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Model Theory: Around Valued Fields and Dependent Theories

Organised by
Andreas Baudisch, Berlin
Anand Pillay, Leeds
Katrin Tent, Münster
Martin Ziegler, Freiburg

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Abstract. The general topic of the meeting was "Valued fields and related structures". It included both applications of model theory, as well as so-called "pure" model theory: the classification of first order structures using new techniques extending those developed in stable theories.

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Introduction by the Organisers

The general topic of the meeting was "Valued fields and related structures". It included both applications of model theory, as well as so-called "pure" model theory: the classification of first order structures using new techniques extending those developed in stable theories.

The interactions of "theory" and "applications" were very visible in the meeting and in the list of participants which included people working in pure model theory, in more applied model theory, and researchers outside model theory hoping to use the machinery developed here. There were 21 long talks of 50 minutes each complemented by an afternoon with four 30 minute talks. The organizers are grateful to Rémy and Thuillier for giving a short course on Berkovich spaces as well as being available for a very active question session in the evening. This provided background for the tutorial given by Hrushovski and Loeser on their recent work analyzing the type space of algebraically closed valued fields.

Valued fields, often henselian, have been, for many years, important examples to which model theory and logic can be applied. This began with work of Ax-Kochen-Ersov in the 1960's, leading to an asymptotic solution to a conjecture of Artin. In the 1980's Denef made use of the model theory of the p-adic field, together with p-adic integration, to answer a question of Serre. In the 1990's the methods were generalized to "motivic" measure and integration by Denef and Loeser (following an idea of Kontsevich) with many applications to algebraic geometry. More recently, Hrushovski and Kazhdan have developed a "geometric" theory of measure and integration in valued fields, based on a detailed analysis of the category of definable sets in algebraically closed valued fields, again with new applications. This answered a question of Kontsevich and Gromov: if X, Y are smooth d-dimensional subvarieties of a smooth projective n-dimensional variety V, with $V \subset X$ and $V \subset Y$ birationally isomorphic, then $X \times \mathbb{A}^{n-d}$ and $Y \times \mathbb{A}^{n-d}$ are birationally equivalent. These developments are also behind the identification of the space of stably dominated types as one which in special cases agrees with the Berkovich space. This might pave the way to extending the Berkovich spaces to other settings using model theoretic language.

Over the same time period there have been important developments in "abstract" or "pure" model theory, based on generalizing the powerful machinery of stability to possibly unstable first order theories. One can distinguish three strands: First, the notion of o-minimal structures was introduced, influenced both by real algebraic geometry and the notion of a strongly minimal set from stability. The main examples are expansions of the field of real numbers by certain analytic functions, as the exponential function, where the abstract theory had many applications. Recently o-minimal structures were considered as special structures with NIP (i.e. having the non-independence property). This has been enhanced by the general theory of o-minimality, related to definably compact groups and measures, and also by the general theory of forking in theories with NIP.

Furthermore, various general notions like C-minimality and P-minimality, attempting to include nice valued fields, have been formulated. This developed into a modern model theory of algebraically closed valued fields, which lead to the work of Hrushovski and Kazhdan mentioned above. The core theories of valued fields are neither o-minimal nor simple, but have the NIP, so that these new methods do apply.

Recently, these strands have lead to a new general theory of "metastability" generalizing the theory of algebraically closed valued fields. There is a wider class of theories (not having Shelah's Tree property of the second kind), which comprises both, o-minimal and simple theories. Both o-minimal as well as simple theories were intensely studied in the 1990's. These new methods start to lead to a more uniform treatment of this wider class of theories. Some new results and conjectures concerning groups definable in o-minimal and NIP theories were given by Hrushovski in his final lecture, influenced by some fascinating analogies between o-minimal theories and valued fields.

Among other related and striking contributions were: Berarducci's talk relating o-minimal and classical homotopy in the context of definably compact groups in o-minimal structures, Scanlon's somewhat conjectural talk about motivic integration and valued difference fields, and Peterzil's talk on uniform definability of generalized exponential maps in o-minimal expansions of the reals. Zilber gave a talk about Quantum Field Theory and Zariski geometries.

There were also several exciting contributions by young researchers on the NIP theories, pairs of structures and stable fields.

Workshop: Model Theory: Around Valued Fields and Dependent Theories

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Abstracts

Bounds on VC density of formulae in some NIP theories

Deirdre Haskell
(joint work with Matthias Aschenbrenner, Alf Dolich, Dugald Macpherson, Sergei Starchenko)

The VC dimension of a formula $\varphi(\bar{x}, \bar{u})$ in a model $\mathcal{M}$ is defined to be the VC dimension of the definable family of sets $\mathcal{P}_\varphi = \{\varphi(\bar{b}, \bar{u}) : \bar{b} \in M^{\ell(\bar{x})}\}$; that is, the least n (if it exists) such that, for all sets A in $M^{\ell(\bar{u})}$ of size n, there is a subset A' of A for which there is no element $\bar{b} \in M^{\ell(\bar{x})}$ with $A' = A \cap \varphi(\bar{b}, \bar{u})$. The property that a formula has finite VC dimension is immediately seen to be the same as that of the formula having NIP (see [2] for more on the relationship between NIP and finite VC dimension). The VC density of a formula with finite VC dimension is defined to be the least r such that

$$\lim_{n\to\infty} \sup_{|A|=n} \{\# \text{ subsets of } A \text{ determined by } \mathcal{P}_\varphi / n^r\}$$

is finite. See [1] for further discussion of this definition.

In the talk, I stated a theorem which gives bounds on the possible VC density of a formula in an NIP theory. We first need the following definition.

Definition 1. We say that the theory T with NIP has the *VCm property* if the following is true: for any finite set of $\mathcal{L}$-formulas $\Delta(x; \bar{u})$ (note that $\ell(x) = 1$) there is a finite set $\mathcal{P}_\Delta = \{p_i(x; \bar{u}_1, \ldots, \bar{u}_m) : i \in I\}$ of definable families of definable Δ-types such that, for any finite set $B \subset M^{\ell(\bar{u})}$, and any $q \in S^\Delta_x(B)$ there are $\bar{b}_1, \ldots, \bar{b}_m \in B$, and $p_i \in \mathcal{P}_\Delta$ such that

$$p_i(x; \bar{b}_1, \ldots, \bar{b}_m) \vdash q.$$

Theorem 2. *Suppose that T has the VCm property. For any finite set $\Delta(\bar{x}; \bar{u})$ of $\mathcal{L}$-formulas there is a natural number constant K such that, for any finite set $A \subset M^{\ell(\bar{u})}$*

$$|S^\Delta_{\bar{x}}(A)| \leq K|A|^{m\ell(\bar{x})}.$$

Hence, in particular, the VC density of the set of formulas is bounded by $m\ell(\bar{x})$.

I discussed how one can see that o-minimal theories have the VC1 property, and P-minimal theories with definable Skolem functions have the VC2 property.

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Model theoretic properties of metric valued fields

Itaï Ben Yaacov

The theory of algebraically closed fields is the archetypal example of a stable (and even $\aleph_0$-stable) theory. Other theories of fields, with "more structure" (e.g., differentially closed, separably closed, pseudo-finite...) are stable, or simple, and consequently admit a particularly elegant theory of independence. On the other hand, a valued field necessarily has the strict order property and therefore cannot be either stable or simple. Alternative structure theories have been developed recently for some valued fields (most significantly, the algebraically closed ones, e.g. [4]). Here, however, we follow a different path, reasoning that using real-valued logic we can force the value group to be (a subgroup of) $(\mathbb{R}^{>0}, \cdot)$, thus eliminating the source of the strict order property and of instability.

We wish to consider complete valued fields (with a multiplicative valuation in $\mathbb{R}$) in the framework *continuous first order logic*, a real-valued logic introduced in [3]. For technical reasons, we prefer to consider instead of a field K its projective line $\mathbb{P}^1(K)$. This is due to the fact that continuous first order logic is most conveniently applied to the study of bounded metric structures, while valued fields are *unbounded*. The trick commonly used in the context of Banach space structures, namely, replacing the structure with its closed unit ball (in this case, the valuation ring) will not work, since the class of valuation rings is not elementary (in fact, contains no $\aleph_0$-saturated structure). On the other hand, every unbounded structure can be made bounded through the addition of a formal point at infinity, which in the case of valued fields amounts exactly to replacing K with $\mathbb{P}^1(K)$ (see [1] for the general construction).

Every member of $\mathbb{P}^1(K)$ can be represented as $[a : a']$ where $|a| \vee |a'| = 1$ (where $\vee$ denotes maximum), and we only consider such representatives. Given a polynomial $P(X_0, X_0', X_1, X_1', \ldots, X_{n-1}, X_{n-1}')$, homogeneous in each pair (X_i, X_i') separately, say over $\mathbb{Z}$, and $[a_0 : a_0'], \ldots, [a_{n-1} : a_{n-1}']$, the value $|P(a_0, a_0', \ldots)|$ belongs to $[0, 1]$ and does not depend on the choice of representatives. We may therefore name $|P(\ldots)|$ as an n-ary predicate, and define $\mathcal{L}_{\mathbb{P}^1}$ to consist of all such predicates. One particular predicate of this form, which we take to be the distance on $\mathbb{P}^1(K)$, is

$$d\big([a : a'], [b : b']\big) = |ab' - ba'|.$$

It is an ultra-metric distance, complete if and only if K is complete. Then,

(1) The valued field K can be recovered from the (bounded, metric) $\mathcal{L}_{\mathbb{P}^1}$-structure $\mathbb{P}^1(K)$.
(2) The class of all $\mathbb{P}^1(K)$ is elementary, with theory MVF.
(3) The class of all $\mathbb{P}^1(K)$ for algebraically closed, non trivially valued K is elementary as well, denoted $ACMVF$.
(4) The theory $ACMVF$ eliminates quantifiers, and is strictly stable (even up to any kind perturbations).

Alternatively, one may represent K as a multi-sorted structure $\mathbb{P}(K) = \big(\mathbb{P}^n(K)\big)_n$. In this case we use a mostly functional language $\mathcal{L}_{\mathbb{P}}$, consisting of function symbols $\otimes\colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$ for the Segre embeddings, a function symbol $A\colon \mathbb{P}^n \to \mathbb{P}^n$ for each $A \in \mathrm{SL}_{n+1}(\mathbb{Z})$, acting naturally, and predicate symbols $[a_0 : \ldots : a_n] \mapsto |a_0|$, where again we only consider representatives such that $|a_0| \vee \ldots \vee |a_n| = 1$.

We show that $\mathbb{P}^1(K)$ and $\mathbb{P}(K)$ are quantifier-free biïnterpretable, in the sense appropriate for continuous logic, and uniformly so in K. Consequently, all the results cited above (axiomatisability, quantifier elimination, stability) pass to (classes of) structures of the form $\mathbb{P}(K)$.

We recall that a subset X of a metric structure is called *definable* if it is closed, and the distance $d(x, X)$ is a definable predicate. (For example, the interpretability of $\mathbb{P}^n(K)$ in $\mathbb{P}^1(K)$ consists, first of all, of identifying $\mathbb{P}^n(K)$ with a quotient of a definable subset of $\big(\mathbb{P}^1(K)\big)^{\frac{n(n+1)}{2}}$.) We show that if K is algebraically closed, then every projective variety $V \subseteq \mathbb{P}^n(K)$, defined over K, is a definable set in this sense. More generally, every complete variety defined over K is interpretable in K. Since compact subsets are always definable, not all definable sets are projective varieties, and the question of characterising all definable sets in $\mathbb{P}^n$ is left open.

Quantifier elimination of $ACMVF$ allows us to show that the type space $\mathrm{S}_{\mathbb{P}^n}(K)$ is naturally homeomoprhic to the Berkovich projective space $\mathbb{P}^{n,an}(K)$. More generally, if V is a complete variety defined over K then $\mathrm{S}_V(K)$ is homeomoprhic to V^{an}.

Also mentioned briefly are real closed valued fields with convex valuation ring. These again form an elementary class, with theory $RCMVF$, which is model complete and dependent.

For more details see the pre-print [2].

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Commutators in groups definable in o-minimal structures

Eric Jaligot

(joint work with Elías Baro, Margarita Otero)

Groups definable in o-minimal structures and groups of finite Morley rank share many properties. Both types of groups are equipped with a finite dimension which is definable and additive, satisfy the descending chain condition on definable subgroups, and have definably connected components. These properties suffice for many developments of the theory of groups of finite Morley rank.

The main difference between groups of finite Morley rank and groups definable in an o-minimal structure is in the behaviour of the dimension. The dimension in the o-minimal case fails the essential property of the Morley rank: for every definable set A, $\dim(A) \geq n+1$ if and only if A contains infinitely many pairwise disjoint definable subsets A_i with $\dim(A_i) \geq i$. This property/definition of the Morley rank is crucial in Zilber's stabilizer argument, and consequently in Zilber's generation lemma on indecomposable sets in the finite Morley rank context. Ultimately, the definability of most commutator subgroups, and in full generality of derived subgroups, depends on this in the finite Morley rank case.

For groups definable in o-minimal structures, derived subgroups need not be definable. Using recent results on central extensions in this case, Annalisa Conversano exhibits in [1, Example 3.1.7] a definably connected group G definable in an o-minimal expansion of the reals, with G' not definable. The most surprising is that G is a central extension, by an infinite center, of the definably simple group $\mathrm{PSL}_2(\mathbb{R})$. In this work we prove that this is essentially the only obstruction to the definability of commutator subgroups in the o-minimal context.

Definition 1. We say that a definably connected group G definable in an o-minimal structure is a *strict* central extension of a definably simple group if $Z(G)$ is infinite and $G/Z(G)$ is infinite nonabelian and definably simple.

We call *section* of a group G any quotient H/K where $K \trianglelefteq H \leq G$, and we speak of *definable* section when both K and H are definable.

Definition 2. We say that a group G definable in an o-minimal structure satisfies *assumption* (*) whenever the derived subgroup $(H/K)'$ is definable for every definable section H/K of G which is a strict central extension of a definably simple group.

Notice, for example, that all solvable groups satisfy assumption $(*)$, and hence our main result below applies in all solvable groups definable in o-minimal structures.

Theorem 3. *Let G be a group definable in an o-minimal structure and satisfying assumption $(*)$, and let A and B be two definable subgroups of G which normalize eachother. Then the subgroup $[A,B]$ is definable and $[A,B]^\circ = [A^\circ, B][A, B^\circ]$. Furthermore, any element of $[A,B]^\circ$ can be expressed as the product of at most $\dim([A,B]^\circ)$ commutators from $[A^\circ, B]$ or $[A, B^\circ]$ whenever A° and B° are solvable.*

Our argument for the proof of Theorem 3 consists mainly in finding very rudimentary forms of Zilber's general stabilizer argument on generation by indecomposable sets in the finite Morley rank context. In fact, our finite bound on the number of commutators in Theorem 3 is ultimately obtained by the fact that, in an abelian group, the subgroup generated by an arbitrary family of definable connected subgroups is the product of finitely many of them.

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NIP and generically stable types

Anand Pillay

I discussed (1) variants of definability (2) Stability, (3) NIP, and (4) generically stable types. We fix a complete theory T and work in a saturated model $\bar{M}^{eq}$.

(1) We have various notions of definability: definable set, type-definable set, $*$-definable set, hyperdefinable set.

(2) Fix a complete type $p(x)$ over a small model M. Call a formula $\phi(x, b)$ (over $\bar{M}$) small for p if $p(x) \cup \{\phi(x, b)\}$ divides over M. Then stability of T is characterized by
(a) the set of small formulas for p is a proper ideal, and (b) for any $\phi(x, b)$, either $\phi(x, b)$ is small for p, or $\neg\phi(x, b)$ is small for p.
So (assuming stability) $p(x)$ has a unique global extension $p'(x)$ which does not contain any small formula (i.e. is a nondividing or nonforking extension of p), and one can conclude that p' is definable over M.

(3) T has NIP if for any indiscernible sequence $(a_i : i < \omega)$ and formula $\phi(x, y)$ there is N_ϕ such that for no b do we have $\models \phi(a_i, b)$ iff $\models \neg\phi(a_{i+1}, b)$ for $i = 0, \ldots, N_\phi$.

Fact A. If T has NIP and $p(x) \in S(M)$ then again the set of small formulas for p is an ideal, and moreover p has at most $2^{|M|}$ global nondividing extensions, each of which is Borel definabl over M.
Question. (NIP) If $p(x) \in S(M)$, and $\phi(x, y) \in L$, is $\{b : \phi(x, b) \text{ is small for } p\}$ Borel definable over M?

(4) Assume NIP. We say that $p \in S(M)$ is generically stable if p has a unique global nondividing extension p' which is moreover definable over M.

Fact B. Suppose $p \in S(M)$ is generically stable. Let $I = (a_i : i < \omega)$ be any Morley sequence in p. Then I is totally indiscernible and for $\phi(x, y) \in L$, and any b, $\phi(x, b) \in p'$ if and only if all but finitely many $a_i \in I$ satisfy $\phi(x, b)$ if and only if at least N_ϕ many $a_i \in I$ satisfy $\phi(x, b)$.

Identify now generically stable types with “global types” or with their sequence of $\phi(x, y)$ definitions, as ϕ varies.

Corollary C. The set of generically stable tyes with variable x, is $*$-definable.
Proof. We use also the fact
(*) that if I is an infinite totally indiscernible sequence then its global average type (given by NIP) is generically stable and I is a Morley sequence for it.

Now by Fact B, for any $\phi(x, y) \in L$ there is $\psi_\phi(y, z)$ such that for any generically stable type $p(x)$, its ϕ-definition is $\psi(y, c)$ for some c. Now the collection of tuples $(c_\phi : \phi \in L)$ which correspond to the defining schema of some generically stable type is $*$-definable by (*) and Fact A.

Some basic objects in Berkovich theory, 1 – Survey talk

Bertrand Rémy

This report sums up the first of the two talks given by A. Thuillier and myself in order to introduce V. Berkovich's theory of analytic geometry over non-archimedean complete fields. It can also be seen as an introduction to the talks given later by E. Hrushovski and F. Loeser on their work in progress dealing with the relationship between Berkovich geometry and model theory [4].

More precisely, the talk presented here intended to introduce two kinds of basic spaces of Berkovich geometry, namely analytic spectra and analytifications of affine spaces (mostly the dimension 1 case).

1. Analytic spectra, after Berkovich.— These spaces are the building blocks of the theory. The starting point is a Banach ring A i.e., a (commutative) ring (with unit element) endowed with a Banach norm $\| \cdot \|_{\mathrm{A}}$ that is *submultiplicative*: for any f and g in A, we have: $\|f \cdot g\|_{\mathrm{A}} \leq \|f\|_{\mathrm{A}} \cdot \|g\|_{\mathrm{A}}$.

1A) Algebraic motivation and the definition. Recall that to the commutative ring A is associated the algebraic spectrum Spec(A), consisting of the prime ideals in A, endowed with the Zariski topology [3, II.§4.3]. It is well-known that Spec(A) is in one-to-one correspondence with the set of equivalence classes of ring homomorphisms from A to an arbitrary field, where two maps are identified if they both factorize through a common third such map.

Definition 1. The analytic (or Berkovich) spectrum of A is the set $\mathscr{M}(\mathrm{A})$ of multiplicative seminorms $\mathrm{A} \to \mathbf{R}_{\geq 0}$ whose restrictions to A are bounded with respect to $\| \cdot \|_{\mathrm{A}}$; this space is endowed with the coarsest topology such that for any $f \in \mathrm{A}$, the evaluation map $x \mapsto x(f)$ is continuous.

For $f \in \mathrm{A}$ and $x \in \mathscr{M}(\mathrm{A})$ it is often useful to write $|f(x)|$ instead of $x(f)$. Note that in this context, the real number $|f(x)|$ exists before the notation $f(x)$ makes sense. The latter is defined as follows: to $x \in \mathscr{M}(\mathrm{A})$ is associated the prime ideal $\mathfrak{p}_x = \{f \in \mathrm{A} : |f(x)| = 0\}$, which by the way defines a map $\mathscr{M}(\mathrm{A}) \to \mathrm{Spec}(\mathrm{A})$. The *complete residue field* of x is by definition the completion of the fraction field $\kappa(x) = \mathrm{Frac}(\mathrm{A}/\mathfrak{p}_x)$ with respect to the quotient norm induced by x on $\kappa(x)$. This complete field is denoted by $\mathcal{H}(x)$ and $f(x)$ is defined to be the class of f in $\mathcal{H}(x)$.

The algebraic identification recalled before the definition has the following analog: a seminorm $x \in \mathscr{M}(\mathrm{A})$ gives rise to a bounded homomorphism $\mathrm{A} \to \mathcal{H}(x)$; conversely, any such homomorphism $\varphi : (\mathrm{A}, |\cdot|_{\mathrm{A}}) \to (k, |\cdot|_k)$ gives a multiplicative bounded seminorm in $\mathscr{M}(\mathrm{A})$, namely the composed map $|\cdot|_k \circ \varphi : \mathrm{A} \to \mathbf{R}_{\geq 0}$.

Useful references for the above material are the very beginnings of [2] and [5].

1B) Non-emptiness and compactness. The following theorem is a fundamental result due to V. Berkovich [2, Theorem 1.2.1].

Theorem 2. *For any* $\mathrm{A} \neq \{0\}$ *as above,* $\mathscr{M}(\mathrm{A})$ *is non-empty and compact.*

Non-emptiness. Pick a maximal ideal $\mathfrak{m}$, choose a minimal (with respect to the order given by pointwise comparison) bounded submultiplicative seminorm on $\mathrm{A}/\mathfrak{m}$, say $|\cdot|$. Consider the completion $\mathrm{B} = \widehat{\mathrm{A}/\mathfrak{m}}$ w.r.t. $|\cdot|$: it is a field and a Banach ring, and it suffices to show that $|\cdot|$ is multiplicative in order to prove non-emptiness. This is done by showing that $|\cdot|$ is *power-multiplicative* (i.e., $|f^n| = |f|^n$ for any $f \in \mathrm{B}$ and $n \in \mathbf{N}$) and that $|f^{-1}| = |f|^{-1}$ for any $f \in \mathrm{B}$. The main tool for this is thus the algebras $\mathrm{B}\langle r^{-1}T\rangle = \{\sum_i b_i T^i : \sum_i |b_i| r^i < \infty\}$ endowed with the norms $\|\sum_i b_i T^i\|_r = \sum_i |b_i| r^i$. For power-multiplicativity, assume that there exist f and n contradicting it, set $r = |f^n|^{\frac{1}{n}}$ and introduce $\varphi : \mathrm{B} \to \mathrm{B}\langle r^{-1}T\rangle/(f - T)$. The map $\|\cdot\|_r \circ \varphi$ then contradicts the minimality of the initial seminorm (there is one point: checking that $f - T$ is non-invertible, which is done by a standard Euclidean division trick, as for "convergence" of submultiplicative real sequences). Inversion is treated similarly, with $r = |f^{-1}|^{-1}$.

Compactness. The idea is to use the natural map $\hat{\cdot} : \mathrm{A} \to \prod_{x \in \mathscr{M}(\mathrm{A})} \mathcal{H}(x)$, called the *Gelfand transform*, and defined by $\hat{f} = \big(f(x)\big)_{x \in \mathscr{M}(\mathrm{A})}$. For the sup norm on the product, it is 1-Lipschitz and provides a map $\mathscr{M}\big(\prod_{x \in \mathscr{M}(\mathrm{A})} \mathcal{H}(x)\big) \to \mathscr{M}(\mathrm{A})$ which is continuous and surjective. It remains then to use the fact that for a collection of complete fields $(\mathrm{K}_i)_{i \in I}$ with I discrete, then $\mathscr{M}\big(\prod_{i \in I} \mathrm{K}_i\big)$ is isomorphic to the Stone-Čech compactification of I.

1C) The analytic spectrum of the ring of integers. The now well-known star-shaped figure of $\mathscr{M}(\mathbf{Z})$ was presented at this point, see [1] and [2]. The idea is to use the map $\mathscr{M}(\mathbf{Z}) \to \mathrm{Spec}(\mathbf{Z})$. For each prime number p, there is exactly one seminorm in the preimage of p: it sends any non-multiple of p to 1 and any multiple of p to 0. The other elements of $\mathscr{M}(\mathbf{Z})$ lie above the generic point of $\mathrm{Spec}(\mathbf{Z})$, hence are norms; they are classified by Ostrowski's theorem [3, VI.§6.3]. Note that the topology is coarser than the "intuitive" one, and it is an easy exercice to see convergence of seminorms in this space.

2. Analytic affine spaces and projective line.— In what follows, k is a field endowed with a complete (possibly trivial) absolute value $|\cdot|_k$.

2A) General set-up. To each algebraic variety V over k is attached a Berkovich analytic space over k, which is denoted by V^{an}. The attachment $V \mapsto V^{\mathrm{an}}$ is functorial and moreover satisfies (see [2, 3.4-3.5] for details):

(i) if V is affine with coordinate ring $k[V]$, then V^{an} consists of all the multiplicative seminorms $k[V] \to \mathbf{R}_+$ extending the absolute value of k;
(ii) if V is projective, then V^{an} is compact.

It follows from (i) that the (underlying space of the) analytic affine n-space $\mathbb{A}_k^{n,\mathrm{an}}$ is the set of multiplicative seminorms $k[T_1, \dots, T_n] \to \mathbf{R}_{\geq 0}$ extending $|\cdot|_k$. We henceforth restrict to the case when $n = 1$.

2B) Affine lines for trivially valued fields. Assume that k is trivially valued: $|\cdot|_k$ sends everybody to 1 (except 0). Then the space $\mathbb{A}_k^{1,\mathrm{an}}$ also has the shape of a star as in 1C (but non-compact this time). The exhaustive description of the seminorms is similar to the case of $\mathscr{M}(\mathbf{Z})$. A degenerate case is that of the seminorms that are not norms: there is one such seminorm $\eta_{P,0}$ associated to each irreducible $P \in k[T]$ sending $f \in k[T]$ to 0 if $P \mid f$ and to 1 otherwise. It remains to treat the case of norms on $k[T]$: they have to be non-archimedean by the boundedness assumption made on the restriction on k. Let x be such a norm. If $|T(x)| \leq 1$ then $\mathfrak{q}_x = \{f \in k[T] : |f(x)| < 1\}$ is a prime ideal; when $\mathfrak{q}_x = \{0\}$, the norm x is the trivial one η_1 (the center of the star $\mathbb{A}_k^{1,\mathrm{an}}$) and otherwise there exist an irreducible $P \in k[T]$ and $r \in]0;1[$ such that x is the norm $\eta_{P,r}$ defined by $\eta_{P,r}(f) = r^{\mathrm{val}_P(f)}$. If $|T(x)| > 1$, the ultrametric inequality implies that, setting $r = |T(x)|$, the norm x is equal to η_r such that $\eta_r(f) = r^{\deg(f)}$. See for instance J. Poineau's PhD [5] for details.

2C) Affine lines for algebraically closed non-archimedean fields. Assume now that k is algebraically closed and that $|\cdot|_k$ is non-trivial and non-archimedean. This case was treated thoroughly in A. Thuillier's talk [6], so we refer to his report for details. We simply mentioned how to produce the desired seminorms. One first way is to use a K-rational point $\varphi : k[T] \to \mathrm{K}$ of the algebraic line $\mathbb{A}_k^1$, with $(\mathrm{K}, |\cdot|_{\mathrm{K}})$ a complete non-archimedean field, and to take $|\cdot|_{\mathrm{K}} \circ \varphi$. One other way is to take a closed ball D in k and to consider the seminorm η_D defined by $\eta_D(f) = \sup_{z \in D} |f(z)|_k$ (checking multiplicativity requires an argument: Gauss lemma). At last, a decreasing sequence $\underline{D} = (D_i)$ of closed balls provides further seminorms $\eta_{\underline{D}}$ by setting $\eta_{\underline{D}}(f) = \lim_i \eta_{D_i}(f)$. Berkovich's classification [2, 1.4.4] says that all points of $\mathbb{A}_k^{1,\mathrm{an}}$ come this way and that given two sequences $\underline{D}$ and $\underline{D}'$, we have $\eta_{\underline{D}} = \eta_{\underline{D}'}$ if and only if either $\bigcap_{D \in \underline{D}} D = \bigcap_{D \in \underline{D}'} D \neq \emptyset$ or both intersections are empty and $\underline{D}$ and $\underline{D}'$ are cofinal. Emptiness of an intersection $\bigcap_{D \in \underline{D}} D$ as above may happen when k is *non-spherically* or *non-maximally complete* [3, VI.§10, ex. 2, p. 193].

Note finally that the connection with the so-called *types* of points of $\mathbb{P}_k^{1,\mathrm{an}}$ was made explicit in A. Thuillier's talk [6], and that the case of a non-algebraically closed field can be treated thanks to a natural Galois action [2, 1.3.5].

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Some basic objects in Berkovich theory, 2 – Survey talk

Amaury Thuillier

This was the second introductory talk devoted to Berkovich theory. Its main purpose was to describe the affine and projective lines over a non-Archimedean field k, previously defined by B. Rémy [5], and to give a rough idea of the approach used by Berkovich in order to analyze the homotopy type of a smooth analytic space [3].

We consider a field k endowed with a complete non-Archimedean absolute value $|\cdot|$ and we denote by $\widetilde{k}$ its residue field. We freely use notation introduced in [5] and we assume that k is algebraically closed for simplicity.

1. Points of the affine line $\mathbb{A}_k^{1,\mathrm{an}}$ are classified according to their completed residue field. They split up into (at most) four types.

Type (1) This the case where $\mathcal{H}(x) = k$. Those are the "classical" points, obtained by taking the value of polynomials at some element of k.

Type (2) Here, $|\mathcal{H}(x)^\times| = |k^\times|$ (same valuation group) and $\widetilde{\mathcal{H}(x)} \simeq \widetilde{k}(\mathrm{T})$.

Type (3) Dually, $|\mathcal{H}(x)^\times| = |k^\times| r^{\mathbb{Z}}$ with $r \in \mathbf{R}_{>0} - |k^\times|$ and $\widetilde{\mathcal{H}(x)} = \widetilde{k}$.

Type (4) This is the case where $\mathcal{H}(x)$ is an immediate extension of k: we have $\mathcal{H}(x) \neq k$ but $|\mathcal{H}(x)^\times| = |k^\times|$ and $\widetilde{\mathcal{H}(x)} = \widetilde{k}$.

Any point of type (2), (3) or (4) lies over the generic point of $\mathbb{P}_k^1$: its completed residue field is a completion of $k(T)$. One can describe the projective line $\mathbb{P}_k^{1,\mathrm{an}}$ as the one-point compactification $\mathbb{A}_k^{1,\mathrm{an}} \cup \{\infty\}$ of the affine line. The point at infinity is of type (1).

Types (3) and (4) are somehow "instable" since they disappear if the field k is replaced by a some maximally complete non-Archimedean extension K satisfying $|K^\times| = \mathbb{R}_{>0}$.

2. Given $a \in k$ and $r \in \mathbb{R}_{\geqslant 0}$, it follows from Gauß lemma that the map

$$k[T] \to \mathbf{R}_{>0}, \quad \sum_{n \in \mathbf{N}} a_n (T-a)^n \mapsto \max_n |a_n| r^n$$

is a multiplicative seminorm. Let $\eta_{a,r}$ denote the corresponding point in $\mathbb{A}_k^{1,\mathrm{an}}$, which is of type (1), (2) or (3) whether $r = 0$, $r \in |k^\times|$ or $r \in \mathbf{R}_{>0} - |k^\times|$.

We can thus draw a continuous path $\eta_{a,-} : [0, \infty[\to \mathbb{A}_k^{1,\mathrm{an}}$ from a to ∞, and it follows readily from the ultrametric inequality that the paths $\eta_{a,-}$ and $\eta_{b,-}$ starting at two elements $a, b \in k$ are disjoint on $[0, |a-b|[$ and coincide on $[|a-b|, \infty[$. This construction displays the structure of a real tree on $\mathbb{P}_k^{1,\mathrm{an}}$, whose ends are points of type (1) and (4) and in which bifurcations occur at points of type (2).

However, the topology on $\mathbb{P}_k^{1,\mathrm{an}}$ is much coarser as the tree topology since this space is compact. For example, if x is a point of type (2), then connected components of $\mathbb{P}_k^{1,\mathrm{an}} - \{x\}$ are in natural one-to-one correspondence with elements of $\mathbb{P}^1(\widetilde{k})$, and one obtains a fundamental system of neighbourhoods of x by deleting a finite number of closed discs in $\mathbb{A}_k^{1,\mathrm{an}} - \{x\}$.

3. For each non-Archimedean field K extending k, there is a natural map $p_K : \mathbb{A}_K^{1,\mathrm{an}} \to \mathbb{A}_k^{1,\mathrm{an}}$ induced by the canonical morphism $k[T] \to K[T]$. It extends to the projective lines by mapping ∞ to ∞.

It is instructive to look at this map when $K = \mathcal{H}(x)$ is the completed residue field of a point $x \in \mathbb{A}_k^{1,\mathrm{an}}$. We set $p_x = p_{\mathcal{H}(x)}$. The canonical map $k[T] \to \mathcal{H}(x)$ defines a $\mathcal{H}(x)$-rational point $\underline{x}$ in $\mathbb{A}_{\mathcal{H}(x)}^{1,\mathrm{an}}$ such that $p_x(\underline{x}) = x$. If x is not of type (1), then the fibre $p_x^{-1}(x)$ contains a non-empty open disc centered in $\underline{x}$: indeed, the real number $r(x) = \inf_{a \in k} |(T-a)(x)|$ is positive (otherwise $x \in k$) and $|\underline{x} - a| = |(T-a)(x)|$, hence

$$|(T-a)(y)| = |(T-\underline{x})(y) + (\underline{x} - a)| = |(T-a)(x)|$$

for any $a \in k$ and $y \in \mathbb{A}_{\mathcal{H}(x)}^{1,\mathrm{an}}$ such that $|y - \underline{x}| < r(x)$. This inequality shows that $p_x^{-1}(x)$ contains the open disc of radius $r(x)$ centered in $\underline{x}$.

This construction gives another way to look at points of type (4), since it is easily checked that they are precisely the points $x \in \mathbb{P}_k^{1,\mathrm{an}}$ such that the fibre $p_x^{-1}(x)$ is isomorphic to the closed unit disc over $\mathcal{H}(x)$.

4. A key ingredient used by Berkovich in his study of the homotopy type of non-Archimedean analytic spaces [2], [3] is the action of a compact torus

$$\mathrm{T}_d = \mathcal{M}(k\langle \mathrm{T}_1^{\pm 1}, \ldots, \mathrm{T}_d^{\pm 1}\rangle) = \{x \in \mathbb{A}_k^{d,\mathrm{an}} \mid |T_1(x)| = \ldots = |T_d(x)| = 1\}$$

on the space under consideration. It should be emphasized that T_d is not a group, rather a *group object* in the (here undefined) category of k-analytic spaces. In particular: for any non-Archimedean extension K/k, the set $\mathrm{T}_d(K)$ of K-rational points of T_d is the maximal bounded subgroup of $(K^\times)^d$.

The first non-trivial example is given by the closed unit disc

$$\mathrm{E}(0,1) = \{x \in \mathbb{A}^{1}{}_k^{\mathrm{an}} \mid |T(x)| \leqslant 1\},$$

which is contractible. This is not a surprise in view of the tree structure on the analytic affine line, albeit one should not forget that the actual topology is strictly coarser than the tree topology. There is a natural action of the one dimensional torus T_1 on $\mathrm{E}(0,1)$ such that $t \cdot x = tx$ for any non-Archimedean extension K/k and any K-rational points t, x of T_1, $\mathrm{E}(0,1)$ respectively. The orbits are precisely the fibres of the continuous map $\mathrm{E}(0,1) \to [0,1]$, $x \mapsto |T(x)|$. Since the path $\eta_{0,-}$ introduced above (2) defines a continuous section of the latter map, we have defined a retraction τ of $\mathrm{E}(0,1)$ onto a closed subset homeomorphic to $[0,1]$.

The construction of a homotopy $H : \mathrm{E}(0,1) \times [0,1] \to \mathrm{E}(0,1)$ between $\mathrm{id}_{\mathrm{E}(0,1)}$ and τ follows easily from two remarks:

- the group T_1 has a natural increasing filtration by closed subgroups

$$\mathrm{T}_{1,r} = \{z \in \mathrm{T}_1 \mid |(T-1)(z)| \leqslant r\}, \quad r \in [0,1];$$

- the orbit of any point $x \in \mathrm{E}(0,1)$ under $\mathrm{T}_{1,r}$ has a unique maximal point $h(x,r)$, where maximal means with respect to evaluation of polynomials:

$$|f(y)| \leqslant |f(h(x,t))|$$

for any $f \in k[T]$ and any $y \in \mathrm{T}_{1,r} \cdot x$.

More generally, one proves in the same way that the closed unit ball in $\mathbb{A}_k^{d,\mathrm{an}}$ is contractible.

5. Contractibility of closed unit balls implies immediately that any smooth k-analytic space X is locally contractible at any "classical" point $x \in \mathrm{X}(k)$, for each such point has a fundamental system of neighbourhoods isomorphic to closed balls. However, unlike what happens in complex geometry, most points in X are non-classical and a precise description of a fundamental system neighbourhoods of them is a delicate problem.

A natural way to attack this question is to look at *models* of X over the valuation ring $k^\circ = \{z \in k \mid |z| \leqslant 1\}$, i.e., (formal) schemes over k° with generic fibre X. If there exists a model with very "mild" singularities, then Berkovich showed in [3] that X has the homotopy type of a CW-complex. Roughly speaking, one obtains a covering of X by elementary pieces which we know how to deform explicitly and one checks that these deformations glue together, the compatibility being deduced from the fact that the local homotopies come from the action of some torus T_d.

Existence of "nice" models is intimately connected with resolution of singularities (at least when X is the analytification of an algebraic variety). A positive answer is known for curves (semi-stable reduction theorem of Deligne-Mumford and Bosch-Lütkebohmmert), and in any dimension if the residue field $\widetilde{k}$ has characteristic zero (semi-stable reduction theorem of Knudsen-Mumford, building on Hironaka's desingularisation). Using J. de Jong's alterations, Berkovich managed to prove local contractibility of smooth k-analytic spaces and conjectured in [3] that V^{an} should have the homotopy type of a finite CW-complex if V is a smooth projective variety over k. A more general statement and model-theoretic proof was recently announced by E. Hrushovski and F. Loeser [4].

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Valued difference fields as a foundation for motivic integration

Thomas Scanlon

With his work in [2] Denef revealed that quantifier elimination theorems for valued fields could be employed to prove rationality theorems about Poincaré series associated to counting problems for algebraic equations over p-adic rings. In later work (see, for example, [3] and [4]) these techniques have been applied to other contexts such as p-adic analytic equations and problems in motivic integration.

Results generalizing the relative completeness and quantifier elimination theorems for henselian fields were proven for *difference* henselian fields in [1] and [8] and in this talk I proposed extending the theory of motivic integration by replacing the target rings built from Grothendieck rings of algebraic varieties by ones constructed from Grothendieck rings of difference varieties.

A difference henselian field is a valued field (K, v) given together with an automorphism $\sigma : K \to K$ satisfying $v(\sigma(x)) = v(x)$ universally. We say that (K, v) is difference henselian if every valuation is represented by an element fixed by σ and the analogue of Hensel's lemma holds for difference polynomials. That is, if $P(X_0, \ldots, X_n) \in \mathcal{O}_K[X_0, \ldots, X_n]$ is a polynomial in $n+1$ variables over the ring of integers $\mathcal{O}_K := \{a \in K : v(a) \geq 0\}$ of K, $a \in \mathcal{O}$, and $v(P(a, \sigma(a), \ldots, \sigma^n(a)) > 0$ while $v(\frac{\partial P}{\partial x_i}(a, \sigma(a), \ldots, \sigma^n(a))$ for some i, then there is some solution $b \in \mathcal{O}_K$ with $P(b, \sigma(b), \ldots, \sigma^n(b)) = 0$ and $v(a - b) > 0$. Newton's method for approximating roots may be adapted to prove the difference Hensel's lemma in every maximally complete valued difference field in whose residue field every nonzero linear difference operator is surjective. In particular, we have two important classes of examples of difference henselian fields.

- If (k, σ) is an existentially closed difference field, then $K := k((t))$, the field of Laurent series over k, with the σ extended via $\sigma(\sum a_i t^i) := \sum \sigma(a_i) t^i$ is difference henselian.
- If k is an algebraically closed field of characteristic $p > 0$, $\tau_q : k \to k$ is the q-power Frobenius map $x \mapsto x^q$ where q is some nonzero power of p, K is the field of fractions of the Witt vectors of k, and $\sigma : K \to K$ is the unique lifting of τ_q to K, then (K, σ) is difference henselian.

The main theorem of [1] asserts that these structures, when the characteristic of K is zero, admit quantifier elimination relative to the theories of the residue field and value group. In these specific cases, the theory of the value group is simply Pressburger arithmetic so that divisibility predicates suffice. In the case of the Laurent series field, the definable sets in models of ACFA are well-understood while in the case of the Witt vectors, one-by-one, we have full quantifier elimination in the residue field. For us, the most important point is that just as with the classical Ax-Kochen-Eršov theorems, it follows from Hrushovski's theorem on the limit theory of the Frobenius that the theory of the Witt vectors with a Witt-Frobenius converges to the theory of a Laurent series field with a difference closed residue field as p tends to infinity [6].

For the time being, we restrict to the case of difference henselian field with value group $\mathbb{Z}$. If X is a definable subset of $\mathcal{O}^n$ and $\pi_m : \mathcal{O} \to \mathcal{O}/\mathfrak{m}^{m+1}$ is the reduction map modulo the $(m+1)^{\mathrm{st}}$ power of the maximal ideal, then $\pi_m(X)$ is definably isomorphic to a set definable in the residue field. As such, one may associate to such a set X a sequence $\langle [\pi_m(X)] : m \in \mathbb{Z}_+ \rangle$ of classes of definable sets in the Grothendieck ring of definable sets over the residue field. in practice, because we wish apply counting methods, we restrict this construction to finite dimensional sets, that is, definable sets X having the property that the transcendence degree of the difference field generated by some point in X in some elementary extension is bounded. Moreover, we consider $[\pi_m(X)]$ as a class in the Grothendieck ring of finite dimensional definable sets over the residue field. From the work of Ryten and Tomašić one knows that this ring admits nontrivial homomorphisms to integral domains.

With the restriction to finite dimensional sets, then away from a lower dimensional set, the maps $\pi_{m+1}(X) \to \pi_m(X)$ are fibrations where the fibres are isomorphic to some fixed finite Cartesian power of points of the fixed field in the affine line of the residue field. In particular, in the case of the Witt vectors, finite dimensional definable sets are locally piecewise p-adic manifolds where "locally" means that it might happen that $\pi_m(X)$ is infinite for some m (for example, consider the equation $\sigma(x) = x^q$), but each fibre is a finite Boolean combination of p-adic manifolds. With these observations, on any finite dimensional difference variety we may define a motivic measure taking values in a localization of the Grothendieck ring of finite dimensional definable sets over the residue field in such a way that the measures specialize to the canonical p-adic measures.

The remaining key ingredient in Denef's proof of the rationality of Poincaré series is the fact that after suitable changes of variables, the valuation of a difference polynomial may be computed piecewise as the valuation of a monomial. A similar result is true for difference polynomials over difference henselian fields and is a key step in the proof of the extension of isomorphisms lemma required for the proof of quantifier elimination. With these ingredients in place, rationality of Igusa integrals is a formal consequence.

Our main theorems on difference henselian fields hold without any restriction on the value group and the description of the definable sets is compatible with the principles elucidated in [7] on how definable sets in algebraically closed valued fields may be analyzed by definable sets in the residue field and in the value group.

However, we have not systematically investigated what the Hrushovski-Kazhdan theory might say in the case of difference valued fields.

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On the differential ring of semi-germs of definable holomorphic functions

Alex J. Wilkie

Let ${}^*\mathbb{R}$ be an extension of $\mathbb{R}$ suitable for nonstandard analysis. For definiteness we may take ${}^*\mathbb{R}$ to be $\mathbb{R}^I/\mathcal{U}$, a sufficiently saturated ultrapower of $\mathbb{R}$. This has the advantage that *all* functions $f:\mathbb{R}^n\to\mathbb{R}^m$ and *all* sets $S\subseteq\mathbb{R}^n$ have canonical extensions to ${}^*\mathbb{R}$, which we denote by ${}^*f$ and ${}^*S$ respectively. Then Łoś's Theorem asserts that ${}^*\mathbb{R}$ is an elementary extension of $\mathbb{R}$ for *any* first order structure on $\mathbb{R}$.

Now let us fix an o-minimal structure on $\mathbb{R}$ expanding the ordered field structure and denote its language by L. It is convenient for us to assume that every $r\in\mathbb{R}$ is named by a constant symbol of L. Then, in particular, $\mathbb{R}\preceq{}^*\mathbb{R}$ as L-structures and ${}^*\mathbb{R}$ is o-minimal. Henceforth "definable" means L-definable in $\mathbb{R}$ or in ${}^*\mathbb{R}$ *with* parameters. This is also applied in the corresponding complex fields using the usual identification of $\mathbb{C}$ with $\mathbb{R}\times\mathbb{R}$ and (using Łoś) ${}^*\mathbb{C}$ with ${}^*\mathbb{R}\times{}^*\mathbb{R}$.

The ring of definable holomorphic semi-germs is constructed as follows. Firstly, fix $n\geq 1$ and let $\bar{r}$ be an n-tuple of positive real numbers, i.e. $\bar{r}$ is a *polyradius*. The *polydisk* of polyradius $\bar{r}$ is the set $\Delta^{(n)}(\bar{r}):=\{\bar{z}\in\mathbb{C}^n : |z_i|<|r_i|, i=1,\ldots,n\}$. So, as discussed above, ${}^*\Delta^{(n)}(\bar{r})$ denotes its extension to ${}^*\mathbb{C}^n$. Note that ${}^*\Delta^{(n)}(\bar{r})$

is definable in $^*\mathbb{R}$ (without parameters). Let us now consider a function $f : {}^*\Delta^{(n)}(\bar{r}) \to {}^*\mathbb{C}$ that is definable in $^*\mathbb{R}$. We call f *definably holomorphic* if, in the sense of the o-minimal structure $^*\mathbb{R}$, the real and imaginary parts of f are continuously differentiable and satisfy the Cauchy-Riemann equations. The set of all such definably holomorphic functions is denoted $\mathcal{F}^n(\bar{r})$. Clearly $\mathcal{F}^n(\bar{r})$ is a differential ring (for the usual derivations) and $\bar{r} < \bar{s}$ (i.e. $r_i < s_i$ for $i = 1, \ldots, n$) implies that the restriction map embeds $\mathcal{F}^n(\bar{s})$ into $\mathcal{F}^n(\bar{r})$. The *differential ring of definable semi-germs*, denoted $\mathcal{F}^n$, is now defined to be the direct limit of the directed set $\{\mathcal{F}^n(\bar{r}) : \bar{r} \in \mathbb{R}^n_+\}$ as $\bar{r} \to \bar{0}$.

In this talk I shall sketch a proof of the fact that if our original o-minimal structure is polynomially bounded, then $\mathcal{F}^n$ is a Noetherian ring. I do not know whether one can do away with this assumption. Certainly the result is false for the corresponding direct limit of all *internal* (in the sense of nonstandard analysis) holomorphic functions, as one can see by considering the ideal generated by all (semi-germs of) functions of the form z^N where N is an infinite positive integer. However, it was shown by Peterzil and Starchenko that no such function is definable in any o-minimal structure (whether polynomially bounded or not).

The polynomial boundedness is needed in order to show that every nonzero element of $\mathcal{F}^n$ has, up to a multiplicative constant (from $^*\mathbb{R}$), a representative f in some $\mathcal{F}^n(\bar{s})$ which is both bounded by 1 and yet takes, for each $\bar{r} < \bar{s}$, a non-infinitesimal value on $^*\Delta^{(n)}(\bar{r})$. This ensures that the standard part of f, which is a standard holomorphic function by one of Abraham Robinson's early results in nonstandard complex analysis, is nonzero. One then pulls back the usual proof (using the Weierstrass Division Theorem) of the Noetherianity of standard rings of germs to $\mathcal{F}^n$.

Finally, I should mention that this work is motivated by Zilber's conjecture on the quasi-minimality of the complex exponential field and the observation that although the complex exponential function is not definable in any o-minimal structure, its restrictian to *any* disk of radius 1 in $^*\mathbb{C}$ *is* so definable from the parameter $\exp(w)$, where w is the centre of the disk.

On stable fields of finite weight

Krzysztof Krupiński
(joint work with A. Pillay)

A longstanding conjecture says that each infinite, stable field is separably closed. There are several well-known results saying that under some stronger assumptions (e.g. superstability or semiregularity) the field is even algebraically closed [4, 1]. In all these situations, a suitable rank with good additive properties is available, which allows one to prove an "exchange property for generics" (if g is generic and $g \in \mathrm{acl}(h)$, then h is also generic), and then, using Galois theory, one gets the desired conclusion that the field is algebraically closed. Such a rank, on the face of it, is unavailable in an arbitrary stable field.

We consider stable fields of finite weight (i.e. whose generic type has finite weight). Weight does not have as good additive properties as Lascar U-rank, and this makes our situation interesting and requiring new methods.

A separably closed field of infinite Ershov invariant is an example of a stable field of weight 1 [2]. So, the strongest conjecture on infinite, stable fields of finite weight that we can make is

Conjecture 1. *An infinite, stable field of finite weight is separably closed.*

Our main result is

Theorem 2. *Each stable field of weight 1 is separably closed.*

One of the essential steps in the proof is to show the following version of the "exchange property for generics".

Lemma 3. *Let K be a stable field in which both the sum and the product of any two non-generics over a subset A are non-generic over A (this is the case when K is stable of weight 1). Let k be a subfield of K. If an element g is generic over k and g is separably algebraic over $k(h_1, \ldots, h_m)$ (of course, in the field-theoretic sense), then one of the h_i's is generic over k.*

Besides Theorem 2, we get some partial results on stable fields of arbitrary finite weight.

Proposition 4. *If K is an infinite, stable field of finite weight, then for almost all primes q, $K^q = K$. More precisely, the number of exceptions is at most the weight of K.*

If one was able to strengthen the above proposition so that the conclusion holds for all primes q different from the characteristic of K, then using Scanlon's result that stable fields are closed under Artin-Schreier extensions [3], Conjecture 1 could be easily proved. Another observation is

Proposition 5. *An infinite, stable field of finite weight and of finite degree of imperfection is perfect.*

Our work is also related to Shelah's question on the structure of strongly dependent fields [5]. In fact, assuming stability, strong dependence of the theory amounts exactly to saying that all types have finite weight. Stable, strongly dependent theories are also called strongly stable. Thus, our context of stable fields of finite weight is more general than the context of strongly stable fields. A reasonable conjecture on strongly stable fields is

Conjecture 6. *Each infinite, strongly stable field is algebraically closed.*

We notice that each strongly stable field is perfect, and so, in virtue of Theorem 2, we get that Conjecture 6 is true in the case when the weight of the field equals 1.

We finish with the following two conjectures, in which we assume that K is an infinite, stable, saturated field with an additional structure.

Conjecture 7. *The following conditions are equivalent:*
(1) K is separably closed.
(2) K satisfies the exchange property for generics formulated in the conclusion of Lemma 3.

Conjecture 8. *The following conditions are equivalent:*
(1') K is algebraically closed.
(2') For any small subfield k of K and $g, h \in K$, if g is generic over k and g is algebraic over $k(h)$ in the field-theoretic sense, then h is generic over k.

From our work, it follows that (2) implies (1). Similarly, one can check that (2') implies (1'). The both converses remain open problems.

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On the homotopy type of definable groups in an o-minimal structure

Alessandro Berarducci
(joint work with Marcello Mamino)

It is known that given a definable group G in a saturated o-minimal expansion of a field, there is a canonical homomorphism from G to a compact real Lie group G/G^{00}, where G^{00} is the "infinitesimal subgroup" of G and G/G^{00} has the "logic topology" [5]. If G is definably compact, we show that the Lie-isomorphism type of G/G^{00} determines the definable homotopy type of G. Our results are based on the study of the o-minimal fundamental group $\pi_1^{\mathrm{def}}(G)$ and the o-minimal fundamental groupoid of G (all definable groups will be assumed to be definably compact and definably connected). Similar results have been independently obtained in [1] by different methods.

Large part of our analysis does not use the group structure of G. So more generally we consider quotients X/E where X is a definable set, E is a type-definable equivalence relation of bounded index, and X/E has the "logic topology". We work under the following general assumptions. (A1): X/E is locally simply connected; (A2): each E-equivalence class x/E is the intersection of a countable decreasing family of definably simply connected definable sets. Both assumptions hold in the case $X/E = G/G^{00}$ (i.e. when $X = G$ and $E = \{(x, y) : xG^{00} = yG^{00}\}$). Indeed the verification of (A1) for G/G^{00} follows at once from its being a Lie group. The verification of (A2) is carried out in [2] and is based on the "compact domination conjecture" established in [9] ([8] for the non-abelian case). We show that, after choosing base points $x_0 \in X$ and $x_0/E \in X/E$, there is a canonical isomorphism $\pi_1(X)^{\mathrm{def}} \cong \pi_1(X/E)$. Moreover this result can be "localized", namely for any open subset U of X/E we obtain, after the appropriate choice of the base points, an isomorphism $\pi_1(V)^{\mathrm{def}} \cong \pi_1(U)$ where V is the preimage of U under the projection $X \to X/E$. In particular we obtain a localizable version of the isomorphism $\pi(G)^{\mathrm{def}} \cong \pi(G/G^{00})$ established in [6].

The isomorphism $\pi_1(X)^{\mathrm{def}} \cong \pi_1(X/E)$ is obtained as the restriction of a homomorphism between the o-minimal fundamental groupoid of X and the fundamental groupoid of X/E. Since the universal cover of a space can be construed as a subset of its fundamental groupoid, we have in particular the following consequence: there is a natural homomorphism from the o-minimal universal cover of G and the univeral cover of G/G^{00} whose kernel is isomorphic to G^{00}. Roughly speaking this says that the "infinitesimal subgroup of the universal cover of G" is naturally isomorphic the infinitesimal subgroup of G. This can be used to show that every Lie group extension of G/G^{00} comes from a definable extension of G.

It is natural to ask whether the above results extend to the higher homotopy groups. Let us first observe that in the case $X/E = G/G^{00}$, we actually have a strong form of (A1) and (A2) with "simply connected" replaced by "contractible". Let us call (A1*) and (A2*) the strong forms. We conjecture that they entail an isomorphism of the higher homotopy groups, namely $\pi_n^{\mathrm{def}}(X) \cong \pi_n(X/E)$ for all n. So far however this has been proved only for $X = G$ and $E = G^{00}$ [4] using that fact that every definably compact group G is an almost direct product of an abelian definable subgroup and a semisimple definable subgroup [8], and that when G is abelian all the higher o-minimal homotopy groups of G vanish [4]. Combining all the above results we finally obtain that the Lie-isomorphism type of G/G^{00} determines the definable homotopy type of G.

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The space of generically stable types, I, II

Ehud Hrushovski

1.

Let T be a NIP theory. A definable type p is *generically stable* if $p(x)\otimes q(y) = p(y)\otimes q(x)$. Given a definable set V, let $\widehat{V}$ be the family of generically stable types on V. As explained in Anand Pillay's talk, $\widehat{V}$ can be construed as a pro-definable set.

Assume now that V carries a definable topology. One can define a topology on $\widehat{V}$, whose basic open sets have the form $\widehat{U}$, where U is a definable open subset of V. A pro-definable subset X of $\widehat{V}$ is *definably compact* if any definable type q on X has a limit point in X. If in addition a specific definable set is designated as a "line", path-connectedness and homotopy between definable functions can be defined.

These notions were developed in work with François Loeser, where we study the topology of $\widehat{V}$ for the theory $T = ACVF$ of algebraically closed valued fields, when V is an algebraic variety carrying the valuation topology. The value group Γ (with an element at ∞ adjoined) serves as a natural definable "line". In this case $\widehat{V}$ can be represented as a projective limit of space of semi-lattices LH (or in the projective case, semi-lattices up to homothety), with H a finite dimensional vector space over the valued field.

The natural operation $p\otimes q : \widehat{U} \times \widehat{V} \to \widehat{U \times V}$ is not continuous, but when restricted to a Γ-internal domain it is continuous. This uses the curve selection theorem, familiar from the o-minimal case, as well as this uses the fact that if $f : [0, \infty] \to LH$ is continuous, then near 0 and near ∞, the lattices $f(t)$ can be diagonalized by a single basis.

In [2] a definable deformation of any definable $X \subseteq \widehat{V}$ to a *skeleton* is constructed. This is a pro-definable subset of $\widehat{V}$ which is definably homeomorphic to a definable subset of Γ_∞^n.

We will illustrate using Abelian groups some uses of the space of stably dominated types (as opposed to a single type.) A definable subgroup B of A is called *generically stable* if it carries a generically stable, translation invariant type p. In this case p is unique. The set of such p is characterized by the property: $p^{-1} = p * p = p$, where $p * q$ is the convolution, i.e. the pushforward under addition of $p\otimes q$. Convolution is thus continuous on any Γ-internal set.

Let A be a Abelian variety defined over a valued F. It can be shown on the definable level, without topological considerations, that there exists a definable homomorphism $h : A \to C$ onto a definably compact Abelian group C defined over Γ, whose kernel B is the union of a definable, Γ-internal, directed system of generically stable subgroups. Since a limit point for a cofinal definable type on this family must exist in the definably compact space $\widehat{A}$, B must itself be generically stable.

For any $c \in C$, let p_c be the unique translate of p_c concentrating on $h^{-1}(c)$. This defines a map $h^* : C \to \widehat{A}$. $h^*(C)$ is in fact a skeleton for A: there exists a definable deformation of $\widehat{A}$ into $h^*(C)$. In fact there exists a definable sequence of subgroups B_t varying along a Γ-interval I, with translation invariant generically stable types p_t on B_t, interpolating continuously between (0) and B. The homotopy can then be taken to be such that $h(a,t)$ is the generic type of $a + B_t$. This can be shown by beginning with any definable deformation $q : \widehat{A} \times I \to \widehat{A}$ with $h_1(\widehat{A})$ contained in a skeleton S. Let $q^m = q * q^{-1} * \cdots * q$. Then for large enough even m, by an argument using finiteness of weight but also the commutativity of the group structure, q^m is the generic type of a generically stable subgroup. When starting from a generic element v of B, the deformation $h(t,v)^m$ traces a path leading from (0) to the generic of B.

2.

As explained in Pierre Simon's talk, a Keisler measure p on a definable set D is *generically stable* if $p(x) \otimes q(y) = p(y) \otimes q(x)$.

The space $M(V)$ of generically stable measures can again be parametrized by a pro-definable set; it forms a hyperdefinable set. While generically stable types (and paths in the space of such types) suffice for ACVF, in o-minimal theories they are scarce, but generically measures may take their place as a structural tool.

We will also consider generically stable measures on $\bigvee$-definable sets $\bigcup_i D_i$; by definition, these are finitely additive real valued measures on the set definable subsets $\bigcup_i D_i$, whose restriction to each D_i is generically stable.

There exists an analogy between sets defined by weak (strict) valuation inequalities in ACVF, and $\bigvee$ (resp. $\bigwedge$-) definable sets in o-minimal theories. In particular the structure $\mathbb{R}((t))$ can be viewed either as a substructure of a model of ACVF, or as an o-minimal ordered field; and any set defined by strict (weak) valuation inequalities for the valued field structure is $\bigvee$ (resp. $\bigwedge$-) definable for the o-minimal structure.

One can begin to see some outlines of a possible unification of o-minimality and metastability under the aegis of NIP. I will again use Abelian groups to sketch what is known, and what is not. A $\bigvee$-definable group is called *generically stable* if it admits a left- translation invariant generically stable measure. In this case, the measure is unique up to multiplication by a constant. For ACVF, we have a structure theorem for interpretable Abelian groups A: There exists a definable familiy $(B_t : t \in S)$ of definable subgroups of A, forming a directed system. Let $L = \bigcup_t B_t$. Then the quotient A/L and the partially ordered set S are both internal to an o-minimal set. The groups B_t are generically stable, dominated by a homomorphism $h : B_t \to D_t$ into a stable group, with kernel C_t; and we have $C_t = \bigcup_{s<t} B_s$. This is more generally valid for metastable groups with appropriate finite rank assumptions.

For o-minimal theories, one can conjecture the analogous statement, with B_t $\bigvee$-definable, C_t $\bigwedge$-definable, $L = A$, D_t compact. The o-minimality assumption on L, S is of course vacuous here.

For NIP theories, one has the following proposition: there exists an $\bigwedge$-definable set S and $\bigwedge$-definable subgroups C_t of A, for $t \in S$, forming a directed system. We have $\bigcup C_t = A$. Each C_t stabilizes a generically stable measure; in particular, if there is no indiscernible linearly ordered family of $\bigvee$-definable subgroups, A itself is generically stable.

Here the $\bigwedge$-definable groups C_t are visible, but not the $\bigvee$-definable generically stable groups. One can conjecture their existence under an appropriate finite weight assumption. A provisional definition with generically stable measures can be given. Anand Pillay found evidence that Shelah's notion of "strong dependence" is closely related to the finite weight required here. At the moment, even at the conjectural level, no analogue is known to the o-minimality of S. The stability or compactness of B_t/C_t may to some extent be replaced by the condition that for any definable subset X of B_t, the relation: "$aX \cap bX$ has measure zero" is a stable relation on $B_t \times B_t$.

To prove the proposition, one considers first any $M(A)$-invariant measure on the M-definable subsets of A, where M is an $\aleph_1$-saturated model. Such measures exist by amenability of A. They can be extended to smooth, $M(A)$-invariant measures μ, by a variant of Keisler's argument. Let C_μ be the stabilizer of μ. Any generically stable measure is conjugate to a measure p based on M; by the fim property of p, p leaves μ invariant, with probability 1. It follows that $C_p \leq C_\mu$; and the set of conjugates C_t of C_μ satisfies the conditions.

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On Γ-internal subsets of $\widehat{V}$

François Loeser
(joint work with Ehud Hrushovski)

In this talk, which followed the one by E. Hrushovski in which he introduced the space of generically stable types in NIP theories, we present some points of our joint work in progress on the topology of these spaces for ACVF. If V is a definable set, we denote by $\widehat{V}$ the space of generically stable (i.e., stably dominated in this case) types on V.

Let K be a valued field. Let $H = K^N$ be a vector space of dimension N. By a lattice in H we mean a free $\mathcal{O}$-submodule of rank N. By a semi-lattice in H we mean an $\mathcal{O}$-submodule u of H, such that for some K-subspace U_0 of H we have $U_0 \subseteq u$ and u/U_0 is a lattice in H/U_0. We define a topology on $L(H)$: the pre-basic open sets are those of the form: $\{u : h \notin u\}$ and those of the form $\{u : h \in \mathfrak{M}u\}$, where h is any element of H. We call this family the linear pre-topology on $L(H)$.

Let $H_{\ell;d}$ be the space of polynomials of degree $\leq d$ in ℓ variables. Here ℓ is fixed, so we suppress the index and write H_d.

Proposition 1. *For p in $\widehat{\mathbb{A}^n}$, the set*

$$J_d(p) = \{h \in H_d : p_*(val(h)) \geq 0\}$$

belongs to $L(H_d)$. There are canonical morphisms

$$J_d : \widehat{\mathbb{A}^n} \to L(H_d),$$

leading to a morphism of profinite spaces

$$J : \widehat{\mathbb{A}^n} \to \lim_{\leftarrow} L(H_d)$$

which induces an homeomorphism between $\widehat{\mathbb{A}}^n$ and its image.

Now, Γ-internal subsets of $L(H)$ have the following very important "diagonal basis" property:

Proposition 2. *Assume K is algebraically closed. Let Y be a Γ-internal subset of $L(H)$. Then there exists a finite number of bases $b^1, \dots, b^\ell$ of H such that each $y \in Y$ is diagonal for some b^i.*

This statement would follow from Theorem 2.4.13 (iii) of [1], except that in this theorem one considers f defined on Γ (or a finite cover of Γ) whereas here Y is the image of Γ^n under some definable function f, which requires some more work.

We ended the talk by sketching the proof of the following statement:

Proposition 3. *Let $X \subseteq \widehat{\mathbb{A}^N}$ be iso-definable and Γ-internal over an algebraically closed valued field F. Then for some d, and finitely many polynomials h_i of degree $\leq d$, the map $p \mapsto (p_*(val(h_i)))_i$ is injective on X.*

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On existentially closed ordered difference fields

Françoise Point

Let RCF denote the theory of real-closed fields; a direct consequence of results of H. Kikyo and S. Shelah, is that the theory of real-closed ordered *difference* fields, RCF_σ does not have a model-companion (see [4]).

In a difference field (K, σ), one has automatically a pair of fields, namely $(K, Fix(\sigma))$, where $Fix(\sigma)$ denotes the subfield of elements of K fixed by σ and if K is real-closed, then so is $Fix(\sigma)$. W. Baur showed that the theory of all pairs of real-closed fields (K, L) with a predicate for a subfield is undecidable ([1]). However, he also showed that the theory of the pairs (K, L) such that, adding to the language of ordered rings a new function symbol for a convex valuation v such that the residue field of L is dense in the residue field of K and each finite-dimensional L-vector space of K has a basis $a_1, \cdots, a_n$ satisfying for all $b_i \in L$ that $v(\sum_i b_i \cdot a_i) = min_i\{v(b_i \cdot a_i)\}$, becomes decidable ([1]).

In [5], we first describe a class of existentially closed totally ordered difference fields (even though it is not an elementary class). We also consider the case of a proper preordering, using former results of A. Prestel and L. van den Dries.

Then, we consider valued ordered difference fields and we assume on one hand that σ is strictly increasing on the set of elements of strictly positive valuation and on the other hand that in the pair $(K, Fix(\sigma))$, the residue field of K and the residue field of $Fix(\sigma)$ coincide. We proceed as for the case of valued difference fields with an ω-increasing automorphism treated by E. Hrushovski ([3]). We show an Ax-Kochen-Ersov type result for those which are real-closed and satisfy in addition a σ-Hensel lemma, which entails that the corresponding class is model-complete. Note that any complete valued ordered difference field with σ strictly increasing satisfies this σ-Hensel Lemma.

Finally, we consider commutative von Neumann regular lattice-ordered rings (ℓ-rings) with a distinguished automorphism σ which fixes the set of its maximal ℓ-ideals and we use transfer results due to S. Burris and H. Werner ([2]) in certain Boolean products in order to describe a class of existentially closed difference ℓ-rings.

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Definability of the theta functions and universal polarized abelian varieties

Ya'acov Peterzil
(joint work with Sergei Starchenko)

In [3] we investigated the uniform definability of 1-dimensional complex tori and the Weierstrass $\wp$-functions, in the o-minimal structure $\mathbb{R}_{an,exp}$. The result was used in the recent work of J. Pila, [4], to create a bridge between the analytic and algebraic description of certain Shimura varieties, within the o-minimal framework. This in turn allowed Pila to apply results with A. Wilkie, [5], on rational points of definable sets in o-minimal structures in order to solve certain cases of the arithmetic Andre-Oort conjecture.

In this talk I describe a recent generalization of our earlier work to the case of abelian varieties of arbitrary dimension.

We first observe that each single n-torus $T_\Pi = \mathbb{C}^n/\Lambda_\Pi$ can be viewed as a definable complex group in the real field, by considering the manifold and group structures on the fundamental domain

$$E_\Pi = \{\sum_{i=1}^{2n} t_i v_i : 0 \leq t_i \leq 1, i = 1, \dots, 2n\}.$$

(Here $\Pi = (v_1, \dots, v_{2n})$ is the $n \times 2n$ period matrix whose columns are $\mathbb{R}$-independent in $\mathbb{C}^n$). Moreover, the family of all n-tori is uniformly definable in the field $\mathbb{R}$.

When $n > 1$, the family of all abelian varieties is a proper sub-collection of all n-tori, which is given by countably many definable subfamilies, $\{\mathcal{F}_D\}$, parameterized by all $n \times n$ diagonal matrices

$$D = Diag(d_1, d_2, \dots, d_n),$$

with $d_1|d_2|...|d_n$ positive integers. Each single family $\mathcal{F}_D$ is parameterized by the Siegel half space $\mathcal{H}_n$, of all $n \times n$ complex symmetric matrices with a positive definite imaginary part, as follows:

$$\mathcal{F}_D = \{T_{(\tau,D)} : \tau \in \mathcal{H}_n\}.$$

Each abelian variety $T_\tau := T_{(\tau,D)}$ in $\mathcal{F}_D$ admits a polarization of type D. Given n and D as above, there is a natural number k such that every T_τ in $\mathcal{F}_D$ can be embedded, via a map Θ_τ, into $\mathbb{P}^k(\mathbb{C})$.

Any two such polarized varieties T_{τ_1} and T_{τ_2} are isomorphic (as polarized varieties) if and only if there is an element g of the symplectic group $\Gamma = Sp_D(2n, \mathbb{Z})$ such that $g \cdot \tau_1 = \tau_2$. Here $Sp_D(2n, \mathbb{Z})$ is the the group of $2n \times 2n$ integral matrices preserving the alternating form

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}.$$

The quotient space $\Gamma \backslash \mathcal{H}_n$ is known to admit an embedding, let's call it $\bar{\vartheta}$, into projective space $\mathbb{P}^\ell(\mathbb{C})$ such that the topological closure of $\bar{\vartheta}(\mathcal{H}_n)$ is an algebraic projective variety (for the classical results in complex analysis we refer to Igusa's book, [2]).

Both $\bar{\vartheta}$ and $\Theta_\tau(z)$ (as a function of z and τ) are given in coordinates by finitely many theta functions, defined by

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) = \sum_{\mathbb{Z}^n} e^{i\pi({}^t(n+a)\tau(n+a)+2{}^t(n+a)(z+b))},$$

for $a, b \in \mathbb{Q}^n$.

Consider the map

$$\alpha : \mathbb{C}^n \times \mathcal{H}_n \longrightarrow \mathbb{P}^\ell(\mathbb{C}) \times \mathbb{P}^k(\mathbb{C}),$$

defined by

$$\alpha(z, \tau) = (\Theta_\tau(z), \bar{\vartheta}(\tau)).$$

Then the topological closure of $\alpha(\mathbb{C}^n \times \mathcal{H}_n)$ in $\mathbb{P}^\ell(\mathbb{C}) \times \mathbb{P}^k(\mathbb{C})$, call it $\chi_{n,D}$, is an algebraic variety (see [1]). This variety can be viewed as the universal polarized abelian variety of polarization type D.

We now turn to the definability content of the above classical constructions. Clearly, the lattices Λ_Π and the discrete groups $Sp_D(2n, \mathbb{Z})$ cannot be definable in an o-minimal structure. Similarly, the holomorphic maps $\Theta_\tau : \mathbb{C}^n \to \mathbb{P}^k(\mathbb{C})$, $\bar{\vartheta} : \mathcal{H}_n \to \mathbb{P}^\ell(\mathbb{C})$ and α cannot be definable in an o-minimal structure because of their infinite periods. However, as pointed out above, each torus T_τ has a semilinear fundamental domain, call it $E_\tau \subseteq \mathbb{C}^n$. Also, it follows from the so-called Siegel reduction theory (see Igusa's book) that the quotient $Sp(2n, \mathbb{Z})\backslash\mathcal{H}_n$ also has a semi-algebraic fundamental domain, call it F_n. Let us define

$$\Omega_n = \{(z, \tau) \in \mathbb{C}^n \times \mathcal{H}_n : \tau \in F_n \& z \in E_\tau\}.$$

Our main theorem states:

Theorem 1. *For every $a, b \in \mathbb{R}^n$, the restriction of $\vartheta \begin{bmatrix} a \\ b \end{bmatrix}$ to Ω_n is definable in the o-minimal structure $\mathbb{R}_{an,exp}$. Hence, the restriction of α to Ω_n is definable in $\mathbb{R}_{an,exp}$.*

In particular, $\bar{\vartheta}|F_n$ and the family $\{\Theta_\tau|E_\tau : \tau \in F_n\}$ are definable in $\mathbb{R}_{an,exp}$.

In very broad terms, the idea of the proof is to present $\vartheta \begin{bmatrix} a \\ b \end{bmatrix}$ as a composition of the complex functions of the form $\tau_{i,j} \mapsto e^{i\pi\tau_{i,j}}$ (defined on a subset of $\mathbb{C}$ with a bounded real part) and an analytic function on a bounded domain U, which can be extended analytically to the closure of U.

As a corollary, one obtains for example:

Corollary 2. *Let $\mathcal{F} = \{A_w : w \in W\}$ be a family of n-dimensional projective abelian varieties which is definable in $\mathbb{R}_{an,exp}$. Then, there is in $\mathbb{R}_{an,exp}$ a definable family of maps $\{f_{w_1,w_2} : w_1, w_2 \in W\}$ such that, for every $w_1, w_2 \in W$ the function $f_{w_1,w_1} : A_{w_1} \to A_{w_2}$ is a real analytic group isomorphism of the two abelian varieties (note that all n-tori are real analytically isomorphic to a $2n$-product of the circle group).*

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No indiscernibles in NIP

Itay Kaplan
(joint work with Saharon Shelah)

We study a conjuncture of Shelah about the existence of indiscernible sequences in dependent (or NIP) theories. The conjuncture states that given a big enough set, it contains an indiscernible sequence.

Definition 1. For a cardinal κ, $n < \omega$ (or an ordinal) and an ordinal δ, $\kappa \to (\delta)_{T,n}$ means: for every sequence $\langle a_\alpha \,|\alpha < \kappa \rangle \in {}^{\kappa}(\mathfrak{C}^n)$, there is a subset $u \subseteq \kappa$ of order type δ such that $\langle a_\alpha \,|\alpha \in u \rangle$ is an indiscernible sequence.

Morley, in [1], proved that for ω-stable T, and for κ regular big enough, $\kappa \to (\kappa)_{T,1}$. In fact, for stable theories, and for $\kappa > 2^{|T|}$ regular, $\kappa \to (\kappa)_{T,n}$ for all $n < \omega$ (or even $n \leq |T|$) (for example by local character of non-forking and Feodor's lemma - see [3, III]). In the dependent context we have the following theorem (from [2]):

Theorem 2. *If T is strongly dependent then $\beth_{|T|^+}(\kappa) \to (\kappa^+)_{T,n}$ for all $n < \omega$.*

The two conditions, T being strongly dependent, and $n < \omega$ seemed at first redundant. We show that they are necessary.

Definition 3. $\kappa \to (\delta)_\theta^{<\omega}$ means: for every coloring $c : [\kappa]^{<\omega} \to \theta$, there is a sub-sequence of κ of length δ, $\langle \alpha_i \,|i < \delta \rangle$ ($\alpha_i < \kappa$) such that for all $n < \omega$, there exists some $c_n \in \theta$ such that $c(\alpha_{i_1}, \ldots, \alpha_{i_n}) = c_n$ for all increasing sequences $i_1 < \ldots < i_n$.

So $\kappa \to (\delta)_\theta^{<\omega}$ says that κ is a (θ, δ)-Erdös cardinal. An easy claim is

Claim 4. If $\kappa \to (\delta)_\theta^{<\omega}$ then for any theory T of size $|T| \leq \theta$, $\kappa \to (\delta)_{T,n}$ for all $n \leq \omega$.

The first theorem shows that under just NIP (as opposed to strong dependence), unless there is a good set theoretical reason (as in the previous claim), indiscernibles need not exist.

Theorem 5. *For any cardinal θ there is a dependent theory T such that $|T| = \theta$, and for any cardinal κ and any limit ordinal δ, $\kappa \to (\delta)_{T,1}$ iff $\kappa \to (\delta)_\theta^{<\omega}$.*

The assumption $n \leq \omega$ in Theorem 2 is also needed:
First, the example in Theorem 5 can be modified so that we get :

Theorem 6. *For any cardinal θ there is a strongly dependent theory T such that $|T| = \theta$, and for any cardinal κ and any limit ordinal δ, $\kappa \to (\delta)_{T,\omega}$ iff $\kappa \to (\delta)^{<\omega}_{\theta}$.*

Moreover, we can reach a similar result for $\mathbb{R}$, whose theory, RCF, is strongly dependent.

Theorem 7. *If $\kappa <$ The smallest inaccessible cardinal, then $\kappa \nrightarrow (\omega)_{RCF,\omega}$.*

A few words about the proofs: The proof of both theorems 5 and 7 uses the same basic idea: use induction on κ, and use properties of trees to generate a witness for $\kappa \nrightarrow_T (\delta)$. Both proofs divide into cases: $\kappa = \aleph_0$ (or θ), κ is a singular cardinal, and κ is regular but not strongly inaccessible.
In Theorem 5 we also deal with the case where κ is strongly inaccessible, and in fact this takes most of the work. We use the assumption that $\kappa \nrightarrow (\delta)^{<\omega}_{\theta}$ and induction to produce a model where the existence of an indiscernible sequence will produce a contradiction.
In order to prove Theorem 7 we prove something a bit stronger, i.e. that for all such κ there is a sequence of intervals $\langle\langle I^{\alpha}_n \,|n < \omega\rangle \,|\alpha < \kappa\rangle$, such that if $\langle\langle b^{\alpha}_n \,|n < \omega\rangle \,|\alpha < \kappa\rangle$ is a sequence of points such that $b^{\alpha}_n \in I^{\alpha}_n$, then it is not indiscernible.

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A Proof in Search of a Theorem

Frank Olaf Wagner
(joint work with Thomas Blossier, Amador Martin Pizarro)

Let $\mathfrak{M}$ be a structure in some language $\mathcal{L}$, and $(\mathfrak{M}_i : i < n)$ a family of reducts of $\mathfrak{M}$ to some sublanguages $\mathcal{L}_i \subset \mathcal{L}$. We shall study the relationship between definable groups in $\mathfrak{M}$ and interpretable groups in the reducts $(\mathfrak{M}_i : i < n)$. There are various natural examples for this type of question:

- The theory of differentially closed fields of characteristic zero as an expansion of the theory of algebraically closed fields of characteristic zero.
- The theory of existentially closed fields with automorphism as an expansion of the theory of algebraically closed fields (of given characteristic).
- The fusion of two strongly minimal sets in disjoint languages [9].

- The fusion of two strongly minimal sets over a common vector space over a finite field [5].
- The coloured fields (black, red and green) [11, 12, 1, 4, 2] as expansions of the underlying algebraically closed field.

We shall suppose that T (and hence the reducts T_i) is stable. Our results remain true if T is merely simple but the reducts remain stable; if the reducts are merely simple, we have to assume geometric elimination of hyperimaginaries, and the simple group configuration theorem only yields almost hyperdefinable groups rather than interpretable ones [6].

With no other hypotheses on the theory or its reducts, we obtain

Theorem 1. [7] *Let G be a T-definable group, and T_0 a reduct of T. Then (possibly after adding parameters) there is a T_0-interpretable group H and a T-definable homomorphism $\varphi_0 : G \to H$ such that for independent T-generic $g, g' \in G$ we have $\mathrm{acl}_T(g), \mathrm{acl}_T(g') \mathop{\smile\!\!\!\!\!\!\mid}^0_{\varphi_0(gg')} \mathrm{acl}_T(gg')$. Moreover, $\varphi_0(gg')$ is 0-interalgebraic with $\mathrm{acl}_0(\mathrm{acl}_T(g), \mathrm{acl}_T(g')) \cap \mathrm{acl}_T(gg')$.*

If T_0 is the reduct to equality, H is the trivial group and the independence condition just means that $\mathrm{acl}_T(g) \cup \mathrm{acl}_T(g')$ and $\mathrm{acl}_T(gg')$ are disjoint. We shall have to suppose some geometric conditions to ensure that the kernel of the homomorphism is small. We shall do this relative to an additionnal closure operator $\langle . \rangle$ contained in $\mathrm{acl}_T(.)$ satisfying

(†) If A ist T-algebraically closed and $b \mathop{\smile\!\!\!\!\!\!\mid}_A c$, then $\langle Abc \rangle \subseteq \bigcap_{i<n} \mathrm{acl}_i(\langle Ab \rangle, \langle Ac \rangle)$.
(‡) If $\bar{a} \in \bigcup_{i<n} \mathrm{acl}_i(A)$, then $\langle \mathrm{acl}_T(\bar{a}), A \rangle \subseteq \bigcap_{i<n} \mathrm{acl}_i(\mathrm{acl}_T(\bar{a}), \langle A \rangle)$.

Definition 2. [7] A theory T is *one-based over* $(T_i : i < n)$ for $\langle . \rangle$ if for all T-algebraically closed $A \subseteq B$ and all $\bar{c}$, if $\langle A\bar{c} \rangle \mathop{\smile\!\!\!\!\!\!\mid}^i_A B$ for all $i < n$, then the canonical base $\mathrm{cb}_T(\bar{c}/B)$ is T-algebraic over A.
T is CM-*trivial over* $(T_i : i < n)$ for $\langle . \rangle$ if for all T-algebraically closed $A \subseteq B$ and all $\bar{c}$, if $\langle A\bar{c} \rangle \mathop{\smile\!\!\!\!\!\!\mid}^i_A B$ for all $i < n$, then the canonical base $\mathrm{cb}_T(\bar{c}/A)$ is T-algebraic over $\mathrm{cb}_T(\bar{c}/B)$.

Condition (†) then ensures that these definitions behave well under adding and forgetting parameters; moreover, we may assume that A and/or B are models.

Example 3.
- The theory DCF_0 of differentially closed fields of characteristic 0 is one-based over the theory ACF_0 of algebraically closed fields of characteristic 0 for the differential closure acl_δ, which satisfies (†) and (‡) [13, 10] .
- The theory $ACFA$ of existentially closed fields with automorphism is one-based over the theory ACF of algebraically closed fields for the σ-closure acl_σ, which satisfies (†) and (‡) [8].

Moreover, we show

Theorem 4. [7] *The theory of the fusion of two strongly minimal theories (in disjoint languages or over a common vector space over a finite field) is CM-trivial* *over the two constituent theories for the self-sufficient closure. The theory of the coloured fields is CM-trivial over ACF for the self-sufficient closure. Moreover, the self-sufficient closure satisfies (†) and (‡).*

We then get

Theorem 5. [7] *Let G be a T-definable group, and $\varphi_i : G \to H_i$ as in Theorem 1. If T is one-based over $(T_i : i < n)$ with respect to a closure operator satisfying (†) and (‡), then $\bigcap_{i<n} \ker(\varphi_i)$ is finite. If T is CM-trivial over $(T_i : i < n)$, then $(\bigcap_{i<n} \ker(\varphi_i))^0 \leq Z(G^0)$.*

Corollary 6. [7] *A simple group, or a field, embeds into a T_i-interpretable group or field for some $i < n$.*

Finally, we want to study arbitrary groups definable in some coloured field. Such a group G gives rise to the usual group configuration (a, b, c, ab, ca, cab), where a, b, c are three independent generic elements of G. We manage to construct elements $\alpha \in \operatorname{acl}_0(\operatorname{acl}_T(c), \operatorname{acl}_T(ca))$, $\beta \in \operatorname{acl}_0(\operatorname{acl}_T(cab), \operatorname{acl}_T(ca))$ and $\alpha\beta \in \operatorname{acl}_0(\operatorname{acl}_T(cab), \operatorname{acl}_T(c))$ such that $(\alpha, \beta, c, \alpha\beta, ca, cab)$ is a group configuration in ACF, and hence gives rise to an algebraic group H. However, the precise relationship between G and H has not yet been completely elucidated.

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Model theoretic connected components of algebraic groups

Jakub Gismatullin

Let G be a (sufficiently saturated) group with some first order structure and $A \subset G$ a small set of parameters. We consider several kinds of *model-theoretic connected components* of G:

(1) G_A^0 — the connected component of G over A, is the intersection of all A-definable subgroups of G with finite index,
(2) G_A^{00} — the type-connected component of G over A, is the smallest subgroup of G type definable over A and with bounded index,
(3) G_A^∞ — the ∞-connected component of G over A, is the smallest $\mathrm{Aut}(G/A)$-invariant subgroup of G with bounded index.

If for every small A, $G_A^\infty = G_\emptyset^\infty$, then we call $G_\emptyset^\infty$ the ∞-connected component of G and denote it by G^∞ (we also say that G^∞ exists in this case). Similarly we define G^{00} (the type-connected component) and G^0 (the connected component).

G^0 is a classical object. G^{00} has been studied widely in model theory. G^∞ was designated as G^{000} by Peterzil, Pillay, Hrushovski and by Shelah for abelian groups with NIP in [7]. It was proved in [7] that for abelian group G with NIP, G^∞ exists. In [3, Theorem 5.3] we extended this result to an arbitrary group with NIP. In general G^∞ may not exists (even when the theory of G is simple). If G^∞ exists, then also G^{00} and G^0 exist and $G_A^\infty \subseteq G_A^{00} \subseteq G_A^0$ are normal subgroups of G (e.g. [3]).

One of the motivations for considering $G_\emptyset^0$, $G_\emptyset^{00}$ and $G_\emptyset^\infty$, is the interplay between them and strong types. In [4] we investigated the following construction: consider the 2-sorted structure $\mathcal{G} = (G, X, \cdot)$, where $\cdot : G \times X \to X$ is a regular action of G on X, and X is a predicate (on G we take its original structure). Then, Lascar, Kim-Pillay and Shelah strong types on the sort X correspond exactly to orbits of $G_\emptyset^0$, $G_\emptyset^{00}$ and $G_\emptyset^\infty$ (resp.) on X. Also, $G/G_\emptyset^\infty$ with "the logic topology" is a quasi-compact topological group, which can be seen as a canonical subgroup of the *Lascar group* $\mathrm{Gal_L}(\mathcal{G})$ of the structure $\mathcal{G}$.

The Lascar group is an abstractly defined invariant of first order theories of classical mathematical content. Lascar showed in [5] that for a very large class of theories, the *G-compact* ones, the group carries a compact Hausdorff topology. Another characterization of G-compactness is the following: T is G-compact if and only if in a saturated model of T, Kim-Pillay and Lascar strong types coincide. Essentially, there is only one known example of non-G-compact theory due to Ziegler [1]. Therefore an example of a group G with $G_\emptyset^{00} \neq G_\emptyset^\infty$ will yield a new kind of *non-G-compact* theory, based on the group structure. Such an example is not currently known. Pillay and Hrushovski showed that $G^{00} = G^\infty$ for definable amenable group G with NIP.

The main result of my talk concerns Chevalley groups and connected perfect linear algebraic groups. For such groups non-G-compactness in the above sense cannot occur.

Theorem 1 ([2]). *If G is a classical Chevalley group over an arbitrary infinite field (see below and [8]) or G is a connected perfect (i.e. $G = [G,G]$) linear algebraic group over an algebraically closed field, then*

$$G^\infty = G^{00} = G^0 = G$$

for an arbitrary first order structure on G (working in a saturated extension).

The components of a non-perfect reductive connected linear group G depend on the related components of $G/[G,G]$. Namely, for an arbitrary first order structure on G, there is some "natural" quotient structure on $G/[G,G]$ such that

$$G^x_\emptyset = j^{-1}\left[(G/[G,G])^x_\emptyset\right], \quad \text{for } x \in \{\infty, 00, 0\},$$

where $j : G \to G/[G,G]$ is the quotient map. The problem of determining the connected components of an abelian group seems to be difficult and related to additive combinatorics.

The classical construction of Chevalley groups is quite technical. It depends on a choice of three parameters: an arbitrary field K, a crystallographic root system $\Phi \subset \mathbb{R}^n$ and Λ_π — the weight lattice of some representation π of a semisimple complex Lie algebra associate to Φ. The resulting group G_{K,Φ,Λ_π} is generated by root subgroups. Every root system Φ decomposes into a finite union of irreducible root subsystems $\Phi = \Phi_1 \cup \cdots \cup \Phi_l$. Consequently G_{K,Φ,Λ_π} can be written as a product of corresponding subgroups $G_{K,\Phi,\Lambda_\pi} = \prod_{1\leq i\leq n} G_{K,\Phi_i,\Lambda_\pi}$. Moreover, irreducible root system are classified: A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 4)$, E_6, E_7, E_8, F_4 and G_2. E.g. when $\Phi = A_n$, or $\Phi = C_n$, G_{K,Φ,Λ_π} is a quotient of $\mathrm{SL}_{n+1}(K)$ or $\mathrm{Sp}_{2n}(K)$, respectively.

There is another characterization of Chevalley groups. If K is algebraically closed, then $G_K = G_{K,\Phi,\Lambda_\pi}$ is a connected semi-simple linear algebraic group over K, defined and split over the prime subfield of K. Any connected semi-simple linear algebraic group over K is isomorphic to one of the Chevalley groups. If F is an arbitrary subfield of K, then $G_F = G_{F,\Phi,\Lambda_\pi}$ is a commutator subgroup of the group $G_K(F)$ of points of G_K that are rational over F: $G_F = [G_K(F), G_K(F)]$.

We sketch the theory from [2] leading to the proof of Theorem 1. In [3] we gave another descriptions of G^∞_A and G^{00}_A. The key notion is the notion of *thick subset* of a group. A subset $X \subseteq G$ is thick if it is symmetric $X = X^{-1}$ and, for some natural N, for every N-sequence $g_1, \ldots, g_n$ from G, there are $1 \leq i < j \leq n$ such that $g_i^{-1} g_j \in X$.

Proposition 2. (1) *G^∞_A is generated by the intersection of all A-definable thick subsets of G: $G^\infty_A = \langle \bigcap\{P : P \subseteq G \text{ is } A\text{-definable and thick}\}\rangle$,*
(2) *$G^{00}_A = \bigcap\{P \cdot Q : P, Q \subseteq G \text{ are } A\text{-definable, thick, and } P \supseteq G^\infty_A \cup Q\}$.*

This Proposition allows us to investigate G^∞_A for certain groups. For instance, consider $G = \mathrm{PSL}_n(K)$, where K is a saturated field. Then G (with the pure group structure) is also saturated and simple (as a group). Since G^∞_A is a normal subgroup of G, $G = G^\infty_A$. Therefore, by compactness and Proposition 2, there is a natural number N, such that for every A-definable and thick $P \subseteq G$,

$P^N = \underbrace{P \cdot \dots \cdot P}_{N \text{ times}} = G$. In fact, a more general result is true:

Let $(G, \cdot)$ be an arbitrary non-trivial group. The following conditions are equivalent:

(1) There exists a natural number N such that, for every thick subset $P \subseteq G$ (not necessarily definable), $P^N = G$.
(2) G is infinite and if G^* is a sufficiently saturated extension of an arbitrary first order expansion $(G, \cdot, \dots)$ of $(G, \cdot)$, then $G^{*\infty}$ exists and $G^* = G^{*\infty}$.

We say that a group G is N-*absolutely connected* (N-ac) if it satisfies condition (1).

In order to prove that certain groups are absolutely connected, we introduce the class of *weakly simple groups*. For a group G and a natural number N define: $\mathcal{G}_N(G) = \left\{ g \in G : \left(g^G \cup g^{-1G} \right)^{\leq N} = G \right\}$. We say that a group G is N-*weakly simple* if $G \setminus \mathcal{G}_N(G)$ is not thick. It can be proved that N-weakly simplicity implies $4N$-absolutely connectedness.

Theorem 3. *Chevalley groups over infinite fields are 3-weakly simple and 12-ac.*

Here is an idea of the proof. Chevalley groups have the structure of a split BN-pair. If g is a regular element from the maximal torus, then the conjugacy class of g generates the whole group in 3 steps (so g is in $\mathcal{G}_3$). Using invertibility of the Cartan matrix, one can construct a sequence $(g_i)_{i \in \mathbb{N}}$ with regular $g_i^{-1} g_j$. $\square$

One can show that every absolutely connected group is perfect. In general the converse is not true, however for connected linear algebraic groups it is true.

Theorem 4. *Let G be a connected linear algebraic group over an algebraically closed field. The following conditions are equivalent: G is weakly simple, G is absolutely connected and G is perfect. Moreover, if the commutator width of G is R and the radical $\mathcal{R}(G)$ is solvable of derived length M, then G is $3(4R)^M$-weakly simple ($12(4R)^M$-ac).*

Let $\mathcal{C}_N$ be the class of N-ac groups and $\mathcal{C}_\infty = \bigcup_{n<\omega} \mathcal{C}_N$. Using Newelski criterion for type definability in groups [6] we have the following important remark:

> If the sequence $\mathcal{C}_N$, $N < \omega$ does not stabilize i.e. $\mathcal{C}_\infty \neq \mathcal{C}_N$ for every N, then there exists a group G with $G^\infty_\emptyset \neq G^{00}_\emptyset$.

Unfortunately, we do not know if there is a universal upper bound for the absolutely connectedness of connected perfect linear groups.

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RV information of definable sets in valued fields

IMMANUEL HALUPCZOK

Let K be a henselian valued field of characteristic $(0,0)$. For simplicity of exposition, we assume $K = k((t))$ where k is any field of characteristic 0. Let $R = k[[t]]$ be the corresponding valuation ring. We write Γ for the value group.

It will be useful to define the valuation and the angular component of tuples $\overline{x} = \sum_{i\in\Gamma} \overline{a}_i t^i \in K^n$ (where $\overline{a}_i \in k^n$): $v(\overline{x}) := \min\{i \in \Gamma \mid \overline{a}_i \neq 0\} = \min\{v(x_1), \ldots, v(x_n)\}$ and $\mathrm{ac}(\overline{x}) := \overline{a}_{v(\overline{x})}$.

For a definable set $X \subset R^n$, we want to understand the residue field and value group information (the "RV information") contained in X. More precisely, we want to describe definable sets "up to RV-isometry", which is defined as follows:

Definition 1. A definable bijection $f\colon X \to Y$ is an *RV-isometry* if for any $\overline{x}, \overline{x}' \in X$, we have $v(f(\overline{x}) - f(\overline{x}')) = v(\overline{x} - \overline{x}')$ (i.e. it is a usual isometry) and $\mathrm{ac}(f(\overline{x}) - f(\overline{x}')) = \mathrm{ac}(\overline{x} - \overline{x}')$.

We will present a theorem which yields a good description of definable sets up to RV-isometry; it implies that large parts of any definable set are, up to RV-isometry, translation invariant in many directions. To make this precise, we need some more definitions.

By a "ball" in R^n, we shall mean a set of the form $B = \overline{x}_0 + t^\lambda R^n = \{\overline{x} \in R^n \mid v(\overline{x} - \overline{x}_0) \geq \lambda\}$.

Call a definable set $X \subset R^n$ *translatable* on a ball B if there exists a direction $\overline{c} \in K^n \setminus \{0\}$ in which it is translation invariant on B, i.e. $(X + K\overline{c}) \cap B = X \cap B$. Call $X \subset R^n$ *almost translatable* on B if there exists an RV-isometry $X \cap B \to Y \subset B$ such that Y is translatable on B.

theorem 1. [1] *For every definable set X, there exists a finite number of sets S_i each of which is either a ball or a point such that for any ball B, X is almost translatable on B if and only if B does not contain any of the sets S_i.*

Each ball S_i yields a finite number of balls B on which X is not almost translatable; each point S_i yields an infinite descending chain of balls. Typically, these points are singularities of X.[2]

[1] "theorem" with lowercase "t" because still work in progress.

[2] In a more general setting, when the value group Γ is not $\mathbb{Z}$, the theorem can still be formulated essentially in the same way. However, then even the balls S_i yield infinite chains of balls B where X is not almost translatable.

This theorem is useful because it reduces understanding X up to RV-isometry for most of the set to lower dimension: suppose that B is a ball where X is almost translatable, i.e. $X \cap B$ is RV-isometric to a set $Y \subset B$ which is translation invariant in direction $\overline{c}$. Then Y is the preimage under π of a set $Y' \subset B'$, where $B' \subset R^{n-1}$ is a ball of the same radius as B but of lower dimension and $\pi \colon B \twoheadrightarrow B'$ is a suitable projection sending $\overline{c}$ to 0. Hence, up to RV-isometry $X \cap B$ is determined by Y' and the direction $\overline{c}$. Now theorem 1 can be recursively applied to Y'. In other words, we obtain that on most of the balls where X is almost translatable, it is even RV-isometric to a set which is translation invariant in two directions, and so on.

This description is a rather strong restriction on possible RV-isometry classes of definable sets. It turns out that indeed, the number of possible RV-isometry classes which are left over is small in a precise sense: the RV-isometry class of a set can be specified using only parameters from the residue field and the value group. Moreover, this works in a definable way. More precisely:

theorem 2. *Let X_s be a definable family of definable sets ($s \in S$). Then there exists a definable map $\psi \colon S \to (k \cup \Gamma)^{\mathrm{eq}}$ such that $\psi(s) = \psi(s')$ if and only iff X_s and $X_{s'}$ are RV-isometric.*

Dimensions, matroids, and dense pairs of structures

Antongiulio Fornasiero

In the following, all structures expands an integral domain.

Definition 1. A structure M is geometric if the algebraic closure is a pregeometry in all M' elementarily equivalent to M ([2] observe that in this case M eliminates the quantifier $\exists^\infty$).

We define a generalisation: structures with an "existential matroid". The main examples are superstable structures of U-rank a power of ω and d-minimal structures:

Definition 2. A d-minimal structure is a structure M with a definable Hausdorff topology, such that every definable subset X of M is the union of an open set and finitely many discrete sets (where the number of discrete sets does not depend on the parameters of definition of X), plus some additional conditions.

O-minimal structures, p-adic fields, and algebraically closed valued fields are d-minimal (and also geometric).

On a structure there can be at most one existential matroid. Ultraproducts of geometric structures, while not geometric in general, do have an unique existential matroid. An existential matroid on M can be extended in a canonical way to a closure operator on M^{eq} (see [6] for the case when M is geometric).

A dimension function dim on M is a function from M-definable sets to the natural numbers, which is additive, definable, and such that $\dim(M) = 1$ [3, 8].

There is a canonical correspondence between dimension functions and existential matroids.

Generalising previous results in [7, 4, 1], we study dense closed pairs of structures with an existential matroid:

Definition 3. $X \subseteq M$ is *dense* if X intersects every definable subset of M of dimension 1.

Given T the theory of a structure with an existential matroid, let T^d be the theory of pairs (B, A) such that $B \models T$ and A is a dense and closed (w.r.t. the matorid) subset of B. Then, T^d is consistent and complete. Moreover, the models of T^d also have an existential matroid, the "small closure": $b \in scl(X)$ if b is in the closure of $A \cup X$. We extend the above result to dense tuples of structures:

Theorem 4. *Let T be the theory of a structure with an existential matroid. Define T^{nd} be the theory of tuples $A_0 \prec A_1 \prec \ldots \prec A_n \models T$, such that each A_i is closed in A_n, and A_0 is dense in A_n. Then, T^{nd} is consistent and complete.*

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Counting types and NIP

Artem Chernikov

Classically stable theories can be characterized as those with few types over models in some cardinalities. We consider a generalization of stability spectrum $S_T^{loc}(\kappa)$ by counting only those types which satisfy local character (do not fork over some subset of the model of size $\leq |T|$). In stable theories it coincides with the usual spectra. Based on results from [1] we show that (modulo NTP_2) the following are equivalent

(1) T is NIP
(2) $S_T^{loc}(\kappa) \leq \kappa$ for some κ
(3) $S_T^{loc}(\kappa) \leq \kappa$ for every $\kappa = \kappa^{|T|}$

It is known that even *o*-minimal theories rarely have models with all types over them definable. However there are always models with few types over them in NIP. More precisely, using results of Baldwin and Benedikt on definability of types over indiscernible sequences indexed by complete linear orders (see [2]) we show that every NIP theory has a model M with $|M| = \kappa$ and $S(M) = \kappa^{|T|}$ for every cardinal κ. In addition we can choose M to be gross (every definable subset is either finite or of size $|M|$, see [3]). This property does not characterize NIP, but we conjecture some strengthening which might.

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Generically stable measures

Pierre Simon

(joint work with Ehud Hrushovski, Anand Pillay)

Measures in model theory, as a generalization of types, were introduced by Keisler in [4] to recover certain phenomena of stability theory in general NIP context. Hrushovski, Peterzil and Pillay resurrected them in [1] in their study of Pillay's conjecture for definably compact groups in o-minimal theories. In [2] and [3], it is shown how, in the NIP context, they behave very much like types and notions such as "finitely satisfiable" or "definable" can make sense for them.

Generically stable types are defined in [2] as invariant types that have a symmetric Morley sequence. Equivalently, they are definable and finitely satisfiable global types. This notion extends naturally to measures. However, whereas generically stable types need not always exist in NIP theories, generically stable measures can always be found.

Two special cases are of interest. First any measure induced by a sigma-additive measure on some model is generically stable. In particular, it is definable. This generalizes previously known results for the Lebesgue measure on an o-minimal structure.

Secondly, a group has an invariant generically stable measure if and only if it is fsg, i.e. there is a type p and a small model M_0 such that all translates of p are finitely satisfiable in M_0. The invariant measure is then unique. Furthermore, if the measure is smooth (has a unique extension to any bigger model), then the group is compactly dominated by $\pi : G \to G/G^{00}$. This means that for any definable set X of G, the set $\{x \in G/G^{00} : \pi^{-1}(x) \cap X \neq \emptyset \text{ and } \pi^{-1}(x) \cap X^c \neq \emptyset\}$ has Haar measure 0.

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Amalgamation functors and the boundary conditions

Byunghan Kim

(joint work with John Goodrick, Alexei Kolesnikov)

This is a joint work with J. Goodrick and A. Kolesnikov continuing their earlier work [2]. As well-known 4-amalgamation is used to produce the hyperdefinable homogeneous space from a group configuration [1][3]. Recently E. Hrushovski pointed out various relationships between n-amalgamation functors and imaginaries generated by definable groupoids in stable theories [4]. He showed that we can essentially assume n-amalgamation for all n in stable theories. Whether the result can be extended to the context of simple theories remains open. On the other hand, we showed that n-uniqueness is equivalent to $(n+1)$-amalgamation in stable theories, which was left open in [4]. Then we studied how much the equivalence of n-amalgamation and a certain definable closure condition in stable theories is preserved in the context of simple theories. Roughly the failure of amalgamation in simple theories is due to either the bad boundary condition or intrinsic instability. Then we tried to develop a certain homology theory to detect such difference. Amalgamation notions are rephrased, and many natural questions come after.

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On Zariski structures, noncommutative geometry and physics

Boris Zilber

In recent developments in the theory of Zariski structures it has been established that many quantum algebras can be represented as (noncommutative) co-ordinate rings of Zariski geometries. In particular, we are interested in the quantum algebra generated by invertible operators U and V satisfying the Weyl commutation relation $VU = qUV$. This is related to the Heisenberg commutation relation $PQ - QP = \hbar$, for the position and momentum operators, via the Campbell-Hausdorff formula. We note that, on the other hand, the corresponding Zariski geometries for general q can be approximated (in a well-defined model-theoretic sense) by Noetherian Zariski structures for q a root of unity of order N. The latter are of a finitary type, where Dirac calculus has a well-defined meaning. Correspondingly, we assume that $\hbar = 1/N$ for an integer N with divisibility properties dictated by the model-theoretic analysis. We use this to give a mathematically rigorous calculation of the Feynman propagator for the free particle

$$\sqrt{\frac{1}{2\pi i\hbar t}}\,\exp i\frac{(x_1-x_0)^2}{2t\hbar}$$

and the quantum harmonic oscillator (for the frequency $\omega = 2\pi$)

$$c_0\sqrt{\frac{1}{\hbar|\sin 2\pi t|}}\,\exp \pi i\frac{(x_1^2+x_2^2)\cos 2\pi t - 2x_1x_2}{\hbar \sin 2\pi t}.$$

p-adic van der Corput Lemma

Raf Cluckers

We present the p-adic analogue of the real van der Corput Lemma with analytic phase which dates back to 1921, see [2]. It concerns p-adic (resp. real) oscillatory integrals with an analytic phase, where the k-th derivative of the phase does not vanish, and where then upper bounds for the norm of the integrals are found in terms of k. Many typical, but often recent, real corollaries also follow in the p-adic case, for example related to Fourier transforms of L^p functions in many variables and their restrictions to "suitably curved" p-adic manifolds, in analogy to real results in [3] and [4]. More generally, in [1], the theory is developed over the p-adics and over $\mathbf{F}_q((t))$ in great analogy to Chapter VIII of [4]. This research fits in a broader project by Cornulier, Louvet, Tessera, Valette, and the author on groups with the Howe-Moore property.

In his lecture, the author made a link to another broader project namely that of developing the theory of motivic harmonic analysis, and noted that this p-adic van der Corput Lemma fills in a gap in the p-adic knowledge where clearly a better understanding of the p-adics will help for developing the broader project.

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Reporter: Antongiulio Fornasiero

Participants

Dr. Hans Adler
Kurt Gödel Research Center
for Mathematical Logic
Universität Wien
Währinger Str. 25
A-1090 Wien

Prof. Dr. Andreas Baudisch
Institut für Mathematik
Humboldt-Universität Berlin
10099 Berlin

Prof. Dr. Itai Ben Yaacov
Institut Camille Jordan
Universite Claude Bernard Lyon 1
43 blvd. du 11 novembre 1918
F-69622 Villeurbanne Cedex

Prof. Dr. Alessandro Berarducci
Dip. di Matematica "L.Tonelli"
Universita di Pisa
Largo Bruno Pontecorvo, 5
I-56127 Pisa

Prof. Dr. Alexander Berenstein
Departamento de Matematicas
Universidad de los Andes
Bogota D.C.
COLOMBIA

Prof. Dr. Elisabeth Bouscaren
Laboratoire de Mathematique
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Prof. Dr. Enrique Casanovas
Departament de Logica, Historia i
Filosofia de la Ciencia
Universitat de Barcelona
Montalegre 6
E-08001 Barcelona

Prof. Dr. Gregory L. Cherlin
Department of Mathematics
Rutgers University
Hill Center, Busch Campus
110 Frelinghuysen Road
Piscataway , NJ 08854-8019
USA

Artem Chernikov
Institut Camille Jordan
Universite Claude Bernard Lyon 1
43 blvd. du 11 novembre 1918
F-69622 Villeurbanne Cedex

Prof. Dr. Raf Cluckers
Laboratoire Paul Painleve, UMR 8524
Universite des Sciences et
Technologie de Lille 1
Cite Scientifique, Bat. M2
F-59655 Villeneuve d'Ascq. Cedex

Prof. Dr. Francoise Delon
U. F. R. de Mathematiques
Case 7012
Universite Paris 7
2, Place Jussieu
F-75251 Paris Cedex 05

Prof. Dr. Clifton Ealy
Department of Mathematics
Western Illinois University
Macomb , IL 61455
USA

Prof. Dr. Pantelis Eleftheriou
Centro de Matematica e Aplicacoes
Fundamentais
2, Av. Prof. Gama Pinto
P-1649003 Lisboa

Prof. Dr. David M. Evans
School of Mathematics
University of East Anglia
GB-Norwich NR4 7TJ

Dr. Antongiulio Fornasiero
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Jakub Gismatullin
Institute of Mathematics
Wroclaw University
pl. Grunwaldzki 2/4
50-384 Wroclaw
POLAND

Dr. Immanuel Halupczok
Institut für Mathematische
Logik und Grundlagenforschung
Universität Münster
Einsteinstr. 62
48149 Münster

Prof. Dr. Deirdre Haskell
Dept. of Mathematics & Statistics
McMaster University
1280 Main Street West
Hamilton , Ont. L8S 4K1
CANADA

Dr. Martin Hils
U.F.R. de Mathematiques
Case 7012
Universite Paris 7
F-75205 Paris Cedex 13

Prof. Dr. Ehud Hrushovski
Department of Mathematics
The Hebrew University
Givat Ram
Jerusalem 91904
ISRAEL

Prof. Dr. Eric Jaligot
Institut Camille Jordan
Universite Claude Bernard Lyon 1
43 blvd. du 11 novembre 1918
F-69622 Villeurbanne Cedex

Itay Kaplan
Einstein Institute of Mathematics
Edmond J. Safra Campus
The Hebrew University of Jerusalem
Givat Ram
Jerusalem 91904
ISRAEL

Prof. Dr. Byunghan Kim
Department of Mathematics
Yonsei University
Seoul 120-749
KOREA

Prof. Dr. Piotr Kowalski
Department of Mathematics
University of Wroclaw
50 384 Wroclaw
POLAND

Prof. Dr. Krzysztof Krupinski
Institute of Mathematics
Wroclaw University
pl. Grunwaldzki 2/4
50-384 Wroclaw
POLAND

Prof. Dr. Salma Kuhlmann
Fachbereich Mathematik u. Statistik
Universität Konstanz
Universitätsstr. 10
78457 Konstanz

Prof. Dr. Francois Loeser
Departement de Mathematiques et
Applications
Ecole Normale Superieure
45, rue d'Ulm
F-75230 Paris Cedex 05

Dr. Fares Maalouf
Institut für Mathematik
Humboldt-Universität zu Berlin
Unter den Linden 6
10099 Berlin

Prof. Dr. H. Dugald MacPherson
School of Mathematics
University of Leeds
GB-Leeds LS2 9JT

Dr. Amador Martin-Pizarro
Institut Camille Jordan
Universite Claude Bernard Lyon 1
43 blvd. du 11 novembre 1918
F-69622 Villeurbanne Cedex

Prof. Dr. Rahim Moosa
Dept. of Pure Mathematics
University of Waterloo
200 University Avenue West
Waterloo , Ont. N2L 3G1
CANADA

Prof. Dr. Ludomir Newelski
Institute of Mathematics
Wroclaw University
pl. Grunwaldzki 2/4
50-384 Wroclaw
POLAND

Gönenc Onay
U.F.R. de Mathematiques
Case 7012
Universite Paris 7
F-75205 Paris Cedex 13

Prof. Dr. Alf Onshuus
Depto. Matematicas
Universidad de Los Andes
Carrera 1a. No. 18-A-10
Apartado Aero 4976-12340
Bogota
COLOMBIA

Prof. Dr. Kobi Peterzil
Dept. of Mathematics & Computer Sciences
University of Haifa
Mount Carmel
Haifa 31905
ISRAEL

Prof. Dr. Anand Pillay
Mathematical Logic
Department of Pure Mathematics
University of Leeds
GB-Leeds LS2 9JT

Prof. Dr. Francoise Point
Equipe Logique Mathematique
U.F.R. de Mathematique
Universite Paris VII
2, Place Jussieu
F-75251 Paris Cedex 05

Prof. Dr. Alexander Prestel
Fachbereich Mathematik u. Statistik
Universität Konstanz
Universitätsstr. 10
78457 Konstanz

Prof. Dr. Bertrand Remy
Institut Camille Jordan
Universite Lyon 1
Bat. Braconnier
21, Avenue Claude Bernard
F-69622 Villeurbanne Cedex

Prof. Dr. Philipp Rothmaler
Department of Mathematics
BCC CUNY
Bronx , NY 10453
USA

Prof. Dr. Thomas W. Scanlon
Department of Mathematics
University of California
Berkeley , CA 94720-3840
USA

Pierre Simon
Analyse Numerique et EDP
Universite de Paris-Sud
Batiment 425
F-91405 Orsay Cedex

Prof. Dr. Sergei S. Starchenko
Department of Mathematics
Notre Dame University
Notre Dame , IN 46556-4618
USA

Prof. Dr. Charles Steinhorn
Vassar College
Box 453
Poughkeepsie , NY 12604
USA

Prof. Dr. Katrin Tent
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Amaury Thuillier
Institut Camille Jordan
Universite Claude Bernard Lyon 1
43 blvd. du 11 novembre 1918
F-69622 Villeurbanne Cedex

Prof. Dr. Alex Usvyatsov
CMAF & University of Lisbon
Centro de Matematica e Aplicacoes
Fundamentais
Av. Prof. Gama Pinto, 2
P-Lisboa 1649-003

Prof. Dr. Frank Olaf Wagner
Institut Camille Jordan
Universite Lyon 1
Bat. Braconnier
21, Avenue Claude Bernard
F-69622 Villeurbanne Cedex

Dr. Roman Wencel
Institute of Mathematics
Wroclaw University
pl. Grunwaldzki 2/4
50-384 Wroclaw
POLAND

Prof. Dr. Alex Wilkie
Department of Mathematics
The University of Manchester
Oxford Road
GB-Manchester M13 9PL

Prof. Dr. Martin Ziegler
Abteilung für Math. Logik und
Grundlagen der Mathematik
Universität Freiburg
Eckerstr. 1
79104 Freiburg

Prof. Dr. Boris I. Zilber
Mathematical Institute
Oxford University
24-29 St. Giles
GB-Oxford OX1 3LB