram–Levine signatures of links L_0 and $L_1, \dots, L_N$ are related by the following Murasugi-like inequality.

Proposition 6. *(see* [Bo, Bo3]*) For almost all $\zeta \in S^1$ we have*

$$\left|\sigma_{L_0}(\zeta) - \sum_{k=1}^N \sigma_{L_k}(\zeta)\right| \le b_1(C_s). \tag{2}$$

We want to integrate the above inequality and make use of the estimate (1). However, this inequality holds only for cuspidal singularities. Therefore we add an assumption

Assumption 7. *There exists an $n \le N$ such that the singular points $z_1, \dots, z_n$ are cuspidal and the remaining $R = N - n$ singular points are ordinary double points.*

An ordinary double point has $\sigma \equiv -1$, so in this case we can rewrite (2) as

$$-\sum_{k=1}^n \sigma_{L_k} + \sigma_{L_0} \le 2g,$$

where g is the geometric genus of C_s (because of our assumptions $b_1(C_s) = 2g+R$). Then, applying (1) and manipulating with formulae we arrive finally at

Theorem 8. *If in the deformation as above, C_s has only double points and cuspidal singularities, then*

$$\sum_{k=1}^n M_k - M_0 < 8g + 2R + \frac{2}{9}. \tag{3}$$

Here M_k and M_0 are corresponding M numbers of singularities.

We would expect that the right hand side of (3) is like $2g$, but the method applied here seems to be insufficient to prove that strong result.

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The δ-constant stratum in the discriminant

Paul Cadman

(joint work with David Mond)

Let $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be a holomorphic germ with an isolated singularity at 0 of Milnor number μ. Let $F : B \times \Lambda \longrightarrow U$ be a representative of a versal deformation of f where $B \subset \mathbb{C}^2, \Lambda \subset \mathbb{C}^\mu, U \subset \mathbb{C}$ are neighbourhoods of the origin in their respective spaces chosen so that F has the required transversality properties and let $(\lambda_1, \ldots, \lambda_\mu)$ be coordinates for Λ.

We study the geometry of the discriminant:

$$D := \{\lambda \in \Lambda \,:\, X_\lambda := F_\lambda^{-1}(0) \text{ is singular}\}$$

Using the δ-invariant for plane curves we define the following strata in the discriminant:

$$D_k := \{\lambda \in \Lambda \,:\, \delta(X_\lambda) \leq k\}$$

Let $\delta := \delta(X_0)$ then we call D_δ the δ-constant stratum of the discriminant.

There is a intersection pairing in $H^1(X_\lambda; \mathbb{C})$ coming from the topological intersection of cycles on the curve X_λ. In [1] Givental and Varchenko show that when f is irreducible (so the intersection pairing is nondegenerate and μ is even) the intersection pairing can be used to define a symplectic form Ω on Λ. Moreover, they show that Ω identifies the stratum D_δ as a Lagrangian subvariety of Λ. In [2] Givental also shows that D_δ is Cohen-Macaulay for A_{2k} singularities.

Assume $\Omega = \sum_{i,j=1}^{\mu} g_{ij} \mathrm{d}\lambda_i \wedge \mathrm{d}\lambda_j$ where $g_{ij} \in \mathcal{O}_{\mathbb{C}^\mu}$ We use Ω to construct a rank 2 maximal Cohen-Macaulay module on the discriminant. To do this we define a skew-symmetric matrix $G = (g_{ij})$ from the coefficients of Ω. Then we define the matrix χ by setting its columns to be the coefficients of the vector fields $\chi_i \ldots \chi_\mu$ that make up a basis of $\mathrm{Der}(-\log(D))$, the logarithmic vector fields tangent to the discriminant.

We view the matrix $S := \chi^t G \chi$ as a presentation matrix for the module M_Ω:

$$\mathcal{O}^\mu_{\mathbb{C}^\mu} \xrightarrow{S} \mathcal{O}^\mu_{\mathbb{C}^\mu} \longrightarrow M_\Omega \longrightarrow 0$$

We remark that the ijth entry of S is equal to $\Omega(\chi_i, \chi_j)$. The vector fields χ_i are tangent to D_δ which is Lagrangian so $\Omega(\chi_i, \chi_j)$ is a function vanishing on D_δ. In fact, this collection of functions define D_δ.

This remark motivates the definition of the following varieties obtained from the presentation matrix of M_Ω:

$$R_{2m} := V(\mathcal{F}_{\mu-2(m+1)}(M_\Omega)) = V(\mathrm{Pf}_{2(m+1)}(S)) \quad m = 0, \ldots, \delta - 1$$

where $\mathcal{F}_{\mu-2(m+1)}(M_\Omega)$ is the $(\mu-2(m+1))$th Fitting ideal of M_Ω and $\mathrm{Pf}_{2(m+1)}(S)$ is the ideal of Pfaffians of $2(m+1) \times 2(m+1)$ symmetrically placed submatrices of S.

The varieties R_{2m} are strata in the discriminant where the intersection pairing in $H^1(X_\lambda; \mathbb{C})$ has rank less than or equal to $2m$. We show that the rank of the intersection pairing determines the genus of the normalisation of the curve and hence its δ-invariant.

Let $n : \overline{X}_\lambda \longrightarrow X_\lambda$ be the normalisation of the curve X_λ. Since X_λ has isolated singularities we can recover it as a quotient of its normalisation $X_\lambda \simeq \overline{X}_\lambda / S$ which glues together on $\overline{X}_\lambda$ the preimage under n of singular points of X_λ. We can show that the intersection pairing I_{X_λ} on $H^1(X_\lambda; \mathbb{C})$ passes to the quotient, i.e.:

$$I_{X_\lambda}(a, b) = I_{\overline{X}_\lambda}(n^*a, n^*b)$$

for $a, b \in H^1(X_\lambda; \mathbb{C})$. When $\lambda \in D_\delta$ the normalisation of X_λ is contractible so I_{X_λ} vanishes on $H^1(X_\lambda; \mathbb{C})$ by the equation above. Since Ω is induced from I_{X_λ} this demonstrates that D_δ is Lagrangian with respect to Ω. Also, since $\overline{X}_\lambda$ is smooth, it shows that the rank of the intersection pairing on $H^1(X_\lambda; \mathbb{C})$ determines the genus of $\overline{X}_\lambda$.

By considering the change in first Betti-number between X_λ and $\overline{X}_\lambda$ we obtain the following relation between the δ-invariants of X_0 and X_λ:

$$\delta(X_0) - \delta(X_\lambda) = g(\overline{X}_\lambda)$$

where $g(\overline{X}_\lambda)$ is the genus of $\overline{X}_\lambda$. We deduce that the genus of $\overline{X}_\lambda$ determines the δ-invariant of X_λ.

From these observations we conclude that the stratifications of the discriminant defined by the rank of Ω and the δ-invariant are the same. The relationship between the two sets of varieties in each stratification is:

$$R_{2m} = D_{\delta-m} \quad m = 0, \ldots, \delta - 1$$

We can now answer questions about D_k via the more computable R_k. For instance in the case of E_6 and E_8 we can show D_δ is Cohen-Macaulay by computing its depth. This is sufficient to prove that D_δ is a rigid Lagrangian singularity for E_6 and E_8 by a result of Sevenheck and van Straten in [3].

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Polar Representations and Symplectic Reduction

Christian Lehn

1. Introduction and Statement of the results

A symplectic singularity in the sense of Beauville [1] is a normal variety together with a symplectic form on its regular part which extends to a regular 2-form on any resolution. The number of known techniques to produce symplectic singularities is rather limited. Here we look at polar representations in the sense of Dadok and Kac [4] and their symplectic reductions. For simple groups the irreducible polar representation are classified in [4].

Let V be a vector space and $G \subseteq \mathrm{GL}(V)$ be reductive. We consider the symplectic double $V \oplus V^*$ together with the induced symplectic G-action and the associated moment map $\mu : V \oplus V^* \longrightarrow \mathfrak{g}^*$. The symplectic reduction is defined by $V \oplus V^* /\!/\!/ G := \mu^{-1}(0) /\!/ G$. In [3] we adress the following

Conjecture 1. (M. Lehn) Let (G, V) be an irreducible, stable, polar representation of a simple algebraic group and $c \subseteq V$ a Cartan subspace. Then there is a subspace $c^\vee \subseteq V^*$ and an isomorphism of schemes

$$c \oplus c^\vee / W \longrightarrow V \oplus V^* /\!/\!/ G,$$

where W is the Weyl group of (G, V).

A consequence would be that $V \oplus V^* /\!/\!/ G$ is an irreducible, normal variety and moreover symplectic, as $c \oplus c^\vee / W$ is. For example the adjoint representation of a semisimple Lie algebra is polar and in that case $\mu^{-1}(0) = \{(x, y) \in \mathfrak{g} \oplus \mathfrak{g} : [x, y] = 0\}$, the *commuting scheme*. By the work of Joseph ([6], Theorem 0.1) we know that the conjecture is true in that case. In [3] we give a proof of Conjecture 1 under the additional assumption, that the nullfibre is a normal variety.

Theorem 2. *Let (G, V) be an irreducible, stable, polar representation and $c \subseteq V$ a Cartan subspace such that $\mu^{-1}(0)$ is a normal variety, the inclusion $c \oplus c^\vee \subseteq V \oplus V^*$ induces an isomorphism*

$$c \oplus c^\vee / W \longrightarrow V \oplus V^* /\!/\!/ G$$

where $c^\vee$ is the Cartan space dual to c (see section 3).

We cannot show the normality of $\mu^{-1}(0)$ in general. In the case of the *commuting scheme* this is a long standing conjecture, but at least we know, that it is irreducible by the work of Richardson [9]. We give an irreducibility criterion in the general case.

Theorem 3. *The nullfibre $\mu^{-1}(0)$ is irreducible if and only if for every $(v, \varphi) \in \mu^{-1}(0)$ with closed orbit both $G.v$ and $G.\varphi$ are closed in V and V^*.*

By a casewise analysis we obtain

Proposition 4. Conjecture 1 is true if (G, V) is one of the following: $(\mathrm{SL}_2, S^3\mathbb{C}^2)$, $(\mathrm{SL}_3, S^3\mathbb{C}^3)$, $(\mathrm{SO}_n, \mathbb{C}^n)$, $(G, \mathfrak{g})$.

The nullfibre is irreducible, if (G, V) is $(\mathrm{SL}_n, \Lambda^2\mathbb{C}^n)$, $n = 2k$ or $(\mathrm{SL}_n, S^2\mathbb{C}^n)$.

2. Framework

As we rely on [4], we give a brief overview of its concepts and results. If not stated otherwise, all results given here are due to Dadok and Kac.

A vector $v \in V$ is called *semisimple*, if its orbit is closed. The representation is called *stable*, if there is a closed orbit among the orbits of maximal dimension. A subspace of the form $c := c_v := \{x \in V : \mathfrak{g}.x \subset \mathfrak{g}.v\}$ for semisimple v is called a *Cartan subspace*, if $\dim c = \dim V/\!\!/G$. The representation (G, V) is called *polar*, if it admits a Cartan subspace.

If we put $N := \{g \in G : g.c \subseteq c\}$ and $H := \{g \in G : g|_c = \mathrm{id}_c\}$ then $W = N/H$ is finite and acts faithfully on c. It is called the *Weyl group* of c. A main property of polar representations is that the inclusion $c \subseteq V$ of a Cartan subspace induces an isomorphism $\mathbb{C}[V]^G \longrightarrow \mathbb{C}[c]^W$. The proof uses the following Theorem of [5]: Let G be reductive, X an affine, normal G-variety and Y a closed subvariety of X such that the following conditions are fulfilled:

(1) Any two G-equivalent points of Y are W-equivalent.
(2) Every closed orbit in X intersects Y non-trivially.
(3) For $y \in Y$ the orbit $G.y$ is closed.

Then the restriction $\mathbb{C}[X] \longrightarrow \mathbb{C}[Y]$ induces an isomorphism $\mathbb{C}[X]^G \overset{\cong}{\to} \mathbb{C}[Y]^W$.

Basically the three conditions say, that the morphism is bijective and then normality takes care of the scheme structure. An easy example, where our Theorem applies is given by the representation $G = \mathrm{SO}_n \circlearrowright \mathbb{C}^n =: V$, which is stable and polar with Cartan space $c = \mathbb{C}.e_1$. The Weyl group is $W = \{\pm 1\} \cong \mathbf{Z}/2$ and hence $\mathbb{C}[c \oplus c]^W \cong \mathbb{C}[a, b, c]/(ab - c^2)$ is an A_1-singularity. The nullfibre is $\mu^{-1}(0) = \{(x, y) \in \mathbb{C}^n \oplus \mathbb{C}^n : \mathrm{rk}(x|y) \leq 1\}$, which is known to be normal, so Theorem 2 can be applied.

3. Methods

As for a polar representation stability is equivalent to the statement $V = c \oplus \mathfrak{g}.c$ (see [4]), it is reasonable to look at

$$c^\vee := \{\varphi \in V^* : \varphi(\mathfrak{g}.c) = 0\} \subseteq V^*,$$

the *Cartan-subspace dual to c*. This satisfies $c \oplus c^\vee \subseteq \mu^{-1}(0)$ so we may hope to make use of the Theorem of [5] referred to above. We are able to verify conditions (1) and (2) and if $\mu^{-1}(0)$ is irreducible, then a dimension argument also shows (3). Assuming normality gives Theorem 2. Let us finally illustrate how Theorem 3 may proof irreducibility of $\mu^{-1}(0)$ by linear algebra methods:

Example 5. $\mathrm{SL}_n \circlearrowright S^2\mathbb{C}^n$

We identify $S^2\mathbb{C}^n$ with the symmetric $n \times n$-matrices on which SL_n acts on $S^2\mathbb{C}^n \oplus (S^2\mathbb{C}^n)^*$ by $g.(A, B) := (gAg^t, (g^{-1})^t Bg^{-1})$. The representation is polar and stable and the moment map is

$$\mu : S^2\mathbb{C}^n \oplus \left(S^2\mathbb{C}^n\right)^* \longrightarrow \mathfrak{sl}_n, \quad (A, B) \mapsto AB - \frac{\mathrm{tr}AB}{n}\mathbf{1}_n,$$

where we identified $\mathfrak{sl}_n$ with $\mathfrak{sl}_n^*$.

In the decisive case $\lambda := \frac{\mathrm{tr}AB}{n} = 0$ we have $AB = BA = 0$ for $(A, B) \in \mu^{-1}(0)$, so by linear algebra we may transform (A, B) into $A' = \mathrm{diag}(\mathbf{1}_r, 0)$, $B' = \mathrm{diag}(0, \star)$. Therefore $0 \in \overline{G.(A, B)}$, since $T(t).(A', B') \stackrel{t \longrightarrow 0}{\longrightarrow} (0, 0)$ for $T(t) = \mathrm{diag}(t^{n-r}\mathbf{1}_r, t^{-r}\mathbf{1}_{n-r})$. Hence the only closed orbit for $\lambda = 0$ is the origin. In particular $\mu^{-1}(0)$ is irreducible due to Theorem 3.

The general proof of normality seems difficult, but in the complete intersection case a strategy similar to [7], section 3 might work. It is also worthwile to note, that in the case $SL_3 \circlearrowleft S^3\mathbb{C}^3$ we proved Conjecture 1 thereby providing an equivalent description of this interesting singularity, cf. [2], [8]. Another related open problem is, that symplectic reduction does not always lead to a symplectic singularity. Can we make precise when it does?

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Poincaré series and Coxeter functors for Fuchsian singularities

Wolfgang Ebeling
(joint work with David Ploog)

Let (X, x) be a normal surface singularity with a good $\mathbb{C}^*$-action. Then the coordinate algebra A is a graded $\mathbb{C}$-algebra $A = \oplus_{k=0}^{\infty} A_k$. We consider the Poincaré series of this algebra

$$p_A(t) = \sum_{k=0}^{\infty} \dim(A_k)\ t^k.$$

Let $(g; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r))$ be the orbit invariants of (X, x). It is known that

$$\phi_A(t) := p_A(t)\psi_A(t), \text{ where } \psi_A(t) := (1-t)^{2-r}(1-t^{\alpha_1}) \cdots (1-t^{\alpha_r}),$$

is a polynomial. It turns out that this algebraic invariant is related to topological invariants of the singularity in a rather mysterious way.

If X is a hypersurface and $g = 0$ (for simplicity) then it was shown in [2] that the Saito dual of $\phi_A(t)$ is the characteristic polynomial of the monodromy of the singularity (X,x). In particular, if $\phi_A(t)$ is self-dual (as for example in the case of the Kleinian singularities), the polynomial $\phi_A(t)$ is the characteristic polynomial of the monodromy. If (X,x) is one of the 14 exceptional unimodal singularities then $\phi_A(t)$ is the characteristic polynomial of the monodromy of the dual singularity corresponding to Arnold's strange duality. More generally, we consider a class of singularities where the polynomial $\phi_A(t)$ is the characteristic polynomial of an abstract Coxeter element. This is the class of Fuchsian singularities.

A Fuchsian singularity is the affine surface singularity obtained from the tangent bundle of the upper half plane by taking the quotient by a Fuchsian group of the first kind and collapsing the zero section. In particular, it has a good $\mathbb{C}^*$-action. The surface can be compactified in a natural manner, leading to additional cyclic quotient singularities of type A_μ on the boundary. After resolving the singularities on the boundary, one gets a star-shaped configuration $\mathcal{E}$ of rational (-2)-curves with a central curve of genus g and self-intersection number $2g-2$.

We denote the abstract lattice corresponding to the dual graph of this configuration by V_-. It is the free $\mathbb{Z}$-module generated by the rational (-2)-curves $E_1,\ldots,E_{n-1}$ and the central curve E, endowed with the symmetric intersection form. Let $U=\mathbb{Z}u+\mathbb{Z}w$ be a unimodular hyperbolic plane and define $V_0=V_-\oplus\mathbb{Z}u$ and $V_+=V_-\oplus U$, where $\oplus$ denotes the orthogonal direct sum. Let τ_0 be the product of the reflections corresponding to $E_1,\ldots,E_{n-1}$ and the Eichler-Siegel transformation $\psi_{u,E}$. This transformation is defined by the formula

$$\psi_{u,E}(x)=x+\langle x,u\rangle E-\langle x,E\rangle u-\frac{1}{2}\langle E,E\rangle\langle x,u\rangle u \text{ for } x\in V_+.$$

Let τ_+ be the product of τ_0 and the reflection corresponding to the vector $u-w$. Denote by $\Delta_0(t)=\det(1-\tau_0^{-1}t)$ and $\Delta_+(t)=\det(1-\tau_+^{-1}t)$ the corresponding characteristic polynomials. Then we have:

Theorem 1. For a Fuchsian singularity one has

$$\psi_A(t)=\Delta_0(t),\qquad \phi_A(t)=\Delta_+(t).$$

This result was already proved in [3]. In the case $g=0$, this theorem also follows from results of H. Lenzing and J. A. de la Peña (see [6]). We give a geometric proof of this result for the case $g=0$ in [4] and for the general case in [5].

We give a geometric interpretation of the lattices V_0 and V_+ and the isometries τ_0 and τ_+ in the case when the singularity (X,x) is negatively smoothable. Then the compactification of the generic fibre of such a smoothing is (after resolving the singularities at the boundary) a smooth K3 surface Y containing the configuration $\mathcal{E}$.

Let $\mathrm{Coh}(Y)$ be the abelian category of coherent sheaves on Y and $K(Y)$ its Grothendieck K-group. Let $N(Y)$ be the numerical K-group which is obtained from $K(Y)$ by dividing out the radical of the Euler form. Denote by $\mathrm{Coh}_{\mathcal{E}}(Y)$ the abelian subcategory of $\mathrm{Coh}(Y)$ consisting of sheaves whose support is contained in $\mathcal{E}$ and let $K_{\mathcal{E}}(Y)$ be its K-group. Then the lattice V_0 can be identified with the image $N_{\mathcal{E}}(Y)$ of $K_{\mathcal{E}}(Y)$ under the map $K(Y)\longrightarrow N(Y)$ and the lattice V_+ with the orthogonal direct sum $N_{\mathcal{E}}(Y)\oplus\mathbb{Z}[\mathcal{O}_Y]$.

We consider the bounded derived category of coherent sheaves on Y, $\mathcal{D}^b(Y)$. Moreover, we consider the full triangulated category $\mathcal{D}_0 := \mathcal{D}^b_{\mathcal{E}}(Y)$ consisting of complexes whose support is contained in $\mathcal{E}$. This is a 2-Calabi-Yau triangulated category. We also consider the smallest full triangulated subcategory $\mathcal{D}_+$ of $\mathcal{D}^b(Y)$ containing $\mathcal{D}_0$ and the structure sheaf $\mathcal{O}_Y$ of Y. Now reflections of the (numerical) K-group of $\mathcal{D}^b(Y)$ lift to spherical twist functors (autoequivalences) of this category. The Eichler-Siegel transformations lift to line bundle twists. In this way one can lift the Coxeter elements τ_0 and τ_+ to autoequivalences of $\mathcal{D}_0$ and $\mathcal{D}_+$ respectively. For more details see [5].

The Fuchsian singularities are the Gorenstein normal surface singularities with a good $\mathbb{C}^*$-action with Gorenstein parameter $R = 1$. We also consider some singularities with Gorenstein parameter $R \neq \pm 1$, namely the 14 exceptional bimodal hypersurface singularities. There is a mirror symmetry between these singularities and some other singularities given by the construction of Berglund and Hübsch [1], i.e. by "transposing" the equation (see Table 1).

Name	μ	R	f	f^t	R^t	μ^t	Dual
E_{18}	18	2	$x^5z + y^3 + z^2$	$x^5 + y^3 + xz^2$	1	12	Q_{12}
E_{19}	19	3	$x^7y + y^3 + z^2$	$x^7 + xy^3 + z^2$	1	15	$Z_{1,0}$
E_{20}	20	5	$x^{11} + y^3 + z^2$	$x^{11} + y^3 + z^2$	5	20	E_{20}
Z_{17}	17	2	$x^4z + xy^3 + z^2$	$x^4y + y^3 + xz^2$	1	14	$Q_{2,0}$
Z_{18}	18	3	$x^6y + xy^3 + z^2$	$x^6y + xy^3 + z^2$	3	18	Z_{18}
Z_{19}	19	5	$x^9 + xy^3 + z^2$	$x^9y + y^3 + z^2$	5	25	E_{25}
Q_{16}	16	2	$x^4z + y^3 + xz^2$	$x^4z + y^3 + xz^2$	2	16	Q_{16}
Q_{17}	17	3	$x^5y + y^3 + xz^2$	$x^5z + xy^3 + z^2$	1	21	$Z_{2,0}$
Q_{18}	18	5	$x^8 + y^3 + xz^2$	$x^8z + y^3 + z^2$	5	30	E_{30}
W_{17}	17	2	$x^5y + z^2 + y^2z$	$x^5 + xz^2 + y^2z$	1	14	$S_{1,0}$
W_{18}	18	3	$x^7 + y^2z + z^2$	$x^7 + y^2 + yz^2$	3	18	W_{18}
S_{16}	16	2	$x^4y + xz^2 + y^2z$	$x^4y + xz^2 + y^2z$	2	16	S_{16}
S_{17}	17	3	$x^6 + xz^2 + y^2z$	$x^6y + z^2 + y^2z$	1	21	$X_{2,0}$
U_{16}	16	2	$x^5 + y^2z + yz^2$	$x^5 + y^2z + yz^2$	2	16	U_{16}

Table 1. Mirror symmetry of the exceptional bimodal singularities

In this case the polynomial $\phi_A(t)$ of an exceptional bimodal singularity is the characteristic polynomial of an operator τ such that τ^{R/R^t} is the monodromy of the dual singularity. This is work in progress.

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From jet schemes to the base scheme – Isomorphism problems (global and local)

Shihoko Ishii

Let k be an algebraically closed field of arbitrary characteristic. To a k-scheme X we associate the m-jet scheme X_m for every $m \in \mathbb{N}$. These schemes are somethings to represent the nature of the geometric properties of the base scheme X. It is well known that if a morphism $f : X \longrightarrow Y$ of the base schemes is isomorphic, then the induced morphism $f_m : X_m \longrightarrow Y_m$ is also isomorphic for every $m \in \mathbb{N}$. Similarly, if a morphism $f : (X, x) \longrightarrow (Y, y)$ of the germs of the base schemes is isomorphic, then the induced morphism $f_m : X_m(x) \longrightarrow Y_m(y)$ of local m-jet schemes is also isomorphic for every $m \in \mathbb{N}$. Here, we think of the opposite implications, i.e.:

Does an isomorphism of the jet schemes induce an isomorphism of the base schemes?

1. Global isomorphism problem

This part is a joint work with Jörg Winkelmann. There are two possible formulation for this problem. The first one is a weaker version.

Question 1. If a morphism $f : X \longrightarrow Y$ is given and the induced morphism $f_m : X_m \longrightarrow Y_m$ is isomorphic for every $m \in \mathbb{N}$, then is f isomorphic?

The answer to this question is “YES” and the statement is rather stronger, i.e.:

Proposition 1. If a morphism $f : X \longrightarrow Y$ is given and the induced morphism $f_m : X_m \longrightarrow Y_m$ is isomorphic for some $m \in \mathbb{N}$, then f is isomorphic.

Next question is a stronger version.

Question 2. If there is an isomorphism $\varphi^{(m)} : X_m \xrightarrow{\sim} Y_m$ for every $m \in \mathbb{N}$ compatible with the truncation morphisms, then is there an isomorphism $f : X \xrightarrow{\sim} Y$, preferably with the property that $\varphi^{(m)} = f_m$?

The answer to this question is “NO”, even in case we do not require the property $\varphi^{(m)} = f_m$.

Example 1. There are non-singular surfaces X, Y with $X \not\simeq Y$ but $X_m \simeq Y_m$ for every $m \in \mathbb{N}$ with the compatibility with the truncation morphism. Concretely, X and Y are hypersurfaces in $\mathbb{C}^3$ defined by $xz - y^2 + 1 = 0$ and $x^2z - y^2 + 1 = 0$, respectively. This is Danielewski’s counterexample of cancellation.

2. Local isomorphism problem

This part is a joint work with Tommaso De Fernex and Lawrence Ein. Let $x \in X$ be a closed point of a k-scheme X. Let $X_m(x)$ be the scheme theoretic fiber of x by the canonical projection $\pi_m : X_m \longrightarrow X$.

Question 3. If a morphism $f : (X, x) \longrightarrow (Y, y)$ of the germ of the base schemes is given and the induced morphism $f_m : X_m(x) \longrightarrow Y_m(y)$ is isomorphic for every $m \in \mathbb{N}$, then is f isomorphic?

In general, the answer is "NO". There is a counter example for non-Noetherian Y. But at this moment the answer is not known for Noetherian case.

In order to study the Noetherian case, we introduce a new notion of "closure" of ideals. Let A be a Noetherian regular local k-algebra and $\mathfrak{a} \subset A$ an ideal. Let $X = \operatorname{Spec} A/\mathfrak{a}$ and $0 \in X$ be the closed point. For an element $f \in A$, let H be the hypersurface in $\operatorname{Spec} A$ defined by $f = 0$. Let

$$\overline{\mathfrak{a}}^m = \{f \in A \mid X_m(0) \subset H_m(0)\}, \quad \text{and} \quad \overline{\overline{\mathfrak{a}}} := \bigcap_{m \in \mathbb{N}} \overline{\mathfrak{a}}^m.$$

Then, we have the following:

Proposition 2. $\overline{\overline{\mathfrak{a}}}$ is an ideal of A and $\mathfrak{a} \subset \overline{\overline{\mathfrak{a}}} \subset \overline{\mathfrak{a}}$, where $\overline{\mathfrak{a}}$ is the integral closure of $\mathfrak{a}$.

As $(\overline{\overline{\overline{\overline{\mathfrak{a}}}}}) = \overline{\overline{\mathfrak{a}}}$, the ideal $\overline{\overline{\mathfrak{a}}}$ is another kind of "closure" of the ideal $\mathfrak{a}$. Question 3 is translated into the following question:

Question 4. For every ideal $\mathfrak{a}$ of A, does the equality $\mathfrak{a} = \overline{\overline{\mathfrak{a}}}$ hold?

At this moment we have $\mathfrak{a} = \overline{\overline{\mathfrak{a}}}$ in case that $\mathfrak{a}$ is reduced, or principal or generated by homogeneous elements. Unfortunately, the final answer to this question for Noetherian case is not yet obtained, but both answers, affirmative or negative, will provide us with a good news. If the answer is yes, then the local isomorphism problem is affirmatively solved. If the answer is no, then we have another non-trivial closure of ideals and a new theory may be developed.

Homological Mirror Symmetry for Cusp Singularities

Atsushi Takahashi

1. Statement and the result

We associate two triangulated categories to a triple $A := (\alpha_1, \alpha_2, \alpha_3)$ of positive integers called a *signature*: the bounded derived category $D^b\mathrm{coh}(X_A)$ of coherent sheaves on a weighted projective line $X_A := \mathbb{P}^1_{\alpha_1,\alpha_2,\alpha_3}$ and the bounded derived category $D^b\mathrm{Fuk}^{\to}(f_A)$ of the directed Fukaya category for a "cusp singularity" $f_A := x^{\alpha_1} + y^{\alpha_2} + z^{\alpha_3} - q^{-1}xyz$, $(q \in \mathbb{C}^*)$. Here, we consider f_A as a *tame* *polynomial* if $\chi_A := 1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 - 1 > 0$ and as a *germ* of a holomorphic function if $\chi_A \leq 0$.

Then, the *Homological Mirror Symmetry* (*HMS*) *conjecture* for cusp singularities can be formulated as follows:

Conjecture 1 ([T1]). *There should exist an equivalence of triangulated categories*

$$D^b\mathrm{coh}(X_A) \simeq D^b\mathrm{Fuk}^{\to}(f_A).$$

□

Combining results in [GL] with known results in singularity theory, one can easily see that the HMS conjecture holds at the Grothendieck group level, i.e., there is an isomorphism

$$(K_0(D^b\mathrm{coh}(X_A)), \chi + {}^t\chi) \simeq (H_2(Y_A, \mathbf{Z}), -I),$$

where Y_A denotes the Milnor fiber of f_A.

The HMS conjecture is shown if $\alpha_3 = 1$ (Auroux-Katzarkov-Orlov [AKO], Seidel [Se1], van Straten, Ueda, ...). Also the cases $A = (3,3,3), (4,4,2), (6,3,2)$, which correspond to two of three simple elliptic hypersurface singularities, are known ([AKO], [U], [T2], ...).

The following is our main theorem:

Theorem 3. *Assume that $\alpha_3 = 2$. Then the HMS conjecture holds.* □

The keys in our proof are; the reduction of surface singularities to curve singularities (the stable equivalence of Fukaya categories given in [Se2] section 17), the use of A'Campo's divide [A1][A2] in order to describe the Fukaya category, and mutations of exceptional collections (distinguished basis of vanishing Lagrangian cycles). We shall give quivers with relations associated to cusp singularities with $\alpha_3 = 2$ obtained from devides attached to them.

2. Devides and quivers with relations

2.1. **A recipe.** First, we consider a curve singularity $\tilde{f}$ which is stable equivalent to the surface singularity f. Then, the following statement holds:

Proposition 2. *There exists a distinguished basis of vanishing cycles $\mathcal{L}_1, \ldots, \mathcal{L}_\mu$ in the Milnor fiber of $\tilde{f}$ and a choice of gradings on $\mathcal{L}_i$ such that $\mathrm{Fuk}^{\to}(\mathcal{L}_i, \mathcal{L}_j)$ is at most one dimensional complex concentrated on degree 0. Hence, there exists a quiver Δ and relations I by Gabriel's theorem such that $D^b\mathrm{Fuk}^{\to}(f) \simeq D^b\mathrm{Fuk}^{\to}(\tilde{f}) \simeq D^b(\text{mod-}\mathbb{C}\vec{\Delta}/I)$.*

The quiver Δ and relations I in the above proposition can be described as follows:

(1) Choose a real Morsification g of $\tilde{f}$.
(2) Draw a picture of $g^{-1}(0)$ in $\mathbb{R}^2$.
(3) Put a vertex • to ODP.

(4) Put a vertex with a sign $\oplus$ ($\ominus$) into each compact connected component of $\mathbb{R}^2\backslash g^{-1}(0)$ if g is positive (resp. negative) on the component.
(5) Draw 1 arrow $\longrightarrow$ from $\oplus$ to $\bullet$ (from $\bullet$ to $\ominus$) if $\bullet$ is on the boundary of the component for $\oplus$ (resp. $\ominus$).
(6) Draw 1 dotted line from $\oplus$ to $\ominus$ if there are 2 paths from $\oplus$ to $\ominus$, which means a commutative relation between them.

Note that the pair (Δ, I) depends on the choice of a real Morsification of g. However, it is known that the derived category $D^b(\text{mod-}\mathbb{C}\vec{\Delta}/I)$, as a triangulated category, is an invariant of the singularity $\tilde{f}$ (and hence f). Indeed, two diffenrent choices of pairs (Δ, I) and (Δ', I') are connected by a sequence of mutations, the braid group action on the set of distinguished basis of vanishing cycles.

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Cohen-Macaulay modules over non-isolated singularities

Igor Burban
(joint work with Yuriy Drozd)

The theory of Cohen-Macaulay modules over the quotient surface singularities was intensively studied in 80-th. As an application, it provides a conceptual explanation of the McKay correspondence in two-dimensional case, see [10, 1, 3, 8, 9] and [13, 5].

In the PhD thesis of Kahn [11], the geometric McKay correspondence was extended to the case of the minimally elliptic singularities. Using Atiyah's classification of vector bundles on elliptic curves [2], he described all Cohen-Macaulay modules over the simply elliptic singularities. Later, Drozd and Greuel generalized his approach on the case of the cusp singularities, see [6] and [7].

For a long time it was believed that the log-canonical surface singularities exhaust all the cases, where the problem of classifying of all Cohen-Macaulay modules is representation-tame. However, in my talk I am going to show that in the case of non-isolated surface singularities called *degenerate cusps*, all Cohen-Macaulay modules can be classified in a very explicit way. The rings $k[\![x, y, z]\!]/xyz$ and $k[\![x, y, u, v]\!]/(xy, uv)$ are examples of degenerate cusps.

We have also discovered a wide class of non-isolated surface singularities, whose category of Cohen-Macaulay modules is representation-discrete (one can view them as some limiting cases of quotient surface singularities). We show that these singularities can have arbitrarily many irreducible components, providing a negative answer to a question posed by F.-O. Schreyer in 1987, see [12]. Our method allows to get a classification of Cohen-Macaulay modules over the singularities $k[\![x, y, z]\!]/xy$ and $k[\![x, y, z]\!]/(x^2y - z^2)$ obtained for the first time by Buchweitz, Greuel and Schreyer in [4], see [5] for more details.

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Monodromy eigenvalues are induced by poles of zeta functions

Wim Veys

1. Let $f : X \to \mathbb{C}$ be a non-constant analytic function on an open part X of $\mathbb{C}^n$. We consider C^∞ functions φ with compact support on X and the corresponding differential forms $\omega = \varphi dx \wedge d\bar{x}$. Here $x = (x_1, \cdots, x_n)$ and $dx = dx_1 \wedge \cdots \wedge dx_n$. For such ω the integral

$$Z(f, \omega; s) := \int_X |f(x)|^{2s} \omega,$$

where $s \in \mathbb{C}$ with $\Re(s) > 0$, has been the object of intensive study. One verifies that $Z(f, \omega; s)$ is holomorphic in s. Either by resolution of singularities or by the theory of Bernstein polynomials, one can show that it admits a meromorphic continuation to $\mathbb{C}$, and that all its poles are among the translates by $\mathbb{Z}_{<0}$ of a finite number of rational numbers. Combining results of Barlet [4] , Kashiwara [7] and Malgrange [11], the poles of (the extended) $Z(f, \omega; s)$ are strongly linked to the eigenvalues of (local) monodromy at points of $\{f = 0\}$.

Theorem*(1) If s_0 is a pole of $Z(f, \omega; s)$ for some diffential form ω, then $\exp(2\pi\sqrt{-1}s_0)$ is a monodromy eigenvalue of f at some point of $\{f = 0\}$.*

(2) All monodromy eigenvalues of f are obtained this way, that is, if λ is a monodromy eigenvalue of f at a point of $\{f = 0\}$, then there exists a differential form ω and a pole s_0 of $Z(f, \omega; s)$ such that $\lambda = \exp(2\pi\sqrt{-1}s_0)$.

2. Let now $f : X \to \mathbb{Q}_p$ be a non-constant ($\mathbb{Q}_p$-)analytic function on a compact open $X \subset \mathbb{Q}_p^n$, where $\mathbb{Q}_p$ denotes the field of p-adic numbers. Let $|\cdot|_p$ and $|dx|$ denote the p-adic norm and the Haar measure on $\mathbb{Q}_p^n$, normalized in the standard way. The p-adic integral

$$Z_p(f; s) := \int_X |f(x)|_p^s |dx|,$$

again defined for $s \in \mathbb{C}$ with $\Re(s) > 0$, is called the (p-adic) Igusa zeta function of f. Using resolution of singularities Igusa showed that it is a rational function of p^{-s}; hence it also admits a meromorphic continuation to $\mathbb{C}$. In this context the analogue of (1) is an intriguing conjecture of Igusa. More precisely, let f be a polynomial in n variables over $\mathbb{Q}$. Then we can consider $Z_p(f; s)$ for all prime numbers p (taking $X = \mathbb{Z}_p^n$).

Monodromy conjecture.*For all except a finite number of p, we have that, if s_0 is a pole of $Z_p(f; s)$, then $\exp(2\pi\sqrt{-1}s_0)$ is a monodromy eigenvalue of $f : \mathbb{C}^n \to \mathbb{C}$ at a point of $\{f = 0\}$.*

This conjecture was proved for $n = 2$ by Loeser [9]. There are by now various other partial results, e.g. [ACLM1], [2], [8], [10], [13], [16].

But, even assuming the conjecture, in general quite few eigenvalues of f are obtained this way. (And considering more general zeta functions involving the p-adic analogues of the C^∞ functions φ, being just locally constant functions with compact support, does not yield more possible poles.)

There are various 'algebro-geometric' zeta functions, related to the p-adic Igusa zeta functions: the motivic, Hodge and topological zeta functions. For those zeta functions a similar monodromy conjecture can be stated; and analogous partial results are valid.

Aiming at an analogue of (2) for these p-adic and related algebro-geometric zeta functions, we rather consider zeta functions associated to f and *algebraic* or *analytic* differential forms ω. (This is natural and useful also in other contexts, see for example [1], [2], [9], [15].)

3. We concentrate further on the topological zeta function, being the easiest one to describe. We first state a formula for it in terms of an embedded resolution π of $f^{-1}\{0\} \cup \operatorname{div} \omega$, which is in fact usually taken as the definition of this zeta function. From now on f and ω are $\mathbb{C}$-analytic in some neighbourhood of $0 \in \mathbb{C}^n$.

Denote by $E_i, i \in S$, the irreducible components of the inverse image $\pi^{-1}(f^{-1}\{0\} \cup \operatorname{div} \omega)$ and by N_i and $\nu_i - 1$ the multiplicities of E_i in the divisor of $\pi^* f$ and $\pi^* \omega$, respectively. We put $E_I^\circ := (\cap_{i \in I} E_i) \setminus (\cup_{j \notin I} E_j)$ for $I \subset S$. So the E_I° form a stratification of the resolution space in locally closed subsets.

Definition. The (local) *topological zeta function* of f and ω (at $0 \in \mathbb{C}^n$) is

$$Z_{top}(f, \omega; s) := \sum_{I \subset S} \chi(E_I^\circ \cap \pi^{-1}\{0\}) \prod_{i \in I} \frac{1}{\nu_i + sN_i},$$

where s is a variable.

In particular the $-\nu_i/N_i, i \in S$, form a complete list of candidate poles. Typically however many of them cancel.

This invariant was introduced by Denef and Loeser in [5] for 'trivial ω', i.e. for $\omega = dx_1 \wedge \cdots \wedge dx_n$. Their original proof that this expression does not depend on the chosen resolution is by describing it as a kind of limit of p-adic Igusa zeta functions. Later they obtained it as a specialization of the intrinsically defined motivic zeta functions [6]. Another technique is applying the Weak Factorization Theorem [3] to compare two different resolutions. For arbitrary ω one can proceed analogously.

Challenge. Find an 'intrinsic' definition of the topological zeta function.

4. We showed in [17] that each eigenvalue of f is induced (as in (2)) by a pole of the topological zeta function of f and some ω. But typically these zeta functions have *other* poles that don't induce monodromy eigenvalues of f. So for those zeta functions the analogue of (1) is (unfortunately) not true. It would be really interesting to have a complete analogue of (1) and (2), roughly saying that the monodromy eigenvalues of f correspond precisely to the poles of the zeta functions associated to f and some collection of *allowed* differential forms ω, including dx. Of course this would be a lot stronger than the (in arbitrary dimension) still wide open monodromy conjecture.

For instance when $f = y^q - x^p$ with $\gcd(p, q) = 1$, a possible collection of such forms is $\{x^{i-1}y^{j-1}dx \wedge dy$ such that $p \nmid i$ and $q \nmid j\}$. The simplicity of this case is however misleading.

5. In joint work with A. Némethi we identified such a collection of allowed forms for an arbitrary f in two variables. More precisely we define *allowed* differential forms ω (depending on f) and show that

(i) if s_0 is a pole of $Z_{top}(f,\omega;s)$, then $\exp(2\pi\sqrt{-1}s_0)$ is a monodromy eigenvalue of f;

(ii) $\omega = dx \wedge dy$ is allowed, and

(iii) all monodromy eigenvalues of f are obtained this way, that is, can be written as $\exp(2\pi\sqrt{-1}s_0)$ for some pole s_0 of a zeta function $Z(f,\omega;s)$ for some allowed ω.

Our definition of allowed forms uses the Eisenbud-Neumann diagram of (the minimal embedded resolution) of f, and the natural splicing of this diagram into star-shaped pieces, yielding a reasonably 'natural' proof of (i). Then the point is that we have enough forms to prove (ii) and (iii). Note that (i) and (ii) provide an alternative proof of the monodromy conjecture for curves.

Maybe surprisingly, but indicating that our concept of allowed form is 'good', our ideas extend to functions f on some normal surface germs. Note that zeta functions are also considered in this setting (see for example [14]). More precisely we can show generalizations of (i), (ii) and (iii) when the Eisenbud-Neumann diagram of such an f satisfies a semi-group condition like the one for the recently much studied splice type singularities of Neumann and Wahl (see e.g. [12]).

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Global Singularities and Betti bounds

Dirk Siersma
(joint work with Mihai Tibăr)

We consider polynomial functions $f : \mathbb{C}^n \longrightarrow \mathbb{C}$. One knows that a general fibre $X_t = f^{-1}(t)$ has the homotopy type of an $(n-1)$-dimensional CW-complex. Denote for t generic the top Betti number by $b_{n-1}(f) := b_{n-1}(X_t)$.

In this talk we consider the question: " What kind of restictions are there on f, depending on this top Betti number ?"

We mention first two important facts about $b_{n-1}(f)$:

1. $b_{n-1}(f)$ is semi-continuous in families [ST2],
2. $b_{n-1}(f)$ is bounded : $b_{n-1}(f) \leq b_{max} := (d-1)^n$. $\quad (d =$ degree of $f)$

The second fact follows from a deformation to polynomials, which are generic at infinity (see below).

We will use a description of the polynomials in terms of the boundary singularities in a compactification of the generic fibres. To be more precise, consider $\mathbb{P}^n$ as the (standard) compactification of $\mathbb{C}^n$. Let:

$\overline{X_t}$ the compactification of X_t in $\mathbb{P}^n$
$\overline{X_t} \cap H$ its intersection with the hyperplane $H = \mathbb{P}^{n-1}$ "at infinity".

At a point $P \in H$ we consider the germ $(\overline{X_t}, \overline{X_t} \cap H)_P$ as boundary pair. The (local) singularity theory of germs of boundary singularities with respect to a hyperplane has been studied by Arnol'd [Ar1]. He studied the concept of isolated boundary singularity, its properties, including a classification of simple singularities.

The boundary pair, mentioned above, has an isolated singularity if both $\overline{X_t}$ and $\overline{X_t} \cap H$ have isolated singularities (which includes the case that one of them is is smooth).

In the isolated case the generic fibre of the polynomial has the homotopy type of a bouquet of spheres of dimension $n-1$ (see e.g Broughton [Br]). Moreover we have the following formula, which expresses the top Betti number in the Betti numbers of the boundary singularities

$$b_{n-1}(f) = (d-1)^n - \sum_P [\mu(\overline{X_t}, P) + \mu(\overline{X_t} \cap H, P)],$$

where μ denotes the Milnor number of a singularity.
The expression $\mu(\overline{X_t}, P) + \mu(\overline{X_t} \cap H, P)$ is the so called boundary Milnor number

at P.
In case that $\overline{X_t}$ is transversal to the hyperplane H we say, that f is *generic at infinity*. In that case we have $b_{n-1}(f) = (d-1)^n$.

We make the following observations:

Proposition 2. In case $b_{max} - b_{n-1}(f) < d$ then for all P the boundary pairs have isolated singularities (or are smooth).

Note that this bound does not depend on n. Moreover it follows, that

$$\sum_P \mu(\overline{X_t}, P) + \mu(\overline{X_t} \cap H, P) < d,$$

so the sum of the boundary Betti numbers over all singular points P is less than d. This makes it possible to start a classification of these boundary singularities (see below). This is also related to Arnold's [Ar2] theory of singularities of fractions.

The bound in proposition 1 is due to non-isolated boundary singularities.
The first types of (local) singularities one meets after the study of isolated singulartities are those with a one dimensional singular set. They were intensively studied. For a survey we refer to [Si3]. For each branch of the singular set one considers a generic transversal slice in a generic point. The restriction of the function to the slice has an isolated singularity on an $(n-1)$-dimensional space, which gives a well defined transversal singularity type for this branch of the singular set.

Prototypes of non-isolated singularities are the so-called *isolated line singularities* [Si1]. Its singular set is a smooth 1-dimensional space and the transversal type is A_1. These singularities have nice topological properties, e.g. the homotopy type of the Milnor fibre is a bouquet of spheres.

The isolated line singularities (and no other non-isolated singularities !) appear in the next range of top Betti numbers:

Proposition 3. In case $b_{max} - b_{n-1}(f) < 2d - 1$ then for all P the boundary pairs are as follows:

a. have isolated singularities (or are smooth), or
b. f has isolated line singularities on a straight smooth affine line in $\mathbb{C}^n$, or
c. $\overline{X_t} \cap H$ has isolated line singularities along a $\mathbb{P}^1 \subset \overline{X_t} \cap H \subset H$

Consider the set of all singularities with a straight smooth affine line and transversal type A_1. The maximum of $b_{n-1}(f)$ in this class is exactly $b_{max} - d$. This shows that the bound in the proposition 1 is sharp.

The proof of the two propositions uses the semi-continuity of the top Betti number, during (repeated) deformations of Yomdin type $f + sw^d$, where w is a generic linear map. Each time the dimension of the singular set decreases with one. Continue until the affine singular set and the singular set at infinity have dimension less than 1 and at least one of them is exactly 1. Next deform by a Yomdin type deformation to isolated singularities. Use the formula for series of singularities [Si2] in order to compute Betti numbers. In that formula occurs the multiplicity of the singular set and the transversal Milnor numbers. To keep their contribution minimal one has to consider low multiplicity and low transversal Milnor number.

Taking the multiplicity 1 and transversal type A_1 will give proposition 2. It is well possible to continue in the same way.

At the end of this note we return to the proposition 1 and list the possible combinations of critical points at infinity, which give generic fibres with Betti numbers near to the maximum $b_{max} = (d-1)^n$.

Betti	boundary type	Arnol'd type
b_{max}	$< A_0 \vert A_0 >$	A_0
$b_{max} - 1$	$< A_0 \vert A_1 >$	A_1
$b_{max} - 2$	$< A_0 \vert A_2 >$	A_2
	$2 < A_0 \vert A_1 >$	$2A_1$
	$< A_1 \vert A_1 >$	B_2
$b_{max} - 3$	$< A_0 \vert A_3 >$	A_3
	$< A_0 \vert A_2 > + < A_0 \vert A_1 >$	$A_2 + A_1$
	$3 < A_0 \vert A_1 >$	$3A_1$
	$< A_1 \vert A_2 >$	C_3
	$< A_1 \vert A_1 > + < A_0 \vert A_1 >$	$B_2 + A_1$
	$< A_2 \vert A_1 >$	B_3

NB. The notation $< X|Y >$ describes the singularity types of $\overline{X_t}$, resp $\overline{X_t} \cap H$ at the given singular point at infinity. The additive notation is used if several special points at infinity play a role. For Arnol'd type fraction notation, cf [Ar2].

Let me mention that this project is a joint research (in progress) with Mihai Tibăr. Part of the work was done during 'Research in Pairs' at the Mathematisches Forschungsinstitut Oberwolfach.

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Reporter: Ágnes Szilárd

Participants

Prof. Dr. Norbert A'Campo
Mathematisches Institut
Universität Basel
Rheinsprung 21
CH-4051 Basel

Prof. Dr. Klaus Altmann
Institut für Mathematik
Freie Universität Berlin
Arnimallee 3
14195 Berlin

Dr. Javier F. de Bobadilla de Olazabal
Depto de Algebra
Facultad de Ciencias Matematicas
Universidad Complutense
Plaza de Ciencias 3
E-28040 Madrid

Michael Bogner
Fachbereich Mathematik
Johannes Gutenberg Universität
Staudingerweg 9
55128 Mainz

Prof. Dr. Maciej Borodzik
Instytut Matematyki
Uniwersytet Warszawski
ul. Banacha 2
02-097 Warszawa
POLAND

Prof. Dr. Ragnar-Olaf Buchweitz
Dept. of Computer & Mathematical Science
University of Toronto Scarborough
1265 Military Trail
Toronto Ont. M1C 1A4
CANADA

Igor Burban
Hausdorff Center for Mathematics
Universität Bonn
Endenicher Allee 62
53115 Bonn

Paul Cadman
Mathematics Institute
University of Warwick
Gibbet Hill Road
GB-Coventry CV4 7AL

Prof. Dr. Jan Arthur Christophersen
Matematisk Institutt
Universitetet i Oslo
P.B. 1053 - Blindern
N-0316 Oslo

Prof. Dr. Raf Cluckers
Algebraic Geometry and Number Theory
Katholieke Universiteit Leuven
Celstijnenlaan 200B
B-3001 Leuven

Prof. Dr. Wolfgang Ebeling
Institut für Algebraische Geometrie
Gottfried Wilh. Leibniz Universität
Welfengarten 1
30167 Hannover

Dr. Anne Frühbis-Krüger
Institut für Algebraische Geometrie
Gottfried Wilh. Leibniz Universität
Welfengarten 1
30167 Hannover

Dr. Mauricio D. Garay
24 avenue Herriot
F-91440 Bures-sur-Yvette

Prof. Dr. Victor Goryunov
Dept. of Mathematical Sciences
University of Liverpool
Peach Street
GB-Liverpool L69 7ZL

Prof. Dr. Claus Hertling
Lehrstuhl für Mathematik VI
Fak. für Mathematik und Informatik
Universität Mannheim
Seminargebäude A 5
68159 Mannheim

Prof. Dr. Shihoko Ishii
Graduate School of Science and Engineering
Tokyo Institute of Technology
2-12-1 Oh-okayama, Meguro-ku
Tokyo 152-8551
JAPAN

Prof. Dr. Maxim E. Kazarian
Department of Algebra
Steklov Mathematical Institute
Gubkina 8
119 991 Moscow
RUSSIA

Cong Trinh Le
Fachbereich Mathematik
T.U. Kaiserslautern
Erwin-Schrödinger-Straße
67653 Kaiserslautern

Eva Leenknegt
Department of Mathematics
KU Leuven
Celestijnenlaan 200 B
B-3001 Heverlee

Christian Lehn
Fachbereich Mathematik
Johannes Gutenberg Universität
Staudingerweg 9
55128 Mainz

Prof. Dr. Manfred Lehn
Institut für Mathematik
Johannes-Gutenberg-Universität Mainz
Staudingerweg 9
55128 Mainz

Dr. Ann Lemahieu
Departement Wiskunde
Faculteit der Wetenschappen
Katholieke Universiteit Leuven
Celestijnenlaan 200B
B-3001 Leuven

Prof. Dr. Ignacio Luengo
Facultad de Matematicas
Depto. de Algebra
Universidad Complutense de Madrid
E-28040 Madrid

Prof. Dr. Laurentiu-G. Maxim
Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Drive
Madison , WI 53706-1388
USA

Prof. Dr. Alejandro Melle Hernandez
Facultad de Matematicas
Depto. de Algebra
Universidad Complutense de Madrid
E-28040 Madrid

Prof. Dr. David Mond
Mathematics Institute
University of Warwick
Zeeman Building
GB-Coventry CV4 7AL

Prof. Dr. Yoshinori Namikawa
Department of Mathematics
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto 606-8502
JAPAN

Prof. Dr. Andras Nemethi
Alfred Renyi Institute of Mathematics
Hungarian Academy of Sciences
P.O.Box 127
H-1364 Budapest

Prof. Dr. Tomohiro Okuma
Department of Education
Yamagata University
1-4-12 Kojirakawa-machi
Yamagata 990-8560
JAPAN

Prof. Dr. Patrick Popescu-Pampu
U.F.R. de Mathematiques
Case 7012
Universite Paris 7
F-75205 Paris Cedex 13

Prof. Dr. Ana J. Reguera
Depto. de Algebra, Geometria y Topologia
Universidad de Valladolid
Facultad de Ciencias
E-47005 Valladolid

Prof. Dr. Richard Rimanyi
Dept. of Mathematics
University of North Carolina
Phillips Hall
Chapel Hill , NC 27599-3250
USA

Prof. Dr. Claude Sabbah
Centre de Mathematiques
Ecole Polytechnique
Plateau de Palaiseau
F-91128 Palaiseau Cedex

Prof. Dr. Morihiko Saito
Research Institute for
Mathematical Sciences
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto 606-8502
JAPAN

Dr. Jan Schepers
Department of Mathematics
KU Leuven
Celestijnenlaan 200 B
B-3001 Heverlee

Dr. Mathias Schulze
Department of Mathematics
Oklahoma State University
401 Math Science
Stillwater , OK 74078-1058
USA

Dr. Jörg Schürmann
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Dr. Christian Sevenheck
Lehrstuhl für Mathematik VI
Fak. für Mathematik und Informatik
Universität Mannheim
Seminargebäude A 5
68159 Mannheim

Prof. Dr. Dirk Siersma
Mathematisch Instituut
Universiteit Utrecht
Budapestlaan 6
P. O. Box 80.010
NL-3508 TA Utrecht

Prof. Dr. Jan Stevens
Department of Mathematics
Chalmers University of Technology
S-412 96 Göteborg

Prof. Dr. Duco van Straten
Fachbereich Mathematik
Universität Mainz
Saarstr. 21
55122 Mainz

Dr. Agnes Szilard
Alfred Renyi Institute of
Mathematics
Hungarian Academy of Sciences
P.O.Box 127
H-1364 Budapest

Dr. Atsushi Takahashi
Dept. of Mathematics
Graduate School of Science
Osaka University
Machikaneyama 1-1, Toyonaka
Osaka 560-0043
JAPAN

Dr. Bernard Teissier
Equipe "Geometrie et dynamique"
Institut Mathematique de Jussieu
175 rue du Chevaleret
F-75013 Paris

Prof. Dr. Victor A. Vassiliev
V.A. Steklov Institute of
Mathematics
Russian Academy of Sciences
8, Gubkina St.
119991 Moscow GSP-1
RUSSIA

Prof. Dr. Wim Veys
Departement Wiskunde
Faculteit der Wetenschappen
Katholieke Universiteit Leuven
Celestijnenlaan 200B
B-3001 Leuven