Mathematisches Forschungsinstitut Oberwolfach

Report No. 49/2009

DOI: 10.4171/OWR/2009/49

Mini-Workshop: Spectrum of Transfer Operators – Recent Developments and Applications

Organised by
Viviane Baladi, Paris
Gerhard Keller, Erlangen
Carlangelo Liverani, Roma

November 8th – November 14th, 2009

Abstract. Transfer operators and their spectral theory provide a unifying framework for studying stochastic properties of chaotic deterministic dynamical systems. The goal of this workshop was to widen the class of systems that can be analysed in this way and to discuss and present new applications.

Mathematics Subject Classification (2000): 37C30, 11K55, 37D20, 37D50, 37L60, 47A30, 47A35, 47B60, 60F17, 81Q50.

Introduction by the Organisers

Transfer operators are linear operators associated to discrete- or continuous-time dynamical systems with some hyperbolicity. Acting on suitable Banach spaces, they often have Perron-Frobenius type spectrum, although it can be quite tricky to find a proper Banach space on which this can be proved and be exploited to obtain statistical information about the dynamical system (SRB measures and other Gibbs states, exponential decay of correlations, statistical stability, probabilistic limit theorems, linear response, . . .). The first results date back to the1960's (D. Ruelle), and the three organisers have contributed to this theory since the 1980's (see [1] for a detailed discussion and results of such a theory). In the last decade a fresh point of view has gradually emerged, where most of the combinatorial (and sometimes quite artificial) constructions used previously, like Markov partitions, Young towers, cluster expansions etc., can be avoided by introducing suitable Banach spaces. This not only allows to give simpler proofs of known results on smooth hyperbolic systems, but also to extend the theory to piecewise smooth systems, partially hyperbolic systems, or systems in infinite dimensions (such as coupled map lattices). What is common to most of these situations is that rather complicated arguments of combinatorial flavour (book-keeping of iterated singularities, spatial dependencies etc.) are captured by norm estimates in judiciously chosen Banach spaces. These techniques not only apply to dynamical systems but also to many Markov chains including various time series models of current interest (see [2]). All participants of the workshop are actively working on the quest for suitable Banach spaces or on exploiting the spectral information thus obtained for a deeper understanding of the dynamical systems under study (or on both).

In order to leave enough time for discussions and cooperations, the number of talks was limited to twelve. They all were followed by long and lively discussions, and some of them were continued the following day by demand of the audience.

For the development of new Banach spaces the dialogue between tools developed in *semiclassical analysis* (related to quantum mechanics) and the techniques introduced recently for transfer operators of chaotic dynamical systems was most fruitful. A number of new ideas originated from this which hopefully will produce interesting results in the near future. But also the detailed comparison of the special advantages and difficulties of the approaches based on the one hand on Triebel-type Banach spaces and on the other hand on more geometrically defined spaces lead to new insights into the particular problems associated with the spectral approach to piecewise hyperbolic systems. At the end of the workshop the road to an operator treatment of the Sinai-billiard flow was visible.

Among the applications of transfer operator theory were a strikingly new and very general approach to prove almost sure invariance principles for partial sum processes generated by chaotic dynamical systems, precise results on linear response theory for Benedicks-Carleson unimodal maps, applications to Poincaré sums and sum-level sets of continued fraction expansions, genericity results on the Ruelle spectrum of transfer operators, and limit theorems for the statistics of systems with shrinking targets.

Talks on complex cone contractions and on self-consistent (nonlinear) Perron-Frobenius operators indicated related directions of research.

In summary it is clear that a new unifying point of view in the study of hyperbolic dynamical systems is emerging. Ideally such a new approach will encompass both discrete and continuous time, both smooth and piecewise smooth systems, both conservative and dissipative phenomena, both finite and infinite dimensional models. We believe that this workshop, by bringing together different approaches to and people working on different, but related, facets of such project, has considerably advanced the above research program and that its influence will be felt in the future development of the field.

References

[1] V. Baladi, *Positive Transfer Operators and Decay of Correlations*, Book Advanced Series Nonlinear Dynamics, Vol 16, World Scientific, Singapore (2000).

[2] H. Hennion, L. Hervé, *Limit Theorems For Markov Chains And Stochastic Properties Of Dynamical Systems By Quasi-compactness*, Lecture Notes in Mathematics vol. 1766, springer-Verlag (2001).

Mini-Workshop: Spectrum of Transfer Operators

Table of Contents

Abstracts

Fourier norms for piecewise hyperbolic maps

Viviane Baladi
(joint work with Sébastien Gouëzel)

The "spectral" or "functional" approach to study statistical properties of dynamical systems with enough hyperbolicity, originally limited to one-dimensional dynamics, has greatly expanded its range of applicability in recent years. The following spectral gap result of Blank–Keller–Liverani [6] appeared in 2002:

Theorem 1. *Let $T : X \to X$ be a C^3 Anosov diffeomorphism on a compact Riemannian manifold, with a dense orbit. Define a bounded linear operator by*

$$\mathcal{L}\omega = \frac{\omega \circ T^{-1}}{|\det DT \circ T^{-1}|}, \quad \omega \in L^\infty(X)\,. \tag{1}$$

Then there exist a Banach space $\mathcal{B}$ of distributions on X, containing $C^\infty(X)$, and a bounded operator on $\mathcal{B}$, coinciding with $\mathcal{L}$ on $\mathcal{B} \cap L^\infty(X)$ and denoted also by $\mathcal{L}$, with the following properties: The spectral radius of $\mathcal{L}$ on $\mathcal{B}$ is equal to one, the essential spectral radius of $\mathcal{L}$ on $\mathcal{B}$ is strictly smaller than one, $\mathcal{L}$ has a fixed point in $\mathcal{B}$. Finally, 1 *is the only eigenvalue on the unit circle, and it is simple.*

It is a remarkable fact that "Perron-Frobenius-type" spectral information as in the above theorem gives simpler proofs of many known theorems, but also new information. Among these consequences, let us just mention: Existence of finitely many physical measures whose basins have full measure, exponential decay of correlations for physical measures and Hölder observables, statistical and stochastic stability, linear response and the linear response formula, central and local limit theorems, location of the poles of dynamical zeta functions and zeroes of dynamical determinants, etc. One of the advantages of this "functional approach" is that it bypasses the construction of Markov partitions and the need to introduce artificial "one-sided" expanding endomorphisms.

Billiards with convex scatterers, also called Sinai billiards, are among the most natural and interesting dynamical systems. They are uniformly hyperbolic, preserve Liouville measure, but they are only piecewise smooth. Analysing the difficulties posed by the singularities has been an important challenge for mathematicians, and it is only in 1998 that L.-S. Young [12] proved that the Liouville measure enjoys exponential decay of correlations for two-dimensional Sinai billiards. It should be noted that these results were in fact obtained for a discrete-time version of the billiard flow. Indeed the question of whether the original two-dimensional continuous-time Sinai billiard enjoys decay of correlations is to this day still open. (Chernov [7] recently obtained stretched exponential upper bounds.) It is well known that the continuous-time case is much more difficult, and it seems that the ideas of Dolgopyat which were exploited in several smooth hyperbolic situations are not compatible with the tools used in [12] for example. We believe that a new, "functional," proof (via a spectral gap result for the transfer operator (1) on a suitable anisotropic Banach space of distributions) of exponential decay of correlations for *discrete-time* surface Sinai billiards will be a key stepping stone towards the expected proof of exponential decay of correlations for the *continuous-time* Sinai billiards.

The recent paper of Demers-Liverani [8] was a first breakthrough in this direction, as we explain next. Since none of the spaces of [9, 10, 4, 5] behave well with respect to multiplication by characteristic functions of sets, they cannot be used for systems with singularities. Demers–Liverani therefore introduced some new Banach spaces, on which transfer operators associated to two-dimensional piecewise hyperbolic systems admit a spectral gap. However, the construction and the argument of [8] are quite intricate, in particular, pieces of stable or unstable manifolds are iterated by the dynamics, and the way they are cut by the discontinuities has to be studied in a very careful way. As a consequence, adapting the approach in [8] to billiards (which are not piecewise hyperbolic, stricto sensu, because their derivatives blow up along the singularity lines) is daunting.

Another progress in the direction of a modern proof of exponential decay of correlations for discrete-time billiards is our previous paper [2]. There, we showed that ideas of Strichartz [11] imply that classical anisotropic Sobolev spaces $H_p^{t,s}$ in the Triebel-Lizorkin class (these spaces had been introduced in dynamics in [1]) are suitable for piecewise hyperbolic systems, under the condition that the system admits a smooth (at least C^1) stable foliation. Unfortunately, although it holds for several nontrivial examples, this condition is pretty restrictive: In general, the foliations are only measurable!

More recently [3], for piecewise smooth and hyperbolic dynamics, we were able to remove the assumption of smoothness of the stable foliation, whenever the hyperbolicity exponents of the system satisfy a *bunching* condition. This is the new result presented in this talk.

The bunching condition is rather standard in smooth hyperbolic dynamics, where it ensures that the dynamical foliations are C^1 instead of the weaker Hölder condition which holds in full generality. The bunching condition is always satisfied in codimension one (in particular, it holds in dimension two, so that our results apply to physical measures of all surface piecewise hyperbolic systems previously covered). Our methods require the dynamics to be $C^{1+\alpha}$ on each (closed) domain of smoothness, and therefore do not apply directly to discrete-time Sinai billiard. However, we expect that it will be possible to adapt them to obtain the desired functional proof of exponential decay of correlations for two-dimensional Sinai billiards. We stress that hyperbolicity is defined in terms of cones and that there is *a priori* no invariant stable distribution, contrary to our previous work [2].

We use the Triebel spaces $H_p^{t,s}$ as building blocks in the construction of our new Banach spaces $\mathbf{H}_p^{t,s}(R)$ and $\mathbf{H}$). As a consequence, we may exploit, as we did in [2], the rich existing theory (in particular regarding interpolation), and use again the results of Strichartz [11]. The new ingredient with respect to [2] is that we define our norm by considering the Triebel norm in $\mathbb{R}^d$ through suitable C^1

charts, taking now the *supremum* over *all* cone-admissible charts. We use the bunching assumption to show that the family is invariant under iteration. *Indeed, this is how we avoid the necessity for a smooth stable foliation.* As in [2], we do not iterate single stable or unstable manifolds (contrary to [12, 8]), and we do not need to match nearby stable or unstable manifolds: Everything follows from an appropriate functional analytic framework.

Our main result is an upper bound on the essential spectral radius of weighted transfer operators associated to cone hyperbolic systems satisfying the bunching condition and acting on a Banach space $\mathbf{H}$ of anisotropic distributions. If hyperbolicity dominates complexity growth and if either $\det DT \equiv 1$, or $d_s = 1$ and $d_u > 0$, then one can choose the Banach space so that at least one of the transfer operator has essential spectral radius strictly smaller than 1, and thus a spectral gap.

References

[1] V. Baladi, *Anisotropic Sobolev spaces and dynamical transfer operators: C^∞ foliations,* in: Algebraic and Topological Dynamics, S. Kolyada, Y. Manin and T. Ward (eds). Contemp. Math., Amer. Math. Soc. (2005) 123–136.

[2] V. Baladi and S. Gouëzel, *Good Banach spaces for piecewise hyperbolic maps via interpolation,* Annales de l'Institut Henri Poincaré / Analyse non linéaire **26** (2009) 1453–1481.

[3] V. Baladi and S. Gouëzel, *Banach spaces for piecewise cone hyperbolic maps,* Arxiv preprint (2009).

[4] V. Baladi and M. Tsujii, *Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms,* Annales de l'Institut Fourier **57** (2007) 127–154.

[5] V. Baladi and M. Tsujii, *Dynamical determinants and spectrum for hyperbolic diffeomorphisms,* in: Probabilistic and Geometric Structures in Dynamics, K. Burns, D. Dolgopyat and Ya. Pesin (eds). Contemp. Math. **469** Amer. Math. Soc., Volume in honour of M. Brin's 60th birthday (2008) 29–68.

[6] M. Blank, G. Keller, and C. Liverani, *Ruelle-Perron-Frobenius spectrum for Anosov maps,* Nonlinearity **15** (2002), 1905–1973.

[7] N. Chernov, *A stretched exponential bound on time correlations for billiard flows,* J. Stat. Phys., **127,** (2007) 21–50.

[8] M. Demers and C. Liverani, *Stability of statistical properties in two-dimensional piecewise hyperbolic maps,* Trans. Amer. Math. Soc. **360** (2008) 4777–4814.

[9] S. Gouëzel and C. Liverani, *Banach spaces adapted to Anosov systems,* Ergodic Theory Dynam. Systems **26** (2006) 189–218.

[10] S. Gouëzel and C. Liverani, *Compact locally maximal hyperbolic sets for smooth maps: fine statistical properties,* J. Differential Geometry **79** (2008) 433–477.

[11] R. Strichartz, *Multipliers on fractional Sobolev spaces,* J. Math. Mech. **16** (1967) 1031–1060.

[12] L.-S. Young, *Statistical properties of dynamical systems with some hyperbolicity,* Ann. of Math. **147** (1998) 585–650.

Old results, new techniques: problems in piecewise smooth dynamics

Jérôme Buzzi

Since [4], there has been exciting developments of the transfer operator method in geometric space – see [1, 2, 3, 13, 15, 16] as a subjective selection geared towards this talk. The first works considered globally smooth, uniformly hyperbolic systems but there are now results for some piecewise smooth dynamics, motivated both by physical examples like the hard ball systems and as one of the simplest forms of non-uniformity. Hence, these new techniques are an invitation to give a fresh look to a number of old (and not so old) results, sometimes painfully obtained in a restricted context.

We would like to draw the attention to three problems.

The first one is the existence and properties of *Sinai-Ruelle-Bowen measures* (i.e., absolutely continuous in the unstable direction) for general piecewise hyperbolic systems. Under some technical assumptions, this has been accomplished by V. Baladi and S. Gouëzel [1, 2]. The assumptions involve (i) the non-conformality of the expanding direction which is related to the regularity defect of the unstable foliation (and does not appear in the expanding case); (ii) the local complexity of the discontinuities. We only consider systems with a finite number of pieces with boundaries as smooth as the map itself.

> *Do all piecewise affine systems preserving complementary stable and unstable cone fields always have a Sinai-Ruelle-Bowen measure? Is it true of most of them?*

This has been shown in the expanding case by M. Tsujii [19]. One deals with (ii) by introducing the Jacobian into the counting of discontinuities. In dimension 2 the argument is quite simple and extends to piecewise real-analytic maps [6, 17] but not piecewise C^r maps for $r < \infty$ [8, 18] ($r = \infty$ is unknown). A less ambitious question is to establish the above for almost all systems, say an open and dense set of systems in a natural topology [5, 12].

The second problem is to link the above ergodic theory with the periodic structure by defining and studying appropriate Ruelle zeta functions. There are classical results for uniformly hyperbolic systems going back to Ruelle. Such results are now being refined and extended (see other talks in this workshop).

> *Can one build meromorphic extensions of the Ruelle zeta functions for the above systems?*

For piecewise expanding systems this was done in [11] by using a Hofbauer tower as an elaborate substitute for the Markov structure. See also [10]. The introduction of good Banach spaces raises the hope that this could be much simplified as was done in dimension 1 with the consideration of sharp traces and determinants (see [14] and the references therein).

The third problem deals with the invariant probability measures with *maximal entropy* (which describe "most" of the topological dynamics and often the distribution of periodic orbits).

> *Do all piecewise affine systems which are (1) uniformly expanding; (2) uniformly hyperbolic possess finitely many ergodic invariant probability measures with maximal entropy? Is it true of most of them?*

This is known for the expanding case under "smallness of the boundary entropy" [7]. It is also known without any uniform assumption for all piecewise affine surface homeomorphisms [5] (existence can fail for piecewise affine surface *maps* [8] and finiteness can obviously fail in higher dimensions). Even for surfaces a simplified proof and more informations about the maximal entropy measures would be very interesting.

References

[1] V. Baladi, S. Gouëzel, *Good Banach spaces for piecewise hyperbolic maps via interpolation*, Ann. IHP Analyse non linéaire **26** (2009), 1453–1481.

[2] V. Baladi, S. Gouëzel, *Banach spaces for piecewise cone hyperbolic maps*, preprint arXiv:0907.1402.

[3] V. Baladi, M. Tsujii, *Dynamical determinants and spectrum for hyperbolic diffeomorphisms,* Geometric and probabilistic structures in dynamics, 29–68, Contemp. Math., 469, Amer. Math. Soc., Providence, RI, 2008.

[4] M. Blank, G. Keller, C. Liverani, *Ruelle-Perron-Frobenius spectrum for Anosov maps*, Nonlinearity **15** (2002), no. 6, 1905–1973

[5] J. Buzzi, *Absolutely continuous invariant measures for generic multi-dimensional piecewise affine expanding maps*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **9** (1999), 1743–1750.

[6] J. Buzzi, *Absolutely continuous invariant probability measures for arbitrary expanding piecewise* $\mathbb{R}$*-analytic mappings of the plane,* Ergodic Theory Dynam. Systems **20** (2000), 697–708.

[7] J. Buzzi, Thermodynamic Al formalism for piecewise invertible maps: absolutely continuous invariant measures as equilibrium states. *Smooth ergodic theory and its applications (Seattle, WA, 1999),* 749–783, Proc. Sympos. Pure Math., **69**, Amer. Math. Soc., Providence, RI, 2001.

[8] J. Buzzi, *No or infinitely many a.c.i.p. for piecewise expanding* C^r *maps in higher dimensions*, Comm. Math. Phys. **222** (2001), 495–501.

[9] J. Buzzi, *Maximal entropy measures for piecewise affine surface homeomorphisms*, Ergodic Theory Dynam. Systems **29** (2009), 1723–1763.

[10] J. Buzzi, *Puzzles of quasi-finite type, zeta functions and symbolic dynamics for multi-dimensional maps,* Ann. Inst. Fourier (to appear).

[11] J. Buzzi, G. Keller, *Zeta functions and transfer operators for multidimensional piecewise affine and expanding maps*, Ergodic Theory Dynam. Systems **21** (2001), 689–716.

[12] W. Cowieson, *Absolutely continuous invariant measures for most piecewise smooth expanding maps*, Ergodic Theory Dynam. Systems **22** (2002), 1061–1078.

[13] M. F. Demers, C. Liverani, *Stability of Statistical Properties in Two-Dimensional Piecewise Hyperbolic Maps*, Trans. Amer. Math. Soc. **32** (1990), 120–140.

[14] S. Gouëzel,*Spectre de l'opérateur de transfert en dimension* 1, Manuscripta Math. **106** (2001), 365–403.

[15] S. Gouëzel, C. Liverani, *Compact locally maximal hyperbolic sets for smooth maps: fine statistical properties*, Journal Diff. Geom. **79** (2008), 433–477.

[16] C. Liverani, M. Tsujii, *Zeta functions and Dynamical Systems*, Nonlinearity **19** (2006), 2467–2473.

[17] M. Tsujii, *Absolutely continuous invariant measures for piecewise real-analytic expanding maps on the plane*, Comm. Math. Phys. **208** (2000), 605–622.

[18] M. Tsujii, *Piecewise expanding maps on the plane with singular ergodic properties*, Ergodic Theory Dynam. Systems **20** (2000), 1851–1857.

[19] M. Tsujii, *Absolutely continuous invariant measures for expanding piecewise linear maps*, Invent. Math. **143** (2001), 349–373.

Shrinking targets and decay of correlations

Stefano Galatolo

Let (X, T, μ) be an ergodic system on a metric space X and fix a target point $x_0 \in X$. For μ-almost every $x \in X$, the orbit of x goes closer and closer to x_0 entering (sooner or later) in each positive measure neighbourhood of the point x_0.

For several applications it is useful to quantify the speed of approaching of the orbit of x to x_0. In the literature this has been done in several ways, with more or less precise estimations or considering different kind of target sets.

A general approach to this kind of problems problem is to consider a family of sets S_r indexed by a real parameter r containing x_0 and give an estimation for the time needed for the orbit of a point x to enter in S_r

$$\tau(x, S_r) = \min\{n \in \mathbb{N}^+ : T^n(x) \in S_r\}. \tag{1}$$

If X is a metric space, the most natural choice is to take $S_r = B_r(x_0)$ (the ball of radius r). In this case several estimations are known for the behaviour of $\tau(x, S_r)$ as $r \to 0$. Another way to look at the same kind of problem which is present in the literature, is to consider the behaviour of the minimum distance after n iterations: $d_n(x, x_0) = \min_{i \le n} \operatorname{dist}(T^i(x)), x_0)$ these two approaches are somewhat equivalent for our purposes (see the appendix of [3] for some precise statements).

It is known that if the system has fast decay of correlations or is a circle rotation with generic arithmetical properties (see [2], [1]) then for a.e. x

$$\lim_{r \to 0} \frac{\log \tau(x, B_r(x_0))}{-\log r} = d_\mu(x_0). \tag{2}$$

This gives an estimation for the scaling behaviour of τ when $r \to 0$ and relates it to the local dimension of the invariant measure at the target point x_0. On the other hand it is worth to remark that there are mixing systems (having particular arithmetical properties and slow decay of correlations) for which $\liminf_{r \to 0} \frac{\log \tau(x, B_r(x_0))}{-\log r} = \infty > d_\mu(x_0)$ (see [3]). For applications it is important to extend this kind of result to a larger class of target sets. It is possible to prove that (see [4])

Proposition 1. *If a system has superpolynomial (faster than any power law) decay of correlations (with respect to Lipschitz observables) then the time needed for a typical point x to enter for the first time a set $S_r = \{x : f(x) \le r\}$ which is a sublevel of a Lipschitz function f satisfies*

$$\lim_{r \to 0} \frac{\log \tau(x, S_r)}{-\log r} = \lim_{r \to 0} \frac{\log \mu(S_r)}{\log r}.$$

As an example of an application of the above result we consider the geodesic flow of a negatively curved manifold, which is known to have exponential decay of correlations ([6]). This gives the following result, similar to the one given in [7]

Proposition 2. *Let M be a C^4, compact manifold of dimension d with strictly negative curvature and T^1M be its unitary tangent bundle. Let $\pi : T^1M \to M$ the* *canonical projection. If T is the time 1 map of the geodesic flow, μ the Liouville measure on T^1M , and $dist()$ the Riemannian distance on M, then for each $p \in M$:*

$$\limsup_{n\to\infty} \frac{-\log dist(p, \pi(T^n x))}{\log n} = \frac{1}{d} \tag{3}$$

holds for almost each $x \in T^1M$.

Finally we mention that if one wants to obtain a law similar to (2) for a flow, it is not necessary to prove that the whole flow has fast decay of correlations. One can work with a Poincaré section (provided the return time is integrable and some more technical assumptions) and prove fast decay of correlation for the induced map on the section. This is what was done in [5] leading to a formula like (2) for a class of Lorenz like flows.

References

[1] Kim D. H. and Seo B. K., *The waiting time for irrational rotations*, Nonlinearity **16** (2003), 1861–1868.

[2] Galatolo S., *Dimension and hitting time in rapidly mixing systems*, Math Res. Lett.14 (2007), no. 5,pp 797–805 (2007).

[3] Galatolo S., Peterlongo P. *Long hitting time, slow decay of correlations and arithmetical properties.*Preprint arXiv:0801.3109

[4] Galatolo S *Hitting time in regular sets and logarithm law for rapidly mixing dynamical systems* Preprint arXiv:0906.3416

[5] Galatolo S., Pacifico M. J. *Lorenz like flows: exponential decay of correlations for the Poincaré map, logarithm law, quantitative recurrence* Preprint arXiv:0901.0574

[6] Liverani C *On contact Anosov flows,* Ann. Math.159, 3, 1275–1312 (2004)

[7] Maucourant F. *Dynamical Borel-Cantelli Lemma for hyperbolic spaces* Israel Journal of mathematics, 152 (2006), p 143–155

Almost sure invariance principle by spectral methods

Sébastien Gouëzel

The almost sure invariance principle is a very strong reinforcement of the central limit theorem: it ensures that the trajectories of a process can be matched with the trajectories of a Brownian motion in such a way that, almost surely, the error between the trajectories is negligible compared to the size of the trajectory (the result can be more or less precise, depending on the specific error term one can obtain).

Definition 1. *For $\lambda \in (0, 1/2]$, an $\mathbb{R}^d$–valued process $(A_0, A_1, \dots)$ satisfies an almost sure invariance principle with error exponent λ and limiting covariance Σ^2 if there exist a probability space Ω, and two processes $(A_0^*, A_1^*, \dots)$ and $(B_0, B_1, \dots)$ on Ω such that*

(1) *The processes $(A_0, A_1, \dots)$ and $(A_0^*, A_1^*, \dots)$ have the same distribution.*
(2) *The random variables $B_0, B_1, \dots$ are independent, distributed as $\mathcal{N}(0, \Sigma^2)$.*

(3) *Almost surely in* Ω,

$$\left|\sum_{\ell=0}^{n-1} A_\ell^* - \sum_{\ell=0}^{n-1} B_\ell\right| = o(n^\lambda). \tag{1}$$

We mainly consider processes originating from dynamical systems: let $T : X \to X$ be a map, let $f : X \to \mathbb{R}^d$ be a function, and let μ be a probability measure on X (invariant or not). We are interested in the almost sure limit theorem for $A_\ell = f \circ T^\ell$. Such results are known for instance in the following cases:

(1) $d = 1$, T a piecewise expanding map of the interval, f of bounded variation [HK82].
(2) $d = 1$, T subshift of finite type or Axiom A, f Hölder continuous [DP84].
(3) $d \geq 1$, T subshift of finite type (or more generally Gibbs-Markov), f locally Hölder continuous [MN09].

All those results rely either on techniques specific to $d = 1$, or on the existence of a nice Markov structure on the space, that makes it possible to approximate an observable by a locally constant one and then apply probabilistic techniques for Rosenblatt–mixing processes and martingales. This does not cover simple dynamical examples where such a natural Markov structure is not available. On the other hand, in such situations, it is often possible to prove the central limit theorem using spectral information. Moreover, this spectral method is so powerful that it should be possible to use it to prove for dependent sequences essentially all the results that are known for i.i.d. sequences. In this talk, we illustrate this philosophy by showing that the almost sure invariance principle follows from the spectral method (and with very good error bounds, contrary to the previous martingale methods).

Definition 2. *The characteristic function of a process* (A_ℓ) *as above is coded by a family of operators* $(\mathcal{L}_t)_{|t|\leq\epsilon_0}$ *on a Banach space* $\mathcal{B}$ *if there exist* $\phi_0 \in \mathcal{B}'$ *and* $u_0 \in \mathcal{B}$ *such that, for any* $t_0, \dots, t_{n-1} \in B(0, \epsilon_0)$,

$$E\left(e^{i\sum_{\ell=0}^{n-1} t_\ell A_\ell}\right) = \langle \phi_0, \mathcal{L}_{t_{n-1}}\mathcal{L}_{t_{n-2}} \cdots \mathcal{L}_{t_1}\mathcal{L}_{t_0} u_0\rangle.$$

This definition is an abstraction of the properties that are commonly used for the spectral method: in most cases, $\mathcal{B}$ is a space of functions on X, u_0 is the function 1 and ϕ_0 is the integral with respect to μ. However, for more complicated dynamical systems, one sometimes needs to take for $\mathcal{B}$ a space of distributions, or measures, or even wilder objects. The previous definition applies in all those settings.

Our main theorem is the following.

Theorem 1 ([Go09, Theorem 2.1])**.** *Let* (A_ℓ) *be a process whose characteristic function is coded by a family of operators* $(\mathcal{L}_t)$, *and bounded in* L^p *for some* $p > 2$. *Assume that*

(1) *One can write* $\mathcal{L}_0 = \Pi + Q$ *where* Π *is a one-dimensional projection and* Q *is an operator on* $\mathcal{B}$, *with* $Q\Pi = \Pi Q = 0$, *and* $\|Q^n\|_{\mathcal{B}\to\mathcal{B}} \leq C\kappa^n$ *for some* $\kappa < 1$.

(2) *For small enough t, we have $\|\mathcal{L}_t^n\|_{\mathcal{B}\to\mathcal{B}} \leq C$ uniformly in n.*

Then there exist $a \in \mathbb{R}^d$ and a matrix Σ^2 such that $\sum_{\ell=0}^{n-1}(A_\ell - a)/\sqrt{n}$ converges to $\mathcal{N}(0, \Sigma^2)$. Moreover, this process satisfies an almost sure invariance principle with limiting covariance Σ^2 for any error exponent larger than $p/(4p-4)$.

This applies for instance to piecewise expanding or hyperbolic maps, or to coupled map lattices.

Contrary to most similar results, we do not assume the existence of a perturbed eigenvalue (at the cost of a boundedness assumption in L^p for some $p > 2$, while the usual assumptions in the central limit theorem is simply for $p = 2$).

The proof of this theorem is mainly probabilistic: we deduce from the spectral assumption in the above theorem a decorrelation estimate on characteristic functions, and then show that this estimate is sufficient by itself to get the almost sure invariance principle.

References

[DP84] Manfred Denker and Walter Philipp, *Approximation by Brownian motion for Gibbs measures and flows under a function*, Ergodic Theory Dynam. Systems **4** (1984), no. 4, 541–552.

[Go09] Sébastien Gouëzel, *Almost sure invariance principle for dynamical systems by spectral methods*, Preprint, 2009.

[HK82] Franz Hofbauer and Gerhard Keller, *Ergodic properties of invariant measures for piecewise monotonic transformations*, Math. Z. **180** (1982), no. 1, 119–140.

[MN09] Ian Melbourne and Matthew Nicol, *A vector-valued almost sure invariance principle for hyperbolic dynamical systems*, Ann. Probab. **37** (2009), 478–505.

Self-consistent Perron-Frobenius operators for globally coupled maps

Gerhard Keller

(joint work with Jean-Baptiste Bardet and Roland Zweimüller)

Globally coupled maps are collections of individual discrete-time dynamical systems (their *units*) which act independently on their respective phase spaces, except for the influence (the *coupling*) of a common parameter that is updated, at each time step, as a function of the *mean field* of the whole system. Systems of this type have received some attention through the work of Kaneko [5, 6] in the early 1990s, who studied systems of N quadratic maps acting on coordinates $x_1, \ldots, x_N \in [0, 1]$, and coupled by a parameter depending in a simple way on $\bar{x} := N^{-1}(x_1 + \cdots + x_N)$. His key observation, for huge *system size* N, was the following: if $(\bar{x}^t)_{t=0,1,2,\ldots}$ denotes the time series of mean field values of the system started in a random configuration $(x_1, \ldots, x_N)$, then, for many parameters of the quadratic map, and even for very small coupling strength, pairs $(\bar{x}^t, \bar{x}^{t+1})$ of consecutive values of the field showed complicated functional dependencies plus some noise of order $N^{-1/2}$, whereas for uncoupled systems of the same size the $\bar{x}^t$, after a while, are constant up to some noise of order $N^{-1/2}$. While the latter observation is not surprising for independent units, the complicated dependencies for weakly coupled systems, a phenomenon Kaneko termed *violation of the law of large numbers*, called for closer investigation. Similar numerical findings were later presented for systems of coupled tent maps [3, 8, 9] and partially supported by mathematical arguments [3, 2]. Rigorous proofs for the "non-violation" of the law of large numbers in the case of expanding circle maps with weak global coupling are given in [4, 7], but no rigorous treatment of a situation where the law of large numbers is "violated" was known. It was the purpose of my talk to present an example of this kind, studied in [1], that can be analysed rather completely.

I first presented the formal framework to study mean field coupled systems, namely the passage from ordered configurations of finite systems to unordered ones, i.e. to empirical measures. This idea is well known in other branches of many particle systems and was described for deterministic dynamical systems in [7]. If the individual units $T_r : X \to X$ are piecewise expanding interval maps satisfying a uniform Lasota-Yorke type inequality on $\mathrm{BV}(X)$, it leads to the problem of studying the dynamics of a nonlinear version of the Perron-Frobenius operator (also called self consistent PFO) when the individual units allow a description in terms of linear PFOs. The dynamics of this self consistent PFO $\tilde{P}$ on the space $\mathcal{D} \subset L^1 := L^1_{\mathrm{Leb}}(X)$ of all probability densities on X determines much of the dynamical properties of systems of N coupled maps for large N. For the example system, all local maps $T_r : X \to X$, $X = [-\frac{1}{2}, \frac{1}{2}]$, $-\frac{2}{3} < r < \frac{2}{3}$, have two linear fractional branches. T_0 s the doubling map, and the general formula is $T_r(x) = \frac{(r+4)x+r+1}{2rx+2}$ "mod X". The N-particle system is then given by $\mathcal{T}_N : X^N \to X^N$, $\mathcal{T}_N x = (T_r x_1, \ldots, T_r x_N)$ with $r = G(\phi(x))$, where $\phi(x) = N^{-1} \sum_i x_i$ and $G(x) = A \tanh(\frac{B}{A} x)$ with parameters $0 < A \leq 0.4$ and $0 \leq B \leq 16$. The corresponding self consistent PFO $\tilde{P} : \mathcal{D} \to \mathcal{D}$ turns out to be $\tilde{P} u = P_{T_r} u$ where $r = G(\phi(u))$ and $\phi(u) = \int_X x\, u(x)\, dx$.

The main results from [1] are the following two theorems:

Theorem 1. *For all $0 < A \leq 0.4$, $0 \leq B \leq 16$ and for all $N \geq 1$ the map $\mathcal{T}_N$ has a unique absolutely continuous invariant measure μ_N. μ_N has a strictly positive, analytic density, and the system $(\mathcal{T}_N, \mu_N)$ shows exponential decay of correlations for Hölder-observables on X^N.*

For the second theorem denote by u_r the unique invariant density of the local map T_r. (Up to normalisation, $u_r(x) = \frac{2r^2}{(rx-1)^2 - r^2}$.)

Theorem 2. *For all $0 < A \leq 0.4$ holds:*

(1) *If $0 \leq B \leq 6$, then $\tilde{P}$ has a unique fixed point u_0 in $\mathcal{D}$. Indeed, $u_0 = 1$, and $\lim_{t\to\infty} \tilde{P}^t u = u_0$ in L^1 for all $u \in \mathcal{D}$.*
(2) *If $6 < B \leq 16$, then $\tilde{P}$ has exactly three fixed points in $\mathcal{D}$, namely u_{-r_*}, u_0, and u_{r_*}, where r_* is the unique positive fixed point of the map $r \mapsto G(\phi(u_r))$. For each $u \in \mathcal{D}$, the sequence $\tilde{P}^t u$ converges in L^1 to one of these three fixed points. The basins of attraction of u_{-r_*} and of u_{r_*} are open in L^1.*

I conjecture that the basin of u_0 is the common boundary of the two other basins, but so far I can only prove that the union of the two open basins is dense in $\mathcal{D}$. Refined information about the structure of the basin of u_0 is relevant for a large deviations analysis (as $N \to \infty$) of the finite systems.

The proof of Theorem 1 relies on classical results for piecewise expanding maps, while the proof of Theorem 2 rests on the fact that PFOs of maps with full fractional-linear branches leave the class of Herglotz-Pick-Nevanlinna functions invariant. This observation can be used to study the action of $\widetilde{P}$ in terms of an iterated function system on the interval $[-\frac{2}{3}, \frac{2}{3}]$ with two fractional-linear branches and place dependent probabilities. In the bistable regime the system is of course not contractive, but it has strong monotonicity properties and special geometric features which allow to prove the theorem.

References

[1] J.-B. Bardet, G. Keller, R. Zweimüller, *Stochastically stable globally coupled maps with bistable thermodynamic limit*, Commun. Math. Phys. **292** (2009), 237–270.

[2] T. Chawanya, S. Morita, *On the bifurcation structure of the mean-field fluctuation in the globally coupled tent map systems*, Physica D **116** (1998), 44–70.

[3] S.V. Ershov, A.B. Potapov, *On mean field fluctuations in globally coupled maps*, Physica D **86** (1995), 523–558.

[4] E. Järvenpää, *An SRB-measure for globally coupled analytic expanding circle maps,* Nonlinearity **10** (1997), 1435–1469.

[5] K. Kaneko, *Globally coupled chaos violates the law of large numbers but not the central limit theorem*, Phys. Rev. Letters **65** (1990), 1391–1394.

[6] K. Kaneko, *Remarks on the mean field dynamics of networks of chaotic elements*, Physica D **86** (1995), 158–170.

[7] G. Keller, *An ergodic theoretic approach to mean field coupled maps*, Progress in Probability, Vol. **46** (2000), 183-208.

[8] N. Nakagawa, T.S. Komatsu, *Collective motion occurs inevitably in a class of populations of globally coupled chaotic elements*, Phys. Rev. E **57** (1998), 1570–1575

[9] N. Nakagawa, T.S. Komatsu, *Confined chaotic behaviour in collective motion for populations of globally coupled chaotic elements*, Phys. Rev. E **59** (1999), 1675–1682.

Poincaré sums, sum-level sets, and uniform distribution

Marc Kesseböhmer

(joint work with Bernd O. Stratmann)

Our main task is to give a detailed measure-theoretical analysis of the following sets $\mathcal{C}_n$, for $n \in \mathbb{N}$, which we will refer to as the sum-level sets:

$$\mathcal{C}_n := \{[a_1, a_2, \ldots] \in [0, 1] : \sum_{i=1}^{k} a_i = n \text{ for some } k \in \mathbb{N}\}.$$

In here, $x = [a_1, a_2, \ldots]$ denotes the regular continued fraction expansion of $x \in [0, 1]$. The first main result is to settle a recent conjecture of Fiala and Kleban [1], which asserts that the Lebesgue measure of these level sets decays to zero, for the level tending to infinity. The second and third main result then give precise asymptotic estimates for this decay. The proofs of these results are based on recent progress in infinite ergodic theory [2] (also relying on [11, 12]) obtained for the Farey map. In particular we show in [6] that

$$\sum_{k=1}^{n} \lambda\left(\mathcal{C}_k\right) \sim \frac{n}{\log_2 n} \quad \text{and} \quad \lambda(\mathcal{C}_n) \sim \frac{1}{\log_2 n},$$

where $a_n \sim b_n$ means that $\lim_{n\to\infty} a_n/b_n = 1$ for two positive sequences (a_n) and (b_n).

Refined results from [3] then allow us to prove that the Stern–Brocot and the Farey sequence are uniformly distributed in $[0,1]$ with respect to certain geometric weights. That is we consider two sequences of subsets of the unit interval: the well-known *Farey sequence* given by $\mathcal{F}_n := \{p/q : 0 < p \leq q \leq n, (p,q) = 1\}$ and the *even Stern–Brocot sequence* $\mathcal{S}_n := \left\{s_{n,2k}/t_{n,2k} : k = 1, \ldots, 2^{n-1}\right\}$ (cf. [10]), where $s_{0,1} := 0$ and $s_{0,2} := t_{0,1} := t_{0,2} := 1$, $s_{n+1,2k-1} := s_{n,k}$ and $t_{n+1,2k-1} := t_{n,k}$, for $k = 1, \ldots, 2^n + 1$, $s_{n+1,2k} := s_{n,k} + s_{n,k+1}$ and $t_{n+1,2k} := t_{n,k} + t_{n,k+1}$ for $k = 1, \ldots, 2^n$, $n \in \mathbb{N}$ (see [4] for more details). More precisely, on the one hand we have

$$\frac{1}{\operatorname{card}\left(\mathcal{F}_n\right)} \sum_{r \in \mathcal{F}_n} \delta_r \xrightarrow{*} \lambda \;\text{ and }\; \frac{1}{\operatorname{card}\left(\mathcal{S}_n\right)} \sum_{r \in \mathcal{S}_n} \delta_r \xrightarrow{*} \nu,$$

where ν denotes the measure of maximal entropy for the Farey map ([9], [5]). (Note that the distribution function of ν is given by the famous Minkowski question mark function.) While on the other hand, in [8] we show that for the weighted measures, for n tending to infinity, we have

$$\frac{\zeta(2)}{\log n} \sum_{p/q \in \mathcal{F}_n} \frac{1}{q^2}\, \delta_{p/q} \xrightarrow{*} \lambda \quad \text{and} \quad 2\log n \sum_{p/q \in \mathcal{S}_n} \frac{1}{q^2}\, \delta_{p/q} \xrightarrow{*} \lambda.$$

Carrying over the above described ideas to the setting of Kleinian groups we are able to derive in [7] estimates for the algebraic growth rate of the Poincaré series for a Kleinian group at its critical exponent of convergence. That is we study the Poincaré series

$$\mathcal{P}(z,w,s) := \sum_{g \in G} e^{-sd(z,g(w))}$$

of a geometrically finite Kleinian group G acting on the $(N+1)$-dimensional hyperbolic space $\mathbb{H}$, for arbitrary $z, w \in \mathbb{H}$. Here, $d(z,w)$ denotes the hyperbolic distance between z and w, and $s \in \mathbb{R}$. It is well known that a group of this type is of δ-divergence type, which means that $\mathcal{P}(z,w,s)$ diverges for s equal to the exponent of convergence $\delta = \delta(G)$ of $\mathcal{P}(z,w,s)$. For $g \in G$ let $|g|$ denote the word norm of G. Then for a geometrically finite, essentially free, zonal Kleinian group G with $r_{\max}$ denoting the maximal rank of the parabolic fixed points of G, and for each $z, w \in \mathbb{H}$, we have

$$\mathcal{P}_n(z,w,\delta) := \sum_{\substack{g \in G \\ |g| \leq n}} e^{-\delta d(z,g(w))} \asymp \begin{cases} n^{2\delta - r_{\max}} & \text{for} \quad \delta < (r_{\max}+1)/2 \\ n/\log n & \text{for} \quad \delta = (r_{\max}+1)/2 \\ n & \text{for} \quad \delta > (r_{\max}+1)/2, \end{cases}$$

where $a_n \asymp b_n$ means that the sequence of quotients (a_n/b_n) is bounded away from zero and infinity.

References

[1] J. Fiala, P. Kleban. Intervals between Farey fractions in the limit of infinite level. Preprint, arXiv:math-ph/0505053v2, 2006.

[2] M. Kesseböhmer, M. Slassi. A distributional limit law for the continued fraction digit sum. *Mathematische Nachrichten* 281 (2008), no. 9, 1294–1306.

[3] M. Kesseböhmer, M. Slassi. Large deviation asymptotics for continued fraction expansions, *Stochastics and Dynamics* 8 (1), 103–113, 2008.

[4] M. Kesseböhmer, B.O. Stratmann. Stern–Brocot pressure and multifractal spectra in ergodic theory of numbers. *Stochastics and Dynamics*, 4 (1):77–84, 2004.

[5] M. Kesseböhmer, B.O. Stratmann. A multifractal analysis for SB–intervals, continued fractions and Diophantine growth rates. *J. reine angew. Math.*, 605, 2007.

[6] M. Kesseböhmer, B.O. Stratmann. On the Lebesgue measure of sum-level sets for continued fractions. Preprint: arXiv: 0901.1787v2.

[7] M. Kesseböhmer, B.O. Stratmann. A note on the algebraic growth rate of Poincaré series for Kleinian groups. Preprint: arXiv:0910.5560.

[8] M. Kesseböhmer, B.O. Stratmann. Uniform distribution: A dichotomy between the Farey and the Stern–Brocot sequence. In preparation.

[9] E. Landau. Bemerkungen zu der vorstehenden Abhandlung von Herr Franel. *Nachr. Ges. Wiss. Göttingen*, Math.-phys. K1., 202–206 (1924).

[10] M.A. Stern. Über eine zahlentheoretische Funktion. *J. reine angew. Math.*, 55:193–220, 1858.

[11] M. Thaler. A limit theorem for the Perron–Frobenius operator of transformations on $[0, 1]$ with indifferent fixed points. *Israel J. Math.*, 91(1-3):111–127, 1995.

[12] M. Thaler. The asymptotics of the Perron-Frobenius operator of a class of interval maps preserving infinite measures. *Studia Math.*, 143(2):103–119, 2000.

Transfer operators of continuous-time dynamics

Carlangelo Liverani

(joint work with Paolo Giulietti)

The study of the transfer operator for hyperbolic systems has received a big impulse lately. Many new techniques have been proposed. I like to illustrate the relation between the *standard pair approach* pioneered by Dmitry Dolgopyat [4] and perfected by Dolgopyat and Chernov [3] and the *Banach space approach* (initiated in [1]) which is the theme of the present workshop. In fact, standard pairs are related to a particular version of the Banach space approach that can be found in [6] and [7].

The standard pairs approach is based on the identification of a set of measures (supported on manifolds close to the unstable foliation) which are invariant under the action of the dynamics. One can then use a version of the *coupling technique* introduced in the field by Lai-Sang Young [13] to obtain results on the convergence to equilibrium, limit theorems and linear response. One can look at [5] to see how the combination of such an approach with the *martingale methods* introduced by Stroock and Varadhan [12] allows to obtain very sharp results concerning the limiting behaviour of Dynamical Systems with fast-slow degree of freedom.

The relation between standard pairs and [6] consists in the possibility to use such a set of measures as the predual of a Banach space. If one proceeds along such lines it is then possible to obtain a Lasota-Yorke inequality and the necessary compactness to apply the usual machinery of transfer operators whereby obtaining much more fine results on the statistical properties of the system than available by the coupling technique or other competing methods.

I illustrate such a procedure both in discrete and continuous time [8, 2]. In the continuous time case I announce some results contained in a work in progress with Paolo Giulietti concerning the meromorphicity of the zeta function for Anosov flows. The methods used to obtain such last results are a combination of the approach in [2] and the tools developed in [9, 10], following in the footsteps of Ruelle [11].

References

[1] M. Blank, G. Keller, C. Liverani, *Ruelle-Perron-Frobenius spectrum for Anosov maps.* Nonlinearity **15** (2002), no. 6, 1905–1973.

[2] O. Butterley, C. Liverani, *Smooth Anosov flows: correlation spectra and stability.* J. Mod. Dyn. **1** (2007), no. 2, 301–322.

[3] N. Chernov, D. Dolgopyat, *Brownian Brownian motion. I.* Mem. Amer. Math. Soc. **198** (2009), no. 927, viii+193 pp.

[4] D. Dolgopyat, *On differentiability of SRB states for partially hyperbolic systems.* Invent. Math. **155** (2004), no. 2, 389–449.

[5] D. Dolgopyat, *Averaging and invariant measures.* Mosc. Math. J. **5** (2005), no. 3, 537–576, 742.

[6] S. Gouëzel, C. Liverani, *Banach spaces adapted to Anosov systems.* Ergodic Theory Dynam. Systems **26** (2006), no. 1, 189–217.

[7] S. Gouëzel, C. Liverani, *Compact locally maximal hyperbolic sets for smooth maps: fine statistical properties.* J. Differential Geom. **79** (2008), no. 3, 433–477.

[8] C. Liverani, *On contact Anosov flows.* Ann. of Math. (2) **159** (2004), no. 3, 1275–1312.

[9] C. Liverani, *Fredholm determinants, Anosov maps and Ruelle resonances.* Discrete Contin. Dyn. Syst. **13** (2005), no. 5, 1203–1215.

[10] C. Liverani, M. Tsujii, *Zeta functions and dynamical systems.* Nonlinearity **19** (2006), no. 10, 2467–2473.

[11] D. Ruelle, *Zeta-functions for expanding maps and Anosov flows.* Invent. Math. **34** (1976), no. 3, 231–242.

[12] D.W. Stroock, S.R.S. Varadhan,*Diffusion processes.* Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. III: Probability theory, pp. 361–368. Univ. California Press, Berkeley, Calif., 1972.

[13] L.-S. Young, *Recurrence times and rates of mixing.* Israel J. Math. **110** (1999), 153–188.

The Ruelle spectrum of generic transfer operators

Frédéric Naud

Let $\Omega \subset \mathbb{C}^d$ be a an open connected bounded non-empty set. Let $U(\Omega)$ denote the Banach space of holomorphic functions $f : \Omega \to \mathbb{C}$ having a continuous extension to the closure $\overline{\Omega}$, endowed with the obvious supremum norm. The set of holomorphic contractions $\gamma : \Omega \to \Omega$ is defined by

$$\mathcal{K}(\Omega) := \{\gamma \in U(\Omega)^d \ : \ \gamma(\overline{\Omega}) \subset \Omega\}.$$

Let $k \geq 1$ be an integer. We now denote by $\mathcal{M}_k(\Omega)$ the product

$$\mathcal{M}_k(\Omega) := \mathcal{K}(\Omega)^k \times U(\Omega)^k.$$

Given $(\phi, w) \in \mathcal{M}_k(\Omega)$ one defines the transfer operator associated to the data (ϕ, w) by the formula

$$\mathcal{L}_{\phi,w} f(z) = \sum_{i=1}^{k} w_i(z)(f \circ \phi_i)(z).$$

This linear operator, when acting on a reasonable function space of holomorphic functions (for example $A^2(\Omega)$ which is the Bergmann Hilbert space of square integrable holomorphic functions on Ω), is a compact trace class operator. Transfer operators of this type have been first considered by Ruelle in [5], arising as complexified Perron-Frobenius operators related to real analytic expanding maps. The eigenvalues of $\mathcal{L}$ form a discrete decreasing sequence denoted by $(\lambda_n(\mathcal{L}))$. Ruelle gave an upper bound on this sequence in dimension $d = 1$ which was extended recently to arbitrary domains Ω by Bandtlow-Jenkinson in [1] :

$$|\lambda_n(\mathcal{L})| \leq A e^{-an^{1/d}},$$

for some constants $A, a > 0$. Except in some affine contraction examples, no general lower bounds on the spectrum is known. In this talk we explain how to obtain lower bounds for a dense set of data by using some tools of potential theory that have been successfully applied in the framework of scattering theory by Tanya Christiansen in [3, 4]. More precisely, we prove the following.

Theorem 1. *Assume that Ω is convex. Then there exists a dense subset $\mathcal{G} \subset \mathcal{M}_k(\Omega)$ such that for all $(\phi, w) \in \mathcal{G}$, for all $\epsilon > 0$, we have*

$$\limsup_{n \to +\infty} \frac{|\lambda_n(\mathcal{L})|}{\exp(-n^{1/d+\epsilon})} = +\infty. \tag{1}$$

This result shows that for a dense set of data, the upper bound is the best possible in terms of exponent. For all d, k, it is possible to build examples of transfer operators with data in $\mathcal{M}_k(\Omega)$ without eigenvalues except possibly 0. We actually show in [2] that a similar statement holds for Perron-Frobenius operators related to piecewise real analytic expanding maps for $d = 1$. Our proof is based on complex analysis and the potential theoretic properties of the order of families of entire functions. Given $(\phi, w) \in \mathcal{M}_k(\Omega)$, we investigate the determinants

$$Z(\zeta) := \det(I - e^{\zeta} \mathcal{L}_{\phi,w}),$$

which are entire functions of order $M(\mathcal{L}) \leq d + 1$, the order being defined by

$$M(\mathcal{L}) := \limsup_{r \to +\infty} \frac{\log \left(\sup_{|\zeta|=r} \log |Z(\zeta)| \right)}{\log r}.$$

The order is closely related to the distribution of zeros of $Z(\zeta)$ and therefore the spectrum of $\mathcal{L}$, and we show that the previous "lower bound" (1) follows from the equality $M(\mathcal{L}) = d+1$. By looking at one dimensional families $\mathcal{L}_z := \mathcal{L}_{\phi_z, w_z}$ where $\phi_z = (1-z)\phi_0 + z\phi_1$, $w_z = (1-z)w_0 + zw_1$, z is in a complex neighbourhood $\mathcal{U} \supset [0,1]$, we actually obtain the following statement.

Theorem 2. *Assume that Ω is convex and that (1) holds for $z = 0$, then there exists a set E of Hausdorff dimension* 0 *such that the lower bound (1) holds for all $z \in \mathcal{U} \setminus E$.*

This result may be viewed as a weak maximum principle that follows from approximating the order $M(\mathcal{L}_z)$ by subharmonic functions. More details can be found in [2].

References

[1] Oscar F. Bandtlow and Oliver Jenkinson. Explicit eigenvalue estimates for transfer operators acting on spaces of holomorphic functions. *Adv. Math.*, 218(3):902–925, 2008.

[2] Frédéric Naud. The Ruelle spectrum of generic transfer operators. *Preprint*, 2009.

[3] T. Christiansen. Several complex variables and the distribution of resonances in potential scattering. *Comm. Math. Phys.*, 259(3):711–728, 2005.

[4] T. J. Christiansen. Several complex variables and the order of growth of the resonance counting function in Euclidean scattering. *Int. Math. Res. Not.*, pages Art. ID 43160, 36, 2006.

[5] David Ruelle. Zeta-functions for expanding maps and Anosov flows. *Invent. Math.*, 34(3):231–242, 1976.

Quantum transfer operators and chaotic scattering

Stéphane Nonnenmacher

Consider a symplectic diffeomorphism T on $T^*\mathbb{R}^d$, which can be generated near the origin by a single function $W(x_1, \xi_0)$, in the sense that the dynamics $(x_1, \xi_1) = T(x_0, \xi_0)$ is the implicit solution of the two equations $\xi_1 = \partial_{x_1} W(x_1, \xi_0)$, $x_0 = \partial_{\xi_0} W(x_1, \xi_0)$. One can associate to T a family of *quantum transfer operators* $M(T, h)$ acting on $L^2(\mathbb{R}^d)$, of the form:

$$[M(T,h)\psi](x_1) = \int a(x_1, \xi_0)\, e^{\frac{i}{h}(W(x_1,\xi_0) - \langle \xi_0, x_0\rangle)}\, \psi(x_0)\, \frac{dx_0\, d\xi_0}{(2\pi h)^d}\,. \qquad (1)$$

Here $a \in C^\infty(T^*\mathbb{R}^d)$ is called the *symbol* of the operator. The "small parameter" $h > 0$ is the typical wavelength on which the integral kernel of the operator oscillates; it is often called "Planck's constant", due to the appearance of such operators in quantum mechanics.

The operator $M(T, h)$ (understood as a family $(M(T, h))_{h\in(0,1]}$) can be interpreted as a "quantisation" of the symplectic map T, for the following reason. Consider a phase space point $(x_0, \xi_0) \in T^*\mathbb{R}^d$. There exist wavefunctions (*quantum states*) $\psi_{x_0,\xi_0,h} \in L^2(\mathbb{R}^d)$ which are localised near the position $x_0 \in \mathbb{R}^d$, and whose h-Fourier transform is localised near the momentum $\xi_0 \in \mathbb{R}^d$ (equivalently, the usual Fourier transform is localised near $h^{-1}\xi_0$). Such wavefunctions are said to be *microlocalised* near (x_0, ξ_0); in some sense, they represent the best quantum approximation of a "point particle" at (x_0, ξ_0). In the semiclassical limit $h \to 0$, the application of stationary phase expansions to the integral (1) shows that the image state $M(T,h)\psi_{x_0,\xi_0,h}$ is microlocalised near the point $(x_1, \xi_1) = T(x_0, \xi_0)$; that is, this operator transports the quantum mass at the point (x_0, ξ_0) to the point $T(x_0, \xi_0)$.

Similar families of operators have appeared in the theory of linear PDEs in the 1960s: the "Fourier integral operators" invented by Hörmander. A modern account (closer to the above definition) can be found in the recent lecture notes of C.Evans & M.Zworski [1]. We are using these operators as nice models for "quantum chaos", that is the study of quantum systems, the classical limits of which are "chaotic". In this framework, these operators (sometimes called "quantum maps") generate a quantum dynamical system:

$$L^2(\mathbb{R}^d) \ni \psi \mapsto M(T,h)\psi \,. \tag{2}$$

These quantum maps provide a discrete time generalisation of the Schrödinger flow $U^t(h) = \exp(-itP(h)/h)$ associated with the Schrödinger equation $ih\partial_t\psi = P(h)\,\psi$, where $P(h)$ is a selfadjoint operator, e.g. of the form $P(h) = -\frac{h^2\Delta}{2} + V(x)$; in that case, the classical evolution is the Hamilton flow ϕ_p^t generated by the classical Hamiltonian $p(x,\xi) = \frac{|\xi|^2}{2} + V(x)$ on $T^*\mathbb{R}^d$.

As usual in dynamics, one is mostly interested in the long time properties of the dynamical system (2). For such a linear dynamics, these properties are encoded in the *spectrum* of $M(T,h)$. Therefore, a major focus of investigation concerns the spectral properties of the operators $M(T,h)$, especially in the semiclassical limit $h \to 0$, where the connection to the classical map is most effective. Quantum maps have mostly been studied in cases where $M(T,h)$ is replaced by a unitary operator on some N-dimensional Hilbert space, with $N \sim h^{-1}$. This is the case if T is a symplectomorphism on a compact symplectic manifold, like the 2-torus [4]. More recently, one has got interested in operators $M(T,h)$ which act unitarily on states microlocalised inside a certain domain of $T^*\mathbb{R}^d$, but "semiclassically kill" states microlocalised outside a larger bounded domain (these properties depend on the choice of the symbol $a(x_1, \xi_0)$). As a result, the spectrum of $M(T,h)$ is contained in the unit disk, and its effective rank is $\leq Ch^{-d}$ (according to the handwaving argument that one quantum state occupies a volume $\sim h^d$ in phase space). Such operators have been called "open quantum maps".

Let us now assume that the map T has chaotic properties: the nonwandering set Γ is a fractal set included inside $B(0,R)$, and T is uniformly hyperbolic on Γ. We may then expect this dynamical structure to imply some form of *quantum decay*: indeed, a quantum state cannot be localised on a ball of radius smaller than $\sqrt{h}$, and such a ball is not fully contained in Γ, so most of the ball will escape to infinity through the map T. On the other hand, quantum mechanics involves *interference effects*, which may balance this purely classical decay. Following old works of M.Ikawa [3] and P.Gaspard & S.Rice [2] in the framework of Euclidean obstacle scattering, one is lead to the following condition for quantum decay:

Theorem 1. *For any $(x,\xi) \in \Gamma$, call $\varphi^u(x,\xi) = -\log|\det DT_{|E^u(x,\xi)}|$ the unstable Jacobian of T at (x,ξ), and consider the corresponding topological pressure $\mathcal{P}(\frac{1}{2}\varphi^u)$.*

If that pressure is negative, then for any $1 > \gamma > \exp\{\mathcal{P}(\frac{1}{2}\varphi^u)\}$, and any small enough $h > 0$, the operator $M(T,h)$ has a spectral radius $\leq \gamma$.

In dimension $d = 1$ (that is, when T acts on T^*R), the negativity of that pressure is equivalent with the fact that the Hausdorff dimension $d_H(\Gamma) < 1$. This equivalence breaks down in higher dimension, but a negative pressure is still correlated with Γ being a "thin" set.

The above theorem has been obtained by M.Zworski and myself in the framework of Euclidean scattering by smooth potentials [6]. The extension to quantum maps $M(T,h)$ is straightforward, and should be part of a work in preparation with J.Sjöstrand and M.Zworski. In general we do not expect the above to be optimal. Following a recent work of V.Petkov & L.Stoyanov [8], one should be able to compare $M(T,h)$ with classical transfer operators of the form $\mathcal{L}_{\frac{1}{2}\varphi^u+i/h}$, apply Dolgopyat's method to the latter to get a spectral radius $\gamma = \exp\{\mathcal{P}(\frac{1}{2}\varphi^u) - \epsilon_1\}$ for the classical and the quantum operators.

Most of the $\mathcal{O}(h^{-d})$ eigenvalues of $M(T,h)$ can be very close to the origin when $h \to 0$. Indeed, the fractal character of the trapped set has a strong influence on the semiclassical density of eigenvalues: any point situated at distance $\gg h^{1/2}$ from Γ will be pushed out of $B(0,R_1)$ through the classical dynamics (either in the past or in the future), within a time $|n| \leq C\log(1/h)$, where semiclassical methods still apply. As a result, the eigenstates of $M(T,h)$ associated with nonnegligible eigenvalues must be "supported" by the tubular neighbourhood of Γ of radius $\sqrt{h}$. A direct volume estimate of this neighbourhood, and the above-mentioned argument on the volume occupied by a quantum states, lead to the following upper bound for the density of eigenvalues:

Theorem 2. *Assume that the hyperbolic trapped set $\Gamma \subset T^*\mathbb{R}^d$ has upper Minkowski dimension $d_M > 0$. Then, for any small $\epsilon, \epsilon' > 0$ and any small enough $h > 0$, one has*

$$\#\{\lambda \in \mathrm{Spec}(M(T,h)),\, |\lambda| > \epsilon\} \leq C_{\epsilon,\epsilon'}\, h^{-d_M/2-\epsilon'}\,. \tag{3}$$

(eigenvalues are counted with multiplicities.)

A similar result has been first obtained by J.Sjöstrand in the case of Euclidean scattering by smooth potentials [10], and has then been refined and generalised to various settings. The case of quantum maps should also appear in the forthcoming work with J.Sjöstrand and M.Zworski.

The "fractal upper bound" (3) is actually conjectured to be an asymptotics. This has been shown numerically in various cases, including hyperbolic scattering [5] as well as quantum maps [9]. This asymptotics has been proved only for a very specific quantum maps [7], and represents an interesting challenge for more realistic systems.

References

[1] L.C. Evans and M.Zworski, Lectures on semiclassical analysis, `http://math.berkeley.edu/ zworski/semiclassical.pdf`

[2] P. Gaspard and S.A. Rice, *Semiclassical quantization of the scattering from a classically chaotic repellor,* J. Chem. Phys. **90**(1989), 2242–2254.

[3] M. Ikawa, *Decay of solutions of the wave equation in the exterior of several convex bodies,* Ann. Inst. Fourier, **38**(1988), 113–146.

[4] M. Degli Esposti and S. Graffi (Eds), *The Mathematical Aspects of Quantum Maps*, Springer,Berlin, 2003

[5] L. Guillopé, K. Lin, and M. Zworski, *The Selberg zeta function for convex co-compact Schottky groups,* Comm. Math. Phys, **245**(2004), 149–176.

[6] S. Nonnenmacher and M. Zworski, *Quantum decay rates in chaotic scattering,* Acta Math. (in press)

[7] S. Nonnenmacher and M. Zworski, *Distribution of resonances for open quantum maps,* Comm. Math. Phys. **269** (2007), 311–365

[8] V. Petkov and L. Stoyanov, *Analytic continuation of the resolvent of the Laplacian and the dynamical zeta function*, C. R. Acad. Sci. Paris, Ser.I, **345** (2007), 567–572

[9] H. Schomerus and J. Tworzydło and, *Quantum-to-classical crossover of quasi-bound states in open quantum systems*, Phys. Rev. Lett. Phys. Rev. Lett. **93**(2004), 154102

[10] J. Sjöstrand, *Geometric bounds on the density of resonances for semiclassical problems*, Duke Math. J., **60**(1990), 1–57; J. Sjöstrand and M. Zworski, *Fractal upper bounds on the density of semiclassical resonances,* Duke Math. J. **137**(2007), 381–459.

Complex Cone Contractions

Hans Henrik Rugh

We present recent developments in which 'contraction of complex cones' is used to obtain 'spectral gaps' for linear operators. The notions was introduced in [3] and has been developed further in [2]. Given a bounded linear operator A on a complex Banach space X our goal is to provide sufficient conditions for a so-called spectral gap. By this we here mean the existence of a non-zero complex value λ which is a simple eigen-value of A and such that the rest of the spectrum is contained in a disc of radius strictly smaller than $|\lambda|$. The central ideas are as follows: Find a proper complex cone $C \subset X$ and a projective metric d_C which satisfies a Uniform Contraction Principle (UCP) with respect to linear maps.

By a proper complex cone we mean a subset which is invariant under multiplication by any non-zero complex number and by properness that if the complex vector space spanned by two vectors x and y is entirely contained in C then the two elements must be co-linear. For the UCP suppose that $A \in L(X)$ maps $C^* = C \setminus \{0\}$ into itself. It then induces a projective map (a homography) on the corresponding subset $P(C)$ of complex projective space $P(X)$. We say that a metric d on $P(C)$ satisfy a uniform contraction principle if any homography A of $P(C)$ is automatically a contraction with respect to the metric and that this contraction is strict whenever the diameter of the image is finite. Thus, if $\Delta = \sup_{x,y \in C^*} d(Ax, Ay) \in [0, +\infty]$ denotes the diameter of the image of the cone, there should be $\eta = \eta(\Delta) \leq 1$ such that $d_C(Ax, Ay) \leq \eta d_C(x, y)$ for all $x, y \in C^*$. And η should be strictly smaller than one if Δ is finite.

For real Banach spaces Birkhoff [1] found that the Hilbert metric indeed verifies such a UCP with $\eta = \tanh(\Delta/4)$. And he used this to give a conceptually new proof (with explicit bounds) for Perron-Frobenius like theorems. We observed in [3] that the same is true for holomorphic maps of hyperbolic subsets of the Riemann sphere. Modulo technical assumptions on the cone (so that we may compare the cone metric and the Banach space norm) this gives rise to conditions for the operator to have a spectral gap. In [2] this point of view has been further developed introducing yet another (and simpler) metric satisfying the UCP.

References

[1] G. Birkhoff *Extensions of Jentzsch's theorem*, Trans. Amer. Math. Soc. **85** (1957), 219–227.
[2] L. Dubois *Projective metrics and contraction principles for complex cones*, J. London Math. Soc. **2** (2009), 719–737.
[3] H.H. Rugh, *Cones and gauges in complex spaces: Spectral gaps and complex Perron-Frobenius theory*, (accepted 2007, to appear in Annals of Math)

Deformations of Benedicks-Carleson unimodal maps

Daniel Smania
(joint work with Viviane Baladi)

We study the linear response problem for unimodal maps satisfying the so called Topological Slow Recurrence (TSR) condition. A S-unimodal map g satisfies the Topological Slow Recurrence (TSR) condition if g is a Collet-Eckmann map, that is, there exists $C > 0$ and $\lambda > 1$ such that

$$|Dg^n(g(c))| \geq C\lambda^n$$

for every $n \in \mathbb{N}$, where c is the critical point of g, and moreover

$$\lim_{\delta\to 0} \limsup_{n\to+\infty} \frac{1}{n} \sum_{\substack{1\leq j\leq n \\ |g^j(c)-c|<\delta}} -\log|Dg(g^j(c))| = 0. \tag{1}$$

In particular, maps satisfying the TSR condition satisfy a strong Benedicks-Carleson condition: For every $\beta > 0$ there exists $C > 0$ such that

$$|g^i(c) - c| \geq Ce^{-\beta i}.$$

Another remarkable fact is that the TSR condition is a topological invariant [4].

Consider a smooth family of TSR S-unimodal maps g_t. It is not difficult to show that g_t is a smooth deformation, that is, there exists a conjugacy h_t between g_t and g_0. Since these maps are Collet-Eckmann, each g_t has a unique absolutely continuous invariant probability μ_t. Given an observable ψ, one can ask about the differentiability of the function

$$t \to \int \psi \; d\mu_t.$$

To study this question, it is necessary to study the differentiability of h_t with respect to the parameter t. We show the following result

Theorem. Given a smooth family g_t of S-unimodal maps, C^3, symmetric and with quadratic critical points, satisfying the TSR condition, then there exists a unique bounded function α such that

$$\partial_t g_t(x)|_{t=0} = \alpha(g_0(x)) - Dg_0(x)\alpha(x)$$

for every x in the phase space. Furthermore α is continuous and $\alpha(x) = \partial_t h_t(x)|_{t=0}$ for every x which is either a periodic point or in the forward orbit of the critical point.

The proof consists in a dynamical resummation of the formal solution of the cohomological equation above. To this end we use the tower defined in [3].

References

[1] V. Baladi, D. Smania, *Linear response formula for piecewise expanding unimodal maps*, Nonlinearity, **21** (2008), 677–711.

[2] V. Baladi, D. Smania, *Smooth deformations of piecewise expanding unimodal maps*, Discrete Contin. Dyn. Syst **23** (2009), 685–703.

[3] V. Baladi, M. Viana, *Strong stochastic stability and rate of mixing for unimodal maps*, Ann. Sci. École Norm. Sup. (4) **29** (1996), 483–517.

[4] S. Luzzatto, L. Wang, *Topological invariance of generic non-uniformly expanding multimodal maps*, Math. Res. Lett. **13** (2006), 343–357.

[5] D. Ruelle, *Structure and f-dependence of the A.C.I.M. for a unimodal map f is Misiurewicz type,* Comm. Math. Phys., **287** (2009), 1039–1070.

Transfer operators for geodesic flows on negatively curved manifolds

Masato Tsujii

In this talk, we discussed about spectrum of transfer operators for geodesic flows on negatively curved manifolds and related analytic properties of the corresponding semi-classical zeta functions.

Let $F^t : M = T_1^*N \to M$ be the geodesic flow of a C^∞ closed Riemannian manifold N with negative sectional curvature. Given a C^∞ multiplicative cocycle $g^t : M \to \mathbb{C}$, we consider a one parameter family of transfer operators

$$L^t : C^\infty(M) \to C^\infty(M), \quad L^t u(x) = g^t(x) \cdot u(F^t(x)).$$

Set

$$M = M(F^t, g^t) = \lim_{t\to\infty} \left(\sup_{x\in M} |g^t(x)| / \sqrt{\det(DF|_{E^u})} \right)^{1/t}$$

$$m = m(F^t, g^t) = \lim_{t\to\infty} \left(\inf_{x\in M} |g^t(x)| / \sqrt{\det(DF|_{E^u})} \right)^{1/t}$$

The main result I gave was

Theorem 1([2]) There exists a Hilbert space $\mathcal{H}$, embedded in the space of distributions $(C^\infty(M))'$ and containing $C^\infty(M)$ as a dense subset, such that L^t for $t \geq 0$ extends to $L^t : \mathcal{H} \to \mathcal{H}$ boundedly and the essential spectral radius of $L^t|_H$ is exactly M^t.

The semiclassical zeta function $\zeta_{sc}(\cdot)$ is a function defined (formally) by

$$\zeta_{sc}(s) = \exp\left(-\sum_{n\geq 1}\sum_{\gamma\in\Gamma} \frac{e^{-sn|\gamma|}}{n} \frac{1}{\sqrt{\det(Id - D_\gamma^n)}} \right)$$

It is known that this function, though the *rhs* is well-defined only for $s \in \mathbb{C}$ with large real part, has meromorphic extension to the whole complex plane. If we restrict ourselves to the case of $\dim N = 2$, we may write the *rhs* of the definition above using dynamical traces of transfer operators with cocycles $g^t(x) = (\det DF^t|_{E^u})^{1/2}$ and $g^t(x) = (\det DF^t|_{E^u})^{-1/2}$, and relate the spectral properties of those transfer operators to analytic properties of $\zeta_{sc}(\cdot)$. (The assumption $\dim N = 2$ is not very essential. If $\dim N > 2$ we have to consider vector-valued transfer operators.) Up to some technical argument, Theorem 1 leads naturally to

Theorem 2 For any $\epsilon > 0$, there are only finitely many zeros of $\zeta_{sc}(\cdot)$ (and no poles if $\dim N = 2$) on the region $\Re(s) > \epsilon$.

Note that, in the case of constant negative curvature ($\equiv -1$), the semi-classical zeta function $\zeta_{sc}(\cdot)$ coincides with the Selberg zeta function $\zeta_{\text{Selberg}}(\cdot)$ up to translation by 1/2: $\zeta_{sc}(s) = \zeta_{\text{Selberg}}(s + 1/2)$, so that the classical result of Selberg[1] implies that the bound in theorem 2 is optimal in a sense.

In the talk, we also discussed about the following statements on the spectra of transfer operators and semi-classical zeta functions that may be inferred from the argument in the proof of theorem 1 and 2:

- the essential spectrum $L^t : \mathcal{H} \to \mathcal{H}$ is contained in the subset
$$\{|z| \leq e^{-\lambda t} M^t\} \cup \{m^t \leq |z| \leq M^t\}$$
where λ is the hyperbolicity exponent of the flow.
- $\zeta_{sc}(s)$ has infinitely many zeros on the strip $|\Re(s)| < \epsilon$ and only finitely many zeros on $-\lambda + \epsilon < \Re(s) < -\epsilon$, for any $\epsilon > 0$.

The proof of Theorem 1 in [2] is based on Littlewood-Paley type decomposition and somewhat complicated. In the talk, we outlined a new proof which uses partial Bargmann transform and has flavour of "semi-classical analysis".

References

[1] McKean, H. P., *Selberg's trace formula as applied to a compact Riemann surface*, Comm. Pure Appl. Math. **25** (1972), 225–246.

[2] Tsujii, M., *Quasi-compactness of transfer operators for contact Anosov flows*, Preprint (2008), Math ArXiv 0806.0732

Reporter: Gerhard Keller

Participants

Prof. Dr. Viviane Baladi
Departement de Mathematiques et Applications
Ecole Normale Superieure
45, rue d'Ulm
F-75230 Paris Cedex 05

Dr. Peter Balint
Institute of Mathematics
Budapest University of Technology and Economics
1.Egry Jozsef U., Bldg. H
H-1111 Budapest

Prof. Dr. Jerome Buzzi
Laboratoire de Mathematique
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Prof. Dr. Stefano Galatolo
Dip. di Matematica Applicata
Universita di Pisa
Via Buonarroti
I-56127 Pisa

Prof. Dr. Paolo Giulietti
Dipartimento di Matematica
"Guido Castelnuovo"
Universita di Roma "La Sapienza"
Piazzale Aldo Moro, 2
I-00185 Roma

Prof. Dr. Sebastien Gouezel
U. F. R. Mathematiques
I. R. M. A. R.
Universite de Rennes I
Campus de Beaulieu
F-35042 Rennes Cedex

Prof. Dr. Gerhard Keller
Department Mathematik
Universität Erlangen-Nürnberg
Bismarckstr. 1 1/2
91054 Erlangen

Prof. Dr. Marc Kesseböhmer
Fachbereich 3
Mathematik und Informatik
Universität Bremen
Bibliothekstr.1
28359 Bremen

Prof. Dr. Carlangelo Liverani
Dipartimento di Matematica
Universita di Roma "Tor Vergata"
Via della Ricerca Scientif. 1
I-00133 Roma

Prof. Dr. Frederic Naud
Laboratoire d'Analyse non lineaire et Geometrie
Universite d'Avignon
33, rue Louis Pasteur
F-84000 Avignon

Dr. Stephane Nonnenmacher
Institut de Physique Theorique
CEA Saclay
F-91191 Gif-sur-Yvette Cedex

Prof. Dr. Hans-Henrik Rugh
Departement de Mathematiques
Universite de Cergy-Pontoise
Site Saint-Martin, BP 222
2, avenue Adolphe Chauvin
F-95302 Cergy-Pontoise Cedex

Prof. Dr. Daniel Smania
ICMC, C.P. 668
Universidade de Sao Paulo
Sao Carlos SP 13560-970
BRAZIL

Damien Thomine
c/o Laurence Vincent
Departement de Mathematiques et
Applications (DMA)
45, rue d'Ulm
F-75230 Paris Cedex 05

Dr. Imre Peter Toth
Institute of Mathematics
Budapest University of Technology and
Economics
1.Egry Jozsef U., Bldg. H
H-1111 Budapest

Dr. Masato Tsujii
Faculty of Mathematics
Kyushu University
Hakozaki 6-10-1
Higashi-ku
Fukuoka 812-8581
JAPAN