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Noncommutative Geometry and Loop Quantum Gravity: Loops, Algebras and Spectral Triples

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Abstract. Spectral triples have recently turned out to be relevant for different approaches that aim at quantizing gravity and the other fundamental forces of nature in a mathematically rigorous way. The purpose of this workshop was to bring together researchers mainly from noncommutative geometry and loop quantum gravity –two major fields that have used spectral triples independently so far– in order to share their results and open issues.

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Introduction by the Organisers

The workshop "Noncommutative Geometry and Loop Quantum Gravity: Loops, Algebras and Spectral Triples" has been organized by Christian Fleischhack (Paderborn), Matilde Marcolli (Pasadena), and Ryszard Nest (Copenhagen). This meeting was attended by 23 researchers from 8 countries, including several younger postdocs and two PhD students. We enjoyed 16 talks lasting about 50 to 75 minutes plus discussions. As there were no "official" talks after lunch until 4 pm and also no talks in the evening, there was a large amount of time left for informal discussions.

The task of defining both a consistent and mathematically rigorous theory of quantum gravity is one of most challenging undertakings in modern theoretical physics. It is widely expected that at Planck scale the usual notions of smooth geometries have to be replaced by something different. Various arguments point towards geometric notions becoming noncommutative, so that geometric measurements should correspond to noncommuting operators.

In fact, noncommutative geometry (NCG) provides a remarkably successful framework for unification of all known fundamental forces. Mathematically, it mainly grounds on the pioneering work of Connes, who related Riemannian spin geometries to a certain class of spectral triples over commutative C^*-algebras. Extending this formalism, Chamseddine and Connes demonstrated that the standard model coupled to gravitation naturally emerges from a spectral triple over an almost commutative C^*-algebra together with a spectral action. This way they even entailed experimentally falsifiable predictions in elementary particle physics. However, although fully implementing the idea of unification, this approach has remained essentially classical. Moreover, as the theory of spectral triples has only been developed for Riemannian manifolds, full general relativity needing Lorentzian geometries has not been tackled.

Loop quantum gravity (LQG), on the other hand, is one of the most successful theories to quantize canonical gravity. Resting on a generalization of Dirac quantization by Ashtekar and Lewandowski, its decisive idea is to break down the quantization to finite-dimensional problems on graphs and then to reconstruct the continuum theory using projective/inductive limits over all graphs. Although the kinematical part of LQG is nicely understood, the dynamical part is vastly open territory – both mathematically and conceptually. This concerns mainly three, related issues: First of all, the spectral analysis of the quantum Hamiltonian constraint, responsible for time evolution, is very immature. Secondly, it is completely unknown how to reconstruct classical general relativity as a semiclassical limit of loop quantum gravity. And, instead of an emergent unification, matter has to be included by hand.

Although NCG and LQG use very similar mathematical techniques – e. g., operator algebras in general, or spectral encoding of geometry to be more specific –, their conceptual problems are rather complementary. Nevertheless, only recently, first steps to join the strengthes of both approaches have been made. In several papers since 2005, Aastrup and Grimstrup, later with one of the organizers (RN), have outlined how to construct a semifinite spectral triple for the full theory out of spectral triples based on a restricted system of nested graphs.

One of the main tasks of the meeting was to bring together researchers from different fields – first of all, noncommutative geometry and loop quantum gravity, but also other fields like spectral triples on its own and axiomatic quantum field theory. For this, there were several introductory talks:

- Hanno Sahlmann and Thomas Thiemann gave an overview on the origins and the current status of loop quantum gravity. Sahlmann focused on physical and kinematical issues, Thiemann on open issues concerning dynamics.
- Giovanni Landi and Walter van Suijlekom presented introductions into noncommutative geometry. Whereas Landi spoke on general issues, Walter

van Suijlekom showed how one can encode the standard model of particle physics within the language of spectral triples.
- Johannes Aastrup and Jesper Grimstrup demonstrated how spectral triples can fruitfully transfer ideas from noncommutative geometry into loop quantum gravity.
- Klaus Fredenhagen and Rainer Verch introduced axiomatic quantum field theories as functors from the category of globally hyperbolic spacetimes into that of C^*-algebras. Fredenhagen concentrated on perturbation theory, i.e., such functors that are formal power series in $\hbar$. Verch used this framework to extend the notion of spectral triples to the Lorentzian case.

Beyond these talks there have been more specialized ones:
- Alan Carey described a generalization of spectral triples, so-called semifinite spectral triples. They arise naturally in the Aastrup-Grimstrup-Nest approach.
- Matilde Marcolli and Jerzy Lewandowski studied further noncommutative structures arising in loop quantum gravity. Marcolli described how extended spin foams define noncommutative coordinate algebras; Lewandowski replaced the underlying structure group $SU(2)$ of LQG by the quantum group $SU_q(2)$.
- Victor Gayral and Thomas Krajewski spoke on quantum groups as well: Gayral from a more generalized perspective, Krajewski inspired by string theory.
- Fedele Lizzi described noncommutative lattices that may lead to emerging spacetime.
- Varghese Mathai and Raimar Wulkenhaar explained different types of deformation quantization. Mathai constructed noncommutative principal bundles and Wulkenhaar outlined why there should be non-perturbative quantum field theories over Moyal deformed $\mathbb{R}^4$.

The atmosphere within the workshop benefited very much from the liveliness of the discussions and questions, which occurred frequently before, during, and after the talks. From this point of view the meeting was very successful, on the one hand for enabling a significant exchange of ideas between researchers in the two major fields, and on the other side for presenting the results of the few scientists that work in the intersection of LQG and NCG. In particular the fact that for every talk usually at least half the audience was no specialist in the field covered in it, resulted in a very effective exchange of knowledge, from which both sides gained profit.

Workshop: Noncommutative Geometry and Loop Quantum Gravity: Loops, Algebras and Spectral Triples

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Introduction to Loop Quantum Gravity – physics background, and kinematics

Hanno Sahlmann

Extended abstract

Loop quantum gravity is non-perturbative approach to the quantum theory of gravity, in which no classical background metric is used (see [1, 2] for recent reviews). In particular, its starting point is not a linearized theory of gravity. As a consequence, while it still operates according to the rules of quantum field theory, the details are quite different of those for field theories that operate on a fixed classical background space-time. It has considerable successes to its credit, perhaps most notably a quantum theory of spatial geometry in which quantities such as area and volume are quantized in units of the Planck length [3, 4, 5], and a calculation of black hole entropy for static and rotating, charged and neutral black holes [6]. But there are also open questions, many of them surrounding the dynamics ("quantum Einstein equations") of the theory.

Loop quantum gravity is, in its original version, a canonical approach to quantum gravity. Nowadays, a covariant formulation of the theory exists in the so called *spin foam models*. One of the canonical variables in loop quantum gravity is an $SU(2)$ connection, the other one a section in an associated vector-bundle [7, 8]. Many distinct technical features (such as the 'loops' in its name) are directly related to the choice of these variables. The quantization is done in two steps: First, the canonical variables are quantized on a kinematical Hilbert space $\mathcal{H}_{\text{kin}}$, then Einstein's equations (technically: Constraint equations on the canonical variables) are implemented as operator equations.

The space $\mathcal{H}_{\text{kin}}$ and operators thereon are of mathematical interest in their own right:$\mathcal{H}_{\text{kin}} = L^2(\overline{\mathcal{A}}, \mathrm{d}\mu)$, where $\overline{\mathcal{A}}$ is a space of distributional connections, and $\mathrm{d}\mu$ is a diffeomorphism and gauge invariant measure. An orthonormal basis on $\mathcal{H}$ is given by so called *spin networks*, generalizations of characters of parallel transport operators.

A distinct feature of these constructions is that no fixed classical geometric structures are used. New techniques had to be developed for this, and the resulting Hilbert spaces look very different than those in standard quantum field theory, with excitations of the fields one- or two-dimensional. But it has also simplified the theory, since can be shown that some choices made in the quantum theory are actually uniquely fixed by the requirement of background independence [9, 10]. In particular, it seems to lead to a theory which is built around a very quantum mechanical gravitational "vacuum", a state with degenerate and highly fluctuating geometry. This is exciting, because it means that when working in loop quantum gravity, the deep quantum regime of gravity is 'at one's fingertips'. However, it also means that to make contact with low energy physics is a complicated endeavor.

The latter problem has attracted a considerable amount of work, but is still not completely solved.

Another (related) challenge is to fully understand the implementation of the dynamics. In loop quantum gravity, the question of finding quantum states that satisfy 'quantum Einstein equations' is reformulated as finding states that are annihilated by the quantum Hamilton constraint. The choices that go into the definition of this constraint are not yet well understood in physical terms.

While these challenges remain, remarkable progress has happened over the last couple of years: The master constraint program has brought new ideas to bear on the implementation of the dynamics [11]. Progress has been made in identifying observables for general relativity that can be used in the canonical quantization [12, 13, 14]. A revision of the vertex amplitudes used in spin foam models has brought them in much more direct contact to loop quantum gravity [15, 16]. And, last not least, in loop quantum cosmology, the application of the quantization strategy of loop quantum gravity to mini-superspace models has become a beautiful and productive laboratory for the ideas of the full theory, in which the quantization program of loop quantum gravity can be tested, and, in many cases, brought to completion [17, 18, 19].

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An introduction to Noncommutative Geometry

Giovanni Landi

The starting data for noncommutative geometry is a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with $\mathcal{A}$ a $*$-algebra represented on the Hilbert space $\mathcal{H}$, and an unbounded self-adjoint operator D on $\mathcal{H}$. There is a list of requirements on these objects (coming from physics). I described the general construction and gave some interesting examples: notably isospectral toric deformations and quantum groups and associated homogenous spaces.

On Spectral Triples of Holonomy Loops I

Johannes Aastrup

Noncommutative geometry (NCG) provides a framework for describing matter coupled to gravity. In the paper [1] it is shown that the action of the standard model coupled to gravity emerges from an almost commutative geometry. In this way NCG provides a unification of the known forces. However the description is classical in the sense that it provides the action. Some kind of quantization scheme has to be applied afterwards.

Loop quantum gravity (LQG) is an attempt to quantize gravity, see [2]. It is based on Ashtekar's reformulation of gravity as a gauge field theory with constraints. However LQG deals with quantization of pure gravity, and matter has to be included by hands, so there is no unification.

It is therefore natural to look for an intersection of NCG with LQG.

In the following I will outline a construction, which intersects NCG with LQG. More precisely I will outline the construction of a semi-finite spectral triple over an algebra of holonomy loops. The physical interpretation of the construction will be discussed by J. Grimstrup. For full details on the construction see [3, 4].

The configuration space in LQG is the space of smooth connections $\mathcal{A}$ in a trivial $SU(2)$ bundle over a 3-dimensional time slice Σ. The algebra over which we want to construct a spectral triple is the following: Let L be a loop in Σ with base point x. Then L gives rise to a function

$$h_L : \mathcal{A} \to M_2,$$

via

$$h_L(\nabla) = Hol(\nabla, L),$$

where $Hol(\nabla, L)$ denotes the holonomy of ∇ around L. Let $\mathcal{B}_x$ be the algebra generated by all the h_L's, i.e. by all the loops with base point x.

In order to construct a Dirac type operator over $\mathcal{B}_x$ we first need to construct $L^2(\mathcal{A})$. This is done via a variation of the Ashtekar-Lewandowski Measure.

Let Γ_0 be a cubic lattice on Σ, and let Γ_n be the n'th subdivision of this lattice. To each Γ_n we associate $\mathcal{A}_n = G^{e(\Gamma_n)}$, where $e(\Gamma_n)$ denotes the number of edges in Γ_n. There are natural maps

$$P_{n,n+1} : \mathcal{A}_{n+1} \to \mathcal{A}_n.$$

Define

$$\overline{\mathcal{A}} = \lim_n \mathcal{A}_n.$$

It is not hard to see

Proposition 0.0.1. *The natural map*

$$\phi : \mathcal{A} \to \overline{\mathcal{A}}$$

is a dense embedding.

This mirrors a similar result for the Ashtekar-Lewandowski construction.

We define "$L^2(\mathcal{A})$" as $\lim_n L^2(\mathcal{A}_n)$.

The idea to construct the Dirac operator is to construct a Dirac operator on each $\mathcal{A}_n$, since these are just classical geometries. Therefore let

$$\mathcal{H} = \lim L^2(\mathcal{A}_n, Cl(T^*\mathcal{A}_n)) \otimes M_2,$$

where $CL(T^*\mathcal{A})$ is the Clifford algebra of $T^*\mathcal{A}$. We construct Dirac operators D_n on each $L^2(\mathcal{A}_n, Cl(T^*\mathcal{A}_n)) \otimes M_2$, which are compatible with the embeddings

$$P^*_{n,n+1} L^2(\mathcal{A}_n, Cl(T^*\mathcal{A}_n)) \otimes M_2 \to L^2(\mathcal{A}_{n+1}, Cl(T^*\mathcal{A}_{n+1})) \otimes M_2.$$

These operators therefore descend to a densely defined operator D on $\mathcal{H}$.

The operator D depends on a sequence a_k of real numbers. k is roughly a labeling of the different copies of G.

We now need to modify our algebra $\mathcal{B}_x$ to get an action on $\mathcal{H}$. We therefore consider the sub algebra $\mathcal{B}_x^\Gamma$ of $\mathcal{B}_x$ generated by loops in $\cup\Gamma_n$.

Finally consider the algebra $\lim_n Cl(T^*_{id}\mathcal{A}_n)$. The weak closure C of the representation of this algebra on the Hilbert space $\lim_n Cl(T^*_{id}\mathcal{A}_n)$ is the CAR-algebra (Canonical Anticommutation Relation algebra), and therefore has a finite trace τ. Let

$$\mathcal{N} = \mathbb{B}(\lim L^2(\mathcal{A}_n) \otimes M_2) \otimes \mathcal{N}.$$

This is a semi-finite von Neumann algebra with semi-finite trace $Tr \otimes \tau$.

Theorem 0.0.2. *The triple* $(\mathcal{B}_x^\Gamma, \mathcal{H}, D)$ *is a semi-finite spectral triple with respect to* $(\mathcal{N}, \tau)$ *when* $|a_k| \to \infty$.

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Perturbative Algebraic Quantum Field theory

Klaus Fredenhagen

Algebraic Quantum Field Theory can be defined as a functor which associates C*-algebras of observables to globally hyperbolic spacetimes and algebra homomorphims to causality preserving isometric embeddings of spacetimes. Examples for such a functor are provided by free quantum field theories, where in the simplest case, the free scalar field, a hyperbolic differential operator, the Klein-Gordon operator, is used to define retarded and advanced propagators whose difference then first defines a Poisson bracket on functionals of the field and then, in a second step, a *-product in the sense of formal deformation quantization. If one restricts oneself to finite sums of exponential functionals, the arising algebra has a unique C*-norm, and one obtains the wanted functor with the required covariance property.

The first step, namely the introduction of the Poisson product, is possible also for interacting classical theories, but for quantum field theory the construction in the sense of C*-algebras was possible up to now only in special cases, in particular for the $P(\varphi)$-theories in two dimensional Minkowski space.

One can, however, construct the functor for generic quantum field theories in the sense of formal power series in $\hbar$. Here one starts from a free theory, equipped with a *-product, but extended now to all sufficiently differentiable functionals, in particular to polynomials of the field, corresponding to Wick polynomials. In this algebra one now introduces a second product, namely the time ordered product, which a priori is not always well defined, but which is, where it exists, equivalent to the pointwise product of functionals and hence in particular commutative and associative. The definition of the time ordered product of n local functionals amounts mathematically to an extension of distributions which are originally defined only outside of some submanifold. Such an extension is always possible and can be done in a functorial way so that all requirements of covariance are satisfied.

The extension is, however, not unique, and the ambiguity amounts to a finite renormalisation. The ambiguity can be described in terms of the renormalisation group which is the group of formal diffeomorphisms, tangent to the identity, on the space of interaction Lagrangeans. Symmetries of the classical theory give rise to cocycles in the renormalisation group, and nontrivial cocycles are the famous anomalies of quantum field theory.

The algebra of observables of the theory can now be constructed, and one obtains a corresponding functor which takes values in a *-algebra of formal power series. The renormalisation group acts on the functor by natural transformations.

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Noncommutative Geometry and Particle Physics

Walter D. van Suijlekom

This is a short survey on the derivation of the Standard model from a noncommutative manifold.

1. Noncommutative manifolds and gauge field theory

The starting point is a noncommutative spin manifold as described by a *spectral triple* [3] $(\mathcal{A}, \mathcal{H}, D)$ consisting of a $*$-algebra $\mathcal{A}$ of bounded operators on a Hilbert space $\mathcal{H}$, and a self-adjoint operator in $\mathcal{H}$. One requires that $[D, a]$ is bounded for any $a \in \mathcal{A}$ and the resolvent of D is compact. This structure can be further enriched by introducing a *grading* γ on $\mathcal{H}$ and an anti-linear isometry J (*real structure*) in $\mathcal{H}$ such that $[a, [D, b]] = [JaJ^{-1}, [D, b]] = 0$ Moreover, we demand that $\gamma D = D\gamma$, $J\gamma = \pm\gamma J$, $J^2 = \pm$ and $JD = \pm DJ$. The $\pm$-signs determine the *KO-dimension*; they can be found in [4].

The main idea is that the above consists of all structure necessary to define a gauge theory. In fact, the group $U(\mathcal{A})$ of unitary elements in the algebra $\mathcal{A}$ naturally acts on the Hilbert space and as intertwiners on the representation of $\mathcal{A}$ and D. More precisely,

$$\psi \mapsto U\psi; \qquad a \mapsto UaU^*; \qquad D \mapsto UDU^*; \qquad (\psi \in \mathcal{H}, a \in \mathcal{A}),$$

where $U = uJuJ^*$ is the adjoint representation of $u \in U(\mathcal{A})$. It is then only natural to look for invariants under this group action and we work with the following combination

$$S_\Lambda[D, \psi] := \langle \psi, D\psi \rangle + \operatorname{Tr} f(D/\lambda)$$

considered as a physical action functional on D and ψ. Here f is an even function, and is such that the trace is well-defined. There are now two ways of introducing gauge fields, the first of physical character and the second of mathematical.

Observe that the unitaries $u \in U(\mathcal{A})$ act as

$$D \mapsto UDU^* = D + u[D, u^*] \pm Ju[D, u^*]J^{-1}.$$

Thus, as usual in minimal coupling, one replaces D by the operator $D + A \pm JAJ^{-1}$ where $A = \sum a_j[D, b_j]$ with a_j, b_j now arbitrary elements in $\mathcal{A}$. This is our gauge field, which transforms in the usual way: $A \mapsto uAu^* + u[D, u^*]$.

From a mathematical point of view there is a nice interpretation of gauge fields as inner fluctuations, generated by Morita equivalence. It is based on the following question: given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and an algebra $\mathcal{B}$ that is Morita equivalent to $\mathcal{A}$, is it possible to construct a spectral triple $(\mathcal{B}, \mathcal{H}', D')$?

Not surprisingly, the answer is yes [4]. We will not give the details here, but note that in the case that $\mathcal{B} = \mathcal{A}$ there is still freedom in choosing D' different from D. These are precisely the *inner fluctuations*, and correspond to choosing a connection one-form of the form $A = \sum_j a_j[D, b_j]$ with $a_j, b_j \in \mathcal{A}$. The operator D then becomes $D_A := D + A \pm JAJ^{-1}$ and A transforms as above.

In the rest of this note, we will give in several examples the leading terms of the spectral action, as an expansion in Λ. For the details, we refer to [2] and [5].

1.1. **Einstein's general theory of relativity.** Consider a compact 4-dimensional Riemannian spin manifold (M, g); then $(C^\infty(M), L^2(M, S), \partial\!\!\!/)$ is canonically a spectral triple, where $\partial\!\!\!/$ is the ordinary Dirac operator on the spinor bundle $S \to M$. Further, there is a grading given by γ_5 and a real structure by charge conjugation J_M. Since the algebra is commutative the inner fluctuations are trivial. The spectral action is computed to be

$$\operatorname{Tr} f(\partial\!\!\!//\Lambda) = \frac{1}{4\pi^2}\int_M \left(2\Lambda^4 f_4 + \frac{\Lambda^2 f_2}{6} R - \frac{f_0}{80} C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\right) + \mathcal{O}(\Lambda^{-2})$$

in terms of the Weyl curvature tensor $C_{\mu\nu\rho\sigma}$. This action is recognized as the Einstein–Hilbert action, plus additional higher-order gravitational terms.

1.2. **Yang–Mills action.** We make the spectral triple of the previous section 'mildly' noncommutative and consider $(C^\infty(M, M_N(\mathbb{C})), L^2(M, S) \otimes M_N(\mathbb{C}), \partial\!\!\!/ \otimes 1)$. In addition, there exist a grading $\gamma = \gamma_5 \otimes 1$ and a real structure $J = J_M \otimes (\cdot)^*$. It turns out that the inner fluctuations are parametrized by a $SU(N)$-gauge field A_μ, the group of unitaries is $C^\infty(M, U(N))$ acting in the adjoint on the Hilbert space. Actually, the fact that the fermions are in the adjoint representation is the origin of supersymmetry in this model as was suggested in [2] and worked out in detail in [1]. One computes that in this case the spectral action contains —in addition to the gravitational terms considered before— the Yang–Mills action

$$S_\Lambda[A, \psi] = -\frac{f_0}{24\pi^2} \int_M \operatorname{Tr} F_{\mu\nu} F^{\mu\nu} + \langle \psi, (\partial\!\!\!/ + i\gamma^\mu \operatorname{ad} A_\mu)\psi \rangle + \mathcal{O}(\Lambda^{-2}).$$

1.3. **The Standard Model of high-energy physics.** The previous model extends to the full Standard Model, including Higgs boson. The spectral triple is

$$(C^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}),\ L^2(M, S) \otimes \mathbb{C}^{96},\ \partial\!\!\!/ \otimes 1 + \gamma_5 \otimes D_F).$$

Here 96 is 2 (particles and anti-particles) times 3 (families) times 4 leptons times 4 quarks with 3 colors each. We write the representation of $\mathcal{A}$ in terms of the suggestive basis of $\mathbb{C}^{96}$: $\left(\nu_L\ e_L\ \nu_R\ e_R\ u_L\ d_L\ u_R\ d_R\ \bar{\nu}_L\ \bar{e}_L\ \bar{\nu}_R\ \bar{e}_R\ \bar{u}_L\ \bar{d}_L\ \bar{u}_R\ \bar{d}_R \right)^t$. Then,

$$\pi(\lambda, q, m) = \begin{pmatrix} q & & & & & \\ & \left[\begin{smallmatrix} \lambda & \\ & \bar{\lambda} \end{smallmatrix}\right] & & & & \\ & & q \otimes 1_3 & & & \\ & & & \left[\begin{smallmatrix} \lambda & \\ & \bar{\lambda} \end{smallmatrix}\right] \otimes 1_3 & & \\ & & & & \lambda 1_4 & \\ & & & & & 1_4 \otimes \bar{m} \end{pmatrix}; \quad ((\lambda, q, m) \in \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}))\,.$$

Here, the quaternion q is considered as a 2×2-matrix. The 96×96-matrix D_F is of the following form: $D_F = \left(\begin{smallmatrix} S & T^* \\ T & \bar{S} \end{smallmatrix} \right)$ where

$$S = \begin{pmatrix} & \left[\begin{smallmatrix} \Upsilon_v & \\ & \Upsilon_e \end{smallmatrix}\right] & & \\ \text{(h.c.)} & & & \\ & & & \left[\begin{smallmatrix} \Upsilon_u \otimes 1_3 & \\ & \Upsilon_d \otimes 1_3 \end{smallmatrix}\right] \\ & & \text{(h.c.)} & \end{pmatrix}; \quad T = \begin{pmatrix} \left[\begin{smallmatrix} 0 & & & \\ & 0 & \Upsilon_R & \\ & & & 0 \end{smallmatrix}\right] & \\ & 0_4 \end{pmatrix}$$

in terms of the 3×3 *Yukawa-mixing-matrices* $\Upsilon_\nu, \Upsilon_e, \Upsilon_u, \Upsilon_d$ and a real constant Υ_R responsible for neutrino mass terms. One can further enrich this spectral triple by a grading γ_F which is $+1$ (-1) on all L (R)-particles; the total grading is then $\gamma_5 \otimes \gamma_F$. The anti-linear operator J is a combination of J_M and the (anti-linear) matrix $J_F = \left(\begin{smallmatrix} & 1_{48} \\ 1_{48} & \end{smallmatrix} \right)$.

The rest then follows from a long calculation; the inner fluctuations are $D_A = \partial\!\!\!/ + i\gamma_\mu A_\mu + \gamma_5(D_F + \mathbb{M}(\Phi))$ with

$$A_\mu = \begin{pmatrix} \frac{g_1}{2} B_\mu - \frac{g_2}{2} W_\mu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & g_1 B_\mu \end{pmatrix} \oplus \begin{pmatrix} -\frac{g_2}{2} W_\mu \otimes 1_3 - \frac{g_1}{6} B_\mu \otimes 3 & 0 & 0 \\ 0 & -\frac{2g_1}{3} B_\mu \otimes 1_3 & 0 \\ 0 & 0 & \frac{g_1}{3} B_\mu \otimes 1_3 \end{pmatrix} - 1_4 \otimes \frac{g_3}{2} V_\mu$$

$$\mathbb{M}(\Phi) = \begin{pmatrix} & \begin{smallmatrix} \Upsilon_\nu \phi_1 & \Upsilon_\nu \phi_2 \\ -\Upsilon_e \bar{\phi}_2 & \Upsilon_e \bar{\phi}_1 \end{smallmatrix} \\ \text{(h.c.)} & \end{pmatrix} \oplus \begin{pmatrix} & \begin{smallmatrix} \Upsilon_u \phi_1 & \Upsilon_u \phi_2 \\ -\Upsilon_d \bar{\phi}_2 & \Upsilon_d \bar{\phi}_1 \end{smallmatrix} \\ \text{(h.c.)} & \end{pmatrix}$$

Here B_μ, W_μ, V_μ are $U(1), SU(2)$ and $SU(3)$-gauge fields, resp. and $\Phi = (\phi_1\ \phi_2)^t$ two scalar (Higgs) fields. The spectral action is modulo gravitational terms:

$$S_\Lambda = \frac{-2af_2\Lambda^2 + ef_0}{\pi^2}\int |\phi|^2 + \frac{f_0}{2\pi^2}\int a|D_\mu\phi|^2 - \frac{f_0}{12\pi^2}\int aR|\phi|^2$$
$$- \frac{f_0}{2\pi^2}\int\left(g_3^2 G^i_\mu G^{\mu i} + g_2^2 F^a_\mu F^{\mu\nu a} + \frac{5}{3}g_1^2 B_\mu B^\mu\right) + \frac{f_0}{2\pi^2}\int b|\phi|^4 + \mathcal{O}(\Lambda^{-2})$$

with a, b, c, d, e constants depending on the Yukawa parameters.

When we add the fermionic term $\langle J\psi, D_A\psi\rangle$ to S_Λ, we obtain the Standard Model Lagrangian, including the Higgs boson, provided we have

$$\frac{g_3^2 f_0}{2\pi^2} = \frac{1}{4} \qquad g_3^2 = g_2^2 = \frac{5}{3}g_1^2.$$

These GUT-type relations between the coupling constants allows for predictions. For example, one identifies the mass of the W as $2M_W = \sqrt{a/2}$ so that the Higgs vacuum reads $2M/g_2$. This allows for a postdiction for the mass of the top quark as $m_t \leq 180$ GeV. Moreover, the mass of the Higgs is $m_H = 8\lambda M^2/g_2^2$ with $\lambda = g_3^2 b/a^2$ resulting in a prediction of $m_H \sim 168$ GeV.

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An overview of semifinite noncommutative geometry

Alan Carey

1. Background

In semifinite NCG one extends the usual point of view of [13] by replacing the bounded operators $B(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$ by certain sub-algebras; namely semifinite von Neumann algebras. These are weakly closed $*$-subalgebras of the bounded operators on a Hilbert space that admit a faithful, normal semifinite trace. Two major motivating examples of this so-called semifinite NCG are the study of foliations [2] and the notion of spectral flow for paths of operators in a semifinite von Neumann algebra [5, 1]. It was Mathai in 1992, [21], who asked the question about whether there is a semifinite spectral flow in connection with Atiyah's L^2-index theorem. Motivated by a fundamental paper of Getzler in 1993, [17], J. Phillips, [22], independently studied analytic approaches to spectral flow in von Neumann algebras.

Semifinite NCG also involves an extension of the theory of Fredholm operators in the sense of Breuer, [3]. If $\mathcal{N}$ is a semifinite von Neumann algebra acting on $\mathcal{H}$ and τ is a fixed faithful normal semifinite trace then to say that $F \in \mathcal{N}$ is τ-Fredholm is shorthand for saying that F is invertible in the τ-Calkin algebra (semifinite Atkinson's theorem). This latter algebra is the quotient of $\mathcal{N}$ by the norm closed ideal $\mathcal{K}_\tau$ generated by the τ-finite projections. A discussion of what is needed in the way of an extension of Breuer's work is contained in [10]. Spectral triples in the semifinite setting naturally involve unbounded self adjoint operators affiliated to $\mathcal{N}$. If D is such an operator, introduce the map $D \to F_D := D(1 + D^*D)^{-1/2}$, and say that D is τ-Fredholm if its bounded image F_D is τ-Fredholm. Now recall the following fundamental definition from [8, 2].

Definition. A semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ consists of a (unital) $*$-subalgebra $\mathcal{A}$ of the semifinite von Neumann algebra on a separable Hilbert space $\mathcal{H}$ and an unbounded self-adjoint operator $\mathcal{D} : \mathrm{dom}\mathcal{D} \subset \mathcal{H} \to \mathcal{H}$ with resolvent in $\mathcal{K}_\tau$ such that $[\mathcal{D}, a]$ extends to a bounded operator for all $a \in \mathcal{A}$. The spectral triple is even if there is a self-adjoint bounded involution $\Gamma \in \mathcal{N}$ which anticommutes with $\mathcal{D}$ and commutes with $\mathcal{A}$; otherwise it is odd.

The key axiom here is the compact resolvent condition. This tells us that $\mathcal{D}$ is an (unbounded) τ-Fredholm operator, a condition natural from the viewpoint of index theory for elliptic operators on both compact manifolds. and in the setting of Atiyah's L^2-index theorem where one has elliptic operators acting on sections of bundles on the universal cover of a compact manifold. These elliptic operators have a resolvent in $\mathcal{K}_\tau$ and they are not compact in the usual sense. This is exactly the setting of Mathai's question of whether the L^2-index theorem for odd dimensional manifolds could be thought of as calculating a kind of spectral flow and it is naturally answered by semifinite NCG. To work in this generality one needs a theory of operator ideals in semifinite von Neumann algebras. This was provided in a fundamental paper of T. Fack and H. Kosaki, [16]. Extending [16] there is a definition of the 'Dixmier ideal' $\mathcal{L}^{(1,\infty)}$ and the Wodzicki residue (see [2, 8]). A survey of all the refinements of the Fack-Kosaki work that semifinite NCG requires is contained in [12].

Finding formulas for spectral flow has been a major motivation. The first such general formula is due to Getzler [17]. The paper [5] initiated the development of a semifinite theory. It introduced the notion of semifinite unbounded Fredhom modules or (in the current terminology) spectral triples and many other techniques needed to produce formulas to calculate spectral flow in the semifinite setting. The next development was in the form of a preliminary version of the manuscript of Benameur-Fack [2]. Motivated by foliations they introduced both odd and even versions of semifinite NCG and their work provided an impetus for developing the semifinite local index formula. A further ingredient comes from the paper [14] and is perhaps the earliest indication of the connection between semifinite Kasparov theory and cyclic cohomology. In [14], Connes and Cuntz show that cyclic n-cocycles for an appropriate algebra A are in one-to-one correspondence with traces on a certain ideal J^n in the free product $A * A$. Assuming some positivity for this trace yields the same kind of 'semifinite Kasparov modules' as are described in [20]. In other words, to realise all the cyclic cocycles for an algebra will, in general, necessitate considering semifinite Fredholm modules and semifinite spectral triples.

2. The local index formula

The two papers, [9, 10] respectively for odd and even semifinite spectral triples, succeed in achieving the extension of the original Connes-Moscovici local index formula to the setting of semifinite NCG. These papers contain a very different approach and proof from simply trying a direct generalisation of [15]. First, for semifinite NCG, we wanted to avoid the discrete dimension spectrum hypothesis of [15]. We were of the view that this hypothesis may be quite hard to check in semifinite spectral triples. This forced us to necessarily avoid their starting point, the JLO cocycle [19]. It was [18] that illustrated a possible different approach introducing a kind of 'resolvent cocycle' as an alternative. Unfortunately the cocycle in [18] does not resolve all of the difficulties presented by starting with the JLO formula and still retains the discrete dimension spectrum assumption. These considerations led us in [9, 10] to our first new proof of the local index formula. The argument is different in the odd and even cases relying respectively on the formula for spectral flow presented in [6] and on a generalised McKean-Singer formula proved in [10]. From these two formulae we derived a new cocycle that we termed the resolvent cocycle. We found that it can be used to express the relevant numerical index pairing between the spectral triple (regarded as determining a class in the K-homology of the algebra $\mathcal{A}$) and the K-theory of $\mathcal{A}$. To show that this pairing is the same as that given by the Chern character we proved that our resolvent cocycle could be homotopied to the Chern character of semifinite Kasparov modules. This is the basis of the proof of the semifinite local index formula in [11]. Both of these proofs avoid the discrete dimension spectrum assumption of [15] replacing it by the minimal assumptions on the singularity structure of the zeta functions that are needed to produce the residue cocycle. It seems highly likely that these minimal assumptions are much easier to check in examples.

We remark that semifinite Kasparov modules and semifinite spectral triples provide information that is different from that of the standard theory [13]. In [20] it is shown that a semifinite spectral triple for $\mathcal{A}$ represents an element of $KK^1(\mathcal{A}, J)$, where J is the separable norm closed algebra of compact operators in $\mathcal{N}$ generated by the resolvent of $\mathcal{D}$ and the commutators $[F_\mathcal{D}, a]$ for $a \in \mathcal{A}$ where $F_\mathcal{D} = \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}$. This Kasparov module picture is the one implied by [14].

3. Open questions

Semifinite spectral triples arise in loop quantum gravity as can be seen from the other abstracts from this meeting. There they are theta summable and one can use the semifinite JLO formula instead of the local index formula [6]. They also arise in the study of foliations [2] and of certain graph C^*-algebras [7]. They may be used to create an index theory associated to KMS states (for references see [4]). However a challenge in all these cases is to extract information from the semifinite index formulas. A promising direction is the study of Mumford curves by NCG methods [4].

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On Spectral Triples of Holonomy Loops II

Jesper Møller Grimstrup

My talk is concerned with an intersection of quantum gravity with noncommutative geometry. The overall theme is to apply the ideas and tools of noncommutative geometry directly to the setup of canonical quantum gravity. The aim is a new approach to quantum gravity which combines mathematical rigor with elements of unification.

So far, we have successfully constructed a semi-finite spectral triple which encodes the kinematical part of quantum gravity and which gives, in a semi-classical limit, the Dirac Hamiltonian in 3+1 dimensions.

In more detail, we have constructed a semi-finite spectral triple over an algebra of holonomy loops. This construction is related to a configuration space of connections which, in turn, can be related to a formulation of gravity, due to Ashtekar, in terms of connections. In this spectral triple construction, the Dirac type operator has a natural interpretation as a global functional derivation operator. In terms of canonical gravity, it is an infinite sum of certain flux operators, conjugate to the holonomy loop operators. The interaction between the Dirac type operator and the algebra reproduces the structure of the Poisson bracket of general relativity. Thus, the spectral triple contains, a priori, the kinematical part of general relativity, in the sense that it involves information tantamount to a representation of the Poisson structure of general relativity.

The construction of the semi-finite spectral triple is based on an inductive system of embedded graphs. Although the construction works for a large class of ordered graphs we find that a system of 3-dimensional nested, cubic lattices has a clear physical interpretation. In fact, we find certain semi-classical states for which the Dirac type operator descents to the Dirac Hamiltonian in 3+1 dimensions. This semi-classical limit only works for cubic lattices and provides an interpretation of the lattices as a choice of a coordinate system. Here, the lapse and shift fields, which encodes the choice of time-variable, can be understood in terms of certain degrees of freedom found in the Dirac type operator. The semi-classical states have a completely natural interpretation related to a choice of basepoint for the algebra of holonomy loops. It is the elimination of this basepoint which brings us to the particular form of these states.

The concreteness of the appearance of the lapse and shift fields raises the hope that this construction might hint towards a formulation of the Hamilton constraint which should implement invariance under the choice of the time-coordinate.

The spectral triple construction raises many question. For instance, we do not yet know how to extract the classical algebra of functions on the (spatial) manifold. Here, the key question is whether the function algebra, should it emerge from the construction in a semi-classical limit, will be commutative, or whether it will pick up a noncommutative factor stemming from the noncommutativity of the holonomy loops. If the latter should be the case it would make contact to the work of Alain Connes on the standard model of particle physics, which is based on almost commutative algebras.

The construction of the semi-finite spectral triple is completed in collaboration with Johannes Aastrup (Münster, Germany) and Ryszard Nest (Copenhagen, Denmark). The construction of the semi-classical states is made in collaboration with Johannes Aastrup, Ryszard Nest and Mario Paschke (Münster, Germany).

Relevant publications

(1) **Emergent Dirac Hamiltonians in Quantum Gravity.**
Johannes Aastrup, Jesper M. Grimstrup, Mario Paschke.
e-Print: arXiv:0911.2404 [hep-th], 2009.

(2) **On Semi-Classical States of Quantum Gravity and Noncommutative Geometry.**
Johannes Aastrup, Jesper M. Grimstrup, Mario Paschke, Ryszard Nest.
e-Print: arXiv: 0907.5510, 2009.

(3) **Holonomy Loops, Spectral Triples & Quantum Gravity.**
Johannes Aastrup, Jesper M. Grimstrup, Ryszard Nest.
Published in Class.Quant.Grav.26:16500, 2009.

(4) **A New Spectral Triple over a Space of Connections.**
Johannes Aastrup, Jesper M. Grimstrup, Ryszard Nest. 2008.
Published in Commun.Math.Phys. 290:389-398, 2009.

(5) **On Spectral Triples in Quantum Gravity II.**
Johannes Aastrup, Jesper M. Grimstrup, Ryszard Nest. 2008.
Published in J.Noncommut.Geom.3:47-8, 2009.

(6) **On Spectral Triples in Quantum Gravity I.**
Johannes Aastrup, Jesper M. Grimstrup, Ryszard Nest . 2008.
Published in Class.Quant.Grav.26:065011, 2009.

(7) **Intersecting Connes Noncommutative Geometry with Quantum Gravity.**
Johannes Aastrup, Jesper M. Grimstrup. 2006.
Published in Int.J.Mod.Phys.A22:1589, 2007.

(8) **Spectral Triples of Holonomy Loops.**
Johannes Aastrup, Jesper M. Grimstrup. 2005.
Published in Commun.Math.Phys.264:657-681, 2006.

Parametrized strict deformation quantization & noncommutative principal torus bundles

Varghese Mathai
(joint work with Keith Hannabuss)

1. C^*-bundles over X

We begin by reviewing the notion of C^*-bundles over X and the special case of noncommutative principal bundles. Then we discuss the fibrewise smoothing of these, which is used in parametrised Rieffel deformation later on.

Let X be a locally compact Hausdorff space and let $C_0(X)$ denote the C^*-algebra of continuous functions on X that vanish at infinity. A *C^*-bundle* $A(X)$ over X in the sense of [1] is exactly a $C_0(X)$-algebra in the sense of Kasparov. That is, $A(X)$ is a C^*-algebra together with a non-degenerate $*$-homomorphism

$$\Phi : C_0(X) \to ZM(A(X)),$$

called the *structure map*, where $ZM(A)$ denotes the center of the multiplier algebra $M(A)$ of A. The *fibre* over $x \in X$ is then $A(X)_x = A(X)/I_x$, where

$$I_x = \{\Phi(f) \cdot a;\, a \in A(X) \text{ and } f \in C_0(X) \text{ such that } f(x) = 0\},$$

and the canonical quotient map $q_x : A(X) \to A(X)_x$ is called the *evaluation map* at x.

NB. This definition does ***not*** require local triviality of the bundle, or for the fibres of the bundle to be isomorphic.

Let G be a locally compact group. One says that there is a *fibrewise action* of G on a C^*-bundle $A(X)$ if there is a homomorphism $\alpha : G \longrightarrow \mathrm{Aut}(A(X))$ which is $C_0(X)$-linear in the sense that

$$\alpha_g(\Phi(f)a) = \Phi(f)(\alpha_g(a)), \qquad \forall g \in G,\, a \in A(X),\, f \in C_0(X).$$

That is α induces an action α^x on the fibre $A(X)_x$ for all $x \in X$.

The first observation is that if $A(X)$ is a C^*-algebra bundle over X with a fibrewise action α of a *Lie group* G, then there is a canonical *smooth $*$-algebra bundle* over X. We recall its definition. A vector $y \in A(X)$ is said to be a *smooth vector* if the map

$$G \ni g \longrightarrow \alpha_g(y) \in A(X)$$

is a smooth map from G to the normed vector space $A(X)$.

Then

$$A^\infty(X) = \{y \in A(X) \,|\, y \text{ is a smooth vector}\}$$

is a $*$-subalgebra of $A(X)$ which is norm dense in $A(X)$. Since G acts *fibrewise* on $A(X)$, it follows that $A^\infty(X)$ is again a $C_0(X)$-algebra which is *fibrewise smooth.*

Let T denote the torus of dimension n. The authors of [1] define a *noncommutative principal T-bundle* (or *NCP T-bundle*) over X to be a separable C^*-bundle $A(X)$ together with a fibrewise action $\alpha : T \to \mathrm{Aut}(A(X))$ such that there is a Morita equivalence,

$$A(X) \rtimes_\alpha T \cong C_0(X, \mathcal{K}),$$

as C^*-bundles over X, where $\mathcal{K}$ denotes the C^*-algebra of compact operators.

The motivation for calling such C^*-bundles $A(X)$ NCP T-bundles arises from a special case of a theorem of Rieffel [5], which states that if $q : Y \longrightarrow X$ is a principal T-bundle, then $C_0(Y) \rtimes T$ is Morita equivalent to $C_0(X, \mathcal{K})$. It is also a special case of the noncommutative torus bundles considered in [3] which arise as T-dual spacetimes to spacetimes that are T-dual to principal torus bundles with H-flux, which appear as compactified spacetimes in string theory.

If $A(X)$ is a NCP T-bundle over X, then we call $\mathcal{A}^\infty(X)$ a *fibrewise smooth noncommutative principal T-bundle* (or *fibrewise smooth NCP T-bundle*) over X.

The noncommutative principal torus bundles $A(X)$ were first classified in [1]. They are determined by a pair of invariants; the first is the Chern class $c_1(A(X)) \in H^1(X, \underline{T})$, and the second is a continuous map $\sigma : X \to Z^2(\widehat{T}, \mathbf{T})$, that is, a continuous family of bicharacters of $\hat{T}$.

In [2], we are able to give an alternate complete classification of fibrewise smooth NCP T-bundles $\mathcal{A}^\infty(X)$ over X via a parametrised version of Rieffel's theory of strict deformation quantization

2. Parametrised strict deformation quantization in a nutshell

In a nutshell, parametrised strict deformation quantization is a functorial extension of Rieffel's strict deformation quantization from algebras A, to $C(X)$-algebras $A(X)$, and in particular to C^*-bundles over X. Unlike Rieffel's deformation theory [4] the version in [2] starts with multipliers via the Landstad–Kasprzak approach.

We generalize the strict deformation quantization of the n-torus T by Rieffel [4], to the case of principal torus bundles $q : Y \to X$ with fibre T. Note that fibrewise smooth functions on Y decompose as a direct sum,

$$C^\infty_{\text{fibre}}(Y) = \widehat{\bigoplus}_{\alpha \in \hat{T}} C^\infty_{\text{fibre}}(X, \mathcal{L}_\alpha)$$
$$\phi = \sum_{\alpha \in \hat{T}} \phi_\alpha$$

where $C^\infty_{\text{fibre}}(X, \mathcal{L}_\alpha)$ is defined as the subspace of $C^\infty_{\text{fibre}}(Y)$ consisting of functions which transform under the character $\alpha \in \hat{T}$, and where $\mathcal{L}_\alpha$ denotes the associated line bundle $Y \times_\alpha \mathbf{C}$ over X. That is, $\phi_\alpha(yt) = \alpha(t)\phi_\alpha(y)$, $\forall\, y \in Y$, $t \in T$.

The direct sum is completed in such a way that the function $\hat{T} \ni \alpha \mapsto ||\phi_\alpha||_\infty \in \mathbf{R}$ is in $\mathcal{S}(\hat{T})$.

In this interpretation of $C^\infty_{\text{fibre}}(Y)$, it is easy to extend to this case, the explicit deformation quantization given in the previous example, which we now briefly outline. For $\phi, \psi \in C^\infty_{\text{fibre}}(Y)$, define a new associative product $\star_\hbar$ on $C^\infty_{\text{fibre}}(Y)$ as follows. For $y \in Y$, $\alpha, \alpha_1, \alpha_2 \in \hat{T}$, let

$$(\psi \star_\hbar \phi)(y, \alpha) = \sum_{\alpha_1 \alpha_2 = \alpha} \psi(y, \alpha_1)\phi(y, \alpha_2)\sigma_\hbar(q(y); \alpha_1, \alpha_2),$$

using the notation $\psi(y,\alpha_1) = \psi_{\alpha_1}(y)$ etc., and where $\sigma_\hbar \in C(X, Z^2(\hat{T}, \mathbf{T}))$ is a continuous family of bicharacters of $\hat{T}$ such that $\sigma_0 = 1$, which is part of the data that we start out with.

We remark that one way to get such a $\sigma_\hbar$ is to choose a continuous family skew-symmetric forms on $\hat{T}$, $\gamma : X \longrightarrow Z^2(\hat{T}, \mathbf{R})$, and define $\sigma_\hbar = \exp(-\pi\hbar\gamma)$. In the case of a principal torus bundle Y, we note that the vertical tangent bundle of T has a symplectic structure, i.e. $\gamma \in \Lambda^2 T^{vert} Y$, which can be naturally interpreted as a continuous family of symplectic structures along the fibre, that is, γ is of the sort considered just previously. We denote the deformed algebra by $C^\infty_{\text{fibre}}(Y)_\hbar$, and we can realize it as a parametrised strict deformation quantization of $C^\infty_{\text{fibre}}(Y)$.

Theorem ([2]). *Given a fibrewise smooth NCPT-bundle $\mathcal{A}^\infty(X)$, there is a defining deformation $\sigma \in C(X, Z^2(\widehat{T}, \mathbf{T}))$ and a principal torus bundle $q : Y \to X$ such that $\mathcal{A}^\infty(X)$ is the parametrised strict deformation quantization of $C^\infty_{\text{fibre}}(Y)$ (continuous functions on Y that are fibrewise smooth) with respect to σ, that is,*

$$\mathcal{A}^\infty(X) \cong C^\infty_{\text{fibre}}(Y)_\sigma.$$

Proof By the construction given earlier, $C^\infty_{\text{fibre}}(Y)_\sigma$ is a fibrewise smooth noncommutative principal torus bundle.

Conversely, if $\mathcal{A}^\infty(X)$ is a fibrewise smooth noncommutative principal torus bundle, then it defines a $\sigma \in C(X, Z^2(\widehat{T}, \mathbf{T}))$. Consider now the deformed algebra $\mathcal{A}^\infty(X)_{\bar{\sigma}}$. It is equivariantly isomorphic to $C^\infty_{\text{fibre}}(Y)$ for some principal torus bundle Y over X, since it is classified by $H^1(X, \underline{T})$. It turns out that parametrized strict deformation quantization [2] shares exactly similar properties to standard case, so we see that the iterated parametrized strict deformation quantization $(\mathcal{A}^\infty(X)_{\bar{\sigma}})_\sigma \cong \mathcal{A}^\infty(X)$ is Morita isomorphic to $C^\infty_{\text{fibre}}(Y)_\sigma$.

3. Discussion

Parametrized strict deformation quantization [2] gives new examples of noncommutative manifolds. In [2] it is used, as outlined here, to classify fibrewise smooth NCP T-bundles. It would be interesting if a similar complete classification can be given for the more general noncommutative torus bundles in [3].

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Reduced phase-space quantisation in Loop Quantum Gravity

Thomas Thiemann

General Relativity is a gauge theory. The gauge group is the group of diffeomorphisms of the underlying 4D manifold. As in other gauge theories such as YM theory, the gauge symmetry of the Lagrangian implies that the Legendre transform to the canonical formulation is singular: Not all velocities can be inverted for the canonical momenta and thus constraints arise. Unlike YM Theory, the canonical Hamiltonian (defined by the (singular) Legendre transform) is a functional linear in those constraints.

This leads to a deep conceptual problem, sometimes called the "problem of time": Observable quantities are gauge invariant and thus have (weakly) vanishing Poisson brackets with the constraints. As a consequence, the canonical Hamiltonian not only is constrained to vanish (unlike YM theory) but also generates trivial evolution for the observables.

Clearly, there is something wrong in interpreting the canonical Hamiltonian as "the" Hamiltonian of the system. In fact, it is a generator of spacetime diffeomorphimsms and thus of gauge transformations and not of any physical time evolution. The resolution of the paradox is that in GR one has to understand evolution relationally as Leibniz envisaged. There is no absolute observer. The operational physical description is observer dependent. Physically, an observer is defined by a material reference system which assigns to mathematical spacetime coordinates a physical meaning (the readings of clocks and rulers).

It turns out that the coupling of a matter reference system to GR gives rise to a physical Hamiltonian which is not constrained to vanish, which is itself an observable and which does not generate gauge transformations but rather non trivial physical time evolution of the observables. In mathematical terms, a material reference system enables one to give a very effective description of the reduced phase space (symplectic manifold) with respect to the (co-isotropic or first class) constraints together with a preferred Hamiltonian.

Traditionally, the quantisation of constrained (or gauge) theories follows the Dirac approach: One looks for a quantisation of the unconstrained phase space which supports the constraints as densely defined and closable operators (after suitable regularisation and renormalisation) and then equips the generalised kernel of the constraints with a new Hilbert space structure. The last step is non trivial in gauge system such as GR where the constraints do not form a Poisson Lie algebra but rather a more general mathematical object that involves (non trivial phase space dependent) structure functions rather than structure constants. As a consequence, standard mathematical techniques from the theory of Rigged Hilbert spaces are not available.

Thus, a reduced phase space approach seems to be more promising for GR. The physical input required is the choice of a suitable material reference system. This is non trivial because the required matter must have nowhere and never vanishing energy density, a non-holonomic requirement that must be gauge invariant. In a seminal work, Brown and Kuchar introduced a system of pressure free dust fields that precisely simulate a congruence of non-interacting particles in geodesic motion. Thus, this system is not only physically motivated but also meets the mathematical requirements.

In our talk we described the classical reduced phase space of GR based on the Brown-Kuchar choice of reference system. Then we presented a quantisation of the resulting Hamiltonian system using methods from Loop Quantum Gravity (LQG). More in detail, the reduced phase space is isomorphic to that of an ordinary SU(2) YM theory together with a physical Hamiltonian which drastically differs from that of YM theory because it encodes of course the Einstein equations rather than the YM equations. Since a description in terms of connections of an SU(2) principal bundle is available, it is natural to employ methods from Lattice gauge theory and to define an algebra of Wilson loop operators. If one insists on unitary implementation of diffeomorphisms on the Hilbert space, then a theorem establishes that there is a unique cyclic representation of the Wilson loop algebra (and their conjugate momenta)! In that representation, the physical Hamiltonian can be implemented as a self- adjoint operator. Moreover, it is possible to construct sets of coherent (resolutions of the identity, Heisenberg uncertainty bound saturating) vectors with respect to which the correct semiclassical limit (the limit of vanishing Planck constant) of the Hamiltonian can be established in the sense of expectation values.

On quantum groups as symmetry groups of noncommutative spaces

Victor Gayral

One aims to have a good definition in the world of C^*-algebras. While it works in the compact, respectively locally compact one, there are many open problems, in particular the question of Haar states. In this talk I gave a class of examples based on non-formal Drinfeld twists coming from quantization of globally symmetric symplectic spaces.

Spin foams and noncommutative geometry

Matilde Marcolli

This talk is based on work in progress, in collaboration with Domenic Denicola.

In the setting of loop quantum gravity (see [2], [17]) spacetime at the Planck scale is discretized, in a background independent formalism, in which the data of the metric structure (the gravitational field) are encoded by spin networks (3-dimensional quantum geometries) and spin foams (4-dimensional cobordisms).

A spin network Ψ in a smooth compact three-manifold M is a triple (Γ, ρ, ι) with:

(1) an oriented embedded graph $\Gamma \subset M$;
(2) a labeling ρ of each edge e of Γ by a representation ρ_e of a Lie group G;

(3) a labeling ι of each vertex v of Γ by an intertwiner

$$\iota_v : \rho_{e_1} \otimes \cdots \otimes \rho_{e_n} \to \rho_{e'_1} \otimes \cdots \otimes \rho_{e'_m},$$

where $e_1, \ldots, e_n$ are the edges incoming to v and $e'_1, \ldots, e'_m$ are the edges outgoing from v.

The ambient isotopy classes of spin networks define elements $|\Psi\rangle$ in a Hilbert space of quantum states. Quantized area and volume operators act on these states, in such a way that vertices and edges of the graph correspond to quanta of volume and area. When one formulates loop quantum gravity in terms of a "sum over histories" (see [2], [17]) the transition amplitudes between different spin networks are given by spin foams.

A spin foam $F : \Psi \to \Psi'$ between spin networks $\Psi = (\Gamma, \rho, \iota)$ and $\Psi' = (\Gamma', \rho', \iota')$, with $\Gamma \subset M$ and $\Gamma' \subset M'$, is a triple $(\Sigma, \tilde{\rho}, \tilde{\iota})$ with:

(1) an oriented 2-complex Σ embedded in a smooth 4-manifold W with $\partial W = M \cup \bar{M}'$, such that $\partial\Sigma = \Gamma \cup \bar{\Gamma}'$,
(2) a labeling $\tilde{\rho}$ of each face f of Σ by a representation $\tilde{\rho}_f$ of G,
(3) a labeling $\tilde{\iota}$ of each edge e of Σ that does not lie in Γ or Γ' by an intertwiner

$$\tilde{\iota}_e : \tilde{\rho}_{f_1} \otimes \cdots \otimes \tilde{\rho}_{f_n} \to \tilde{\rho}_{f'_1} \otimes \cdots \otimes \tilde{\rho}_{f'_m}$$

where $f_1, \ldots, f_n$ are the faces incoming to e (that is, $e \in \partial f_i$) and $f'_1, \ldots, f'_m$ are the faces outgoing from e (that is, $\bar{e} \in \partial f_i$).
(4) the labeling $\tilde{\rho}_f$ and $\tilde{\iota}_e$ of Σ agree with the labeling ρ_e and ι_v of Γ ($\rho_{e'}$ and $\iota_{v'}$ of Γ', respectively) for faces and edges of Σ adjacent to Γ (to Γ', respectively).

Notice that, although the construction is background independent as far as the metric is concerned, the information about the metric tensor being carried by the holonomies of the connection along loops around the edges of the graph, it is *not* background independent as far as the topology is concerned. One does fix a background topology by choosing the 3-manifold M in which the graphs Γ are embedded.

The partition function of Euclidean quantum gravity should consist of a "sum over geometries" (see [13]), which in fact means both a sum over topologies and one over metric structures. To overcome the problem of the topological background dependence, we propose a modified version of the above notions of spin networks and spin foams, which encodes the topology as well as the metric data.

This is based on the fact that all PL (hence smooth) 3-manifolds and 4-manifolds can be realized as branched coverings of the 3-sphere, branched along an embedded graph, respectively of the 4-sphere, branched along an embedded 2-complex (see [3], [11], [16]). The data of the branch coverings is completely specified by a representation of the fundamental group of the complement of the branch locus in the symmetric group on a number of elements equal to the order of the covering. In turn, the fundamental group of the complement is explicitly described by a Wirtinger presentation, in terms of generators and relations.

Thus, one can work with spin foams embedded in S^3 and spin networks embedded in $S^3 \times [0,1]$, where in addition to the labeling described above, one also has a labeling of the edges (resp. faces) by permutations, with Wirtinger relations at vertices (resp. edges) and crossings. Two such data determine the same 3-manifold (resp. 4-manifold) if and only if they can be obtained from one another by a finite sequence of covering moves [3]. The data of representations and intertwiners can be chosen consistently with covering moves. Thus, one obtains a refined class of spin networks and spin foams which encode both the topology and the metric structure of 3-manifolds M and M' and of 4-manifolds W with $\partial W = M \cup \bar{M}'$.

One then can see that, in addition to the usual composition of spin foams, given by gluing two spin foams together along a common boundary spin network, one has another composition, which corresponds to the fibered products of the 3-manifolds and 4-dimensional cobordisms, as branched coverings of S^3 and $S^3 \times [0,1]$. This composition is more similar to the KK-product used in the context of D-branes geometries (see [5], [10]).

Using the results of [15], one can construct an associative noncommutative algebra of coordinates on the space of all 4-dimensional geometries (encoded by spin foams with additional topological labeling as above) and with the two compositions of gluing and fiber product. The resulting algebraic structure is the algebra associated to a 2-category.

One can then construct time evolutions on this algebra. This develops an analogy described in the last chapter of [7] between the quantum statistical mechanical systems associated to number fields in [8], [12], [14]. As in the arithmetic case, one has a algebra of all (possibly degenerate) geometries, and a time evolution, whose low temperature KMS states (see [4]) automatically select, via a symmetry breaking phenomenon, the nondegenerate geometries. The time evolutions in this came come from the quantized area and volume operators and from the Wirtinger relations for the additional topological data on the spin networks and spin foams.

Additional data of matter fields should then be implemented as in the setting of noncommutative geometry models for particle physics [6], by considering almost commutative geometries, obtained as product of a spectral triple associated to the spin networks and spin foams as in [1] and a finite noncommutative space.

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Noncommutative Lattices

Fedele Lizzi

In my talk I discussed some old work done in various collaborations with A.P. Balachandran, G. Bimonte, E. Ercolessi, G. Landi, G. Sparano and P. Teotonio-Sobrinho over a period of about four years 1994-1998. The reason I discussed this relatively old work is that it contains the description of non Hausdorff space from which, in a canonical way, spacetime emerges as a projective limit. The topic has become timely again because the emergence of spacetime as a limit is one of the problems faced by the attempts at a quantum gravity, including the ones based on noncommutative geometry or loop quantum gravity.

Noncommutative lattices stem from an original idea of Sorkin [1] to substitute to a continuous topological space a finite (or countable) number of "points" with a non trivial (non Hausdorff) topology, so to maintain some information of the original topology, which would be totally lost if one were to use a normal lattice of points. This non Hausdorff space is a noncommutative geometry, in the sense that the points and their topology emerge as the set of prime ideals of a noncommutative algebra [2, 3]. The algebra is approximatively finite, i.e. it is the norm limit of finite rank algebras, and in this sense it can be approximated at the algebraic level by matrix algebras.

The limit process of adding more and more points can be handled at the algebraic level as an inductive limit of algebras. The original commutative algebra emerges then as the centre of the C^* completion of the limit [4]. This is in contrast with the limit of usual lattices which unavoidably give the cantor set [5].

The construction hints at the presence of pregeometric substratum of spacetime [6] where the number of degrees of freedom is drastically reduces, but the description must be via a noncommutative geometry.

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Dirac field on Lorentzian non-commutative spacetime

Rainer Verch

A unified framework for non-commutative Lorentzian spacetime is being proposed which suggests a generalization of spectral geometry (developed by A. Connes) to Lorentzian signature. Tentative conditions for Lorentzian spectral triples (LOSTs) are given, together with a conjectural statement of a reconstruction theorem, akin to the Riemannian situation, of a Lorentzian spin geometry in the classical situation where the algebra appearing in the spectral triple data is commutative. Furthermore, we investigate a concrete example, the Lorentzian Moyal plane. Here we consider the second quantization of the spectral triple data, which yields the Dirac field on Minkowski spacetime. The observables of the Dirac field depending on the "observables" of the noncommutative spacetime are obtained by scattering of the Dirac field by a noncommutative potential via Bogoliubov's formula. Unitary implementability of the scattering transformation in the Dirac field's vacuum representation is proved. The observables of the Dirac field arising from the scattering transformation are the same as for the usual, commutative potential scattering, but with the Moyal-Rieffel product between quantum field operators instead of the usual operator product. The ideas on LOSTs are based on a collaboration with M. Paschke. The results on Dirac field scattering by a non-commutative potential has been obtained together with M. Borris.

Quantum Group Connections

Jurek Lewandowski
(joint work with Andrzej Okołow)

The underlying algebra of Loop Quantum Gravity is the Ashtekar-Isham algebra [1]-[8]. This is a commutative, unital C^*-algebra which consists of the so called cylindrical functions defined on the space of $SU(2)$-connections over a given 3-manifold Σ. This algebra is dual, in the Gel'fand-Naimark sense, to a compactification of the space of the $SU(2)$-connections. The Ashtekar-Isham algebra has an equivalent definition which does not involve the connections at all [7, 6]. It can be defined by gluing the algebras $\{C^{(0)}(SU(2))^\gamma\}$ assigned to graphs $\{\gamma\}$ embedded in the manifold Σ. In this construction, the natural partial ordering in the family of graphs is applied to define suitable inductive family of the algebras. The Ashtekar-Isham algebra is useful in LQG, because it admits the action of the diffeomorphisms of Σ, and, moreover, a natural, invariant state generated by the Haar measure [9]. The corresponding Hilbert space serves as the kinematical Hilbert space of LQG. The generalization of the Ashtekar-Isham algebra to an arbitrary compact group (still classical) is quite natural. The goal of our current work [10] is a generalization of the Ashtekar-Isham algebra to a *compact quantum group*, in particular to the $SU_q(2)$ group.

Applications of quantum groups in the context of quantum gravity or field theory on the lattice are known in the literature. For example, quantum group spin-networks play important role in 2+1 quantum gravity [11], in quantization of Chern-Simons theory [12], there is quantum group Yang-Mills theory on the lattice [13] and a lattice gauge theory based on quantum group [14]. Another class of works presents constructions of spectral triples for spaces of connections by using the inductive family approach [15, 16]. Our generalization goes in a third direction. We are not satisfied by assigning an algebra to a single lattice or a graph as in the generalizations of the lattice gauge theory or constructions of quantum group spin-networks. The key difficulty we address is admitting sufficiently large family of embedded graphs and defining the gluing of the algebras in a way consistent with the possible overlappings and other relations between the embedded graphs. In Loop Quantum Gravity that consistency ensures the so called "continuum limit" of the theory: despite of using lattices and graphs the full theory is considered continuous rather than discrete. We will maintain that continuum limit property in our quantum group generalization.

Our motivation to initiate this research was rather mathematical. Since the G-bundle connections can be defined in an alternative way by using the Ashtekar-Isham algebra which, on the other hand, can be seen as constructed from (i) graphs embedded in a manifold and (ii) the C^*-algebra of continuous functions on a compact Lie group equipped with the comultiplication and the antipode, it was tempting to replace the commutative C^*-algebra by a noncommutative one and, along the lines of noncommutative geometry, promote the result to quantum group connections. In the context of LQG it was also interesting to see what kind of quantum generalization of the Lie group $SU(2)$ is consistent with the inductive limit framework. We found that Woronowicz's quantum group $SU_q(2)$ [17] matches the framework in a natural way.

With regard to possible applications of our result let us recall that LQG coupled to a Yang-Mills field uses two Ashtekar-Isham algebras: one for LQG connection and another one for the Yang-Mills connection [8]. Our result could be applied to couple LQG with a quantum group Yang-Mills field.

We begin our lecture with introducing the ingredients that will be used in the presented constructions: the set of embedded graphs directed by a suitable relation, and a compact quantum group.

An embedded graph in a (semi)analytic manifold Σ is a finite set of oriented (semi)analytic edges in Σ, such that every two distinct edges can share only one or the both endpoints. Given two graphs γ', γ we say that γ' is bigger than γ, $\gamma' \geq \gamma$, if every edge of γ can be composed from some edges from the set $\{e_1, \ldots, e_N, e_1^{-1}, \ldots, e_N^{-1}\}$, where the first N edges constitute the graph γ', and e_I^{-1} denotes the edge obtained from e_I by the change of its orientation. One can show that thanks to (semi)analyticity of the edges the set of all graphs with the relation $\geq$ is a directed set.

A Woronowicz's compact quantum group [18] is a pair $(\mathcal{C}, \Phi)$, where $\mathcal{C}$ is a unital (separable) C^*-algebra, and a *comultiplication* $\Phi : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ is a C^*-algebra unital homomorphism such that

$$(\Phi \otimes \mathrm{id})\Phi = (\mathrm{id} \otimes \Phi)\Phi \tag{1}$$

and the sets

$$\{ \ (a \otimes I)\Phi(b) \mid a, b \in \mathcal{C} \ \}$$
$$\{ \ (I \otimes a)\Phi(b) \mid a, b \in \mathcal{C} \ \}$$

(where I is the unit of $\mathcal{C}$) are linearly dense subsets of $\mathcal{C} \otimes \mathcal{C}$.

At the beginning of the lecture we also recall the graph construction of the classical group Ashtekar-Isham algebra, whose generalization is our goal. This constructions starts from assigning to every edge $e \subset \Sigma$ a C^*-algebra

$$C^e \ := \ C^0(G^e), \tag{2}$$

of continuous complex functions on G^e, where G^e is a set of all maps from $\{e\}$ to a compact connected Lie group G equipped with a topology induced by a natural bijection from G^e onto G. Next, to each embedded graph γ we assign the C^*-algebra

$$C^\gamma \ := \ C^0(G^\gamma) \tag{3}$$

where G^γ is the set of all maps from $\gamma = \{e_1, ..., e_N\}$ to G equipped with a topology induced by a natural bijection from G^γ onto G^N. Clearly, $G^\gamma \cong G^{e_1} \times \ldots \times G^{e_N}$, hence (by virtue of Stone-Weierstrass theorem)

$$C^\gamma \ = \ \bigotimes_{e \in \gamma} C^e. \tag{4}$$

Given a pair of graphs such that $\gamma' \geq \gamma$, there is a naturally defined injective unital $*$-homomorphism

$$p_{\gamma'\gamma} : C^{\gamma} \to C^{\gamma'} \tag{5}$$

— in general, the homomorphism is constructed by means of the comultiplication and the antipode on $C^0(G)$ and the unit element of $C^0(G)$.

It is easy to show that the family $\{C^{\gamma}, p_{\gamma'\gamma}\}$ is an inductive family whose inductive limit is the Ashtekar-Isham algebra build over the manifold Σ and the Lie group G.

Our first result is generalization of the construction of the Ashtekar-Isham algebra to a compact quantum group $(\mathcal{C}, \Phi)$ equipped with some extra structure. Given edge e, we associate with it an algebra $\mathcal{C}^e \equiv \mathcal{C}$; consequently we associate with a graph γ a C^*-algebra $\mathcal{C}^{\gamma}$ given by the tensor product of C^*-algebras associated with the edges $\{e_1, \ldots, e_N\}$ of γ:

$$\mathcal{C}^{\gamma} := \bigotimes_{e \in \gamma} \mathcal{C}^e.$$

Then, given a pair $\gamma' \geq \gamma$, we construct an injective unital $*$-homomorphism

$$p_{\gamma'\gamma} : \mathcal{C}^{\gamma} \to \mathcal{C}^{\gamma'} \tag{6}$$

using the comultiplication Φ (provided it is injective), the unit element of $\mathcal{C}$ and (instead of the antipode, which in the noncommutative case is not suitable for our purpose) a C^*-algebra isomorphism $\xi : \mathcal{C} \to \mathcal{C}$ which is an involution and anti-comultiplicative. We call the isomorphism ξ an *internal framing* and this is the extra structure mentioned above[1]. The resulting family $\{\mathcal{C}^{\gamma}, p_{\gamma'\gamma}\}$ again turns out to be an inductive family and its inductive limit is the desired generalization of the Ashtekar-Isham algebra. We name it a *quantum group connection space*.

Next, we characterize the set of the internal framings. We also study the dependence of a quantum group connection space on the internal framing and formulate conditions upon which two different internal framings lead to isomorphic quantum group connection spaces:

Lemma 1. *Let $(\mathcal{C}, \Phi)$ be a compact quantum group. Suppose that, given two internal framings ξ, ξ' there exists an automorphism ρ of the quantum group such that $\xi' = \rho \circ \xi \circ \rho^{-1}$. Then there is a C^*-algebra isomorphism uniquely determined by ρ between the quantum group connection space built over a manifold Σ, the quantum group and ξ and the one built over the same manifold and quantum group and ξ'.*

Lemma 2. *Let $(\mathcal{C}, \Phi)$ be a compact quantum group and Σ a semianalytic manifold which admits an analytic atlas. Then, for every two internal framings ξ, ξ' in the quantum group, the corresponding quantum group connection spaces are isomorphic to each other.*

[1] One could say, that this is the price payed for non-introducing in Σ any framing of edges of the graphs, as it is usually done in the definition of the quantum group spin-networks.

Next we focus on the $SU_q(2)$ quantum group connection spaces. We find all the internal framings in $SU_q(2)$. They form a family which has a natural structure of the circle. Every two internal framings are conjugate to each other by the action of the automorphism group of $SU_q(2)$. Therefore, all the $SU_q(2)$ connection spaces (corresponding to different framings) are isomorphic to each other.

Finally, we formulate yet another, equivalent up to a C^*-algebra isomorphism, definition of a $SU_q(2)$ connection space which does not distinguish any of the internal framings. The construction democratically uses all the internal framings of the $SU_q(2)$ quantum group.

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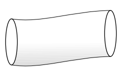
Quasi-quantum groups from strings in 3-form backgrounds

Thomas Krajewski

A peculiar quasi-Hopf algebra $D_\omega[G]$ based on a finite group G and a $U(1)$ valued 3-cocycle ω on G has been introduced in the 1990's by R. Dijkgraaf, V. Pasquier and P. Roche [1], in relation with conformal field theory. In this talk, we show how $D_\omega[G]$ arises as the analog of magnetic translations T for the twisted sectors of a string in a 3-from magnetic background. For the sake of brevity, we do not give here all the technical details and refer the reader to [2] for a detailed account. In particular, let us mention that all the fields are only locally defined and we used the technics pioneered by K. Gawędzki [3] (see also [4]).

First recall that a closed string moving on a manifold $\mathcal{M}$ couples to a magnetic field H which is a closed 3-from on $\mathcal{M}$ with periods in $2\pi\mathbb{Z}$, as is the case for the Wess-Zumino-Witten model where $\mathcal{M}$ is a semi-simple Lie group and H the suitably normalized Maurer-Cartan 3-form. In general H fails to be exact and defines an abelian gerbe with connection on $\mathcal{M}$. As the string moves in $\mathcal{M}$ it sweeps a surface Σ and the corresponding magnetic term to be inserted in the path integral is the holonomy of the corresponding gerbe, which we write symbolically as

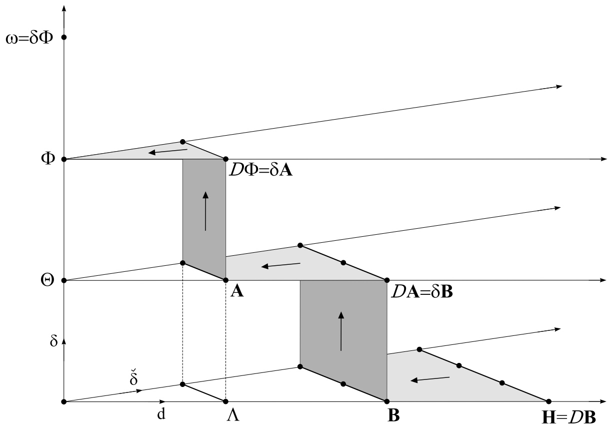
(1)  $\rightarrow \quad \mathrm{e}^{\mathrm{i}\int_\Sigma \mathcal{B}}$

where $\mathcal{B} = (B_i, B_{ij}, f_{ijk})$ is a collection of locally defined fields on a good open cover of $\mathcal{M}$ that define the gerbe with connection.

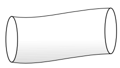
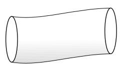
Now let us consider a finite group G acting on $\mathcal{M}$ such that H is preserved, $g^*H = H$ for any $g \in G$. This does not mean that the gerbe with connection is equivariant with respect to the action of G, but assuming that $\mathcal{M}$ is cohomologically trivial in dimension 1 and 2, the gerbes defined by $\mathcal{B}$ and $g^*\mathcal{B}$ are isomorphic. To handle the action of G on gerbes and on their isomorphisms, it is convenient to introduce a tricomplex made of three commuting differentials: the de Rham differential d, the Čech coboundary $\check{\delta}$ and the group coboundary δ in the r direction. Then, the isomorphisms between $\mathcal{B}$ and $g^*\mathcal{B}$ yield a series of cohomological equations yielding a degree 1 term $\mathbf{A}$ and a degree 0 term Φ in the Deligne subcomplex with differential $\mathcal{D}$ as well as a three cocycle ω with values in $U(1)$.

The quasi-quantum group $D_\omega[G]$ arises as the algebra generated by the operators that realize the action of G on the sections of a line bundle on the twisted sectors, i.e. strings that close up to their winding $X(2\pi) = X(0)\cdot w$, with $w \in G$. These operators $T_g^w : \mathcal{H}_{w^g} \to \mathcal{H}_w$ are defined by

(2) $$T_g^w \Psi(X) = \Gamma_{w,g}(x)\, \mathrm{e}^{-\mathrm{i}\int_x^{xw} \mathbf{A}_g}\, \Psi(X\cdot g),$$



with $\Gamma_{w,g} = \Phi_{g,w^g}\Phi^{-1}_{w,g}$ and $w^g = g^{-1}wg$. The actual form of these operators is dictated by their commutation with propagation,

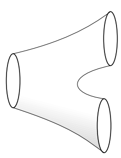
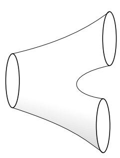
(3) T  $=$  $T.$

These operators obey the following multiplication rule

$$T^w_g T^v_h = \delta_{v,w^g}\,\frac{\omega_{w,g,h}\,\omega_{g,h,w^{gh}}}{\omega_{g,w^g,h}}\,T^w_{gh}, \tag{4}$$

which is nothing but the multiplication rule of the $D_\omega[G]$.

The geometric nature of the string interaction provides $D_\omega[G]$ with additional algebraic structures. The commutation with the pair of pants process $\mathcal{H}_{vw} \to \mathcal{H}_v \otimes \mathcal{H}_w$

(5) T  $=$  $\Delta T.$

imposes

$$\Delta(T^u_g) = \sum_{vw=u} \frac{\omega_{v,w,g}\,\omega_{g,v^g,w^g}}{\omega_{v,g,w^g}}\,T^v_g \otimes T^w_g, \tag{6}$$

which provides the coproduct of $D_\omega[G]$. However, this coproduct is not associative

$$(\mathrm{id}\otimes\Delta)\circ\Delta = \Omega\big((\Delta\otimes\mathrm{id})\circ\Delta\big)\Omega^{-1}, \tag{7}$$

where Ω is the Drinfel'd associator, related to the 3-cocycle ω by

$$\Omega = \sum_{u,v,w} \omega^{-1}_{u,v,w}\,T^u_e \otimes T^v_e \otimes T^w_e. \tag{8}$$

Together with the braiding and the antipode, it provides an example of a quasi-Hopf symmetry, along the lines presented by G. Mack and V. Schomerus in [5].

Finally, we note that this structure also appears in the study of defects in conformal field theory [6].

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Quantum field theory on noncommutative geometries

Raimar Wulkenhaar

Constructive renormalisation of quantum field theories was very successful in low dimensions but a complete failure in four dimensions. The reason is that the only candidate, Yang-Mills theory, is too complicated. Simplifications such as QED or ϕ^4_4-theory would be treatable, but they do not exist due to the Landau ghost problem (resp. triviality). In previous work with Harald Grosse we noticed that if the ϕ^4_4-model is put on a (particular) noncommutative Euclidean space, the β-function is modified so that the model should exist non-perturbatively. There is a realistic chance to prove this.

We consider the quantum field theory defined by the action

$$S = \int d^4x \Big(\frac{1}{2}\phi(-\Delta + \Omega^2\tilde{x}^2 + \mu^2)\phi + \frac{\lambda}{4}\phi\star\phi\star\phi\star\phi\Big)(x)\ . \tag{1}$$

Here, $\star$ refers to the Moyal product parametrised by the antisymmetric 4×4-matrix Θ, and $\tilde{x} = 2\Theta^{-1}x$. We have shown in [1] that (1) gives rise to a renormalisable quantum field theory. The action is covariant under the Langmann-Szabo duality transformation and becomes self-dual at $\Omega = 1$. Evaluation of the β-functions for the coupling constants Ω, λ in first order of perturbation theory leads to a coupled dynamical system which indicates a fixed-point at $\Omega = 1$, while λ remains bounded [2, 3]. The vanishing of the β-function at $\Omega = 1$ was next proven in [4] at three-loop order and finally by Disertori, Gurau, Magnen and Rivasseau [5] to all orders of perturbation theory. It implies that there is no infinite renormalisation of λ, and a non-perturbative construction seems possible. The Landau ghost problem is solved. The action (1) also arises by sign-reversal of μ^2 in the spectral action for the harmonic oscillator spectral triple [6, 7].

The action functional (1) is most conveniently expressed in the matrix base of the Moyal algebra [1]. For $\Omega = 1$ it simplifies to

$$S = \sum_{m,n\in\mathbb{N}^2_\Lambda} \frac{1}{2}\phi_{mn}H_{mn}\phi_{nm} + V(\phi)\;, \tag{2}$$

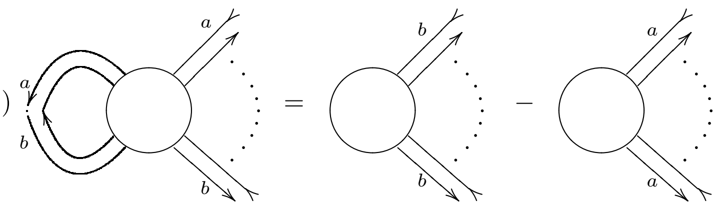
$$H_{mn} = Z\big(\mu^2_{bare} + |m| + |n|\big)\;, \qquad V(\phi) = \frac{Z^2\lambda}{4}\sum_{m,n,k,l\in\mathbb{N}^2_\Lambda} \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\;, \tag{3}$$

The model only needs wavefunction renormalisation $\phi \mapsto \sqrt{Z}\phi$ and mass renormalisation $\mu_{bare} \to \mu$, but no renormalisation of the coupling constant [5] or of $\Omega = 1$. All summation indices $m, n, \ldots$ belong to $\mathbb{N}^2$, with $|m| := m_1 + m_2$, and $\mathbb{N}^2_\Lambda$ refers to a cut-off in the matrix size.

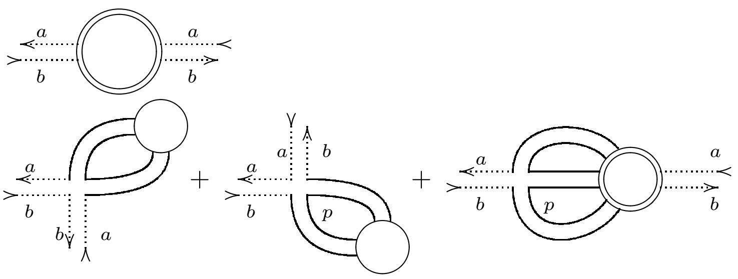
The key step in the proof [5] that the β-function vanishes is the discovery of a Ward identity induced by inner automorphisms $\phi \mapsto U\phi U^\dagger$. Inserting into the connected graphs one special insertion vertex

$$V^{ins}_{ab} := \sum_n (H_{an} - H_{nb})\phi_{bn}\phi_{na} \tag{4}$$

is the same as the difference of graphs with external indices b and a, respectively, $Z(|a| - |b|)G^{ins}_{[ab]\ldots} = G_{b\ldots} - G_{a\ldots}$:

(5) $Z(|a| - |b|)$ 

The Schwinger-Dyson equation for the one-particle irreducible two-point function Γ^{ab} reads

(6) $\Gamma_{ab} =$ 

The sum of the last two graphs can be reexpressed in terms of the two-point function with insertion vertex,

$$\Gamma_{ab} = Z^2\lambda\sum_p \Big(G_{ap} + G_{ab}^{-1}G^{ins}_{[ap]b}\Big) = Z^2\lambda\sum_p \Big(G_{ap} - G_{ab}^{-1}\frac{G_{bp}-G_{ba}}{Z(|p|-|a|)}\Big)$$
$$= Z^2\lambda\sum_p\Big(\frac{1}{H_{ap}-\Gamma_{ap}} + \frac{1}{H_{bp}-\Gamma_{bp}} - \frac{1}{H_{bp}-\Gamma_{bp}}\frac{(\Gamma_{bp}-\Gamma_{ab})}{Z(|p|-|a|)}\Big)\;. \tag{7}$$

This is a closed equation for the two-point function alone. It involves the divergent quantities Γ_{bp} and Z, μ_{bare} in H (3). Introducing the renormalised planar two-point function Γ^{ren}_{ab} by Taylor expansion $\Gamma_{ab} = Z\mu^2_{bare}-\mu^2+(Z-1)(|a|+|b|)+\Gamma^{ren}_{ab}$, with $\Gamma^{ren}_{00} = 0$ and $(\partial\Gamma^{ren})_{00} = 0$, we obtain a coupled system of equations for Γ^{ren}_{ab}, Z and μ_{bare}. It leads to a closed equation for the renormalised function Γ^{ren}_{ab} alone, which is further analysed in the integral representation.

We replace the indices in $a, b, \dots \mathbb{N}$ by continuous variables in $\mathbb{R}_+$. Equation (7) depends only on the length $|a| = a_1 + a_2$ of indices. The Taylor expansion respects this feature, so that we replace $\sum_{p\in\mathbb{N}^2_\Lambda}$ by $\int_0^\Lambda |p|\,dp$. After a convenient change of variables $|a| =: \mu^2\frac{\alpha}{1-\alpha}$, $|p| =: \mu^2\frac{\rho}{1-\rho}$ and

$$\Gamma^{ren}_{ab} =: \mu^2\frac{1-\alpha\beta}{(1-\alpha)(1-\beta)}\Big(1-\frac{1}{G_{\alpha\beta}}\Big)\;, \tag{8}$$

and using an identity resulting from the symmetry $G_{0\alpha} = G_{\alpha 0}$, we arrive at [8]:

Theorem 1. *The renormalised planar connected two-point function $G_{\alpha\beta}$ of self-dual noncommutative ϕ^4_4-theory satisfies the integral equation*

$$\begin{aligned} G_{\alpha\beta} = 1 + \lambda\Big(&\frac{1-\alpha}{1-\alpha\beta}\big(\mathcal{M}_\beta - \mathcal{L}_\beta - \beta\mathcal{Y}\big) + \frac{1-\beta}{1-\alpha\beta}\big(\mathcal{M}_\alpha - \mathcal{L}_\alpha - \alpha\mathcal{Y}\big) \\ &+ \frac{1-\beta}{1-\alpha\beta}\Big(\frac{G_{\alpha\beta}}{G_{0\alpha}}-1\Big)\big(\mathcal{M}_\alpha - \mathcal{L}_\alpha + \alpha\mathcal{N}_{\alpha 0}\big) \\ &- \frac{\alpha(1-\beta)}{1-\alpha\beta}\big(\mathcal{L}_\beta + \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\alpha 0}\big) + \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta}(G_{\alpha\beta}-1)\mathcal{Y}\Big)\;, \end{aligned} \tag{9}$$

where $\alpha, \beta \in [0, 1)$,

$$\mathcal{L}_\alpha := \int_0^1 d\rho\,\frac{G_{\alpha\rho}-G_{0\rho}}{1-\rho}\;,\quad \mathcal{M}_\alpha := \int_0^1 d\rho\,\frac{\alpha\, G_{\alpha\rho}}{1-\alpha\rho}\;,\quad \mathcal{N}_{\alpha\beta} := \int_0^1 d\rho\,\frac{G_{\rho\beta}-G_{\alpha\beta}}{\rho-\alpha}\;,$$

and $\mathcal{Y} = \lim_{\alpha\to 0}\frac{\mathcal{M}_\alpha-\mathcal{L}_\alpha}{\alpha}$.

The non-trivial renormalised four-point function fulfils a linear integral equation with the inhomogeneity determined by the two-point function [8].

These integral equations are the starting point for a perturbative solution. In this way, the renormalised correlation functions are directly obtained, without Feynman graph computation and further renormalisation steps. On the other hand, the implicit function theorem in Banach spaces or the Nash-Moser theorem in Fréchet spaces might be used to prove existence and uniqueness of the solution in a neighbourhood of the free theory.

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