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Mini-Workshop: Valuations and Integral Geometry

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January 17th – January 23rd, 2010

Abstract. As a generalization of the notion of measure, valuations have long played a central role in the integral geometry of convex sets. In recent years there has been a series of striking developments. Several examples were presented at this meeting, e.g. the work of Bernig and Fu on the integral geometry of groups acting transitively on the unit sphere, that of Hug and Schneider on kinematic and Crofton formulas for tensor valued valuations and a series of results by Ludwig and Reitzner on classifications of affine invariant notions of surface areas and of convex body valued valuations.

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Introduction by the Organisers

The workshop *Valuations and Integral Geometry*, organized by Semyon Alesker, Andreas Bernig, and Franz Schuster was held from January 17th to January 23rd, 2010. The meeting was attended by 16 participants working in different areas such as convex and differential geometry or geometric measure theory. The program involved 3 lecture series by Fu, Ludwig, and Reitzner as well as several one hour and shorter lectures built around them. Some highlights of the program will be described in the following.

In a 3 hour lecture series, Joseph Fu presented intriguing recent results about the product and convolution structures on the space of continuous translation invariant valuations. This new algebraic machinery has been the key tool for a fuller understanding of the kinematic formulas for groups acting transitively on the unit sphere obtained by Alesker, Bernig, and Fu. Fu described in detail the hermitian case and presented a mysterious conjecture concerning the integral geometry in complex space forms.

In a related vein, Judit Abardia presented her joint work with Eduardo Gallego and Gil Solanes on the integral geometry in complex space forms. Some starting points for an integral geometry in Hermitian symmetric spaces were discussed by Hiroyuki Tasaki. Daniel Hug gave a very clear talk about kinematic formulas for tensor valuations which he recently obtained in a joint work with Rolf Schneider. Rolf Schneider talked about zonoids and Crofton formulas in Minkowski spaces, and Wolfgang Weil about translative kinematic formulas. Closely related to these developments is a generalization of the notion of valuations to smooth manifolds which was explained by Semyon Alesker in an impromptu evening lecture.

On a different line of research, Monika Ludwig gave an extremely interesting 3 hour lecture series about her characterizations of convex body valued valuations compatible with affine transformations. These results are deeply connected with the theory of isoperimetric inequalities. Here, the valuation point of view has shed new light on some classical affine isoperimetric inequalities which were shown to hold for larger classes of valuations by Haberl and Schuster. These inequalities have led to new affine L_p Sobolev inequalities and an affine symmetrization principle presented in a one hour lecture by Christoph Haberl.

Matthias Reitzner gave a 3 hour lecture series on his joint work with Monika Ludwig concerning their breakthrough in the characterization of upper semicontinuous $\mathrm{SL}(n)$ invariant valuations. Their results classified both of the classical $\mathrm{SL}(n)$ invariant notions of affine surface area - affine surface area and centro-affine surface area - which date back to Blaschke's school of affine differential geometry. In fact, all of the L_p affine surface areas introduced by Lutwak in early 1990's were completely characterized.

The program of the workshop also involved several excellent talks by young researchers. Thomas Wannerer spoke about his extension of Ludwig's characterization of the projection operator, one of the key concepts introduced by Minkowski for the study of projections of convex bodies. Andy Tsang gave a talk about his work on valuations defined on L_p function spaces, which presents a particularly exciting new area in the theory of valuations. Gautier Berck presented joint work with Juan-Carlos Álvarez on the use of Crofton formulas in the metrisability problem on Finsler manifolds. Gil Solanes spoke about Crofton and Gauss-Bonnet formulas which are invariant under the action of the Möbius group.

Mini-Workshop: Valuations and Integral Geometry

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Convex body valued valuations

Monika Ludwig

Let $\mathcal{K}^n$ denote the space of convex bodies (compact, convex sets) in $\mathbb{R}^n$ equipped with the usual topology induced by the Hausdorff metric. If $\mathcal{S}^n \subset \mathcal{K}^n$ and $\langle \mathbb{A}, + \rangle$ is an abelian semigroup, then $\mathrm{Z} : \mathcal{S}^n \to \langle \mathbb{A}, + \rangle$ is called a valuation if

$$\mathrm{Z}\, K + \mathrm{Z}\, L = \mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L),$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{S}^n$. Many important operators on convex bodies are valuations. Maybe the most important such operator is the projection operator Π which associates with a convex body K its projection body $\Pi\, K$ (defined via its support function as $h(\Pi\, K, u) = \mathrm{vol}_{n-1}(K|u^\perp)$ for $u \in S^{n-1}$, where vol_{n-1} denotes $(n-1)$-dimensional volume, $u^\perp$ the hyperplane orthogonal to u, $K|u^\perp$ the image of the orthogonal projection of K on $u^\perp$, $h(K, v) = \max\{v \cdot x : x \in K\}$, $v \in \mathbb{R}^n$, and $v \cdot x$ is the standard inner product of v and x). In this series of three lecture, we show that using convex body valued valuations it is possible to classify operators on convex bodies and, as a consequence, obtain strengthened isoperimetric inequalities.

There are several important additions on $\mathcal{K}^n$ and we consider here Minkowski addition, which is defined for $K, L \in \mathcal{K}^n$ by $K + L = \{x + y : x \in K,\ y \in L\}$ (for Blaschke addition, a complete classification was recently obtained by Haberl [1]). An operator $\mathrm{Z} : \mathcal{K}^n \to \mathcal{K}^n$ is called $\mathrm{SL}(n)$ contravariant, if $\mathrm{Z}(\phi K) = \phi^{-t}\, \mathrm{Z}\, K$ for all $\phi \in \mathrm{SL}(n), K \in \mathcal{K}^n$.

Theorem 1 ([6])**.** *An operator $\mathrm{Z} : \mathcal{K}^n \to \langle \mathcal{K}^n, + \rangle$, $n \geq 2$, is a continuous, translation invariant, $\mathrm{SL}(n)$ contravariant valuation if and only if there is a constant $c \geq 0$ such that $\mathrm{Z} = c\, \Pi$.*

The corresponding classification problem for continuous, translation invariant, $\mathrm{O}(n)$ covariant valuations was studied by Schneider [15], Kiderlen [4], and Schuster [16, 17, 18]. For even (i.e., $\mathrm{Z}(-K) = \mathrm{Z}(K)$ for all $K \in \mathcal{K}^n$), $(n-1)$-homogeneous valuations a complete classification was obtained by Schuster [16]. In general, this interesting problem remains open but general representation theorems were recently obtained by Schuster [18].

The above theorem was extended to valuations $\mathrm{Z} : \mathcal{K}_0^n \to \langle \mathcal{K}^n, + \rangle$ that are $\mathrm{GL}(n)$ covariant. Here $\mathcal{K}_0^n$ is the space of convex bodies containing the origin in their interiors and $\mathrm{Z} : \mathcal{K}_0^n \to \mathcal{K}^n$ is called $\mathrm{GL}(n)$ covariant if for some $q \in \mathbb{R}$, $\mathrm{Z}(\phi K) = |\det \phi|^q \phi\, \mathrm{Z}\, K$ for all $\phi \in \mathrm{GL}(n), K \in \mathcal{K}_0^n$.

Theorem 2 ([8])**.** *An operator $\mathrm{Z} : \mathcal{K}_0^n \to \langle \mathcal{K}^n, + \rangle$, $n \geq 3$, is a continuous, non-trivial, $\mathrm{GL}(n)$ covariant valuation if and only if either there are constants $c_0 \geq 0$ and $c_1 \in \mathbb{R}$, not both zero, such that*

$$\mathrm{Z}\, K = c_0\, \mathrm{M}\, K + c_1\, m(K)$$

for every $K \in \mathcal{K}_0^n$ *or there is a constant* $c_0 > 0$ *such that*

$$\mathrm{Z}\, K = c_0\, \Pi(K^*)$$

for every $K \in \mathcal{K}_0^n$*.*

Here, a valuation is called trivial if it is a linear combination of the identity and the central reflection, K^* denotes the polar body of $K \in \mathcal{K}_0^n$, $m(K) = \int_K x\, dx$ its moment vector, and $\mathrm{M}\, K$ its moment body (defined via its support function as $h(\mathrm{M}\, K, u) = \int_K |x \cdot u|\, dx$).

Theorem 2 was recently extended to L^p Minkowski valuations for $p > 1$ [7], that is, valuations $\mathrm{Z} : \mathcal{K}_0^n \to \langle \mathcal{K}_0^n, +_p \rangle$, where $+_p$ denotes L^p Minkowski addition (defined by $h^p(K +_p L, \cdot) = h^p(K, \cdot) + h^p(L, \cdot)$). The resulting two-parameter family of operators (first defined in [6]) is a generalization of the family of symmetric operators given by L^p projection bodies and L^p centroid bodies, which were introduced by Lutwak [10] and Lutwak and Zhang [14]. Using these new asymmetric operators, Haberl and Schuster [3] obtained L_p affine isoperimetric inequalities that strengthen and imply the inequalities by Lutwak and Zhang [14] and Lutwak, Yang, and Zhang [11]. In [2], Haberl and Schuster used these inequalities to prove strengthened versions of the sharp affine L_p Sobolev inequality of Lutwak, Yang, and Zhang [12].

The relation between Sobolev inequalities and valuations is further explained by the following result. Let $\mathcal{K}_c^n$ denote the space of origin-symmetric convex bodies in $\mathbb{R}^n$ and $W^{1,1}(\mathbb{R}^n)$ the Sobolev space of functions $f \in L^1(\mathbb{R}^n)$ whose weak partial derivatives are in $L^1(\mathbb{R}^n)$. An operator $\mathrm{Z} : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$ is called affinely contravariant if it is translation invariant, 1-homogeneous and such that for some $q \in \mathbb{R}$, $\mathrm{Z}(f \circ \phi^{-1}) = |\det \phi|^q \phi^{-t}\, \mathrm{Z}(f)$ for all $\phi \in \mathrm{GL}(n), f \in W^{1,1}(\mathbb{R}^n)$ and a valuation if $\mathrm{Z}(f \vee g) + \mathrm{Z}(f \wedge g) = \mathrm{Z}(f) + \mathrm{Z}(g)$ for all $f, g \in W^{1,1}(\mathbb{R}^n)$, where $f \vee g = \max\{f, g\}$ and $f \wedge g = \min\{f, g\}$.

Theorem 3 ([9])**.** *An operator* $\mathrm{Z} : W^{1,1}(\mathbb{R}^n) \to \langle \mathcal{K}_c^n, + \rangle$*,* $n \geq 3$*, is a continuous, affinely contravariant valuation if and only if there is a constant* $c \geq 0$ *such that*

$$\mathrm{Z}(f) = c\, \Pi\langle f \rangle$$

for every $f \in W^{1,1}(\mathbb{R}^n)$*.*

For $f \in W^{1,1}(\mathbb{R}^n)$, the convex body $\langle f \rangle$ is (up to normalization) the unit ball of the optimal Sobolev norm of f and the solution of the functional Minkowski problem associated to f. These important notions were introduced by Lutwak, Yang, and Zhang [13]. The operator $\Pi\langle f \rangle$ is a critical ingredient of Zhang's proof of his affine Sobolev inequality [19] and

$$h(\Pi\, \langle f \rangle, u) = \frac{1}{2} \int_{\mathbb{R}^n} |u \cdot \nabla f(x)|\, dx$$

for all $f \in W^{1,1}(\mathbb{R}^n)$.

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Affine invariant notions of surface area

Matthias Reitzner

The classical notion of surface area, the $n-1$ dimensional Hausdorff measure $\mathcal{H}_{n-1}(\partial K)$ of the boundary of a set convex set $K \subset \mathbb{R}^n$, is an continuous, monotone, homogenous, translation and rotation invariant valuation. Yet it is not affine invariant. In this series of lectures affine invariant notions of surface area have been investigated.

In the first lecture we introduced the following affine invariant notions of surface area:

- **Minimal Surface-Area**:

$$S_{min}(K) = \min_{A \in SL(n)} S(AK)$$

This natural notion of surface is by definition affine invariant, yet it is not a valuation. A beautiful reverse isoperimetric inequality for S_{min} was proved by Ball [1].

- **Volume of the Polar Projection Body:** The projection body ΠK is defined by $h(\Pi K, v) = V_{n-1}(K|_{v^\perp})$. By Cauchy's surface area formula, its first intrinsic volume equals - up to a constant - the classical surface area,

$$V_1(\Pi K) = \int_{S^{n-1}} V_{n-1}(K|_{v^\perp}) dv = c_n V_{n-1}(K)$$

which is not affine invariant. Changing the exponent "1" to $-n$ we obtain

$$V(\Pi^o K) = \frac{1}{n} \int_{S^{n-1}} V_{n-1}(K|_{v^\perp})^{-n} dv$$

which is affine invariant. Although ΠK itself is a valuation, $V(\Pi^o K)$ is not a valuation.

- **Classical Affine Surface Area:**

$$\Omega(K) = \int_{\partial K} \kappa(x)^{\frac{1}{n+1}} \, d\mathcal{H}(x)$$

is an affine invariant valuation [5], [9], [11]. Since it vanishes on polytopes it cannot be continuous. It turns out that this funcitonal is upper semi-continuous.

- **Centro-Affine Surface Area:**

$$\Omega_n(K) = \int_{\partial K} \left(\frac{\kappa(x)}{h(x)^{n+1}} \right)^{\frac{1}{2}} h(x) \, d\mathcal{H}(x)$$

is even a GL(n) invariant valuation [10], which in addition is invariant under polarization. Again, since it vanishes on polytopes it is only upper semi-continuous. In addition, it is also not translation invariant .

- L_p **Affine Surface Area:** Classical affine surface area and centro-affine surface area are on particular cases of

$$\Omega_p(K) = \int_{\partial K} \left(\frac{\kappa(x)}{h(x)^{n+1}} \right)^{\frac{p}{n+p}} h(x) \, d\mathcal{H}(x),$$

which for all $p > 0$ is an affine invariant upper semi-continuous valuation, which vanishes on polytopes [10],[3]. It is not translation invariant.

- **Maximal Affine Surface Area:** One possibility to obtain a continuous affine surface area is given by

$$\Omega^{\max}(K) = \max_{L \subseteq K} \Omega(L) \ .$$

This is an affine invariant continuous notion of surface area, yet it is not a valuation. It was introduced by Bárány [2]. $\Omega^{\max}(K)$ is the affine surface area of the limit shape of certain random polytopes in K.

As an open question we raise the question to determine the convex set K_{min} such that for all n-dimensional convex $K \subset I\!R^n$ we have

$$\Omega^{max}(K_{min}) \leq \Omega^{max}(K) \leq \Omega^{max}(B).$$

The right hand side follows from the affine isoperimetric inequality.

Let $\mathcal{K}_0^n$ denote the set of all convex bodies containing the origin in their interior, and denote by $\mathcal{P}_0^n$ the polytopes in $\mathcal{K}_0^n$. Among all the affine invariant notions of surface area mentioned above, the L_p affine surface areas play a special role. This is due to their characterization as the only upper semicontinuous, SL(n) invariant homogeneous valuation, which vanish on polytopes. Valuations not vanishing on polytopes with these properties are volume, Euler characteristic, and the volume of the polar body [6]. In the second lecture a sketch of the proof of this recent characterization of affine surface areas [8] was presented. More general, define

$$\text{Conc}(0,\infty) = \{\phi : I\!R_0^+ \to I\!R_0^+,\ \text{concave}, \lim_{t\to 0}\phi(t) = \lim_{t\to\infty}\phi(t)/t = 0\}$$

Theorem: *$\Phi : \mathcal{K}_0^n \to I\!R$ upper semicontinuous, SL(n) invariant valuation, vanishing on polytopes*

$$\Updownarrow$$

$$\Phi(K) = \int_{\partial K} \phi\left(\frac{\kappa(x)}{h(x)^{n+1}}\right) h(x)\, d\mathcal{H}(x)$$

with $\phi \in$ Conc$(0,\infty)$, where $\kappa(x)$ denotes the generalized Gaussian curvature and $h(x) = x \cdot u(K,x)$.

Two interesting open problems are:

Conjecture: If $\Psi : \mathcal{P}_0^n \to I\!R$ is a Borel measurable, SL(n) invariant valuation, then

$$\Psi(P) = c_0\, V_0(P) + c_n\, V(P) + c_{-n}\, V(P^o)$$

Conjecture: If $\Phi : \mathcal{K}_0^n \to I\!R$ is a SL(n) invariant, simple weak valuation of 1st Baire class, homogeneous of degree q, vanishing on polytopes then

$$\Phi(K) = c\, \Omega_p(K)$$

for every $K \in \mathcal{K}_0^n$ where $p = n(n-q)/(n+q)$.

In the third lecture we presented recent developments concerning the ϕ-affine surface areas

$$\Phi(K) = \int_{\partial K} \phi\left(\frac{\kappa(x)}{h(x)^{n+1}}\right) h(x)\, d\mathcal{H}(x)$$

with $\phi \in \text{Conc}(0,\infty)$. They behave nicely under polarization [4], [7]

$$\Omega_\phi(K^o) = \Omega_{\phi_o}(K),$$

with $\phi_o(t) = t\phi(1/t) \in \text{Conc}(0,\infty)$, vanish for most convex bodies, and satisfy an L_ϕ-isoperimetric inequality [10], [12], [7]. This class of valuations can be complemented by two additional classes of valuations introduced by Ludwig [7]. It remains an open question to characterize all upper-semicontinuous SL(n)-invariant homogeneous valuations which take values in $I\!R^+ \cup \{\infty\}$.

Open questions:

- Is there a Steiner formula for affine surface area $\Omega(K+\varepsilon B^n)$ (partial answers due to Colesanti), or for $\Omega^{\max}(K+\varepsilon B^n)$?
- $\Phi(K)=\int_{\mathcal{L}_i^n}\Omega(K|_E)dE$ is an upper semicontinuous, translation and SO(n) invariant valuation, which vanishes on polytopes. Is it possible to determine Φ more explicitly?
- More general: Are there affine kinematic formulae?

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Crofton formulae and zoonoids

Rolf Schneider

Motivated by the classical Crofton formula of Euclidean integral geometry, one understands by a *Crofton formula* any result of the type

$$k\text{-area}(M)=\int_{(n-k)\text{-flats}}\operatorname{card}(M\cap E)\,\varphi(\mathrm{d}E). \tag{1}$$

If this holds for a given notion of k-area and a given class of k-dimensional surfaces M, with some signed measure φ on a space of $(n-k)$-flats, then φ is called a *Crofton measure* for the considered k-area. Investigations of this type were initiated by Busemann, who suggested an integral-geometric approach to Hilbert's fourth problem, concerning the determination of all metrics for which lines are minimizing, and who later extended this to a study of areas for which flats are minimizing. He related this to the existence of positive Crofton measures. A proper framework for investigating such questions is provided by projective Finsler spaces. We describe work in this direction, roughly of the last decade, with special emphasis on general (non-smooth) Finsler spaces and the application of the theory of (generalized) zonoids.

A (generalized) Finsler metric on an open convex subset $C \subset \mathbb{R}^n$ is a continuous function $F : C \times \mathbb{R}^n \to [0, \infty)$ such that $F(x, \cdot)$ is a norm, for each $x \in C$. Curve length and induced metric are defined as usual. The pair (C, F) is a *projective Finsler space* if line segments are shortest curves connecting their endpoints. The classical examples are the Minkowski spaces ($\mathbb{R}^n$ with a norm) and the Hilbert geometries in bounded convex domains. For Minkowski spaces, Busemann has defined a general notion of Minkowskian k-area; it extends naturally to generalized Finsler spaces, in the following way. A k-area is defined by a positive function α_k on the set of k-dimensional, origin-symmetric convex bodies which is invariant under linear transformations, continuous, suitably normalized, and satisfies an additional convexity assumption if $k = n - 1$. The corresponding k-area of a k-surface M in the projective Finsler space (C, F) is then given, up to a factor, by

$$\int_M \alpha_k(B_x \cap T_x M)\, \mathcal{H}_F^k(\mathrm{d}x).$$

Here $B_x := \{\xi \in \mathbb{R}^n : F(x, \xi) \leq 1\}$, $T_x M$ is the tangent space of M at x (identified with a subspace of $\mathbb{R}^n$), and $\mathcal{H}_F^k$ denotes the k-dimensional Hausdorff measure coming from the metric induced by the Finsler structure (see [13] for this representation of the k-area). The most prominent examples of such areas are the *Busemann area* (obtained for constant α_k) and the *Holmes–Thompson area* (obtained from the volume product, also definable as a symplectic volume).

In smooth projective Finsler spaces, Crofton formulae (with smooth signed densities) for Holmes–Thompson areas were obtained by Álvarez and Fernandes [2, 3, 7, 5], see also [4]. They employed the symplectic structure on the space of geodesics and later double fibrations and Gelfand transforms.

Exploiting the theory of zonoids, one can obtain various results on Crofton measures in non-smooth projective Finsler spaces. This was initiated in [8], where Crofton formulae (with positive measures) for Holmes–Thompson areas in hypermetric Minkowski spaces were obtained. A first explanation for the role of zonoids is the following result. (All considered spaces from now on have dimension n.)

Theorem (by several authors) (a) *For a Minkowski space, the following conditions are equivalent:*
(1) *there exists a positive Crofton measure for the curve length,*
(2) *the dual unit ball is a zonoid,*
(3) *the metric induced by the norm is a hypermetric.*

(b) *For a Minkowskian* $(n-1)$*-area in* $(\mathbb{R}^n, \|\cdot\|)$*, a Crofton measure* (*a positive Crofton measure*) *exists if and only if the corresponding isoperimetrix is a generalized zonoid* (*a zonoid*).

Here, the isoperimetrix is the suitably normalized solution of the isoperimetric problem.

The following results on the existence or non-existence of Crofton measures are all based on finer properties of zonoids (more information is contained in [14]).

Theorem [9] *There exist Minkowski spaces, for example* ℓ_∞^n *and* ℓ_1^n*, for which the only Minkowskian* $(n-1)$*-area admitting a positive Crofton measure is the Holmes–Thompson area (up to a factor).*

Theorem [11]
(1) *There are Minkowski spaces arbitrarily close (in the Banach–Mazur metric) to* ℓ_2^n*, but not Euclidean, in which there exists a positive Crofton measure for the Busemann* $(n-1)$*-area.*
(2) *There are Minkowski spaces arbitrarily close to* ℓ_2^n *in which there exists no positive Crofton measure for the Busemann* $(n-1)$*-area.*
(3) *If* $n = 3$ *or* n *is sufficiently large, then a full neighbourhood of* ℓ_∞^n *consists of Minkowski spaces in which there is no positive Crofton measure for the Busemann* $(n-1)$*-area.*

On the other hand, in every sufficiently smooth Minkowski space there is a (signed) Crofton measure for the Busemann $(n-1)$-area. This is not generally true in smooth projective Finsler spaces, as shown by Álvarez and Berck [1].

Conjecture. *In the space of* n*-dimensional Minkowski spaces, there is a dense subset of spaces in which there is no positive Crofton measure for the Busemann* $(n-1)$*-area.*

Holmes–Thompson k-areas, which we now denote by vol_k, behave much better with respect to Crofton formulae, as demonstrated by the following observations. In every Minkowski space, there is a positive Crofton measure for vol_{n-1}. If in a Minkowski space there exists a Crofton measure (a positive Crofton measure) for vol_1, then there also exists a Crofton measure (a positive Crofton measure) for vol_k, $k \in \{2, \dots, n-2\}$. Based on Holmes–Thompson areas, one can also define valuations extending the intrinsic volumes, as done for hypermetric Minkowski spaces in [10], and for smooth projective Finsler spaces by Bernig [6].

The theory of generalized zonoids can further be used, in combination with Pogorelov's approach to Hilbert's fourth problem, to obtain the to date most general Crofton formula in smooth projective Finsler spaces.

Theorem [13] *Let* (C, F) *be a smooth projective Finsler space. For* $j \in \{1, \dots, n-1\}$*, there exists a signed measure* η_j *on the space* $A(n, j)$ *of* j*-flats such that, for* $k \in \{n-j, \dots, n\}$ *and every* $(\mathcal{H}^k, k)$ *rectifiable Borel set* $M \subset C$*,*

$$\mathrm{vol}_k(M) = \mathrm{const} \cdot \int_{A(n,j)} \mathrm{vol}_{k+j-n}(M \cap E)\, \eta_j(\mathrm{d}E).$$

The line measure η_1 *is positive.*

The existence of positive Crofton measures for vol_k if $k > 1$ requires stronger assumptions. The following can be proved by approximation arguments.

Theorem [12]
(1) *In every hypermetric projective Finsler space, there exists a positive Crofton measure for* vol_k*,* $k = 1, \dots, n-1$*.*

(2) *In every projective Finsler space, there exists a positive Crofton measure for* vol_{n-1}.

The method of proof is responsible for the fact that these are merely existence results. Explicit representations of the line measure (the Crofton measure for the Holmes–Thompson $(n-1)$-area) in Minkowski spaces have been known for a long time. More recently, an explicit construction of the line measure was given for polytopal Hilbert geometries [15]. We doubt whether this can be extended to Crofton measures for vol_k, $k < n-1$; indeed, we close with the following

Conjecture. *For the Hilbert geometry in a convex body C, a positive Crofton measure for* vol_1 *exists only if the geometry is hyperbolic, that is, if C is an ellipsoid.*

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Local Functionals and Translative Integral Formulas

Wolfgang Weil

A functional φ on the class $\mathcal{P}$ of convex polytopes in $\mathbb{R}^d$ is called *local*, if there is a measurable, measure valued functional Φ which is translation covariant, locally defined and satisfies $\Phi(P, \mathbb{R}^d) = \varphi(P), P \in \mathcal{P}$ (compare [1, Section 11.1]). The following result is shown.

Theorem. **(a)** *A local functional φ on $\mathcal{P}$ with local extension Φ has a unique decomposition*

$$\varphi(P) = \sum_{j=0}^{d-1} \varphi^{(j)}(P) + c_d V_d(P)$$

with j-homogeneous local functionals $\varphi^{(j)}$ on $\mathcal{P}$ and $c_d \in \mathbb{R}$.

(b) *Each $\varphi^{(j)}$ has a local extension $\Phi^{(j)}$ of the form*

$$\Phi^{(j)}(P, \cdot) = \sum_{F \in \mathcal{F}_j(P)} f_j(n(P, F))\lambda_F$$

with a function f_j (uniquely determined by Φ) on the class of $(d-j-1)$-dimensional spherical polytopes and the Lebesgue measure λ_F on the j-dimensional faces F of P.

(c) *Any additive, translation invariant, measurable functional φ on $\mathcal{P}$ is local.*

(d) *There exist local functionals $\varphi_m^{(j)}$ on $\mathcal{P} \times \mathcal{P}$, such that*

$$\int_{\mathbb{R}^d} \varphi^{(j)}(P \cap Q^x)\lambda_d(dx) = \varphi^{(j)}(P)V_d(Q) + \sum_{m=j+1}^{d-1} \varphi_m^{(j)}(P, Q) + V_d(P)\varphi^{(j)}(Q).$$

The functional $\varphi_m^{(j)}(P, Q)$ is homogeneous of degree m in P and $d-j+m$ in Q.

There is a corresponding result for the local extension Φ yielding mixed measures $\Phi_m^{(j)}$. Both translative formulas (global and local) can be iterated. There are explicit representations of the mixed functionals $\varphi_m^{(j)}$ (resp. mixed measures $\Phi_m^{(j)}$).

For applications in stochastic geometry it is important that local functionals φ can be extended to finite unions of convex polytopes which are pairwise in mutual general position by the inclusion-exclusion formula. Consequently, formulas for the mean values of φ on stationary Boolean models with polytopal grains result.

Two open problems mentioned in the talk: (1) Does a local functional φ on $\mathcal{P}$ with suitable continuity properties allow an extension to the class $\mathcal{K}$ of all convex bodies? (2) Since the local extension Φ of φ need not be unique, describe all possible local extensions (this is already an important problem for the intrinsic volumes $\varphi = V_j$).

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Algebraic integral geometry

Joseph H.G. Fu

Over the last ten years or so, S. Alesker has shown that the space of continuous convex valuations enjoys a rich algebraic structure. Restricting to valuations invariant under appropriate group actions, this structure turns out to encode an enormous amount of integral geometric information, shedding light not only on the classical integral geometry of the euclidean group (due originally to Blaschke) but also to heretofore unexplored territory, such as the recent work [2] of A. Bernig and the author on "hermitian integral geometry", i.e. the integral geometry of $\mathbb{C}^n$ under the unitary group $U(n)$.

The prototype for these developments was the approach to the Blaschke kinematic formulas developed by Hadwiger. Denote by $\mathrm{Val} = \mathrm{Val}(\mathbb{R}^n)$ the vector space of continuous translation-invariant convex valuations on $\mathbb{R}^n$, and $\mathrm{Val}^{SO(n)}$ the subspace of continuous valuations invariant under the full isometry group $\overline{SO(n)} := SO(n) \ltimes \mathbb{R}^n$. Hadwiger showed that the intrinsic volumes the intrinsic volumes $\mu_0, \ldots, \mu_n$, constitute a basis of $\mathrm{Val}^{SO(n)}$. It follows that, given $k \in \{0, \ldots, n\}$, there exist universal constants $c_{n,k,i,j}$

$$\int_{\overline{SO(n)}} \mu_k(K \cap \bar{g}L)\, d\bar{g} = \sum_{i+j=n+k} c_{n,k,i,j}\mu_i(K)\mu_j(L), =: k_{SO(n)}(\mu_k)(K,L) \tag{1}$$

for all compact convex sets $K, L \subset \mathbb{R}^n$, where $d\bar{g}$ is the Haar measure and $k_{SO(n)} : \mathrm{Val}^{SO(n)} \to \mathrm{Val}^{SO(n)} \otimes \mathrm{Val}^{SO(n)}$ is the **kinematic operator** of $\overline{SO(n)}$. The coefficients $c_{n,k,i,j}$ may then be evaluated using the **template method**, i.e. by explicitly evaluating the integral for enough conveniently chosen K, L and solving the resulting linear equations.

Alesker showed that if $G \subset SO(n)$ acts transitively on the sphere S^{n-1} then the subspace Val^G of $\overline{G} := G \ltimes \mathbb{R}^n$-invariant is again finite-dimensional. It follows that there is a kinematic operator $k_G : \mathrm{Val}^G \to \mathrm{Val}^G \otimes \mathrm{Val}^G$ encoding the integral geometry of the group G. A prominent example is the case where $\mathbb{R}^n = \mathbb{R}^{2m} = \mathbb{C}^m$ and $G = U(m)$. Furthermore he defined a natural commutative graded product on Val under which each Val^G is closed, and that there is a natural Poincaré duality operator $p : \mathrm{Val}^G \to \mathrm{Val}^{G^*}$. It turns out that the kinematic operator k_G carries precisely the same information as the restriction $m_G : \mathrm{Val}^G \otimes \mathrm{Val}^G \to \mathrm{Val}^G$ of the multiplication.

Theorem 1 (Fundamental theorem of algebraic integral geometry, or ftaig)**.** Put $m_G^* : \mathrm{Val}^{G^*} \to \mathrm{Val}^{G^*} \otimes \mathrm{Val}^{G^*}$ for the adjoint of the Alesker product map. Then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Val}^G & \xrightarrow{p} & \mathrm{Val}^{G^*} \\ k_{\overline{G}}\big\downarrow & & m_G^*\big\downarrow \\ \mathrm{Val}^G \otimes \mathrm{Val}^G & \xrightarrow{p\otimes p} & \mathrm{Val}^{G^*} \otimes \mathrm{Val}^{G^*} \end{array} \tag{2}$$

The groups G acting transitively on spheres have been classified. Using the ftaig the integral geometry has been worked out for the special cases $(\mathbb{C}^n, U(n))$ [1, 2], $(\mathbb{C}^n, SU(n))$ [4], $(\mathbb{C}^4, Spin(7))$ and $(\mathbb{R}^7, G_2)$ [3]. Of these, the case of hermitian integral geometry $(\mathbb{C}^n, U(n))$ is in many ways the most interesting, as the $SU(n)$ case may be regarded as a variation on it, and the other two cases as variations on the classical $SO(n)$ case.

The earliest definite results in the hermitian case were due to Tasaki [7]. The starting point of the present approach is the following characterization of the algebra of $U(n)$-invariant valuations on $\mathbb{C}^n$. Let s, t be variables of formal degrees 2, 1 respectively, and define the polynomials $f_i(s,t)$ by $\sum_i f_i(s,t) = \log(1+s+t)$. Then

$$\mathrm{Val}^{\overline{U(n)}}(\mathbb{C}^n) \simeq \mathbb{R}[s,t]/(f_{n+1}, f_{n+2}). \tag{3}$$

In view of the Alesker-Poincaré duality for $\mathrm{Val}^{\overline{U(n)}}$, this may be proved in a completely formal way from the evaluations

$$s^{n-k}t^{2k}(\text{unit ball} \subset \mathbb{C}^n) = \binom{2k}{k} \tag{4}$$

and the classical Pfaff-Saalschütz identities.

The integral geometry of certain homogeneous spaces may be approached in the same way, using Alesker's theory of valuations on manifolds. In this setting the resulting algebra of valuations is no longer graded, but filtered. Alesker and Bernig [5] have proved the ftaig for any compact space M whose group of isometries acts transitively on the sphere bundle SM. Furthermore the "transfer principle" of R. Howard ensures that certain similarities between spaces M are reflected in their integral geometry. In fact one way to establish the identities (4) is via transfer from $\mathbb{C}P^n$: there are analogous valuations s, t on $\mathbb{C}P^n$ such that (4) holds with the total space $\mathbb{C}P^n$ replacing the unit ball. In fact, using the calculations of Gray [6] of the Lipschitz-Killing curvatures of Kähler manifolds, it is easy to compute that

$$s^k t^{2l}(\mathbb{C}P^n) = \binom{2l}{l}\binom{n-k+1}{l+1} \tag{5}$$

By Alesker-Poincaré duality again, the evaluations (5) are enough to completely determine the structure of the algebra of invariant valuations on $\mathbb{C}P^n$. Furthermore, the product formula of [5] implies that we may use analytic continuation to compute this algebra for all complex space forms, whether positively or negatively curved. This leads to the following conjecture, which we have checked by symbolic machine calculation through $n = 16$:

Conjecture 2. Let s, t, λ be variables of formal degrees $2, 1, -2$ respectively. Define the formal series $\bar{f}_k(s,t,\lambda)$ to be the sum of the terms of weighted degree k in the expansion of

$$\begin{aligned}&\log(1+s+t+\lambda+3\lambda^2+13\lambda^3+...)\\&\qquad=\log(1+s+t+\sum\left[\binom{4n+1}{n+1}-9\binom{4n+1}{n-1}\right]\lambda^n).\end{aligned}$$

Then the filtered algebra $\mathcal{V}^{U(n)}(\mathbb{C}M^n_\lambda)$ of invariant valuations on the complex space form of holomorphic sectional curvature 4λ is isomorphic to

$$\mathbb{R}[s,t]/(\bar{f}_{n+1},\bar{f}_{n+2},t^{2n+1},st^{2n-1},\dots,s^nt). \tag{6}$$

The filtration on $\mathcal{V}^{U(n)}(\mathbb{C}M^n_\lambda)$ is induced by the degrees of s,t, without reference to λ.

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$SL(n)$-contravariant Minkowski valuations

Thomas Wannerer

(joint work with Franz E. Schuster)

The projection body of a convex body K, originally introduced by Minkowski, is a convex body ΠK which is determined by the $(n-1)$-dimensional volume of projections of K on hyperplanes through the origin. More precisely,

$$h(\Pi K,u)=\mathrm{Vol}_{n-1}(K^u),\quad u\in S^{n-1},$$

where $h(L,x)=\max\{\langle x,y\rangle : y\in L\}$ is the support function of a convex body L, $\mathrm{Vol}_{n-1}(K^u)$ denotes the $(n-1)$-dimensional volume of the orthogonal projection of K on the subspace orthogonal to u and $S^{n-1}\subset\mathbb{R}^n$ is the Euclidean unit sphere. Projection bodies turned out to be a very useful concept, having applications in many different areas ranging from stochastic geometry to Sobolev inequalities, cf. [1], [3],[6], [7] and [8].

Denote by $\mathcal{K}^n$ the set of all nonempty, compact, convex subsets of $\mathbb{R}^n$. A map Z of $\mathcal{K}^n$ into $\mathcal{K}^n$ which satisfies

$$Z(K\cup L)+Z(K\cap L)=ZK+ZL$$

whenever $K, L, K \cup L \in \mathcal{K}^n$ is called a Minkowski valuation (here we agree that $Z\emptyset = \{0\}$). We say that Z is translation invariant if $Z(K + x) = ZK$ for $x \in \mathbb{R}^n$ and Z is called homogeneous of degree $d \in \mathbb{R}$ if $Z(\lambda K) = \lambda^d ZK$ for $\lambda > 0$. If Z satisfies

$$Z(\phi K) = \phi^{-t} ZK, \qquad K \in \mathcal{K}^n,\ \phi \in SL(n)$$

we say that Z is $SL(n)$-contravariant. Continuity of maps of $\mathcal{K}^n$ into $\mathcal{K}^n$ means continuity with repect to the topology induced by the Hausdorff metric.

One can show that the projection body map Π is a continuous, translation invariant, homogeneous, $SL(n)$-contravariant Minkowski valuation, cf. [2]. It was shown by Ludwig in [5] (and under stronger assumptions on Z already in [4]) that Π is characterized by these properties.

Theorem 1 ([5]). *Let $Z : \mathcal{K}^n \to \mathcal{K}^n$ be a continuous, translation invariant, homogeneous, $SL(n)$-contravariant Minkowski valuation. Then there exists a number $c \geq 0$ such that*

$$ZK = c\Pi K$$

for $K \in \mathcal{K}^n$.

If one considers valuations Z defined only on the smaller subset $\mathcal{K}^n_o = \{K \in \mathcal{K}^n : 0 \in K\}$, one can omit the assumption of translation invariance in the previous theorem. However, translation invariance is essential in Theorem 1. To see this, consider the map $\Pi_o : \mathcal{K}^n \to \mathcal{K}^n$ defined by

$$\Pi_o K = \Pi(\mathrm{conv}(\{0\} \cup K)).$$

From the relations

$$\mathrm{conv}(\{0\} \cup (K \cup L)) = \mathrm{conv}(\{0\} \cup K) \cup \mathrm{conv}(\{0\} \cup L)$$

and

$$\mathrm{conv}(\{0\} \cup (K \cap L)) = \mathrm{conv}(\{0\} \cup K) \cap \mathrm{conv}(\{0\} \cup L) \qquad \text{if } K \cup L \in \mathcal{K}^n$$

it follows that Π_o is indeed a Minkowski valuation. As a consequence of the properties of Π, the map Π_o is continuous, homogeneous and $SL(n)$-contravariant. Obviously, Π_o is not translation invariant.

Surprisingly, Π and this new, not translation invariant valuation Π_o are enough to exhaust all examples of continuous, homogeneous, $SL(n)$-contravariant Minkowski valuations.

Theorem 2. *Let $Z : \mathcal{K}^n \to \mathcal{K}^n$ be a continuous, homogeneous, $SL(n)$-contravariant Minkowski valuation. Then there exist numbers $a_1, a_2 \geq 0$ such that*

$$ZK = a_1 \Pi K + a_2 \Pi_o K$$

for $K \in \mathcal{K}^n$.

The proof of this theorem uses ideas developed by Ludwig in [5] and a non-trivial result on generalized zonoids.

There are several important volume inequalities for projection bodies. One of them is the Petty projection inequality, which states that the affinely invariant functional $V_n(\Pi^*K)V_n(K)^{n-1}$ is maximized exactly by ellipsoids:

$$V_n(\Pi^*K)V_n(K)^{n-1} \leq V_n(\Pi^*B^n)V_n(B^n)^{n-1},$$

where we write Π^*K for $(\Pi K)^*$, $K^* = \{x : \langle x, y\rangle \leq 1 \text{ for } y \in K\}$ denotes the polar body of K and B^n is the Euclidean unit ball. Using Theorem 2 it is possible to generalize this inequality to continuous, homogeneous, $SL(n)$-contravariant Minkowski valuations.

Theorem 3. *Let $Z : \mathcal{K}^n \to \mathcal{K}^n$ be a non-trivial, continuous, homogeneous, $SL(n)$-contravariant Minkowski valuation. If $\dim K = n$ then*

$$V_n(Z^*K)V_n(K)^{n-1} \leq V_n(Z^*B^n)V_n(B^n)^{n-1}.$$

If Z is not a multiple of Π, there is equality if and only if K is an ellipsoid containing the origin, otherwise equality holds if and only if K is an ellipsoid.

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Valuations and Sobolev inequalities

Christoph Haberl

(joint work with Franz E. Schuster, Jie Xiao)

For $p \geq 1$ and $n \geq 2$, let $W^{1,p}(\mathbb{R}^n)$ denote the space of real-valued L^p functions on $\mathbb{R}^n$ with weak L^p partial derivatives. The classical Pólya–Szegö principle [9] states that the L^p norm of the gradient of a function on $\mathbb{R}^n$ does not increase under symmetric rearrangement. To be precise, if $f \in W^{1,p}(\mathbb{R}^n)$ for some $p \geq 1$, then $f^\star \in W^{1,p}(\mathbb{R}^n)$ and

$$\|\nabla f^\star\|_p \leq \|\nabla f\|_p.$$

Here, the L_p norm of the gradient of a function f is defined by

$$\|\nabla f\|_p = \left(\int_{\mathbb{R}^n} |\nabla f|^p \, dx\right)^{1/p},$$

where $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^n$. The *symmetric decreasing rearrangement* $f^\star$ of a function f is defined as follows. For $f \in W^{1,p}(\mathbb{R}^n)$, denote by $\mu_f : [0,\infty) \to [0,\infty]$ the distribution function of the absolute value of f. The decreasing rearrangement $f^* : [0,\infty) \to [0,\infty]$ of f is defined to be zero for $s \geq \mu_f(0)$ and

$$f^*(s) = \sup\{t > 0 : \ \mu_f(t) > s\} \qquad \text{for} \qquad s < \mu_f(0).$$

Now, the symmetric decreasing rearrangement $f^\star : \mathbb{R}^n \to [0,\infty]$ is given by

$$f^\star(x) = f^*(\kappa_n |x|^n),$$

where $\kappa_n = \pi^{n/2}/\Gamma(1 + \frac{n}{2})$ denotes the volume of the Euclidean unit ball in $\mathbb{R}^n$.

The geometric core of the Pólya–Szegö principle is the isoperimetric inequality. The Petty projection inequality [8] is the classical affine isoperimetric inequality which connects the volume of a convex body with that of its polar projection body. It is stronger than (i.e., directly implies) the classical isoperimetric inequality. Using this affine isoperimetric inequality as well as its symmetric L_p analog [6], Zhang [11], Lutwak, Yang, and Zhang [7] and Cianchi et al. [2] proved an affine version of the Pólya–Szegö principle: For every function $f \in W^{1,p}(\mathbb{R}^n)$

$$\mathcal{E}_p(f^\star) \leq \mathcal{E}_p(f).$$

Here, the L^p *affine energy* $\mathcal{E}_p(f)$ is defined by

$$\mathcal{E}_p(f) = c_{n,p} \left(\int_{S^{n-1}} \|\mathrm{D}_u f\|_p^{-n} \, du \right)^{-1/n}$$

where $c_{n,p} = (n\kappa_n)^{1/n} (\frac{n\kappa_n \kappa_{p-1}}{2\kappa_{n+p-2}})^{1/p}$ and $\mathrm{D}_u f$ is the directional derivative of f in direction u.

Recent characterizations of valuations by Ludwig [5] showed that for $p > 1$, the geometric operator behind the affine Pólya–Szegö principle is only one of a whole family of possible L_p analogs of the projection body operator. Using this insight, it is proved in [4], that for every function $f \in W^{1,p}(\mathbb{R}^n)$ the inequality

$$\mathcal{E}_p^+(f^\star) \leq \mathcal{E}_p^+(f)$$

holds. The *asymmetric* L^p *affine energy* $\mathcal{E}_p^+(f)$ of a function f is defined by

$$\mathcal{E}_p^+(f) = d_{n,p} \left(\int_{S^{n-1}} \|\mathrm{D}_u^+ f\|_p^{-n} \, du \right)^{-1/n}$$

where $d_{n,p} = 2^{1/p} c_{n,p}$ and $\mathrm{D}_u^+ f(x) = \max\{\mathrm{D}_u f(x), 0\}$ denotes the positive part of the directional derivative of f in direction u. Since

$$\|\nabla f^\star\|_p = \mathcal{E}_p(f^\star) = \mathcal{E}_p^+(f^\star) \leq \mathcal{E}_p^+(f) \leq \mathcal{E}_p(f) \leq \|\nabla f\|_p,$$

the asymmetric affine Pólya–Szegö principle strengthens and directly implies the symmetric affine Pólya–Szegö principle as well as the classical one.

The asymmetric affine Pólya–Szegö inequality gives rise to affine versions of several Sobolev inequalities. For example, an asymmetric affine version of the sharp L_p Sobolev inequality due to Aubin [1] and Talenti [10] is established in [3] (see also [4]).

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Valuations on manifolds

Semyon Alesker

Let X be a smooth manifold, $n = \dim X$. Let $\mathcal{P}(X)$ denote the family of compact submanifolds with corners of X.

Definition 1. *A smooth valuation on X is a functional*

$$\phi\colon \mathcal{P}(X) \to \mathbb{C}$$

of the following form: there exists a smooth measure μ on X and a smooth $n-1$-form ω on the spherical cotangent bundle $\mathbb{P}_+(T^*X)$ such that for any $P \in \mathcal{P}(X)$ one has

$$\phi(P) = \mu(P) + \int_{N(P)} \omega$$

where $N(P) \subset \mathbb{P}_+(T^*X)$ is the normal cycle of P.

Example 2. *(1) Any smooth measure on X is a smooth valuation (indeed take $\omega = 0$).*

(2) The Euler characteristic χ is a smooth valuation. (This is less trivial and follows from a version of the Gauss-Bonnet theorem of Chern [9]*.)*

We denote by $V^\infty(X)$ the space of smooth valuations. Naturally $V^\infty(X)$ is a Fréchet space. Let $V_c^\infty(X)$ denote the subspace of compactly supported valuations. We have the *integration functional*

$$\int : V_c^\infty(X) \to \mathbb{C}$$

given by $[\phi \mapsto \phi(X)]$.

A rather non-trivial and important structure is the *product* on valuations. There exists a continuous bilinear map

$$V^\infty(X) \times V^\infty(X) \to V^\infty(X)$$

which makes $V^\infty(X)$ a commutative associative algebra with the unit (= the Euler characteristic χ); see [1], [2], [8], [7].

The product satisfies a version of the Poincaré duality: the bilinear form $V^\infty(X) \times V_c^\infty(X) \to \mathbb{C}$ given by $(\phi, \psi) \mapsto \int \phi \cdot \psi$ is a perfect pairing. In other words the induced map $V^\infty(X) \to (V_c^\infty(X))^*$ is injective and has a dense image in the weak topology.

We denote $V^{-\infty}(X) := (V_c^\infty(X))^*$ and call it the space of generalized valuations. Thus $V^\infty(X) \subset V^{-\infty}(X)$ is dense.

Let us denote by $\mathcal{F}(X)$ the space of *constructible* functions, e.g. finite linear combinations with complex coefficients of the indicator functions of compact submanifolds with corners. The natural linear map $\mathcal{F}(X) \to V^{-\infty}(X)$ given by $\sum_i \lambda_i \mathbb{1}_{P_i} \mapsto [\phi \mapsto \sum_i \lambda_i \phi(P_i)]$ is injective and has a dense image. Thus

$$V^\infty(X) \subset V^{-\infty}(X) \supset \mathcal{F}(X).$$

Now let us explain the meaning of the product on valuations. Recently the speaker and Bernig [7] have defined a partial product on generalized valuations $V^{-\infty}(X)$ which extends, on one hand, the above mention product on smooth valuations, and restricts, on the other hand, to the pointwise product of constructible functions. (We omit the relevant precise but technical statement.)

Let us discuss the push-forward on valuations. We discuss a general idea which has been made rigorous in some special cases under technical assumptions in [6]. Let $f : X \to Y$ be a smooth proper map. The push-forward map $f_* : V(X) \to V(Y)$ (here we omit the superscript indicating what regularity we require since it may vary from one situation to another) is "defined" by

$$(f_*\phi)(P) = \phi(f^{-1}(P)).$$

Once this definition is made rigorous, the push-forward of smooth measures considered as smooth valuations coincides with the usual push-forward on measures. Moreover the push-forward on constructible functions considered as generalized valuations should coincide with the well known operation of integration with respect to the Euler characteristic along the fibers.

The pull-back operation $f^* : V(Y) \to V(X)$ should be considered as the dual map to f_*. If we restrict f^* to constructible functions, we get the usual pull-back on functions. Let us restrict f^* to measures. Consider the special case when

$f\colon \mathbb{R}^n \to \mathbb{R}^k$ is the orthogonal projection ($k < n$). Consider the pull-back of the Lebesgue measure $f^*(vol_k)$. It is a well known valuation such that its value on any convex compact set $K \subset \mathbb{R}^n$ is $(f^*vol_k)(K) = vol_k(f(K))$, i.e. the volume of the projection.

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Valuations on L_p spaces

Andy Tsang

If $(X, \mathfrak{F}, \mu)$ is a measure space then the L^p-space, $L^p(\mu)$ where $1 \le p < \infty$, is the collection of μ-measurable functions $f : X \to [-\infty, \infty]$ that satisfies

$$\int_X |f|^p \, d\mu < \infty.$$

For an $f \in L^p(\mu)$, the L^p-norm of f denoted by $\|f\|_p$, is defined as

$$\|f\|_p = \Big(\int_X |f|^p \, d\mu \Big)^{1/p}.$$

The functional $\| \cdot \|_p : L^p(\mu) \to \mathbb{R}$ is a semi-norm. If functions in $L^p(\mu)$ that are equal almost everywhere with respect to μ (a.e. $[\mu]$) are identified, then $\| \cdot \|_p : L^p(\mu) \to \mathbb{R}$ becomes a norm and $L^p(\mu)$ becomes a normed linear space. From now on $L^p(\mu)$ will be the identified space. One can easily check that $L^p(\mu)$ is a lattice of functions, that is $f \vee g, f \wedge g \in L^p(\mu)$ whenever $f, g \in L^p(\mu)$, where $f \vee g = \max\{f, g\}$ and $f \wedge g = \min\{f, g\}$. A function $\Phi : L^p(\mu) \to \mathbb{R}$ is called a valuation if $\Phi(0) = 0$ and

$$\Phi(f \vee g) + \Phi(f \wedge g) = \Phi(f) + \Phi(g),$$

for all $f, g \in L^p(\mu)$. The following theorem provides a large class of continuous valuations on the lattice $L^p(\mu)$.

Theorem 1.1 [1] *If $h : \mathbb{R} \to \mathbb{R}$ is a continuous function with the properties that $h(0) = 0$ and there exist real numbers $\gamma, \delta \geq 0$ such that $|h(\alpha)| \leq \gamma|\alpha|^p + \delta$ for all $\alpha \in \mathbb{R}$, then the functional $\Phi : L^p(\mu) \to \mathbb{R}$ defined by*

$$\Phi(f) = \int_X h \circ f \, d\mu$$

for $f \in L^p(\mu)$, is a continuous valuation provided that $\delta = 0$ if $\mu(X) = \infty$.

A measure space $(X, \mathfrak{F}, \mu)$ is called non-atomic if for every $E \in \mathfrak{F}$ with $\mu(E) > 0$, there exists $F \in \mathfrak{F}$ with $F \subseteq E$ and $0 < \mu(F) < \mu(E)$. The following is an integral representation theorem for a certain class of continuous valuations on L^p-spaces.

Theorem 1.2 [1] *Let $(X, \mathfrak{F}, \mu)$ be a non-atomic measure space and let $\Phi : L^p(\mu) \to \mathbb{R}$ be a continuous valuation. If there exists a continuous function $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ such that $\Phi(\alpha\chi_E) = h(\alpha)\mu(E)$ for all $\alpha \in \mathbb{R}$ and all $E \in \mathfrak{F}$ with $\mu(E) < \infty$ then there exist real numbers $\gamma, \delta \geq 0$ such that $|h(\alpha)| \leq \gamma|\alpha|^p + \delta$ for all $\alpha \in \mathbb{R}$ and*

$$\Phi(f) = \int_X h \circ f \, d\mu$$

for all $f \in L^p(\mu)$. In addition, if $\mu(X) = \infty$ then $\delta = 0$.

Theorem 1.1 and Theorem 1.2 give two characterization theorems. But first some notations and definitions. If $X = \mathbb{R}^n$, $\mathfrak{F} = \mathfrak{M}$ where $\mathfrak{M}$ is the collection of Lebesgue measurable sets in $\mathbb{R}^n$ and $\mu = m$ where m is Lebesgue measure then we usually write $L^p(\mathbb{R}^n)$ instead of $L^p(m)$. Also, if $f \in L^p(\mathbb{R}^n)$, it is customary to write $\int_{\mathbb{R}^n} f(x)\, dx$ in place of $\int_{\mathbb{R}^n} f\, dm$. For simplicity, we will always write measurable functions instead of m-measurable functions. Denote by S^{n-1} the unit sphere in $\mathbb{R}^n$ and denote $\mathfrak{S}$ to be the σ-algebra defined as

$$\mathfrak{S} = \{E : \, E \subseteq S^{n-1}, \, \{\lambda x : \, x \in E, \, 0 \leq \lambda \leq 1\} \in \mathfrak{M}\}.$$

Also denote by σ the Lebesgue spherical measure. We will use the notation $L^p(S^{n-1})$ for $L^p(\sigma)$. Also, if $f \in L^p(S^{n-1})$, it is customary to write $\int_{S^{n-1}} f(u)\, du$ in place of $\int_{S^{n-1}} f\, d\sigma$. The collection of translations on $\mathbb{R}^n$ will be denoted by $T(n)$. If $\tau \in T(n)$ and $f \in L^p(\mathbb{R}^n)$ then τf is defined as $\tau f = f \circ \tau^{-1}$. It should be noted that $\tau f \in L^p(\mathbb{R}^n)$. A valuation $\Phi : L^p(\mathbb{R}^n) \to \mathbb{R}$ is called translation invariant if $\Phi(\tau f) = \Phi(f)$ for every $f \in L^p(\mathbb{R}^n)$ and every $\tau \in T(n)$. The collection of (proper) rotations on $\mathbb{R}^n$ will have the standard notation $SO(n)$. If $\theta \in SO(n)$ and $f \in L^p(S^{n-1})$ then θf is defined as $\theta f = f \circ \theta^{-1}$. It should be noted that $\theta f \in L^p(S^{n-1})$. A valuation $\Phi : L^p(S^{n-1}) \to \mathbb{R}$ is called rotation invariant if $\Phi(\theta f) = \Phi(f)$ for every $f \in L^p(S^{n-1})$ and every $\theta \in SO(n)$. Now we are ready to state the two characterization theorems.

Theorem 1.3 [1] *A functional* $\Phi : L^p(\mathbb{R}^n) \to \mathbb{R}$ *is a continuous translation invariant valuation if and only if there exists a continuous function* $h : \mathbb{R} \to \mathbb{R}$ *with the property that there exists a real number* $\gamma \geq 0$ *such that* $|h(\alpha)| \leq \gamma|\alpha|^p$ *for all* $\alpha \in \mathbb{R}$ *and*

$$\Phi(f) = \int_{\mathbb{R}^n} (h \circ f)(x)\, dx$$

for all $f \in L^p(\mathbb{R}^n)$.

Theorem 1.4 [1] *A functional* $\Phi : L^p(S^{n-1}) \to \mathbb{R}$ *is a continuous rotation invariant valuation if and only if there exists a continuous function* $h : \mathbb{R} \to \mathbb{R}$ *with the properties that* $h(0) = 0$ *and there exist real numbers* $\gamma, \delta \geq 0$ *such that* $|h(\alpha)| \leq \gamma|\alpha|^p + \delta$ *for all* $\alpha \in \mathbb{R}$ *and*

$$\Phi(f) = \int_{S^{n-1}} (h \circ f)(u)\, du$$

for all $f \in L^p(S^{n-1})$.

Perhaps one can find more applications of Theorems 1.1 and 1.2.

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Integral geometry in complex space forms

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(joint work with Eduardo Gallego, Gil Solanes)

One of the classical problems in integral geometry consists on expressing the measure of the set of planes intersecting a convex domain in terms of the geometry of the convex set, namely in terms of curvature integrals. These expressions are known as Crofton formulas.

In the space of constant sectional curvature k, $\mathbb{M}^n(k)$, Santaló [9, p. 310] found these expressions in terms of the mean curvature integrals. That is, if $\mathcal{L}_r$ denotes the space of r-dimensional geodesic planes in $\mathbb{M}^n(k)$, dL_r is a measure on $\mathcal{L}_r$ invariant under the isometry group of $\mathbb{M}^n(k)$, $\Omega \subset \mathbb{M}^n(k)$ is a compact domain with smooth boundary, and $r = 2l$, then

$$\int_{\mathcal{L}_{2l}} \chi(\Omega \cap L_{2l}) dL_{2l} = c_0 \mathrm{vol}(\Omega) + \sum_{i=1}^{l} c_i k^{l-i} M_{2i-1}(\partial\Omega), \tag{1}$$

where c_i are known coefficients depending only on n, r and i, while $M_j(\partial\Omega)$ denotes the j-th mean curvature integral. An analogous formula holds in the case of odd-dimensional planes.

The main goal of this work is to extend expression (1) to complex space forms. Complex space forms - denoted by $\mathbb{CK}^n(\epsilon)$ - are simply connected Kähler manifolds with constant holomorphic curvature 4ϵ. (In a Kähler manifold, the holomorphic curvature is the sectional curvature of a complex line.) A complex space form is isometric to the standard Hermitian space, $\mathbb{C}^n$, if $\epsilon = 0$, to a complex projective space, $\mathbb{CP}^n$, if $\epsilon > 0$ or to a complex hyperbolic space, $\mathbb{CH}^n$, if $\epsilon < 0$. For a more extended description of these spaces see, for instance, [6] and [7].

In the standard Hermitian space $\mathbb{C}^n$, some developments in the field of integral geometry have been done recently. Alesker [3] computed the dimension of the space of continuons translation invariant valuations, invariant also under the grup $U(n)$, and also gave two bases of valuations. In [5], Fu obtained the algebraic structure of this space, and together with Bernig [4], they obtained an explicit formula of the principal kinematic formula in terms of some new basis of valuations. From this result, one can obtain the Crofton formulas in $\mathbb{C}^n$, i.e. an expression for

$$\int_{\mathcal{L}_{k,q}} \chi(\Omega \cap L_{k,q}) dL_{k,q},$$

where $\mathcal{L}_{k,q}$ denotes the space of totally geodesic planes $L_{k,q} \cong \mathbb{C}^q \oplus \mathbb{R}^{k-2q}$, in terms of one of the introduced basis. The elements of this basis were called *hermitian intrinsic volumes* $\{\mu_{k,p}\}_{\max\{0,k-n\} \leq p \leq k/2 \leq n/2}$. They can be represented as curvature integrals of the boundary of the convex domain, and satisfy $\mathrm{Kl}_{\mu_{k,p}}(L_{k',p'}) = \delta_{k,q}^{k',q'}$ where Kl is the Klain function (see [4] for definitions).

In our work, we obtain in any complex space form the expression for

$$\phi_r(\Omega) := \int_{\mathcal{L}_r^{\mathbb{C}}} \chi(\Omega \cap L_r) dL_r, \tag{2}$$

in terms of the hermitian intrinsic volumes, which can be generalized to any complex space form, as in [8]. In the case $\epsilon = 0$ we get the result in [4] but in a different way, leading to simplified expressions of the same coefficients. In the complex projective and hyperbolic space, it makes also sense to study the integrals over the spaces of totally real totally geodesic submanifolds. This is done in [1], and this completes the study of the Crofton formulas in complex space forms.

The obtained expression is (cf. [2])

$$\phi_r = \sum_{k=n-r}^{n} \epsilon^{k-(n-r)} \frac{\omega_{2n-2k}}{\binom{n}{k}\omega_{2r}} \left(\sum_{q=\max\{0,2k-n\}}^{k-1} \frac{\binom{2k-2q}{k-q}}{4^{k-q}} \mu_{2k,q} + (k+r-n+1)\mu_{2k,k} \right)$$

where $\mu_{k,q}$ denote the hermitian intrinsic volumes.

These coefficients, as Fu pointed out, correspond to the Maclaurin expansion of $1/\sqrt{1-x}$. A geometric interpretation of this fact is not known.

In order to prove the result we obtain the first variation formula for both the integral (2), using a similar method as in [10], and the hermitian intrinsic volumes, extending a result in [4] for $\mathbb{C}^n$ to the other complex space forms.

On the other hand, using the same variational method, we can express the Euler characteristic of a domain in $\mathbb{CP}^n$ or $\mathbb{CH}^n$ in terms of the hermitian intrinsic volumes, obtaining an extrinsic expression in terms of the curvatures of the boundary (cf. [2]).

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Metrizability of Path Geometries

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(joint work with Juan Carlos Álvarez)

Our goal is to understand how to construct Finsler surfaces with prescribed geodesics. This question was addressed by Busemann [5, 6, 7], Ambartzumian [2], Alexander [1], Matsumoto [8], and Arcostanzo [3].

We stress that we work with the classical definition of (reversible) Finsler metrics:

Definition. A Finsler metric on a manifold M is a continuous function $F : TM \to [0\,..\,\infty)$ that is homogeneous of degree one, smooth outside the zero section and satisfies the *quadratic convexity* condition: at every tangent space T_xM the Hessian of $F^2(x,\cdot)$ (computed using any affine coordinates on T_xM) at any nonzero tangent vector is a positive-definite quadratic form.

Indeed, in studying inverse problems in Finsler geometry our predecessors have either weakened the condition of quadratic convexity to the condition that the restriction of F to each tangent space be a norm ([3]), considered wider generalizations of Finsler metrics such as G-spaces ([5, 6, 7]), or considered integrands that are homogeneous of degree one and not necessarily defined in all tangent directions ([8]). If we insist on the quadratic convexity condition it is because without it we cannot speak of the geodesic flow of the Finsler metric nor define important invariants such as the flag curvature (see [4]).

Consider a manifold M together with a family of hypersurfaces such that through every point $x \in M$ and every tangent hyperplane ζ_x in T_xM there is a unique hypersurface passing through x tangent to ζ_x. Let us assume that this family is parameterized by a smooth manifold Γ and that the projection that sends a tangent hyperplane ζ_x to the unique hypersurface that is tangent to it is a submersion from the space PT^*M of contact elements on M to the parameter manifold Γ.

Theorem A. If μ is a smooth positive measure on Γ, there is a (unique) Finsler metric $F : TM \to [0\,..\,\infty)$ such that for any piecewise smooth curve c on M we have the Crofton-type formula

$$\int F(\dot{c}(t))\, dt = \int_{\gamma\in\Gamma} \#(\gamma \cap c)\, d\mu(\gamma). \tag{1}$$

In general, the authors do not know what is the precise relationship between the family of hypersurfaces and the Finsler metrics associated to it. However, in two dimensions the relationship is simple enough:

Theorem B. If M is a two-dimensional manifold, the Finsler metrics defined by Eq. (1) are precisely those whose geodesics are the curves of the family parameterized by Γ.

In particular, *any path geometry in a two-dimensional manifold is locally the system of geodesics of a Finsler metric.*

An interesting remark is that while the construction in Theorem A extends to families of cooriented hypersurfaces such as the family of horospheres in hyperbolic space, Theorem B does not. Indeed, if we allow different curves of Γ to be tangent on the condition that their coorientations at the point of tangency be different, then not only do those curves fail to be geodesics of the Finsler metric defined by Eq. (1), but metrics associated to different measures may have different unparameterized geodesics. We demonstrate this by taking Γ to be the family of horocycles in the hyperbolic plane and using Eq. (1) to construct two Riemannian metrics — the hyperbolic metric and a metric conformal to it — with different sets of unparameterized geodesics.

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Integral geometry of tensor valuations

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(joint work with Rolf Schneider, Ralph Schuster)

Valuations and integral geometric results for real valued functionals have been studied extensively in the literature. The present purpose is to present a survey of recent integral geometric results for tensor valued valuations. Here the situation turns out to be substantially more complicated than in the real valued case. The use of algebraic methods in this context may shed some new light on the results obtained so far and hopefully leads to new results in the future.

We write $\mathbb{T}^p$ for the vector space of symmetric tensors of rank p over $\mathbb{R}^n$, $p \in \mathbb{N}_0$, $ab = a \odot b$ for the symmetric tensor product of tensors a and b, and Q for the metric tensor, that is $Q(x, y) = \langle x, y\rangle$ for $x, y \in \mathbb{R}^n$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product. We write B^n for the Euclidean unit ball, κ_n for its volume and $\mathbb{S}^{n-1}$ for its boundary.

1. The classical case, $p = 0$

For any nonnegative $\varepsilon \geq 0$, we have the Steiner formula

$$V_n(K + \varepsilon B^n) = \sum_{j=0}^{n} \varepsilon^{n-j} \kappa_{n-j} V_j(K).$$

The functionals V_j on the space $\mathcal{K}^n$ of convex bodies (nonempty compact convex sets) are called intrinsic volumes or Minkowski functionals. They are isometry invariant, continuous valuations and V_j is homogeneous of degree j. By Hadwiger's classical characterization theorem, all functionals which are isometry invariant, continuous valuations are linear combinations of the intrinsic volumes. As a consequence of Hadwiger's theorem, one can deduce the Crofton formula (CF)

$$\int_{\mathbf{A}(n,k)} V_j(K \cap E)\, \mu_k(\mathrm{d}E) = a_{njk}\, V_{n+j-k}(K)$$

and the principal kinematic formula (PKF)

$$\int_{\mathbf{G}_n} V_j(K \cap gM)\, \mu(\mathrm{d}g) = \sum_{k=j}^{n} a_{njk} V_{n+j-k}(K) V_k(M),$$

where $K, M \in \mathcal{K}^n$, $\mathbf{A}(n,k)$ is the set of k-flats in $\mathbb{R}^n$, $\mathbf{G}_n$ is the motion group and μ_k and μ are suitably normalized Haar measures. The constants a_{njk} are well known and easy to determine. A surprisingly general result is Hadwiger's general integral geometric (GIG) theorem which states that for any continuous valuation φ on the space of convex bodies, we have

$$\int_{\mathbf{G}_n} \varphi(K \cap gM)\,\mu(\mathrm{d}g) = \sum_{k=0}^{n} \int_{\mathbf{A}(n,k)} \varphi(K \cap E)\,\mu_k(\mathrm{d}E) V_k(M).$$

2. Vector valued valuations, $p = 1$

As a starting point for the investigation of vector valued valuations one can replace the volume functional by the moment vector

$$M_n(K) := \int_K x\,\mathrm{d}x.$$

The Steiner formula for the moment vector

$$M_n(K + \varepsilon B^n) = \sum_{j=0}^{n} \varepsilon^{n-j} \kappa_{n-j} M_j(K)$$

leads to vector valued functionals $M_j : \mathcal{K}^n \to \mathbb{R}^n$, which are isometry covariant, continuous valuations. A characterization theorem due to Hadwiger and Schneider states that any isometry covariant, continuous vector valued valuation is a linear combination of $M_0, \dots, M_n$. For M_j there exist (CF) and (PKF). An efficient approach to these results uses curvature measures $\Phi_j(K, \cdot)$ of a convex body K, which are Borel measures on $\mathbb{R}^n$ and image measures of the support measures $\Lambda_j(K, \cdot)$ under the projection map $\mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}^n$, $(x,u) \mapsto x$. Local versions of (CF) and (PKF) then are

$$\int_{\mathbf{A}(n,k)} \int_E f(x)\,\Phi_j(K \cap E, \mathrm{d}x)\,\mu_k(\mathrm{d}E) = a_{njk} \int_{\mathbb{R}^n} f(x)\,\Phi_{n+j-k}(K, \mathrm{d}x),$$

$$\int_{\mathbf{G}_n} \int f(x)\,\Phi_j(K \cap gM, \mathrm{d}x)\,\mu(\mathrm{d}g) = \sum_{k=j}^{n} a_{njk} \int f(x)\,\Phi_{n+j-k}(K, \mathrm{d}x) V_k(M).$$

The choice $f(x) = x$ yields the (CF) and the (PKF) for vector valuations.

3. General tensor valuations

Next we consider the moment tensor

$$\Psi_r(K) := \frac{1}{r!} \int_K x^r\,\mathrm{d}x.$$

We obtain

$$\Psi_r(K + \varepsilon B^n) = \sum_{j=0}^{n+r} \varepsilon^{n+r-j} \kappa_{n+r-j} V_j^{(r)}(K),$$

where
$$V_j^{(r)}(K) = \sum_s \Phi_{j-r+s,r-s,s}(K)$$
with
$$\Phi_{k,r,s}(K) := \frac{\omega_{n-k}}{r!s!\omega_{n-k+s}} \int x^r u^s \, \Lambda_k(K, \mathrm{d}(x,u)),$$
for $k \in \{0, \ldots, n-1\}$ and $r, s \in \mathbb{N}_0$, and
$$\Phi_{n,r,0}(K) := \frac{1}{r!} \int x^r \, \Lambda_n(K, \mathrm{d}x),$$
where $\Lambda_n(K, \cdot)$ is the restriction of Lebesgue measure to K.

Theorem 3.1 (Alesker)**.** Let $p \in \mathbb{N}_0$ and $\varphi : \mathcal{K}^n \to \mathbb{T}^p$ be a continuous isometry covariant tensor valuation. Then φ is a linear combination of the basic tensor functionals $Q^l \Phi_{j,r,s}$ with $p = 2l + r + s$.

The basic tensor valuations are not linearly independent, since
$$2\pi \sum_s s\Phi_{j-r+s,r-s,s} = Q \sum_s \Phi_{j-r+s,r-s,s-2},$$
which is due to McMullen. The following theorem is from [1].

Theorem 3.2. Every linear relation among the basic tensor valuations is obtained by multiplying McMullen's relations by Q^l, for some $l \in \mathbb{N}_0$, and by taking linear combinations of relations obtained in this way.

4. Integral geometric results

It is of considerable interest to obtain integral geometric formulas for the basic tensor valuations. In various special cases, the formulas still take a simple form. This is the case for tensor valuations of rank 2. Another example follows from translative integral formulas for support measures, which yield the (PKF)
$$\int_{\mathbf{G}_n} \Phi_{n-1,r,s}(K \cap gM) \, \mu(\mathrm{d}g) = \Phi_{n-1,r,s}(K)V(M) + \delta(n,s)Q^{\frac{s}{2}} \Phi_{n,r,0}(K)V_{n-1}(M),$$
where $\delta(n,s)$ is explicitly known and zero iff s is odd. The formula
$$\int_{\mathbf{A}(n,n-1)} \Phi_{n-1,r,s}(K \cap E) \, \mu_{n-1}(\mathrm{d}E) = \delta(n,s)Q^{\frac{s}{2}} \Phi_{n,r,0}(K)$$
can be obtained as a consequence of the preceding (PKF)and Hadwiger's (GIG) theorem. As a third example, we mention
$$\int_{\mathbf{A}(n,n-2)} \Phi_{n-2,r,s}(K \cap E) \, \mu_{n-2}(\mathrm{d}E) = \alpha_{n,n-2,s} \, Q^{\frac{s}{2}} \, \Phi_{n,r,0}(K),$$
where $\alpha_{n,n-2,s}$ is explicitly known and zero if s is odd, and
$$\int_{\mathbf{A}(n,n-1)} \Phi_{n-2,r,s}(K \cap E) \, \mu_{n-1}(\mathrm{d}E) = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \alpha(n,s,m) \, Q^m \, \Phi_{n-1,r,s-2m}(K),$$

where $\alpha(n,s,m)$ is explicitly known. All these results are special cases of a general Crofton formula stated in [2]. The approach there is essentially based on translative integral geometry and presently requires methods of geometric measure theory. It is desirable to simplify the approach to and the statement of these results.

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The intersection of two real forms in Hermitian symmetric spaces of compact type

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(joint work with Makiko Sumi Tanaka)

This is based on [12] and a joint paper [11] with Tanaka.

The 1-dimensional Hermitian symmetric space of compact type is the complex projecive line $\mathbb{C}P^1$. If we regard $\mathbb{C}P^1$ as the 2-dimensional sphere, then its real form is a great circle. Two different great circles intersect at just two points and their intersection is always a pair of antipodal points. The purpose of this talk is to generalize this phenomenon to the intersection of two real forms in any Hermitian symmetric space of compact type. This study has not yet reached to any result of integral geometry, however I think an exact information on the intersection of fundamental submanifolds is important for formulation of several integral formulas in integral geometry. This is one of my motivations of this study.

1. Main results

Let $\bar{M}$ be a Hermitian symmetric space. A submanifold M is called a *real form* of $\bar{M}$, if there exists an involutive anti-holomorphic isometry σ of $\bar{M}$ satisfying

$$M = \{x \in \bar{M} \mid \sigma(x) = x\}.$$

Any real form M is a totally geodesic Lagrangian submanifold of $\bar{M}$. Leung [4] and Takeuchi [9] classified real forms of Hermitian symmetric spaces of compact type.

A subset S in a Riemannian symmetric space M is called an *antipodal set*, if $s_x y = y$ for any points x and y in S, where s_x is the geodesic symmetry with respect to x. The 2-*number* $\#_2 M$ of M is the supremum of the cardinalities of antipodal sets of M. We call an antipodal set in M *great* if its cardinality attains $\#_2 M$. These were introduced by Chen and Nagano [2]. Takeuchi [10] proved

$$\#_2 M = \dim H_*(M, \mathbb{Z}_2)$$

for any symmetric R-space M, where $H_*(M, \mathbb{Z}_2)$ is the homology group of M with coefficient $\mathbb{Z}_2$. A compact Riemannian symmetric space is called a *symmetric R-space*, if its maximal torus has an orthonormal basis of the lattice for a suitable invariant metric. He also showed that any real form of Hermitian symmetric spaces of compact type is a symmetric R-space in [9].

Now se can state our main results.

Theorem 1.1 ([11])**.** Let M be a Hermitian symmetric space of compact type. If two real forms L_1 and L_2 of M transversally intersect, then $L_1 \cap L_2$ is an antipodal set of L_1 and L_2.

Two submanifolds in a Hermitian symmetric space are *congruent*, if one is transformed to another by a holomorphic isometry.

Theorem 1.2 ([11])**.** Let M be a Hermitian symmetric space of compact type and let L_1 and L_2 be two real forms of M which are congruent and transversally intersect. Then $L_1 \cap L_2$ is a great antipodal set of L_1 and L_2. That is, $\#(L_1 \cap L_2) = \#_2 L_1 = \#_2 L_2$.

Theorem 1.3 ([11])**.** Let M be an irreducible Hermitian symmetric space of compact type and let L_1 and L_2 be two real forms of M which transversally intersect.

(1) If $M = G_{2m}^{\mathbb{C}}(\mathbb{C}^{4m})$ $(m \geq 2)$, L_1 is congruent to $G_m^{\mathbb{H}}(\mathbb{H}^{2m})$ and L_2 is congruent to $U(2m)$, then
$$\#(L_1 \cap L_2) = 2^m < \binom{2m}{m} = \#_2 L_1 < 2^{2m} = \#_2 L_2.$$
(2) Otherwise, $L_1 \cap L_2$ is a great antipodal set of one of L_i's whose 2-number is less than or equal to another and we have
$$\#(L_1 \cap L_2) = \min\{\#_2 L_1, \#_2 L_2\}.$$

We call a Lagrangian submanifold L of a Hermitian symmetric space M *globally tight*, if L satisfies
$$\#(L \cap g \cdot L) = \dim H_*(L, \mathbb{Z}_2)$$
for any holomorphic isometry g of M with the property that L transversally intersects with $g \cdot L$ (Oh [6]). We obtain the following corollary from Theorem 1.2.

Corollary 1.4. Any real form of a Hermitian symmetric space of compact type is a globally tight Lagrangian submanifold.

2. Outline of the proofs

We need the following lemma to start the proofs.

Lemma 2.1 ([12])**.** Let M be a compact Kähler manifold with positive holomorphic sectional curvature. If L_1 and L_2 are totally geodesic compact Lagrangian submanifolds in M, then $L_1 \cap L_2 \neq \emptyset$.

The proof of this lemma is similar to that of a result of Frankel [3] concerning the intersection of two totally geodesic submanifolds in a Riemannian manifold with positive sectional curvature.

Since Hermitian symmetric spaces of compact type have positive holomorphic sectional curvature, we can apply Lemma 2.1 to real forms of Hermitian symmetric spaces of compact type. Hence two real forms of them always intersect.

According to a result by Takeuchi [8] on maximal tori of compact symmetric spaces and a result by Sakai [7] on cut loci of compact symmetric spaces, we can prove Theorem 1.1.

Let M be a compact connected Riemannian symmetric space. We decompose the fixed point set $F(s_o, M)$ of the geodesic symmetry s_o at the origin o to the disjoint union of its connected components:

$$F(s_o, M) = \bigcup_{j=0}^{r} M_j^+.$$

We call each connected component M_j^+ a *polar* of M. The notion of polar was introduced and investigated by Chen and Nagano [1], [5].

If M is a Hermitian symmetric space of compact type, then each polar M^+ is also a Hermitian symmetric space of compact type. If L is a real form through o and if $L \cap M^+$ is not empty, then the intersection $L \cap M^+$ is a real form of M^+.

We assume that M, L_1, L_2 are manifolds stated in Theorem 1.2 or 1.3. We can suppose $o \in L_1 \cap L_2$. By Theorem 1.1, $L_1 \cap L_2$ is an antipodal set of L_1 and L_2, so $L_1 \cap L_2$ is an antipodal set of M, too. Hence $L_1 \cap L_2 \subset F(s_o, M)$. Therefore we have the equality:

$$L_1 \cap L_2 = \bigcup_{j=0}^{r} \{(L_1 \cap M_j^+) \cap (L_2 \cap M_j^+)\}.$$

The intersection of two real forms in M is reduced to that of two real forms in M_j^+. We can prove Theorem 1.2 by induction of the polars. In order to prove Theorem 1.3 we use the classification of irreducible Hermitian symmetric spaces of compact type and their real forms.

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Integral geometry under the Möbius group

Gil Solanes
(joint work with Jun O'Hara)

The aim of this talk is to show that integral geometry can be developed, to some extent, in homogeneous spaces with a non-compact isotropy group. Concretely, we study the case of the Möbius group acting on the extended plane $\mathbb{R}^2 \cup \{\infty\} \equiv \mathbb{S}^2$.

To begin with, we consider the space Γ of circles in the plane. This space admits a natural measure $d\gamma$ which is invariant under the Möbius group. Unfortunately, the measure of the set of circles intersecting a given curve is infinite. Langevin and O'Hara solved this problem by restricting the integration to circles intersecting more than twice. Namely, for a smooth curve $C \subset \mathbb{R}^2$ one has (cf.[1])

$$\int_\Gamma \binom{\#(\gamma \cap C)/2}{2} d\gamma = c \int_{C\times C} (\theta \cos\theta - \sin\theta) \frac{dxdy}{\|y-x\|^2}.$$

where the constant c depends only on the normalization of $d\gamma$, and θ is the angle between the two circles through x, y tangent to C at x and y respectively. We notice that $dxdy/\|y-x\|^2$ is the modulus of the following Möbius invariant complex valued form ω called the *infinitesimal cross-ratio* (cf.[1])

$$\omega = (x, x+dx; y, y+dy) = \frac{(dx_1 + idx_2) \wedge (dy_1 + idy_2)}{(y-x)^2}.$$

In a recent work with Jun O'Hara we considered the case of 0-dimensional spheres. Indeed, the space of point pairs admits the following Möbius invariant measure: $dP_{(w,z)} = dwdz/\|w-z\|^4$, where dw, dz denote area elements. Given $\Omega \subset \mathbb{R}^2$ with nonempty interior, the integral of dP over $\Omega \times \Omega$ is divergent. In fact, for Ω compact with differentiable boundary $C = \partial\Omega$, one has the following Laurent expansion

$$\int_{\Omega\times\Omega\setminus\Delta_\epsilon} dP = \frac{\pi A(\Omega)}{\epsilon^2} - \frac{2L(C)}{\epsilon} + E(\Omega) + O(\epsilon)$$

where $\Delta_\epsilon = \{(w,z) \in \Omega\times\Omega \mid \|w-z\| < \epsilon\}$, and A, L denote the area, and the length respectively. We call the degree zero term $E(\Omega)$ the *renormalized energy.* Using the fact that $2dP = \Re e\omega \wedge \Re e\omega$ one can show that

$$E(\Omega) = -\int_{C\times C} \sin\theta_x \sin\theta_y \frac{dxdy}{\|y-x\|^2},$$

where θ_x, θ_y are the oriented angles between the vector $y-x$ and the positve tangent vector of C at x, y respectively.

It is also easy to see that

$$E(\Omega) = \lim_{\epsilon\to 0}\left(\frac{2L(C)}{\epsilon} - \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}^2}\lambda^2(w,z)dP\right) \tag{1}$$

where λ is the indicator function of $\Omega\times\Omega\cup\Omega^c\times\Omega^c$.

By the invariance of dP, it is natural to expect the renormalized energy to be Möbius invariant. In order to prove this we consider $\mathbb{R}^2$ as the ideal boundary $\partial_\infty\mathbb{H}^3$ of Poincaré model of hyperbolic space $\mathbb{H}^3$. Given $\Omega\subset\mathbb{R}^2$ as above, let $S\subset\mathbb{H}^3$ be a complete surface orthogonal to $\partial_\infty\mathbb{H}^3$ along C. Assume further that $S\cup C$ is a smooth surface with boundary. Such an S was said to have *cone-like ends* in [2]. Considering $S_\epsilon = S\cap\{(x,y,z)|z>\epsilon\}$, we have the following Gauss-Bonnet theorem for the integral of the extrinsic curvature K of S:

$$\begin{aligned}\infty > \int_S KdS &= 2\pi\chi(S) + \lim_{\epsilon\to 0}\left(\mathcal{A}(S_\epsilon) - \frac{L(C)}{\epsilon}\right)\\ &= 2\pi\chi(S) + \lim_{\epsilon\to 0}\left(\frac{2}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\#(\ell\cap S_\epsilon)dP - \frac{L(C)}{\epsilon}\right)\\ &= 2\pi\chi(S) + \lim_{\epsilon\to 0}\left(\frac{2}{\pi}\int_{\Delta_\epsilon^c}\#(\ell\cap S)dP - \frac{8L(C)}{\pi\epsilon}\right)\end{aligned}$$

where $\mathcal{A}$ denotes the hyperbolic area, and ℓ is the geodesic defined by a point pair in $\partial_\infty H^3$. Combining the equation above with (1) we get

$$\int_S K = 2\pi\chi(S) + \frac{2}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}(\#(\ell(w,z)\cap S) - \lambda^2(w,z))dP_{(w,z)} - \frac{4}{\pi}E(\Omega). \tag{2}$$

This shows the Möbius invariance of $E(\Omega)$. An invariant expression of $E(\Omega)$ has been obtained in [2]. Another easy consequence of (2) is that $E(\Omega)\geq\pi^2\chi(\Omega)/2$. This follows by taking the convex hull of Ω, and observing that its boundary has vanishing extrinsic curvature.

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