Mathematisches Forschungsinstitut Oberwolfach

Report No. 45/2009

DOI: 10.4171/OWR/2009/45

Arbeitsgemeinschaft: Minimal Surfaces

Organised by
William H. Meeks, Amherst
Matthias Weber, Bloomington

October 3rd – October 9th, 2009

Abstract. The theory of Minimal Surfaces has developed rapidly in the past 10 years. There are many factors that have contributed to this development:

- Sophisticated construction methods [14,29,31] have been developed and have supplied us with a wealth of examples which have provided intuition and spawned conjectures.
- Deep curvature estimates by Colding and Minicozzi [3] give control on the local and global behavior of minimal surfaces in an unprecedented way.
- Much progress has been made in classifying minimal surfaces of finite topology or low genus in $\mathbb{R}^3$ or in other flat 3-manifolds. For instance, all properly embedded minimal surfaces of genus 0 in $\mathbb{R}^3$, even those with an infinite number of ends, are now known [21, 23, 25].
- There are still numerous difficult but easy to state open conjectures, like the genus-g helicoid conjecture: *There exists a unique complete embedded minimal surface with one end and genus g for each $g \in \mathbb{N}$*, or the related Hoffman-Meeks conjecture: *A finite topology surface with genus g and $n \geq 2$ ends embeds minimally in $\mathbb{R}^3$ with a complete metric if and only if $n \leq g + 2$.*
- Sophisticated tools from 3-manifold theory have been applied and generalized to understand the geometric and topological properties of properly embedded minimal surfaces in $\mathbb{R}^3$.
- Minimal surfaces have had important applications in topology and play a prominent role in the larger context of geometric analysis.

Mathematics Subject Classification (2000): 53A10,49Q05, 53C42.

Introduction by the Organisers

The Arbeitsgemeinschaft *Minimal Surfaces*, organised by William Meeks (Amherst) and Matthias Weber (Bloomington) was held October 3rd – October 9th, 2009.

This meeting was well attended with about 40 participants with broad geographic representation. Both well-established researchers, postdocs, and graduate students were present. As is customary for the style of the Arbeitsgemeinschaft, the majority of the talks was given by graduate students and young postdocs.

Arbeitsgemeinschaft: Minimal Surfaces

Table of Contents

Matthias Weber
Introduction: Examples and Methods 2549

Rob Kusner
Conjugate surfaces, Jenkins-Serrin graphs and Scherk towers 2551

M. Magdalena Rodríguez
The classification of doubly periodic minimal tori with parallel ends 2551

Robert W. Neel
Universal superharmonic functions and their application to the conformal type of proper minimal surfaces in $\mathbb{R}^3$ 2553

William H. Meeks
Unsolved problems in minimal surface theory 2556

John M. Sullivan
Meeks' proof of Osserman's Theorem 2561

Keomkyo Seo
Estimates for stable minimal hypersurfaces 2564

Rami Younes
Construction of triply periodic minimal surfaces 2565

Emanuele Spadaro
Barrier construction 2565

Giuseppe Tinaglia
Colding-Minicozzi's one-sided curvature estimate 2566

Jacob Bernstein
Colding-Minicozzi Theory: The Calabi-Yau conjectures for embedded surfaces. 2566

Christine Breiner
Colding-Minicozzi Theory: Weak chord-arc bounds for embedded minimal disks 2568

Thilo Kuessner
Rescaling arguments and minimal laminations 2570

José Miguel Manzano
Harmonic function theory on minimal surfaces 2571

Joaquín Pérez
Complete embedded minimal ends with infinite total curvature and uniqueness of the helicoid . . . 2572

Hojoo Lee
Minimal Graphs in Nil^3 . . . 2575

Karsten Große-Brauckmann
The halfspace theorem in Nil . . . 2578

Julia Plehnert
Minimal Surfaces in $\mathbb{H}^2 \times \mathbb{R}$ . . . 2579

Abstracts

Introduction: Examples and Methods

Matthias Weber

This talk serves as a leisurely introduction, showing the evolution of minimal surfaces from classical examples to the most recent developments.

In particular, I described the singly and doubly periodic Scherk surfaces, how they form conjugate pairs and limit to catenoids and helicoids. Similarly, I showed the Riemann family [28] limiting in a horizontal foliation of $\mathbb{R}^3$ by parallel planes.

By a recent theorem of Meeks, Perez and Ros [22], all properly embedded, complete minimal surfaces of genus 0 belong to the Riemann family or its limits.

I then listed the all known examples of genus one, and briefly explained what methods were used for their construction. We have the Costa surface and its deformation to the Costa-Hoffman-Meeks family, the genus one helicoid [11–13], the Riemann-Costa fusion of Hauswirth and Pacard [10], and a similar conjectural example.

Recent deep theorems and general conjectures like the Hoffman-Meeks conjecture and the helicoid conjectures indicate that this might be all there is of genus one.

Finally, I spent some time with doubly periodic surfaces, mentioning the classification results by Lazard-Holly and Meeks for genus 0 and for genus one in the case of parallel ends by Perez, Rodriguez and Traizet [27].

References

[1] T. H. Colding and W. P. Minicozzi II. Minimal disks that are double spiral staircases. *Notices of the AMS*, 50(3):327–339, 2003. MR1954009, Zbl pre02115046.

[2] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold IV; Locally simply-connected. *Ann. of Math.*, 160:573–615, 2004. MR2123933, Zbl 1076.53069.

[3] T. H. Colding and W. P. Minicozzi II. Embedded minimal disks. In *Global theory of minimal surfaces*, pages 405–438. American Mathematical Society, Providence, RI, for the Clay Mathematics Institute, Cambridge, MA, edited by D. Hoffman, 2005. MR2167253, Zbl 1078.53002.

[4] T. H. Colding and W. P. Minicozzi II. Shapes of embedded minimal surfaces. *Proc. National Academy of Sciences*, 103:11106–11111, 2006. MR2242650, Zbl pre05272457.

[5] T. H. Colding and W. P. Minicozzi II. The Calabi-Yau conjectures for embedded surfaces. *Ann. of Math.*, 167:211–243, 2008. MR2373154, Zbl 1142.53012.

[6] P. Collin and H. Rosenberg. Construction of harmonic diffeomorphisms and minimal graphs. Preprint, 2007.

[7] B. Daniel. Isometric immersions into 3-dimensional homogeneous manifolds. *Comment. Math. Helv.*, 82(1):87–131, 2007. MR2296059, Zbl 1123.53029.

[8] B. Daniel and L. Hauswirth. Half-space theorem, embedded minimal annuli and minimal graphs in the Heisenberg group. *Proc. Lond. Math. Soc. (3)*, 98(2):445–470, 2009. MR2481955, Zbl pre05530823.

[9] I. Fernandez and P. Mira. Holomorphic quadratic differentials and the Berstein problem in Heisenberg space. To appear in Trans. Amer. Math. Soc., arxiv:0705.1436v2.

[10] L. Hauswirth, F. Pacard, *Higher genus Riemann minimal surfaces*, Inventiones Mathematicae **169** (2007), 569–620.
[11] D. Hoffman, H. Karcher, F. Wei, *Adding handles to the helicoid*, Bulletin of the AMS, New Series **29** (1993), 77–84.
[12] D. Hoffman, M. Weber, M. Wolf, *An embedded genus-one helicoid*, Ann. of Math. **169** (2009), 347–448.
[13] D. Hoffman and B. White, *Genus-One Helicoids from a Variational Point of View*, preprint.
[14] N. Kapouleas. Complete embedded minimal surfaces of finite total curvature. *J. Differential Geom.*, 47(1):95–169, 1997. MR1601434, Zbl 0936.53006.
[15] H. Karcher. Embedded minimal surfaces derived from Scherk's examples. *Manuscripta Math.*, 62:83–114, 1988. MR0958255, Zbl 658.53006.
[16] H. Karcher. Construction of minimal surfaces. *Surveys in Geometry*, pages 1–96, 1989. University of Tokyo, 1989, and Lecture Notes No. 12, SFB256, Bonn, 1989.
[17] H. Karcher. Introduction to conjugate Plateau constructions. In *Global theory of minimal surfaces*, pages 489–524. American Mathematical Society, Providence, RI, for the Clay Mathematics Institute, Cambridge, MA, edited by D. Hoffman, 2005. MR2167253, Zbl 1078.53002.
[18] L. Mazet and M. Traizet. A quasi-periodic minimal surface. *Commentarii mat. Helv.*, 83:573–601, 2008.
[19] W. H. Meeks III and J. Pérez. Embedded minimal surfaces of finite topology. Work in progress.
[20] W. H. Meeks III and J. Pérez. Finite type annular ends for harmonic functions. Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.
[21] W. H. Meeks III and J. Pérez. Properly embedded minimal planar domains with infinite topology are riemann minimal examples. To appear in Current Developments in Mathematics 2008 Conference Proceedings, Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.
[22] W.H. Meeks III, J. Pérez, A. Ros, *Properly embedded minimal planar domains*, preprint.
[23] W. H. Meeks III, J. Pérez, and A. Ros. The embedded Calabi-Yau conjectures for finite genus. Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.
[24] W. H. Meeks III and H. Rosenberg. The global theory of doubly periodic minimal surfaces. *Invent. Math.*, 97:351–379, 1989. MR1001845, Zbl 676.53068.
[25] W. H. Meeks III and H. Rosenberg. The uniqueness of the helicoid. *Ann. of Math.*, 161:723–754, 2005. MR2153399, Zbl 1102.53005.
[26] W. H. Meeks III and H. Rosenberg. The minimal lamination closure theorem. *Duke Math. Journal*, 133(3):467–497, 2006. MR2228460, Zbl 1098.53007.
[27] J. Pérez, M. Rodríguez, and M. Traizet. The classification of doubly periodic minimal tori with parallel ends. *Journal of Diff. Geometry*, 69(3):523–577, 2005. MR2170278, Zbl pre05004289.
[28] B. Riemann, *Über die Fläche vom kleinsten Inhalt bei gegebener Begrenzung*, Abh. Königl, d. Wiss. Göttingen, Mathem. Cl. **13** (1867), 3–52.
[29] M. Traizet. Construction of minimal surfaces by gluing Weierstrass Representations. In *Global theory of minimal surfaces*, pages 439–452. American Mathematical Society, Providence, RI, for the Clay Mathematics Institute, Cambridge, MA, edited by D. Hoffman, 2005. MR2167253, Zbl 1078.53002.
[30] M. Traizet. Adding handles to Riemann minimal examples. *J. Differential Geom.*, 79:243–275, 2008. MR2420019, Zbl pre05294962.
[31] M. Weber and M. Wolf. Teichmüller theory and handle addition for minimal surfaces. *Ann. of Math.*, 156:713–795, 2002. MR1954234, Zbl 1028.53009.

Conjugate surfaces, Jenkins-Serrin graphs and Scherk towers

Rob Kusner

The goal of this introductory talk was to develop the basic tools needed to construct the Scherk tower M_α which is a singly-periodic minimal surface that scales down to a pair of planes meeting at angle $\alpha \in (0, \frac{\pi}{2}]$. It has been conjectured [7–9] that any connected minimal surface with this property is an M_α (or possibly a catenoid in the limiting case $\alpha = 0$). The right-angled Scherk tower $M_{\frac{\pi}{2}}$ can be constructed directly by solving the Dirichlet problem for a minimal graph $x_3 = u(x_1, x_2)$ over a unit strip $\{-\infty < x_1 < \infty,\ \ 0 < x_2 < 1\}$ with boundary values $u = |x_1|$, then using Schwarz reflection; it arises as the limit of various finite topology minimal surfaces $[2, 6, 10]$ in S^3 and $\mathbf{R}^3$. The more general towers M_α used in desingularization constructions such as [4] require conjugate surface methods [5], especially Krust's theorem [1], applied to Jenkins-Serrin graphs [3] over a unit rhombus of angle α; and these results depend, in turn, on basic force-balancing ideas (going back, essentially, to Archimedes).

References

[1] K. Grosse-Brauckmann, M. Wolgemuth, *The gyroid is embedded and has constant mean curvature companions*, Calc. Var. **4** (1996), 499-523

[2] D. Hoffman, W. H. Meeks III, *Complete embedded minimal surfaces of finite total curvature*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), 134-136

[3] H. Jenkins, J. Serrin, *Variational problems of minimal surface type II. Boundary value problems for the minimal surface equation*, Arch. Rat. Mech. Anal. **21** (1966), 321-342

[4] N. Kapouleas, *Complete embedded minimal surfaces of finite total curvature*, J. Differential Geom. **47** (1997), 95169

[5] H. Karcher, *Embedded minimal surfaces derived from Scherk's examples*, Manuscripta Math. **62** (1988), 83-114

[6] H. B. Lawson, *Complete minimal surfaces in S^3*, Ann. Math. **92** (1970), 335-374

[7] W. H. Meeks III, *Topological properties of properly embedded minimal surfaces in $\boldsymbol{R}^3$*, Topology and Geometry of Manifolds: 2001 Georgia International Topology Conference, A. M. S. Proc. Symp. Pure Math. **71** (2003), 69-79

[8] W. H. Meeks III, M. Wolf, *Minimal surfaces with the area growth of two planes; the case of infinite symmetry* J. Amer. Math. Soc. (to appear).

[9] R. Schoen, *Uniqueness, symmetry and embeddedness of minimal surface*, J. Differential Geom. **18** (1983), 791-809

[10] M. Weber, M. Wolf, *Teichmüller theory and handle addition for minimal surfaces*, Ann. Math. **156** (2002), 713-795

The classification of doubly periodic minimal tori with parallel ends

M. Magdalena Rodríguez

Scherk [8] found the first (connected, orientable) properly embedded minimal surface in $\mathbb{R}^3$, invariant by two linearly independent translations (we will shorten by saying a *doubly periodic minimal surface*). This surface fits naturally into a 1-parameter family $\mathcal{F} = \{F_\theta\}_\theta$ of examples, called doubly periodic Scherk minimal surfaces. Assume that one of the periods points to the x_2-axis. In the quotient by the period lattice generated by its shortest period vectors, each F_θ has genus zero and four asymptotically flat annular ends: two left and two right ones. This kind of annular ends are called *Scherk-type ends*. The parameter θ in this family $\mathcal{F}$ is the angle between left and right ends. Lazard-Holly and Meeks [4] proved that these are the only doubly periodic minimal surfaces in $\mathbb{R}^3$ which have genus zero in the quotient by their periods, up to translations, rotations and homotheties. Moreover, the angle map $\theta : \mathcal{F} \to (0, \pi)$ is a diffeomorphism.

We construct in [7] a 3-parameter family $\mathcal{K}$ of doubly periodic minimal surfaces in $\mathbb{R}^3$ with genus one and four parallel Scherk-type ends in the quotient, called *KMR examples*, including the examples given by Karcher [2,3] and by Meeks and Rosenberg [5]. We prove that $\mathcal{K}$ is a 3-dimensional real analytic manifold and the degenerate limits of sequences in $\mathcal{K}$ are the catenoid, the helicoid, any singly or doubly periodic Scherk minimal surface and any Riemann minimal example. Furthermore $\mathcal{K}$ is self-conjugate, in the sense that the conjugate surface of any element in $\mathcal{K}$ also belongs to $\mathcal{K}$.

Let $\mathcal{S}$ be the space of singly periodic minimal surfaces, which is on the boundary of $\mathcal{K}$, and define $\widetilde{\mathcal{K}} = \mathcal{K} \cup \mathcal{S}$. We consider the map $\mathcal{C} : \widetilde{\mathcal{K}} \to \mathbb{R}^+ \times \mathbb{R}^2$, which associates to each surface in $\widetilde{\mathcal{K}}$ two geometric invariants: the length of the period at its ends and the horizontal part of the flux along a nontrivial homology class with vanishing period vector. We prove that $\mathcal{C}$ is a diffeomorphism, obtaining a description of the space $\mathcal{K}$ of KMR examples.

Theorem 1. *[7] $\mathcal{K}$ is diffeomorphic to* $(\mathbb{R}^2 - \{(0,0)\}) \times \mathbb{R}$.

The main focus of the talk will be on proving the following uniqueness of the KMR examples.

Theorem 2. *[6] If M is a doubly periodic minimal surface of $\mathbb{R}^3$ with parallel ends and genus one in the quotient, then $M \in \mathcal{K}$.*

We remark that Theorem 2 does not hold if we remove the hypothesis on the ends to be parallel, as demonstrate the 4-ended tori discovered by Hoffman, Karcher and Wei [1].

Since any KMR example is invariant by the translation of $\frac{1}{2}(T_1 + T_2)$, where T_1, T_2 generate the period lattice, then Theorem 2 also gives a classification of all properly embedded minimal Klein bottles with parallel ends in doubly periodic quotients of $\mathbb{R}^3$.

References

[1] D. Hoffman, H. Karcher, F. Wei, *The singly periodic genus-one helicoid*, Comment. Math. Hel. **74** (1999), 248-279.

[2] H. Karcher, *Embedded minimal surfaces derived from Scherk's examples*, Manuscripta Math. **62** (1988), 83–114.

[3] H. Karcher, *Construction of minimal surfaces*, Surveys in Geometry, University of Tokyo (1989), 1–96; and Lecture Notes No. 12, SFB256, Bonn (1989).

[4] H. Lazard-Holly and W. H. Meeks III, *Classification of doubly-periodic minimal surfaces of genus zero*, Inventiones Math. **143** (2001), 1–27.

[5] W. H. Meeks III and H. Rosenberg, *The global theory of doubly periodic minimal surfaces*, Invent. Math., **97** (1989), 351–379.
[6] J. Pérez, M. M. Rodríguez, and M. Traizet, *The classification of doubly periodic minimal tori with parallel ends*, J. Diff. Geom. **69** (2005), 523-577.
[7] M. M. Rodríguez *The space of doubly periodic minimal tori with parallel ends: Standard Examples*, Michigan Math. J., **55** (2007), 103-122.
[8] H. F. Scherk, *Bemerkungen über die kleinste Fläche innerhalb gegebener Grenzen*, J. R. Angew. Math. **13** (1835), 185–208.

Universal superharmonic functions and their application to the conformal type of proper minimal surfaces in $\mathbb{R}^3$

Robert W. Neel

The material in all but the last section of this abstract can be found in [3], especially section 6. Many of the underlying ideas can be traced back to [1].

The author would like to thank Bill Meeks, Matthias Weber, and the Mathematisches Forschungsinstitut Oberwolfach for inviting him to speak at the Minimal Surfaces Arbeitsgemeinschaft.

1. Basic notions

Underlying any minimal surface is a Riemann surface. This is part of the Weierstrass data, and the determination of the underlying Riemann surface of a minimal surface, either exactly or in terms of some broad class, is an important problem in the field.

Recall that harmonic functions are invariant under a conformal change of metric, so the question of whether or not a surface admits a non-constant, bounded harmonic function depends only on the conformal structure of the manifold. We refer to this question as the question of the conformal type of the surface (note that this term is often used for other, similar properties).

There are a few ways of introducing Brownian motion on a manifold; intuitively, we think of it as the continuous version of an isotropic random walk on a manifold. More precisely, it solves the martingale problem for half the Laplacian; that is, if B_t is Brownian motion and f is smooth and compactly supported, then $f(B_t) - f(x_0) - \int_0^t \frac{1}{2}\Delta f(B_s)\, ds$ is a martingale. (See [5] or [2] for more background on Brownian motion on Riemannian manifolds.) Brownian motion on a surface M is called *recurrent* if any of the following equivalent conditions hold:

- There exists an open, precompact set $A \subset M$ and a point $x \in M$ (with $x \notin \overline{A}$) such that Brownian motion started at x almost surely hits A.
- For any $x \in M$ and open, precompact $A \subset M$, Brownian motion started at x almost surely hits A.
- Brownian motion returns infinitely often to any (equivalently, some) open, precompact A, almost surely.

If M is not recurrent, it is *transient*. In this case, Brownian motion almost surely has a last time in any compact set. We note that recurrence and transience depend only on the conformal structure of M.

Our next task is to consider the relationship between bounded harmonic functions and Brownian motion. If M is recurrent, M admits no non-constant bounded harmonic functions. On the other hand, if M is transient, it may or may not admit a non-constant bounded harmonic function. Fortunately, we when consider surfaces with non-empty boundaries, the natural notions for bounded harmonic functions and Brownian motion are equivalent. In particular, a surface M with non-empty boundary ∂M is *parabolic* if any of the following equivalent conditions hold:

- Any bounded harmonic function on M is determined by its boundary values on ∂M.
- There exists a point $x \in M$ such that Brownian motion started from x hits ∂M almost surely.
- Brownian motion started from any point hits ∂M almost surely.

Note that if a surface M is recurrent or parabolic (depending on whether ∂M is empty), then M with a compact set added or removed is also recurrent or parabolic.

2. Universal superharmonic functions

Recall that a function is *superharmonic* if its Laplacian is everywhere non-positive. It is well known that if a surface M with non-empty boundary admits a positive, proper superharmonic function, then M is parabolic.

Definition 1. *Let U be a non-empty, open subset of $\mathbb{R}^3$. A function $f : U \to \mathbb{R}$ is a* universal superharmonic function on U *if the restriction of f to any minimal surface (possibly with boundary) in U is superharmonic.*

For example the x_i are universal superharmonic functions on all of $\mathbb{R}^3$. More interestingly, let $r = \sqrt{x_1^2 + x_2^2}$. Then, for any minimal surface M,

$$|\Delta_M \log r| \leq \frac{|\nabla_M x_3|^2}{r^2} \quad \text{on } M \setminus \{x_3\text{-axis}\}.$$

It follows that

- $\log r - x_3^2$ is a universal superharmonic function on $\left\{r \geq 1/\sqrt{2}\right\}$
- $\log r - x_3 \arctan x_3 + \frac{1}{2}\log\left(x_3^2+1\right)$ is a universal superharmonic function on $\left\{r \geq \sqrt{1+x_3^2}\right\}$

Consider the slab $S(C) = \{0 \leq x_3 \leq C\}$ for some $C > 0$. Using that $\log r - x_3^2 + C^2$ is a proper, positive, superharmonic function on any properly immersed minimal surface contained in $S(C) \cap \{r \geq 1\}$, we outline the proof of (see Theorem 6.7 of [3] and Theorem 3.1 of [1])

Theorem 2. *Let M be a properly immersed minimal surface, possibly with boundary, contained in $\{x_3 \geq 0\}$. If $\partial M = \emptyset$, then $M = \{x_3 = c\}$ for some $c \geq 0$. If*

$\partial M \neq \emptyset$, then M is parabolic. In particular, if a properly immersed minimal surface (without boundary) intersects any plane in a compact set, it is recurrent.

3. A more geometric application: area growth

We now wish to see how the universal superharmonic function $f = \log r - x_3^2$ can be used to control the growth of the area of a properly immersed minimal surface-with-boundary, which is contained in a slab. Such a surface arises when considering certain ends of properly embedded minimal surfaces. In particular, we sketch the proof of the fact that such a minimal surface-with-boundary, which we denote M, has quadratic area growth. That is,

$$\int_{M\cap\{r\leq t\}} dA = Ct^2 + o\left(t^2\right).$$

The argument relies on the divergence theorem and the relationship between $\Delta_M f$ and two more geometric quantities, namely $|\nabla_M x_3|^2$ and $\Delta_M \log r$.

We note that analogous results, namely parabolicity and quadratic area growth, can be proven for minimal surfaces-with-boundary contained between two half-catenoids, rather than contained in a slab, by using the "other" universal superharmonic function mentioned above,

$$f = \log r - x_3 \arctan x_3 + \frac{1}{2}\log\left(x_3^2 + 1\right).$$

Recall that a minimal surface is contained between two half-catenoids if $|x_3| \leq C \log r$ for large r.

4. More Brownian motion

We again consider a properly immersed minimal surface M, possibly with boundary, contained in the halfspace $\{x_3 \geq 0\}$. This time we wish to use Brownian motion to understand bounded harmonic functions. Using that x_3 composed with Brownian motion on M is a martingale, we give an alternative proof of Theorem 2 (see the proof of Theorem 2.2 of [4] for the basic approach). In particular, the argument seems to rely on the same underlying structure as the proof mentioned above, but doesn't make any use of universal superharmonic functions.

In light of this last point, it might be of some interest to better understand the relationship between these two approaches. For example, is there a similar Brownian motion-based proof of the analogous result for a minimal surface contained between two half-catenoids? More generally, does Brownian motion encode other information about ends of properly embedded minimal surfaces contained between two plane or half-catenoids?

References

[1] Pascal Collin, Robert Kusner, William H. Meeks, III, and Harold Rosenberg, *The topology, geometry and conformal structure of properly embedded minimal surfaces*, J. Differential Geom. **67** (2004), no. 2, 377–393.

[2] Elton P. Hsu, *Stochastic analysis on manifolds*, Graduate Studies in Mathematics, vol. 38, American Mathematical Society, Providence, RI, 2002.

[3] William H. Meeks, III and Joaquín Pérez, *The classical theory of minimal surfaces*, preprint.
[4] Robert W. Neel, *A martingale approach to minimal surfaces*, J. Funct. Anal. **256** (2009), no. 8, 2440–2472.
[5] Daniel W. Stroock, *An introduction to the analysis of paths on a Riemannian manifold*, Mathematical Surveys and Monographs, vol. 74, American Mathematical Society, Providence, RI, 2000.

Unsolved problems in minimal surface theory

William H. Meeks

There are many interesting and important unsolved problems in the classical theory of minimal surfaces in $\mathbb{R}$ and in other homogeneous 3-manifolds. The discussion of some of these problems is the topic of this note.

1. Outstanding problems and conjectures.

I have listed in the statement of each conjecture the principal researchers to whom the conjecture might be attributed. All of these problems appear in my survey article [12] with Joaquin Perez and some appear in [8] or in [9], along with further discussions. Also see the author's 1978 book [10] for a long list of conjectures in the subject.

Conjecture 1 (Convex Curve Conjecture, Meeks)**.** *Two convex Jordan curves in parallel planes cannot bound a compact minimal surface of positive genus.*

There are some partial results on the Convex Curve Conjecture, under the assumption of some symmetry on the curves (see [19, 24, 25]). Also, the results of Meeks and White [19, 20] indicate that the Convex Curve Conjecture probably holds in the more general case where the two convex planar curves do not necessarily lie in parallel planes, but rather lie on the boundary of their convex hull; in this case, the planar Jordan curves are called *extremal.* Recent results by Ekholm, White and Wienholtz [6] imply that every compact, orientable minimal surface that arises as a counterexample to the Convex Curve Conjecture is embedded.

Conjecture 2 (4π-Conjecture, Meeks, Yau, Nitsche)**.** *If* Γ *is a simple closed curve in* $\mathbb{R}^3$ *with total curvature at most* 4π*, then* Γ *bounds a unique compact, orientable, branched minimal surface and this unique minimal surface is an embedded disk.*

Nitsche [22] proved that a regular analytic Jordan curve in $\mathbb{R}^3$ whose total curvature is at most 4π bounds a unique minimal disk and Meeks and Yau [21] demonstrated the conjecture if Γ is a C^2-extremal curve. Concerning this weakening of Conjecture 2 by removing the orientability assumption on the minimal surface spanning Γ, we mention the following generalized conjecture due to Ekholm, White and Wienholtz [6]:

> *Besides the unique minimal disk given by Nitsche's Theorem [22], only one or two Möbius strips can occur; and if the total curvature of* Γ *is at most* 3π*, then there are no such Möbius strip examples.*

Gulliver and Lawson [7] proved that if Σ is an orientable, stable minimal surface with compact boundary that is properly embedded in the punctured unit ball $\mathbb{B}(1) - \{\vec{0}\}$ of $\mathbb{R}^3$, then its closure is a compact, embedded minimal surface. If Σ is not stable, then the corresponding result is not known. Recent results by Meeks, Pérez and Ros [14, 16] indicate that a more general result might hold. In fact, Meeks, Pérez and Ros have conjectured that the set $\{\vec{0}\}$ can be replaced by any closed set in $\mathbb{R}^3$ with zero 1-dimensional Hausdorff measure (see Conjecture 4 below). It is elementary to prove that the following conjecture holds for any minimal surface of finite topology (in fact, with finite genus).

Conjecture 3 (Isolated Singularities Conjecture, Gulliver, Lawson)**.** *The closure of a properly embedded minimal surface with compact boundary in the punctured ball* $\mathbb{B}(1) - \{\vec{0}\}$ *is a compact, embedded minimal surface.*

Conjecture 4 (Fundamental Singularity Conjecture, Meeks, Pérez, Ros)**.** *If* $A \subset \mathbb{R}^3$ *is a closed set with zero* 1*-dimensional Hausdorff measure and* $\mathcal{L}$ *is a minimal lamination of* $\mathbb{R}^3 - A$*, then* $\mathcal{L}$ *extends to a minimal lamination of* $\mathbb{R}^3$*.*

Conjecture 5 (Connected Graph Conjecture, Meeks)**.** *A minimal graph in* $\mathbb{R}^3$ *with zero boundary values over a proper, possibly disconnected domain in* $\mathbb{R}^2$ *can have at most two non-planar components. If the graph also has sublinear growth, then such a graph with no planar components is connected.*

Tkachev [26] proved that the number of disjointly supported minimal graphs is at most three.

In the discussion of the conjectures that follow, it is helpful to fix some notation for certain classes of complete embedded minimal surfaces in $\mathbb{R}^3$.

- Let $\mathcal{C}$ be the space of connected, complete, embedded minimal surfaces.
- Let $\mathcal{P} \subset \mathcal{C}$ be the subspace of properly embedded surfaces.
- Let $\mathcal{M} \subset \mathcal{P}$ be the subspace of surfaces with more than one end.

Conjecture 6 (Finite Topology Conjecture I, Hoffman, Meeks)**.** *An orientable surface* M *of finite topology with genus* g *and* k *ends,* $k \neq 0, 2$*, occurs as a topological type of a surface in* $\mathcal{C}$ *if and only if* $k \leq g + 2$*.*

Conjecture 7 (Finite Topology Conjecture II, Meeks, Rosenberg)**.** *For every non-negative integer* g*, there exists a unique non-planar* $M \in \mathcal{C}$ *with genus* g *and one end.*

The Finite Topology Conjectures I and II together propose the precise topological conditions under which a non-compact orientable surface of finite topology can be properly minimally embedded in $\mathbb{R}^3$. What about the case where the non-compact orientable surface M has infinite topology? In this case, either M has infinite genus or M has an infinite number of ends. By work in [5], such an M must have at most two limit ends. Work in [14] implies that such an M cannot have one limit end and finite genus. We claim that these restrictions are the only ones.

Conjecture 8 (Infinite Topology Conjecture, Meeks). *A non-compact, orientable surface of infinite topology occurs as a topological type of a surface in $\mathcal{P}$ if and only if it has at most one or two limit ends, and when it has one limit end, then its limit end has infinite genus.*

Conjecture 9 (Liouville Conjecture, Meeks). *If $M \in \mathcal{P}$ and $h\colon M \to \mathbb{R}$ is a positive harmonic function, then h is constant.*

Conjecture 10 (Multiple-End Recurrency Conjecture, Meeks). *If $M \in \mathcal{M}$, then M is recurrent for Brownian motion.*

Conjecture 11 (Isometry Conjecture, Choi, Meeks, White). *If $M \in \mathcal{C}$, then every intrinsic isometry of M extends to an ambient isometry of $\mathbb{R}^3$. Furthermore, if M is not a helicoid, then it is minimally rigid, in the sense that any isometric minimal immersion of M into $\mathbb{R}^3$ is congruent to M.*

The Isometry Conjecture is known to hold if either $M \in \mathcal{M}$ (Choi, Meeks and White [2]), M is doubly-periodic (Meeks and Rosenberg [17]), M is periodic with finite topology quotient (Meeks [11] and Pérez [23]) or M has finite genus. One can reduce the validity of the Isometry Conjecture to checking that whenever $M \in \mathcal{P}$ has one end and infinite genus, then there exists a plane in $\mathbb{R}^3$ that intersects M in a set that contains a simple closed curve.

The One-Flux Conjecture below implies the Isometry Conjecture.

Conjecture 12 (One-Flux Conjecture, Meeks, Pérez, Ros).
Let $M \in \mathcal{C}$ and let $\mathcal{F} = \{F(\gamma) = \int_\gamma \mathrm{Rot}_{90^\circ}(\gamma') \mid \gamma \in H_1(M,\mathbb{Z})\}$ be the abelian group of flux vectors for M. If $\mathcal{F}$ has rank at most 1, then M is a plane, a helicoid, catenoid, a Riemann minimal example or a doubly-periodic Scherk minimal surface.

Conjecture 13 (Scherk Uniqueness Conjecture, Meeks, Wolf). *If M is a connected, properly immersed minimal surface in $\mathbb{R}^3$ and $\mathrm{Area}(M \cap \mathbb{B}(R)) \leq 2\pi R^2$ holds in balls $\mathbb{B}(R)$ of radius R, then M is a plane, a catenoid or one of the singly-periodic Scherk minimal surfaces.*

Conjecture 14 (Unique Limit Tangent Cone Conjecture, Meeks). *If $M \in \mathcal{P}$ is not a plane and has quadratic area growth, then $\lim_{t\to\infty} \frac{1}{t}M$ exists and is a minimal, possibly non-smooth cone over a finite balanced configuration of geodesic arcs in the unit sphere, with common ends points and integer multiplicities. Furthermore, if M has area not greater than $2\pi R^2$ in balls of radius R, then the limit tangent cone of M is either the union of two planes or consists of a single plane of multiplicity two passing through the origin.*

If $M \in \mathcal{C}$ has finite topology, then M has finite total curvature or is asymptotic to a helicoid [1, 4, 12, 18]. It follows that for any such a surface M, there exists a constant $C_M > 0$ such that the injectivity radius function $I_M\colon M \to (0,\infty]$ satisfies

$$I_M(p) \geq C_M\|p\|, \quad p \in M.$$

Recent work of Meeks, Pérez and Ros in [15, 16] indicates that this linear growth property of the injectivity radius function should characterize the examples in $\mathcal{C}$ with finite topology.

Conjecture 15 (Injectivity Radius Growth Conjecture, Meeks, Pérez, Ros)**.** *An $M \in \mathcal{C}$ has finite topology if and only if its injectivity radius function grows at least linearly with respect to the extrinsic distance from the origin.*

Conjecture 16 (Negative Curvature Conjecture, Meeks, Pérez, Ros)**.** *If $M \in \mathcal{C}$ has negative curvature, then M is a catenoid, a helicoid or one of the singly or doubly-periodic Scherk minimal surfaces.*

Conjecture 17 (Four Point Conjecture, Meeks, Pérez, Ros)**.**
Suppose $M \in \mathcal{C}$. If the Gauss map of M omits 4 points on $\mathbb{S}^2(1)$, then M is a singly or doubly-periodic Scherk minimal surface.

The following conjecture is related to the Calabi-Yau conjectures.

Conjecture 18 (Finite Genus Properness Conjecture, Meeks, Pérez, Ros)**.** *If $M \in \mathcal{C}$ and M has finite genus, then $M \in \mathcal{P}$.*

In [13], Meeks, Pérez and Ros proved Conjecture 18 under the additional hypothesis that M has a countable number of ends, thereby generalizing the result in the case of finite topology by Colding and Minicozzi [3].

Conjecture 19 (Embedded Calabi-Yau Conjectures)**.** *(Martín, Meeks, Nadirashvili, Pérez, Ros)*

- *1.: A connected, open surface M properly minimally embeds in every smooth bounded domain of $\mathbb{R}^3$ as a complete surface if and only if it is orientable and every end of M have infinite genus.*
- *2.: A connected, non-orientable open surface M properly minimally embeds in some bounded domain in $\mathbb{R}^3$ as a complete surface if and only if every end of M has infinite genus.*

Any end of a surface $M \in \mathcal{C}$ with finite total curvature is C^2-asymptotic to the end of a plane or catenoid ([25]). The next conjecture can be viewed as a generalization of this result.

Conjecture 20 (Standard Middle End Conjecture, Meeks)**.** *If $M \in \mathcal{M}$ and $E \subset M$ is a one-ended representative for a middle end of M, then E is C^0-asymptotic to the end of a plane or catenoid. In particular, if M has two limit ends, then each middle end is C^0-asymptotic to a plane.*

References

[1] J. Bernstein and C. Breiner, *Conformal structure of minimal surfaces with finite topology.* Preprint available at `http://arxiv.org/abs/0810.4478v1`.

[2] T. Choi and W. H. Meeks III and B. White, *A Rigidity Theorem for Properly Embedded Minimal Surfaces in $\mathbb{R}^3$*, J. Differential Geom. **32** (1990), 65–76.

[3] T. H. Colding and W. P. Minicozzi II, *The Calabi-Yau conjectures for embedded surfaces*, Ann. of Math. **167**(2008), 211–243.

[4] P. Collin, *Topologie et courbure des surfaces minimales de* $\mathbb{R}^3$, Ann. of Math. (2) **145** (1997), 1–31.

[5] P. Collin, R. Kusner, W. H. Meeks III and H. Rosenberg, *The Geometry, Conformal Structure and Topology of minimal surfaces with infinite topology*, J. Differential Geom. **67**(2004), 377–393.

[6] T. Ekholm, B. White and D. Wienholtz, *Embeddedness of minimal surfaces with total curvature at most* 4π, Ann. of Math.**155** (2002), 209–234.

[7] R. Gulliver and H. B. Lawson, *The structure of minimal hypersurfaces near a singularity*, Proc. Symp. Pure Math.**44** (1986), 213–237.

[8] W. H. Meeks III, *Geometric results in classical minimal surface theory*, in Surveys in Differential Geometry **8**, pages 269–306. International Press, edited by S.T. Yau, 2003.

[9] W. H. Meeks III, "Global problems in classical minimal surface theory", in *Global theory of minimal surfaces*, pages 453–470. American Mathematical Society, Providence, RI, for the Clay Mathematics Institute, Cambridge, MA, edited by D. Hoffman, 2005.

[10] W. H. Meeks III, *Lectures on Plateau's Problem.* Instituto de Matematica Pura e Aplicada (IMPA), Rio de Janeiro, Brazil, 1978.

[11] W. H. Meeks III, *The Theory of Triply-Periodic Minimal Surfaces*, Indiana Univ. Math. J. **39** No. 3 (1990), 877–936.

[12] W. H. Meeks III and J. Pérez, *Embedded minimal surfaces of finite topology*, Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[13] W. H. Meeks III, J. Pérez and A. Ros, *The embedded Calabi-Yau conjectures for finite genus*, Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[14] W. H. Meeks III, J. Pérez and A. Ros, *The geometry of minimal surfaces of finite genus II; nonexistence of one limit end examples*, Invent. Math. **158** (2004), 323–341.

[15] W. H. Meeks III, J. Pérez and A. Ros, *The local picture theorem on the scale of topology*, Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[16] W. H. Meeks III and J. Pérez and A. Ros, *Local removable singularity theorems for minimal and H-laminations*, Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[17] W. H. Meeks III and H. Rosenberg, *The Global Theory of Doubly Periodic Minimal Surfaces*, Invent. Math. **97** (1989), 351–379.

[18] , W. H. Meeks III and H. Rosenberg, *The uniqueness of the helicoid*, Ann. of Math. **161** (2005), 723–754.

[19] W. H. Meeks III and B. White, *Minimal Surfaces Bounded By Convex Curves in Parallel Planes*, Comment. Math. Helvetici **66** (1991), 263–278.

[20] W. H. Meeks III and B. White, *The Space of Minimal Annuli Bounded by an Extremal Pair of Planar Curves*, Communications in Analysis and Geometry **1** No. 3 (1993), 415–437.

[21] W. H. Meeks III, and S. T. Yau, *The Existence of Embedded Minimal Surfaces and the Problem of Uniqueness*, Math. Z. **179**(1982), 151–168.

[22] J.C.C. Nitsche, *A new uniqueness theorem for minimal surfaces*, Arch. Rat. Mech. Anal. **52**, (1973), 319–329.

[23] J. Pérez, *A rigidity theorem for periodic minimal surfaces*, Comm. in Analysis and Geom. **7** No. 1 (1999), 95–104.

[24] A. Ros, *Embedded minimal surfaces: forces, topology and symmetries*, Calc. Var.**4** (1996),469–496.

[25] R. Schoen, *Uniqueness, Symmetry, and Embeddedness of Minimal Surfaces*, J. Differential Geom.,**18** (1983), 791–809.

[26] V. G. Tkachev, *Disjoint Minimal Graphs*, Annals of Global Analysis and Geometry **35** No. 2 (2009), 139–155.

Meeks' proof of Osserman's Theorem

John M. Sullivan

In 1964, Bob Osserman focused attention on minimal surfaces of finite total curvature by proving [4] the following:

Theorem 1. *If Σ is a complete oriented immersed minimal surface in $\mathbb{R}^3$ with finite total curvature, then Σ has finite conformal type and its Gauss map extends holomorphically across each end. Thus the total curvature of Σ is a multiple of 4π.*

Here, *finite conformal type* means that Σ has finite topology – finite genus g and a finite number k of (annular) ends – and furthermore that each end is conformally a punctured disk. That is, Σ is conformally a compact surface $\overline{\Sigma}$ of genus g with k punctures. Osserman's theorem says the Gauss map G extends to $\overline{G}\colon \overline{\Sigma} \to \hat{\mathbb{C}}$, so the total Gauss curvature is 4π times the degree of this map. (In the same paper, Osserman also proved that the rest of the Weierstrass data extends appropriately to $\overline{\Sigma}$, and considered analogous results for $\Sigma^2 \subset \mathbb{R}^n$.)

Osserman's proof depends on the machinery developed by Huber for subharmonic functions. Huber [2] showed that any complete surface metric whose curvature is nonnegative outside a compact set has finite conformal type. Osserman could apply Huber's result after constructing a complete flat metric on each end.

In 1995, Bill Meeks gave a new, more self-contained proof [3] of Osserman's theorem. My talk at the Arbeitsgemeinschaft in Oberwolfach outlined (as much as possible in one hour) his proof. This report (with its even more restrictive page limit) focuses in particular on two small steps – Lemma 3 and Proposition 6 – which were left implicit in Meeks' paper and which were of interest to the audience at Oberwolfach.

Meeks starts by showing that an integral bound on curvature implies a pointwise bound. (Results of this nature are not surprising for solutions to elliptic PDEs.)

Proposition 2. *For any $\varepsilon > 0$ there exists $\delta > 0$ with the following property. Suppose M is a compact minimal surface (with boundary) in $\mathbb{R}^3$ and let $d\colon M \to \mathbb{R}$ denote the intrinsic distance to ∂M. If the Gauss image of M includes no spherical cap of area δ (in particular if the total curvature is less than δ) then the Gauss curvature satisfies $|K(p)| \le \varepsilon/d(p)^2$.*

The proof proceeds by contradiction. Given a sequence of surfaces forming a counterexample, we rescale each surface around a point maximizing the scale-invariant quantity $d^2|K|$ and show we can take a limit. To get the strong form of the result – involving not just the area of the Gauss image but also its shape – we need the following:

Lemma 3. *Suppose $f\colon M \to N$ is a conformal map of riemannian manifolds whose conformal factor is bounded below by λ on a geodesic ball $B_r(p) \subset M$. Then*

$$f\big(B_r(p)\big) \supset B_{\lambda r}\big(f(p)\big).$$

Proof. Given any geodesic ray from $f(p)$, we can find in its preimage an arc starting at p; this can be continued for length at least r while staying in $B_r(p)$, so the image geodesic continues for length at least λr. □

Another key ingredient is the ability to truncate each end of a minimal surface along a closed geodesic. The next proposition was originally proved by Freedman, Hass and Scott [1].

Proposition 4. *Suppose γ is a nontrivial simple closed curve on an oriented riemannian surface M. Any closed geodesic of least length in the free homotopy class $[\gamma]$ is injective.*

Proposition 5. *Consider a complete minimal surface M in $\mathbb{R}^n$. Each nontrivial free homotopy class contains a unique closed geodesic (the curve of least length).*

The proofs Meeks gives for these propositions both involve the tower of coverings $\widehat{M} \to \widetilde{M} \to M$, in which $\widehat{M}$ is simply connected and $\pi_1\big(\widetilde{M}\big) = \mathbb{Z}$ is generated by the free homotopy class $[\gamma]$ under consideration.

These results imply that any minimal surface of finite total curvature has finite topology with a pants decomposition along geodesics. In particular we can find a geodesic ∂E bounding each end E. We then set up polar coordinates on E by foliating it with geodesics perpendicular to ∂E; we let γ_r denote the circle at distance r from ∂E. The fact that E has finite total curvature implies that it has quadratic area growth, so the length of γ_r grows linearly (bounded by Cr).

The curvature bound now implies that the length of the Gauss image $G(\gamma_r) \subset \mathbb{S}^2$ converges to zero: Given $\varepsilon > 0$ we choose δ as in Proposition 2 and then choose r such that the total curvature of the end beyond γ_r is less than δ. The pointwise curvature bound shows that $|K| < \varepsilon/r^2$ along γ_{2r}, implying that $\operatorname{len}\big(G(\gamma_{2r})\big)$ is less than $2rC\sqrt{\varepsilon}/r$, which of course can be made arbitrarily small.

The proof of Osserman's theorem now follows from some topological facts about holomorphic maps, in particular from Corollary 8 below. We recall that non-constant holomorphic maps between Riemann surfaces are both *open* (open sets have open images) and *light* (points have totally disconnected preimages). Indeed Stoïlow showed [5] that any light open map between surfaces is holomorphic with respect to some complex structure; see also [6–8]. (Light open maps have also been called *interior maps.*)

Proposition 6. *Let $A := (0, 1] \times \mathbb{S}^1$ be the annulus and set $\gamma_t := \{t\} \times \mathbb{S}^1$. Suppose $F\colon A \to M$ is a light open map to a closed surface M with the property that the length of $F(\gamma_t)$ goes to zero with t. Then F has a well-defined limit as $t \to 0$.*

Proof. Set $\ell_t := \operatorname{len}\big(F(\gamma_t)\big)$. By compactness the condition $\ell_t \to 0$ is independent of the metric on M, but of course we fix a metric for the proof. Choose $\delta > 0$ smaller than the injectivity radius of M and smaller than $\operatorname{diam}(M)/4$. Discarding the outer part of A if necessary, we may assume that $\ell_t < \delta$ for all t. In particular, each $F(\gamma_t)$ is homotopically trivial and lies in a closed ball of radius $\ell_t/2 < \operatorname{diam}(M)/8$ around any of its points. We now claim that each pair of curves

$F(\gamma_t)$ and $F(\gamma_s)$ intersects. The desired convergence then follows since for $\ell_t < \varepsilon$, the image $F\big((0,t] \times \mathbb{S}^1\big)$ is contained in the ε-ball around any point of $F(\gamma_t)$.

To prove the claim, suppose $F(\gamma_t)$ and $F(\gamma_s)$ are disjoint and consider $Z := [s,t] \times \mathbb{S}^1$. Because F is open, each component of $M \setminus F(\partial Z)$ is either contained in or disjoint from $F(Z)$. Note that there is a unique "large" component C which touches both $F(\gamma_s)$ and $F(\gamma_t)$. Since $F(Z)$ is connected, it must include C. By the lemma below, C contains the open ball of radius $\delta/2$ around some $p \in C$. For some $r \in (s,t)$, we have $p \in F(\gamma_r)$, implying that $F(\gamma_r) \subset B_{\delta/2}(p) \subset C$. In particular, $F(\gamma_r)$ is disjoint from $F(\gamma_s)$ and $F(\gamma_t)$. But now we can repeat the argument restricting to $[s,r] \times \mathbb{S}^1$. Even here, F covers "most of" M. By induction, each point in M has infinitely many preimages in the compact space Z, contradicting the lightness of F. □

Lemma 7. *Suppose M is a metric space and $\varepsilon \le \operatorname{diam}(M)/8$. Then for any $a, b \in M$ there exists $p \in M$ such that the ε-ball $B_\varepsilon(p)$ is disjoint from $B_\varepsilon(a) \cup B_\varepsilon(b)$.*

Proof. Equivalently, we must find $p \notin B_{2\varepsilon}(a) \cup B_{2\varepsilon}(b)$. But if these two balls covered M, then for any q in their intersection, M would be covered by $B_{4\varepsilon}(q)$, contradicting the choice of ε. □

Corollary 8. *If $F\colon A \to \mathbb{S}^2$ is a nonconstant holomorphic map, and the length of $F(\gamma_t)$ goes to zero with t, then A is conformally a punctured disk and F extends holomorphically across the puncture.*

Proof. Suppose A were conformally $\{r < |z| \le 1\}$ for some $r > 0$. By Proposition 6, F extends to a constant along the inner boundary $|z| = r$. By Schwarz reflection we would get a holomophic function constant along an interior arc, contradicting the fact that F is nonconstant. Thus A is a punctured disk and F has a removable singularity at the puncture. □

References

[1] M. Freedman, J. Hass and P. Scott, *Closed geodesics on surfaces*, Bull. London Math. Soc. **14** (1982), 385–391.

[2] A. Huber, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv. **32** (1957), 13–72.

[3] W.H. Meeks III, *Proofs of some classical theorems in minimal surface theory*, Indiana Univ. Math. J. **54** (2005), 1031–1045.

[4] R. Osserman, *Global properties of minimal surfaces in E^3 and E^n*, Ann. of Math. (2) **80** (1964), 340–364.

[5] S. Stoïlow, *Leçons sur les principes topologiques de la théorie des fonctions analytiques*, Gauther-Villars, 1938.

[6] G.T. Whyburn, *Topological Analysis* (2nd Ed.), Princeton Univ. Press, 1964.

[7] G.T. Whyburn, *Open mappings on 2-dimensional manifolds*, J. Math. Mech. **10**, 181–197 (1961).

[8] W.P. Ziemer, *The structure of quasi-open maps*, Duke Math. J. **32** (1965) 661–671.

Estimates for stable minimal hypersurfaces

Keomkyo Seo

In this talk, we discuss the estimates for three geometric quantities of stable minimal hypersurfaces. Firstly, we consider curvature estimates for compact stable minimal surfaces in terms of the intrinsic distance from the boundary of the surface. When a surface is a minimal graph, it was proved by E. Heinz [3], E. Hopf [4], and R. Osserman [6] that

Theorem. Let $D_R := \{x^2 + y^2 < R\}$ be a disk in $\mathbb{R}^2$ of radius R centered at 0. Suppose that $u : D_R \to \mathbb{R}$ satisfies the minimal surface equation. Let $p = u(0)$ and let d be the intrinsic distance from p to the boundary $\partial u(D_R)$. Then the Gaussian curvature $K(p)$ of the graph $u(D_R)$ satisfies

$$K(p) \leq \frac{c}{d^2 W(0)^2},$$

where c is an absolute constant and $W^2 := 1 + |\nabla u|^2 \geq 1$.

In [2], this result was generalized to embedded minimal disk with bounded density or bounded total curvature, which played an important role in analyzing the local structure of embedded minimal surfaces. Since minimal graphs are stable, curvature estimate for stable minimal surfaces can be thought of as a generalization of the above estimate for graphs. For stable minimal surfaces, R. Schoen [8] proved the following.

Theorem. There exists a constant $c > 0$ such that for any orientable stable immersed minimal surface $\Sigma \subset \mathbb{R}^3$ and $p \in \Sigma$ the Gaussian curvature

$$|K(p)| < \frac{c}{d(p)^2},$$

where $d(p)$ is an intrinsic distance from p to the boundary of Σ.

The above theorem was reproved by W. Meeks [5] recently.

Secondly, we consider the following area estimates for stable minimal surfaces which was proved by Pogorelov [7], Colding-Minicozzi [1], and Meeks [5]. If $D \subset \Sigma$ is a stable minimal disk of geodesic radius r_0 on a minimal surface $\Sigma \subset \mathbb{R}^3$, then

$$\pi r_0^2 \leq Area(D) \leq \frac{4}{3}\pi r_0^2.$$

Thirdly, we provide the following estimate of the first Dirichlet eigenvalue of the Laplace operator on a complete stable minimal hypersurface M in the hyperbolic space which has finite L^2-norm of the second fundamental form on M. (See [9] for details.)

Theorem. Let M be a complete stable minimal hypersurface in $\mathbb{H}^{n+1}$ with $\int_M |A|^2 dv < \infty$. Then we have

$$\frac{(n-1)^2}{4} \leq \lambda_1(M) \leq n^2.$$

References

[1] T. Colding and W. Minicozzi, *Estimates for parametric elliptic integrands*, Int. Math. Res. Not. (2002), no. 6, 291–297.
[2] T. Colding and W. Minicozzi, *The space of embedded minimal surfaces of fixed genus in a 3-manifold. II. Multi-valued graphs in disks* Ann. of Math. (2) **160** (2004), no. 1, 69–92.
[3] E. Heinz, *Uber die Losungen der Minimalflachengleichung*, Nachr. Akad. Wiss. Gottingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. (1952), 51–56.
[4] E. Hopf, *On an inequality for minimal surfaces $z = z(x, y)$*, J. Rational Mech. Anal. **2** (1953), 519–522.
[5] W. Meeks, *Proofs of some classical theorems in minimal surface theory*, Indiana Univ. Math. J. **54** (2005), no. 4, 1031–1045.
[6] R. Osserman, *On the Gauss curvature of minimal surfaces*, Trans. Amer. Math. Soc. **96** (1960), 115–128.
[7] A. Pogorelov, *On the stability of minimal surfaces*, Dokl. Akad. Nauk SSSR **260** (1981), no. 2, 293–295.
[8] R. Schoen, *Estimates for stable minimal surfaces in three-dimensional manifolds* Seminar on minimal submanifolds, Ann. of Math. Stud. **103** (Princeton Univ. Press, Princeton, NJ, 1983) 111–126,
[9] K. Seo, *Stable minimal hypersurfaces in the hyperbolic space*, To appear in J. Korean Math. Soc.

Construction of triply periodic minimal surfaces

Rami Younes

Given a tiling $\mathcal{T}$ of the plane by straight edge polygons, which is invariant by two independent translations, we construct a family of embedded triply periodic minimal surfaces which desingularizes $\mathcal{T} \times \mathbb{R}$. For this purpose, inspired by the work of Martin Traizet, we open the nodes of singular Riemann surfaces to glue together simply periodic Karcher saddle towers, each placed at a vertex of the tiling in such a way that its wings go along the corresponding edges of the tiling ending at that vertex.

Barrier construction

Emanuele Spadaro

We presented a barrier construction for solutions to Plateau's problem due to Meeks and Yau.

Given a mean-convex domain $\Omega \subset \mathbb{R}^3$ and a null-homotopic (in Ω) curve $\Gamma \subset \partial\Omega$, then there is an area-minimizing embedded disk with boundary Γ contained in $\bar{\Omega}$.

The proof proceeds in two steps: modifying the standard Euclidean metric outside of Ω, one can force Morrey's solutions to the least-area problem to intersect Ω; then, an easy computation on the Laplacian of the distance function from $\partial\Omega$ and the maximum principle for harmonic functions conclude.

This result allows one to construct stable minimal surfaces constrained to lie "between two given minimal surfaces". As a consequence of this construction, using the maximum principle and the curvature estimates, we showed the following strong halfspace theorem due to Hoffman and Meeks.

Strong halfspace Theorem. *Every two disjoint, properly immersed (possibly branched) minimal surfaces in* $\mathbb{R}^3$ *are parallel planes.*

References

[1] D. Hoffman, W. Meeks, *The strong halfspace theorem for minimal surfaces*, Invent. Math. **101** (1991), 373-377

[2] W. Meeks, L. Simon, S. T. Yau, *The existence of embedded minimal surfaces, exotic spheres and positive Ricci curvature*, Ann. of Math. **116** (1982), 221-259

[3] W. Meeks, S. T. Yau, *The classical Plateau problem and the topology of three-dimensional manifolds*, Topology **21** (1982), 409-442

[4] W. Meeks, S. T. Yau, *The existence of embedded minimal surfaces and the problem of uniqueness*, Math. Z. **179** (1982), 151-168

[5] W. B. Morrey, *The problem of Plateau on a Riemannian manifold*, Ann. of Math. **49** (1948), 807-851

Colding-Minicozzi's one-sided curvature estimate

Giuseppe Tinaglia

In this talk I will outline the proof of Colding-Minicozzi one-sided curvature estimate. Loosely speaking, it says that if an embedded minimal disk gets sufficiently close to and stays on one side of a plane, then the curvature of such disk is bounded in the interior. This estimate can be though of an effective version of Hoffman-Meeks half-space theorem.

Colding-Minicozzi Theory: The Calabi-Yau conjectures for embedded surfaces.

Jacob Bernstein

In this talk we discuss some of the results of [2] regarding the Calabi-Yau conjectures for complete embedded minimal surfaces in $\mathbb{R}^3$. We defer the details of the arguments to Christine Breiner's talk. In particular, we address the following hierarchy of statements about complete minimal immersions into $\mathbb{R}^3$:

(1) There exists a complete minimal immersion $F : \Sigma \to \mathbb{R}^3$ so that $F(\Sigma) \subset B_1(0)$.
(2) There exists a non-flat complete minimal immersion $F : \Sigma \to \mathbb{R}^3$ so that $F(\Sigma) \subset \{x_3 \geq 0\}$.
(3) There exists a complete minimal immersion $F : \Sigma \to \mathbb{R}^3$ that is not properly immersed.

Notice that by translation, a surface that satisfies the first condition must satisfy the second, Furthermore, by the strong half-space theorem of Hoffman-Meeks [11] a surface satisfying the second condition satisfies the third. Examples satisfying the third condition have been known for some time, but the status of the first two was somewhat less clear. Indeed, Calabi, in [1], conjectured that there were no examples satisfying the second condition (and hence none satisfying the first condition). However, in 1980, Jorge-Xavier [7] constructed an example of a complete minimal immersion lying in a slab – disproving Calabi's conjecture. Furthermore, in 1996, Nadirashvili [10] constructed a complete minimally immersed disk lying inside the unit ball. Thus, even a weakening of Calabi's conjecture is false for immersed surfaces.

Nevertheless, one expects *embedded* surfaces to be much more rigid objects and the conjecture might hold when restricted to this class. This turns out to be the case, at least with additional finiteness assumption. Indeed, in [2], Colding and Minicozzi show something even stronger. Namely, there are no embedded minimal surfaces of finite topology (i.e. are diffeomorphic to a finitely punctured compact surface) satisfying the third condition. In other words:

Theorem 1. *Let Σ be a complete, minimally embedded surface in $\mathbb{R}^3$ of finite topolgoy, then Σ is properly embedded.*

Using one of the key propositions of [2], Meeks and Rosenberg [9] (see also [8]) generalize this to the following result:

Theorem 2. *Let Σ be a complete, minimally embedded surface in $\mathbb{R}^3$ with uniform lower bound on the injectivity radius, then Σ is properly embedded.*

Colding and Minicozzi approach the problem in [2] by showing certain chord-arc bounds for embedded minimal disks. Recall, the intrinsic distance between two points on a surface in $\mathbb{R}^3$ is always bounded below by the extrinsic distance (in $\mathbb{R}^3$) between them. On the other hand, extrinsic distance, in general, does not control intrinsic distance. However, using their description of the structure of embedded minimal disks, Colding and Minicozzi show that in fact such a reverse control does exist (in some sense) for embedded minimal disks. The heart of their paper is devoted to showing a weak form of this. Namely, the following *weak chord-arc bound*:

Theorem 3. *There exists a $\delta > 0$ so that: Let Σ be an embedded minimal disk. Suppose that that the intrinsic ball $\mathcal{B}_R(x)$ is a subset of $\Sigma \backslash \partial\Sigma$. Then the component $\Sigma_{x,\delta R}$ of $\Sigma \cap B_{\delta R}(x)$ containing x is a subset of $\mathcal{B}_{R/2}(x)$.*

An important consequence of this is that $\partial\Sigma_{x,\delta R} \subset \partial B_{\delta R}(x)$. The proof the weak chord-arc bound uses a blow-up argument and the results on the structure of embeded minimal disks from [3–6]. By combining this result with the one-sided curvature estimate of [5] (discussed in Giuseppe Tinaglia's talk) one obtains the following *strong chord-arc bound*:

Theorem 4. *There exists a $C > 0$ so that: Let $0 \in \Sigma$ be an embedded minimal disk. Suppose that $\mathcal{B}_R(0) \subset \Sigma \backslash \partial\Sigma$. Then if $\sup_{\mathcal{B}_{r_0}(0)} |A|^2 \geq r_0^{-2}$ for $R/2 \geq r_0 > 0$ then for any $x \in \mathcal{B}_{R/2}(0)$ one has $C dist_\Sigma(x, 0) \leq |x| + r_0$.*

Notice that the curvature normalization is neccesary, as can be seen by looking far from the axis of a helicoid. This immediately proves that a complete embedded minimal disk is properly embedded. Arguing in a similar fashion, one shows that annulur ends are properly embedded, which proves the more general result of Colding and Minicozzi.

References

[1] E. Calabi, "Problems in differential geometry", in *Proceedings of the United States-Japan Seminar in Differential Geometry, Kyoto, Japan, 1965*, page 170. Nippon Hyoronsha Co., Ltd., Tokyo, edited by S. Kobayashi and J. Eells Jr., 1966

[2] T.H. Colding and W.P. Minicozzi II, *The Calabi-Yau Conjectures for the Embedded Surfaces* Ann. of Math. (2) **167** (2008), no. 1, 211–243.

[3] T.H. Colding and W.P. Minicozzi II, *The space of embedded minimal surfaces of fixed genus in a 3-manifold. II. Multi-valued graphs in disks* Ann. of Math. (2) **160** (2004), no. 1, 69–92.

[4] T.H. Colding and W.P. Minicozzi II, *The space of embedded minimal surfaces of fixed genus in a 3-manifold. I. Estimates of the axis for disks* Ann. of Math. (2) **160** (2004), no. 1, 27-68.

[5] T.H. Colding and W.P. Minicozzi II, *The space of embedded minimal surfaces of fixed genus in a 3-manifold. IV. Locally Simply Connected* Ann. of Math. (2) **160** (2004), no. 2, 573–615.

[6] T.H. Colding and W.P. Minicozzi II, *The space of embedded minimal surfaces of fixed genus in a 3-manifold. III. Planar Domains* Ann. of Math. (2) **160** (2004), no. 2, 523–572.

[7] L. Jorge and F. Xavier, *On the existence of complete bounded minimal surfaces in* $\mathbb{R}^n$, Bol. Soc. Brasil. Mat. **10** (1979), no. 2, 171–173.

[8] W. Meeks III, J. Perez, and A. Ros, *The geometry of minimal surfaces of finite genus I; Curvature estimates and quasiperiodicity*, J. Diff. Geom., to appear.

[9] W. Meeks, H. Rosenberg, *The minimal lamination closure theorem*, Duke Mathematical Journal **133** (2006), 467–497

[10] N. Nadirashvili, *Hadamards and Calabi-Yaus conjectures on negatively curved and minimal surfaces*, Invent. Math. **126** (1996) 457–465.

[11] D. Hoffman and W. Meeks III, *The strong halfspace theorem for minimal surfaces*, Invent. Math. **101** (1990) 373–377.

Colding-Minicozzi Theory: Weak chord-arc bounds for embedded minimal disks

Christine Breiner

This talk was a continuation of the talk by Jacob Bernstein; the goal of both was to ultimately prove a chord-arc bound for embedded minimal disks in Euclidean space (see [1]). The thrust of this talk was to prove a weak chord-arc bound for embedded disks, without a priori knowledge of the position of the boundary. The precise statement is as follows:

Theorem 1. *Let* $\Sigma \subset \mathbb{R}^3$ *be an embedded minimal disk. Then there exists* $\delta > 0$, *independent of* Σ *such that if* $\mathcal{B}_R(x) \subset \Sigma \backslash \partial\Sigma$, *then* $\Sigma_{x,\delta R} \subset \mathcal{B}_{R/2}(x)$.

Here $\mathcal{B}_R$ represents the intrinsic ball of radius R and $\Sigma_{x,\delta R}$ is the component of $B_{\delta R}(x) \cap \Sigma$ that contains x. Note also that δ is independent of R.

Throughout the proof, we rely on a similar result to Theorem 1, but for embedded disks with a certain boundary condition. We place it here so we can refer to it with convenience.

Proposition 2. *Let $\Sigma \subset B_R \subset \mathbb{R}^3$ be an embedded minimal disk such that $\partial\Sigma \subset \partial B_R$. Then there exists δ_1, independent of Σ, such that $\Sigma_{0,\delta_1 R} \subset \mathcal{B}_{R/2}(0)$.*

To begin understanding the proof of the theorem, we need a few definitions that explain the scale on which a δ-weak chord arc bound holds at a given point.

Definition 3. *Let $\mathcal{B}_s(x) \subset \Sigma\backslash\partial\Sigma$. We say $\mathcal{B}_s(x)$ is δ-weakly chord arc if $\Sigma_{x,\delta s} \subset \mathcal{B}_{s/2}(x)$.*

We let $R_\delta(x)$ denote the supremum of the scales on which all lower scales are δ-weakly chord arc. That is,

Definition 4. $R_\delta(x) = \sup\{R|\mathcal{B}_r(x) \subset \Sigma\backslash\partial\Sigma$ *is δ-weakly chord arc for all* $r \leq R\}$.

Note that the definition of $R_\delta(x)$ follows that of Meeks and Rosenberg in [3], and I adapt the proof from [1] to use this new definition. Using the one-sided curvature estimate, Proposition 2, and curvature estimates for 1/2-stable surfaces, one can prove the following nice result.

Proposition 5. *Let $\Sigma \subset \mathbb{R}^3$ be an embedded minimal disk. There exists a constant $C > 1$, independent of Σ so that if $\mathcal{B}_{CR_0}(y) \subset \Sigma\backslash\partial\Sigma$ is an intrinsic ball and*

- *every intrinsic subball $\mathcal{B}_{R_0}(z) \subset \mathcal{B}_{CR_0(y)}$ is δ_2-weakly chord-arc*

then for every $s \leq 5R_0$, the intrinsic ball $\mathcal{B}_s(y)$ is δ_2-weakly chord-arc. That is, $R_{\delta_2}(y) \geq 5R_0$.

Based upon Proposition 2 and up to a rescaling, this tells us that for a sufficiently large intrinsic ball $\mathcal{B}_C(y) \subset \Sigma\backslash\partial\Sigma$ for which intersections with extrinsic balls of radius 1 are compact, one can show that $B_5(y) \cap \Sigma$ is also compact. Intuitively, it means that one cannot have an arbitrarily long geodesic in an extrinsic ball.

The proof follows by contradiction. It reduces to showing that $\Sigma_{y,5} \subset \mathcal{B}_C(y)$ for some C and then appealing to Proposition 2. Assuming no such C exists, one can choose points on a geodesic through y that are intrinsically far apart but are fixed extrinsically in $B_5(y)$. For a long enough geodesic, two of these points must be extrinsically very close. The geodesic disks around these two points satisfy δ_2-weak chord arc bounds (and thus their intersections with small extrinsic balls of radius δ_2 are compact). Moreover, these geodesic disks are very close at their centers. These criteria are enough to let us appeal to the one-sided curvature estimates of Colding and Minicozzi [2]. That is, when the centers are sufficiently close, one gets good curvature bounds on sub-disks. With small curvature bounds, one can write one geodesic disk as a graph over the other with small norms for $|u|, |\nabla u||A|$. Bounding these norms ultimately gives that each of these surfaces are 1/2-stable. (A surface Ω is 1/2-stable if for all $\phi \in C_0^{0,1}(\Omega)$, one has $1/2 \int |A|^2\phi^2 \leq \int |\nabla\phi|^2$.)

If the initial centers are sufficiently close, one can use a Harnack inequality to bound the extrinsic distance between two points on the boundary of slightly smaller sub-disks. One can then iterate this process to get large, 1/2-stable geodesic disks. The curvature estimates then force them to leave $B_5(y)$, which gives the necessary contradiction. If $\partial\Sigma = \emptyset$, Proposition 5 would be enough to prove the Theorem

1. Unfortunately, the boundary complicates matters. As is usually the case, one deals with this by a blow-up argument.

Then, the thrust of the problem is to find a δ for which the function $G(x) = dist_\Sigma(x, \partial\Sigma)/R_\delta(x)$ is bounded. Let $a_\delta = \sup_{x\in\Sigma} G(x)$, where G is defined for that δ. A standard blow-up result shows

Proposition 6. *Let $\Sigma \subset \mathbb{R}^3$ be a compact, embedded minimal disk that is smooth up to the boundary. For a constant $0 < \delta < 1/2$, there exists $y \in \Sigma$ and $R_0 > 0$ such that*

(1) *$R_\delta(x) > R_0$ for every x such that $\mathcal{B}_{R_0}(x) \subset \mathcal{B}_{a_\delta R_0}(y)$;*
(2) *$R_\delta(y) < 5R_0$.*

Notice if $a_\delta > C$ where C is from Proposition 5 and $\delta \leq \delta_1$ of Proposition 2, Proposition 6 gives a contradiction to Proposition 5. Thus, G is a bounded function and we have the chord-arc bound we desire.

References

[1] T.H. Colding and W.P. Minicozzi II, *The Calabi-Yau Conjectures for the Embedded Surfaces* Ann. of Math. (2) **167** (2008), no. 1, 211–243.

[2] T.H. Colding and W.P. Minicozzi II, *The space of embedded minimal surfaces of fixed genus in a 3-manifold. IV. Locally Simply Connected* Ann. of Math. (2) **160** (2004), no. 2, 573–615.

[3] W. Meeks, H. Rosenberg, *The minimal lamination closure theorem*, Duke Mathematical Journal **133** (2006), 467–497

Rescaling arguments and minimal laminations

Thilo Kuessner

We indicated proofs of the following Theorems:

Lamination Closure Theorem: *If M is a minimal surface in a 3-manifold N with injectivity radius bounded away from zero ($inj \geq C > 0$), then $\overline{M}$ is a minimal lamination.*

The main technical part of the proof (by Meeks and Rosenberg) is to show that $inj \geq C > 0$ implies that the second fundamental form is uniformly locally bounded (in extrinsic balls). The latter implies that M is locally a union of graphs, and leaves of $\overline{M}$ are then constructed as limits of these minimal graphs.

Stable Limit Leaf Theorem: *If $\mathcal{F}$ is a minimal lamination, then two-sided limit leaves are stable.*

The proof is well-known in the foliation case, where it is just an application of Gauss Divergence Theorem: let $\Delta \subset L$ be a domain in a leaf, Δ' a small variation, $\partial\Omega = \Delta \cup \Delta'$, let W be the unit normal field to $\mathcal{F}$ and V the unit normal field to $\partial\Omega$, then $div\,(W) = 2H = 0$, hence

$$0 = \int_\Omega div\,(W) = \int_{\partial\Omega} < V, W > \leq area\,(\Delta') - area\,(\Delta)\,.$$

The proof in the lamination case (by Meeks-Perez-Ros) locally interpolates laminations (in a chart around a limit leaf) by foliations with mean curvature $H_t = o(t)$ and then adapts the above argument.

More generally one can prove that leaves of CMC-foliations are stable if they locally maximize $| H |$.

Curvature Estimates: *There exists a universal constant C such that, for all complete 3-manifolds N with $| sec | \leq 1$ and all CMC-foliations, the second fundamental form of leaves is bounded by C: $| A |_{\mathcal{F}} \leq C$.*

(If $\partial N \neq \emptyset$, then $| A |_{\mathcal{F}} \leq \frac{C}{d(.,\partial N)}$.)

The proof is by a rescaling argument: let N_n be a sequence with $\lambda_n = | A |_{\mathcal{F}_n}$ maximal at p_n, and assume by contradiction $\lambda_n \to \infty$. Then $\lambda_n B(p_n, 1)$ converges to flat R^3, with a (weak) CMC-foliation that satisfies $| A |_{\mathcal{F}} \leq 1$ and $| A |_{\mathcal{F}} = 1$ at some point. The latter implies that there is some non-flat leaf maximizing $| H |$, that is a non-flat stable leaf. But stable leaves in flat R^3 are known to be flat planes, giving a contradiction.

Corollary (Meeks): *Every CMC-foliation of the flat R^3 is a foliation by parallel flat planes.*

References

[1] W. Meeks, H. Rosenberg, *The minimal lamination closure theorem*, Duke Mathematical Journal **133** (20056), 467-497

[2] W. Meeks, H. Rosenberg, *The uniqueness of the Helicoid*, Annals of Mathematics **161** (2005), 727-758

[3] W. Meeks, J. Perez, A. Ros, *Stable constant mean curvature surfaces*, Handbook of geometric analysis. No. 1, Adv. Lect. Math. 7 (2008), 301-380

Harmonic function theory on minimal surfaces

José Miguel Manzano

In this talk we describe the notion of an annular end of a Riemannian surface being of finite type with respect to some harmonic function (i.e. there exists a level set of the function which has a finite number of ends) and prove some theoretical results relating this property to the conformal structure of such an annular end. We then apply these results to understand and characterize properly immersed minimal surfaces in R^3 of finite total curvature, in terms of their intersections with two nonparallel planes.

Complete embedded minimal ends with infinite total curvature and uniqueness of the helicoid

Joaquín Pérez

This talk was devoted to address the following question:

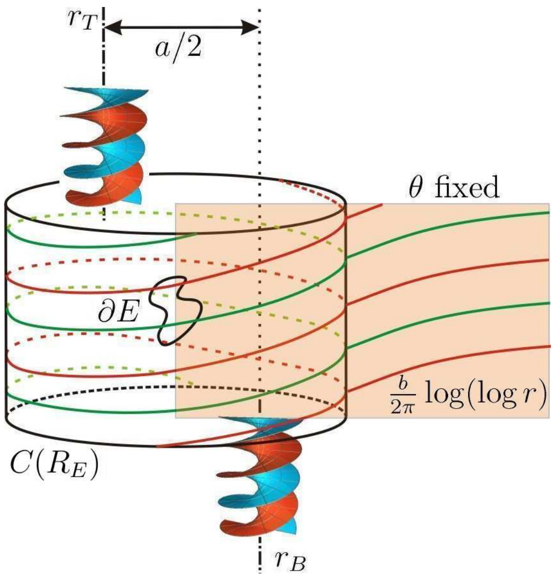
> *What are the possible shapes at infinity for a complete, embedded minimal annular end in $\mathbb{R}^3$ with infinite total curvature?*

The above question can be justified in several ways. From a historical point of view, Schoen [9] characterized the end of a plane and of a half-catenoid as the unique asymptotic models for complete, embedded minimal annular ends in $\mathbb{R}^3$ with finite total curvature. Later on, Meeks and Rosenberg [6, 7] did the same job with periodic minimal surfaces, finding that the only possibilities are planar, helicoidal, Scherk-type ends a less known asymptotic geometry of a sort of mixed catenoidal-helicoidal end, which cannot be part of a minimal surface without boundary by flux reasons (we will encounter again this phenomenon in our setting to be explained below). The case of an embedded annular minimal end with infinite total curvature was studied by Hauswirth, Pérez and Romon [3], with additional assumptions about flux and height differential of the surface. To finish this brief historical tour about the question above, we should mention that in the last section of their celebrated paper [8], Meeks and Rosenberg described how their proof of the uniqueness of the helicoid could be modified to prove that

> ($\star$) *Every nonplanar, properly embedded, one-ended minimal surface M in $\mathbb{R}^3$ with finite genus and infinite total curvature is conformally a compact Riemann surface $\overline{M}$ punctured at a single point, M is asymptotic to a helicoid and it can be expressed analytically in terms of meromorphic data on $\overline{M}$.*

Nevertheless, the original proof of the uniqueness of the helicoid was long and technical and so, a variety of arguments needed to be shown to hold in order to give a rigorous proof of ($\star$) in the positive genus case. The first rigorous proof of this fact has been recently given by Bernstein and Breiner [1]. In this talk, we will overview a recent joint work by the author and Meeks [5] where they tackle the more general problem of describing the asymptotic behavior, conformal structure and analytic representation of an annular end of a complete, injectively immersed minimal surface M in $\mathbb{R}^3$ with *compact boundary* and finite topology. Allowing the surface to have compact boundary introduces a geometrical quantity, namely the flux vector (which is zero in the case of ($\star$) by the Stokes theorem). We will see that this vector essentially parameterizes the moduli space of all possible asymptotics for a surface as in the question posed at the beginning. We also remark that completeness implies properness in this setting with finite topology and compact boundary, as follows from the results in Colding and Minicozzi [2].

In order to state the main result in this talk, we need the following notation. Given $R > 0$, let $C(R) = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leq R^2\}$. A multigraph over $D(\infty, R) = \{x_3 = 0\} \cap [\mathbb{R}^3 - \text{Int}(C(R))]$ is the graph $\Sigma = \{(re^{i\theta}, u(r, \theta)\} \subset \mathbb{C} \times \mathbb{R} \equiv \mathbb{R}^3$ of a function $u = u(r, \theta)$ defined on the universal cover $\widetilde{D}(\infty, R) =$



$\{(r,\theta) \mid r \geq R,\ \theta \in \mathbb{R}\}$ of $D(\infty, R)$. The image in the figure describes how the flux vector $(a, 0, -b)$ of the surface $E_{a,b}$ appearing in Theorem 1 below influences its geometry.

Theorem 1. *Given $a \geq 0$ and $b \in \mathbb{R}$, there exist a positive number $R = R_E = R(a,b)$ and a properly embedded minimal annulus $E_{a,b} \subset \mathbb{R}^3$ with compact boundary and flux vector $(a, 0, -b)$ along its boundary, such that the following statements hold.*

(1) *$E_{a,b} - C(R)$ consists of two disjoint multigraphs Σ_1, Σ_2 over $D(\infty, R)$ of smooth functions $u_1, u_2 \colon \widetilde{D}(\infty, R) \to \mathbb{R}$ such that their gradients satisfy $\nabla u_i(r,\theta) \to 0$ as $r \to \infty$ and the separation function $w(r,\theta) = u_1(r,\theta) - u_2(r,\theta)$ between both multigraphs converges to π as $r + |\theta| \to \infty$. Furthermore for θ fixed and $i = 1, 2$,*
$$\lim_{r\to\infty} \frac{u_i(r,\theta)}{\log(\log(r))} = \frac{b}{2\pi}.$$

(2) *The translated surfaces $E_{a,b} + (0, 0, -2\pi n - \frac{b}{2\pi}\log n)$ (resp. $E_{a,b} + (0, 0, 2\pi n - \frac{b}{2\pi}\log n)$) converge as $n \to \infty$ to a vertical helicoid H_T (resp. H_B) such that $H_B = H_T + (0, a/2, 0)$. Note that this last equation together with item (1) above imply that for different values of a, b, the related surfaces $E_{a,b}$ are not asymptotic after a rigid motion and homothety.*

(3) *The annulus $E_{0,0}$ is the end of a vertical helicoid, and the annuli $E_{0,b}$ are each invariant under reflection across the x_3-axis l and $l \cap E_{0,b}$ contains two infinite rays.*

(4) *Every complete, embedded minimal annulus E in $\mathbb{R}^3$ with compact boundary and infinite total curvature satisfies the following properties:*

(a) *E is properly embedded in $\mathbb{R}^3$ and conformally diffeomorphic to a punctured disk.*
(b) *After replacing E by a subend and applying a suitable homothety and rigid motion to E, then:*
   (i) *The holomorphic height differential $dh = dx_3 + idx_3^*$ of E is $dh = (1 + \frac{\lambda}{z-\mu})\, dz$, defined on $D(\infty, R) = \{z \in \mathbb{C} \mid R \leq |z|\}$ for some $R > 0$, where $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{C}$. In particular, dh extends meromorphically across infinity with a double pole.*
   (ii) *The stereographic projection $g\colon D(\infty, R) \to \mathbb{C} \cup \{\infty\}$ of the Gauss map of E can be expressed as $g(z) = e^{iz+f(z)}$ for some holomorphic function f in $D(\infty, R)$ with $f(\infty) = 0$.*
   (iii) *E is asymptotic to the end $E_{a,b}$ where $(a, 0, -b)$ is the flux vector of E along its boundary; in particular, E satisfies the properties (1),(2) above, and E is asymptotic to the end of a helicoid if and only if it has zero flux.*

The existence of the "canonical model" $E_{a,b}$ given in Theorem 1 is based on the classical Weierstrass representation. Namely, in the case of nonvertical flux (i.e. $a \neq 0$) one consider the following data for $E_{a,b}$:

$$(1) \qquad g(z) = t\, e^{iz} \frac{z - A}{z}, \qquad dh = \left(1 + \frac{B}{z}\right) dz, \qquad z \in D(\infty, R),$$

where $t > 0$, $A \in \mathbb{C} - \{0\}$, $B \in \mathbb{R}$ and $R > |A|$ are to be determined. It can be proved that the parameters A, B, t can be adjusted so that the corresponding period problem is solved, and at the same time one can prescribe arbitrarily the flux vector $(a, 0, -b)$ (with $a \neq 0$) along the boundary $\{|z| = R\}$. Similarly, in the vertical flux case $E_{0,b}$, one takes

$$(2) \qquad g(z) = e^{iz} \frac{z - A}{z - \overline{A}}, \qquad dh = \left(1 + \frac{B}{z}\right) dz, \quad z \in D(\infty, R),$$

where $A \in \mathbb{C} - \{0\}$ and $R > |A|$. After closing the period, one notices that the conformal map $z \overset{\Phi}{\mapsto} \overline{z}$ in the parameter domain $D(R, \infty)$ of $E_{0,b}$ satisfies $g \circ \Phi = 1/\overline{g}$, $\Phi^* dh = \overline{dh}$, which implies that up to a translation in $\mathbb{R}^3$, Φ produces a 180°-rotation of $\mathbb{R}^3$ around the x_3-axis which leaves $E_{0,b}$ invariant.

To prove the items in Theorem 1 one argues directly with a complete embedded minimal annulus E with compact boundary and infinite total curvature (which could be in particular $E_{a,b}$, assuming embeddedness for this last surface). The two multigraph structure in item (1) of Theorem 1 essentially follows from Colding-Minicozzi theory; some special care is needed here since we are dealing with compact boundary. The crucial point here is that, with the notation of item (1) of the theorem, the slope $\frac{du_i}{d\theta}$ is strictly positive (up to change of orientation) on the spiraling curves obtained after intersecting E with a vertical cylinder of radius large enough, a fact already observed by Bernstein and Breiner in [1]. From here is not difficult to derive that horizontal planes sufficiently high or low intersect E exactly in a transverse proper arc. This property implies that the conformal structure of E is parabolic, that the (stereographically projected) Gauss map of E can be written as $g(z) = z^k e^{H(z)}$ for some $k \in \mathbb{Z}$ and some holomorphic function in $D(\infty, R)$, and that the height differential $dh = dx_3 + i dx_3^*$ extends holomorphically across ∞ with a double pole at ∞ (Meeks and Pérez [4]). The next step consists of proving that $k = 0$, which follows by finding a curve $\Gamma \subset D(\infty, 1)$ homologous to $\{|z| = R\}$ such that the winding number of g along Γ is zero. Again the spiraling curves described above are essential here. Once we know that $k = 0$, then a suitable change of coordinates and rotation in $\mathbb{R}^3$ produce the Weierstrass data for E given in items (4)-(b)-(i) and (4)-(b)-(ii) of Theorem 1. The remaining properties stated in Theorem 1 (in particular, the embeddedness of $E_{a,b}$) follow from a careful analysis of the geometry of E based on the Weierstrass representation.

References

[1] J. Bernstein & C. Breiner, *Conformal structure of minimal surfaces with finite topology*, Preprint available at http://arxiv.org/abs/0810.4478v1.

[2] T. H. Colding W. P. Minicozzi II, *The Calabi-Yau conjectures for embedded surfaces*, Ann. of Math. **167** (2008) 211–243.

[3] L. Hauswirth, J. Pérez & P. Romon, *Embedded minimal ends of finite type*, Transactions of the AMS **353** (2001) 1335–1370.

[4] W. H. Meeks III & J. Pérez, *Finite type annular ends for harmonic functions*, Preprint available at http://www.ugr.es/local/jperez/papers/papers.htm.

[5] W. H. Meeks III & J. Pérez, *Embedded minimal surfaces of finite topology*, Preprint available at http://www.ugr.es/local/jperez/papers/papers.htm.

[6] W. H. Meeks III & H. Rosenberg, *The Geometry of Periodic Minimal Surfaces*, Comment. Math. Helvetici **68** (1993) 538–578.

[7] W. H. Meeks III & H. Rosenberg, *The Global Theory of Doubly Periodic Minimal Surfaces*, Invent. Math. **97** (1989) 351–379.

[8] W. H. Meeks III & H. Rosenberg, *The uniqueness of the helicoid*, Ann. of Math. **161** (2005) 723–754.

[9] R. Schoen, *Uniqueness, Symmetry, and Embeddedness of Minimal Surfaces*, J. Differential Geom. **18** (1983) 791–809.

Minimal Graphs in Nil^3

Hojoo Lee

The Heisenberg group Nil^3 with bundle curvature $\frac{1}{2}$ admits a Riemannian fibration over the Euclidean plane. More explicitly, after identifying Nil^3 as $\mathbb{R}^3$ endowed with the Riemannian metric

$$ds^2 = dx^2 + dy^2 + \left[\frac{1}{2}(ydx - xdy) + dz\right]^2,$$

the standard projection $(x, y, z) \mapsto (x, y)$ becomes a Riemannian fibration. The coordinate (x, y) is geometric and so the notion of vertical graph $z = f(x, y)$ is natural. Fernández and Mira in [5] obtained a characterization of the moduli space of entire minimal graphs in Nil^3.

Theorem 1. *Let $\mathcal{Q}$ be a holomorphic quadratic differential on $\mathbb{D}$ or a non-zero holomorphic quadratical differential on $\mathbb{C}$. Then, there exists a 2-parameter family of (generically non-congruent) entire minimal graphs in* Nil^3*. Conversely, all entire minimal graphs in* Nil^3 *belong to these families.*

One of the enlightening ideas we encounter in the Fernández-Mira solution of the Bernstein problem in Nil^3 is the explicit duality between minimal surface in Nil^3 and spacelike surface with constant mean curvature $\frac{1}{2}$ in $\mathbb{L}^3$.

Theorem 2. *Given a conformal immersion of the minimal surface in* Nil^3

$$X = (F, h) : \Sigma \to \mathrm{Nil}^3,$$

let $\mathcal{Q}$ denote its Abresch-Rosenberg differential on the Riemann surface Σ. Applying Daniel's isometric correspondence in [4], we obtain a conformal immersion of its sister surface with constant mean curvature $\frac{1}{2}$ in the product space $\mathbb{H}^2 \times \mathbb{R}$

$$X^{sister} = (F^{sister}, h^{sister}) : \Sigma \to \mathbb{H}^2 \times \mathbb{R}.$$

Then, we can choose $\epsilon \in \{-1, 1\}$ having the property that

$$X^{twin} = (F, \epsilon h^{sister}) : \Sigma \to \mathbb{L}^3$$

becomes the conformal immersion of the unique (up to positive isometries) spacelike surface with constant mean curvature $\frac{1}{2}$ in Minkowski space $\mathbb{L}^3$. The induced Hopf differential of the immersion X^{twin} is $-\mathcal{Q}$.

Then, we are able to exploit the well-known theory ([2], [12], [13]) of spacelike surfaces with constant mean curvature $\frac{1}{2}$ in $\mathbb{L}^3$ to describe the moduli space of entire minimal graphs in Nil^3.

Now, one may naturally ask about the existence of the dualities for surfaces with constant mean curvature in more general ambient spaces. We define the Bianchi-Cartan-Vranceanu space $\mathbb{E}^3(\kappa, \tau)$ by

$$\mathbb{E}^3(\tau, \kappa) = \left(V, \frac{dx^2 + dy^2}{{\delta_\kappa(x,y)}^2} + \left[\tau \left(\frac{ydx - xdy}{\delta_\kappa(x,y)} \right) + dz \right]^2 \right),$$

where $\delta_\kappa(x,y) = 1 + \frac{\kappa}{4}\left(x^2 + y^2\right)$ and $\mathrm{V} = \{(x,y,z) \mid \delta_\kappa(x,y) > 0\}$. We introduce the Lorentzian Bianchi-Cartan-Vranceanu space $\mathbb{E}^3(\kappa, \tau)$ as follows.

$$\mathbb{L}^3(\tau, \kappa) = \left(\mathrm{V},\ \frac{dx^2 + dy^2}{{\delta_\kappa(x,y)}^2} - \left[\tau \left(\frac{ydx - xdy}{\delta_\kappa(x,y)} \right) + dz \right]^2 \right).$$

The following theorem in [9] generalizes the classical duality ([1], [6], [11]) between minimal graphs in $\mathbb{R}^3$ and maximal graphs in $\mathbb{L}^3$.

Theorem 3. *There exists the duality between the moduli space of graphs with constant mean curvature H in $\mathbb{E}^3(\kappa, \tau)$ and the moduli space of spacelike graphs with constant mean curvature τ in $\mathbb{L}^3(\kappa, H)$.*

For instance, we obtain the correspondence between the maximal graphs in the Lorentzian Heisenberg space $\mathrm{Nil}_1^3\left(\frac{1}{2}\right) = \mathbb{L}^3\left(0, \frac{1}{2}\right)$ and the graphs of constant mean curvature $\frac{1}{2}$ in $\mathbb{R}^3$.

Motivated by this duality, we studied in [10] maximal surfaces in the Lorentzian Heisenberg space $\mathrm{Nil}_1^3(\tau) = \mathbb{L}^3(0, \tau)$ with $\tau \neq 0$. We proved that their Gauss maps becomes harmonic maps into $\mathbb{S}^2$ and that their Abresch-Rosenberg differentials are holomorphic. (See also Daniel's results in [3].)

Employing the above duality and applying Chern's theorem that there is no entire spacelike graph of constant mean curvature $H \neq 0$ in $\mathbb{R}^3$, we can solve the Bernstein problem in $\mathrm{Nil}_1^3(\tau)$ with $\tau \neq 0$.

Theorem 4. *There exists no entire spacelike graph with zero mean curvature in the Lorentzian Heisenberg space* $\mathrm{Nil}_1^3(\tau)$ *with* $\tau \neq 0$.

The well-known self-duality in the moduli space of timelike graphs with zero mean curvature in $\mathbb{L}^3$ is also generalized in [9].

Theorem 5. *There exists the duality between the moduli space of timelike graphs with constant mean curvature* H *in* $\mathbb{L}^3(\kappa, \tau)$ *and the moduli space of timelike graphs with constant mean curvature* τ *in* $\mathbb{L}^3(\kappa, H)$.

The geometries of our PDE graph duality appeared in Theorem 3 and Theorem 5 and their various applications will be fully discussed in [7] and [8].

References

[1] E. Calabi, *Examples of Bernstein problems for some non-linear equations*, Proc. Sympos. Pure Math. **15** (1970), Amer. Math. Soc., Providence, RI, 223–230.

[2] S.Y. Cheng, S.T. Yau, *Maximal spacelike hypersurfaces in the Lorentz- Minkowski spaces*, Ann. of Math., **104** (1976), 407–419.

[3] B. Daniel, *The Gauss map of minimal surfaces in the Heisenberg group*, preprint.

[4] B. Daniel, *Isometric immersions into* 3*-dimensional homogeneous manifolds*, Comment. Math. Helv. **82** (2007), No. 1, 87–131.

[5] I. Fernández and P. Mira, *Holomorphic quadratic differentials and the Bernstein problem in Heisenberg space*, Trans. Am. Math. Soc. **361** (2009), 5737–5752.

[6] H. Jenkins and J. Serrin, *Variational problems of minimal surface type II. Boundary value problems for the minimal surface equation*, Arch. Rational Mech. Anal., **21** (1966), 321–342.

[7] H. Lee, *Duality between* H*-graphs in* $\mathbb{E}^3(\kappa, \tau)$ *and spacelike* τ*-graphs in* $\mathbb{L}^3(\kappa, H)$, preprint.

[8] H. Lee, *Duality between timelike* H*-graphs in* $\mathbb{E}^3(\kappa, \tau)$ *and timelike* τ*-graphs in* $\mathbb{L}^3(\kappa, H)$, preprint.

[9] H. Lee, *Extension of the duality between minimal surfaces and maximal surfaces*, preprint.

[10] H. Lee, *Maximal surfaces in Lorentzian Heisenberg space*, preprint.

[11] M. Shiffman, *On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes*, Ann. of Math. **63** (1956), 77–90.

[12] T. Y. Wan, *Constant mean curvature surface harmonic map and universal Teichmuller space*, J. Differential Geom. **35** (1992), 643–657.

[13] T. Y. Wan, T.K. Au, *Parabolic constant mean curvature spacelike surfaces*, Proc. Amer. Math. Soc. **120** (1994), 559–564.

The halfspace theorem in Nil

Karsten Grosse-Brauckmann

I reported on the recent paper [2] by Benoît Daniel and Laurent Hauswirth about minimal surfaces in Heisenberg space Nil_3. This space is a homogeneous Riemannian 3-manifold which has the structure of a Riemannian fibration $\pi\colon \mathrm{Nil}_3 \to \mathbb{R}^2$ with geodesic fibres. Hence the notion of a *graph* makes sense: This is a section of the bundle over some subset of the base. More general are *multigraphs* which are proper immersions transversal to the fibres.

The main result of [2] is a Bernstein theorem for Nil_3 under the weak assumption of a multigraph:

Theorem. *A complete minimal multigraph in* Nil_3 *is an entire graph and hence classified by the work of Fernandez and Mira [3].*

Consequently we have, unlike the Euclidean case: The entire minimal graphs in Nil_3 form an infinite-dimensional space, classified by the Hopf differentials on the complex plane or disk.

Let us define a *vertical plane* as the preimage of π of a geodesic in the base $\mathbb{R}^2$; it is minimal. Also, let a *vertical slab* be the preimage of a closed strip bounded by two disjoint geodesics. We say the *slab is perpendicular* to a vector in $\mathbb{R}^2$ if the vector is orthogonal to the bounding geodesics of the projection of the slab. Then we can quote the main technical result of the paper:

Lemma 1. *There exists a family of properly embedded minimal annuli, called* horizontal catenoids*, whose degenerate limit is a double cover of a punctured vertical plane. Moreover, there is an* axis vector *in* $\mathbb{R}^2$ *such that each vertical slab perpendicular to this direction meets each catenoid of the family only in a compact set.*

Note that the only rotation isometries about horizontal geodesics of Nil_3 are halfturns, and so for a horizontal axis a surface of revolution cannot be defined. Nevertheless, certain ODE solutions lead to generalized Weierstrass data for the horizontal catenoids in Nil_3.

Given the catenoid family of Lemma 1, the proof of the Hoffman-Meeks Halfspace Theorem immediately carries over to the case of Nil_3 and shows:

Vertical Halfspace Theorem in Nil_3**.** *The only proper minimal immersions* Nil_3 *disjoint to vertical planes are vertical planes.*

Implicit in the paper by Daniel and Hauswirth is the following statement:

Lemma 2. *The projection of a multigraph in* Nil_3 *to the base* $\mathbb{R}^2$ *is an immersion whose boundary is the union of entire geodesics (possibly empty).*

For instance, the domain of a graph can only be a strip, a halfspace or the entire plane; the same is claimed for multigraphs. The proof of Lemma 2 literally follows a proof by Hauswirth, Rosenberg and Spruck [1]. In my talk, I indicated that Lemma 2 holds in greater generality for minimal surfaces or constant mean curvature surfaces in Riemannian fibrations with geodesic fibres over simply connected constant mean curvature bases. I plan to write a short note containing the precise statement.

If there is a bounding geodesic in Lemma 2, then it contradicts the Halfspace Theorem. Hence a multigraph can only be entire, which proves the theorem.

Entire minimal graphs and vertical planes foliate and hence are stable; Bill Meeks asked after the talk if these can be shown to be all complete stable minimal surfaces in Nil_3. One could also ask if there are any other properly embedded minimal disks. Another question is if properly embedded annuli in Nil_3 can be characterized in general, or constructed by desingularization.

References

[1] L. Hauswirth, H. Rosenberg and J. Spruck [1]. On complete mean curvature 1/2 surfaces in Harold Rosenberg space, Comm. Anal. Geom. to appear

[2] B. Daniel, L. Hauswirth, *Half-space theorem, embedded minimal annuli and minimal graphs in the Heisenberg group*, Proc. London Math. Soc. **98** (2009), 445–470

[3] I. Fernandez, P. Mira: Harmonic maps and constant mean curvature surfaces in $\mathbb{H}^2 \times \mathbb{R}$. Amer. J. Math. **129** (2007), 1145–1181.

Minimal Surfaces in $\mathbb{H}^2 \times \mathbb{R}$

Julia Plehnert

In this talk we proved the following theorem:

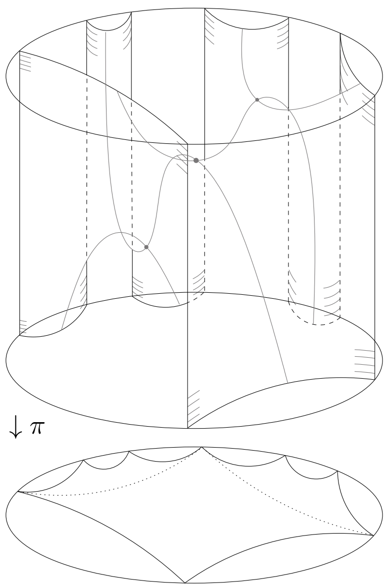
Theorem (Collin/Rosenberg): *In $\mathbb{H}^2 \times \mathbb{R}$ there exist entire minimal graphs over $\mathbb{H}^2$ which are conformally the complex plane $\mathbb{C}$.*

The proof is devided into three steps:

(1) Construct minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$ over an ideal quadrilateral with a generalized Jenkins-Serrin Theorem.
(2) Show that these graphs are conformally $\mathbb{C}$.
(3) Extend graphs to $\mathbb{H}^2$ such that the extension is conformally $\mathbb{C}$.

Collin and Rosenberg start with an ideal regular quadrilateral and consider the Dirichlet problem for the minimal surface equation with values alternating between plus and minus infinity over consecutive edges. This generalizes a construction by Jenkins and Serrin. The graph is simply connected, complete and has finite total curvature. By a theorem of Huber the graph is conformally $\mathbb{C}$.

To get the desired graph over $\mathbb{H}^2$ Collin and Rosenberg inductively construct a sequence of Jenkins-Serrin graphs by replacing each ideal boundary edge with an ideal quadrilateral. By choosing the extending quadrilateral to be very close to a regular ideal quadrilateral, they can assert that the extended solution is almost vertical over the original edges. This lets the extended solution stay as close as desired to the original one, and leaves the conformal type of the extension unchanged.



The limit of the sequence is an entire minimal graph over $\mathbb{H}^2$ which is conformally equivalent to the complex plane. The vertical projection of the graph gives a surjective harmonic diffeomorphism, and so provides an answer to a question posed by R. Schoen:

Corollary (Collin/Rosenberg): *There exist harmonic diffeomorphisms from* $\mathbb{C}$ *onto the hyperbolic plane* $\mathbb{H}^2$*.*

References

[1] P. Collin, H. Rosenberg, *Construction of harmonic diffeomorphisms and minimal graphs*, arXiv:math/0701547.

[2] A. Huber, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv. **32** (1957), p.13-72.

[3] A. Vassil'ev, *Moduli of families of curves for conformal and quasiconformal mappings*, Lecture Notes in Mathematics **1788** (2002), Springer-Verlag, Berlin.

Reporter: Jacob Bernstein

Participants

Prof. Dr. Uwe Abresch
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstr. 150
44801 Bochum

Dr. Jacob Bernstein
Department of Mathematics
Stanford University
Stanford , CA 94305
USA

Dr. Christine Breiner
Department of Mathematics
Massachusetts Institute of
Technology
77 Massachusetts Avenue
Cambridge , MA 02139-4307
USA

Dr. Maria Calle
Departamento de Matematicas
Universidad Autonoma de Madrid
Ciudad Universitaria de Cantoblanco
E-28049 Madrid

Sebastien Cartier
Laboratoire d'Analyse et de
Mathematiques Appliquees
Universite de Marne la Vallee
F-77454 Marne la Vallee Cedex 2

Prof. Dr. Jaigyoung Choe
School of Mathematics
Korea Inst. for Advanced Study
87 Hoegiro
Dongdaemun-gu
Seoul 130-722
KOREA

Hagyun Choi
School of Mathematical Sciences
Seoul National University
Daehak-dong, Gwanak-gu
Seoul 151-747
KOREA

Dr. Andrew Clarke
Laboratoire de Mathematiques
Universite de Nantes
2 rue de la Houssiniere
F-44322 Nantes Cedex 03

Antoine Coutant
Laboratoire d'Analyse et de
Mathematiques Appliquees
LAMA UMR-CNRS 8050
61, av. du General de Gaulle
F-94010 Creteil Cedex

Dr. Casey Douglas
Dept. of Maths. and Comp. Science
Saint Mary's College of Maryland
Saint Mary's City , MD 20686
USA

Prof. Dr. Gerd Faltings
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn

PD Dr. Steffen Fröhlich
Institut für Mathematik
Freie Universität Berlin
Arnimallee 6
14195 Berlin

Dr. Nadine Grosse
Naturwissenschaftliche Fakultät I
Mathematik
Universität Regensburg
93040 Regensburg

Prof. Dr. Karsten Grosse-Brauckmann
Fachbereich Mathematik
TU Darmstadt
Schloßgartenstr. 7
64289 Darmstadt

Prof. Dr. David Hoffman
Department of Mathematics
Stanford University
Stanford , CA 94305-2125
USA

Dr. Thilo Kuessner
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Robert B. Kusner
Dept. of Mathematics
University of Massachusetts
Amherst , MA 01003-9305
USA

Dr. Hojoo Lee
Department of Mathematical Sciences
Seoul National University
San 56-1, Shinrim-dong
Kwanak-gu
Seoul 151-747
KOREA

Dr. Jose Miguel Manzano
Dept. of Geometry and Topology
University of Granada
Avenida Fuentenueva s/n
E-18071 Granada

Prof. Dr. Francisco Martin
Dept. of Geometry and Topology
University of Granada
Avenida Fuentenueva s/n
E-18071 Granada

Prof. Dr. William H. Meeks
Dept. of Mathematics
University of Massachusetts
Amherst , MA 01003-9305
USA

Sunghong Min
Department of Mathematical Sciences
Seoul National University
San 56-1, Shinrim-dong
Kwanak-gu
Seoul 151-747
KOREA

Dr. Robert Neel
Department of Mathematics
Lehigh University
Christmas-Saucon Hall
14 E. Packer Avenue
Bethlehem , PA 18015-3174
USA

Tuan Khanh Hoan Nguyen
SFB 478
Geom. Strukturen in der Mathematik
Universität Münster
Hittorfstr. 27
48149 Münster

Dr. Sung-ho Park
KIAS Hoegiro87(207-43Cheongnyangni2-dong
Dongdaemun-gu
Seoul 130-722
South Korea

Prof. Dr. Joaquin Perez
Dept. of Geometry and Topology
University of Granada
Avenida Fuentenueva s/n
E-18071 Granada

Julia Plehnert
Fachbereich Mathematik
TU Darmstadt
Schloßgartenstr. 7
64289 Darmstadt

Dr. Juncheol Pyo
Department of Mathematical Sciences
Seoul National University
San 56-1, Shinrim-dong
Kwanak-gu
Seoul 151-747
KOREA

Dr. Magdalena Rodriguez-Perez
Depto. Algebra
Facultad de Ciencias Matematicas
Universidad Complutense de Madrid
Plaza de las Ciencias, 3
E-28040 Madrid

Prof. Dr. Keomkyo Seo
Department of Mathematics
Sookmyung Women's University
Hyochangwongil 52, Yongsan-ku
Seoul , 140-742
South Korea

Dr. Graham A.C. Smith
Centre de Ricerca Matematica
Departament de Matematicas
Universitat Autonoma de Barcelona
E-08193 Bellaterra Barcelona

Dr. Emanuele Spadaro
Institut für Mathematik
Universität Zürich
Winterthurerstr. 190
CH-8057 Zürich

Prof. Dr. John M. Sullivan
Fakultät II-Institut f. Mathematik
Technische Universität Berlin
Sekr. MA 3-2
Straße des 17. Juni 136
10623 Berlin

Dr. Giuseppe Tinaglia
Mathematics Institute
University of Warwick
Gibbet Hill Road
GB-Coventry CV4 7AL

Dr. Martin Traizet
Departement de Mathematiques
Faculte des Sciences
Universite de Tours
Parc de Grandmont
F-37200 Tours

Dr. Matthias Weber
Department of Mathematics
Indiana University at Bloomington
Rawles Hall
Bloomington , IN 47405-5701
USA

Dr. Adam G. Weyhaupt
Department of Mathematics and Statistics
Southern Illinois University
Edwardsville
Science Building 1316
Edwardsville IL 62026
USA

Dr. Rami Younes
Departement de Mathematiques
Faculte des Sciences
Universite de Tours
Parc de Grandmont
F-37200 Tours