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Complex Algebraic Geometry

Organised by
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Abstract. The Conference focused on several classical and novel theories in the realm of complex algebraic geometry, such as Algebraic surfaces, Moduli theory, Minimal Model Program, Abelian Varieties, Holomorphic Symplectic Varieties, Homological algebra, Kähler manifolds theory, Holomorphic dynamics, Quantum cohomology.

Mathematics Subject Classification (2000): 14xx, 18xx, 32xx, 53xx.

Introduction by the Organisers

The Workshop Komplexe algebraische Geometrie, organized by Fabrizio Catanese (Bayreuth), Yujiro Kawamata (Tokyo), Gang Tian (Princeton), and Eckart Viehweg (Essen), drew together 49 participants in spite of the ghost of the swine flue.

There were several PhD students and other young PostDocs in their 20 ‘s and early 30 ‘s, together with established leaders of the fields related to the thematic title of the workshop. There were 21 talks which lasted 50 minutes, and other 4 talks by junior participants which lasted 30 minutes. All the talks were followed by a lively 10 minutes discussion. The schedule left sufficiently ample time for the exchange of mathematical ideas.

As usual at an Oberwolfach Meeting, the mathematical discussions continued outside the lecture room throughout the day and the night.

The Conference was fully successful in setting in contact younger researchers with elder ones. There were fruitful exchanges between mathematicians with different specializations and backgrounds.

New fashionable topics were presented, alongside with new insights on long standing classical open problems, and also cross-fertilizations with other research topics: as algebraic geometry in positive characteristic and $\mathscr{D}$- modules (talk by Esnault), quantum cohomology and enumerative geometry (talk by Jun Li), and application of the Ricci flow techniques to Minimal Model Program (talk by Zhou).

A central role was occupied by the recent developments around the Minimal Model Program: a simpler proof of the finite generation of canonical rings (Corti), progress towards Shokurov's ACC conjecture (Ein, Mustata), and the Zariski decomposition problem (Birkar).

Moduli spaces and their compactifications were a central theme too: compactified moduli spaces of stable varieties appeared from a theoretical viewpoint in the new results presented by Kovacs, and more concretely in the talks by Pardini and Rollenske.

Moduli spaces also played a dominant role: as in the talks by Farkas (moduli of curves) and Bauer (moduli spaces of 'special' surfaces of general type).

Algebraic surfaces, of special and of general type, appeared throughout in several talks, by Bauer, Li, Mukai, Pardini, Pignatelli, Rollenske.

Homological algebra and derived categories, with applications to classification theory, were covered by talks by Schreyer, Ishii, Nakaoka, Lazarsfeld.

The talk by Lazarsfeld built a new bridge between homological algebra and Kähler manifold theory, applying the Bernstein-Gelfand-Gelfand correspondence to obtain powerful new extensions of the classical inequalities by Castelnuovo for irregular varieties.

In the classical theory of Abelian and Modular varieties there were interesting expositions of new results by van der Geer and Hulek.

There were also other very interesting topics treated:

- Holomorphic dynamics: Siegel disks on rational and K3 surfaces (Oguiso);
- Hyperdiscriminants, Chow forms and Mabuchi energy of Kähler manifolds (Paul);
- Mori theory and Fano varieties (Totaro);
- holomorphic symplectic varieties (Debarre and Namikawa).

The variety of striking results and the very interesting and challenging proposals made the participation in the workshop very rewarding. We hope that these abstracts will convey our enthusiasm to the readers, and we are sure that they will be quite useful to the mathematical community.

Workshop: Complex Algebraic Geometry

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Abstracts

Surfaces with geometric genus zero: their fundamental groups and moduli spaces

Ingrid Bauer
(joint work with Fabrizio Catanese, Fritz Grunewald, Roberto Pignatelli)

Surfaces of general type with vanishing geometric genus (i.e., $p_g(S) = h^0(S, \Omega_S^2)$) had a remarkable revival in the past few years yielding many new constructions and examples of such surfaces. In our joint papers [4], [5] we contribute to the current knowledge giving many new examples of surfaces of general type with $p_g = 0$.
In loc. cit. we analyze the following situation:

- C_1, C_2 compact complex curves of respective genera g_1, $g_2 \geq 2$;
- G a finite group acting faithfully on each C_i;
- $X := (C_1 \times C_2)/G$ the quotient by the diagonal action;
- $S \to X$ the minimal resolution of the singularities of X.

We call surfaces as above *product-quotient surfaces*. For the details of the systematic computer aided search for these surfaces under the additional hypothesis that $p_g(S) = q(S) = 0$ and the obtained classification results we refer to [9].
The main tool for distinguishing the constructed surfaces (among themselves and from old examples) is the following structure theorem on the fundamental group of product-quotient surfaces.

Theorem 1. *([5]) Let S be a product-quotient surface. Then $\pi_1(S)$ contains a normal subgroup of finite index isomorphic to $\Pi_g \times \Pi_{g'}$, g, $g' \geq 0$, where Π_g, $\Pi_{g'}$ are the fundamental groups of a smooth curve of genus g resp. g'.*

The proof of the above theorem is algebraically and indirect. E.g., we cannot keep track of the index of the normal subgroup isomorphic to a product of surface groups.
We would like to explain in the sequel in a concrete case that a geometric understanding of the fundamental group can give rise to a complete understanding of the corresponding connected component in the moduli space of surfaces of general type.

We reproduce below an excerpt of the classification table (of quotients as above by a non free action of G, but with canonical singularities) in [5].

K^2	Sing X	T_1	T_2	G	N	$H_1(S,\mathbb{Z})$	$\pi_1(S)$
4	$1/2^4$	2^5	2^5	$\mathbb{Z}_2^3$	1	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	$1 \to \mathbb{Z}^4 \to \pi_1 \to \mathbb{Z}_2^2 \to 1$
4	$1/2^4$	$2^2, 4^2$	$2^2, 4^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	1	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	$1 \to \mathbb{Z}^4 \to \pi_1 \to \mathbb{Z}_2^2 \to 1$

Once we found out that the fundamental groups of the above two families were isomorphic to the fundamental groups of surfaces constructed by Keum and Naie

([6], [8]), the most natural question was whether all these surfaces would belong to a unique irreducible component of the moduli space.
This turns out to be correct, but a good understanding of the fundamental group allows to prove more. In fact, the main theorem of [1] is the following:

Theorem 2. *Let S be a smooth complex projective surface which is homotopically equivalent to a Keum - Naie surface. Then S is a Keum - Naie surface. The connected component of the Gieseker moduli space corresponding to Keum - Naie surfaces is irreducible, normal, unirational of dimension six.*

In order to prove this, we resort first of all to a slightly different construction of Keum - Naie surfaces.
We start with a $(\mathbb{Z}/2\mathbb{Z})^2$ - action on the product of two elliptic curves $E_1' \times E_2'$. This action has 16 fixed points and the quotient is an 8 - nodal Enriques surface. Instead of constructing S as the double cover of the Enriques surface (as done by Keum and Naie), we consider an étale $(\mathbb{Z}/2\mathbb{Z})^2$ - covering $\hat{S}$ of S, whose existence is guaranteed from the structure of the fundamental group of S. $\hat{S}$ is obtained as a double cover of $E_1' \times E_2'$ branched in a $(\mathbb{Z}/2\mathbb{Z})^2$ - invariant divisor of type $(4, 4)$, and S is recovered as the quotient of $\hat{S}$ by the action of $(\mathbb{Z}/2\mathbb{Z})^2$ on it.
The structure of this $(\mathbb{Z}/2\mathbb{Z})^2$-action and the geometry of the covering $\hat{S}$ of S is essentially encoded in the fundamental group $\pi_1(S)$, which is described as an affine group $\Gamma \in \mathbb{A}(2, \mathbb{C})$. In particular, it follows that the Albanese map of $\hat{S}$ is the above double cover $\hat{\alpha} : \hat{S} \to E_1' \times E_2'$.
If S' is now homotopically equivalent to a Keum - Naie surface S, then we have a corresponding étale $(\mathbb{Z}/2\mathbb{Z})^2$ - covering $\hat{S}'$ which is homotopically equivalent to $\hat{S}$. Since we know that the degree of the Albanese map of $\hat{S}$ is equal to two (by construction), we can conclude the same for the Albanese map of $\hat{S}'$ and this allows to deduce that also $\hat{S}'$ is a double cover of a product of elliptic curves branched in a $(\mathbb{Z}/2\mathbb{Z})^2$ - invariant divisor of type $(4, 4)$.
It was surprising for us that the same method works for Burniat-Inoue surfaces with $K^2 = 6$ (*primary Burniat surfaces*). The main result of [2] is the following

Theorem 3. *Let S be a smooth complex projective surface which is homotopically equivalent to a primary Burniat surface. Then S is a Burniat surface.*

We can then use this result to give an alternative, and less involved proof of the following result due to Mendes-Lopes and Pardini ([7]).

Theorem 4. *The subset of the Gieseker moduli space corresponding to primary Burniat surfaces is an irreducible connected component, normal, unirational and of dimension equal to 4.*

At this point we got interested to describe the irreducible (or even connected) components of the moduli space of surfaces of general type corresponding to Burniat surfaces with $2 \leq K^2 \leq 5$. It turns out that there is one Burniat surface with $K^2 = 2$, which is a standard Campedelli surface with torsion $(\mathbb{Z}/2\mathbb{Z})^3$, so this case is well known.

For Burniat surfaces with $4 \leq K^2 \leq 5$ instead we can prove the following two results (cf. [3]):

Theorem 5. *The subset of the Gieseker moduli space corresponding to secondary Burniat surfaces with $K^2 = 5$ is an irreducible connected component, normal, unirational and of dimension equal to 3.*

For the case $K^2 = 4$ there are two cases: the *non nodal case* and the *nodal case*. Burniat surfaces with $K^2 = 4$ are described as smooth bidouble covers of a weak Del Pezzo surface Y of degree 4. In the non nodal case, Y is in fact a Del Pezzo surface, whereas in the nodal case Y contains a rational (-2)-curve.

Theorem 6. *The subset of the Gieseker moduli space corresponding to secondary Burniat surfaces with $K^2 = 4$ consists of two irreducible connected components, each of dimension equal to 2. One connected component corresponds to Burniat surfaces with $K^2 = 4$ of non nodal type and the other one to Burniat surfaces with $K^2 = 4$ of nodal type.*

The techniques to prove the last two theorems are different from those to prove theorems 3,4.

The openness of the components corresponding to secondary Burniat surfaces is shown by local deformation theory. The arguments are quite standard, except in the case $K^2 = 4$ of nodal type.

For the closedness, the essential arguments are the following:

1) the $(\mathbb{Z}/2\mathbb{Z})^2$-action on Burniat surfaces X_t, $t \neq 0$, can be extended to the limit X_0;
2) let X_0 be the limit of (the canonical model of a) secondary Burniat surface, then consider $Y_0 := X_0/(\mathbb{Z}/2\mathbb{Z})^2$; Y_0 is a normal Gorenstein Del Pezzo surface;
3) use a combinatorial argument on the number of lines in the branch locus to show that Y_0 cannot have worse singularities as Y_t for $t \neq 0$;
4) this shows that X_0 is again a secondary Burniat surface of the same type as X_t, for $t \neq 0$.

References

[1] I. Bauer, F. Catanese, *The moduli space of Keum-Naie surfaces*, arxiv:0909.1733.

[2] I. Bauer, F. Catanese, *Burniat surfaces I: fundamental groups and moduli of primary Burniat surfaces*, arxiv:0909.3699 (2009).

[3] I. Bauer, F. Catanese, *Burniat surfaces II: secondary Burniat surfaces form three connected components of the moduli space*, arXiv:0911.1466 (2009).

[4] I. Bauer, F. Catanese, F. Grunewald, *The classification of surfaces with $p_g = q = 0$ isogenous to a product of curves*, Pure Appl. Math. Q. **4** (2008), 547–586.

[5] I. Bauer, F. Catanese, F. Grunewald, R. Pignatelli, *Quotient of product of curves, new surfaces with $p_g = 0$ and their fundamental groups*: arxiv.0809.3420.

[6] Y.H. Keum, *Some new surfaces of general type with $p_g = 0$*. Unpublished manuscript.

[7] M. Mendes Lopes, R. Pardini, *A connected component of the moduli space of surfaces with $p_g = 0$*. Topology 40 (2001), no. 5, 977–991.

[8] D. Naie, *Surfaces d'Enriques et une construction de surfaces de type général avec $p_g = 0$*. Math. Z. 215 (1994), no. 2, 269–280.

[9] R. Pignatelli, *Surfaces with $p_g = 0$: computer aided constructions.* In: Komplexe Algebraische Geometrie; Oberwolfach, 27.9.-3.10.2009, Report No. 44/2009.

On the Zariski decomposition problem

Caucher Birkar

We work over an algebraically closed field k of characteristic zero, and in the relative situation, that is, when we have a projective morphism $X \to Z$ of normal quasi-projective varieties written as X/Z.

There have been many attempts to generalise the Zariski decomposition for surfaces to higher dimensions. Here we mention two of them.

Fujita-Zariski decomposition. Let D be an $\mathbb{R}$-Cartier divisor on a normal variety X/Z. A Fujita-Zariski decomposition/Z for D is an expression $D = P+N$ such that

(1) P and N are $\mathbb{R}$-Cartier divisors,
(2) P is nef/Z, $N \geq 0$, and
(3) if $f\colon W \to X$ is a projective birational morphism from a normal variety, and $f^*D = P' + N'$ with P' nef $/Z$ and $N' \geq 0$, then $P' \leq f^*P$.

CKM-Zariski decomposition. Let D be an $\mathbb{R}$-Cartier divisor on a normal variety X/Z. A Cutkosky-Kawamata-Moriwaki-Zariski (CKM-Zariski for short) decomposition/Z for D is an expression $D = P + N$ such that

(1) P and N are $\mathbb{R}$-Cartier divisors,
(2) P is nef/Z, $N \geq 0$, and
(3) the morphisms $\pi_*\mathscr{O}_X(\llcorner mP \lrcorner) \to \pi_*\mathscr{O}_X(\llcorner mD \lrcorner)$ are isomorphisms for all $m \in \mathbb{N}$ where π is the fixed given morphism $X \to Z$.

Let $(X/Z, B)$ be a lc pair. It is well-known that if we have a log minimal model for this pair, then birationally there is a Zariski decomposition for $K_X + B$ according to both definitions of Zariski decomposition. However, the converse is known only in some special cases. Based on works of Moriwaki [3], Kawamata [2] proved that for a klt pair $(X/Z, B)$ with $K_X + B$ being $\mathbb{Q}$-Cartier and big/Z, existence of a CKM-Zariski decomposition for $K_X + B$ implies existence of a log canonical model for $(X/Z, B)$.

The problem with both definitions is that it is frequently very hard to construct such decompositions because of condition (3). To deal with this issue we introduce a very weak type of Zariski decomposition.

Weak Zariski decomposition. Let D be an $\mathbb{R}$-Cartier divisor on a normal variety X/Z. A *weak Zariski decomposition/Z* for D consists of a projective birational morphism $f\colon W \to X$ from a normal variety, and a numerical equivalence $f^*D \equiv P + N/Z$ such that

(1) P and N are $\mathbb{R}$-Cartier divisors,
(2) P is nef/Z, and $N \geq 0$.

Note that this is much weaker than the other definitions. For example, if $D \geq 0$, then by taking f to be the identity, $P = 0$ and $N = D$ we already have a weak Zariski decomposition. Of course, a natural thing to ask is the following:

Question. Does every pseudo-effective/Z $\mathbb{R}$-Cartier divisor D on a normal variety X/Z have a weak Zariski decomposition/Z?

The main result concerning the relation between Zariski decompositions and existence of minimal models is the following theorem which is proved in [1].

Theorem 1. *Assume the log minimal model program for $\mathbb{Q}$-factorial dlt pairs in dimension $d-1$. Let $(X/Z, B)$ be a lc pair of dimension d. Then, the following are equivalent:*

(1) $K_X + B$ has a weak Zariski decomposition/Z,
(2) $K_X + B$ birationally has a CKM-Zariski decomposition/Z,
(3) $K_X + B$ birationally has a Fujita-Zariski decomposition/Z,
(4) $(X/Z, B)$ has a log minimal model.

References

[1] C. Birkar, *On existence of log minimal models and weak Zariski decompositions*, arXiv:0907.5182v1.
[2] Y. Kawamata; *The Zariski decomposition of log-canonical divisors.* Algebraic geometry: Bowdoin 1985. Proceedings of Symposia in Pure Mathematics, 46 (1987), 425-434.
[3] A. Moriwaki; *Semi-ampleness of the numerically effective part of the Zariski decomposition.* J. Math. Kyoto Univ,. 26 (1986), 465-481.

Finite generation of the canonical ring after V. Lazić

Alessio Corti

The finite generation of the canonical ring of (nonsingular, projective) algebraic varieties in characteristic 0 is now a theorem [1]. In this talk I outline a new direct proof by Lazić [2], based on the hyperplane section principle and induction on dimension.

Let X be a nonsingular projective variety, Λ a finitely generated abelian semigroup and $D\colon \Lambda \to \operatorname{Div} X$ an additive map to the space of (integral, say, or rational) divisors on X. A *divisorial ring* on X is a Λ-graded ring of the form

$$R(X, D) = \bigoplus_{\lambda \in \Lambda} H^0\big(X, D(\lambda)\big)$$

A divisorial ring is a *klt adjoint ring* if

$$D(\lambda) = r(\lambda)\big(K + B(\lambda)\big)$$

where $r\colon \Lambda \to \mathbb{Q}_+$ is an additive map and $B\colon \Lambda \to \operatorname{Div} X$ is a map such that the pair $\big(X, B(\lambda)\big)$ is klt for all $\lambda \in \Lambda$. Note that the function $B\colon \Lambda \to \operatorname{Div} X$ is *homogeneous of degree zero*; that is, $B(r\lambda) = B(\lambda)$ for all $\lambda \in \Lambda$ and $r \in \mathbb{N}$.

Theorem 1. [1], [2] *Let $R = R(X; D)$ be a klt adjoint ring. If $B(\lambda)$ is big for every $\lambda \in \Lambda$, then R is finitely generated.*

In [2], this theorem is proved by induction on the dimension of X together with the following:

Theorem 2. *Let X be a nonsingular projective variety and $B = \sum B_i \subset X$ a snc divisor on X. Also fix an ample $\mathbb{Q}$-divisor A on X, such that the pair (X, A) is klt and A meets transversally every component of B. denote by $\mathscr{B}$ the "box" $\{\Theta = \sum b_i B_i \mid 0 \le b_i \le 1\}$. Then, for every component G of B:*

(1) $\mathscr{P}^G_A = \{\Theta \in \mathscr{B} \mid G \not\subset \mathbf{B}(K + A + \Theta)\}$ is a finite rational polyhedron. (Where, for a divisor D, $\mathbf{B}(D)$ denotes the stable base locus.)

(2) $\Theta \in \mathscr{P}^G_A(\mathbb{Q})$ if and only if the 'Lelong number:'

$$\nu_G\|K + A + \Theta\| := \lim_{n\to\infty} \frac{1}{n} mult_G\, |n(K + A + \Theta)| = 0.$$

The proof of both theorems is a transparent (though not easy) induction on the dimension. Hence, these ideas constitute a new approach to finite generation not relying on the detailed machinery of the minimal model program. Statements closely related to Theorem B can be found in [1] and in the work of Mihai Paun [3].

References

[1] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, James McKernan, *Existence of minimal models for varieties of log general type*, arXiv:math/0610203.

[2] Vladimir Lazić, *Adjoint rings are finitely generated*, arXiv:0905.2707.

[3] Mihai Paun, *Relative critical exponents, non-vanishing and metrics with minimal singularities*, arXiv:0807.3109.

A new family of symplectic fourfolds

Olivier Debarre

(joint work with Claire Voisin)

It follows from work of Beauville ([1]) and Bogomolov that any smooth complex compact Kähler manifold M with $c_1(M) = 0$ has a finite étale cover which is a product of (Kähler) manifolds of one of the following types:

- complex tori;
- Calabi-Yau manifolds, i.e., simply connected projective manifolds X with $H^0(X, \Omega^p_X) = 0$ for $0 < p < \dim(X)$;

- irreducible symplectic manifolds, i.e., (simply connected) compact even-dimensional Kähler manifolds X with an everywhere non-degenerate 2-form ω such that $H^0(X, \Omega_X^p)$ is 0 for p odd and generated by $\omega^{p/2}$ for p even.

It is very easy to construct Calabi-Yau manifolds, for example by taking complete intersections in Fano manifolds. Irreducible symplectic manifolds are much rarer. Beauville constructed in [1], in each dimension $2n$, two series of such varieties:

- the n-th punctual Hilbert scheme $S^{[n]}$ for a K3 surface S (it has $b_2 = 23$);
- the inverse image $K^n(A)$ of the origin by the sum morphism $A^{[n+1]} \to A$, where A is a 2-dimensional torus (it has $b_2 = 7$);

and O'Grady constructed two other families in dimensions 6 and 10.

Beauville's examples all have, in dimension at least 4, Picard number ≥ 2, while a very general algebraic deformation has Picard number 1, hence is not of the same type. There are very few explicit geometric descriptions for these deformations: only three such families are explicitly described, each of which is 20-dimensional and parametrizes general polarized deformations of the second punctual Hilbert scheme of a K3 surface:

(1) Beauville and Donagi proved in [2] that the variety of lines $F(X)$ on a smooth cubic hypersurface $X \subset \mathbf{P}^5$ is an algebraic symplectic fourfold. This gives a 20-dimensional moduli space of fourfolds, and along an explicitly described hypersurface in this moduli space (corresponding to "Pfaffian" cubics), $F(X)$ is isomorphic to the second punctual Hilbert scheme of a general K3 surface S of genus 8.

(2) Iliev and Ranestad proved in [7] that the variety $V(X)$ of sum of powers of a general cubic $X \subset \mathbf{P}^5$ as above is another algebraic symplectic fourfold, with 20 moduli. Along another hypersurface in the moduli space (corresponding to "apolar" cubics), $V(X)$ is also isomorphic to $S^{[2]}$. However, it is shown in [8] that the polarization on $V(X)$ is in general numerically different from the Plücker polarization on $F(X)$. This guarantees that we have two different families of deformations of $S^{[2]}$.

(3) O'Grady constructed in [12] a 20-parameter family of symplectic algebraic fourfolds. They are quasi-étale double covers of certain sextic hypersurfaces constructed by Eisenbud, Popescu, and Walter, and are deformations of the second punctual Hilbert scheme of a general K3 surface of genus 6.

We construct another family of symplectic fourfolds, which is close in spirit to the Beauville-Donagi family: it is related to the geometry of Grassmannians, and there is an associated Fano hypersurface which plays the role of the cubic hypersurface in [2].

The Grassmannian considered here is $G(6, V_{10})$, which parametrizes vector subspaces of dimension 6 of a fixed complex vector space V_{10} of dimension 10. Our starting point, which came to us following a discussion with Peskine, is a 3-form $\sigma \in \bigwedge^3 V_{10}^*$. A dimension count shows that the moduli space of such σ is 20-dimensional.

We associate with σ the variety

$$Y_\sigma = \{[W_6] \in G(6, V_{10}) \mid \sigma|_{W_6} \equiv 0\}.$$

Theorem 1. *For σ general, Y_σ is an irreducible symplectic fourfold.*

Unfortunately, unlike Beauville and Donagi, we were unable to identify special σ for which Y_σ is actually isomorphic to an $S^{[2]}$. Instead, we prove the following result, using ideas of Huybrechts ([6]).

Theorem 2. *The polarized manifolds $(Y_\sigma, \mathscr{O}_{Y_\sigma}(1))$ are the general deformations of $S^{[2]}$, where S is a general K3 surface of genus 12 with an explicit line bundle.*

Gritsenko, Hulek, and Sankaran proved in [4] that polarized irreducible symplectic fourfolds which are deformation equivalent to an $S^{[2]}$ with some polarization h admit a quasi-projective coarse moduli space $\mathscr{M}_h$ which is finite over a dense open subset of a locally symmetric modular variety $\mathscr{S}_h$. There are two "types" of polarizations; when h is "of split type," $\mathscr{S}_h$ (hence also every component of $\mathscr{M}_h$) is of general type for $d := \frac{1}{2}q(h) \geq 12$ and of nonnegative Kodaira dimension for $d = 9$ or 11 (q is the Beauville-Bogomolov quadratic form; [1]). In our case, h is of "nonsplit type" and $d = 11$, and our construction proves that one component of $\mathscr{M}_h$ (hence also $\mathscr{S}_h$) is unirational.

Problem 1. *Find specific σ for which Y_σ is actually isomorphic to an $S^{[2]}$.*

Problem 2. *Find other globally generated homogeneous vector bundles on rational homogeneous spaces for which the zero-set of a general section is an irreducible symplectic manifold.*

References

[1] Beauville, A., Variétés kählériennes dont la première classe de Chern est nulle, *J. Differential Geom.* **18** (1984), 755–782.

[2] Beauville, A., Donagi, R., La variété des droites d'une hypersurface cubique de dimension 4, *C. R. Acad. Sci. Paris Sér. I Math.* **301** (1985), 703–706.

[3] Bogomolov, F. A., Hamiltonian Kählerian manifolds (in Russian), *Dokl. Akad. Nauk SSSR* **243** (1978), 1101–1104. English transl.: *Soviet Math. Dokl.* **19** (1978), 1462–1465 (1979).

[4] Gritsenko, V., Hulek, K., Sankaran, G. K., Moduli spaces of irreducible symplectic manifolds, `arXiv:0802.2078`.

[5] Guan, D., On the Betti numbers of irreducible compact hyperkähler manifolds of complex dimension four, *Math. Res. Lett.* **8** (2001), 663–669.

[6] Huybrechts, D., Compact hyper-Kähler manifolds: basic results, *Invent. Math.* **135** (1999), no. 1, 63–113. Erratum: *Invent. Math.* **152** (2003), 209–212.

[7] Iliev, A., Ranestad, K., K3 surfaces of genus 8 and varieties of sums of powers of cubic fourfolds, *Trans. Amer. Math. Soc.* **353** (2001), 1455–1468.

[8] Iliev, A., Ranestad, K., Addendum to "K3 surfaces of genus 8 and varieties of sums of powers of cubic fourfolds", *C. R. Acad. Bulgare Sci.* **60** (2007), 1265–1270.

[9] Mukai, S., Polarized K3 surfaces of genus thirteen, *Moduli spaces and arithmetic geometry (Kyoto, 2004),* 315–326, Adv. Stud. Pure Math. **45**, Math. Soc. Japan, Tokyo, 2006.

[10] O'Grady, K., A new six-dimensional irreducible symplectic variety, *J. Algebraic Geom.* **12** (2003), 435–505.

[11] O'Grady, K., Desingularized moduli spaces of sheaves on a K3, *J. reine angew. Math.* **512** (1999), 49–117.

[12] O'Grady, K., Irreducible symplectic 4-folds and Eisenbud-Popescu-Walter sextics, *Duke Math. J.* **134** (2006), 99–137.

Shokurov's ACC Conjecture for log canonical thresholds on smooth varieties

Lawrence Ein

(joint work with Tommaso de Fernex, Mircea Mustaţă)

Let k be an algebraically closed field of characteristic zero. Log canonical varieties are varieties with mild singularities that provide the most general context for the Minimal Model Program. More generally, one considers the log canonicity condition on pairs $(X, \mathfrak{a}^t)$, where $\mathfrak{a}$ is a proper ideal sheaf on X (most of the times, it is the ideal of an effective Cartier divisor), and t is a nonnegative real number. Given a log canonical variety X over k, and a proper nonzero ideal sheaf $\mathfrak{a}$ on X, one defines the *log canonical threshold* $\mathrm{lct}(\mathfrak{a})$ of the pair $(X, \mathfrak{a})$. This is the largest number t such that the pair $(X, \mathfrak{a}^t)$ is log canonical. One makes the convention $\mathrm{lct}(0) = 0$ and $\mathrm{lct}(\mathscr{O}_X) = \infty$. One also defines a local version of the log canonical threshold at a point $p \in X$, which we denote by $\mathrm{lct}_p(\mathfrak{a})$. The log canonical threshold is a fundamental invariant in birational geometry. It plays an important role in the studying birational rigidity of smooth hyper-surfaces of degree n in $\mathbf{P}^n$, in the study of Bernstein polynomial from $D-$module theory and in the study of space of arcs [10] and [11]. See also [7], [6], or [9].

Shokurov's ACC Conjecture [12] says that the set of all log canonical thresholds on varieties of any fixed dimension satisfies the ascending chain condition, that is, it contains no infinite strictly increasing sequences. This conjecture attracted considerable interest due to its implications to the Termination of Flips Conjecture (see [1] for a result in this direction).

Consider the following set.

$$T_n^{\mathrm{sm}} := \{\mathrm{lct}(\mathfrak{a}) \mid X \text{ is smooth}, \dim X = n,\ \mathfrak{a} \subsetneq \mathscr{O}_X\}$$

Using ultra-filters and nonstandard model theory, deFernex and Mustaţă proved that T_n is a closed subset of the real numbers [5]. Kollár replaces the nonstandard techniques in [5] by more traditional methods and reproved the result. Further using the powerful results of [2]. he shows that the set of accumulation points in T_n is precisely T_{n-1} [8]. In a recent preprint, deFernex, Mustaţă and the PI are able to prove the ACC conjecture for smooth varieties using relatively elementary methods. It also provides a simpler proof for the accumulation result of Kollár.

Theorem 1. [3] *For every n, the set*

$$T_n^{\mathrm{sm}} := \{\mathrm{lct}(\mathfrak{a}) \mid X \textit{ is smooth}, \dim X = n,\ \mathfrak{a} \subsetneq \mathscr{O}_X\}$$

of log canonical thresholds on smooth varieties of dimension n satisfies the ascending chain condition.

Log canonical threshold on a variety with quotient singularities can be written as a log canonical threshold on a smooth variety of the same dimension. Therefore for every n the set

$$T_n^{\mathrm{quot}} := \{\mathrm{lct}(\mathfrak{a}) \mid X \text{ has quotient singularities, } \dim X = n,\, \mathfrak{a} \subsetneq \mathscr{O}_X\}$$

is equal to T_n^{sm}, and thus the ascending chain condition also holds for log canonical thresholds on varieties with quotient singularities.

Using inversion of adjunction for local complete intersection [6], we are also able to prove a similar result for local complete intersection varieties.

Theorem 2. [3] *For every n, the set*

$$T_n^{\mathrm{l.c.i.}} := \{\mathrm{lct}(\mathfrak{a}) \mid X \textit{ is l.c.i.}, \dim X = n,\, \mathfrak{a} \subsetneq \mathscr{O}_X\}$$

of log canonical thresholds on l.c.i. varieties of dimension n satisfies the ascending chain condition.

Let $\mathfrak{a}_i$ be a sequence of ideals in the formal power series ring of $n-$variables. Generalizing a construction of Kollár for principal ideals, in [3] the PI and his coworker construct a generalized limit of this sequence of ideals, which play a similar role of the limit of a converging sequence of numbers of a given sequence of numbers. The generalized limit $\mathfrak{a}$ is an ideal in a formal power series of n variable over an extension field of k. If $\lim_i \mathrm{lct}(\mathfrak{a}_i) = c$, then $\mathrm{lct}(\mathfrak{a}) = c$.

A key ingredient is the following effective $\mathfrak{m}$-adic semi-continuity property for log canonical thresholds.

Theorem 3. *Let X be a log canonical variety, and let $\mathfrak{a} \subsetneq \mathscr{O}_X$ be a proper ideal. Suppose that E is a prime divisor over X computing $\mathrm{lct}(\mathfrak{a})$, and consider the ideal sheaf $\mathfrak{q} := \{h \in \mathscr{O}_X \mid \mathrm{ord}_E(h) > \mathrm{ord}_E(\mathfrak{a})\}$. If $\mathfrak{b} \subseteq \mathscr{O}_X$ is an ideal such that $\mathfrak{b} + \mathfrak{q} = \mathfrak{a} + \mathfrak{q}$, then after possibly restricting to an open neighborhood of the center of E we have $\mathrm{lct}(\mathfrak{b}) = \mathrm{lct}(\mathfrak{a})$.*

References

[1] C. Birkar, Ascending chain condition for log canonical thresholds and termination of log flips, Duke Math. J. **136** (2007), 173–180.

[2] C. Birkar, P. Cascini, C. Hacon and J. McKernan, Existence of minimal models for varieties of log general type, preprint available at `arXiv:math/0610203`.

[3] , T. de Fernex, L. Ein, M. Mustaţă) Shokurov's ACC conjecture for log canonical thresholds on smooth varieties, AG 0905.3775v2 to appear in Duke Math. J.

[4] Demailly, J.-P., *Complex Analytic and Differential Geometry*, `http://www-fourier.ujf-grenoble.fr/d̃emailly/manuscripts/agbook.pdf`

[5] T. de Fernex and M. Mustaţă, Limits of log canonical thresholds, to appear in Ann. Sci. École Norm. Sup., preprint available at `arXiv:0710.4978`.

[6] L. Ein and M. Mustaţă, Invariants of singularities of pairs, Proceeding of the ICM 2006 Madrid ICM, Vol. 2 583-602.

[7] J. Kollár, Singularities of pairs, in *Algebraic Geometry, Santa Cruz 1995*, volume **62** of Proc. Symp. Pure Math Amer. Math. Soc. 1997, 221–286.

[8] J. Kollár, Which powers of holomorphic functions are integrable?, preprint available at `arXiv:0805.0756`.

[9] R. Lazarsfeld, Positivity in algebraic geometry, I- II, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Vols. **48**–**49**, Springer-Verlag, Berlin, 2004.
[10] M. Mustaţă, Singularities of pairs via jet schemes, J. Amer. Math. Soc. **15** (2002), 599–615.
[11] M. Mustaţă, Jet schemes of locally complete intersection canonical singularities, with an appendix by D. Eisenbud and E. Frenkel, Invent. Math. **145** (2001), 397–424.
[12] V. V. Shokurov, Three-dimensional log perestroikas. With an appendix in English by Yujiro Kawamata, Izv. Ross. Akad. Nauk Ser. Mat. **56** (1992), 105–203, translation in Russian Acad. Sci. Izv. Math. **40** (1993), 95–202.
[13] V. V . Shokurov, *Letters of a bi-rationalist. V. Minimal log discrepancies and termination of log flips*, Proc. Steklov Inst. Math. **246** (2004), 315–336.

Relation between the étale fundamental group and $\mathscr{O}_X$-coherent $\mathscr{D}_X$-modules

Hélène Esnault

If X is a complex smooth algebraic variety, its $\mathbb{C}$-valued points $X(\mathbb{C})$ define a topological manifold, and one has the topological fundamental group $\pi_1^{\mathrm{top}}(X,x)$ based in a complex point x. This an abstract group, which moreover is of finite type. Its profinite completion $\widehat{\pi_1^{\mathrm{et}}(X,x)}$ is identified via Riemann existence theorem with Grothendieck's étale fundamental group $\pi_1^{\mathrm{et}}(X,x)$. The homomorphism $\pi_1^{\mathrm{top}}(X,x) \to \pi_1^{\mathrm{et}}(X,x)$ factors through the proalgebraic completion $\left(\pi_1^{\mathrm{top}}(X,x)\right)^{\mathrm{alg}} = \varprojlim H$, where H is the Zariski closure of the monodromy group of a complex linear representation $\pi_1^{\mathrm{top}}(X,x) \to GL(n,\mathbb{C})$. This yields a surjective homomorphism $\left(\pi^{\mathrm{top}}(X,x)\right)^{\mathrm{alg}} \twoheadrightarrow \pi_1^{\mathrm{et}}(X,x)$. The classical Malcev-Grothendieck theorem ([5], [4]) asserts that if the profinite completion of an abstract group Γ of finite type is trivial, so is its algebraic completion. Using the Riemann-Hilbert correspondence [1], the theorem implies for $\Gamma = \pi_1^{\mathrm{top}}(X,x)$ that if $\pi_1^{\mathrm{et}}(X,x) = \{1\}$, then $\left(\pi^{\mathrm{top}}(X,x)\right)^{\mathrm{alg}} = \{1\}$, so there are no nontrivial algebraic bundles with an integrable connection, or, what is equivalent, no nontrivial $\mathscr{O}_X$-coherent $\mathscr{D}_X$-module (with regular singularities at ∞ if X is not proper).

Gieseker conjectured in [3], p. 8, that the same should hold true in characteristic $p > 0$. We show

Theorem 1. *(*[2, Theorem 1,1]*) Let X be a smooth projective variety defined over a perfect field k of characteristic $p > 0$. If $\pi_1^{\mathrm{et}}(X \otimes_k \bar{k}, x) = \{1\}$, then there are non nontrivial $\mathscr{O}_X$-coherent $\mathscr{D}_X$-modules.*

References

[1] Deligne, P.: *Équations Différentielles à Points Singuliers Réguliers*, Lecture Notes in Mathematics **163** (1970), Springer Verlag Berlin-New York.
[2] Esnault, H., Mehta, V.: *Simply connected projective manifolds in characteristic $p > 0$ have no nontrivial stratified bundles*, `http://de.arxiv.org/pdf/math/0907.3375`
[3] Gieseker, D.: *Flat vector bundles and the fundamental group in non-zero characteristics*, Ann. Scu. Norm. Sup. Pisa, 4. série, tome 2, no 1 (1975), 1–31.
[4] Grothendieck, A.: *Représentations linéaires et compactifications profinies des groupes discrets*, Manuscripta mathematica, **2** (1970), 375–396.

[5] Malcev, A.: *On isomorphic matrix representations of infinite groups*, Rec. Math. [Mat. Sbornik] N.S. **8** (50) (1940), 405–422.

Rational maps between moduli spaces of curves and Gieseker-Petri divisors

Gavril Farkas

The main issue we address is a detailed intersection theoretic study of a rational map between two different moduli spaces of curves. We fix $g := 2s+1 \geq 3$. Since $\rho(2s+1, 1, s+2) = 1$ we can define a rational map between moduli spaces of curves

$$\phi : \overline{\mathscr{M}}_{2s+1} - - > \overline{\mathscr{M}}_{1+\frac{s}{s+1}\binom{2s+2}{s}}, \quad \phi([C]) := [W^1_{s+2}(C)],$$

where $W^1_{s+2}(C) := \{L \in \mathrm{Pic}^{s+2}(C) : h^0(C, L) \geq 2\}$ is the so-called *Brill-Noether curve* of C. It is known that ϕ is generically injective (cf. [10], [1]). Since ϕ is the only-known rational map between two moduli spaces of curves and one of the very few natural examples of a rational map admitted by $\overline{\mathscr{M}}_g$, its study is clearly of independent interest. We carry out a detailed enumerative study of ϕ and among other things, we determine the pull-back map $\phi^* : \mathrm{Pic}(\overline{\mathscr{M}}_{g'}) \to \mathrm{Pic}(\overline{\mathscr{M}}_g)$:

Theorem 1. *We consider the rational map* $\phi : \overline{\mathscr{M}}_g - - > \overline{\mathscr{M}}_{g'}, \quad \phi[C] = [W^1_{s+2}(C)]$, *where*

$$g := 2s+1 \quad \text{and} \quad g' := 1 + \frac{s}{s+1}\binom{2s+2}{s}.$$

We then have the following description of the map $\phi^* : \mathrm{Pic}(\overline{\mathscr{M}}_{g'}) \to \mathrm{Pic}(\overline{\mathscr{M}}_g)$*:*

$$\phi^*(\lambda') = n_0\Big(\frac{6s^4+20s^3-s^2-20s-2}{(s+2)(2s-1)}\lambda - \frac{s(s^2-1)}{2s-1}\delta_0 -$$

$$-\frac{2s(s-1)(6s^2+10s+1)}{(s+2)(4s-2)}\delta_1 - \sum_{i=2}^{[g'/2]} b_i\delta_i\Big),$$

where $b_i \geq \frac{s(s^2-1)}{2s-1}$ *for* $2 \leq i \leq [g/2]$,

$\phi^*(\delta_0') = n_0 \cdot \delta_0 + [\overline{\mathscr{GP}}^1_{2s+1,s+2}]$, $\phi^*(\delta_1') = n_0 \cdot \delta_1$ *and* $\phi^*(\delta_j') = 0$ *for* $2 \leq j \leq [g'/2]$.

Here n_0 denotes the Catalan number of linear series $\mathfrak{g}^1_{s+1}$ on a general curve of genus $2s$.

In particular we have the following formula concerning slopes of divisor classes pulled back from $\overline{\mathscr{M}}_{g'}$ (For the definition of the slope function $s : \mathrm{Eff}(\overline{\mathscr{M}}_g) \to \mathbb{R} \cup \{\infty\}$ on the cone of effective divisors we refer to [8] or [5]):

Theorem 2. *We set* $g := 2s+1$ *and* $g' := 1 + \frac{s}{s+1}\binom{2s+2}{s}$. *For any divisor class* $D \in \mathrm{Pic}(\overline{\mathscr{M}}_{g'})$ *having slope* $s(D) = c$, *we have the following formula for the slope of* $\phi^*(D) \in \mathrm{Pic}(\overline{\mathscr{M}}_g)$*:*

$$s(\phi^*(D)) = 6 + \frac{8s^3(c-4) + 5cs^2 - 30s^2 + 20s - 8cs - 2c + 24}{s(s+2)(cs^2 - 4s^2 - c - s + 6)}.$$

We use this formula to describe the cone $\mathrm{Mov}(\overline{\mathcal{M}}_g)$ of *moving divisors*[1] inside the cone $\mathrm{Eff}(\overline{\mathcal{M}}_g)$ of effective divisors. The cone $\mathrm{Mov}(\overline{\mathcal{M}}_g)$ parametrizes rational maps from $\overline{\mathcal{M}}_g$ in the projective category while the cone $\mathrm{Nef}(\overline{\mathcal{M}}_g)$ of numerically effective divisors, parametrizes regular maps from $\overline{\mathcal{M}}_g$. A fundamental question is to estimate the following slope invariants associated to $\overline{\mathcal{M}}_g$:

$$s(\overline{\mathcal{M}}_g) := \inf_{D\in \mathrm{Eff}(\overline{\mathcal{M}}_g)} s(D) \quad \text{and} \quad s'(\overline{\mathcal{M}}_g) := \inf_{D\in \mathrm{Mov}(\overline{\mathcal{M}}_g)} s(D).$$

The formula of the class of Brill-Noether divisors $\overline{\mathcal{M}}^r_{g,d}$ when $\rho(g,r,d) = -1$ shows that $\lim_{g\to\infty} s(\overline{\mathcal{M}}_g) \leq 6$ (cf. [3]). In [4] we provided an infinite sequence of genera of the form $g = a(2a+1)$ with $a \geq 2$ for which $s(\overline{\mathcal{M}}_g) < 6 + 12/(g+1)$, thus contradicting the Slope Conjecture [8]. There is no known example of a genus g such that $s(\overline{\mathcal{M}}_g) < 6$. We have the following estimate of the moving cone:

Theorem 3. $s'(\overline{\mathcal{M}}_g) < 6 + 16/(g-1)$.

References

[1] C. Ciliberto, J. Harris and M. Teixidor, *On the endomorphisms of* $\mathrm{Jac}(W^1_d(C))$ *when* $\rho = 1$ *and* C *has general moduli*, in: Classification of irregular varieties, Springer LNM **1515** (1992), 41-67.

[2] D. Eisenbud and J. Harris, *Limit linear series: basic theory*, Inventiones Math. **85** (1986), 337-371.

[3] D. Eisenbud and J. Harris, *The Kodaira dimension of the moduli space of curves of genus* ≥ 23, Inventiones Math. **90** (1987), 359-387.

[4] G. Farkas, *Koszul divisors on moduli spaces of curves*, Amer. J. Mathematics, **131** (2009), 819-867.

[5] G. Farkas and M. Popa, *Effective divisors on* $\overline{\mathcal{M}}_g$*, curves on* $K3$ *surfaces and the Slope Conjecture*, J. Algebraic Geometry, **14** (2005), 241-267.

[6] D. Gieseker, *Stable curves and special divisors*, Inventiones Math. **66** (1982), 251-275.

[7] D. Hyeon and Y. Lee, *Log minimal model for the moduli space of stable curves of genus* 3, math.AG/07003093.

[8] J. Harris and I. Morrison, *Slopes of effective divisors on the moduli space of curves*, Inventiones Math. **99** (1990), 321-355.

[9] Y. Hu and S. Keel, *Mori dream spaces and GIT*, Michigan Math. J. **48** (2000), 331-348.

[10] G. Pirola and M. Teixidor, *Generic Torelli for* W^r_d, Mathematische Zeitschrift **209** (1992), 53-54.

The Limit of the Fourier-Mukai Transform

Gerard van der Geer

(joint work with Alexis Kouvidakis)

This is a report on joint work ([3, 4]) with Alexis Kouvidakis (University of Crete). The Chow ring $\mathrm{CH}^*_{\mathbf{Q}}(X)$ of a principally polarized abelian variety X of dimension g over an algebraically closed field is graded by codimension $\mathrm{CH}^*_{\mathbf{Q}}(X) = \oplus_{i=0}^{g} \mathrm{CH}^i_{\mathbf{Q}}(X)$ and carries an intersection product $(x,y) \mapsto x\cdot y$ that makes it a

[1]Recall that an effective $\mathbb{Q}$-Cartier divisor D on a normal projective variety X is said to be moving, if the stable base locus $\bigcap_{n\geq 1} \mathrm{Bs}|\mathcal{O}_X(nD)|$ has codimension at least 2 in X.

commutative ring. But it is also provided with a second structure of commutative ring via the Pontryagin product $(x, y) \mapsto x \star y$. The relation is given by the Fourier-Mukai transform

$$F : \mathrm{CH}^*_{\mathbf{Q}}(X) \longleftrightarrow \mathrm{CH}^*_{\mathbf{Q}}(X^t)$$

with X^t the dual abelian variety that we identify with X using the principal polarization. The map F is defined by

$$x \mapsto p_{2*}(e^{c_1(P)} \cdot p_1^*(x)),$$

where p_1 and p_2 are the two projections $X \times X \to X$ and P is the Poincaré bundle on $X \times X$. The Fourier-Mukai transform is a bijection transforming the usual intersection product into the Pontryagin product: $F(x \cdot y) = F(x) \star F(y)$. This Fourier-Mukai transform is a powerful tool for probing the structure of the Chow ring. Using it Beauville constructed in [1] a second grading

$$\mathrm{CH}^*_{\mathbf{Q}}(X) = \oplus_{i,j} \mathrm{CH}^i_{(j)}(X),$$

where

$$\mathrm{CH}^i_{(j)}(X) = \{x \in \mathrm{CH}^i(X) : n^*(x) = n^{2i-j}\, x \quad \text{for all } n \in \mathbf{Z}\}.$$

We have $F(\mathrm{CH}^i_{(j)}(X)) = \mathrm{CH}^{g-i+j}_{(j)}(X)$.

The quotient ring $A^*(X)$ of $\mathrm{CH}^*_{\mathbf{Q}}(X)$ modulo algebraic equivalence inherits this double grading from $\mathrm{CH}^*_{\mathbf{Q}}(X)$.

We now look at a degenerating abelian variety $\mathscr{X} \to S$ over a discrete valuation ring with residue field k so that the generic fibre is a ppav and the special fibre X_0 is a semi-abelian variety of torus rank 1:

$$1 \to \mathbb{G}_m \to X_0 \to B \to 0,$$

where B is a $g-1$-dimensional principally polarized abelian variety and the extension class is $\beta \in B$. We assume that $\mathscr{X}$ has a compactification $\bar{\mathscr{X}}$ whose special fibre $\bar{X}_0$ has as normalization a $\mathbf{P}^1$-bundle $\mathbf{P} = \mathbf{P}(O_B \oplus J)$ over B with J the line bundle $O(\Theta - \Theta_\beta)$ with Θ defining the polarization on B. Then $\bar{X}_0$ is obtained by gluing the two natural sections of the $\mathbf{P}^1$-bundle by a translations over β.

If c_η is an algebraic cycle on X_η we can take the Fourier-Mukai transform $\varphi_\eta := F(c_\eta)$ and consider the limit cycle (specialization) φ_0 of φ_η. A natural question is:

Question 1. *What is the limit φ_0 of φ_η?*

If $q : \mathbf{P} \to B$ denotes the natural projection of the $\mathbf{P}^1$-bundle, the Chow ring of $\mathbf{P}$ is the extension $\mathrm{CH}^*(B)[\lambda]/(\lambda^2 - \lambda \cdot q^* c_1(J))$ with $\lambda = c_1(O_{\mathbf{P}}(1))$. We denote by c_0 the specialization of the cycle c_η on $\bar{X}_0$. We can write c_0 as $\nu_*(\gamma)$ with $\gamma = q^*z + q^*w \cdot \lambda$, where $\nu : \mathbf{P} \to \bar{X}_0$ is the normalization map.

Theorem 1. *Let c_η be a cycle on X_η with $c_0 = \nu_*(q^*z + q^*w \cdot \lambda)$. The limit φ_0 is up to algebraic equivalence given by*

$$\varphi_0 \stackrel{a}{=} \nu_*(q^*F_B(w) - q^*F_B(z) \cdot \lambda)\,.$$

with F_B the Fourier-Mukai transform of B.

Modulo rational equivalence the answer is more involved. The limit φ_0 of the Fourier-Mukai transform $\varphi_\eta = F(c_\eta)$ is given by $\varphi_0 = \nu_*(q^*a + q^*b \cdot \lambda)$ with

$$a = F_B(w) + \sum_{n=0}^{2g-2} \sum_{m=0}^{n} \frac{(-1)^m}{(n+2)!} F_B[(z + w \cdot c_1(J)) \cdot c_1^m(J)] \cdot c_1^{n-m+1}(J)$$

and

$$b = \sum_{n=0}^{2g-2} \sum_{m=0}^{n} \frac{(-1)^m}{(n+2)!} F_B[(((-1)^{n+1} - 1)z - w \cdot c_1(J)) \cdot c_1^m(J)] \cdot c_1^{n-m}(J)\,,$$

One can apply this to prove non-vanishing results. For example, suppose that $c_\eta = \sum c^{(j)}$ with $c^{(j)} \in A^i_{(j)}(X)$ with corresponding decomposition $\varphi_\eta = \sum \varphi^{(j)}$ with $\varphi^{(j)} \in A^{g-i+j}(X)$.

Corollary 1. *Suppose $\varphi_0^{(j)} \neq 0$ with $\varphi_0^{(j)}$ the codimension $g-i+j$-part of φ_0. Then $c^{(j)} \neq 0$ (modulo algebraic equivalence).*

For example, one can take for the degenerating abelian variety the Picard variety of the Fano surface of lines on a cubic threefold that degenerates to a generic one-nodal cubic threefold. In this case the special fibre is the $\mathbb{G}_m$-extension of the Jacobian of a general curve C of genus 4 and the extension class β is given by the difference of the two g_3^1's on C. We can embed the Fano surface Σ in $\mathrm{Pic}^0(\Sigma)$ by choosing a base point s_0 and sending s to $[D_s - D_{s_0}]$ with D_s the divisor of lines on the cubic threefold that intersect the line corresponding to s. The cycle class of the Fano surface Σ of the general fibre in $\mathrm{Pic}^0(\Sigma)$ has a Beauville decomposition

$$[\Sigma] = \Sigma^{(0)} + \Sigma^{(1)} + \Sigma^{(2)}$$

with $\Sigma^{(j)} \in A^3_{(j)}(X)$. Then the claim is that $\Sigma^{(1)}$ is not algebraically equivalent to 0 because it degenerates to the class with Fourier-Mukai transform

$$\varphi_0^{(1)} = \nu_*(q^*[(F_B(C^{(0)}) \cdot F_B(C^{(1)})] - q^*F_B(C^{(1)}) \cdot \lambda)$$

with $C = C^{(0)} + C^{(1)}$ the Beauville decomposition of C in $A^*(\mathrm{Jac}(C))$ and this is not zero since we know by Ceresa's classical result [2] that for sufficiently general C the class $C^{(1)}$ is not zero.

But maybe the reader can come up with more important applications of this limit.

References

[1] A. Beauville: Sur l'anneau de Chow d'une variété abélienne. *Math. Annalen* **273** (1986), 647–651.

[2] G. Ceresa: C is not algebraically equivalent to C^- in its Jacobian. *Ann. Math.* **117** (1983), 285–291.

[3] G. van der Geer, A. Kouvidakis: The rank-one limit of the Fourier-Mukai transform. `math.AG/0906.3793`

[4] G. van der Geer, A. Kouvidakis: A note on Fano surfaces of nodal cubic threefolds. `math.AG/0902.3877`

Characters of orthogonal groups and the fundamental group of modular varieties

Klaus Hulek
(joint work with Valeri Gritsenko, Gregory Kumar Sankaran)

1. Introduction

The main question considered in this talk is the following: what can one say about the abelianisation of arithmetic groups, in particular of orthogonal type? This question has applications to the existence of possible characters of automorphic forms and to the fundamental group of modular varieties.

A first example is the group $\mathrm{SL}(2,\mathbb{Z}) = \mathrm{Sp}(1,\mathbb{Z})$. It is well known that the abelianisation of this group is cyclic of order 12, i.e. $\mathrm{SL}(2,\mathbb{Z})^{\mathrm{ab}} = \mathbb{Z}/12\mathbb{Z}$. The generator of this group is the character of the square η^2 of the Dedekind η-function. Related to this, Mumford has shown that this is also the Picard group of the moduli stack of elliptic curves, i.e. $\mathrm{Pic}(\underline{\mathscr{A}}_1) = \mathbb{Z}/12\mathbb{Z}$.

2. Directory of groups

We denote by L an even non-degenerate integral lattice and by $\mathrm{O}(L)$ its group of orthogonal transformations. Examples which are of relevance for us include

- $L_{\mathrm{K3}} = 3U \oplus 2E_8(-1)$, the K3-lattice, where U denotes the hyperbolic plane and $E_8(-1)$ is the unique negative definite, even unimodular lattice of rank 8.
- $L_{2d} = 2U \oplus 2E_8(-1) \oplus \langle -2d \rangle$, the lattice which arises in the moduli problem for degree $2d$-polarised K3-surfaces.
- $L_{2d}^{\text{non-split}} = 2U \oplus 2E_8(-1) \oplus \begin{pmatrix} -(d+1)/2 & 1 \\ 1 & -2 \end{pmatrix}$ (here $d \equiv -1 \mod 4$), a lattice which arises in the moduli problem of polarised irreducible symplectic manifolds of non-split type (see the talk of Debarre in this report).
- $M = U \oplus U(2) \oplus E_8(-2)$, a lattice which plays an important role for moduli of Enriques surfaces.

Let $L^\vee$ be the dual lattice of L. The discriminant $D(L) = L^\vee/L$ carries a quadratic form with values in $\mathbb{Q}/2\mathbb{Z}$. The kernel $\widetilde{\mathrm{O}}(L)$ of the natural map $\mathrm{O}(L) \to \mathrm{O}(D(L))$ is called the *stable* orthogonal group and we set $\widetilde{\mathrm{SO}}(L) = \widetilde{\mathrm{O}}(L) \cap \mathrm{SO}(L)$. We denote the group of orthogonal transformations of *real* spinor norm 1 (for a definition see [2, Section 1]) by $\mathrm{O}^+(L)$ and set $\mathrm{SO}^+(L) = \mathrm{O}^+(L) \cap \mathrm{SO}(L)$, $\widetilde{\mathrm{O}}^+(L) = \widetilde{\mathrm{O}}(L) \cap \mathrm{O}^+(L)$ and $\widetilde{\mathrm{SO}}^+(L) = \widetilde{\mathrm{SO}}(L) \cap \mathrm{O}^+(L)$. Finally we define the *spinorial kernel* $\mathrm{O}'(L)$ to be the group of transformations in $\mathrm{SO}(L)$ of *rational* spinor norm 1. Note that every root $a \in L$ (i.e., element of length -2) defines a reflection $\sigma_a \in \widetilde{\mathrm{O}}^+(L)$ by setting $\sigma_a(x) = x + (a,x)a$.

3. A result of Kneser

The following result of Kneser is important for our considerations.

Theorem 1 (Kneser)**.** *Let L be an even lattice which satisfies Kneser's conditions i.e.,*

(i) *The real Witt rank* $\mathrm{Witt}_{\mathbb{R}}(L) \geq 2$*,*
(ii) *L represents -2,*
(iii) $\mathrm{rank}_2(L) \geq 6$ *and* $\mathrm{rank}_3(L) \geq 5$ *where* $\mathrm{rank}_p(L)$ *denotes the p-rank of L.*

Then

$$\mathrm{O}'(L) = \langle \sigma_a \sigma_b \mid a^2 = b^2 = -2 \rangle.$$

From this it is not hard to conclude the following

Theorem 2. *Let L be an even lattice satisfying Kneser's conditions. Then $\widetilde{\mathrm{O}}^+(L)^{\mathrm{ab}}$ and $\widetilde{\mathrm{SO}}^+(L)^{\mathrm{ab}}$ are 2-groups whose order divides 2^N resp. 2^{N-1} where N is the number of orbits of roots with respect to the groups $\widetilde{\mathrm{O}}^+(L)$ and $\widetilde{\mathrm{SO}}^+(L)$ respectively.*

Using this and Eichler's criterion one obtains

Corollary 1. *Let L be an even unimodular lattice of rank at least 6 containing two hyperbolic planes. Then*

$$\widetilde{\mathrm{SO}}^+(L)^{\mathrm{ab}} = \{1\}, \quad \widetilde{\mathrm{O}}^+(L)^{\mathrm{ab}} = \mathbb{Z}/2\mathbb{Z} = \langle \det \rangle.$$

4. The main theorem

In many geometric applications the lattices in question are not unimodular. This was the main motivation for our main result which can be seen as both a strengthening of Theorem 2 and a generalisation of Corollary 1.

Theorem 3. *Let L be an even lattice containing at least two hyperbolic planes such that* $\mathrm{rank}_2(L) \geq 6$ *and* $\mathrm{rank}_3(L) \geq 5$*. Then*

$$\widetilde{\mathrm{SO}}^+(L)^{\mathrm{ab}} = \{1\}, \quad \widetilde{\mathrm{O}}^+(L)^{\mathrm{ab}} = \mathbb{Z}/2\mathbb{Z} = \langle \det \rangle.$$

This result is indeed stronger than Theorem 2 as can be seen from the example of the lattice $L_{2d} = 2U \oplus 2E_8(-1) \oplus \langle -2d \rangle$, which plays an important role in the theory of polarised K3-surfaces. I $d \equiv 1 \mod 4$ then the number N of orbits of roots is 2. Hence Theorem 2 gives that the order $|\widetilde{\mathrm{O}}^+(L_{2d})^{\mathrm{ab}}|$ equals 2 or 4, whereas Theorem 4 gives $|\widetilde{\mathrm{O}}^+(L_{2d})^{\mathrm{ab}}| = 2$.

Examples show that the conditions on the 2- and 3-rank are necessary. For example the lattice $L = 2U \oplus A_2(-3)$ does not fulfill the condition on the 3-rank and indeed has a character of order 3 as shown by Desreumaux [1].

The proof of the main theorem uses the following tools: Eichler transvections, Kneser's solution of the principal congruence subgroup problem and strong approximation.

5. Application to fundamental groups

Let $\mathscr{D}$ be a homogeneous domain on which an arithmetic group Γ acts properly discontinuously with quotient $X = \Gamma\backslash\mathscr{D}$. It is well known that there is an epimorphism $\rho : \Gamma \twoheadrightarrow \pi_1(X)$ and that every element $\gamma \in \Gamma$ which has a fixed point is contained in the kernel of ρ. Moreover, if $\tilde{X}$ is a smooth projective model of the quasi-projective variety X, then ρ factors as $\rho : \Gamma \twoheadrightarrow \pi_1(\tilde{X}) \twoheadrightarrow \pi_1(X)$.

Now let L be a lattice of signature $(2, n)$. This defines a homogeneous domain consisting of two connected components

$$\Omega_L = \{[x] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (x, x) = 0,\ (x, \bar{x}) > 0\} = \mathscr{D}_L \cup \mathscr{D}'_L.$$

One defines

$$\mathscr{F}(L) = \widetilde{\mathrm{O}}^+(L)\backslash\mathscr{D}_L, \quad \mathscr{S}\mathscr{F}(L) = \widetilde{\mathrm{SO}}^+(L)\backslash\mathscr{D}_L.$$

In many cases such modular varieties have an interpretation as a moduli space.

Theorem 4. *Let L be as in the main theorem. Then $\mathscr{F}(L)$ and $\mathscr{S}\mathscr{F}(L)$ as well as all smooth projective models of these varieties are simply connected.*

The proof of this theorem follows easily from the fact that the arithmetic groups in questions are generated by reflections (which have fixed points).

To illustrate this example we consider the following two cases. First let $L = L_{2d} = 2U \oplus 2E_8(-1) \oplus \langle -2d\rangle$ and $\mathscr{F}_{2d} = \mathscr{F}(L_{2d}) = \widetilde{\mathrm{O}}^+(L_{2d})\backslash\mathscr{D}_{L_{2d}}$. This is the moduli space of degree $2d$ pseudo-polarised K3-surfaces. By Theorem 4 this moduli space as well as any smooth projective model is simply connected. Another example is given by the lattices $M = U \oplus U(2) \oplus E_8(-2)$, resp. $M' = U(2) \oplus U \oplus E_8(-1)$. It is not hard to see that $\mathscr{M} = \mathrm{O}^+(M)\backslash\mathscr{D}_M = \mathrm{O}^+(M')\backslash\mathscr{D}_{M'}$. By the Torelli theorem for Enriques surfaces this is the union of the moduli space of Enriques surfaces, which is an open subset of $\mathscr{M}$, and a divisor, the discriminant, i.e.,

$$\mathscr{M} = \mathscr{M}_{\mathrm{Enriques}} \cup \Delta.$$

The group $\mathrm{O}(M')$ is also generated by elements with non-empty fixed locus and hence it follows from Theorem 4 that $\mathscr{M}$ as well as any smooth model is simply connected. The latter fact can also be deduced from the rationality of the moduli space of Enriques surfaces as proved by Kondo.

Similar applications exist for moduli spaces of non-principally polarised abelian surfaces, cubic fourfolds and modular varieties related to moduli spaces of polarised irreducible symplectic manifolds.

References

[1] C. Desreumaux, *Construction de formes automorphes réflectives sur un espace de dimension* 4. J. Théorie des Nombres de Bordeaux **18** (2006), 89–111.
[2] V. Gritsenko, K. Hulek, G. K. Sankaran *Abelianisation of orthogonal groups and the fundamental group of modular varieties*, Journal of Algebra **322** (2009), 463–478.
[3] M. Kneser, *Erzeugung ganzzahliger orthogonaler Gruppen durch Spiegelungen*, Math. Ann. **255** (1981), 453-462.
[4] M. Kneser, *Normalteiler ganzzahliger Spingruppen*, J. reine angew. Math. **311** (1979), 191-214.

Dimer models and tilting bundles

Akira Ishii
(joint work with Kazushi Ueda)

1. Dimer models

A *dimer model* is a bicolored graph on a torus $T = \mathbb{R}^2/\mathbb{Z}^2$ consisting of a set $B \subset T$ of black nodes, another set $W \subset T$ of white nodes, and a set E of edges consisting of embedded line segments connecting vertices of different colors.

A connected component of the complement $T \setminus E$ is called a *face* of the graph. A bicolored graph on T is said to be a *dimer model* if any face is simply-connected.

A dimer model (B, W, E) encodes the information of a *quiver* (=oriented graph) Γ as in Figure 2: the vertices are the faces and the arrows are the edges. The directions of the arrows are determined by the colors of the vertices of the graph, so that the white vertex $w \in W$ is on the right of the arrow. In other words, the quiver is the dual graph of the dimer model equipped with an orientation given by rotating the white-to-black flow on the edges of the dimer model by minus 90 degrees.

Γ is naturally equipped with *relations*: For an arrow $a \in A$, there exist two paths $p_+(a)$ and $p_-(a)$ from the target of a to the source of a, the former going around the white vertex connected to $a \in E$ clockwise and the latter going around the black vertex connected to a counterclockwise. Then the relation of Γ is the two-sided ideal $\mathscr{I}$ of the path algebra generated by $p_+(a) - p_-(a)$ for all $a \in A$. As an example, consider the dimer model in Figure 1. The corresponding quiver is shown in Figure 2, whose relations are given by

$$\mathscr{I} = (dbc - cbd, dac - cad, adb - bda, acb - bca).$$

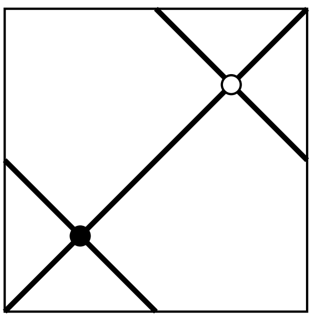


Figure 1. A dimer model

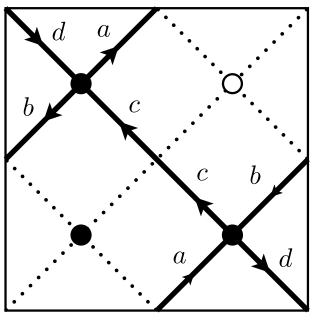


Figure 2. The corresponding quiver

2. Tilting bundles on crepant resolutions

We can consider the moduli space $\mathscr{M}_\theta$ of θ-stable representations of the associated quiver with relations with dimension vector $(1,\dots,1)$, where θ is a parameter for stability.

A perfect matching of a dimer model is a set D of edges such that every node is contained in exactly one edge in D. It corresponds to a torus invariant divisor on $\mathscr{M}_\theta$ for some θ. Considering all the perfect matchings, we can construct a lattice polygon Δ. We say a dimer model is *non-degenerate* if every edge is contained in some perfect matching.

Theorem 1 ([6]). *If a dimer model is non-degenerate, then $\mathscr{M}_\theta$ is a crepant resolution of the three-dimensional affine toric variety associated with the cone over Δ.*

There is a universal representation parameterized by $\mathscr{M}_\theta$, which is given by line bundles corresponding to vertices and their maps corresponding to arrows. We call these line bundles the tautological bundles on $\mathscr{M}_\theta$. We can define the notion of *consistency* for a dimer model in a combinatorial way [5], which is stronger than non-degeneracy. See also [8, 3, 2] for more about consistency conditions on dimer models. A *tilting bundle* on a smooth variety X is a vector bundle $\mathscr{E}$ such that $D^b(\operatorname{coh} X)$ is equivalent to $D^b(\operatorname{mod} \operatorname{End}(\mathscr{E}))$.

Theorem 2 ([5]). *If a dimer model is consistent, then the direct sum of the tautological bundles is a tilting bundle. The endomorphism algebra of the direct sum is isomorphic to the path algebra of the quiver with relations.*

This implies that $\mathscr{M}_\theta$ is derived equivalent to the path algebra of the quiver with relations. The following shows that consistency is not a too strong condition:

Theorem 3 ([5]). *For any lattice polygon Δ, there is a consistent dimer model corresponding to it.*

To obtain these results, we use the 'special McKay correspondence' for 2-dimensional cyclic quotient singularities.

3. Exceptional collections on toric weak Fano stacks

Let X be a smooth variety (or a Deligne-Mumford stack). An object $\mathscr{E} \in D^b(\operatorname{coh} X)$ is *exceptional* if it satisfies

$$\operatorname{Ext}^p(\mathscr{E},\mathscr{E}) = \begin{cases} 0 & p \neq 0 \\ \mathbb{C} & p = 0. \end{cases}$$

A sequence of exceptional objects $(\mathscr{E}_1,\dots,\mathscr{E}_n)$ is an *exceptional collection* if $\operatorname{Ext}^p(E_i,E_j) = 0$ for any p and for any $i > j$. It is *strong* if it further satisfies $\operatorname{Ext}^p(E_i,E_j) = 0$ for any $p \neq 0$ and for any $i \neq j$. An exceptional collection is *full* if it generates $D^b(\operatorname{coh} X)$. If $(\mathscr{E}_1,\dots,\mathscr{E}_n)$ is a full strong exceptional collection, then $\mathscr{E} := \bigoplus_i \mathscr{E}_i$ is a *tilting object* on X, i.e., $D^b(\operatorname{coh} X)$ is equivalent to the bounded derived category of finitely generated modules over $End(\mathscr{E})$.

It is proved by Kawamata that for any projective smooth toric Deligne-Mumford stack X, there is a full exceptional collection consisting of sheaves. King [7] conjectured that every smooth complete toric variety has a full strong exceptional collection consisting of line bundles, which turned out to be false [4]. The conjecture might still be true if we further impose Fano condition.

Let Σ be a stacky fan in $N = \mathbb{Z}^d$, i.e., a rational simplicial fan together with a lattice point for each ray. Σ determines a d-dimensional smooth toric Deligne-Mumford stack $\mathscr{X}_\Sigma$. It is Fano if such lattice points are vertices of a simplicial convex polytope. It is weak Fano (nef-Fano) if they are on the boundary of their convex hull.

Conjecture 1 ([1]). *Every smooth toric weak Fano stack has a full strong exceptional collection consisting of line bundles.*

Borisov and Hua [1] prove that for a smooth toric Fano stack of dimension two or of Picard number at most two, there exists a full strong exceptional collection consisting of line bundles. Dimer models allow us to prove a slightly stronger statement in two-dimensional cases:

Theorem 4. *There is a full strong exceptional collection of line bundles on any two dimensional toric weak Fano stack. We can describe the endomorphism algebra* $\mathrm{End}(\mathscr{E})$ *in terms of quiver with relations by using dimer models.*

The above description of $\mathrm{End}(\mathscr{E})$ should be relevant to describing the homological mirror symmetry for these stacks.

References

[1] Lev Borisov and Zheng Hua. On the conjecture of King for smooth toric Deligne-Mumford stacks. arXiv:0801.2812.
[2] Nathan Broomhead. Dimer models and Calabi-Yau algebras. arXiv:0901.4662.
[3] Ben Davison. Consistency conditions for brane tilings. arXiv:0812.4185.
[4] L. Hille and M. Perling. A counterexample to King's conjecture. *Compos. Math.*, 142(6):1507–1521, 2006.
[5] Akira Ishii and Kazushi Ueda. Dimer models and the special Mckay correspondence. arXiv:0905.0059.
[6] Akira Ishii and Kazushi Ueda. On moduli spaces of quiver representations associated with dimer models. *RIMS Kokyuroku Bessatsu*, (B9):127–142, 2008. arXiv:0710.1898.
[7] Alastair King. Tilting bundles on some rational surfaces. preprint, 1997.
[8] Sergey Mozgovoy and Markus Reineke. On the noncommutative Donaldson-Thomas invariants arising from brane tilings. arXiv:0809.0117.

Deformations of Du Bois singularities and applications

Sándor Kovács
(joint work with János Kollár)

Classification of algebraic varieties is one of the most fundamental questions in algebraic geometry. It is far from being completed, but we have a relatively detailed plan on how to proceed.

First, one obtains a canonical model in order to find a natural polarization on a birational model of the given variety. This is usually done via the Minimal Model Program, first producing a minimal model and then its canonical model using base point freeness. There have been spectacular advances in this theory recently, the main example being [1]. There are other approaches to constructing the canonical model, e.g., [3, 4], but in any case the purpose of this note is to discuss something else, so I will leave it to the reader to explore the details.

Once the canonical model is found, it may be embedded into a projective space via some power of the canonical bundle. The necessary power only depends on the Hilbert polynomial due to Matsusaka's Big Theorem [5, 6, 7] (for smooth canonical models). Then the moduli space is constructed by taking a quotient of an appropriate subscheme of the corresponding Hilbert scheme.

One may consider the produced moduli space and the procedure to determine the moduli point of any given variety the answer to the classification problem. The moduli space is, in some sense, a (non-discrete) "list" of preferred models in a class of varieties one aims to classify.

A major technical difficulty arises from the fact that the canonical model of a smooth projective variety is usually singular. But even if one restricted to the study of smooth models, a meaningful theory should include information on degenerations. In other words, one would like to obtain a *compact* moduli space. In this case there is no way out; one must work with singular spaces.

Fortunately, the singularities that are necessary to consider can be controlled and stay relatively "mild". Nevertheless, it makes the treatment technical and to some extent perhaps even threatening for a newcomer.

The main purpose of this talk was to discuss some of the singularities that occur in this program, their relationships and significance. One of the main applications discussed was the following joint result with János Kollár proved in [2]:

Theorem 1. *Let $\phi : X \to B$ be a flat projective morphism such that all fibers are log canonical. Then the cohomology sheaves $h^i(\omega_\phi^\bullet)$ are flat over B, where $\omega_\phi^\bullet$ denotes the relative dualizing complex of ϕ.*

Corollary 1. *Under the same hypothesis, assume that one of the fibers of ϕ is Cohen-Macaulay. Then so are all the fibers.*

References

[1] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan: *Existence of minimal models for varieties of log general type*, preprint, 2006. arXiv:math.AG/0610203

[2] J. Kollár and S. J. Kovács: *Log canonical singularities are Du Bois*, preprint, 2009. arXiv:0902.0648v2 [math.AG]

[3] V. Lazić: *Towards finite generation of the canonical ring without the MMP*, 2008. arXiv:0812.3046v1 [math.AG]

[4] V. Lazić: *Adjoint rings are finitely generated*, 2008. arXiv:0905.2707v2 [math.AG]

[5] T. Matsusaka: *On canonically polarized varieties*, Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), Oxford Univ. Press, London, 1969, pp. 265–306. MR0263815 (41 #8415a)

[6] T. Matsusaka: *On canonically polarized varieties. II*, Amer. J. Math. **92** (1970), 283–292. MR0263816 (41 #8415b)

[7] T. Matsusaka: *On polarized normal varieties. I*, Nagoya Math. J. **104** (1986), 175–211. MR868444 (88e:14011)

BGG correspondence and the cohomology of compact Kähler manifolds

Robert Lazarsfeld

(joint work with Mihnea Popa)

Let X be a compact Kähler manifold of dimension d, and consider the cohomology

$$Q_X = H^*\big(X, \omega_X\big) = \oplus H^i\big(X, \omega_X\big)$$

of the canonical bundle of X. Via cup product, we may view this as a graded module over the exterior algebra $E = \Lambda^* H^1\big(X, \mathscr{O}_X\big)$ on $H^1\big(X, \mathscr{O}_X\big)$. In recent years, there has been considerable interest in the study of graded modules over the exterior algebra of a vector space, and the so-called Bernstein-Gel'fand-Gel'fand (BGG) correspondence between these and linear complexes over a symmetric algebra ([1],[2]). I discussed some results from [6] showing that a body of work involving generic vanishing theorems ([3], [4], [5],[7], [8]) implies that this picture takes a particularly clean form in the case of Q_X, and that it allows one to deduce some surprising connections between the algebraic properties of this module and the geometry of X. Furthermore, a vector bundle arising from the BGG correspondence to establish, under mild additional hypotheses, a number of inequalities on Hodge numbers and the holomorphic Euler characteristic of X.

Turning to details, we grade E and Q_X by declaring that E is generated in degree -1, and that $H^i\big(X, \omega_X\big)$ has degree $= -i$. We say that Q_X is k-regular if it is generated in degrees $0, \ldots, -k$, and the p^{th} syzygies among the generators appears in degrees $-p, \ldots, -p-k$.

Our first main result asserts that the regularity of Q_X is governed by the generic fibre dimension of the Albanese mapping $a_X : X \longrightarrow \mathrm{Alb}(X)$ of X.

Theorem 1. *Set*

$$k = k(X) = \dim X - \dim a_X(X).$$

Then Q_X is k-regular as an E-module. In particular, if X has maximal Albanese dimension (i.e. $k = 0$), then Q_X is generated in degree 0 *and has a linear free resolution.*

It is natural to ask what additional geometric data Q_X determines. Of course the dimensions of its graded pieces are the Hodge numbers $h^{d,i}(X)$ of X, but the module in question turns out to contain also more subtle information. Recall that in classical terminology, a *paracanonical divisor* on X is an effective divisor algebraically equivalent to a canonical divisor. The set of all such is parametrized by the Hilbert scheme (or Douady space) $\mathrm{Div}^{\{\omega\}}(X)$, which admits an Abel-Jacobi mapping

$$u : \mathrm{Div}^{\{\omega\}}(X) \longrightarrow \mathrm{Pic}^{\{\omega\}}(X)$$

to the corresponding component of the Picard torus of X. The projective space $|\,\omega_X\,|$ parametrizing all canonical divisors sits as a subvariety of $\mathrm{Div}^{\{\omega\}}(X)$: it is the fibre of u over the point $[\,\omega_X\,] \in \mathrm{Pic}^{\{\omega\}}(X)$. Our second result asserts that Q_X dictates the infinitesimal geometry of $\mathrm{Div}^{\{\omega\}}(X)$ along $|\,\omega_X\,|$.

Theorem 2. *One can read off from Q_X (and its structure as an E-module) the projectivized normal cone to $|\,\omega_X\,|$ inside* $\mathrm{Div}^{\{\omega\}}(X)$.

In fact, Q_X determines via the BGG-correspondence a coherent sheaf $\mathscr{F}$ on the projectivized tangent space $\mathbf{P}$ to $\mathrm{Pic}^{\{\omega\}}(X)$ at $[\,\omega_X\,]$, and we show that the normal cone in question is identified with $\mathbf{P}(F)$.

Under additional hypotheses the BGG sheaf F is locally free. We discussed finally how one could then use the geometry of vector bundles on projective space to obtain some inequalities on certain numerical invariants of X.

References

[1] D. Eisenbud, *The geometry of syzygies*, Springer 2005.

[2] D. Eisenbud, G. Fløystad and F.-O. Schreyer, Sheaf cohomology and free resolutions over the exterior algebra, Trans. Amer. Math. Soc. **355** (2003), no. 11, 4397–4426.

[3] M. Green and R. Lazarsfeld, Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville, Invent. Math. **90** (1987), 389–407.

[4] M. Green and R. Lazarsfeld, Higher obstructions to deforming cohomology groups of line bundles, J. Amer. Math. Soc. **1** (1991), no.4, 87–103.

[5] Ch. Hacon, A derived category approach to generic vanishing, J. Reine Angew. Math. **575** (2004), 173–187.

[6] R. Lazarsfeld and M. Popa, BGG correspondence, cohomology of compact Kähler manifolds, and numerical invariants, to appear.

[7] G. Pareschi and M. Popa, *GV*-sheaves, Fourier-Mukai transform, and Generic Vanishing, preprint math/0608127.

[8] G. Pareschi and M. Popa, Strong Generic Vanishing and a higher dimensional Castelnuovo-de Franchis inequality, preprint arXiv:0808.2444, to appear in Duke Math. J.

Degeneration of Hilbert schemes of ideal sheaves and a proof of Goettsche conjecture

Jun Li

Degeneration is a powerful tool in algebraic geometry; equally so in the study of moduli spaces. In this talk, I outlined the recent construction of ideal degenerations of Hilbert scheme of ideal sheaves by Wu and the author [8, 5], and its application to prove Geottsche's conjecture on enumerating nodal curves on surfaces by Tzeng [7].

Let $X \to C$ be a simple degeneration of smooth projective varieties, meaning that X is smooth; π is projective and is smooth over $C^* = C - 0$; the central fiber X_0 has normal crossing singularity with smooth singular loci $D \subset X_0$. We fix a polynomial $p(\cdot)$, and denote $X^* = X \times_C C^*$. Our goal is to extend the usual relative Hilbert scheme $\mathrm{Hilb}^{p(\cdot)}_{X^*/C^*}$ to an *ideal degeneration* $\mathfrak{Hilb}^{p(\cdot)}_{X/C}$.

Here is the *principle* of an *ideal degeneration*:

(1) the family $\mathfrak{Hilb}^{p(\cdot)}_{X/C}$ is an extension of $\mathrm{Hilb}^{p(\cdot)}_{X^*/C^*}$, and is proper over C; its total space has the same complexity as $\mathrm{Hilb}^{p(\cdot)}_{X_t}$ for $t \neq 0$;

(2) the central fiber $\mathfrak{Hilb}^{p(\cdot)}_{X_0}$ of $\mathfrak{Hilb}^{p(\cdot)}_{X/C}$ has the complexity that is a combination of normal crossing singularity and the complexity of $\mathrm{Hilb}^{p(\cdot)}_{X_t}$;

(3) let $\tilde{X}_0$ be the normalization of X_0 and $\tilde{D}_0 \subset \tilde{X}_0$ be the preimage of the singular loci of X_0. Then there is a similarly constructed relative Hilbert scheme $\mathfrak{Hilb}^{q(\cdot)}_{(\tilde{X},\tilde{D})}$ such that $\mathfrak{Hilb}^{p(\cdot)}_{X_0} = \cup_{q(\cdot)} \mathfrak{Hilb}^{q(\cdot)}_{(\tilde{X}_0,\tilde{D})}/\text{gluing}$, where the sum is the irreducible components decomposition modulo the complexity of $\mathrm{Hilb}^{p(\cdot)}_{X_t}$.

Some further clarifications are in order. Since in general Hilbert schemes of ideal sheaves are singular, it is difficult to phrase in general how ideal degenerations should satisfy. In the writing above, I used a vague "complexity of $\mathrm{Hilb}^{p(\cdot)}_{X_t}$" indicate the general behavior of $\mathrm{Hilb}^{p(\cdot)}_{X_t}$. For instance, in case X/C is a family of surfaces and $p(\cdot) = p$ is an integer. Then $\mathrm{Hilb}^{p}_{X_t}$ is smooth. Thus an ideal degeneration should have smooth total space $\mathfrak{Hilb}^{p}_{X/C}$, local complete intersection central fibers and that each irreducible component of $\mathfrak{Hilb}^{p}_{X_0}$ can be reconstructed using the gluing of relative Hilbert schemes $\mathfrak{Hilb}^{q_1,q_2}_{(\tilde{X}_0,\tilde{D})}$, where $p = q_1 + q_2$ are ordered partitions of p.

The main result of Wu and the author [5] is

Theorem 1. *Let X/C be as stated. Then there is an ideal degeneration $\mathfrak{Hilb}^{p(\cdot)}_{X/C}$ extending the family of relative Hilbert schemes $\mathrm{Hilb}^{p(\cdot)}_{X^*/C^*}$. The family $\mathfrak{Hilb}^{p(\cdot)}_{X/C}$ is proper and of finite type over C*

For more precise statement of the theorem, please see [8, 5]. We comment that the theorem is proved for $\dim X/C \le 3$ in [8]. The construction follows the construction of degeneration of relative stable morphisms by the author [4].

It will be useful to describe the closed points of the central fiber $\mathfrak{Hilb}^{p(\cdot)}_{X_0}$. For simplicity, we consider the case X_0 is the union of two smooth varieties Y_1 and Y_2 intersecting transversally along a smooth divisor D. Toward our goal, we form a ruled variety $\Delta = \mathbf{P}(N_{D/Y_1} \oplus 1)$ with two distinguished sections $\sigma_\pm$ so that the degree of the normal bundle to σ_+ (resp. σ_-) is the same as the degree of the normal bundle to $D \subset Y_1$ (resp. $D \subset Y_2$). We form $X[k]_0$ that is the gluing of Y_1 with Δ along $D \cong \sigma_-$, gluing the σ_+ of this Δ to the σ_- of a new (the second) Δ, repeating this procedure, and lastly gluing the σ_+ of the k-th Δ to $D \subset Y_2$.

Closed points of $\mathfrak{Hilb}^{p(\cdot)}_{X_0}$ are subschemes $z \subset X[k]_0$ for some k that satisfy the following three conditions. The first is that z is flat along the normal directions of the singular loci of $X[k]_0$ in $X[k]_0$; the second is that the polynomial $\chi(\mathscr{O}_z \otimes \pi^* H^n) = p(n)$, where $\pi : X[k]_0 \to X_0$ is the projection and H is the ample line bundle on X/C. Given two such subschemes $z \subset X[k]_0$ and $z' \subset X[k']_0$, they are equivalent if $k = k'$ and there is an isomorphism $\xi : X[k]_0 \cong X[k']_0$ that commutes with $X[k]_0$ and $X[k']_0 \to X_0$ so that $\xi(z) = z'$. The equivalence relations of z with itself defines the automorphism group of z. The third condition is that z has finite automorphism group.

We remark that requiring $z \subset X[k]_0$ flat along the normal directions of the singular loci of $X[k]_0$ guarantees that the complexity of $\mathfrak{Hilb}^{p(\cdot)}_{X/C}$ is similar to that of $\mathrm{Hilb}^{p(\cdot)}_{X_t}$. The freedom to add arbitrarily many Δ's makes $\mathfrak{Hilb}^{p(\cdot)}_{X/C}$ proper over C. Finally, the requirement that the automorphism group of z is finite ensures that for the fixed $p(\cdot)$, the number of Δ's appearing in this construction is universally bounded, which implies that $\mathfrak{Hilb}^{p(\cdot)}_{X/C}$ if of finite type over C.

This construction by adding ruled varieties was used earlier by Gieseker [1] in his degeneration of moduli of vector bundles, by Harris-Mumford [2] in their admissible cover construction, and in gauge theory using degenerations.

Applying this degeneration theory, Tzeng [7] provided an algebro-geometric proof of a folklore conjecture on enumerating nodal curves in a linear series of a surface. (cf. for symplectic approach see [6].)

Theorem 2. *For every integer r, there exist universal polynomials $T_r(x, y, z, t)$ of degree r with the following property: given r and a pair of a sufficiently ample line bundle L on a smooth surface S, a general r-dimensional sublinear system in $|L|$ contains exactly $T_r(L^2, LK, c_1(S)^2, c_2(S))$ many r-nodal curves.*

Based on this, she proved the Goettsche-Yau-Zaslow formula, which was motivated by the work of Yau-Zaslow on enumerating rational curves on K3 surfaces and originally conjectured by Goettsche.

Theorem 3.

$$\sum_{r\in Z} T_r(L^2, LK_S, c_1(S)^2, c_2(S))(DG_2(\tau)^r = \frac{(DG_2(\tau)/q)^{\chi(L)} B_1(q)^{K_S^2} B_1(q)^{LK_S}}{(\Delta(\tau)D^2G_2(\tau)/q^2)^{\chi(\mathscr{O}_S)/2}}.$$

Here $D = q\frac{d}{dq}$, G_2 is the weight 2 quasi-modular form and Δ is the discriminant.

Tzeng proved theorem B by first worked out the cobordism theory of pairs (S, L), following the algebraic cobordism theory of Levine-Pandharipande [3]. She then defined a homomorphism from $\mathbb{Z}\{(S, L)\}$ to $\mathbb{Q}[\![t]\!]^\times$ that sends (S, L) to the generating function $\gamma_S^L(t) = \sum_{r\geq 1} d_r(S, L)t^r$, where $d_r(S, L)$ is an integration over the Hilbert scheme of points on S that enumerating r-nodal curves in $|L|$ when L is sufficiently ample. Applying degeneration of Hilbert schemes constructed in Theorem A, she proved that this homomorphism factors through a quotient

$$\mathbb{Z}\{(S, L)\} \longrightarrow \mathbb{Q}^{\oplus 4},$$

where $\mathbb{Q}^{\oplus 4}$ is spanned by L^2, LK_S, $c_1(S)^2$ and $c_2(S)$. This implies the Theorem B. Theorem C follows.

References

[1] D. Gieseker, *A degeneration of the moduli space of stable bundles*, J. Differential Geom. **19** (1984), no. 1, 173–206.
[2] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67** (1982), no. 1, 23–88.
[3] M. Levine and R. Pandharipande, *Algebraic cobordism revisited*, Invent. Math. **176** (2009), no. 1, 63–130.
[4] J. Li, *A degeneration formula of GW-invariants*, J. Differential Geom. **60** (2002), no. 2, 199–293.
[5] J. Li and B.S. Wu, *Degeneration of Hilbert scheme of ideal sheaves*, in preparation.
[6] A.K Liu, *Family blowup formula, admissible graphs and the enumeration of singular curves. I*, J. Differential Geom. **56** (2000), no. 3, 381–579.
[7] Y.R. Tseng, *A proof of Goettsche-Yau-Zaslow formula*, in preparation.
[8] B.S. Wu, *The moduli stack of stable relative ideal sheaves*, math.AG/0701074.

Numerically reflective involutions of Enriques surfaces

Shigeru Mukai

A (holomorphic) automorphism of an Enriques surface S is *numerically reflective* (resp. *numerically trivial*) if it acts on the $\mathbb{Q}$-cohomology group $H^2(S, \mathbb{Q})(\simeq \mathbb{Q}^{10})$ by reflection (resp. trivially). For K3 surfaces we have

- a numerically trivial automorphism is trivial, and
- no automorphisms are numerically reflective.

But these are no more true for Enriques surfaces. In my talk I summarized the classification and gave a very rough sketch of the proof. The details will be published elsewhere.

1. Numerically trivial involutions

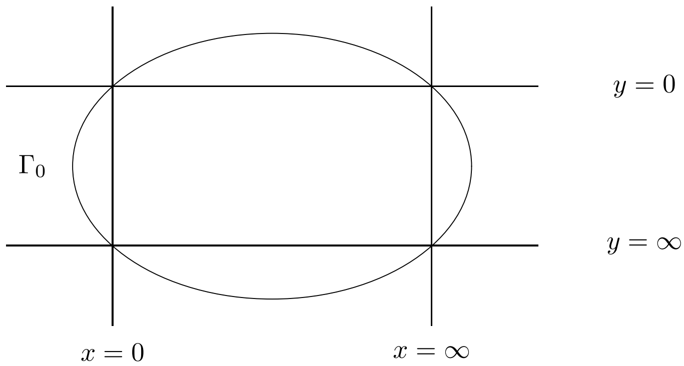
Let X_{BP} be the minimal model of the function field

$$\mathbb{C}\left(x, y, \sqrt{a(x + \frac{1}{x}) + b(y + \frac{1}{y}) + 2c}\right) \tag{1}$$

of two variables, where $a, b \in \mathbb{C}^\times$ and $c \in \mathbb{C}$ are constants. X_{BP} is the minimal resolution of the double $\mathbb{P}^1 \times \mathbb{P}^1$ with branch the union of the coordinate quadrilateral and the curve

$$\Gamma_0 : a(x^2 + 1)y + bx(y^2 + 1) + 2cxy = 0 \tag{2}$$

of bidegree $(2, 2)$.



Assume further that $a \pm b \pm c \neq 0$. Then the involution $\varepsilon : (x, y, \sqrt{\ }) \mapsto (1/x, 1/y, -\sqrt{\ })$ has no fixed points on X_{BP}. Hence the quotient $S_{BP} = X_{BP}/\varepsilon$ is an Enriques surface. Let σ_{BP} be the involution of S_{BP} induced from the covering involution $\sqrt{\ } \mapsto -\sqrt{\ }$ of X_{BP}. Then σ_{BP} is *homologically trivial*, that is, its acts on the $\mathbb{Z}$-homology group $H_2(S_{BP}, \mathbb{Z})$ trivially ([1, (4.8)], [4, Exmaple 2]).

Theorem 1 *Every homologically trivial automorphism of an Enriques surface is either trivial or the above involution* $\sigma_{BP} \curvearrowright S_{BP}$.

Theorem 2 ([2]) *Let* σ *be a numerically trivial involution of an Enriques surface, and assume that* σ *is neither trivial nor* σ_{BP}. *Then the universal cover is a Kummer surface* $Km(E_1 \times E_2)$ *of product type and* σ *is either of Liberman type (*[4, Exmaple 1]*) or Kondo-Mukai type (*[4, Exmaple 2]*, see also §2).*

2. Numerically reflective involutions

Let X_{GBP} be the minimal model of the field

$$\mathbb{C}\left(x, y, \sqrt{a(x + \frac{1}{x}) + b(y + \frac{1}{y}) + c(\frac{x}{y} + \frac{y}{x}) + 2d}\right), \tag{3}$$

where $a, b, c \in \mathbb{C}^\times$ and $d \in \mathbb{C}$ are constants. X_{GBP} is the minimal resolution of the double $\mathbb{P}^2$ with branch the union of the coordinate triangle and the cubic curve

$$\Gamma_1 : a(x^2 + 1)y + bx(y^2 + 1) + c(x^2 + y^2) + 2dxyz = 0. \tag{4}$$

Assume further that

$$(a + b + c + d)(a + b - c - d)(a - b + c - d)(a - b - c + d) \neq 0. \tag{5}$$

Then the involution $\varepsilon : (x, y, \sqrt{\ }) \mapsto (1/x, 1/y, -\sqrt{\ })$ has no fixed points and we obtain the Enriques quotient $S_{GBP} := X_{GBP}/\varepsilon$. The involution σ_{GBP} of S_{GBP} induced from the covering involution is numerically reflective if (4) is irreducible and numerically trivial otherwise. By [2, Remark 9], σ_{GBP} is equivalent to [2, Example 2] in the latter case.

Let C be a curve of genus 2 and G a Göpel subgroup of the 2-torsion group of its Jacobian $J(C)$. For a non-bielliptic pair (C, G), we constructed an Enriques surface $Km(C)/\varepsilon_G$ and a numerically reflective involution $\sigma_G \curvearrowright Km(C)/\varepsilon_G$ in [3], where $Km(C)$ is the Kummer surface of $J(C)$.

Theorem 3 *Let* σ *be a numerically reflective involution of an Enriques surface* S. *Then either*

(1) σ *is isomorphic to the involution* σ_{GBP}*, or*
(2) the universal cover of S *is isomorphic to the Jacobian Kummer surface* $Km(C)$ *and* (S, σ) *is isomorphic to* $(Km(C)/\varepsilon_G, \sigma_G)$ *for a curve* C *of genus 2 and a Göpel subgroup* G.

References

[1] Barth, W. and Peters, C.: Automorphism of Enriques surfaces, Invent. math., **73**(1983), 383–411.

[2] Mukai, S.: Numerically trivial involutions of Enriques surfaces, RIMS preprint #1544, May 2006.

[3] Mukai, S.: Kummer's quartics and numerically reflective involutions of Enriques surfaces, RIMS preprint #1633, June 2008 (http://www.kurims.kyoto-u.ac.jp/preprint/file/RIMS1633.pdf).

[4] Mukai, S. and Namikawa, Y.: Automorphisms of Enriques surfaces which act trivially on the cohomology groups, Invent. math., **77**(1984), 383–397.

Asymptotic invariants of graded sequences of ideals

Mircea Mustaţă

(joint work with Mattias Jonsson, Robert Lazarsfeld)

We study asymptotic versions of invariants of singularities such as the log canonical threshold, or more generally, the jumping numbers for the multiplier ideals. Let X be a fixed smooth variety over an algebraically closed field of characteristic zero. Recall that if $\mathfrak{a}$ is a nonzero ideal on X, then one defines the *log canonical threshold* $\mathrm{lct}(\mathfrak{a})$ of $\mathfrak{a}$ in terms of a log resolution of singularities of the pair $(X, \mathfrak{a})$. In terms of the multiplier ideals $\mathscr{J}(\mathfrak{a}^{\lambda})$ of $\mathfrak{a}$, this can be described as the smallest λ such that $\mathscr{J}(\mathfrak{a}^{\lambda}) \neq \mathscr{O}_X$ (with the convention that $\mathrm{lct}(\mathfrak{a}) = \infty$ if $\mathfrak{a} = \mathscr{O}_X$). More generally, if $\mathfrak{q}$ is an auxiliary ideal, then $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a})$ is the smallest λ such that $\mathfrak{q} \not\subseteq \mathscr{J}(\mathfrak{a}^{\lambda})$. If we let $\mathfrak{q}$ vary, then the numbers obtained in this way are precisely the jumping numbers for the multiplier ideals of $\mathfrak{a}$.

Given a prime divisor E over X, we have an associated valuation ord_E of the function field of X. The log discrepancy $A(\mathrm{val}_E)$ of this valuation is the coefficient of E in $K_{Y/X}$, plus one (here Y is a model such that E is a divisor on Y). It follows from definition that

$$\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}) = \min_E \frac{A(\mathrm{ord}_E) + \mathrm{ord}_E(\mathfrak{q})}{\mathrm{ord}_E(\mathfrak{a})}.$$

In fact, the minimum is achieved by some divisor E on a log resolution of $\mathfrak{a} \cdot \mathfrak{q}$.

Suppose now that $\mathfrak{a}_\bullet = (\mathfrak{a}_p)_p$ is a graded sequence of ideals on X, that is, $\mathfrak{a}_p \cdot \mathfrak{a}_q \subseteq \mathfrak{a}_{p+q}$ for all $p, q > 0$. The main geometric example is the following: X is projective, and $\mathfrak{a}_m$ is the ideal defining the base locus of L^m, where $L \in \mathrm{Pic}(X)$ is a line bundle with $h^0(L) \geq 1$. It is easy to see that if ν is a valuation of the function field of X, then

$$\nu(\mathfrak{a}_\bullet) := \inf_m \frac{\nu(\mathfrak{a}_m)}{m} = \lim_{m\to\infty} \frac{\nu(\mathfrak{a}_m)}{m}.$$

Similarly, if $\mathfrak{q}$ is a fixed ideal, then

$$\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) := \sup_m m \cdot \mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_m) = \lim_{m\to\infty} m \cdot \mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_m).$$

One can show using the asymptotic multiplier ideals discussed below that we have

$$\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \inf_E \frac{A(\mathrm{ord}_E) + \mathrm{ord}_E(\mathfrak{q})}{\mathrm{ord}_E(\mathfrak{a}_\bullet)}.$$

However, it is easy to see that in this setting the above infimum might not be achieved. It turns out that the right notion in this setting is that of *quasi-monomial valuation* (also called *Abhyankar valuation*) of the function field of X. One can check that we may alternatively take the above infimum over the quasi-monomial valuations. In this work in progress, our main interest is in the following
Conjecture. Given $\mathfrak{a}_\bullet$ and $\mathfrak{q}$ as above, there is a quasi-monomial valuation ν of the function field of X such that

$$\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \frac{A(\nu) + \nu(\mathfrak{q})}{\nu(\mathfrak{a}_\bullet)}.$$

The conjecture follows when $\dim(X) = 2$ from the work of Favre and Jonsson [3].

A key tool in the study of graded sequences is given by the asymptotic multiplier ideals of $\mathfrak{a}_\bullet$: one puts

$$\mathfrak{b}_m = \mathscr{J}(\mathfrak{a}_\bullet^m) := \mathscr{J}(\mathfrak{a}_{mp}^{1/p}),$$

for p divisible enough. These ideals satisfy the following subadditivity property, due to Demailly, Ein and Lazarsfeld:

$$\mathfrak{b}_{mp} \subseteq \mathfrak{b}_m^p \text{ for all } m, p > 0. \tag{1}$$

As in the case of graded sequences, one can show that if ν is a valuation of the function field of X, then

$$\nu(\mathfrak{b}_\bullet) := \sup_m \frac{\nu(\mathfrak{b}_m)}{m} = \lim_{m\to\infty} \frac{\nu(\mathfrak{b}_m)}{m}.$$

One can easily deduce from the definition of multiplier ideals that for every quasi-monomial valuation ν and every m, the following holds

$$\nu(\mathfrak{a}_\bullet) - \frac{A(\nu)}{m} < \frac{\nu(\mathfrak{b}_m)}{m}. \tag{2}$$

This immediately implies that $\nu(\mathfrak{a}_\bullet) = \nu(\mathfrak{b}_\bullet)$. Furthermore, one deduces that if

$$\mathrm{lct}^{\mathfrak{q}}(\mathfrak{b}_\bullet) := \inf_m m \cdot \mathrm{lct}^{\mathfrak{q}}(\mathfrak{b}_m),$$

then $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{b}_\bullet) = \mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$.
Given a sequence of ideals $\mathfrak{b}_\bullet$ satisfying (1) and (2), and such that $\mathfrak{b}_p \subseteq \mathfrak{b}_q$ for $p > q$, one can make an analogue of the above conjecture for $\mathfrak{b}_\bullet$. This version (for ideals in $\mathbf{C}[\![x_1, \ldots, x_n]\!]$) would give a positive answer to a conjecture of Demailly and Kollár from [2] on integrability exponents of plurisubharmonic functions.
At this point, we can reduce the above conjecture to the case when the graded sequence of ideals $\mathfrak{a}_\bullet$ satisfies

$$\mathfrak{m}^{pN} \subseteq \mathfrak{a}_p \text{ for some } N \geq 1, \text{ and for all } p > 0,$$

where $\mathfrak{m}$ is the ideal of a point $x \in X$. The advantage once we are in this setting is that one can use the valuation space from [1] to construct a valuation ν of the function field of X centered at x, and having the property that

$$\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \frac{A(\nu) + \nu(\mathfrak{q})}{\nu(\mathfrak{a}_\bullet)}.$$

A basic question is whether any such ν would automatically be quasi-monomial (this is the case in dimension two, by the work of Favre and Jonsson). One can rephrase this as follows:

Question. Let ν be a valuation of the function field of X centered at x, such that for every other such valuation μ (that can be taken quasi-monomial) with the property that $\nu(f) \leq \mu(f)$ for all $f \in \widehat{\mathscr{O}_{X,x}}$, we have

$$A(\nu) + \nu(\mathfrak{q}) \leq A(\mu) + \mu(\mathfrak{q}).$$

Does it follow that ν is a quasi-monomial valuation ?

References

[1] S. Boucksom, C. Favre, and M. Jonsson, Valuations and plurisubharmonic singularities, Publ. Res. Inst. Math. Sci. 44 (2008), 449–494.

[2] J.-P. Demailly and J. Kollár, Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds, Ann. Sci. École Norm. Sup. (4) 34 (2001), 525–556.

[3] C. Favre and M. Jonsson, Valuations and multiplier ideals, J. Amer. Math. Soc. 18 (2005), 655–684.

General heart construction on a triangulated category

Hiroyuki Nakaoka

As shown in [4] (or [3], in 2-CY case), the quotient of a triangulated category by a cluster tilting subcategory becomes an abelian category. On the other hand, as is well known since 1980s, the heart of any t-structure is abelian. To unify these two constructions, we construct an abelian category from any torsion pair without shift condition.

We fix a triangulated category $\mathscr{C}$. Throughout this note, $(\mathscr{U}, \mathscr{V})$ denotes a pair of full additive thick subcategories of $\mathscr{C}$, satisfying the following conditions. These are saying that $(\mathscr{U}, \mathscr{V}[1])$ is a torsion pair without the shift-closedness ([2]).

(1) $\mathrm{Ext}^1(\mathscr{U}, \mathscr{V}) = 0$.

(2) For any $C \in \mathrm{Ob}(\mathscr{C})$, there exists a (not necessarily unique) distinguished triangle

$$U \to C \to V[1] \to U[1]$$

satisfying $U \in \mathrm{Ob}(\mathscr{U}), V \in \mathrm{Ob}(\mathscr{V})$.

Two extremal cases of such pairs are t-structures and cluster tilting subcategories:

(t) $\mathscr{V} \subseteq \mathscr{V}[1]$ if and only if $(\mathscr{U}, \mathscr{V}) = (\mathscr{T}^{\leq -1}, \mathscr{T}^{\geq 1})$ for a t-structure $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$.

(c) $\mathscr{V} = \mathscr{U}$ if and only if $(\mathscr{U}, \mathscr{V}) = (\mathscr{T}, \mathscr{T})$ for a cluster tilting subcategory $\mathscr{T}$.

Also, there are examples of such pairs arising from n-cluster tilting subcategories. By definition ([3]), an *n-cluster tilting subcategory* $\mathscr{T} \subseteq \mathscr{C}$ is a full additive thick subcategory satisfying

- $\mathscr{T}$ is functorially finite,
- An object C belongs to $\mathscr{T}$ if and only if $\mathrm{Ext}^{\ell}(C, \mathscr{T}) = 0 \ \ (0 < {}^{\forall}\ell < n)$,
- An object C belongs to $\mathscr{T}$ if and only if $\mathrm{Ext}^{\ell}(\mathscr{T}, C) = 0 \ \ (0 < {}^{\forall}\ell < n)$.

For any $(n+1)$-cluster tilting subcategory $\mathscr{T}$ and for any integer ϖ satisfying $0 \leq \varpi \leq n-1$, define full additive subcategories $\mathscr{T}_{\varpi}$ and $\mathscr{T}^{\varpi}$ of $\mathscr{C}$ by

$$\begin{aligned} \mathrm{Ob}(\mathscr{T}_{\varpi}) &= \{C \in \mathrm{Ob}(\mathscr{C}) \mid \mathrm{Ext}^{\ell}(C, \mathscr{T}) = 0 \ \ (\varpi + 1 \leq {}^{\forall}\ell \leq n)\}, \\ \mathrm{Ob}(\mathscr{T}^{\varpi}) &= \{C \in \mathrm{Ob}(\mathscr{C}) \mid \mathrm{Ext}^{\ell}(\mathscr{T}, C) = 0 \ \ (1 - \varpi \leq {}^{\forall}\ell \leq 1)\}. \end{aligned}$$

Then it can be shown that the pair $(\mathscr{T}_{\varpi}, \mathscr{T}^{\varpi})$ satisfies the conditions (1) and (2) (cf. Theorem 3.1 in [2]).

As a main theorem, we associate an abelian category $\underline{\mathscr{H}}$ to any pair $(\mathscr{U}, \mathscr{V})$. We call $\underline{\mathscr{H}}$ *the heart* of $(\mathscr{U}, \mathscr{V})$. First, we define full subcategories $\mathscr{C}^{\pm}$ of $\mathscr{C}$ as follows. Put $\mathscr{W} := \mathscr{U} \cap \mathscr{V}$.

(+) $C \in \mathrm{Ob}(\mathscr{C}^{+})$ if and only if any distinguished triangle

$$U \to C \to V[1] \to U[1] \quad (U \in \mathrm{Ob}(\mathscr{U}), V \in \mathrm{Ob}(\mathscr{V}))$$

satisfies $U \in \mathrm{Ob}(\mathscr{W})$.

(−) $C \in \mathrm{Ob}(\mathscr{C}^{-})$ if and only if any distinguished triangle

$$V[-1] \to U[-1] \to C \to V \quad (U \in \mathrm{Ob}(\mathscr{U}), V \in \mathrm{Ob}(\mathscr{V}))$$

satisfies $V \in \mathrm{Ob}(\mathscr{W})$.

Put $\mathscr{H} := \mathscr{C}^{+} \cap \mathscr{C}^{-}$. Since $\mathscr{H} \supseteq \mathscr{W}$, we have an additive category

$$\underline{\mathscr{H}} := \mathscr{H}/\mathscr{W}.$$

In the case of t-structures and cluster tilting subcategories, this gives back the following abelian categories:

(t) If $(\mathscr{U}, \mathscr{V}) = (\mathscr{T}^{\leq -1}, \mathscr{T}^{\geq 1})$ where $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ is a t-structure, then we have

$$\begin{aligned} \mathscr{C}^{-} &= \mathscr{U}[-1] = \mathscr{T}^{\leq 0}, \\ \mathscr{C}^{+} &= \mathscr{V}[1] = \mathscr{T}^{\geq 0}, \\ \underline{\mathscr{H}} &= \mathscr{H} = \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq 0}. \end{aligned}$$

Thus the definition of the heart agrees with that of t-structure $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$, and thus it is abelian (cf. [1]).

(c) If $\mathscr{U} = \mathscr{V} = \mathscr{T}$ is a cluster tilting subcategory of $\mathscr{C}$, then we have

$$\begin{aligned} \mathscr{C}^{+} = \mathscr{C}^{-} = \mathscr{H} &= \mathscr{C}, \\ \underline{\mathscr{H}} &= \mathscr{C}/\mathscr{T}. \end{aligned}$$

Thus $\underline{\mathscr{H}}$ becomes an abelian category also in this case by [4].

Generalizing these two cases, we can show the following by a purely diagrammatic argument.

Theorem 1. *For each pair* $(\mathscr{U}, \mathscr{V})$*, its heart* $\underline{\mathscr{H}}$ *is an abelian category.*

Additionally, as for the existence of enough projectives or injectives in this abelian category, we can show:

(p) If $(\mathscr{U}, \mathscr{V})$ satisfies $\mathscr{U} \subseteq \mathscr{V}$, then $\underline{\mathscr{H}}$ has enough projectives.

(i) If $(\mathscr{U}, \mathscr{V})$ satisfies $\mathscr{V} \subseteq \mathscr{U}$, then $\underline{\mathscr{H}}$ has enough injectives.

This generalizes 1. and 2. of Theorem 4.3 in [4].

References

[1] Beĭlinson, A. A.; Bernstein, J.; Deligne, P.: *Faisceaux pervers* (French) [Perverse sheaves] Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque, **100**, Soc. Math. France, Paris, 1982.

[2] Iyama, O.; Yoshino, Y.: *Mutation in triangulated categories and rigid Cohen-Macaulay modules* (English summary), Invent. Math. **172** (2008), no. 1, 117–168.

[3] Keller, B.; Reiten, I.: *Cluster-tilted algebras are Gorenstein and stably Calabi-Yau*, Adv. Math. **211** (2007), no. 1, 123–151.

[4] Koenig, S.; Zhu, B.: *From triangulated categories to abelian categories: cluster tilting in a general framework* (English summary), Math. Z. **258** (2008), no. 1, 143–160.

[5] Nakaoka, H.: *General heart construction on triangulated categories (I): Unifying t-structures and cluster tilting subcategories*, arXiv:0907.2080.

Poisson deformations of symplectic varieties

Yoshinori Namikawa

A symplectic variety (X, ω) is a pair of a normal algebraic variety X and a holomorphic symplectic 2-form ω on the regular part X_{reg} of X such that ω extends to a (not necessarily non-degenerate) holomorphic 2-form on a resolution $\tilde{X}$ of X. Then X_{reg} admits a natural Poisson structure induced by ω. By the normality of X, this Poisson structure uniquely extends to a Poisson structure on X. In this lecture, I talked on the Poisson deformation of $(X, \{\ ,\ \})$ obtained from a symplectic variety (X, ω). One can define the Poisson deformation functor PD_X from the category of local Artin **C**-algebras with residue field **C** to the category of sets. The first main theorem is:

Theorem 1. *Let* (X, ω) *be an affine symplectic variety. Then* PD_X *is unobstructed.*

Let (X, ω) be the same as in Theorem 1. By Birkar, Cascini, Hacon and McKernan, one can take a **Q**-factorial terminalization $\pi : Y \to X$. By definition, Y has only **Q**-factorial terminal singularities and π is a birational, crepant, projective morphism. The symplectic 2-form ω is pulled-back to a symplectic 2-form on $\pi^{-1}(X_{reg})$. Note that $\pi^{-1}(X_{reg})$ is contained in the rular locus Y_{reg} of Y.

Since π is semi-small, $\pi^*(\omega)$ further extends to a holomorphic symplectic 2-form ω' on Y_{reg} and (Y, ω') becomes a symplectic variety. Therefore, Y has a Poisson structure, and we get the Poisson deformation functor PD_Y. It is relatively easy to prove that PD_Y is unobstructed. Since X has rational singularities, there is a natural blowing-down map of functors $\pi_*; PD_Y \to PD_X$. The map π_* is a finite Galois covering. Let R and S be the pro-representable hulls of PD_X and PD_Y respectively. Then there are formal universal Poisson deformations $\mathscr{X}_{formal}$ and $\mathscr{Y}_{formal}$ over the base spaces $Spec(R)$ and $Spec(S)$ respectively. The birational map π induces a birational map $\mathscr{Y}_{formal} \to \mathscr{X}_{formal}$. It is not clear at all that these are algebraizable. So, we assume the following condition

(*): X has a $\mathbf{C}^*$-action with positive weights and ω is also positively weighted with respect to the action.

Then everything can be algebraized. As a corollary of this construction, we have the following remarkable result:

Theorem 2. *Under the assumption (*), the following are equivalent:*
(a): X has a crepant resolution.
(b): X has a smoothing by a Poisson deformation.

Salem numbers, Siegel disks and automorphisms

Keiji Oguiso

In the talk, I remarked the following two new phenomena in complex dynamics of automorphisms of compact complex surfaces. These results and their proofs are entirely inspired by impressive works of McMullen [3], [4], [5], [1] and Mathematica programs.

Theorem 1. *There is a pair (S, g) of a complex K3 surface S and its automorphism g such that:*

(1) The topological entropy $h(g)$ is the logarithm of the third smallest known Salem number

$$h(g) = \log 1.200026523...;$$

(2) The fixed point set S^g consists of one smooth rational curve and 8 isolated points, say Q_i $(1 \le i \le 7)$ and Q. The 7 points Q_i are in the union of all the complete curves $\cup_{k=0}^{7} C_k$ on S but Q is not in $\cup_{k=0}^{7} C_k$, and g has a Siegel disk at Q and g has no Siegel disk at any other point.

Theorem 2. *There does not exist a pair (S, g) of a complex Enriques surface S and its automorphism g such that*

$$h(g) = \log 1.17628081... .$$

Here, the right hand side is the logarithm of the Lehmer number, i.e., the logarithm of the smallest known Salem number.

See e.g. [7] for the terms in these Theorems. In the rest, we shall remark a few differences between our results and some of preceding known results and more recent results.

In [4], McMullen constructed the first examples of surface automorphisms with Siegel disks. They are K3 surface automorphisms arising from certain Salem numbers of degree 22, including the 9-th smallest known one. In his construction, the resulting K3 surfaces are of Picard number 0. So, they have no complete curve, whence, no pointwise fixed curve as well. the first theorem tells us that it is also possible to have both a Siegel disk and a pointwise fixed curve, necessarily smooth rational, at the same time.

Let S be a rational surface obtained by blowing up at n points on $\mathbf{P}^2$ and g be an automorphism of S. Then, $g^*(K_S) = K_S$ and g naturally acts on the orthogonal complement $K_S^{\perp}$ of the canonical class in $H^2(S, \mathbf{Z})$. The lattice $K_S^{\perp}$ is isomorphic to the lattice $E_n(-1)$, i.e., the lattice represented by the Dynkin diagram with n vertices s_k $(0 \le k \le n-1)$ of self-intersection -2 such that $n-1$ vertices s_1, s_2, $\cdots$, s_{n-1} form Dynkin diagram of type $A_{n-1}(-1)$ in this order and the remaining vertex s_0 is joined to only the vertex s_3 by a simple line. (See [5], Section 2, Figure 2.) The lattice $E_n(-1)$ is of signature $(1, n-1)$ when $n \ge 10$. Then, g naturally induces an orthogonal action $g^*|E_n(-1)$ (after fixing a marking). By Nagata (see e.g. [5], Theorem (12.4) for the statement), $g^*|E_n(-1)$ is an element of the Weyl group $W(E_n(-1))$, i.e., the group generated by the reflections r_k $(0 \le k \le n-1)$ corresponding to the vertices s_k. The Weyl group $W(E_n(-1))$ has a special conjugacy class called the *Coxeter class*. It is the conjugacy class of the product (in any order in this case) of the reflections $\Pi_{k=0}^{n-1} r_k$. McMullen ([5], Theorem (1.1)) shows that, when $n \ge 10$, the Coxeter class is realized *geometrically* by a rational surface automorphism. That is, $\Pi_{k=0}^{n-1} r_k = g^*|E_n(-1)$ (under a suitable marking) for an automorphism g of S with suitably chosen n blown up points. When $n = 10$, i.e., for $E_{10}(-1)$, the characteristic polynomial of the Coxeter class is exactly the Lehmer polynomial, i.e., the minimal polynomial of the Lehmer number over $\mathbf{Z}$. In this way, McMullen realized the logarithm of the Lehmer number as the topological entropy of some rational surface automorphisms with $K_S^{\perp} \simeq E_{10}(-1)$. Note that the Lehmer number is the smallest known Salem number. See [2] and the home page quoted there, for the list of the smallest 47 known Salem numbers. Being also based on his preceding result [3], Theorem (1.1), McMullen ([5], Theorem (A.1)) also shows that the logarithm of the Lehmer number is in fact the minimal positive entropy of automorphisms of complex surfaces. So, the Lehmer number plays a very special role in automorphisms of compact complex surfaces.

On the other hand, lattice $E_{10}(-1)$ is also isomorphic to the free part of $H^2(S, \mathbf{Z})$ of an Enriques surface S. So, it is natural to ask if the logarithm of the Lehmer number can also be realized as the topological entropy of an Enriques surface automorphism or not. The second Theorem says that it is *not*. This may sound negative. However, I believe that such an impossibility result is also of its own interest.

Later, in [6], McMullen realized the Lehmer number as the exponential of the topological entropy of an automorphism of a non-projective K3 surface. In [8], we also show, as the first example in dimension ≥ 3, that there are higher dimensional rational manifolds with automorphisms of positive entropy having arbitrarily high number of/exactly one Siegel disk(s).

References

[1] Gross B. H., McMullen C. T.: *Automorphisms of even unimodular lattices and unramified Salem numbers*, J. Algebra **257** (2002) 265–290.

[2] Flammang V., Grandcolas M., Rhin G.:*Small Salem numbers. Number theory in progress*, Vol. 1 (Zakopane-Kościelisko 1997), de Gruyter, Berlin (1999) 165–168.

[3] McMullen C. T.: *Coxeter groups, Salem numbers and the Hilbert metric*, Publ. Math. Inst. Hautes Études Sci. **95** (2002) 151–183.

[4] McMullen C. T.: *Dynamics on K3 surfaces: Salem numbers and Siegel disks*, J. Reine Angew. Math. **545** (2002) 201–233.

[5] McMullen C. T.: *Dynamics on blowups of the projective plane*, Publ. Math. Inst. Hautes Études Sci. **105** (2007) 49–89.

[6] McMullen C. T.: *K3 surfaces, entropy and glue*, preprint (2009), avaiable at his home page.

[7] Oguiso K.: *The third smallest Salem number in automorphisms of K3 surfaces*, to appear in Adv. Stud. Pure Math (Algebraic Geometry in East Asia -Seoul 2008), arXiv:0905.2396.

[8] Oguiso K. and Perroni F.: *Automorphisms of rational manifolds of positive entropy with Siegel disks*, arXiv:0905.4257.

Some components of the compactified moduli space of surfaces of general type

Rita Pardini

(joint work with Valery Alexeev)

The coarse moduli space $\mathscr{M}_{K^2,\chi}$ of canonical models of surfaces of general type X with fixed numerical invariants $K_X^2 = K^2$ and $\chi(\mathscr{O}_X) = \chi$ is a quasi projective scheme ([9]). The existence of a modular compactification $\mathscr{M}_{K^2,\chi} \subset \overline{\mathscr{M}}_{K^2,\chi}$ was shown at the beginning of the 1990's as a result of the work of several authors ([10], [1], [2], see also [3]). Roughly speaking, $\overline{\mathscr{M}}_{K^2,\chi}$ is obtained from $\mathscr{M}_{K^2,\chi}$ by adding points corresponding to smoothable *stable surfaces*, i.e. surfaces with semilogcanonical ("slc") singularities and ample canonical class. The singularities that can occur on these surfaces are listed in [10].

However, up to now no component of $\mathscr{M}_{K^2,\chi}$ had been described explicitly, apart from trivial cases such as rigid surfaces or products of curves. Only very recently some examples, also related to the moduli space of curves, have been studied in [15].

In [5] we describe explicitly the stable surfaces occurring as boundary points in the closure in $\overline{\mathscr{M}}_{K^2,\chi}$ of some irreducible components U of $\mathscr{M}_{K^2,\chi}$. In all our examples U is a family of $\mathbb{Z}_2^r$-covers of (a blow up) of $\mathbb{P}^2$ branched on a union of lines. The closure $\overline{U}$ is constructed explicitly using (a variation of) the construction of the compactification of the moduli space of weighted slc arrangements of hyperplanes in $\mathbb{P}^n$ ([4]) and a generalization to the non normal case ([6]) of the theory of abelian covers ([14]).

The families of surfaces we study are:

(1) Campedelli surfaces with $\pi_1 = \mathbb{Z}_2^3$: by definition, a Campedelli surface is a minimal surface of general type with $K^2 = 2$ and $p_g = q = 0$ (and so $\chi = 1$). By [13] (cf. also [12]) all Campedelli surfaces with $\pi_1 = \mathbb{Z}_2^3$ are $\mathbb{Z}_2^3$-covers of $\mathbb{P}^2$ branched on 7 lines. The covering map is induced by the bicanonical system. These surfaces give an irreducible 6-dimensional open subset of the moduli space.

(2) Uniform Line covers: this is a generalization of (1). For $r \geq 4$ we consider $\mathbb{Z}_2^r$-covers of $\mathbb{P}^2$ branched on the union of $2^r - 1$ lines.

(3) Burniat surfaces with $K^2 = 6$: these are the minimal desingularizations of certain $\mathbb{Z}_2^2$-covers of the plane branched on a union of 9 lines with three 4-tuple points (see [8] for a precise description, cf. also [11] or [5]). The covering map pulls back to a $\mathbb{Z}_2^2$-cover of the smooth Del Pezzo surface of degree 6, which is induced by the bicanonical system. The numerical invariants are $K^2 = 6$, $p_g = q = 0$. These surfaces give an irreducible 4-dimensional open subset of the moduli space.

In cases (1) and (3) we describe explicitly the stable surfaces occurring on the boundary. Since the construction of the compactification is purely combinatorial, in principle it is possible to do the same also for the families of surfaces in case (2). In case (1), all surfaces on the boundary are $\mathbb{Z}_2^3$-covers of the plane, while in case (3) one gets also $\mathbb{Z}_2^2$-covers of reducible surfaces with up to 6 components. All the singularities that occur have index 2, and we get examples for all the classes of index 2 singularities listed in [10]. In case (3) some of the surfaces on the boundary can be deformed to reducible stable surfaces that are not limits of Burniat surfaces. We do not know whether these reducible surfaces admit a smoothing.

Finally, as a byproduct of our method of construction, we are able to show the following:

Theorem 1. *All the families of surfaces in (1), (2) and (3) give connected components of the moduli space of surfaces of general type.*

This was known for Campedelli surfaces with $\pi_1 = \mathbb{Z}_2^3$ ([13]) and Burniat surfaces with $K^2 = 6$ ([11], reproven in [7] by a different method), but it is a new result in the case (2) of Uniform Line covers.

References

[1] V. Alexeev, *Log canonical singularities and complete moduli of stable pairs*, arXiv alg-geom/9608013 (1996).

[2] ———, *Moduli spaces $M_{g,n}(W)$ for surfaces*, Higher-dimensional complex varieties (Trento, 1994), de Gruyter, Berlin, 1996, pp. 1–22.

[3] ———, *Higher-dimensional analogues of stable curves*, Proceedings of the International Congress of Mathematicians, Vol. II (Madrid, 2006), European Math. Soc. Pub. House, 2006.

[4] ———, *Weighted grassmannians and stable hyperplane arrangements*, arXiv: 0806.0881.

[5] V. Alexeev and R. Pardini, *Explicit compactifications of moduli spaces of Campedelli and Burniat surfaces*, arXiv:0901.4431.
[6] V. Alexeev and R. Pardini, *Non normal abelian covers*, in preparation.
[7] *Burniat surfaces I: fundamental groups and moduli of primary Burniat surfaces*, I. Bauer, F. Catanese. arXiv:0909.3699.
[8] P. Burniat, *Sur les surfaces de genre $P_{12} > 0$*, Ann. Mat. Pura Appl., IV Ser. **71** (1966), 1–24.
[9] D. Gieseker, *Global moduli for surfaces of general type*, Invent. Math. **43** (1977) 233-282.
[10] J. Kollár and N. I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math. **91** (1988), no. 2, 299–338.
[11] M. Mendes Lopes and R. Pardini, *A connected component of the moduli space of surfaces with $p_g = 0$*, Topology **40** (2001), no. 5, 977–991.
[12] M. Mendes Lopes, R. Pardini, M. Reid, *Campedelli surfaces with fundamental group of order 8*, Geom. Ded. **139** (1) 2009, 49-55.
[13] Y. Miyaoka, *On numerical Campedelli surfaces*, Complex analysis and algebraic geometry, Iwanami Shoten, Tokyo, (1977), 113–118.
[14] R. Pardini, *Abelian covers of algebraic varieties*, J. Reine Angew. Math. **417** (1991), 191–213.
[15] S. Rollenske, *Compact moduli for certain Kodaira fibrations*, arXiv:0905.2536.

Hyperdiscriminant Polytopes, Chow Polytopes, and Mabuchi Energy Asymptotics

Sean Timothy Paul

Let $X^n \longrightarrow \mathbb{P}^N$ be a smooth complex projective variety of degree $d \geq 2$ embedded by a very ample complete linear system. Fix any Hermitian metric on $\mathbb{C}^{N+1}$ and let ω_{FS} denote the associated Fubini-Study Kähler form. We set $\omega := \omega_{FS}|_X$. To $\sigma \in G$ (the automorphism group of $\mathbb{P}^N$) we associate the *Bergman potential* $\varphi_\sigma \in C^\infty(X)$

$$\sigma^*\omega = \omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi_\sigma > 0 \ .$$

Let ν_ω denote the Mabuchi energy of (X, ω). For any $\sigma \in G$ we define

$$\nu_\omega(\sigma) := \nu_\omega(\varphi_\sigma) \ .$$

Let $\lambda : \mathbb{C}^* \longrightarrow G$ be an algebraic one parameter subgroup of G. We shall refer to such maps, and their associated potentials $\varphi_{\lambda(t)}$, as *degenerations*. Three basic problems in the field of Kähler geometry are the following.

Problem 1. *Give a* ***complete*** *description of the behavior of the Mabuchi energy along all degenerations. That is, describe*

$$\lim_{|t| \longrightarrow 0} \nu_\omega(\lambda(t)) \quad t \in \mathbb{C}^* \ .$$

Problem 2. *Provide necessary and sufficient conditions in terms of the geometry of the embedding $X \longrightarrow \mathbb{P}^N$ which insure that ν_ω is* ***bounded below*** *along all degenerations.*

Problem 3. *Provide necessary and sufficient conditions in terms of the geometry of the embedding which insure that ν_ω is **proper** along all degenerations.*

The talk described the solution to all of these problems. The solution is given in terms of the X-*resultant* (the Cayley-Chow form of X) and the X-*hyperdiscriminant* of format $(n-1)$ (the defining polynomial of the variety of tangent hyperplanes to $X \times \mathbb{P}^{n-1}$ in the Segre embedding). That the X-resultant appears in the K-energy is not new and is due to Gang Tian . The author's original contribution is the discovery that the X-hyperdiscriminant *also* appears in the Mabuchi energy of an algebraic manifold . In fact, it is the hyperdiscriminant that *reflects the presence of the Ricci curvature.* The Chow form does not.

Theorem 1. *Let $X^n \hookrightarrow \mathbb{P}^N$ be a smooth, linearly normal complex algebraic variety of degree $d \geq 2$. Let R_X denote the **X-resultant** (the Cayley-Chow form of X) . Let $\Delta_{X\times\mathbb{P}^{n-1}}$ denote the **X-hyperdiscriminant** of format $(n-1)$ (the defining polynomial for the dual of $X \times \mathbb{P}^{n-1}$ in the Segre embedding) Then the Mabuchi energy restricted to the Bergman metrics is given as follows*

$$\nu_\omega(\varphi_\sigma) = \deg(R_X)\log\frac{||\sigma\cdot\Delta_{X\times\mathbb{P}^{n-1}}||^2}{||\Delta_{X\times\mathbb{P}^{n-1}}||^2} - \deg(\Delta_{X\times\mathbb{P}^{n-1}})\log\frac{||\sigma\cdot R_X||^2}{||R_X||^2} \ . \quad (1)$$

Remark 1. The Mabuchi energy restricted to G is not manifestly, and most likely not, a convex function .

It follows from Theorem 1 that the asymptotic expansion of the Mabuchi energy along any algebraic one parameter subgroup of H (a maximal algebraic torus of G)[1] is completely determined by the *Chow polytope* $\mathscr{N}(R_X)$ and the *hyperdiscriminant polytope* $\mathscr{N}(\Delta_{X\times\mathbb{P}^{n-1}})$. We remark that these are compact convex lattice polytopes inside $M_{\mathbb{R}} := M_{\mathbb{Z}}(H)\otimes_{\mathbb{Z}}\mathbb{R} \cong \mathbb{R}^N$, where $M_{\mathbb{Z}} = M_{\mathbb{Z}}(H)$ denotes the rank N lattice of rational characters of H. In the statement of 2 below l_λ denotes the integral linear functional on $M_{\mathbb{R}}$ corresponding to the degeneration $\lambda \in N_{\mathbb{Z}} := M_{\mathbb{Z}}^\vee$ (dual lattice) .

Theorem 2. *There is an asymptotic expansion as $|t| \to 0$*

$$\nu_\omega(\lambda(t)) = F_P(\lambda)\log(|t|^2) + O(1) \ , \quad (2)$$

$$F_P(\lambda) := \deg(R_X)min_{\{x\in\mathscr{N}(\Delta_{X\times\mathbb{P}^{n-1}})\}}l_\lambda(x) - \deg(\Delta_{X\times\mathbb{P}^{n-1}})min_{\{x\in\mathscr{N}(R_X)\}}l_\lambda(x).$$

In particular, $\nu_\omega(\lambda(t))$ has a logarithmic singularity as $|t| \to 0$, and the coefficient of blow up is an integer.

[1] G always denotes $SL(N+1,\mathbb{C})$.

Theorem 2 provides a complete solution to Problem 1 .

Theorem 3. *The Mabuchi energy of* $(X, \omega_{FS}|_X)$ *is bounded from below along all degenerations in* G *if and only if for all maximal tori* H *the hyperdiscriminant polytope dominates the Chow polytope*

$$\deg(\Delta_{X\times\mathbb{P}^{n-1}})\mathscr{N}(R_X) \subseteq \deg(R_X)\mathscr{N}(\Delta_{X\times\mathbb{P}^{n-1}}) \ . \tag{3}$$

Theorem 3 provides a complete solution to Problem 2 .

Theorem 4. *The Mabuchi energy of* $(X, \omega_{FS}|_X)$ *is* ***proper*** *along all degenerations in* G *if and only if for all* $0 < \varepsilon << 1$ *and all maximal tori* H *we have*

$$(1-\delta\varepsilon)\deg(\Delta_{X\times\mathbb{P}^{n-1}})\mathscr{N}(R_X) + \varepsilon d^2(n+1)\mathscr{S}_N \subseteq \deg(R_X)\mathscr{N}(\Delta_{X\times\mathbb{P}^{n-1}}) \ . \tag{4}$$

Theorem 4 provides a complete solution to Problem 3 .

In the statement of Theorem 4 we have defined $\delta := d/\deg(\Delta_{X\times\mathbb{P}^{n-1}})$ and $\mathscr{S}_N$ is the standard N-simplex in $\mathbb{R}^N$. The addition on the left side of (4) denotes Minkowski summation of polyhedra .

The next result provides a weak form of the numerical criterion for the Mabuchi K-energy map.

Theorem 5. *Let* H *be any maximal algebraic torus of* G*. Assume that there is a sequence* $\{\tau_i\} \subset H$ *such that*

$$\liminf_{i\longrightarrow\infty} \nu_\omega(\varphi_{\tau_i}) = -\infty \ .$$

Then there exists a one parameter subgroup $\lambda : \mathbb{C}^* \longrightarrow G$ *such that*

$$\lim_{|t|\longrightarrow 0} \nu_\omega(\lambda(t)) = -\infty \ .$$

Applications of Theorem 1 to canonical Kähler metrics are as follows, the precise definition of K-(semi)stability is new and due to the speaker .

Corollary 1. (i) *If a polarized manifold* (X, L) *admits a metric of constant scalar curvature in the class* $c_1(L)$ *then it is K-semistable with respect to all embeddings* $X \xrightarrow{L^m} \mathbb{P}^{N_m}$.

(ii) *In particular a Fano manifold* $(X, -K_X)$ *admits a Kähler Einstein metric only if all pluri-anticanonical models are K-semistable.*

(iii) *If* $(X, -K_X)$ *has a discrete symmetry group and admits a Kähler Einstein metric then it is K-**stable**.*

We single out the following special cases.

Corollary 2. (i) *Any canonically polarized manifold* (X, K_X) *is K-**stable** with respect to its pluricanonical embeddings.*

(ii) *Any polarized Calabi-Yau manifold* (X, L) *is K-**stable** with respect to all embeddings* $X \xrightarrow{L^m} \mathbb{P}^{N_m}$.
(iii) *Any compact homogeneous Kähler manifold is K-semistable with respect to its plurianticanonical embeddings.*

References

[1] S. T. Paul *Hyperdiscriminant polytopes, Chow Polytopes, and Mabuchi energy asymptotics*, arXiv:math/0811.2548v2

Surfaces with $p_g = 0$: computer aided constructions

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(joint work with Ingrid Bauer, Fabrizio Catanese, Fritz Grunewald)

The surfaces of general type with $p_g = 0$ have been recently object of intensive research since they arise naturally from many different directions, as (*e.g.*) the Chow groups (because of the Bloch conjecture) and the analysis of the "exceptional" behaviors of the pluricanonical maps of the surfaces of general type.
We are interested in the following situation (#):

- C_1, C_2 be compact complex curves of respective genera g_1, $g_2 \geq 2$;
- G be a finite group acting faithfully on each C_i;
- $X := (C_1 \times C_2)/G$ be the quotient by the diagonal action;
- $S \to X$ be the minimal resolution of the singularities of X.

We have constructed many new surfaces of general type S with $p_g(S) = 0$ by performing a systematic search of surfaces as in (#). It is remarkable that by a result of Kimura for all these surfaces the Bloch conjecture holds.

Theorem 1. *([2]) There are exactly 17 families of smooth surfaces of general type* $X = C_1 \times C_2/G$, *with* G *finite and* $p_g(X) = 0$. *They form* 17 *connected components of the moduli space of the surfaces of general type.*

Theorem 2. *([3]) There are exactly 27 families of surfaces as in (#) such that* S *is of g.t. with* $p_g(S) = 0$, X *is singular and has canonical singularities.*

Theorem 3. *([4]) There are exactly 32 families of surfaces as in (#) such that* S *is minimal of g.t. with* $p_g(S) = 0$, *and* X *has at least a non-canonical singularity.*

We use the following algebraic recipe.
Let C be a curve and $p_1, \ldots, p_r$ be the branching points of a G-cover $\xi \colon C \to C' := C/G$ of respective branching indices $m_1, \ldots, m_r$. We assume $C' \cong \mathbb{P}^1$, although the argument works with minor modifications when $g(C') > 0$, which is relevant for constructing irregular surfaces (cf. [5], [6], [7]).
ξ has a monodromy representation $\psi \colon \pi_1(\mathbb{P}^1 \setminus \{p_1, \ldots, p_r\}) \to G$, which factors through the map $\varphi \colon \Pi(0; m_1, \ldots, m_r) := \langle c_1, \ldots, c_r | c_i^{m_i}, c_1 \cdots c_r \rangle \to G$ defined by $\varphi(c_i) := a_i := \psi(\gamma_i)$ where the γ_i are geometric loops around the p_i chosen so

that $\prod \gamma_i = 1$. The a_i generate G, $\prod a_i = 1$ and each a_i has order m_i: for short $[a_1, \dots, a_r]$ is a *sequence of spherical generators* of G of signature $(m_1, \dots, m_r)$. By the Riemann Existence Theorem ξ is determined by the p_i, the γ_i and the a_i. Therefore the surfaces as in (#) are given by two finite subsets of $\mathbb{P}^1$, loops around these points as above, and two sequences of spherical generators $[a_1, \dots, a_r]$ and $[b_1, \dots, b_s]$ (of respective signatures, say, $(m_1, \dots, m_r)$ and $(n_1, \dots, n_s)$).

Lemma 1. *There are numbers D^2, M, R and B, explicit functions (only) of the singularities of X such that*

i) *if $\chi(\mathcal{O}_S) = 1$, then $K_S^2 = 8 - B$;*
ii) *$r \le R$ and $\forall i$, $m_i \le M$;*
iii) $|G| = \frac{K_S^2 - D^2}{2\left(-2+\sum_1^r\left(1-\frac{1}{m_i}\right)\right)\left(-2+\sum_1^s\left(1-\frac{1}{n_s}\right)\right)}$.

It follows an algorithm to compute all surfaces S as in (#) with $p_g(S) = q(S) = 0$ and a given fixed value of K_S^2:

1) find all possible *baskets* of singularities with $B = 8 - K_S^2$;
2) for each basket list all signatures respecting the inequalities in ii);
3) for each pair of signatures, search all groups of the order predicted by iii) for sequences of spherical generators of the prescribed signatures;
4) check the resulting surfaces: most of them will be too singular, and not even of general type!

Some remarks:

- In few dozens of cases, the computer can't perform step 3) since the predicted $|G|$ is too big, and no database contains all the necessary groups: we proved theoretically that these cases do not occur.
- The algorithm is heavy and in this form there is little chance that a computer can complete it for small values of K_S^2. We proved and inserted in the algorithm much stronger conditions on Sing X and on the signatures, to obtain the full list of surfaces with $K_S^2 \ge 1$.
- If X has a non-canonical singularity, K_S may be not nef, and therefore K_S^2 may be nonpositive: we may have missed some nonminimal surfaces.

To understand if these surfaces are topologically pairwise distinct, we compute their fundamental groups. We proved the following theorem.

Theorem 4. ([3]) *For every surface S as in (#), $\pi_1(S)$ contains a normal subgroup of finite index isomorphic to $\Pi_g \times \Pi_{g'}$, where Π_g, $\Pi_{g'}$ are the fundamental groups of a smooth curve of genus g resp. g' (here $g, g' \ge 0$).*

The proof is purely algebraic and indirect. Theorem 4 suggests a *geometrical* description of $\pi_1(S)$ (as in table 1) which can be used to study the deformations of these surfaces. This has been already done in a case: see [1].
We listed in table 1, for each family from Theorem 2 or 3, K_S^2, the singularities (where q/n^k means "k points of type 1/n(1,q)") the signatures T_i (with an analogous *exponential* notation), G, the number N of different families we obtain with these data, $H_1(S, \mathbb{Z})$ and $\pi_1(S)$. More details are in [3] and [4].

K^2	Sing X	T_1	T_2	G	N	$H_1(S,\mathbb{Z})$	$\pi_1(S)$
6	$1/2^2$	$2^3,4$	$2^4,4$	$\mathbb{Z}_2\times D_4$	1	$\mathbb{Z}_2^2\times\mathbb{Z}_4^2$	$1\to\mathbb{Z}^2\times\Pi_2\to\pi_1\to\mathbb{Z}_2^2\to 1$
6	$1/2^2$	$2^4,4$	$2,4,6$	$\mathbb{Z}_2\times\mathfrak{S}_4$	1	$\mathbb{Z}_2^3\times\mathbb{Z}_4$	$1\to\Pi_2\to\pi_1\to\mathbb{Z}_2\times\mathbb{Z}_4\to 1$
6	$1/2^2$	$2,5^2$	$2,3^3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_3\times\mathbb{Z}_{15}$	$\mathbb{Z}^2\rtimes\mathbb{Z}_{15}$
6	$1/2^2$	$2,4,10$	$2,4,6$	$\mathbb{Z}_2\times\mathfrak{S}_5$	1	$\mathbb{Z}_2\times\mathbb{Z}_4$	$\mathfrak{S}_3\times D_{4,5,-1}$
6	$1/2^2$	$2,7^2$	$3^2,4$	PSL(2,7)	2	$\mathbb{Z}_{21}$	$\mathbb{Z}_7\times\mathfrak{A}_4$
6	$1/2^2$	$2,5^2$	$3^2,4$	$\mathfrak{A}_6$	2	$\mathbb{Z}_{15}$	$\mathbb{Z}_5\times\mathfrak{A}_4$
5	$1/3,2/3$	$2,4,6$	$2^4,3$	$\mathbb{Z}_2\times\mathfrak{S}_4$	1	$\mathbb{Z}_2^2\times\mathbb{Z}_4$	$1\to\mathbb{Z}^2\to\pi_1\to D_{2,8,3}\to 1$
5	$1/3,2/3$	$2^4,3$	$3,4^2$	$\mathfrak{S}_4$	1	$\mathbb{Z}_2^2\times\mathbb{Z}_8$	$1\to\mathbb{Z}^2\to\pi_1\to\mathbb{Z}_8\to 1$
5	$1/3,2/3$	$4^2,6$	$2^3,3$	$\mathbb{Z}_2\times\mathfrak{S}_4$	1	$\mathbb{Z}_2\times\mathbb{Z}_8$	$1\to\mathbb{Z}^2\to\pi_1\to\mathbb{Z}_8\to 1$
5	$1/3,2/3$	$2,5,6$	$3,4^2$	$\mathfrak{S}_5$	1	$\mathbb{Z}_8$	$D_{8,5,-1}$
5	$1/3,2/3$	$3,5^2$	$2^3,3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_2\times\mathbb{Z}_{10}$	$\mathbb{Z}_5\times Q_8$
5	$1/3,2/3$	$2^3,3$	$3,4^2$	$\mathbb{Z}_2^4\rtimes\mathfrak{S}_3$	1	$\mathbb{Z}_2\times\mathbb{Z}_8$	$D_{8,4,3}$
5	$1/3,2/3$	$3,5^2$	$2^3,3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_2\times\mathbb{Z}_{10}$	$\mathbb{Z}_2\times\mathbb{Z}_{10}$
4	$1/2^4$	2^5	2^5	$\mathbb{Z}_2^3$	1	$\mathbb{Z}_2^3\times\mathbb{Z}_4$	$1\to\mathbb{Z}^4\to\pi_1\to\mathbb{Z}_2^2\to 1$
4	$1/2^4$	$2^2,4^2$	$2^2,4^2$	$\mathbb{Z}_2\times\mathbb{Z}_4$	1	$\mathbb{Z}_2^3\times\mathbb{Z}_4$	$1\to\mathbb{Z}^4\to\pi_1\to\mathbb{Z}_2^2\to 1$
4	$1/2^4$	2^5	$2^3,4$	$\mathbb{Z}_2\times D_4$	1	$\mathbb{Z}_2^2\times\mathbb{Z}_4$	$1\to\mathbb{Z}^2\to\pi_1\to\mathbb{Z}_2\times\mathbb{Z}_4\to 1$
4	$1/2^4$	$3,6^2$	$2^2,3^2$	$\mathbb{Z}_3\times\mathfrak{S}_3$	1	$\mathbb{Z}_3^2$	$\mathbb{Z}^2\rtimes\mathbb{Z}_3$
4	$1/2^4$	$3,6^2$	$2,4,5$	$\mathfrak{S}_5$	1	$\mathbb{Z}_3^2$	$\mathbb{Z}^2\rtimes\mathbb{Z}_3$
4	$1/2^4$	2^5	$2,4,6$	$\mathbb{Z}_2\times\mathfrak{S}_4$	1	$\mathbb{Z}_2^3$	$\mathbb{Z}^2\rtimes\mathbb{Z}_2$
4	$1/2^4$	$2^2,4^2$	$2,4,6$	$\mathbb{Z}_2\times\mathfrak{S}_4$	1	$\mathbb{Z}_2^2\times\mathbb{Z}_4$	$\mathbb{Z}^2\rtimes\mathbb{Z}_4$
4	$1/2^4$	2^5	$3,4^2$	$\mathfrak{S}_4$	1	$\mathbb{Z}_2^2\times\mathbb{Z}_4$	$\mathbb{Z}^2\rtimes\mathbb{Z}_4$
4	$1/2^4$	$2^3,4$	$2^3,4$	$\mathbb{Z}_2^4\rtimes\mathbb{Z}_2$	1	$\mathbb{Z}_4^2$	$G(32,2)$
4	$1/2^4$	$2,5^2$	$2^2,3^2$	$\mathfrak{A}_5$	1	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$
4	$1/2^4$	$2^2,3^2$	$2^2,3^2$	$\mathbb{Z}_3^2\rtimes Z_2$	1	$\mathbb{Z}_3^3$	$\mathbb{Z}_3^3$
4	$2/5^2$	$2^3,5$	$3^2,5$	$\mathfrak{A}_5$	1	$\mathbb{Z}_2\times\mathbb{Z}_6$	$\mathbb{Z}_2\times\mathbb{Z}_6$
4	$2/5^2$	$2,4,5$	$4^2,5$	$\mathbb{Z}_2^4\rtimes D_5$	3	$\mathbb{Z}_8$	$\mathbb{Z}_8$
4	$2/5^2$	$2,4,5$	$3^2,5$	$\mathfrak{A}_6$	1	$\mathbb{Z}_6$	$\mathbb{Z}_6$
3	$1/5,4/5$	$2^3,5$	$3^2,5$	$\mathfrak{A}_5$	1	$\mathbb{Z}_2\times\mathbb{Z}_6$	$\mathbb{Z}_2\times\mathbb{Z}_6$
3	$1/5,4/5$	$2,4,5$	$4^2,5$	$\mathbb{Z}_2^4\rtimes D_5$	3	$\mathbb{Z}_8$	$\mathbb{Z}_8$
3	$1/3,1/2^2,2/3$	$2^2,3,4$	$2,4,6$	$\mathbb{Z}_2\times\mathfrak{S}_4$	1	$\mathbb{Z}_2\times\mathbb{Z}_4$	$\mathbb{Z}_2\times\mathbb{Z}_4$
3	$1/5,4/5$	$2,4,5$	$3^2,5$	$\mathfrak{A}_6$	1	$\mathbb{Z}_6$	$\mathbb{Z}_6$
2	$1/3^2,2/3^2$	$2,6^2$	$2^2,3^2$	$\mathbb{Z}_2^3\rtimes\mathbb{Z}_3$	1	$\mathbb{Z}_2^2$	Q_8
2	$1/2^6$	4^3	4^3	$\mathbb{Z}_4^2$	1	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
2	$1/2^6$	$2^3,4$	$2^3,4$	$\mathbb{Z}_2\times D_4$	1	$\mathbb{Z}_2\times\mathbb{Z}_4$	$\mathbb{Z}_2\times\mathbb{Z}_4$
2	$1/3^2,2/3^2$	$2^2,3^2$	$3,4^2$	$\mathfrak{S}_4$	1	$\mathbb{Z}_8$	$\mathbb{Z}_8$
2	$1/3^2,2/3^2$	$3^2,5$	$3^2,5$	$\mathbb{Z}_5^2\rtimes\mathbb{Z}_3$	2	$\mathbb{Z}_5$	$\mathbb{Z}_5$
2	$1/2^6$	$2,5^2$	$2^3,3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_5$	$\mathbb{Z}_5$
2	$1/2^6$	$2^3,4$	$2,4,6$	$\mathbb{Z}_2\times\mathfrak{S}_4$	1	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
2	$1/3^2,2/3^2$	$3^2,5$	$2^3,3$	$\mathfrak{A}_5$	1	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
2	$1/2^6$	$2,3,7$	4^3	PSL(2,7)	2	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
2	$1/2^6$	$2,6^2$	$2^3,3$	$\mathfrak{S}_3\times\mathfrak{S}_3$	1	$\mathbb{Z}_3$	$\mathbb{Z}_3$
2	$1/2^6$	$2,6^2$	$2,4,5$	$\mathfrak{S}_5$	1	$\mathbb{Z}_3$	$\mathbb{Z}_3$
2	$1/4,1/2^2,3/4$	$2,4,7$	$3^2,4$	PSL(2,7)	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$
2	$1/4,1/2^2,3/4$	$2,4,5$	$3^2,4$	$\mathfrak{A}_6$	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$
2	$1/4,1/2^2,3/4$	$2,4,6$	$2,4,5$	$\mathfrak{S}_5$	2	$\mathbb{Z}_3$	$\mathbb{Z}_3$
1	$1/3,1/2^4,2/3$	$2^3,3$	$3,4^2$	$\mathfrak{S}_4$	1	$\mathbb{Z}_4$	$\mathbb{Z}_4$
1	$1/3,1/2^4,2/3$	$2,3,7$	$3,4^2$	PSL(2,7)	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$
1	$1/3,1/2^4,2/3$	$2,4,6$	$2^3,3$	$\mathbb{Z}_2\times\mathfrak{S}_4$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$

TABLE 1. The minimal surfaces of general type S as in (#) with $p_g(S)=0$ and X singular

References

[1] I. Bauer, F. Catanese, *The moduli space of Keum-Naie surfaces*, arxiv:0909.1733.
[2] I. Bauer, F. Catanese, F. Grunewald, *The classification of surfaces with $p_g = q = 0$ isogenous to a product of curves*, Pure Appl. Math. Q. **4** (2008), 547–586.
[3] I. Bauer, F. Catanese, F. Grunewald, R. Pignatelli, *Quotient of product of curves, new surfaces with $p_g = 0$ and their fundamental groups*: arxiv.0809.3420.
[4] I. Bauer, R. Pignatelli, *The classification of minimal standard isotrivial fibrations with* $p_g = 0$. Preliminary version (2009).
[5] F. Polizzi, *Standard isotrivial fibrations with $p_g = q = 1$*, J. Algebra **321** (2009), 1600–1631.
[6] E. Mistretta, F. Polizzi, *Standard isotrivial fibrations with $p_g = q = 1$. II*, arxiv: 0805.1424. To appear in the Journal of Pure and Applied Algebra.
[7] M. Penegini, *The classification of isotrivial fibred surfaces with $p_g = q = 2$*, arxiv:0904.1352.

Compact moduli for certain Kodaira fibrations

Sönke Rollenske

It is a general fact that moduli spaces of *nice* objects in algebraic geometry, say smooth varieties, are often non-compact. But usually there is a modular compactification where the boundary points correspond to related but more complicated objects.

Such a modular compactification has been known for the moduli space $\mathscr{M}_g$ of smooth curves of genus g for a long time and in [6] Kollár and Shepherd-Barron made the first step towards the construction of a modular compactification $\bar{\mathfrak{M}}$ for the moduli space $\mathfrak{M}$ of surfaces of general type via so called stable surfaces; the boundary points arise from a stable reduction procedure. An overview over the technical issues arising in the construction can be found in [5]; it has later been extended to pairs and and stable maps (see [2]).

But even 20 years later very few explicit descriptions of compact components of $\bar{\mathfrak{M}}$ have been published. The main idea in all approaches is to relate the component of the moduli space one wishes to study to some other moduli space, where a suitable compactification is known. Products of curves and surfaces isogenous to a product of curves have been treated by van Opstall [8, 9] and a recent paper of Alexeev and Pardini [3] studies Burniat and Campedelli surfaces relating them to hyperplane arrangements in (a blow-up of) $\mathbb{P}^2$.

Our aim was to explicitly construct the stable surfaces in $\bar{\mathfrak{M}}$ that arise as stable degenerations of very simple Galois double Kodaira fibrations: let S be a compact complex surface of general type such that a finite group G acts on S and

- $S/G \cong C \times C$ for a smooth curve of genus at least 2,
- the quotient map $\psi : S \to C \times C$ is a ramified covering,
- there exist a set of automorphisms $\mathscr{S} \subset \mathrm{Aut}(C)$ such that the branch divisor is union of their graphs
$$B = \sum_{\sigma \in \mathscr{S}} \Gamma_\sigma \subset C \times C$$
and $\Gamma_\sigma \cap \Gamma_{\sigma'} = \emptyset$ for $\sigma \neq \sigma'$.

On the first glance these surfaces seem quite special but in joint work with Fabrizio Catanese we gave in [4] an effective method of construction, proving in addition the following:

Theorem 1 ([4], Theorem 6.5)**.** *Let* $\psi : S \to C \times C$ *be a very simple Kodaira fibration,* $\mathscr{S} \subset \mathrm{Aut}(C)$ *as above. Let* H *be the subgroup of* $\mathrm{Aut}(C)$ *generated by* $\mathscr{S}$*. Then the connected component* $\mathfrak{N}$ *of the moduli space of surfaces of general type containing* S *contains only very simple Kodaira fibrations and is isomorphic (as a set) to the moduli space of curves with automorphisms* $\mathscr{M}_{g(C)}(H)$*.*

We show that the structure as ramified covers extends also to the stable degenerations of such surfaces. This enables us to give an explicit description of their stable degeneration:

Theorem 2. *Let* $\overline{\mathfrak{N}}$ *be the closure of* $\mathfrak{N}$ *in the moduli space of stable surfaces and* $[S_0] \in \overline{\mathfrak{N}} \setminus \mathfrak{N}$*. Then there are maps* $\psi_0 : S_0 \to Y$ *and* $\pi : Y \to X$ *such that*

- $X = C_0 \times C_0$ *for a stable curve* C_0*,*
- π *is birational replacing some explicitly determined degenerate cusps of* X *with smooth* $\mathbb{P}^1$*'s in* Y*,*
- $S_0 \to Y$ *is a ramified Galois covering with group* G*.*

Moreover, S_0 *is a local complete intersection and its normalisation is smooth.*

More details can be found in [7]. Starting from this description it would now be very interesting to know if $\overline{\mathfrak{N}}$ is a connected component of the moduli space of stable surfaces or if the degenerations have *unexpected* deformations.

References

[1] D. Abramovich and B. Hassett, *Stable varieties with a twist*, 2009, arXiv:0904.2797.

[2] V. Alexeev, *Higher-dimensional analogues of stable curves,* In International Congress of Mathematicians. Vol. II, pages 515–536. Eur. Math. Soc., Zürich, 2006.

[3] V. Alexeev and R. Pardini. *Explicit compactifications of moduli spaces of Campedelli and Burniat surfaces*, 2009, arXiv:0901.4431.

[4] F. Catanese and S. Rollenske, *Double Kodaira fibrations*, J. Reine Angew. Math., 628:205–233, 2009, math.AG/0611428.

[5] S. Kovács, *Young person's guide to moduli of higher dimensional varieties*, In Proceedings of the AMS Summer Research Institute held at the University of Washington, Seattle, WA, July 25 - August 12, 2005, Proceedings of Symposia in Pure Mathematics. American Mathematical Society, 2005.

[6] J. Kollár and N. Shepherd-Barron. *Threefolds and deformations of surface singularities*, Invent. Math., 91(2):299–338, 1988.

[7] Sönke Rollenske. *Compact moduli for certain Kodaira Fibrations*, 2009, arXiv:0905.2536v1.

[8] M. van Opstall, *Moduli of products of curves*, Arch. Math. (Basel), 84(2):148–154, 2005.

[9] M. van Opstall, *Stable degenerations of surfaces isogenous to a product of curves*, Proc. Amer. Math. Soc., 134(10):2801–2806 (electronic), 2006.

Cohomology of sheaves and numerical invariants of free resolutions

Frank-Olaf Schreyer
(joint work with David Eisenbud)

Let M be a graded module over the polynomial ring $S = K[x_0, \ldots, x_n]$. The Hilbert polynomial $p_M(d) \in \mathbb{Q}[d]$ is an important numerical invariant which can be computed from the minimal free resolution

$$M \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots \leftarrow F_{n+1} \leftarrow 0$$

where $F_i = \oplus S(-j)^{\beta_{ij}}$ as

$$p_M(d) = \sum_{i=0}^{n+1} (-1)^i \beta_{ij} \binom{d-j+n}{n}.$$

Thus we may regard the collection of Betti numbers

$$\beta(M) = (\beta_{ij}) \in \bigoplus_{j=-\infty}^{\infty} \mathbb{Q}^{n+2}$$

as a refined numerical invariant. If we consider the associated sheaf $\mathscr{E} = \widetilde{M}$ on $\mathbb{P}^n$ then the dimensions of the cohomology groups

$$h\mathscr{E} : \mathbb{Z} \times \{0, \ldots, n\} \to \mathbb{Q}, (d, i) \mapsto h^i\mathscr{E}(a)$$

form a perhaps even more natural refinement of $p_M(d) = \chi\mathscr{F}(d)$.

Which Betti tables are possible for graded modules? Which cohomology tables are possible for coherent sheaves?

Boij and Söderberg [1] discovered that relaxed versions of these questions might have a complete answer. Consider the rational cone $\mathbb{B} \subset \bigoplus_{j=-\infty}^{\infty} \mathbb{Q}^{n+2}$ generated by Betti tables, and similarly the cone generated by cohomology tables of coherent sheaves. Both cones are described in terms of extremal rays.

Definition 1. *A free resolution $F_0 \leftarrow F_1 \leftarrow \ldots \leftarrow F_c \leftarrow 0$ is called **pure**, if $M = \text{coker}\ (F_1 \to F_0)$ is a Cohen-Macaulay module (with support of codimension c) and each F_i is generated in a single degree:*

$$F_i = S(-d_i)^{\beta_i} \text{ with } \beta_i = \beta_{i,d_i}$$

We call $d_0, \ldots, d_c$ the type of the pure resolution.

Herzog and Kühl [11] observed that the Betti numbers β_i of a pure resolution are determined by the type up to a common factor

$$\beta_i = r \prod_{k \neq i} \frac{1}{\mid d_k - d_i \mid}$$

Theorem 1. [5],[2].

(1) Each type $d_0 < d_1 < \ldots < d_c$ is realized by some pure resolution.

(2) *The Betti tables of pure resolutions generate all extremal rays of the cone Betti tables. More precisely: For every module there exist a unique chain of degree sequence such that $\beta(M)$ is a unique positive rational linear combination of corresponding pure resolutions.*

Here ‘chain of degree sequences refers to the natural order defined by

$$(d_0, \dots, d_c) \leq (d'_0, \dots, d'_{c'}) \Leftrightarrow c \geq c' \text{ and } d_i \leq d'_i \,\forall\, i \leq c'.$$

The extremal rays in the cone of cohomology tables come from what we call supernatural sheaves.

Definition 2. *A coherent sheaf $\mathscr{E}$ on $\mathbb{P}^n$ has **natural cohomology** if for each d at most one of the groups $H^i\mathscr{E}(d) \neq 0$. It has **supernatural cohomology** if in addition the Hilbert polynomial*

$$\chi\mathscr{E}(d) = \frac{\operatorname{rank}\mathscr{E}}{s!}\prod_{j=1}^{s}(d - z_j)$$

has s distinct integral roots $z_1, \dots, z_s$ where $s = \dim \mathscr{E}$.

Theorem 2. [5].

(1) *Each integral zero sequence $z_0 > z_1 > \dots > z_s$ occurs as root sequence of a supernatural sheaf on $\mathbb{P}^n$.*
(2) *The cone of cohomology tables of vector bundles is spanned by the cohomology tables of supernatural vector bundles.*

The proof of both Theorems has two parts:

(1) Existence of pure resolutions and existence of supernatural sheaves.
(2) Description of the facets of the cones.

The equations defining facets of the cones are derived from the following functionals. For $\beta = (\beta_{i,k}) \in \oplus_k \mathbb{Q}^{n+2}$ a Betti table and $\gamma = (\gamma_{j,k}) = (h^j\mathscr{E}(k)) \in \prod \mathbb{Q}^{n+1}$ a cohomology table we define

$$\langle \beta, \gamma \rangle = \sum_{i \leq j} (-1)^{j-i} \sum_k \beta_{i,k}\gamma_{j,-k}$$

Theorem 3. [5],[6]. *For arbitrary free resolutions F and arbitrary coherent sheaves $\mathscr{E}$*

$$\langle \beta(F), \gamma(\mathscr{F}) \rangle \geq 0$$

Modification of these functionals for certain supernatural sheaves define the facets of the cone of Betti tables, and conversely, a modification for certain pure resolutions of zero dimensional modules defines the cone of cohomology tables of vector bundles.

For arbitrary sheaves it is no longer true that the cohomology table is a finite linear combination of cohomology tables of supernatural sheaves. Instead one has to consider infinite series of tables and the closure of the cone in $\prod_k \mathbb{R}^{n+1}$ with its weak topology. Let γ^z denote the cohomology table of a supernatural sheaf with Hilbert polynomial $\prod_{i=1}^{s}(d - z_i)$. Then we have

Theorem 4. [6]. *For each coherent sheaf $\mathscr{E}$ on $\mathbb{P}^n$ there exists a unique chain $Z = \{z\}$ of degree sequences and unique sequence $(q_z)_{z\in Z}$ of positive numbers such that*

$$\gamma(\mathscr{E}) = \sum_{z\in Z} q_z \gamma^z.$$

If $\mathscr{E}$ is a torsion free sheaf then all γ^z are Hilbert polynomials of supernatural vector bundles and all q_z are rational.

We do not know whether conversely if all γ^z are Hilbert polynomials of vector bundles, $\mathscr{E}$ is necessarily torsion free. Also we have no example where irrational coefficients q_z occur.

References

[1] M. Boij and J. Söderberg. *Graded betti numbers of Cohen-Macaulay modules and the multiplicity conjecture.* J. Lond. Math. Soc. 78 (2008) 85–106.

[2] M. Boij and J. Söderberg. *Betti numbers of graded modules and the Multiplicity Conjecture in the non-Cohen-Macaulay case.* Preprint: arXiv:0803.1645.

[3] D. Eisenbud, G. Fløystad and J. Weyman. *The existence of pure free resolutions.* arXiv:0709.1529

[4] D. Eisenbud and F.-O. Schreyer. *Resultants and Chow forms via exterior syzygies.* J. Amer. Math. Soc. **16** (2003) 537–579.

[5] D. Eisenbud and F.-O. Schreyer. *Betti Numbers of Graded Modules and Cohomology of Vector Bundles* To appear in J. Amer. Math. Soc. (2009).

[6] D. Eisenbud and F.-O. Schreyer. *Cohomology of Coherent Sheaves and Series of Supernatural Bundles* to appear in Jour. Euro. Math. Soc. Preprint: arXiv:0902.1594

[7] D. Erman. *The Semigroup of Betti Diagrams.* Algebra and Number Theory 3 (2009) 341–365. Preprint: arXiv:0806.4401

[8] D. Erman. *A special case of the Buchsbaum-Eisenbud-Horrocks rank conjecture.* Preprint: arXiv:0902.0316

[9] R. Hartshorne and A. Hirschowitz. *Cohomology of a general instanton bundle.* Ann. Sci. de l'École Normale Sup. (1982) 365–390.

[10] J. Herzog and H. Srinivasan. *Bounds for Multiplicities.* Trans. Am. Math. Soc. (1998) 2879–2902.

[11] J. Herzog and M. Kühl. *On the betti numbers of finite pure and linear resolutions.* Comm. in Alg. 12 **13** (1984) 1627–1646.

Jumping of the nef cone for Fano varieties

Burt Totaro

Throughout we consider **Q**-factorial varieties over the complex numbers. A divisor D on a projective variety X is said to be *movable* if the base locus of the linear system $|D|$ has codimension at least 2 in X. Let $N^1(X)$ denote the Néron-Severi vector space, that is, the space of divisors with real coefficients on X modulo numerical equivalence. This is a finite-dimensional real vector space, which can be identified with the subspace of $H^2(X, \mathbf{R})$ spanned by divisors. Define the *movable cone* $\text{Mov}(X)$ in $N^1(X)$ to be the closed convex cone spanned by movable divisors.

Define a *modification* of a projective variety X to be another **Q**-factorial projective variety Y together with a birational map $X \dashrightarrow Y$ which is an isomorphism in codimension 1. That is, we are given an isomorphism from X minus some closed subset of codimension at least 2 to Y minus some closed subset of codimension at least 2. Nontrivial modifications only occur in dimension at least 3. Some examples of modifications are the flips and flops used in minimal model theory. A modification determines a canonical identification $N^1(X) = N^1(Y)$ which preserves the movable cones, $\mathrm{Mov}(X) = \mathrm{Mov}(Y)$. Moreover, we have a canonical identification $H^0(X, O(D)) = H^0(Y, O(D))$ for all Weil divisors D on X, which is compatible with multiplication of sections.

For any projective variety X, the cone $\mathrm{Nef}(X)$ of nef divisors is a closed convex subcone of the movable cone. But for a nontrivial modification $X \dashrightarrow Y$, the nef cones $\mathrm{Nef}(X)$ and $\mathrm{Nef}(Y)$ are *different* subcones of $\mathrm{Mov}(X)$; in fact, the interiors of $\mathrm{Nef}(X)$ and $\mathrm{Nef}(Y)$ are disjoint. To see this, we use Kleiman's theorem that the interior of the nef cone is the ample cone. So suppose we have an ample divisor D on X which is also ample on the modified variety Y. Then we would have

$$X = \operatorname{Proj} R(X, O(D)) = \operatorname{Proj} R(Y, O(D)) = Y,$$

a contradiction. Thus, as emphasized by Kawamata, the modifications of a projective variety X are in one-to-one correspondence with a collection of subcones of $\mathrm{Mov}(X)$ whose interiors are disjoint, the nef cones of the modifications.

Birkar, Cascini, Hacon, and M$^{\mathrm{c}}$Kernan showed that the picture is particularly simple for a Fano variety X, meaning that $-K_X$ is ample [1]. Assume that X is klt (for example, smooth). Then:

(1) The movable cone of X is rational polyhedral. That is, $\mathrm{Mov}(X)$ is the convex cone in $N^1(X)$ spanned by finitely many Cartier divisors.

(2) X has only finitely many modifications, $X = X_1, \dots, X_r$.

(3) For $i = 1, \dots, r$, the nef cone of X_i is rational polyhedral.

(4) $\mathrm{Mov}(X) = \cup_{i=1}^r \mathrm{Nef}(X_i)$.

Thus, many properties of a Fano variety X are encoded by some combinatorial data: a rational polyhedral cone (the movable cone) with a chamber decomposition (the nef cones of X and its modifications). The faces of the nef cone parametrize all contractions of X (morphisms from X onto other projective varieties), while the rest of the movable cone describes all rational contractions of X (rational maps from X which do not extract any divisors), as Hu and Keel described [3].

Using extension theorems building on those of Hacon-M$^{\mathrm{c}}$Kernan and Siu, de Fernex and Hacon showed that the movable cone of a $\mathbf{Q}$-factorial terminal Fano variety X_0 remains constant under deformations of X_0 [2]. They asked whether the chamber decomposition of the movable cone remains constant under deformations of a $\mathbf{Q}$-factorial terminal Fano variety. They proved this for X_0 of dimension at most 3, and when X_0 is also Gorenstein in dimension 4.

The question by de Fernex and Hacon would say in particular that the nef cone of a $\mathbf{Q}$-factorial terminal Fano variety remains constant under deformations. Wiśniewski showed that the nef cone of a *smooth* Fano variety of any dimension remains constant under deformations [7, 8]. Dually, the cone of curves of a smooth Fano variety remains constant under deformations. For example, in dimension 2 this is the classical fact that the set of (-1)-curves on a del Pezzo surface X does not change in $N_1(X) = H_2(X, R)$ as we vary the surface. Geometrically, the (-1)-curves (for example, the 27 lines on a cubic surface) vary continuously as we vary the surface.

We give a negative answer to de Fernex and Hacon's question, as follows [6].

Theorem. There is a variety X and a flat projective morphism $t : X \to A^1$ with the following properties. The fibers X_t for $t \neq 0$ are isomorphic to the blow-up of $\mathbf{P}^4$ along a line, and the nef cone of X_0 is a proper subset of the nef cone of X_t. The fiber X_0 is a $\mathbf{Q}$-factorial terminal Fano 4-fold.

Therefore the results by de Fernex and Hacon on deformations of 3-dimensional Fanos are best possible. The example is based on the existence of high-dimensional flips which deform to isomorphisms, generalizing the Mukai flop [4, 5]. This phenomenon will be common, and we give a family of examples in various dimensions, including a Gorenstein example in dimension 5. The examples also disprove the "volume criterion for ampleness" on $\mathbf{Q}$-factorial terminal Fano varieties [2, Question 5.5].

References

[1] C. Birkar, P. Cascini, C. Hacon, and J. McKernan, *Existence of minimal models for varieties of log general type*, http://math.mit.edu/~mckernan/Papers/papers.html

[2] T. de Fernex and C. Hacon, *Deformations of canonical pairs and Fano varieties*, J. Reine Angew. Math., to appear.

[3] Y. Hu and S. Keel, *Mori dream spaces and GIT*, Michigan Math. J. **48** (2000), 331–348.

[4] Y. Kawamata, *On the extension problem of pluricanonical forms*, Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), ed. P. Pragacz, 193–207, AMS (1999).

[5] S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*, Invent. Math. **77** (1984), 101–116.

[6] B. Totaro, *Jumping of the nef cone for Fano varieties*, arXiv:0907.3617

[7] J. Wiśniewski, *On deformation of nef values*, Duke Math. J. **64** (1991), 325–332.

[8] J. Wiśniewski, *Rigidity of Mori cone for Fano manifolds*, arXiv:0905.3485

Kähler-Ricci Flow with applications in Algebraic Geometry

Zhou Zhang

(joint work with Xiuxiong Chen and Gang Tian)

1. Set-up

The differential manifold X is compact without boundary. Ricci flow is, as introduced by R. Hamilton in [2],

$$\frac{\partial g(t)}{\partial t} = -2\mathrm{Ric}(g(t)), \ g(0) = g_0, \tag{1}$$

with g_0 being a Riemannian metric over X. If g_0 is Kähler w.r.t (X, J), the manifold X with a complex structure J, then the flow metric $g(t)$ stays Kähler w.r.t. (X, J). Hence, *Kähler-Ricci flow is nothing but Ricci flow with the initial metric being Kähler.*

1.1. **Evolution of cohomology class and optimal existence result.** The following scaled version of the flow is the most convenient for our purpose,

$$\frac{\partial \widetilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\widetilde{\omega}_t) - \widetilde{\omega}_t, \ \widetilde{\omega}_0 = \omega_0. \tag{2}$$

By studying an ODE in $H^{1,1}(X;\mathbb{C}) \cap H^2(X;\mathbb{R})$, one can see that $[\widetilde{\omega}_t]$, i.e. the cohomology class of $\widetilde{\omega}_t$, would be the same as $[\omega_t]$ for

$$\omega_t = \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)$$

where ω_∞ is a real, smooth, closed $(1,1)$-form such that $[\omega_\infty] = K_X$. Let's point out that $[\omega_t] = e^{-t}\omega_0 + (1-e^{-t})K_X$, i.e. convex combination of the initial class $[\omega_0]$ and K_X. Using this, the Kähler-Ricci flow can be reduced to a scalar potential flow, which is the parabolic version of complex Monge-Ampère equation. We begin with the optimal existence result in [5].

Theorem 1 (Cascini-La Nave, Tian-Z.) The Kähler-Ricci flow exists as long as the class $[\widetilde{\omega}_t] = [\omega_t]$ remains to be Kähler.

This result has the following important implication, i.e. we now know exactly when the Kähler-Ricci flow meets singularity. The information is contained completely in (finitely dimensional) cohomology data. Needlessly to say, the flow singularity should have close relation with the degeneration of the cohomology class (as a Kähler class). On the other hand, we have a more detailed picture about the cohomology degeneration, i.e. a metric picture, which comes from a natural geometric construction, Ricci flow. This nice intersection of fields in mathematics has been providing interesting problems and motivating powerful techniques.

1.2. **Non-degenerate situation.** We have the following result might convince people that we are not trying something too wild.

Theorem 2 (Tian-Z.) If K_X is Kähler, then with any choice of initial Kähler metric, the Kähler-Ricci flow (2) exists forever and converges exponentially fast in the smooth topology to the unique Kähler-Einstein metric at infinity.

1.3. **Kähler-Ricci flow for general class.** H. Tsuji observed that Kähler-Ricci flow can also be applied to analyze general classes. In [5], we also looked at

$$\frac{\partial \widetilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\widetilde{\omega}_t) - \widetilde{\omega}_t + \mathrm{Ric}(\Omega) + L, \ \widetilde{\omega}_0 = \omega_0, \tag{3}$$

where L is a real, smooth and closed $(1,1)$-form over X. Set $\omega_t = L + e^{-t}(\omega_0 - L)$ and we still have $[\widetilde{\omega}_t] = [\omega_t]$. In fact, (2) is the special case of (3) when $L = -\mathrm{Ric}(\Omega)$. For this flow, we have $[\omega_\infty] = [L]$, which can be any class we want to study in principle. Theorem 1 still holds for this flow. Theorem 2 also holds with the limit not being Kähler-Einstein in general.

2. Main interest: degenerate case

The major advantage by allowing the Kähler class to evolve along Kähler-Ricci flow is to be able to study the boundary of Kähler cone.

2.1. **Infinite time singularity.** We have infinite time singularity if $[\omega_\infty]$ is on the boundary of Kähler cone. In this case, we have the following result in [5] and continuation.

Theorem 3 (Tian-Z.) Suppose K_X nef. and big, i.e. X being minimal and of general type. Then (2) exists forever and $\widetilde{\omega}_t \to \widetilde{\omega}_\infty$ locally as $t \to \infty$ out of E, the stable base locus set of K_X, in C^∞-topology to a smooth metric $\widetilde{\omega}_\infty$ satisfying

$$-\mathrm{Ric}(\widetilde{\omega}_\infty) - \widetilde{\omega}_\infty = 0.$$

Over X, the convergence is weak in the sense of current and the limiting current over X, still denoted by $\widetilde{\omega}_\infty$, is a real, closed and positive $(1,1)$-current representing the class $[\omega_\infty] = K_X$ and having Lelong number 0. This limit is canonical and has continuous local over X. Furthermore, scalar curvature is uniformly bounded along the flow.

Later, Yuguang Zhang applied this to give an alternative proof of the classic Miyaoka-Yao Inequality for minimal manifold of general type.

2.2. **Finite time singularity.** There is also this case that the flow meets singularity at some finite time $T < \infty$ where $[\omega_T]$ is on the boundary of Kähler cone. We have the following result as in [5] and continuation.

Theorem 4 (Tian-Z.) Suppose K_X is not nef. but still big (i.e. X is of general type). Then Kähler-Ricci flow (2 exists in $[0, T)$ for some finite maximal time interval and $\widetilde{\omega}_t \to \widetilde{\omega}_T$ locally as $t \to T$ out of E, the stable base locus set of $[\omega_T]$, in C^∞-topology to a smooth metric $\widetilde{\omega}_T$. Over X, the convergence is weak in the sense of current and the limiting current over X, still denoted by $\widetilde{\omega}_T$, is a real, closed and positive $(1,1)$-current representing the class $[\omega_T]$ and having Lelong number 0. If $[\omega_T]$ is semi-ample, then the limit $\widetilde{\omega}_T$ has continuous local potential over X. In this case, the scalar curvature would have to blow up.

This situation of not being nef. and having finite time singularity is not very satisfying from either algebraic geometry or geometry analysis standing points. So the discussion below comes up very naturally.

3. Algebraic surgery and weak flow

In the finite time singularity case, the limit doesn't satisfy a geometric identity (as Kähler-Einstein equation). The limit would depend on the initial metric. This provides the motivation to "continue" the flow to time infinity. As for Ricci flow, we need to do surgery. In the current setting, the surgery should be of more algebraic geometry flavor, i.e. blow-up, flip and so on. There are two types of work for this purpose. One is to learn more about the flow metric singularity. This helps to come up better way to continue the flow in some weak sense. There are some related results regarding this direction. For example, in the setting of Theorem 3, we already know the scalar curvature is uniformly controlled for all time ([6]), while in the finite time singularity case, the scalar curvature always blows up ([7]). The other direction is to construct weak Kähler-Ricci flow in general. The following result is obtained in ([1]).

Theorem 5 (Chen-Tian-Z.) For $\varphi_0 \in PSH_{\omega_0}(X) \cap L^\infty(X)$ with volume form in $L^p(X)(p \geq 3)$, there is a unique smooth solution $g(t)$ of (2) for $t \in (0,T)$ such that $\frac{\omega^n_{g(t)}}{\omega^n} \to \frac{\omega^n_{\varphi_0}}{\omega^n}$ as $t \to 0$ in L^2-topology.

We can already give a nice description for the case of minimal surface of general type, which is a very special case of G. Tian's program ([4]): *for any initial metric* ω_0*, Kähler-Ricci flow (2) has a (possibly singular) solution* $\widetilde{\omega}_t$ *which converges to a (possibly singular) metric in a suitable sense as* $t \to \infty$*. Moreover, this limiting metric may be singular but should be independent of the choice of the initial metric.* In fact, it is further expected that *all singularities of this limiting metric are of rational type.* See [3] for related discussion.

References

[1] Chen, Xiuxiong; Tian, Gang; Zhang, Zhou: On the weak Kähler-Ricci flow. To appear in Transactions of AMS.

[2] Hamilton, Richard S.: Three-manifolds with positive Ricci curvature. J. Differential Geom. 17 (1982), no. 2, 255-306.

[3] Song, Jian; Tian, Gang: The Kähler-Ricci flow on surfaces of positive Kodaira dimension. Invent. Math. 170 (2007), no. 3, 609–653.

[4] Tian, Gang: New results and problems on Kähler-Ricci flow. To appear in Asterisque.

[5] Tian, Gang; Zhang, Zhou: On the Kähler-Ricci flow on projective manifolds of general type. Chinese Annals of Mathematics - Series B, Volume 27, Number 2, 179–192.

[6] Zhang, Zhou: Scalar curvature bound for Kähler-Ricci flows over minimal manifolds of general type. International Mathematics Research Notices 2009; doi: 10.1093/imrn/rnp073.

[7] Zhang, Zhou: Scalar curvature behavior for finite time singularity of Kähler-Ricci flow. To appear in Michigan Mathematical Journal.

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