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Interactions between Algebraic Geometry and Noncommutative Algebra

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Abstract. The aim of this workshop was to communicate the most current developments in the field of noncommutative algebra and its interactions with algebraic geometry and representation theory.

Mathematics Subject Classification (2000): 16xx.

Introduction by the Organisers

This meeting had 45 participants from 10 countries (Australia, Belgium, Canada, China, France, Germany, Japan, Norway, UK and the US) and 23 lectures were presented during the five day period. The sponsorship of the European Union and other organizations allowed the organizers to invite and secure the participation of a number of young investigators. Some of these young mathematicians presented thirty-minute lectures. As always, there was a substantial amount of mathematical activity outside the formal lecture sessions. This meeting explored the applications of ideas and techniques from algebraic geometry to noncommutative algebra and vice-versa . A number of lectures presented open problems. Areas covered include

- noncommutative projective algebraic geometry,
- quantum groups,
- combinatorial ring theory,
- representation theory of quivers and preprojective algebras
- applications of categorical techniques in representation theory

A number of advances in the above areas were presented and possible starting points for further research proposed. The breadth of the conference is illustrated by the abstracts.

Workshop: Interactions between Algebraic Geometry and Noncommutative Algebra

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Abstracts

A primitive or PI dichotomy for domains of quadratic growth

Jason Bell

Given a field k and a finitely generated k-algebra A, a k-subspace V of A is called a *frame* of A if V is finite-dimensional, $1 \in V$, and V generates A as a k-algebra.

We say that A has *quadratic growth* if there exist a frame V of A and constants $C_1, C_2 > 0$ such that

$$C_1 n^2 \ \leq\ \dim_k(V^n)\ \leq\ C_2 n^2 \qquad \text{for all } n \geq 1.$$

We note that an algebra of quadratic growth has Gelfand-Kirillov dimension 2. More generally, the *Gelfand-Kirillov* dimension (GK dimension, for short) of a finitely generated k-algebra A is defined to be

$$\mathrm{GKdim}(A)\ =\ \limsup_{n\to\infty} \frac{\log(\dim(V^n))}{\log n},$$

where V is a frame of A. While algebras of quadratic growth have GK dimension 2, it is not the case that an algebra of GK dimension 2 necessarily has quadratic growth. Constructions of algebras of GK dimension two that do not have quadratic growth tend to be contrived and are generally viewed as being pathological. For instance, there are currently no examples of domains, simple rings, or prime noetherian rings of GK dimension 2 that do not also have quadratic growth. Indeed, Smoktunowicz [8] has shown that a graded domain whose GK dimension is in the interval $[2, 3)$ must have quadratic growth. For this reason, quadratic growth is viewed as being, for all intents and purposes, the same as GK dimension two for domains.

GK dimension can be viewed as a noncommutative analogue of Krull dimension in the following sense: if A is a finitely generated commutative k-algebra then the Krull dimension of A and the GK dimension of A coincide. Thus the study of noncommutative finitely generated domains of quadratic growth can be viewed as the noncommutative analogue of the study of regular functions on affine surfaces.

Our main result is the following dichotomy theorem, which shows that a finitely generated prime Goldie algebra of quadratic growth is either very close to being commutative or it is primitive. Given a field k, we say that a k-algebra A satisfies a *polynomial identity* if there is a nonzero noncommutative polynomial $p(t_1, \ldots, t_d) \in k\{t_1, \ldots, t_d\}$ such that $p(a_1, \ldots, a_d) = 0$ for all $(a_1, \ldots, a_d) \in A^d$. We note that a commutative ring satisfies the polynomial identity $t_1t_2 - t_2t_1 = 0$. In general, polynomial identity algebras behave very much like commutative algebras; in fact, a finitely generated prime k-algebra satisfying a polynomial identity always embeds in a matrix ring over a field. Primitive algebras (i.e., algebras with a faithful simple left module), on the other hand, are very different from commutative algebras; indeed, a commutative algebra that is primitive is a field and a theorem of Kaplansky [6, Theorem 13.3.8] generalizes this, showing that a primitive algebra that satisfies a polynomial identity is a matrix ring over a division algebra and, moreover, the division algebra is finite-dimensional over its centre.

There are many dichotomy results in the literature, which show that an algebra with certain specified properties is either primitive or satisfies a polynomial identity [1, 2, 4, 5, 7]. Occasionally, a trichotomy is proved in which one adds the possibility that the algebra may have a nonzero Jacobson radical. Most of these dichotomies require severe restrictions on the algebra that make it easier to study. Our dichotomy result for prime Goldie algebras is less restrictive than most of these other dichotomies, requiring only quadratic growth and an uncountable base field.

Our main theorem is the following: we show that if k is an uncountable field and A is a finitely generated prime Goldie k-algebra of quadratic growth. Then either A is primitive or A satisfies a polynomial identity.

In fact, we show that over any field k, if A is a finitely generated prime Goldie k-algebra of quadratic growth, then either the set of prime ideals P for which A/P has GK dimension 1 is finite or A satisfies a polynomial identity.

The way this result is proved is by studying prime ideals P in A for which A/P has GK dimension 1. We show there are only finitely many such primes unless A satisfies a polynomial identity. This result was proved by the author and Smoktunowicz [4] in the case that A is a prime monomial algebra of quadratic growth using combinatorial techniques. Here we use centralizers to obtain this result. This intermediate result does not require an uncountable base field. We then use an argument due to Farkas and Small [5] to show that if A is a finitely generated prime Goldie algebra of GK dimension 2 over an uncountable base field, and A has only finitely many prime ideals P for which A/P has GK dimension 1, then A must be primitive.

We note that the author and Smoktunowicz [4] constructed a finitely generated prime algebra A of GK dimension 2 that does not satisfy a polynomial identity and has infinitely many primes P such that A/P has GK dimension 1. We note, however, this algebra does not have quadratic growth. The author [3] has also constructed examples of prime rings of GK dimension 2 (but again not of quadratic growth) that do not satisfy the ascending chain condition on prime ideals. Thus without some prime Goldie hypothesis, one cannot expect the conclusion of the statement of our theorem to hold.

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Trees, Amalgams and Calogero-Moser Spaces

Yuri Berest
(joint work with Alimjon Eshmatov and Farkhod Eshmatov)

The theory of infinite-dimensional algebraic groups goes back to the work of Shafarevich [Sh] (see also [Sh1]). Typically, such groups arise as automorphism groups of affine surfaces, and their algebraic structure is determined by their structure (presentation) as a discrete group (see [GD]).

Two interesting examples of different nature arise from noncommutative algebra: these are the group $G_0 := \mathrm{Aut}_\omega\, \mathbb{C}\langle x,\, y\rangle$ of "symplectic" (i.e. preserving $\omega = xy - yx$) automorphisms of the free algebra on two generators and the automorphism group $\mathrm{Aut}\, A_1$ of the Weyl algebra $A_1 = \mathbb{C}\langle x,\, y\rangle/(xy - yx - 1)$. By a theorem of Makar-Limanov [ML], these two groups are naturally isomorphic as discrete groups (the corresponding isomorphism is induced by the canonical projection $\mathbb{C}\langle x,\, y\rangle \to A_1$); however, their algebraic structures are *different* (see [BW], [G]).

In this talk, we will discuss more examples of this phenomenon. Instead of $\mathrm{Aut}\, A_1$, we consider the automorphism groups of algebras (domains) Morita equivalent to A_1. It is known (see [K] and [BW1]) that such algebras are classified, up to isomorphism, by a single integer $n \geq 0$, and the corresponding isomorphism classes can be represented by the endomorphism rings $D_n := \mathrm{End}_{A_1}(M_n)$ of the right ideals $M_n = x^{n+1}A_1 + (xy + n)\, A_1$. In particular, note that $D_0 \cong A_1$.

The groups G_n for $n > 0$ are defined geometrically, in terms of a natural action of G_0 on the *Calogero-Moser spaces*

$$\mathcal{C}_n := \{\, (X,\, Y) \in \mathtt{Mat}_n(\mathbb{C}) \times \mathtt{Mat}_n(\mathbb{C}) \ :\ \mathtt{rk}([X,\, Y] + I_n) = 1 \,\}/\, \mathtt{PGL}_n(\mathbb{C}) \ ,$$

where $\mathtt{PGL}_n(\mathbb{C})$ operates on matrices (X, Y) by simultaneous conjugation (see [W]). Precisely, the action of G_0 on $\mathcal{C}_n$ is given by

$$(X,\, Y) \mapsto (\sigma^{-1}(X),\, \sigma^{-1}(Y)) \ , \quad \sigma \in G_0 \ . \tag{1}$$

It is known that $\mathcal{C}_n$ is a smooth affine variety with a natural symplectic structure, and it is easy to check that G_0 preserve that structure.

Now, for each $n \geq 0$, we fix a basepoint $(X_0,\, Y_0) \in \mathcal{C}_n$, with

$$X_0 = \sum_{k=1}^{n-1} E_{k+1,k} \quad , \qquad Y_0 = \sum_{k=1}^{n-1} (k - n)\, E_{k,k+1} \ ,$$

where $E_{i,j}$ denotes the elementary matrix with (i,j)-entry 1, and let

$$G_n := \mathtt{Stab}_{G_0}(X_0, Y_0)\ , \quad n \geq 0\ . \tag{2}$$

Thus, by definition, the groups $G_n \subseteq G_0$ are the stabilizers of points of the Calogero-Moser spaces under the action (1).

The following result can be viewed as a generalization of the above-mentioned theorem of Makar-Limanov [ML].

Theorem 1. *There is a natural isomorphism of groups* $G_n \xrightarrow{\sim} \mathrm{Aut}(D_n)$.

Theorem 1 is a simple consequence of the main results of [BW]: in fact, it is shown in [BW] that G_0 acts *transitively* on each $\mathcal{C}_n$ and there is a natural G_0-equivariant bijection (called the Calogero-Moser correspondence) between $\bigsqcup_{n\geq 0} \mathcal{C}_n$ and the space of isomorphism classes of right ideals of A_1. Under this bijection, the points $(X_0, Y_0) \in \mathcal{C}_n$ correspond precisely to the classes of the ideals M_n.

We will use Theorem 1 to give a geometric presentation for the groups $\mathrm{Aut}(D_n)$. To this end, we associate to each space $\mathcal{C}_n$ a graph Γ_n consisting of orbits of some simple subgroups of G_0 and identify G_n with the *fundamental group* $\pi_1(\mathbf{\Gamma}_n, *)$ of a graph of groups $\mathbf{\Gamma}_n$ defined by the stabilizers of points of those orbits over Γ_n. The Bass-Serre theory of groups acting on graphs [Se] gives then an explicit formula for $\pi_1(\mathbf{\Gamma}_n, *)$ in terms of generalized amalgamated products, see (3) below.

To define Γ_n we consider the following subgroups of G_0: A is the group of affine symplectic transformations

$$(x,\, y) \mapsto (ax + by + e,\, cx + dy + f)\ , \quad a, b, \dots, f \in \mathbb{C}\ , \quad ad - bc = 1\ ,$$

B is the group of triangular (Jonquières) transformations

$$(x,\, y) \mapsto (ax + p(y),\, a^{-1}y + h)\ , \quad a \in \mathbb{C}^*,\ h \in \mathbb{C}\ , \quad p(y) \in \mathbb{C}[y]\ ,$$

and U is the intersection of A and B: $(x,\, y) \mapsto (ax + by + e,\, a^{-1}y + h)$. Being subgroups of G_0, A, B and U act on each $\mathcal{C}_n$, and we define Γ_n to be the graph of their orbits. Precisely, Γ_n is an oriented bipartite graph, with vertex and edge sets

$$\mathtt{Vert}(\Gamma_n) := (A\backslash\mathcal{C}_n) \bigsqcup (B\backslash\mathcal{C}_n)\ , \quad \mathtt{Edge}(\Gamma_n) := U\backslash\mathcal{C}_n\ ,$$

and the incidence maps $\mathtt{Edge}(\Gamma_n) \to \mathtt{Vert}(\Gamma_n)$ given by the canonical projections $i : U\backslash\mathcal{C}_n \to A\backslash\mathcal{C}_n$ and $\tau : U\backslash\mathcal{C}_n \to B\backslash\mathcal{C}_n$.

Now, on each orbit in $A\backslash\mathcal{C}_n$, $B\backslash\mathcal{C}_n$ and $U\backslash\mathcal{C}_n$ we choose a basepoint and elements $\sigma \in G_0$ moving these basepoints to the basepoint $(X_0,\, Y_0)$ of $\mathcal{C}_n$. Following a standard construction in the Bass-Serre theory, we then assign to the vertices and edges of Γ_n the stabilizers $A_\sigma = G_n \cap \sigma A \sigma^{-1}$, $B_\sigma = G_n \cap \sigma B \sigma^{-1}$, $U_\sigma = G_n \cap \sigma U \sigma^{-1}$ of the corresponding elements σ in the graph of right cosets of G_0 under the action of G_n. These data together with (properly constructed) homomorphisms $a_\sigma : U_\sigma \hookrightarrow A_\sigma$ and $b_\sigma : U_\sigma \hookrightarrow B_\sigma$ define a graph of groups $\mathbf{\Gamma}_n$ over Γ_n, and its fundamental group $\pi_1(\mathbf{\Gamma}_n,\, T)$ relative to a maximal tree $T \subseteq \Gamma_n$ has canonical presentation (see [Se], Sect. 5.1):

$$\pi_1(\mathbf{\Gamma}_n,\, T) = \frac{A_\sigma *_{U_\sigma} B_\sigma * \dots * \langle\, \mathtt{Edge}(\Gamma_n \setminus T)\, \rangle}{(\, e^{-1} a_\sigma(g)\, e = b_\sigma(g)\ :\ \forall\, e \in \mathtt{Edge}(\Gamma_n \setminus T),\, \forall\, g \in U_\sigma\,)}\ . \tag{3}$$

In (3), the amalgams $A_\sigma *_{U_\sigma} B_\sigma * \ldots$ are taken along the stabilizers of edges of the tree T, while $\langle\, \mathtt{Edge}(\Gamma_n \setminus T)\,\rangle$ denotes the free group based on the set of edges of Γ_n in the complement of T.

Our main result is the following

Theorem 2. *For each $n \geq 0$, the group G_n is isomorphic to $\pi_1(\mathbf{\Gamma}_n, T)$. In particular, G_n has an explicit presentation of the form* (3).

Theorems 1 and 2 reduce the problem of describing the groups $\mathrm{Aut}(D_n)$ to a purely geometric problem of describing the structure of the orbit spaces of A and B and U on the Calogero-Moser varieties $\mathcal{C}_n$. Using the earlier results of [W] and [BW] and some basic geometric invariant theory, one can obtain much information about these orbits (and hence about the groups G_n). In particular, the graph Γ_n can be completely described for small n; it turns out to be a tree for $n = 0,\, 1,\, 2$ (see below) but has cycles for $n \geq 3$.

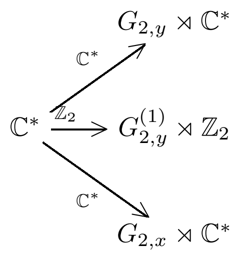
Examples. For $n = 0$, the space $\mathcal{C}_0$ is just a point, and so are a fortiori its orbit spaces. The graph Γ_0 is thus the segment, and the corresponding graph of groups $\mathbf{\Gamma}_0$ is given by $[\, A \xrightarrow{U} B\,]$. Formula (3) then says that $G_0 = A *_U B$, which is a well-known result of [ML] and [Cz] (see also [Co]).

For $n = 1$, we have $\mathcal{C}_1 \cong \mathbb{C}^2$, with (X_0, Y_0) corresponding to the origin. Since each of the groups A, B and U contains translations $(x, y) \mapsto (x + a, y + b)$, $a, b \in \mathbb{C}$, they act transitively on $\mathcal{C}_1$. So again Γ_1 is just the segment, and $\mathbf{\Gamma}_1$ is given by $[\, A_1 \xrightarrow{U_1} B_1\,]$, where $A_1 := G_1 \cap A$, $B_1 := G_1 \cap B$ and $U_1 := G_1 \cap U$. Since, by definition, G_1 consists of all $\sigma \in G_0$ preserving $(0, 0)$, the groups A_1, B_1 and U_1 are obvious:

$$
\begin{aligned}
A_1 \; &: \quad (x,\, y) \mapsto (ax + by,\, cx + dy)\ , \quad a,\, b,\, c,\, d \in \mathbb{C}\ ,\ ad - bc = 1\ , \\
B_1 \; &: \quad (x,\, y) \mapsto (ax + p(y),\, a^{-1}y)\ , \quad a \in \mathbb{C}^*\ ,\ p \in \mathbb{C}[y]\ ,\ p(0) = 0\ , \\
U_1 \; &: \quad (x,\, y) \mapsto (ax + by,\, a^{-1}y)\ , \quad a \in \mathbb{C}^*\ ,\ b \in \mathbb{C}\ .
\end{aligned}
$$

It follows from (3) that $G_1 = A_1 *_{U_1} B_1$.

For $n = 2$, the situation is more interesting. A simple calculation shows that U has three orbits in $\mathcal{C}_2$: two closed orbits of dimension 2 and one open orbit of dimension 4. Moreover, the B-orbits coincide with the U-orbits. Combinatorially, this means that A acts transitively, and the graph Γ_2 is a tree with one nonterminal (the A-orbit) and three terminal (the B-orbits) vertices. The corresponding graph of groups $\mathbf{\Gamma}_2$ is given by



Here $G_{2,x}$ and $G_{2,y}$ are the subgroups of G_0 consisting of transformations $\Phi_p : (x,y) \mapsto (x, y + p(x))$ and $\Psi_q : (x,y) \mapsto (x + q(y), y)$, with $p \in \mathbb{C}[x]$ and $q \in \mathbb{C}[y]$ satisfying $p(0) = p'(0) = 0$ and $q(0) = q'(0) = 0$ respectively, and $G_{2,y}^{(1)} := \{ \Phi_{-x} \Psi_q \Phi_x \in G_0 \; : \; q \in \mathbb{C}[y] \, , \; q(\pm 1) = 0 \}$. Formula (3) yields the presentation

$$G_2 = (G_{2,x} \rtimes \mathbb{C}^*) *_{\mathbb{C}^*} (G_{2,y} \rtimes \mathbb{C}^*) *_{\mathbb{Z}_2} (G_{2,y}^{(1)} \rtimes \mathbb{Z}_2) \, .$$

In particular, G_2 is generated by its subgroups $G_{2,x},\ G_{2,y},\ G_{2,y}^{(1)}$ and $\mathbb{C}^*$.

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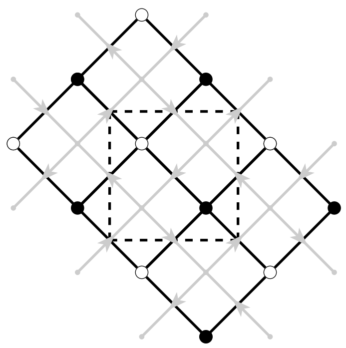
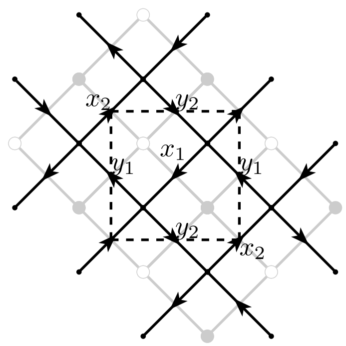
Dimer models and Calabi-Yau algebras

Nathan Broomhead

Using dimer models, as introduced in string theory [3], we can produce non-commutative crepant resolutions (NCCRs) of all toric Gorenstein affine three-folds. In this report we define dimer models, and describe how to construct a non-commutative algebra A and ring R, which is the coordinate ring of a toric Gorenstein affine three-fold, from a given dimer model. We then describe a 'consistency' condition under which A is an NCCR of R.

The theory begins with a finite bipartite tiling of a 2-torus T, that is, a polygonal cell decomposition of T whose vertices can be coloured black and white in such a way that all edges join a black vertex to a white vertex. We often consider





it as a doubly periodic tiling of the plane. We call such a tiling a dimer model. Given a dimer model one can construct the dual tiling (or dual cell decomposition). Crucially, its edges inherit a consistent choice of orientation since the dimer model is bipartite. Thus the dual graph is a quiver Q with the additional structure that it provides a tiling of the torus T with oriented faces. We will refer to the faces of the quiver dual to black/white vertices of the dimer model, as black/white faces.

In the usual way, we denote by Q_0 and Q_1 the sets of vertices and arrows of the quiver and by $\mathbb{C}Q$ the path algebra of the quiver. The additional set Q_2 of oriented faces encodes a 'superpotential'

$$W = \sum_{f \in Q_2} (-1)^f \partial f, \tag{1}$$

a linear combination of cycles in the quiver Q, given by the boundaries of all the faces. The function $(-1)^f$ takes the value $+1$ on black faces of Q, and -1 on white faces. By taking 'cyclic derivatives' of W with respect to each of the arrows we obtain relations. Explicitly, since each arrow $a \in Q_1$ occurs in precisely two oppositely oriented faces $f^+, f^- \in Q_2$, each relation $\frac{\partial}{\partial a} W$ is the difference of two paths $p_a^+ - p_a^-$, where $p_a^\pm$ is the path from the head of a around the boundary of $f^\pm$ to the tail. The quotient of the path algebra $\mathbb{C}Q$ by the ideal I_W generated by these relations is called the superpotential (or 'Jacobian') algebra

$$A = \mathbb{C}Q/I_W.$$

From an algebraic point of view, this is the output of a dimer model.

Example: *We consider the tiling of the torus by squares. The figures show both the bipartite tiling and the dual quiver, drawn together so it is clear how they are related. The dotted line indicates a fundamental domain. The superpotential has two cyclic terms corresponding to the black face and the white face:*

$$W = (x_1 y_2 x_2 y_1) - (x_1 y_1 x_2 y_2).$$

Applying the cyclic derivative with respect to each arrow, we can obtain four relations, for example $\frac{\partial W}{\partial x_2} = y_1 x_1 y_2 - y_2 x_1 y_1 = 0$.

Now we construct the toric variety associated to a dimer model. Using the fact that the 'quiver with faces' Q forms a cell decomposition of the torus T, we may write down a cochain complex

$$\mathbb{Z}^{Q_0} \xrightarrow{d} \mathbb{Z}^{Q_1} \xrightarrow{d} \mathbb{Z}^{Q_2}. \tag{2}$$

Note that, because of the way the faces are oriented, the coboundary map $d\colon \mathbb{Z}^{Q_1} \to \mathbb{Z}^{Q_2}$ simply sums any function of the edges around each face (without any signs). We consider the one-parameter subgroups $\rho\colon \mathbb{C}^* \to \mathrm{Aut}(\mathbb{C}Q)$ arising from an action on the arrows $\rho(t)\colon a \mapsto t^{v_a}a$, for some $v \in \mathbb{Z}^{Q_1}$. Since the coboundary $dv \in \mathbb{Z}^{Q_2}$ gives precisely the weights of the ρ-action on the terms in the superpotential W, ρ is a well-defined map to $\mathrm{Aut}(A)$ when $dv = \lambda\underline{1}$ for some $\lambda \in \mathbb{Z}$, where $\underline{1}$ is the function taking value 1 on every face. We then call λ the *degree* of ρ. We write $N = d^{-1}(\mathbb{Z}\underline{1}) \subset \mathbb{Z}^{Q_1}$ and let $N^+ = N \cap \mathbb{N}^{Q_1}$ be the cone in N defining non-negative gradings of A. There is an exact sequence of lattices

$$0 \to \mathbb{Z} \to \mathbb{Z}^{Q_0} \xrightarrow{d} N \to N_o \to 0 \tag{3}$$

where N_o is the cokernel of d. Note that the lattice N_o is rank 3 and fits in the short exact sequence

$$0 \to H^1(T;\mathbb{Z}) \longrightarrow N_o \xrightarrow{deg} \mathbb{Z} \to 0 \tag{4}$$

We define N_o^+ to be the saturation of the image of N^+ in N_o. Using the machinery of toric geometry this cone defines a three dimensional affine toric variety. In particular it has coordinate ring $R := \mathbb{C}[M_o^+]$, where $M_o^+ := (N_o^+)^\vee$ is the dual cone.

We look for a more explicit combinatorial description of N_o^+. In the literature, a *perfect matching* on a bipartite graph is usually defined as a collection of edges such that each vertex is the end point of precisely one edge (see for example [4]). We take here the dual point of view and consider a perfect matching to be a 1-cochain $\pi \in \mathbb{Z}^{Q_1}$, with all values in $\{0,1\}$, such that $d\pi = \underline{1}$. Thus perfect matchings are actually the degree 1 elements of N^+. Moreover N^+ is integrally generated by perfect matchings (Lemma 2.11 in [1]). We can therefore use the perfect matchings to describe N_o^+. Since they are degree 1 elements their images in N_o^+ define a polygon in a rank 2 affine sublattice, and ${N_o}^+$ is the cone on this polygon. Note that this implies that the toric variety constructed is Gorenstein. Translations of the polygon into $H^1(T;\mathbb{Z}) \cong \mathbb{Z}^2$ may be com! puted by various explicit methods, e.g using the Kastelyn determinant as in [3].

We now construct a second non-commutative algebra from the dimer model. By dualising diagram (3) we obtain the following:

$$\begin{array}{ccccccccc} 0 \longleftarrow & \mathbb{Z} \longleftarrow & \mathbb{Z}Q_0 & \xleftarrow{\delta} & M & \longleftarrow & M_o & \longleftarrow 0 \\ & & & & \cup & & \cup & \\ & & & & M^+ & \longleftarrow & M_o^+ & \end{array}$$

where $M^+ := (N^+)^\vee$ and $M_o^+ := (N_o^+)^\vee$ come from the dual cones. We consider the rank 2 affine sublattices $M_{ij} := \delta^{-1}(j-i)$ of M for $i,j \in Q_0$ and define

$M_{ij}^+ := M_{ij} \cap M^+$. Then we consider the algebra

$$B = \bigoplus_{i,j \in Q_0} \mathbb{C}[M_{ij}^+]$$

where $\mathbb{C}[M_{ij}^+]$ is the vector space with a basis of monomials of the form x^m for $m \in M_{ij}^+$. We call this the *non-commutative (affine) toric algebra*, associated to the data $\{\mathbb{Z}^{Q_0} \xrightarrow{d} N \supset N^+\}$. Note that the ring $R = \mathbb{C}[M_o^+] = \mathbb{C}[M_{ii}^+]$ for all $i \in Q_0$, is the coordinate ring of the toric variety associated to the dimer model. Furthermore, this ring is the centre of algebra B. By construction there is an algebra morphism $\varphi : A \to B$ and we call a dimer model *algebraically consistent* if this map is an isomorphism.

Theorem(Theorem 8.5 in [1]) Given an algebraically consistent dimer model, the algebra A obtained from it is an NCCR of the ring R associated to that dimer model.

There are several different definitions of 'consistency' in the literature. One of these, which we call geometric consistency, is in practice easier to check and we can show that it implies algebraic consistency (Theorem 6.1 in [1]). This gives a way of constructing examples of algebraically consistent dimer models. In particular, Gulotta [2] and Stienstra [5] show that there exists a geometrically consistent dimer model associated to every Gorenstein affine toric threefold, yielding the following:

Corollary (Theorem 8.6 in [1]) Every Gorenstein affine toric threefold admits an NCCR which can be constructed from a dimer model.

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From Lie theory to algebraic geometry and back

DAMIEN CALAQUE

(joint work with Andrei Caldararu and Junwu Tu)

Given a finite dimensional Lie algebra $\mathfrak{g}$ over a field k of zero characteristic, Duflo's Theorem [5] asserts that the restriction of the symmetrization map (also known as PBW isomorphism)

$$\sigma : \mathrm{S}(\mathfrak{g}) \xrightarrow{\sim} \mathrm{U}(\mathfrak{g})$$

to $\mathfrak{g}$-invariants, precomposed with the contraction against the series

$$\partial := \det\left(\sqrt{\frac{e^{ad}-1}{ad}}\right) = \exp\left(\sum_{k\geq 1} c_k \mathrm{tr}(ad^k)\right) \in \widehat{\mathrm{S}}(\mathfrak{g}^*)^{\mathfrak{g}}$$

is an algebra isomorphism[1].

Analogously, given a smooth algebraic variety over k, we can consider the Hochschild-Kostant-Rosenberg (HKR) isomorphism

$$\bigoplus_k \Lambda^k(\mathcal{T}_X)[-k] \xrightarrow{\sim} p_{1*}\left(\mathrm{R}\mathcal{H}om_{X\times X}(\mathcal{O}_X, \mathcal{O}_X)\right)$$

in $\mathcal{D}(\mathcal{O}_X-\mathrm{mod})$. Like in the above situation the (sheaf of) algebra on the right is not (graded) commutative in $\mathcal{D}(\mathcal{O}_X-\mathrm{mod})$[2], but its image under $\mathrm{R}\Gamma(-)$ is. Here again, we need to precompose with the contraction against an element

$$\partial := \det\left(\sqrt{\frac{at}{e^{at}-1}}\right) \in \bigoplus_k H^k(X, \Omega_X^k)$$

to get the following result, first guessed by Kontsevich [9]:
Theorem ([3]). *$HKR \circ \partial\cdot$ is an algebra isomorphism.*

The element $at \in H^1\left(X, \Omega_X^1 \otimes \mathcal{E}nd(\mathcal{T}_X)\right)$ is the Atiyah class of the tangent bundle. Recall that the Atiyah class of a vector bundle $E \to X$ is the obstruction against the existence of a connection on E. More abstractly it is the class of the extension

$$0 \to \Omega_X^1 \otimes E \to J_X^1(E) \to E \to 0\,,$$

and can be viewed as a map $\mathcal{T}_X[-1] \otimes E \to E$ in $\mathcal{D}(\mathcal{O}_X-\mathrm{mod})$. One can prove (see e.g. [8]) that when E is $\mathcal{T}_X[-1]$ this turns $\mathfrak{g} = \mathcal{T}_X[-1]$ into a Lie algebra object in $\mathcal{D}(\mathcal{O}_X-\mathrm{mod})$, and that $\mathcal{D}(\mathcal{O}_X-\mathrm{mod})$ is tautologically equivalent to the representation category of this Lie algebra object. Later on it was proved (see e.g. [10]) that $\mathrm{U}(\mathfrak{g}) \cong p_{1*}\left(\mathrm{R}\mathcal{H}om_{X\times X}(\mathcal{O}_X, \mathcal{O}_X)\right)$. This construction actually becomes more or less tautological, and also works for singular varieties, if one considers the (co)tangent complex [7] instead.

From this we observe that HKR **is** PBW, and that the above Theorem is a straightforward translation of Duflo's result. Namely,

$$\mathrm{R}\Gamma(-) = \mathrm{RHom}_X(\mathcal{O}_X, -) = \mathrm{Hom}_{\mathrm{Rep}(\mathfrak{g})}(\mathbf{1}, -) = (-)^{\mathfrak{g}}\,.$$

Going back to Lie algebras, there are (conjectural) generalizations of Duflo's result. They concern homogeneous spaces, or (at the infinitesimal level) inclusions $\mathfrak{h} \subset \mathfrak{g}$ of finite dimensional Lie algebras. More precisely, under the assumption that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ Duflo conjectured [6] that the Poisson center of $\mathrm{S}(\mathfrak{m})^{\mathfrak{h}}$ is isomorphic (as an algebra) to the center of $\left(\mathrm{U}(\mathfrak{g})/\mathfrak{h}\mathrm{U}(\mathfrak{g})\right)^{\mathfrak{h}}$. This conjecture seems

[1] Here $ad \in \mathfrak{g}^* \otimes \mathrm{End}(\mathfrak{g})$ is the adjoint action.
[2] While it is in $\mathcal{D}(k_X-\mathrm{mod})$.

far too much difficult for us[3]. We will therefore concentrate on an easier question:
Question. *Under what assumption do we have an isomorphism of* $\mathfrak{h}$*-modules*

$$\mathrm{S}\,(\mathfrak{g}/\mathfrak{h}) \xrightarrow{\sim} \mathrm{U}(\mathfrak{g})/\mathfrak{h}\mathrm{U}(\mathfrak{g})\ ?$$

For this purpose let us rewrite

$$\left(\mathrm{U}(\mathfrak{g})/\mathfrak{h}\mathrm{U}(\mathfrak{g})\right)^{\mathfrak{h}} = \mathrm{Hom}_{\mathrm{Rep}(\mathfrak{h})}\big(\mathbf{1}, \mathrm{Res}\circ\mathrm{Ind}(\mathbf{1})\big) = \mathrm{Hom}_{\mathrm{Rep}(\mathfrak{g})}\big(\mathrm{Ind}(\mathbf{1}), \mathrm{Ind}(\mathbf{1})\big)\,.$$

Given a closed embedding $i : X \hookrightarrow Y$ of algebraic varieties, we are going to consider the following Lie algebras in $\mathcal{D}(\mathcal{O}_X-\mathrm{mod})$: $\mathfrak{h} = \mathcal{T}_X[-1] \subset \mathcal{T}_Y\big|_X[-1] = \mathfrak{g}$. Then $\left(\mathrm{U}(\mathfrak{g})/\mathfrak{h}\mathrm{U}(\mathfrak{g})\right)^{\mathfrak{h}}$ becomes

$$\mathrm{Ext}_Y(X,X) := \mathrm{RHom}_Y(i_*\mathcal{O}_X, i_*\mathcal{O}_X) = \mathrm{RHom}_X(i^*i_*\mathcal{O}_X, \mathcal{O}_X)\,.$$

Therefore the above question translates into asking under what assumption we do have an isomorphism

$$\bigoplus_k \Lambda^k(\mathcal{N}_{X,Y})[-k] \xrightarrow{\sim} \mathrm{R}\mathcal{H}om_X(i^*i_*\mathcal{O}_X, \mathcal{O}_X)$$

in $\mathcal{D}(\mathcal{O}_X-\mathrm{mod})$. To answer this question let us consider the normal bundle exact sequence

$$0 \to \mathcal{T}_X \to \mathcal{T}_Y\big|_X \to \mathcal{N}_{X,Y} \to 0\,,$$

which gives a map $\mathcal{N}_{X,Y} \to \mathcal{T}_X[1]$. By tensoring with $\mathcal{N}_{X,Y}$ and composing with the Atiyah "class" of $\mathcal{N}_{X,Y}$ we get an extension $\alpha_{X,Y} \in \mathrm{Ext}^2_X(\mathcal{N}_{X,Y}^{\otimes 2}, \mathcal{N}_{X,Y})$:

$$\mathcal{N}_{X,Y} \otimes \mathcal{N}_{X,Y} \to \mathcal{T}_X[-1] \otimes \mathcal{N}_{X,Y}[2] \to \mathcal{N}_{X,Y}[2]\,.$$

Theorem (Arinkin-Caldararu [1]**).** *The following conditions are equivalent:*

(1) $\alpha_{X,Y} = 0$.
(2) $\mathcal{N}_{X,Y}$ *admits an extension to the first infinitesimal neighbourhood* $X^{(1)}$ *of* X *in* Y.
(3) *the answer to the question is YES.*

Going back once again to Lie algebras, it is now very natural to take a look at the exact sequence $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$ of $\mathfrak{h}$-modules, and the induced map $\mathfrak{g}/\mathfrak{h} \to \mathfrak{h}[1]$ in $\mathcal{D}(\mathfrak{h}-\mathrm{mod})$. Inspired by the geometric situation, we tensor with $\mathfrak{g}/\mathfrak{h}$ and then compose with the $\mathfrak{h}$-action to obtain a class $\alpha_{\mathfrak{h},\mathfrak{g}} \in \mathrm{Ext}^1\big((\mathfrak{g}/\mathfrak{h})^{\otimes 2}, \mathfrak{g}/\mathfrak{h}\big)$:

$$\mathfrak{g}/\mathfrak{h} \otimes \mathfrak{g}/\mathfrak{h} \to \mathfrak{h} \otimes \mathfrak{g}/\mathfrak{h}[1] \to \mathfrak{g}/\mathfrak{h}[1]\,.$$

By complete analogy with Arinkin-Caldararu result we can prove that:
Theorem ([2]**).** *The following conditions are equivalent:*

(1) $\alpha_{\mathfrak{h},\mathfrak{g}} = 0$.
(2) $\mathfrak{g}/\mathfrak{h}$ *"admits an extension to the first infinitesimal neighbourhood* $\mathfrak{h}^{(1)}$ *of* $\mathfrak{h}$ *in* $\mathfrak{g}$*".*
(3) *the answer to the question is YES.*

[3]Even in the symmetric space case, when $\mathfrak{h}$ is the fixed point subalgebra of an involution on $\mathfrak{g}$, the conjecture is not solved despite some very good improvements by Cattaneo-Torossian [4].

We now end this short note by explaining the meaning of condition (2) in the Theorem. First of all we define $\mathfrak{h}^{(1)}$ to be the Lie algebra freely generated by $\mathfrak{g}$ and subjected to the relations

$$[h, g] = [h, g]_{\mathfrak{g}}\,, \quad h \in \mathfrak{h}\,, \quad g \in \mathfrak{g}\,.$$

There is a Lie algera inclusion $\mathfrak{h} \hookrightarrow \mathfrak{h}^{(1)}$, and we say that an $\mathfrak{h}$-module M "admits an extension to $\mathfrak{h}^{(1)}$" if there exists an $\mathfrak{h}^{(1)}$-module $M^{(1)}$ such that $\mathrm{Res}(M^{(1)}) = M$. It can be proved that $\mathcal{T}_{X^{(1)}}\big|_X[-1]$ **is truely** isomorphic to $\mathfrak{h}^{(1)}$ as a Lie algera object in $\mathcal{D}(\mathcal{O}_X\text{-mod})$ (but since $X^{(1)}$ is not smooth, we have to consider the tangent complex instead of the tangent sheaf). E.g. when $X = \{0\} \subset \mathbb{A}^n = Y$ we have that the shifted tangent complex of $X^{(1)}$ is a free Lie algebra in n odd generators.

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Twisted rings and moduli stacks of fat points

Daniel Chan

In 1990, Artin-Tate-Van den Bergh introduced the notion of point modules in [1] to study 3-dimensional Sklyanin algebras. These point modules are essentially torsion-free graded modules with constant Hilbert function 1. Their basic method of study can be described as follows. Let A be a 3-dimensional Skylanin algebra. Then they show there is a Hilbert scheme Y parametrising point modules which is a cubic curve in $\mathbb{P}^2$. Moreover, the shift functor on point modules induces an automorphism σ of Y and associated to the construction of the Hilbert scheme is a line bundle $\mathcal{L}$ on Y. From this data, one can construct a twisted version of the homogeneous co-ordinate ring, namely

$$B = \bigoplus_{i \geq 0} H^0(Y, \mathcal{L} \otimes \sigma^* \mathcal{L} \otimes \ldots \otimes \sigma^{*(i-1)} \mathcal{L}).$$

Finally, there is a natural map $A \longrightarrow B$ and one can try to extract information about A by studying the twisted ring B. In [3], Rogalski-Zhang extended this method to the case where A is a strongly noetherian connected graded algebra generated in degree one. By [2], the hypotheses ensure the existence of Hilbert schemes. They then produce a canonical map from A to a twisted ring on the Hilbert scheme of point modules.

In this talk we look at an extension of these ideas to "fat" point modules which are essentially torsion-free graded modules with constant Hilbert function $m > 1$ which we dub m-points. The key difference with the point module case is that one is forced to consider the moduli stack $\mathcal{Y}$ of simple m-points. We consider now a graded algebra A generated in degree one, satisfying some nice homological properties, namely, strong χ, finite cohomological dimension, CM_1 and projectively connected. Then we show that $\mathcal{Y}$ is isomorphic to $[H/GL_m]$ where H is quasi-projective and is a PGL_m-torsor over some algebraic space Y. The shift functor gives an automorphism σ of the stack $\mathcal{Y}$. This data can be used to construct a twisted ring as follows. The PGL_m-torsor correponds to an Azumaya algebra $\mathcal{A}$ on Y and σ corresponds to an invertible $\mathcal{A}$-bimodule $\mathcal{B}$. We thus obtain a twisted ring

$$\bigoplus_{i \geq 0} H^0(Y, \mathcal{B}^{\otimes i}).$$

Finally, we show there is a canonical map from A to this twisted ring. This result is used to show that a non-commutative projective surface with a surface worth of fat points is birationally PI.

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Macdonald primitivity and D-modules

Iain Gordon

Let V be an n-dimensional complex vector space, $G = GL(V)$, $\mathfrak{g} = \mathfrak{gl}(V)$, and set $\mathfrak{t}$ to be the subalgebra consisting of diagonal matrices. Let $B \leq G$ be the Borel subgroup of upper triangular matrices. The Weyl group, $W = \mathfrak{S}_n$, acts on $\mathfrak{t}$.

Let $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be the commutator. The commuting variety, $\mathfrak{C}$, is the scheme-theoretic fibre $\kappa^*(0)$. Set $\mathfrak{T} = \mathfrak{t} \times \mathfrak{t}$. Simultaneous conjugation provides an action of G on $\mathfrak{C}$ such that the algebraic geometric quotient $\mathfrak{C}/G$ is isomorphic to $\mathfrak{T}/W$. Let $\mathfrak{X} = [\mathfrak{C} \times_{\mathfrak{T}/W} \mathfrak{T}]_{\mathrm{red}}$, the reduced *isospectral commuting variety*, and let $\mathfrak{X}_{\mathrm{norm}}$ be its normalisation with morphism $\psi : \mathfrak{X}_{\mathrm{norm}} \longrightarrow \mathfrak{X}$. There is a projection morphism $p_{\mathfrak{C}} : \mathfrak{X} \longrightarrow \mathfrak{C}$ and an induced morphism on the normalisations $p : \mathfrak{X}_{\mathrm{norm}} \longrightarrow \mathfrak{C}_{\mathrm{norm}}$.

Let $\tilde{\mathfrak{g}} = G \times_B \mathfrak{b}$ be the Grothendieck-Springer resolution. It admits morphisms $\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$ and $\nu : \tilde{\mathfrak{g}} \to \mathfrak{t}$ defined by $(g, x) \mapsto {}^g x$, respectively $(g, x) \mapsto x \bmod [\mathfrak{b}, \mathfrak{b}]$. Let $\mathcal{M} = \int_{\mu \times \nu} \mathcal{O}_{\tilde{\mathfrak{g}}}$, the Hotta-Kashiwara sheaf, a holonomic $D_{\mathfrak{g}\times\mathfrak{t}}$-module.

There is an action of G on $\mathfrak{X}$ induced from $\mathfrak{C}$, of $\mathbb{C}^* \times \mathbb{C}^*$ by dilation in both sets of variables, and of W from the diagonal action on $\mathfrak{T}$. All these lift to $\mathfrak{X}_{\mathrm{norm}}$.

We discussed the following remarkable results of Ginzburg.

Theorem 1. [1, Theorem 2.4.1, Theorem 1.6.3, Proposition 1.6.4]

(1) *The Hodge filtration on* $\mathcal{M}$ *is such that* $gr\mathcal{M} \cong \psi_* \mathcal{O}_{\mathfrak{X}_{norm}}$.
(2) $\mathfrak{X}_{norm}$ *is Cohen-Macaulay and Gorenstein.*
(3) *Set* $\mathcal{R} = p_* \mathcal{O}_{\mathfrak{X}_{norm}}$. *Over the smooth locus of* $\mathfrak{C}$, $\mathcal{R}$ *is a* $G \times W \times \mathbb{C}^* \times \mathbb{C}^*$*-equivariant vector bundle whose fibres carry the regular representation of* W.

Following this we explained a new approach to the positivity of Macdonald polynomials. Recall the (transformed) Macdonald polynomials $\tilde{H}_\mu(z; q, t)$ are symmetric functions with coefficients that are rational functions of two parameters q and t. Expanding these in terms of Schur functions $\tilde{H}_\mu(z; q, t) = \sum_\lambda \tilde{K}_{\lambda,\mu}(q, t) s_\lambda(z)$, Macdonald positivity is the statement that the coefficients $\tilde{K}_{\lambda,\mu}(q, t)$ all belong to $\mathbb{N}[q, t]$. In [2], Haiman confirmed this by proving the $n!$ theorem.

We outlined a new proof of Haiman's theorem using the combinatorics of $\mathcal{R}$. The two crucial points are Hotta and Kashiwara's interpretation of the Springer correspondence in terms of $\mathcal{M}$, [3], and the following well-known commutative diagram which relates the commuting variety to the Hilbert scheme of points on the plane.

$$\mathfrak{C}^\circ \longleftarrow \mathcal{S} \longrightarrow \mathsf{Hilb}^n \mathbb{C}^2$$

Here $\mathfrak{C}^\circ$ is the subvariety of $\mathfrak{C}$ consisting of pairs of commuting matrices X, Y which admit a cyclic vector v (i.e. a vector such that $\mathbb{C}[X, Y] \cdot v = V$) and $\mathcal{S} = \{(X, Y, v) : \kappa(X, Y) = 0, \mathbb{C}[X, Y] \cdot v = V\}$. Essentially the Macdonald polynomials appear as the fibres of $\mathcal{R}$ above the principal nilpotent pairs in $\mathfrak{C}$.

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Rationality of Brauer-Severi Varieties of Sklyanin Algebras

Colin Ingalls

Let k be a perfect field. We say that k has cohomological dimension less than or equal to one $\mathrm{cd}(k) \leq 1$, if $\mathrm{Br}(K) = 0$ for all algebraic extensions K of k. This include finite fields and fields of rational functions on algebraic curves over an algebraically closed field.

Theorem 1. *Let k be a perfect field with $\mathrm{cd}(k) \leq 1$ and characteristic prime to $n+1$. Let $\pi : X \to \mathbb{P}^1_k$ be a morphism of varieties such that the geometric general fibre of π is $\mathbb{P}^n$ and the degree of the non-smooth locus of π is ≤ 2. Then X is rational over k.*

This Theorem has been proved by Iskovskih [3] for $n = 1$. As a corollary we get the following result.

Corollary 1. *Let A be three dimensional Sklyanin algebra that is a finitely generated module over its centre. Then the Brauer-Severi variety of A is rational.*

We refer the reader to [1] and [6] for the necessary definitions.

The proof of the Theorem begins with several reductions. Let η be the generic point of the base $\mathbb{P}^1_k$. The Brauer-Severi variety X_η is rational over any twisted linear subvariety by a result of Roquette [4]. So we may replace X with a non-empty twisted linear subvariety of minimal dimension. The Fadeev exact sequence, Corollary 6.4.6 [2], shows that if the non-smooth locus of π has degree ≤ 1 then X is rational so we may assume the non-smooth locus of π has degree two. Next we find the division algebra $\mathcal{A}_\eta$ corresponding to X_η and choose a maximal order $\mathcal{A}$ in $\mathcal{A}_\eta$. We replace X with the the Brauer-Severi variety of $\mathcal{A}$. We extend work of Artin that uses the structure of maximal orders over discrete valuation rings to see that étale locally X is the blow up at a flag of $\mathbb{P}^n \times \Delta$ where Δ is an étale neighbourhood of a point in $\mathbb{P}^1_k$.

Next we pass to $\overline{k}$, an algebraic closure of k, and analyse $\overline{\pi} : \overline{X} \to \mathbb{P}^1_{\overline{k}}$. We can contract $\overline{X} \to Y \to \mathbb{P}^1_{\overline{k}}$ to a $\mathbb{P}^n$ bundle over $\mathbb{P}^1$ with two complete flags marked in distinct fibres. We need the following Lemma.

Lemma 1. *Let k be a field and let Y be a $\mathbb{P}^n$ bundle over $\mathbb{P}^1$ with marked flags in two distinct fibres. Then Y can be given the structure of a toric variety such that the flags are toric invariant.*

With the toric structure we can begin a combinatorial analysis. We study the normal bundles of the toric invariant sections.

Lemma 2. *There is a section s of $\overline{\pi}$ so that its normal bundle has no non-zero global sections.*

The following Lemma is the main technical part of the proof.

Lemma 3. *The variety X has a non-trivial twisted linear subspace or $\overline{X}$ is the blow up at two complementary flags in distinct fibres of $\mathbb{P}^n \times \mathbb{P}^1$.*

The span of the Galois orbit of the section s descends to a twisted linear subvariety of X. So X is rational unless all toric invariant sections are isomorphic. So all normal bundles of the toric invariant sections are the same. The only possibility is the one stated in the result above.

We then can use the Galois invariant divisors K and a fibre F to construct a complete linear system $|-K-F|$ which defines a rational map $\overline{X} \dashrightarrow (\mathbb{P}^1)^{n+1}$. This map descends to a map $X \dashrightarrow Z$ where Z is a form of $(\mathbb{P}^1)^{n+1}$ which has a point since k is perfect field with $\mathrm{cd}(k) \leq 1$, Corollary 1 [5]. Lastly, we use the point p and the divisor H giving the Segre embedding to show that Z is rational by mapping with the complete linear system $|H - np|$.

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Stable categories of Cohen-Macaulay modules and cluster categories

Osamu Iyama

(joint work with Claire Amiot, Idun Reiten)

Let $\mathcal{T}$ be a k-linear triangulated category with the suspension functor [1] over a field k. For an integer n, we say that $\mathcal{T}$ is *n-Calabi-Yau* (*n-CY*) if there exists a functorial isomorphism $\mathrm{Hom}_{\mathcal{T}}(X,Y) \simeq D\mathrm{Hom}_{\mathcal{T}}(Y,X[n])$ for any $X,Y \in \mathcal{T}$, where $D = \mathrm{Hom}_k(-,k)$ is the k-dual. In representation theory, there are two important classes of n-CY triangulated categories. One is the *generalized n-cluster categories* [BMRRT, Am, G] appearing in study of Fomin-Zelevinsky cluster algebras. The other is the *stable categories* of Cohen-Macaulay modules over Gorenstein isolated singularities [Au1]. The aim of this paper is to compare these two classes of categories. We will show that the stable categories of Cohen-Macaulay modules over certain Gorenstein isolated singularities are triangle equivalent to generalized n-cluster categories (Theorem 1).

1. Preliminaries

Let $n \geq 1$. A key notion in n-CY triangulated categories $\mathcal{T}$ is *n-cluster tilting* objects $M \in \mathcal{T}$ defined by $\mathrm{add}M = \{X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(M, X[i]) = 0 \ (0 < i <$

$n)\}$. They are certain analogue of tilting objects, and 1-cluster tilting objects are nothing but additive generators of $\mathcal{T}$.

1.1. **Cluster categories.** Let $n \geq 2$, and let A be a finite dimensional k-algebra with $\mathrm{gl.dim}A \leq n$. We denote by $\mathcal{D}_A$ the bounded derived category of the category $\mathrm{mod}A$ of finitely generated A-modules, and by $\nu := - \overset{\mathbf{L}}{\otimes}_A DA : \mathcal{D}_A \to \mathcal{D}_A$ the Nakayama functor. We have Auslander-Reiten-Serre duality $\mathrm{Hom}_{\mathcal{D}_A}(X,Y) \simeq D\mathrm{Hom}_{\mathcal{D}_A}(Y, \nu X)$ for any $X, Y \in \mathcal{D}_A$ [Ha]. Let $\nu_n := \nu \circ [-n] : \mathcal{D}_A \to \mathcal{D}_A$. If $\mathrm{gl.dim}A \leq 1$, then the orbit category $\mathcal{C}_A^{(n)} := \mathcal{D}_A/\nu_n$ forms an n-CY triangulated category called the *n-cluster category* [BMRRT, K1]. This is not the case for $\mathrm{gl.dim}A \geq 2$, and the *generalized n-cluster category* $\mathcal{C}_A^{(n)}$ is defined in [K1, Am, G] as a 'triangulated hull' of the orbit category $\mathcal{D}_A/\nu_n$ under the assumption that the functor $H^0(\nu_n) : \mathrm{mod}A \to \mathrm{mod}A$ is nilpotent. This is an n-CY triangulated category with a triangle functor $\pi : \mathcal{D}_A \to \mathcal{C}_A^{(n)}$ satisfying a certain universal property and has an n-cluster tilting object $\pi A \in \mathcal{C}_A^{(n)}$.

1.2. **Stable categories.** Let R be a complete local Gorenstein ring of Krull dimension d. We denote by $\mathrm{CM}(R) := \{X \in \mathrm{mod}R \mid \mathrm{Ext}^i_R(X, R) = 0 \ (0 < i)\}$ the category of maximal Cohen-Macaulay R-modules, and by $\underline{\mathrm{CM}}(R)$ its stable category. It is known that $\underline{\mathrm{CM}}(R)$ forms a triangulated category [Ha], and is triangle equivalent to $\mathcal{D}_R/\mathrm{per}R$ [B]. Assume that R is an isolated singularity. Then $\underline{\mathrm{CM}}(R)$ forms a $(d-1)$-CY triangulated category by a classical result due to Auslander [Au1]. If $M \in \underline{\mathrm{CM}}(R)$ is $(d-1)$-cluster tilting, then $\Gamma := \mathrm{End}_R(R \oplus M)$ satisfies $\mathrm{gl.dim}\Gamma = d$ and $\Gamma \in \mathrm{CM}(R)$ [I2]. In particular Γ is a non-commutative crepant resolution in the sense of Van den Bergh [V]. The existence of a $(d-1)$-cluster tilting object in $\underline{\mathrm{CM}}(R)$ is closely related to the geometry of resolutions of the singularity $\mathrm{Spec}R$.

Let $S := k[[x_1, \cdots, x_d]]$ be the formal power series ring over a field k of characteristic zero, and let G be a finite subgroup of $\mathrm{SL}_d(k)$. If the quotient singularity $R := S^G$ is isolated, then $S \in \underline{\mathrm{CM}}(R)$ is $(d-1)$-cluster tilting [I1]. In particular, if $d = 2$, we have $\underline{\mathrm{CM}}(R) = \mathrm{add}S$ and so R is representation-finite [Au2, He].

2. Main results

Let k be a field of characteristic zero. Let $G = \frac{1}{n}(a_1, \cdots, a_d)$ be a cyclic subgroup of $\mathrm{SL}_d(k)$ generated by a diagonal matrix $g = \mathrm{diag}(\zeta^{a_1}, \cdots, \zeta^{a_d})$ with a primitive n-th root ζ of unity and integers a_i satisfying $0 < a_i < n$, $(n, a_i) = 1$ and $\sum_{i=1}^d a_i = n$. Let $S = k[x_1, \cdots, x_d]$ be a polynomial algebra of d variables. Then S has a $\frac{\mathbb{Z}}{n}$-graded algebra structure $S = \bigoplus_{i \geq 0} S_{\frac{i}{n}}$ defined by $\deg x_i := \frac{a_i}{n}$. The invariant subring $R := S^G = \bigoplus_{i \geq 0} S_i$ is a Gorenstein isolated singularity. For $0 \leq j < n$, we define a $\mathbb{Z}$-graded R-module $T^j := \bigoplus_{i \geq 0} (T^j)_i$ by $(T^j)_i := S_{i + \frac{j}{n}}$. Let $T := \bigoplus_{j=0}^{n-1} T^j$. Then $B := \mathrm{End}_R(T)$ has a $\mathbb{Z}$-graded algebra structure $B = \bigoplus_{i \geq 0} B_i$ with the degree zero part $A := B_0 = \mathrm{End}_R^{\mathbb{Z}}(T)$. Let e be the idempotent of A corresponding to the direct summand T^0 of T, and $\underline{A} := A/\langle e\rangle$. Our main result is the following [AIR]:

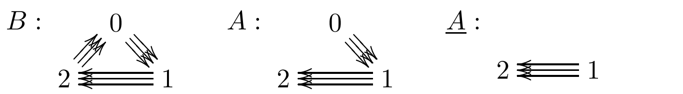
Theorem 1 We have a triangle equivalence $\underline{\mathrm{CM}}(R) \simeq \mathcal{C}^{(d-1)}_{\underline{A}}$.

Remark 2 (a) B is isomorphic to the skew group algebra $S*G$ [Au2], whose quiver is given by the McKay quiver of G. The relations are given by higher derivative of a potential [BSW].

(b) A related result is given in [DV].

(c) Theorem 1 is an analogue of Ueda's equivalence $\underline{\mathrm{CM}}^{\mathbb{Z}}(R) \simeq \mathcal{D}_{\underline{A}}$ [U].

Example 3 Let $G = \frac{1}{3}(1,1,1)$. The algebras B, A and $\underline{A}$ are presented by quivers



Thus $\underline{\mathrm{CM}}(R)$ is triangle equivalent to the cluster category of $2 \overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}} 1$, and we recover a result by Keller and Reiten [KR].

Theorem 1 is a special case of the following result:

Let $B = \bigoplus_{i\geq 0} B_i$ be a graded k-algebra such that $\dim_k B_i < \infty$.

- B is a bimodule d-Calabi-Yau algebra of Gorenstein parameter 1, i.e. $B \in \mathrm{per} B^{\mathrm{e}}$ and $\mathbf{R}\mathrm{Hom}_{B^{\mathrm{e}}}(B, B^{\mathrm{e}})[d] \simeq B(1)$.
- $A := B_0$ has an idempotent e such that $eA(1-e) = 0$.
- B is noetherian and $\underline{B} := B/\langle e\rangle$ is a finite dimensional k-algebra.
- $C := eBe$ satisfies $\mathrm{End}_C(Be) = B$ and $\mathrm{End}_{C^{\mathrm{op}}}(eB) = B$.

Theorem 4 We have a triangle equivalence F and the commutative diagram:

$$\begin{array}{ccccc} \mathcal{D}_{\underline{A}} & \longrightarrow & \mathcal{D}_A & \xrightarrow{-\overset{\mathbf{L}}{\otimes}_A Be} & \mathcal{D}_C \\ \downarrow & & & & \downarrow \\ \mathcal{C}^{(d-1)}_{\underline{A}} & & \xrightarrow{\quad F \quad} & & \underline{\mathrm{CM}}(C) \end{array}$$

The key observation is the following.

Lemma 5 There exists a triangle in $\mathcal{D}(\mathrm{mod}^{\mathbb{Z}}(A^{\mathrm{op}} \otimes_k B))$:

$$A[-1] \to \mathbf{R}\mathrm{Hom}_{A^{\mathrm{e}}}(A, A^{\mathrm{e}}) \overset{\mathbf{L}}{\otimes}_A B(-1)[d-1] \to B \to A$$

As an application of Lemma 5, the *derived d-preprojective DG algebra* [K2] of A is B. In particular A is $(d-1)$*-representation-infinite* in the sense of [IO] or a *quasi* $(d-1)$*-Fano algebra* in the sense of [MM].

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From total positivity to quantum algebras

Stéphane Launois
(joint work with Ken Goodearl and Tom Lenagan)

In recent publications, the same combinatorial description has arisen from three separate objects of interest: totally nonnegative cells in the totally nonnegative grassmannian [14, 15]; torus orbits of symplectic leaves in the classical grassmannian [3, 10]; and torus invariant prime ideals in the quantum grassmannian [5, 13]. The reasons for this coincidence were recently explored in the matrix case in collaboration with Ken Goodearl and Tom Lenagan [8, 9, 12]. Before stating our main results, we give a bit of background on the three areas involved.

A real matrix is *totally nonnegative* (tnn for short) if all of its minors are nonnegative. The theory of tnn matrices was pioneered in the 1930's by Gantmacher, Krein and Schoenberg. Since then this theory has found numerous applications not only in pure mathematics, but also in statistics, game theory, mathematical economics, mathematical biology, etc. (See, for instance, [1, 6, 7].) One can specify a cell decomposition of the space of tnn matrices by specifying exactly which minors are to be zero/nonzero. However, some cells are empty. Recently, in [14], Postnikov classified the nonempty cells by means of a bijection with certain diagrams, known as *Le-diagrams*. By using this bijection, Williams [15] was able to count the number of nonempty cells. One natural question is then: what are the families of minors that define nonempty cells?

The algebra of $m \times p$ quantum matrices is a noncommutative deformation of the coordinate ring of the variety of $m \times p$ complex matrices. This algebra is endowed with a natural action by a torus and a key ingredient in the study of the structure of this algebra is an understanding of the torus invariant objects. For instance, the Stratification Theory of Goodearl and Letzter [2] shows that, in the generic case, a complete understanding of the prime spectrum of quantum matrices would start by classifying the (finitely many) torus invariant prime ideals. In [5], Cauchon succeeded in counting the number of torus invariant prime ideals in quantum matrices. His method involved a bijection between certain diagrams, now known as *Cauchon diagrams*, and the torus invariant primes. It was then proved in [11] that every torus invariant prime ideal is generated by so-called *quantum minors* (some distinguished elements of the algebra of quantum matrices). One natural question here is: what are the families of quantum minors that generate torus invariant prime ideals?

The semiclassical limit of quantum matrices is the classical coordinate ring of the variety of matrices endowed with a Poisson bracket that encodes the nature of the quantum deformation which leads to quantum matrices. As a result, the variety of matrices is endowed with a Poisson structure, and so the variety of matrices is a disjoint union of symplectic submanifolds called *symplectic leaves*. Again, a natural torus action leads to a stratification of the variety via torus orbits of symplectic leaves. In [3], Brown, Goodearl and Yakimov showed that there are finitely many such torus orbits of symplectic leaves. The classification is given in terms of certain "restricted" permutations from the relevant symmetric group with restrictions arising from the Bruhat order. Moreover each torus orbit is defined by certain rank conditions on submatrices, so that the closure of a torus orbit of symplectic leaves is defined by the vanishing of a family of minors. A natural question is then: what are these families of minors?

The interesting observation from the point of view of this work is that in each of the above three sets of results the combinatorial objects that arise turn out to be the same, although the methods employed to obtain the results are very different! The definitions of Cauchon diagrams and Le-diagrams are the same, and the restricted permutations arising in the Brown-Goodearl-Yakimov study can be seen to lead to Cauchon/Le diagrams via the notion of pipe dreams. Postnikov's work is largely combinatorial, Brown-Goodearl-Yakimov employ algebraic geometry, while Cauchon's work is mainly noncommutative algebra. This coincidence was first observed in [13], where the work of Cauchon on quantum matrices was extended to the quantum grassmannian. More precisely, it was shown that torus invariant prime ideals in the quantum grassmannian are in 1:1 correspondence with nonempty cells in the totally nonnegative grassmannian–the space of points in the real grassmannian whose Plücker coordinates are all nonnegative.

More recently the connection between these three areas have been studied in the matrix case [8, 9, 12]. The main result can be expressed as follows.

Theorem 1. *Let $\mathcal{F}$ be a family of minors in the coordinate ring of $M_{m,p}(\mathbb{C})$, and let $\mathcal{F}_q$ be the corresponding family of quantum minors in $O_q(M_{m,p}(\mathbb{C}))$. Then the following are equivalent:*

(1) *The totally nonnegative cell associated to $\mathcal{F}$ is non-empty.*
(2) *$\mathcal{F}$ is the set of minors that vanish on the closure of a torus orbit of symplectic leaves in $M_{m,p}(\mathbb{C})$.*
(3) *$\mathcal{F}_q$ is the set of quantum minors that belong to a torus invariant prime in $O_q(M_{m,p}(\mathbb{C}))$.*

The above families of (quantum) minors can then be described explicitly in the Poisson case, and so in all three areas thanks to the above transfer theorem. As torus invariant primes are generated by quantum minors in the generic case, this allows us to describe explicitly the families of quantum minors that generate torus invariant prime ideals. Recently and independently, Yakimov [16] also described explicit families of quantum minors that generate torus invariant primes. However his families of quantum minors are smaller than the ones appearing in Theorem 1 (3). The problem of deciding whether a given quantum minor belongs to the torus invariant prime associated to a given Cauchon diagram has been studied recently by Casteels [4] who gave a combinatorial criterion inspired by Lindström's Lemma.

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Unipotent cells and preprojective algebras

Bernard Leclerc
(joint work with Christof Geiß and Jan Schröer)

The aim of the talk was to give an interpretation of the *Chamber Ansatz* of Berenstein, Fomin and Zelevinsly [2, 3] in terms of preprojective algebras (see also [6]).

Let Q be a finite quiver with vertex set $\{1,\dots,n\}$ and without oriented cycles. Denote by Λ the preprojective algebra of Q. Let $\mathfrak{g}$ be the Kac-Moody Lie algebra with Cartan datum given by Q, and let W be the Weyl group of $\mathfrak{g}$. The graded dual $U(\mathfrak{n})^*_{\rm gr}$ of the universal enveloping algebra $U(\mathfrak{n})$ of the positive part $\mathfrak{n}$ of $\mathfrak{g}$ can be identified with the coordinate ring $\mathbb{C}[N]$ of an associated pro-unipotent pro-group N with Lie algebra $\mathfrak{n}$.

For $w \in W$, let $N^w := N \cap (B_- w B_-)$ be the corresponding *unipotent cell* in N. Here B_- denotes the standard negative Borel subgroup of a Kac-Moody group attached to $\mathfrak{g}$. Let $x_i(t)$ denote the one-parameter subgroup of N associated to the simple root α_i. For each reduced expression $\mathbf{i} = (i_r,\dots,i_1)$ of w, the map

$$\mathbf{x_i} : (t_r,\dots,t_2,t_1) \mapsto x_{i_r}(t_r)\cdots x_{i_2}(t_2)x_{i_1}(t_1) = x$$

gives a birational isomorphism from $(\mathbb{C}^*)^r$ to N^w.

In [5] we have described a cluster algebra structure on $\mathbb{C}[N^w]$ in terms of the representation theory of Λ. For a nilpotent Λ-module X and $\mathbf{a} = (a_r,\dots,a_1) \in \mathbb{N}^r$, let $\mathcal{F}_{\mathbf{i},\mathbf{a},X}$ be the projective variety of flags

$$X_\bullet = (0 = X_r \subseteq \cdots \subseteq X_1 \subseteq X_0 = X)$$

of submodules of X such that $X_{k-1}/X_k \cong S_{i_k}^{a_k}$ for all $1 \le k \le r$, where S_j denotes the one-dimensional Λ-module supported on the vertex j of Q. The varieties $\mathcal{F}_{\mathbf{i},\mathbf{a},X}$ were first introduced by Lusztig [7] for his Lagrangian construction of $U(\mathfrak{n})$. Dualizing Lusztig's construction, we can associate with X a regular function $\varphi_X \in \mathbb{C}[N]$ satisfying

$$\varphi_X(\mathbf{x_i}(\mathbf{t})) = \sum_{\mathbf{a}\in\mathbb{N}^r} \chi(\mathcal{F}_{\mathbf{i},\mathbf{a},X})\mathbf{t}^{\mathbf{a}}.$$

Here $\mathbf{t} = (t_r,\dots,t_1) \in \mathbb{C}^r$, $\mathbf{t}^{\mathbf{a}} := t_r^{a_r}\cdots t_2^{a_2}t_1^{a_1}$, and χ denotes the topological Euler characteristic.

Buan, Iyama, Reiten, and Scott [1] have attached to w a 2-Calabi-Yau Frobenius subcategory $\mathcal{C}_w$ of the category of finite-dimensional nilpotent Λ-modules. (The same categories were studied independently in [4] for special elements w called adaptable.) In [5] we showed that the $\mathbb{C}$-span of

$$\{\varphi_X \mid X \in \mathcal{C}_w\}$$

is a subalgebra of $\mathbb{C}[N]$, which becomes isomorphic to $\mathbb{C}[N^w]$ after localization at the multiplicative subset

$$\{\varphi_P \mid P \text{ is } \mathcal{C}_w\text{-projective-injective}\}.$$

Moreover, we showed that this provides a cluster algebra structure on $\mathbb{C}[N^w]$, whose cluster variables are of the form φ_X for indecomposable modules X in $\mathcal{C}_w$ without self-extension.

The category $\mathcal{C}_w$ comes with a remarkable module $V_{\mathbf{i}}$ for each reduced expression $\mathbf{i}$ of w (see [1, Section III.2], [5, Section 2.4]). The φ-functions of the indecomposable direct summands of $V_{\mathbf{i}}$ are some generalized minors on N which form a natural initial cluster of $\mathbb{C}[N^w]$. We introduce the new module

$$W_{\mathbf{i}} := I_w \oplus \Omega_w(V_{\mathbf{i}}),$$

where $\Omega_w = \tau_w^{-1}$ is the inverse Auslander-Reiten translation of $\mathcal{C}_w$, and I_w is the direct sum of the indecomposable $\mathcal{C}_w$-projective-injectives. For a Λ-module X, the set $\mathrm{Ext}^1_\Lambda(W_{\mathbf{i}}, X)$ is in a natural way a left module over the stable endomorphism algebra

$$\underline{\mathcal{E}} := \underline{\mathrm{End}}_{\mathcal{C}_w}(W_{\mathbf{i}})^{\mathrm{op}} \cong \underline{\mathrm{End}}_{\mathcal{C}_w}(V_{\mathbf{i}})^{\mathrm{op}}.$$

Denote by $\mathrm{Gr}^{\underline{\mathcal{E}}}_{\mathbf{d}}(\mathrm{Ext}^1_\Lambda(W_{\mathbf{i}}, X))$ the variety of $\underline{\mathcal{E}}$-submodules of $\mathrm{Ext}^1_\Lambda(W_{\mathbf{i}}, X)$ with dimension vector $\mathbf{d}$, a so-called *quiver Grassmannian.* Our first main result is

Theorem 1. For $X \in \mathcal{C}_w$ and all $\mathbf{a} \in \mathbb{N}^r$, there is an isomorphism of projective varieties

$$\mathcal{F}_{\mathbf{i},\mathbf{a},X} \cong \mathrm{Gr}^{\underline{\mathcal{E}}}_{d_{\mathbf{i},X}(\mathbf{a})}(\mathrm{Ext}^1_\Lambda(W_{\mathbf{i}}, X)).$$

Here, $d_{\mathbf{i},X}$ denotes an explicit combinatorial bijection from $\{\mathbf{a} \mid \mathcal{F}_{\mathbf{i},\mathbf{a},X} \neq \emptyset\}$ to $\{\mathbf{d} \mid \mathrm{Gr}^{\underline{\mathcal{E}}}_{\mathbf{d}}(\mathrm{Ext}^1_\Lambda(W_{\mathbf{i}}, X)) \neq \emptyset\}$.

It follows easily that the set $\{\mathbf{a} \mid \mathcal{F}_{\mathbf{i},\mathbf{a},X} \neq \emptyset\}$ has a unique element if and only if $\mathrm{Ext}^1_\Lambda(W_{\mathbf{i}}, X) = 0$. Now by construction, $W_{\mathbf{i}}$ is a cluster-tilting module of $\mathcal{C}_w$, that is, $\mathrm{Ext}^1_\Lambda(W_{\mathbf{i}}, X) = 0$ if and only if X belongs to the additive hull $\mathrm{add}(W_{\mathbf{i}})$ of $W_{\mathbf{i}}$. Moreover, in this case $\mathcal{F}_{\mathbf{i},\mathbf{a},X}$ is reduced to a point. Hence Theorem 1 has the following important consequence:

Theorem 2. For $X \in \mathcal{C}_w$, the polynomial function $\mathbf{t} \mapsto \varphi_X(\mathbf{x_i}(\mathbf{t}))$ is reduced to a single monomial $\mathbf{t}^{\mathbf{a}}$ if and only if $X \in \mathrm{add}(W_{\mathbf{i}})$.

Let $W_{\mathbf{i},1}, \dots, W_{\mathbf{i},r}$ denote the indecomposable direct summands of $W_{\mathbf{i}}$. The r-tuple of regular functions $(\varphi_{W_{\mathbf{i},1}}, \dots, \varphi_{W_{\mathbf{i},r}})$ is a cluster of $\mathbb{C}[N^w]$, and it follows from Theorem 2 that the $\varphi_{W_{\mathbf{i},k}}(\mathbf{x_i}(\mathbf{t}))$ are monomials in the variables $t_1, \dots, t_r$. Inverting this monomial transformation yields expressions of the t_k's as explicit rational functions on N^w, a result originally called *Chamber Ansatz* by Berenstein, Fomin and Zelevinsky [2] in type A_n, because of a convenient description of these formulas in terms of chambers in a wiring diagram. To present these formulas in the general Kac-Moody setting, we need more notation. By construction, the summands $V_{\mathbf{i},k}$ of $V_{\mathbf{i}}$ are related to the modules $W_{\mathbf{i},k}$ by short exact sequences

$$0 \to W_{\mathbf{i},k} \to P(V_{\mathbf{i},k}) \to V_{\mathbf{i},k} \to 0$$

where for $X \in \mathcal{C}_w$, $P(X)$ denotes the projective cover in $\mathcal{C}_w$. We set

$$\varphi'_{V_{\mathbf{i},k}} := \frac{\varphi_{W_{\mathbf{i},k}}}{\varphi_{P(V_{\mathbf{i},k})}},$$

a Laurent monomial in the $\varphi_{W_{\mathbf{i},k}}$ (since $\mathrm{add}(W_{\mathbf{i}})$ contains all $\mathcal{C}_w$-projectives).

Denote by $q(i,j)$ the number of edges between two vertices i and j of the underlying unoriented graph of the quiver Q. For $1 \leq k \leq r$, put

$$C_{\mathbf{i},k} := \frac{1}{\varphi'_{V_{\mathbf{i},k}}\varphi'_{V_{\mathbf{i},k^-(i_k)}}} \cdot \prod_{j=1}^{n} \left(\varphi'_{V_{\mathbf{i},k^-(j)}}\right)^{q(i_k,j)}, \tag{1}$$

where $k^-(j) := \max\{0, 1 \leq s \leq k-1 \mid i_s = j\}$ and $V_{\mathbf{i},0}$ is by convention the zero module.

Theorem 3 (**Chamber Ansatz**)**.** For $1 \leq k \leq r$ and $\mathbf{t} = (t_r, \dots, t_1)$ we have $C_{\mathbf{i},k}(\mathbf{x}_{\mathbf{i}}(\mathbf{t})) = t_k$. Therefore, for $X \in \mathcal{C}_w$ we get an equality in $\mathbb{C}[N^w]$:

$$\varphi_X = \sum_{\mathbf{a} \in \mathbb{N}^r} \chi(\mathcal{F}_{\mathbf{i},\mathbf{a},X}) C_{\mathbf{i},r}^{a_r} \cdots C_{\mathbf{i},2}^{a_2} C_{\mathbf{i},1}^{a_1}. \tag{2}$$

For $x \in N^w$, the intersection $N \cap (B_- w x^T)$ consists of a unique element, which, following [2, 3, 5], we denote by $\eta_w(x)$. The map η_w is in fact a regular automorphism of N^w, and we denote by $(\eta_w^*)^{-1}$ the $\mathbb{C}$-algebra automorphism of $\mathbb{C}[N^w]$, defined by

$$((\eta_w^*)^{-1} f)(x) = f(\eta_w^{-1}(x)), \qquad (f \in \mathbb{C}[N^w]).$$

Theorem 4. For every $X \in \mathcal{C}_w$, we have

$$(\eta_w^*)^{-1}(\varphi_X) = \frac{\varphi_{\Omega_w(X)}}{\varphi_{P(X)}}.$$

Thus, the regular functions $\varphi'_{V_{\mathbf{i},k}}$ occuring in Theorem 3 are obtained by twisting the generalized minors $\varphi_{V_{\mathbf{i},k}}$ with η_w^{-1}, in agreement with [2, 3] in the Dynkin case.

What Theorem 4 is saying is that the automorphism η_w^{-1} of N^w is "induced" by the auto-equivalence Ω_w of $\underline{\mathcal{C}_w}$, via the map $X \mapsto \varphi_X$. It would be interesting to find other examples of automorphisms of N^w which can be "lifted" to auto-equivalences of $\underline{\mathcal{C}_w}$.

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Deformations of linear sites and projective schemes

Wendy Lowen

The starting point for this talk is the philosophy that Grothendieck categories are good models for non-commutative spaces. In joint work with Michel Van den Bergh in [6], a deformation theory for abelian categories was developed, and in the same paper we prove that the deformation of a Grothendieck category remains Grothendieck. We have the following basic result:

Proposition 1. [6] *For a linear category* $\mathfrak{a}$*, there is a deformation equivalence*

$$\mathrm{Def}_{\mathrm{lin}}(\mathfrak{a}) \longrightarrow \mathrm{Def}_{\mathrm{ab}}(\mathsf{Mod}(\mathfrak{a})) : \mathfrak{b} \longrightarrow \mathsf{Mod}(\mathfrak{b})$$

from linear deformations of $\mathfrak{a}$ *to abelian deformations of* $\mathsf{Mod}(\mathfrak{a})$*.*

Remark 1.
(1) Deformations are infinitesimal, in the direction of Artin local k-algebras R with maximal ideal m, and flat in appropriate senses (see [6]).
(2) If $\mathfrak{a}$ has a single object and hence reduces to an algebra, linear deformations of $\mathfrak{a}$ are simply algebra deformations of A.
(3) By definition, an abelian category $\mathcal{C}$ is embedded in a deformation $\mathcal{D}$ in such a way that $\mathcal{C}$ is equivalent to $\mathcal{D}_k = \{D \in \mathcal{D} \mid mD = 0\}$.

The main point in the proof is to associate a linear deformation of $\mathfrak{a}$ to a given abelian deformation $\mathcal{D}$ of $\mathcal{C} = \mathsf{Mod}(\mathfrak{a})$. Considering the objects $A \in \mathfrak{a}$ as objects of $\mathcal{C}$, we make essential use of the following two facts:

(1) $\mathrm{Ext}^{1,2}_{\mathcal{C}}(A, A) = 0$ (in order to obtain unique flat lifts of the individual objects of $\mathfrak{a}$ along the left adjoint $k \otimes_R -$ of the embedding $\mathcal{C} \longrightarrow \mathcal{D}$);
(2) $\mathrm{Ext}^1_{\mathcal{C}}(A, A) = 0$ (in order to organize the lifted object as a linear deformation $\mathfrak{b} \subseteq \mathcal{D}$ of $\mathfrak{a}$).

Proposition 1 tells us that the non-commutative deformation theory of affine schemes is entirely controlled by Gerstenhaber's deformation theory for algebras.

Next, we look into the situation for projective schemes. Consider a projective scheme $X = \mathrm{Proj}(A)$ for some $\mathbb{N}$-graded algebra A. By Serre's theorem, we have $\mathsf{Qch}(X) \cong \mathsf{Qgr}(A)$. We turn A into a linear category $\mathfrak{a}$ with $\mathsf{Gr}(A) = \mathsf{Mod}(\mathfrak{a})$ by putting $\mathrm{Ob}(\mathfrak{a}) = \mathbb{Z}$ and $\mathfrak{a}(n, m) = A_{n-m}$. Since A is $\mathbb{N}$-graded, we have $\mathfrak{a}(n, m) = 0$ unless $n \geq m$. Thus, $\mathfrak{a}$ is a positively graded $\mathbb{Z}$-algebra in the sense of [2]. From now on, we let $\mathfrak{a}$ be an arbitrary positively graded $\mathbb{Z}$-algebra.

The localization $\mathsf{Qgr}(A)$ of $\mathsf{Gr}(A)$ is usually captured by quotienting out torsion modules. Here, we take a different approach. We will define a localization of $\mathsf{Mod}(\mathfrak{a})$ by means of a linear topology on $\mathfrak{a}$ (see eg. [4]). This topology is naturally induced by a Grothendieck topology on the underlying poset $\mathcal{U} = (\mathbb{Z}, \geq)$. For

$n \geq m$, define the sieve $R_m^{\geq n}$ covering m by $R_m^{\geq n} = \{k \in \mathbb{Z} \mid k \geq n\}$ and define the *tails topology* tails on $\mathcal{U}$ by

$$\mathrm{tails}(m) = \{R_m^{\geq n} \mid n \geq m\} = \{R \text{ sieve} \mid R \neq \varnothing\}.$$

In the set theoretic world, this almost discrete topology is not very exciting, and in fact its sheaf category is simply Set. However, when we "linearize" to $\mathfrak{a}$, we do obtain something interesting. More precisely, let $(R_m^{\geq n})^{\mathfrak{a}}$ be the linear sieve covering m with

$$(R_m^{\geq n})^{\mathfrak{a}} = \{a : k \to m \text{ in } \mathfrak{a} \mid k \geq n\}$$

and put

$$\mathcal{L}_{\mathrm{tails}}^{\mathfrak{a}} = \{R \mid \exists\, (R_m^{\geq n})^{\mathfrak{a}} \subseteq R\}.$$

Then $\mathcal{L}_{\mathrm{tails}}^{\mathfrak{a}}$ automatically satisfies the identity and pullback axioms for a topology, and we define $\mathcal{T}_{\mathrm{tails}}^{\mathfrak{a}}$ to be the closure of $\mathcal{L}_{\mathrm{tails}}^{\mathfrak{a}}$ under glueings. Finally, we put

$$\mathsf{Qmod}(\mathfrak{a}) = \mathsf{Sh}(\mathfrak{a}, \mathcal{T}_{\mathrm{tails}}^{\mathfrak{a}}),$$

the category of linear sheaves on $\mathfrak{a}$ with respect to $\mathcal{T}_{\mathrm{tails}}^{\mathfrak{a}}$. On the other hand, let $\mathsf{Tors}(\mathfrak{a}) \subseteq \mathsf{Mod}(\mathfrak{a})$ be the subcategory of filtered colimits of right bounded modules. In joint work with my PhD student Olivier De Deken, we prove:

Proposition 2. [3]

(1) $\mathcal{L}_{\mathrm{tails}}^{\mathfrak{a}} = \mathcal{T}_{\mathrm{tails}}^{\mathfrak{a}}$ *if and only if* $\mathsf{Tors}(\mathfrak{a})$ *is localizing, and in this case* $\mathsf{Qmod}(\mathfrak{a}) \cong \mathsf{Mod}(\mathfrak{a})/\mathsf{Tors}(\mathfrak{a})$.
(2) *If all the* $(R_m^{\geq n})^{\mathfrak{a}}$ *are finitely generated in* $\mathsf{Mod}(\mathfrak{a})$*, then* $\mathcal{L}_{\mathrm{tails}}^{\mathfrak{a}} = \mathcal{T}_{\mathrm{tails}}^{\mathfrak{a}}$.
(3) *If A is a connected $\mathbb{N}$-graded algebra with associated $\mathfrak{a}$, then all the* $(R_m^{\geq n})^{\mathfrak{a}}$ *are finitely generated if and only if A is finitely generated as an algebra.*

It is known that an Artin-Zhang theorem [1] for $\mathbb{Z}$-algebras exists. Such a theorem has been stated in [8] in the noetherian context, and in [7], a version of the theorem under weaker coherence assumptions was obtained. In [3], we prove our own version of the theorem, refining some of the techniques from [4].

Theorem 1. [3] *Let $(\mathcal{O}(n))_n$ be a sequence of objects in a Grothendieck category $\mathcal{C}$, and put*

$$\mathfrak{a}(n,m) = \begin{cases} \mathcal{C}(\mathcal{O}(-n), \mathcal{O}(-m)) & \text{if } n \geq m \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $\mathcal{L}_{\mathrm{tails}}^{\mathfrak{a}} = \mathcal{T}_{\mathrm{tails}}^{\mathfrak{a}}$*. The following are equivalent:*

(1) $\mathfrak{a} \longrightarrow \mathcal{C}$ *induces* $\mathcal{C} \cong \mathsf{Qmod}(\mathfrak{a})$.
(2) $\mathfrak{a} \longrightarrow \mathcal{C}$ *induces a localization* $\mathcal{C} \longrightarrow \mathsf{Mod}(\mathfrak{a})$ *and* $\mathcal{T}_{\mathcal{C}} = \mathcal{T}_{\mathrm{tails}}^{\mathfrak{a}}$.
(3) (a) $(\mathcal{O}(n))_n$ *generates* $\mathcal{C}$.
  (b) $(\mathcal{O}(n))_n$ *is* tails*-projective (i.e. every element in* $\mathrm{Ext}_{\mathcal{C}}^i(\mathcal{O}(-n), M)$ *can be effaced on a* tails*-cover of n).*
  (c) $(\mathcal{O}(n))_n$ *is* tails*-finitely presented.*
  (d) *for $m \leq n$, there is an epimorphism* $\oplus_i \mathcal{O}(-n_i) \longrightarrow \mathcal{O}(m)$ *with every* $n_i \geq n$.

Remark 2. If $\mathcal{C}$ is locally finitely presented and the objects $\mathcal{O}(n)$ are finitely presented, condition (c) is automatic, and (a) and (d) can be combined into the more familiar

(ad) $(\mathcal{O}(n))_n$ is *ample*, i.e. for every finitely presented M there is an n_0 such that for all $n \geq n_0$ there is an epimorphism $\oplus_i \mathcal{O}(-n_i) \longrightarrow M$ with every $n_i \geq n$.

Next, we turn to deformations. Suppose $(\mathcal{O}(n))_n$ induces $\mathcal{C} \cong \mathsf{Qmod}(\mathfrak{a})$. There is a natural

$$\mathrm{Def}_{\mathrm{lin}}(\mathfrak{a}) \longrightarrow \mathrm{Def}_{\mathrm{ab}}(\mathcal{C}) : \mathfrak{b} \longmapsto \mathsf{Qmod}(\mathfrak{b})$$

and we are interested when this is in fact a deformation equivalence. In analogy with the situation in Proposition 1, there are two main types of Ext vanishing conditions involved. The first one is:

(1) $\mathrm{Ext}^{1,2}_{\mathcal{C}}(\mathcal{O}(n), \mathcal{O}(n)) = 0$ for all $n \in \mathbb{Z}$.

The second one, involving Ext's between different objects, allows for a relaxation provided we strengthen tails-projectivity in the following sense. We call $(\mathcal{O}(n))_n$ *strongly tails projective* if for all $m \in \mathbb{Z}$ and $M \in \mathcal{C}$ there is an $n_0 \geq m$ such that for all $n \geq n_0$ we have $\mathrm{Ext}^i_{\mathcal{C}}(\mathcal{O}(-n), M) = 0$.

Remark 3.

(1) The combination of ampleness and strong tails projectivity is called *strong ampleness* in the papers [9, 10]. These papers were one of the motivations for the development of the deformation theory in [6].
(2) In the classical geometric situation where $(\mathcal{O}(n))_n$ is obtained from an ample line bundle, the sequence is strongly ample.
(3) For our purpose, we only need the strong tails projectivity condition for all $m \in \mathbb{Z}$ and $M = \mathcal{O}(-m)$.

Suppose $(\mathcal{O}(n))_n$ induces $\mathcal{C} \cong \mathsf{Qmod}(\mathfrak{a})$ and is strongly tails projective, and consider $\mathcal{U} = (\mathbb{Z}, \geq)$. For all m, fix an $n_m \geq m$ such that for all $n \geq n_m$, we have $\mathrm{Ext}^i_{\mathcal{C}}(\mathcal{O}(-n), \mathcal{O}(-m)) = 0$. Then we define a new order on $\mathbb{Z}$ by

$$n \geq' m \iff [n \geq n_m \ \vee \ n = m].$$

We obtain a new site $\mathcal{U}' = (\mathbb{Z}, \geq')$ with induced tails topology tails$'$ (still consisting of all nonempty sieves), and an induced $\mathfrak{a}' \subseteq \mathfrak{a}$ with $\mathsf{Qmod}(\mathfrak{a}') \cong \mathsf{Qmod}(\mathfrak{a})$. We have the following

Proposition 3. *Suppose* $(\mathcal{O}(n))_n$ *induces* $\mathcal{C} \cong \mathsf{Qmod}(\mathfrak{a})$, *is strongly tails projective and satisfies* $\mathrm{Ext}^{1,2}_{\mathcal{C}}(\mathcal{O}(n), \mathcal{O}(n)) = 0$ *for all* n. *Let* $\mathfrak{a}'$ *be as above. Then*

$$\mathrm{Def}_{\mathrm{lin}}(\mathfrak{a}') \longrightarrow \mathrm{Def}_{\mathrm{ab}}(\mathcal{C}) : \mathfrak{b} \longrightarrow \mathsf{Qmod}(\mathfrak{b})$$

is a deformation equivalence.

Remark 4.

(1) The fact that there is a deformation equivalence $\mathrm{Def}_{\mathrm{lin}}(\mathfrak{a}') \longrightarrow \mathrm{Def}_{\mathrm{ab}}(\mathcal{C})$ is an immediate application of [6]. To see that this map has the correct prescription, the tails topology turns out to be very natural (see [3] in the situation where we can take $\mathfrak{a}' = \mathfrak{a}$).

(2) The fact that, regardless of the ground ring towards which we're deforming, the underlying non-linear tails Grothendieck topology controlls the entire deformation process, is analogous to the situation in [5] where deformations of categories of sheaves of $\mathcal{O}_X$-modules are controlled by the standard (prototypical) Grothendieck topology on the underlying space X. An important difference, however, is that in our current setup $\mathfrak{a}$ is not fibered over $(\mathbb{Z}, \geq)$. A more general treatment of deformations of linear, and in particular linearized sites, is work in progress.

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Restricted rational Cherednik algebras and Hecke algebras

Maurizio Martino

Let G be a complex reflection group, with reflection representation V. Let $S \subset G$ be the set of reflections in G. Given a G-invariant function $c : S \to \mathbb{C}$ one can define the rational Cherednik algebra (at $t = 0$) H_c. These algebras form a flat family of deformations of the smash product algebra $\mathbb{C}[V \oplus V^*] * G$, which were introduced in [1] and have interesting connections to representation theory, geometry, integrable systems and combinatorics.

The H_c are PI algebras and we are interested in their representation theory and how this relates to the geometry of the centre Z_c of H_c. In particular, one goal is to understand simple modules for H_c. Schur's Lemma ensures that all simple modules are annihilated by a maximal ideal of Z_c, and so the strategy is to consider the finite dimensional quotients H_c/mH_c, where $m \lhd Z_c$ is a maximal ideal. Unfortunately both the centre Z_c and the quotient algebras H_c/mH_c are, in general, poorly understood. One approach is instead to consider the inclusion of algebras

$$A := \mathbb{C}[V]^G \otimes \mathbb{C}[V^*]^G \subset Z_c \subset H_c,$$

where each algebra is module-finite over each subalgebra. Then one studies quotients of the form H_c/nH_c, where $n \lhd \mathbb{C}[V]^G \otimes \mathbb{C}[V^*]^G$ is a maximal ideal. This situation is somewhat more manageable since $\mathbb{C}[V]^G \otimes \mathbb{C}[V^*]^G$ is well-understood - it is a polynomial ring in 2 Dim$V$ variables - and the quotient algebras have a more transparent structure. The finite-dimensional quotients H_c/nH_c are called reduced rational Cherednik algebras. Of particular interest is the *restricted rational Cherednik algebra* $\overline{H}_c := H_c/n_+H_c$, where $n_+ \lhd \mathbb{C}[V]^G \otimes \mathbb{C}[V^*]^G$ is the maximal ideal consisting of polynomials with zero constant term. These algebras are $\mathbb{Z}$-graded and have simple modules indexed by the irreducible modules of G.

Our main result concerns the block structure of the algebra $\overline{H}_c$ in the case where G is a wreath product $S_n \ltimes \mathbb{Z}_m^n$, where S_n denotes the symmetric group on n letters. In this situation, the irreducible modules of G can be parametrised by the set of *m-multipartitions of n*, $\mathcal{P}(m,n)$. Thus the partition into blocks can be thought of as a partition of the set $\mathcal{P}(m,n)$. In particular, one can use combinatorics to describe the blocks. To any multipartition $\lambda \in \mathcal{P}(m,n)$ (or more precisely its Young diagram), one can associate an element of the group algebra of $(\mathbb{C},+)$ over $\mathbb{Z}$, that is, a polynomial which is a sum of monomials x^a with $a \in \mathbb{C}$, which is called the *residue* of λ and is denoted $\mathrm{Res}(\lambda)$. Given an element $d \in \mathbb{C}^m$, one can define a shifted residue $\mathrm{Res}^d(\lambda)$ by multiplying certain summands of $\mathrm{Res}(\lambda)$ by x^{d_i}.

Theorem. *There exists an explicit linear function f mapping c into $\mathbb{C}^m$ such that $\lambda, \mu \in \mathcal{P}(m,n)$ lie in the same block if and only if* $\mathrm{Res}^{f(c)}(\lambda) = \mathrm{Res}^{f(c)}(\mu)$.

Let us say a few words about this result. It has its origins in the work of Gordon, [2], who proved this theorem under the condition that $f(c)$ has rational entries and who used geometric techniques. Our proof is algebraic, and relies on finding a large enough subalgebra of Z_c, such that the blocks of $\overline{H}_c$ are determined by this subalgebra; the proof then boils down to combinatorial calculations for representations of the group G. It has been noted in [3] and [4] that this block decomposition is related to the so-called *Rouquier blocks* of a cyclotomic Hecke algebra for G. This observation only makes sense when $f(c)$ has rational entries (since the Hecke algebra is only well-defined in this sitaution). One can however compare the theorem with the block decomposition for a *degenerate* cyclotomic Hecke algebra at parameters $f(c)$, [5], and one finds that these two descriptions are equivalent. In fact, it is a type of degenerate Hecke algebra which gives rise to the central subalgebra used in the proof of the theorem above; the exact relationship, however, between $\overline{H}_c$ and Hecke algebras remains unclear for the moment.

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A categorification of the quantum Frobenius on U^+

Kevin McGerty

Let k be a field of charactistic $p > 0$ and G be an affine algebraic group over k. Then, as a variety over k, the group G can be equipped with a Frobenius morphism $F\colon G \to G$, which moreover is in fact an endomorphism. The existence of this map is of fundamental importance in the study of the representation theory and geometry of G (see for example [14], [1]).

Let $\mathbf{U}_v(\mathfrak{g})$ be the quantum group attached to a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$. In the late 80s, Lusztig [9], [10] discovered that when the parameter v is specialised to ε an ℓ-th root of unity, there is a homomorphism Fr from the resulting algebra $\mathbf{U}_\varepsilon(\mathfrak{g})$ to the integral form of the enveloping algebra $\mathcal{U}_\mathbb{Z}(\mathfrak{g})$, providing a "$q$-analogue" of the Frobenius morphism in positive characteristic. In fact the relation is more precise: Lusztig's map Fr gives an integral lift of the classical Frobenius morphism: if $\ell = \mathrm{char}(\mathsf{k})$, then after base changing to k one obtains the transpose of the map F on the hyperalgebra of G. (Here of course our integers are the ring of cyclotomic integers $\mathbb{Z}[\varepsilon]$).

The existence of this map was a basic ingredient in the program, constructed by Lusztig [11], to compute the characters of irreducible representations of algebraic groups in fields of positive characteristic [6], [2], [5]. More recently, Kumar and Littelmann [8], [7] succeeded in obtaining proofs of many theorems on the geometry of Schubert varieties, (including for example their normality) via the quantum Frobenius map and its splitting. Finally, the existence of the quantum Frobenius is also used in establishing the connection between quantum groups at a root of unity and perverse sheaves on the affine Grassmannian [3].

The classification of quantum groups (or indeed Kac-Moody Lie algebras) allows one to attach to any such algebra a graph, generalising Dynkin's graphs for semisimple Lie algebras. Another central discovery in the theory of quantum groups was Ringel's realisation [13] that the positive part of a quantum group could be constructed as a Hall algebra built from the isomorphism classes of representations of the quiver whose underlying graph is this graph. As well as building a remarkable bridge between the representation theory of finite-dimensional algebras and quantum groups, this work led Lusztig to the discovery of the canonical basis, by “categorifying” the Hall algebra construction: that is, lifting the Hall algebra construction to a convolution operation on perverse sheaves on the moduli space of representations of the quiver.

In my talk I wish to describe a new perspective on the quantum Frobenius which shows how the map can be constructed in this sheaf-theoretic context. The construction is based a localisation known as “hyperbolic localisation” and Lusztig’s work on quivers with automorphisms. It both “categorifies” the quantum Frobenius, and at the same time gives a essentially computation free proof of its existence. At the present time, however, only in the case where the root lengths are coprime to the integer ℓ – Lusztig’s later work [12] deals also with the special cases where condition does not hold, but we hope that our construction can be extended to include all cases.

The realisation of the quantum Frobenius in the context of perverse sheaves also raises the question of how it acts on the canonical basis. We also discuss the question of what compatibility might be expected between the two structures. More precisely, there are a number of combinatorial constructions which are shadows of the existence of the canonical basis, and many of these display a kind of ”scaling” phenomenon (see for example [4]) which appears to be related to the quantum Frobenius.

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Point modules on naive Blow-ups

Tom Nevins
(joint work with Susan J. Sierra)

1. Introduction

One of the important achievements of noncommutative projective geometry is the classification of noncommutative projective planes, or equivalently graded rings that are noncommutative analogs of polynomial rings in three variables; see [SV01] for a discussion. The technique of Artin-Tate-Van den Bergh for classifying such noncommutative rings relies heavily on the study of *point modules*, or cyclic graded modules with Hilbert series $1 + t + t^2 + t^3 + \dots$. The existence and description of a moduli scheme for the point modules allow those authors to construct a homomorphism to a well-understood ring, which provides a first step in describing the structure of the noncommutative plane itself.

A noetherian algebra R is said to be *strongly noetherian* if for any commutative noetherian algebra C, the tensor product $C \otimes_k R$ is again noetherian. A general result of Artin-Zhang shows that if R is any strongly noetherian, connected graded algebra generated in degree one, then its point modules are parametrized by a projective scheme. More recently, Rogalski-Zhang have used this to extend the method of Artin-Tate-Van den Bergh to study general strongly noetherian graded algebras R.

Although it was believed for a time that all connected graded noetherian algebras would be strongly noetherian, Keeler-Rogalski-Stafford [KRS05, RS07] geometrically constructed a beautiful class of new examples, called *naive blow-ups*, of noncommutative graded k-algebras that are noetherian but not strongly noetherian. Along the way, Keeler-Rogalski-Stafford showed that families of R-point modules for naive blow-ups—viewed as objects of noncommutative projective geometry, in a way we make precise below—cannot behave well in families: there can be no fine moduli scheme of finite type for point modules.

In joint work with S. J. Sierra, we systematically develop the moduli theory of point modules for the naive blow-ups R of [KRS05, RS07]. Roughly speaking, we show that there is an analog of a "Hilbert scheme of one point on $\mathrm{Proj}(R)$" that is an infinite blow-up of a projective variety. This infinite blow-up is quasi-compact and noetherian as an algebraic stack. Furthermore, we prove that there is a "coarse moduli space for one point on $\mathrm{Proj}(R)$" that is an ordinary projective variety. These are the first descriptions of moduli structures for the point modules on a naive blow-up.

More precisely, let k be an uncountable algebraically closed field and let X be an irreducible projective k-variety of dimension at least 2. Let Z be a zero-dimensional subscheme of X. Let $\sigma \in \mathrm{Aut}(X)$ be an automorphism and assume that all points in the support of Z have *critically dense orbit*: each such orbit is infinite and every infinite subset of it has Zariski closure equal to X. Given such data together with a σ-ample invertible sheaf $\mathcal{L}$ on X, Rogalski-Stafford define a graded noncommutative algebra $R = R(X, Z, \mathcal{L}, \sigma)$, the *naive blow-up* associated to $(X, Z, \mathcal{L}, \sigma)$. For simplicity, we assume R is generated in degree one.

An *ℓ-shifted embedded R-point module* is a graded module M with Hilbert series $t^\ell + t^{\ell+1} + t^{\ell+2} + \dots$ together with a choice of surjection $R_{\geq \ell} \to M$. Two such quotients M and M' are isomorphic if there is an R-module isomorphism from M to M' that intertwines the maps from $R_{\geq \ell}$. Forgetting the map from $R_{\geq \ell}$ to M, we call the module M an *ℓ-shifted point module*. We call a 0-shifted (embedded) point module simply an *(embedded) point module*.

Our first main theorem characterizes moduli of embedded point modules. Recall that $\widetilde{X}$ is a *fine moduli space* for embedded point modules if there is an R-module quotient $R \otimes_k \mathcal{O}_{\widetilde{X}} \to M$ making M an $\widetilde{X}$-flat family of embedded R-point modules, with the property that, if $R \otimes_k C \to M'$ is any C-flat family of embedded point modules for a commutative k-algebra C, then there is a morphism $\mathrm{Spec}(C) \xrightarrow{f} \widetilde{X}$ and an isomorphism $f^*M \cong M'$ of families of embedded R-point modules. Let $\mathcal{I}_1 = \mathcal{I}_Z$ be the ideal sheaf of Z and set $\mathcal{I}_n = \mathcal{I}_1 \otimes \sigma^*\mathcal{I}_1 \otimes \cdots \otimes (\sigma^{n-1})^*\mathcal{I}_1$. Let $X_n = \mathrm{Bl}_{\mathcal{I}_n} X$. We get an inverse system $\cdots \to X_n \to X_{n-1} \to \dots$ of schemes. Let $\widetilde{X} = \varprojlim X_n$; this inverse limit exists as a stack. In fact:

Theorem 1 ([NS1]). *The inverse limit $\widetilde{X}$ is a noetherian algebraic stack. The morphism $\widetilde{X} \to X$ is quasicompact. Moreover, $\widetilde{X}$ is a fine moduli space for embedded R-point modules.*

Note that the stack $\widetilde{X}$ is discrete: its points have no stabilizers. Thus, $\widetilde{X}$ is actually a *k-space* in the terminology of [LM]; in particular, this justifies our use of the phrase "fine moduli space" in the statement of the theorem. However, $\widetilde{X}$ does not seem to have an étale cover by a scheme, and hence does not have the right to be called an algebraic space.

Associated to R there is a noncommutative projective scheme: by definition, this means the quotient category $\mathrm{Qgr}\text{-}R = \mathrm{Gr}\text{-}R/\mathrm{Tors}\text{-}R$ of graded right R-modules by the full subcategory of locally bounded modules. A *point object* in $\mathrm{Qgr}\text{-}R$ is the image of an ℓ-shifted point module for some ℓ.

Let $\ell \gg 0$ and let F be the moduli functor of embedded ℓ-shifted point modules over R. Define an equivalence relation $\sim$ on $F(C)$ by saying that $M \sim N$ if (their images) are isomorphic in $\mathrm{Qgr}\text{-}R_C$. Define a functor G : Affine schemes $\longrightarrow$ Sets by sheafifying (in the étale topology) the presheaf G^{pre} of sets defined by $\mathrm{Spec}\, C \mapsto F(C)/\sim$.

A scheme M is a *coarse moduli scheme* for point objects if it corepresents the functor G: that is, there is a natural transformation $G \to \mathrm{Hom}(-, M)$ that is universal for natural transformations from G to schemes.

Theorem 2 ([NS1]). *The variety X is a coarse moduli scheme for point objects in* $\mathrm{Qgr}\text{-}R$.

Note that, by Theorem 10.4 of [KRS05], there can be no fine moduli scheme of finite type for point objects of $\mathrm{Qgr}\text{-}R$.

Corollary 2. *There is a fine moduli space $\widetilde{X}$ for embedded R-point modules but only a coarse moduli scheme X for point objects of* Qgr- R.

It may be helpful to compare the phenomenon described by the corollary to a related, though quite different, commutative phenomenon. Namely, let S be a smooth projective (commutative) surface. Fix $n \geq 1$. Let $R = \mathbb{C}[S]$ denote a homogeneous coordinate ring of S (associated to a sufficiently ample line bundle on S), and, for appropriate fixed $\ell = \ell(n)$, consider graded quotient modules $R_{\geq \ell} \to M$ for which $h_M(t) = nt^{\ell} + nt^{\ell+1} + \dots$ and M is generated by M_{ℓ}. By a general theorem of Serre, the moduli space for such quotients is the *Hilbert scheme of n points on S*, denoted $\mathrm{Hilb}^n(S)$. This is a smooth projective variety of dimension $2n$. Alternatively, remembering only the corresponding objects $[M]$ of Qgr- $R \simeq \mathrm{Qcoh}(S)$, and imposing the further S-equivalence relation (see Example 4.3.6 of [HL97]), we get the moduli space $\mathrm{Sym}^n(S)$ for semistable length n sheaves on S, which equals the nth symmetric product of S. The latter moduli space is only a coarse moduli space for semistable sheaves. One has the Hilbert-Chow morphism $\mathrm{Hilb}^n(S) \to \mathrm{Sym}^n(S)$ which is defined by taking a quotient $R_{\geq \ell} \to M$ to the equivalence class of M. It is perhaps helpful to view $\widetilde{X} \to X$, in light of the theorems stated above, as a kind of "noncommutative Hilbert-Chow morphism of one point" for a naive blow-up algebra $R(X, P, \sigma, \mathcal{L})$.

In a work in preparation [NS2], we prove a kind of converse theorem that generalizes the recent work of Rogalski-Zhang. Namely, suppose R is a connected, graded noetherian ring generated in degree one; and that R has a fine moduli space $\widetilde{X}$ for embedded point modules, that R has a projective coarse moduli scheme X for point objects of Qgr- R, and that the spaces $\widetilde{X}$ and X and the morphism $\widetilde{X} \to X$ between them have geometric properties similar to those of the spaces we encountered in the theorems above. Then, we show, there exist an automorphism σ of X, a zero-dimensional subscheme $Z \subset X$ supported on points with critically dense orbits, an ample and σ-ample line bundle $\mathcal{L}$ on σ, and a homomorphism $\phi : R \to R(X, Z, \mathcal{L}, \sigma)$ from R to the naive blow-up associated to $(X, Z, \mathcal{L}, \sigma)$; furthermore, ϕ is surjective in large degree. This construction gives a new tool for analyzing the structure of rings that are noetherian but not strongly noetherian. Details will appear in [NS2].

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Quivers and Donaldson-Thomas type invariants

Markus Reineke

If X is a smooth irreducible projective complex variety of dimension three, then by a result of Cheah, we have

$$\sum_{n=0}^{\infty} \chi(X^{[n]})t^n = \prod_{i\geq 1}(1-t^i)^{-i\chi(X)},$$

where $X^{[n]}$ is the Hilbert scheme of n points in X, and χ denotes topological Euler characteristic.

We are interested in a noncommutative analogue of this result, motivated by the framework of [1] for the definition of Donaldson-Thomas type invariants of noncommutative Calabi-Yau threefolds.

We start with a finite quiver Q with set of vertices I and arrows $\alpha : i \to j$; its Euler form is denoted by $\langle d, e\rangle$ for $d, e \in \mathbf{Z}I$. We choose a functional $\Theta \in (\mathbf{Q}I)^*$ and define the slope of $d = \sum_i d_i i \in \mathbf{N}I \setminus \{0\}$ as $\mu(d) = \Theta(d)/\dim d$, where $\dim d = \sum_i d_i$. This allows us to define a notion of (semi-)stability for complex finite-dimensional representations M of Q.

For dimension vectors $d, n \in \mathbf{N}I$, we can then define moduli spaces $M_d^{\Theta-st}(Q)$ parametrizing isomorphism classes of stable representations of Q of dimension vector d up to isomorphism, and moduli spaces $M_{d,n}^{\Theta}(Q)$ parametrizing pairs (M, f) consisting of a semistable representation of dimension vector d and a map f from the projective representation $\bigoplus_i P_i^{n_i}$ to M such that $\mu(U) < \mu(M)$ for all proper subrepresentations U of M containing the image of f.

The family of moduli spaces $(M_d^{\Theta-st}(Q))_d$ might be viewed as a "noncommutative (projective) variety", and the family $(M_{d,n}^{\Theta}(Q))_d$ might be viewed as a replacement for the Hilbert schemes of points.

For $i \in I$, $d \in \mathbf{N}I$ and $\mu \in \mathbf{Q}$, we define generating series of Euler characteristic

$$Q_\mu^i(x) = \sum_{d:\mu(d)=\mu} \chi(M_{d,i}^{\Theta}(Q))x^n \in \mathbf{Z}[[x_i : i \in I]]$$

and $S_\mu^d(x) = \prod_{j\in I} Q_\mu^j(x)^{-\langle d,j\rangle}$. We have the following noncommutative analogue of Cheah's result:

Theorem 1. *The formal series $S_\mu^d(X)$ are determined by the system of functional equations*

$$S_\mu^d(x) = \prod_{e:\mu(e)=\mu} (1 - t^e S_\mu^e(x))^{\langle d,e\rangle \cdot \chi(M_e^{\Theta-st}(Q))}.$$

As an example, we consider the quiver Q with a single vertex and m loops, thus we consider moduli spaces for finite-dimensional representations of free algebras $A = \mathbf{C}\langle x_1, \ldots, x_m\rangle$. The analogues of Hilbert schemes of points are the Hilbert schemes $\mathrm{Hilb}^d(A)$ parametrizing codimension d left ideals in A. These admit a complex cell decomposition, whose cells are parametrized by m-ary trees with d

nodes; the generating function $F(t) = \sum_{d=0}^{\infty} \chi(\mathrm{Hilb}^d(A))t^d \in \mathbf{Z}[[t]]$ is thus determined by the functional equation $F(t) = (1 - tF(t)^{m-1})^{-1}$. We have the following explicit Euler product factorization:

Theorem 2. *We have* $F(t) = \prod_{i\geq 1}(1 - ((-1)^{m-1}t)^i)^{-i\mathrm{DT}_i}$*, where*

$$\mathrm{DT}_i = \frac{1}{(m-1)i^2}\sum_{j|i}\mu(i/j)(-1)^{(m-1)j}\binom{mj-1}{j} \in \mathbf{Z}.$$

The integers DT_i might be viewed as Donaldson-Thomas type invariants of "noncommutative affine space".

The relation to [1] is given by the following "wall-crossing" formula:

Assume that Q has no oriented cycles, and label the vertices $I = \{i_1, \ldots, i_n\}$ of Q such that $k > l$ if there exists an arrow $i_k \to i_l$. Let Φ be the Coxeter element defined by $\langle \Phi(d), e\rangle = -\langle e, d\rangle$. Define a Poisson algebra $B = \mathbf{Q}[[x_i : i \in I]]$ with Poisson bracket $\{x^d, x^e\} = \{d, e\}x^{d+e}$, where $\{,\}$ denotes the antisymmetrization of the Euler form. In the group Γ of Poisson automorphisms of B, we have elements T_d for $d \in \mathbf{N}I$ defined by $T_d(x^e) = x^e(1 + x^d)^{\{d,e\}}$.

Theorem 3. *In* Γ*, we have the following identity:*

$$T_{i_1} \cdot \ldots \cdot T_{i_n} = \overleftarrow{\prod_{\mu\in\mathbf{Q}}}(T_\mu : x^d \mapsto x^d \cdot S_\mu^{(\mathrm{id}+\Phi)(d)}(x)).$$

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Torsion pairs and matrix categories

Claus Michael Ringel

Matrix categories were introduced by Drozd in 1972 in order to provide an abstract setting for dealing with matrix problems (as considered at that time intensively in Kiev and elsewhere), but results concerning the use of matrix categories are hidden in the literature. The theme of the lecture (in particular, its start) was devoted to a surely folklore result concerning the partial reconstruction of an abelian category from a torsion pair. By definition, a torsion pair $(\mathcal{F}, \mathcal{G})$ in an abelian category $\mathcal{A}$ is a pair of two full subcategories which are closed under isomorphisms such that first of all $\mathrm{Hom}(G, F) = 0$ for all objects G in $\mathcal{G}$ and F in $\mathcal{F}$, and second any object A in $\mathcal{A}$ has a subobject tA which belongs to $\mathcal{G}$ such that A/tA belongs to $\mathcal{G}$ (note that we use the convention that the first entry of the pair $(\mathcal{F}, \mathcal{G})$ is the subcategory $\mathcal{F}$ of "torsionless" objects, the second entry the subcategory $\mathcal{G}$ of "torsion" objects; this corresponds to the vision that non-zero maps are drawn from left to right whenever this is possible).

Given two additive categories $\mathcal{A}$ and $\mathcal{B}$, an $\mathcal{A}$-$\mathcal{B}$-bimodule ${}_{\mathcal{A}}E_{\mathcal{B}}$ is by definition a bilinear functor $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \operatorname{mod} k$. Given such an $\mathcal{A}$-$\mathcal{B}$-bimodule $E = {}_{\mathcal{A}}E_{\mathcal{B}}$, we consider the matrix category of E as introduced by Drozd: its objects are triples (A, B, m), where A is an object of $\mathcal{A}$, B is an object of $\mathcal{B}$ and $m \in E(A, B)$, and a morphism $(A, B, m) \to (A', B', m')$ is a pair (α, β), where $\alpha : A \to A'$ and $\beta : B \to B'$ are morphisms in $\mathcal{A}$, and $\mathcal{B}$ respectively, such that $m\beta = \alpha m'$.

Proposition. *Let $(\mathcal{F}, \mathcal{G})$ be a torsion pair in the abelian category $\mathcal{A}$. Given $A \in \mathcal{A}$, let ϵ_A be the equivalence class of the canonical exact sequence $0 \to tA \to A \to A/tA \to 0$ with tA in $\mathcal{G}$ and A/tA in $\mathcal{F}$, this is an element of the group $\operatorname{Ext}^1(A/tA, tA)$. Then $\eta(A) = (A/tA, tA, \epsilon_A)$ defines a functor η from the category $\mathcal{A}$ to the matrix category of the $\mathcal{F}$-$\mathcal{G}$-bimodule $\operatorname{Ext}^1(\mathcal{F}, \mathcal{G})$ which is full and dense and its kernel is the ideal generated by all maps $\mathcal{F} \to \mathcal{G}$.*

In particular, in case $\mathcal{A}$ is a Krull-Remak-Schmidt category, then we see that the kernel of the functor η lies in the radical of $\mathcal{A}$ and therefore η provides a bijection between the isomorphism classes of indecomposable objects of $\mathcal{A}$ and of the matrix category of the bimodule $\operatorname{Ext}^1(\mathcal{F}, \mathcal{G})$.

The second part of the lecture was devoted to an application concerning the module category of a cluster-tilted algebra as introduced by Buan, Marsh and Reiten. Here, one begins with a tilted algebra B, thus with a tilting module T over a finite-dimensional hereditary algebra A so that B is the endomorphism ring of T. Note that the global dimension of B is at most 2 and we may consider the B-B-bimodule $I = \operatorname{Ext}^2(DB, B)$, where D is the k-duality. The corresponding cluster-tilted algebra B^c may be defined as the trivial extension of B by the bimodule I. The tilting module T provides a torsion pair $(\mathcal{F}, \mathcal{G})$ in the module category $\operatorname{mod} A$, as well as a torsion pair $(\mathcal{Y}, \mathcal{X})$ in $\operatorname{mod} B$, and the latter is even split (this means that any indecomposable B-module belongs either to $\mathcal{Y}$ or to $\mathcal{X}$)), such that $\mathcal{G}$ is equivalent to $\mathcal{Y}$ and $\mathcal{F}$ is equivalent to $\mathcal{X}$. The pair $(\mathcal{Y}, \mathcal{X})$ is still a torsion pair in $\operatorname{mod} B^c$, but usually no longer split: the indecomposable B^c-modules which are not B-modules (thus those not annihilated by I) are neither torsion nor torsionfree; it is the multiplication by I which is responsible for obtaining non-trivial extensions of torsion modules by torsionfree modules, and these are the modules we are interested in!

Now the proposition asserts that the category $\operatorname{mod} A/\langle \operatorname{Hom}(\mathcal{F}, \mathcal{G})\rangle$ is equivalent to the matrix category for the $\mathcal{F}$-$\mathcal{G}$-bimodule $\operatorname{Ext}^1(\mathcal{F}, \mathcal{G})$. Some calculations show that for $\operatorname{mod} B_c$, one may invoke in a similar way the matrix category of the $\mathcal{G}$-$\mathcal{F}$-bimodule $\operatorname{Hom}(\mathcal{G}, \tau\mathcal{F})$. In this way, we see that the module categories $\operatorname{mod} A$ and $\operatorname{mod} B_c$ are related to each other via the bimodule $\operatorname{Ext}^1(\mathcal{F}, \mathcal{G})$ as well as its dual $\operatorname{Hom}(\mathcal{G}, \tau\mathcal{F})$, the duality being one of the basic assertions of the Auslander-Reiten theory (note that the algebra A we start with is assumed to be hereditary).

The relationship shows that for k algebraically closed and T a preprojective tilting module (so that B is a concealed and B^c a "cluster-concealed" algebra) the dimension vectors of the indecomposable B^c-modules are the absolute values of the roots of the corresponding Kac-Moody root system. The proof relies on the one hand on a separation property or the support of torsion and torsionless B-modules, and on the other hand on an old result of de la Peña and Simson: they have shown that the indecomposable objects of the matrix category of a bimodule E correspond bijectively to the positive roots of some quadratic form r_E provided the matrix category of E is directed.

In the special case when A (and therefore also B^c) is representation-finite, we show in this way that the indecomposable B^c-modules are determined by their dimension vectors (this result has been independently obtained by Geng and Peng, and it provides a proof of the Fomin-Zelevinsky denominator conjecture for cluster algebras of simply laced Dynkin type).

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Birationally commutative projective surfaces of GK-dimension 4

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(joint work with Dan Rogalski)

We construct a family of counterexamples to a conjecture of Rogalski and Stafford, and show that they have many other unexpected properties.

We begin by summarizing the work of Rogalski and Stafford [RS06] on birationally commutative graded algebras. We work throughout over an uncountable algebraically closed field k. A k-algebra S is *connected graded* if $S = \bigoplus_{n\geq 0} S_n$, where $S_0 = k$ and $\dim_k S_n < \infty$ for all n. If S is a connected graded noetherian domain (or Ore domain, more generally) we may invert the homogeneous elements of S to obtain a *graded quotient ring*

$$Q_{\mathrm{gr}}(S) \cong D[t, t^{-1}; \phi]$$

for some division ring D and $\phi \in \mathrm{Aut}_k(D)$. If D is commutative, and thus $D \cong k(X)$ for some projective variety X, then we say that S is *birational to* X, or more generally *birationally commutative.* If ϕ is induced from an automorphism of X, we say that S is *geometric.*

Subject to the condition that S is geometric, [RS06] (and more generally [Sie09], [Sie10]) classify birationally commutative projective surfaces: graded algebras birational to a commutative surface. They show:

Theorem 1. [RS06, Theorem 1.1] *Let S be a connected graded noetherian domain that is generated in degree 1, birational to a commutative surface, and geometric. Then S is falls into one of two classes, and, up to finite dimension, may be written explicitly in terms of geometric data. In particular, S is a subalgebra of a twisted homogeneous coordinate ring that has the same graded quotient ring as S.*

Rogalski and Stafford conjecture [RS06, p. 6] that the conclusions of Theorem 1 hold without the assumption that S is geometric. We show their conjecture is false, and give a family of counterexamples.

To explain, we reframe Rogalski and Stafford's conjecture. It turns out that the hypotheses of Theorem 1 place restrictions on the GK-dimension of S. This follows from a theorem of Rogalski, using work of Diller and Favre [DF01] and Gizatullin [Giz80] on the dynamics of bimeromorphic maps of surfaces.

Theorem 2. [Rog07] *Let $Q = K[t, t^{-1}; \phi]$, where K is a finitely generated field extension of k of transcendence degree 2. Then every connected graded Ore domain S with graded quotient ring Q has the same GK-dimension $d \in \{3, 4, 5, \infty\}$. All these d do occur. If $d = \infty$, then S is not noetherian. If $d < \infty$, then $d \in \{3, 5\}$ if and only if S is geometric.*

Thus we may reframe Rogalski and Stafford's conjecture as

Conjecture 1. [Rogalski-Stafford] Suppose that S is a connected graded domain of GK-dimension 4, generated in degree 1, that is birational to a commutative surface. Then S is not noetherian.

In contrast, we show

Theorem 3. *Let $\alpha, \beta \in k$, and let*

$$R := R(\alpha, \beta) := k\langle x_1, x_2, x_3, x_4 \rangle / (f_1, \dots, f_6),$$

where

$$\begin{aligned} f_1 &= x_1(\alpha x_1 - x_3) + x_3(x_1 - \alpha x_3) \\ f_2 &= x_1(\alpha x_2 - x_4) + x_3(x_2 - \alpha x_4) \\ f_3 &= x_2(\alpha x_1 - x_3) + x_4(x_1 - \alpha x_3) \\ f_4 &= x_2(\alpha x_2 - x_4) + x_4(x_2 - \alpha x_4) \\ f_5 &= x_1(\beta x_1 - x_2) + x_4(x_1 - \beta x_2) \\ f_6 &= x_1(\beta x_3 - x_4) + x_4(x_3 - \beta x_4). \end{aligned}$$

If α, β are algebraically independent over the prime subfield of k, then R is a noetherian domain of GK-dimension 4 that is birational to $\mathbb{P}^2$.

In contrast, $R(0, 0)$ is the non-noetherian algebra studied in [YZ06, Proposition 7.6].

We show also that the algebras R have other interesting properties.

Theorem 4. *Let α, β be algebraically independent over the prime subfield of k, and let R be defined as in Theorem 3. Then the trivial module k has a resolution*

$$0 \to R[-4] \to R[-3]^4 \to R[-2]^6 \to R[-1]^4 \to R \to k \to 0.$$

In consequence, R is Koszul of global dimension 4 and has Hilbert series $1/(1-t)^4$. However, R is not AS-Gorenstein and therefore not AS-regular. Further, the Auslander-Buchsbaum equality *fails for R: we have*

$$\text{depth } k + \text{p.dim } k = 0 + 4 > \text{depth } R.$$

We note that if α and β are generic, then $R(\alpha, \beta)$ is not contained in any twisted homogeneous coordinate ring, so Theorem 1 fails completely in this case. We conjecture that the augmentation ideal R_+ is the only nontrivial graded prime ideal of R and that R has no nontrivial map to any twisted homogenous coordinate ring.

To prove Theorems 3 and 4, we work geometrically, with sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$. Let α, β be algebraically independent over the prime subfield. We have

$$R \subseteq Q_{\text{gr}}(R) \cong k(u, v)[t, t^{-1}; \phi] \tag{1}$$

for some $\phi = \phi(\alpha, \beta) \in \text{Aut}_k(k(u, v))$. The automorphism $\phi(0, 0)$ sends

$$u \mapsto uv$$

$$v \mapsto v$$

and induces a birational map of $\mathbb{P}^1 \times \mathbb{P}^1$. For general α, β, we perturb $\phi(0, 0)$ by a suitably defined element $\tau(\alpha, \beta) \in \mathbb{P}GL_2(k) \times \mathbb{P}GL_2(k)$.

One way to see the embedding (1) is to identify R_1 with $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 1)) \cdot t$. That is, we identify

$$x_1 = t \qquad x_2 = ut \qquad x_3 = vt \qquad x_4 = uvt$$

inside $k(u, v) \cdot t$. Our identifications allow us to construct a globally generated quasicoherent sheaf $\mathcal{R} = \bigoplus_{n \geq 0} \mathcal{R}_n$ on $\mathbb{P}^1 \times \mathbb{P}^1$ so that $R = H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{R})$. However, working with the sheaves $\mathcal{R}_n$ is quite delicate; in particular, they do not form an ample sequence in the sense of [Van96]. In fact, in the course of proving Theorems 3 and 4, we establish new geometric results on regularity of ideal sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$, which may be of independent interest.

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Cuspidal $\mathfrak{sl}_n$-modules and deformation of Brauer tree algebras

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(joint work with Catharina Stroppel)

Assume we are given a finite dimensional complex semisimple Lie algebra $\mathfrak{g}$, the category of finite dimensional modules is completely understood, semi-simple and therefore not very interesting. A more interesting category generalizing this example is the highest weight category $\mathcal{O}^{\mathfrak{p}}$ where we fixed a Borel subalgebra $\mathfrak{b}$ inside a parabolic $\mathfrak{p}$ and consider finitely generated modules which have a weight space decomposition and where the parabolic acts locally finitely. This category is well-studied and agrees with the choice $\mathfrak{p} = \mathfrak{g}$ with the category of finite dimensional modules and in the other extreme case with the Bernstein-Gelfand-Gelfand category $\mathcal{O}$. The classification of simple modules in this case is just given by highest weights.

A more general very natural category is the category of *(generalized) weight modules*. That means of modules which have finite dimensional (generalized) weight spaces. The classification of irreducible weight modules is here harder. Already from the results of S. Fernando ([Fe]) and V. Futorny ([Fu1, Fu2]) it was known that simple weight modules with finite dimensional weight spaces fall into two types:

- the so-called *cuspidal* modules, that is the ones which are not parabolically induced modules or equivalently on which all root vectors of the Lie algebra act bijectively ([Mat, Cor 1.4, Cor 1.5]); and
- the simple quotients of generalized Verma modules, parabolically induced from cuspidal modules.

The second type forms the bulk of simple weight modules (and also of the literature on weight modules); they are easy to classify, and their structure and Kazhdan-Lusztig type combinatorics is now relatively well understood, see [MS1], [BFL], [Maz2] and references therein. From [Fe] (see also [Mat, Prop. 1.6]) it is known that cuspidal modules only exist for the Lie algebras $\mathfrak{sl}_n$ (type A) and $\mathfrak{sp}_{2n}$ (type C), and it is the classification of simple cuspidal modules for these two series of Lie algebras, which was completed by Mathieu in [Mat].

The problem we want to consider is an explicit description of the category of cuspidal (generalized) weight modules.

Now, the category of weight modules for $\mathfrak{sp}_{2n}$ is equivalent to $\mathbb{C}$-mod, the category of finite dimensional vector spaces ([BKLM]) and therefore well-understood.

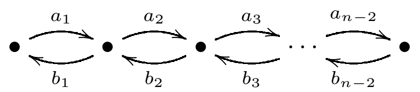
The following result is not surprising (but note that the proof gives a nice concrete description of the simple modules)

Theorem 1. [MS2] *Every nontrivial block of generalized weight modules for* $\mathfrak{sp}_{2n}$ *is equivalent to the category of finite dimensional* $\mathbb{C}[[t_1, t_2, \ldots, t_n]]$*-modules.*

Hence the only interesting remaining case is the category of generalized weight modules for $\mathfrak{sl}_n$. Our main results are then the following: let $\mathcal{C}, \hat{\mathcal{C}}$ be the category of finitely generated, cuspidal, weight (respectively generalized weight) $\mathfrak{sl}_n$-modules (for $n \geq 2$ fixed).

Theorem 2. [MS2]

(1) *Every non-integral or singular block of* $\mathcal{C}$ *is equivalent to the category of finite dimensional* $\mathbb{C}[[x]]$*-modules.*
(2) *Every non-integral or singular block of* $\hat{\mathcal{C}}$ *is equivalent to the category of finite dimensional* $\mathbb{C}[[x_1, x_2, \ldots, x_n]]$*-modules.*
(3) *For* $n > 2$ *every integral regular block of* $\mathcal{C}$ *is equivalent to the category of finite dimensional modules over a flat one-parameter deformation of* A^{n-1} *which is non-trivial as infinitesimal deformation, where* A^{n-1} *is the path algebra of the following quiver with* $n - 1$ *vertices*



modulo the relations $a_{i+1}a_i = 0 = b_i b_{i+1}$ *and* $b_i a_i = a_{i-1} b_{i-1}$ *(whenever the expression makes sense) in the case* $n > 3$ *and* $a_1 b_1 a_1 = 0 = b_1 a_1 b_1$ *in the case* $n = 3$*. The path length induces a non-negative* $\mathbb{Z}$*-grading on* A^{n-1}*. Then the deformation in question is the unique (up to rescaling of the deformation parameter) non-trivial graded one-parameter deformation of* A^{n-1}*. This deformation is the completion of a Koszul algebra with respect to the graded radical.*
(4) *For* $n > 2$ *every integral regular block of* $\hat{\mathcal{C}}$ *is equivalent to the category of finite dimensional modules over a flat* n*-parameter deformation of* A^{n-1}*. The associative algebra of this deformation is isomorphic to the tensor product of the deformation described in the previous claim* (3) *and the algebra* $\mathbb{C}[[x_2, x_3, \ldots, x_n]]$.

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Numerically finite hereditary categories with Serre duality

Adam-Christiaan van Roosmalen

Abelian categories occur both in algebraic geometry and in representation theory of algebras. We present a classification (up to derived equivalence) of hereditary abelian categories which satisfy Serre duality and are numerically finite. In this way, we accomplish a step in the ongoing classification project of hereditary categories (cf. [2, 3]).

Important examples of numerically finite hereditary categories with Serre duality are given by categories of finite dimensional representations of a finite quiver without cycles, and categories of coherent sheaves on a smooth projective curve.

1. Definitions

We start with some definitions. Throughout, let k be an algebraically closed field and let $\mathcal{A}$ be a k-linear abelian category. To avoid set-theoretical issues, we will assume that $\mathcal{A}$ is essentially small.

- We say $\mathcal{A}$ is *hereditary* if $\operatorname{Ext}^i_{\mathcal{A}}(-,-) = 0$, for all $i \geq 2$.
- We say $\mathcal{A}$ is *Ext-finite* if $\dim_k \operatorname{Ext}^i_{\mathcal{A}}(X,Y) < \infty$, for all $X, Y \in \operatorname{Ob}\mathcal{A}$, and for all $i \geq 0$.
- An Ext-finite abelian category $\mathcal{A}$ is said to have *Serre duality* if there is an autoequivalence $\mathbb{S} : \mathrm{D}^b \mathcal{A} \to \mathrm{D}^b \mathcal{A}$ such that for all $X, Y \in \operatorname{Ob}\mathcal{A}$ there is an isomorphism
$$\operatorname{Hom}_{\mathrm{D}^b \mathcal{A}}(X,Y) \cong \operatorname{Hom}_{\mathrm{D}^b \mathcal{A}}(Y, \mathbb{S}X)^*$$
natural in X and Y, where $(-)^*$ is the vector space dual.
- A category is said to be *indecomposable* if it is not a direct product of two nonzero categories.

Example 1. For any smooth projective variety V, the category $\operatorname{Coh} V$ is abelian, Ext-finite, and has Serre duality. If V is a curve, then $\operatorname{Coh} V$ is also hereditary.

The category $\operatorname{rep} Q$ of finite dimensional representations of a finite quiver Q is a hereditary category with Serre duality.

Numerical Grothendieck group. We will define an additional smallness property. Let $\mathcal{A}$ be an abelian category with Serre duality and denote the Grothendieck group of $\mathcal{A}$ by $K_0(\mathcal{A})$. The *Euler form* $\chi : \operatorname{Ob}\mathcal{A} \times \operatorname{Ob}\mathcal{A} \to \mathbb{Z}$ is given by

$$\chi(X,Y) = \sum_{i\geq 0} (-1)^i \dim_k \operatorname{Ext}^i(X,Y)$$

for $X, Y \in K(\mathcal{A})$. The sum is finite due to Serre duality. The Euler form lifts to a map $K_0(\mathcal{A}) \otimes_{\mathbb{Z}} K_0(\mathcal{A}) \to \mathbb{Z}$ also denoted by $\chi(-,-)$.

The *radical of the Euler form* is defined to be

$$\begin{aligned} \operatorname{rad}\chi &= \{X \in K_0(\mathcal{A}) \mid \chi(X,-) = 0\} \\ &= \{X \in K_0(\mathcal{A}) \mid \chi(-,X) = 0\} \end{aligned}$$

where the equality between the first and the second line is given by Serre duality.

The *numerical Grothendieck group* is defined as

$$\operatorname{Num}\mathcal{A} = K_0(\mathcal{A}) / \operatorname{rad}\chi.$$

We will say an abelian category with Serre duality is *numerically finite* if and only if the the numerical Grothendieck group is a free abelian group of finite rank.

Example 2. For any smooth projective variety V, the abelian category $\operatorname{Coh} V$ has Serre duality and is numerically finite.

Example 3. The abelian category $\operatorname{rep} Q$ where Q is a finite quiver without cycles (but possibly with relations) has Serre duality and is numerically finite.

Example 4. Let Q be an A_∞-quiver with zig-zag orientation, then $\operatorname{rep} Q$ is a hereditary category with Serre duality, but it is not numerically finite.

Perpendicular subcategories [1]**.** Let $\mathcal{A}$ be an abelian category and let $X \in \operatorname{Ob}\mathcal{A}$ be any object. We define the *right perpendicular subcategory* $X^\perp$ of $\mathcal{A}$ as the full subcategory given by the objects

$$\operatorname{Ob} X^\perp = \{Y \in \operatorname{Ob}\mathcal{A} \mid \operatorname{Ext}^i(X,Y) = 0, \forall i \geq 0\}.$$

If $\mathcal{A}$ is a hereditary category with Serre duality, then $X^\perp$ is again an abelian hereditary category with Serre duality. If X is furthermore an *exceptional* object (i.e. $\operatorname{Ext}^1(\mathrm{X},\mathrm{X}) = 0$), then $\operatorname{rank}(\operatorname{Num} X^\perp) = \operatorname{rank}(\operatorname{Num}\mathcal{A}) - 1$.

2. Results

We wish to classify the hereditary abelian categories with Serre duality which are numerically finite. Our main tool is to reduce the rank of the numerical Grothendieck group by considering subcategories orthogonal to exceptional objects.

Theorem 1. *Let $\mathcal{A}$ be a nonzero indecomposable hereditary category with Serre duality. If $\mathcal{A}$ does not have any exceptional objects, then $\mathcal{A}$ is derived equivalent to either*

(1) *the category of finite dimensional representations of $k[[x]]$, or*

(2) *the category of coherent sheaves over a smooth projective curve of genus at least 1.*

Thus any nonzero indecomposable hereditary category with Serre duality category which does not belong in the previous characterization has an exceptional object. We may then consider a perpendicular category on such an object to obtain a smaller and simplier category. If we restrict ourselves to numerically finite categories, then this process ends after finitely many steps.

The next theorem is our main result.

Theorem 2. *Let $\mathcal{A}$ be an indecomposable numerically finite hereditary category with Serre duality, then $\mathcal{A}$ is derived equivalent to either*

(1) *the category of nilpotent representations of an $\tilde{A}_n$ quiver ($n \leq 0$) with cyclic orientation, or*
(2) *the category of finite dimensional representations of a finite quiver without cycles, or*
(3) *the category of coherent modules over a hereditary $\mathcal{O}_X$-order over a smooth projective curve X.*

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Contraction of curves and mutation

Michael Wemyss

(joint work with O. Iyama)

The aim of the talk was to present some of the algebraic statements in [1] whilst also presenting some of the (still conjectural) links with the geometrical statements that motivated the work.

Non-commutative crepant resolutions were introduced by Van den Bergh [3] to axiomize various noncommutative structures appearing in resolutions of singularities, in particular the skew group ring $\mathbb{C}[x, y, z]\#G$.

Definition 1. *Let R be a Gorenstein ring, then an NCCR of R is $\operatorname{End}_R(M)$ where M is a reflexive R-module, $\operatorname{End}_R(M) \in \mathrm{CM}R$ and $\mathrm{gl.dim}\operatorname{End}_R(M) < \infty$.*

When $\dim R = 3$ a NCCR need not exist since for it to do so necessarily $\mathrm{Spec} R$ must have a crepant resolution [3]. Thus instead if we aim for the maximal crepant *partial* resolution of $\mathrm{Spec} R$, we are motivated to define the following:

Definition 2. *Let R be a Gorenstein ring, then we call a reflexive R-module M a modifying module if* $\mathrm{End}_R(M) \in \mathrm{CM}R$ *and further we call M a maximal modifying (MM) module if*

$$\mathrm{add}M = \{X \in \mathrm{refl}R : \mathrm{Hom}_R(M \oplus X, M \oplus X) \in \mathrm{CM}R\}.$$

If M is a maximal modifying module, we call $\mathrm{End}_R(M)$ *a maximal modification algebra (MMA).*

A maximal modification algebra may or may not have finite global dimension. Note that if R has only isolated singularities and M is a reflexive R-module containing R as a summand, then M is a modifying module if and only if it is rigid (i.e. $\mathrm{Ext}^1_R(M,M) = 0$) whereas M is maximal modifying if and only if it is maximal rigid. However when R is not isolated the two concepts differ; the Ext-vanishing property turns out to be too strong.

MMAs recover the notion of NCCRs:

Theorem 3. *Let R be a normal Gorenstein three-dimensional ring with* $\dim R = \dim R_{\mathfrak{m}}$ *for all* $\mathfrak{m} \in \mathrm{Max}R$. *Assume that R has a NCCR, then the* $M \in \mathrm{refl}R$ *giving NCCRs are precisely the MM modules.*

They are also all derived equivalent:

Theorem 4. *Let R be a normal Gorenstein three-dimensional ring with* $\dim R = \dim R_{\mathfrak{m}}$ *for all* $\mathfrak{m} \in \mathrm{Max}R$. *Assume that R has a MMA, then all MMAs are derived equivalent.*

We then develop a theory of mutation as a method to produce modifying modules from a given one. As a special case of this if R is an arbitrary Gorenstein normal 3-fold as above and $\Lambda := \mathrm{End}_R(M) = kQ/R$ is an arbitrary NCCR written as a quiver with relations, then we are able to mutate at an arbitrary vertex of Λ regardless of loops and 2-cycles.

For a modifying R-module M denote $M^* := \mathrm{Hom}_R(M, R)$, then given N such that $0 \neq \mathrm{add}N \subset \mathrm{add}M$ we consider

(1) a right $\mathrm{add}N$-approximation of M, i.e. $N_0 \stackrel{a}{\to} M$ with $N_0 \in \mathrm{add}N$ such that $\mathrm{Hom}_R(N, N_0) \to \mathrm{Hom}_R(N, M) \to 0$ is exact.
(2) a right-$\mathrm{add}N^*$-approximation of M^*, namely $N_1^* \stackrel{b}{\to} M^*$ with $N_1^* \in \mathrm{add}N^*$ such that $\mathrm{Hom}_R(N^*, N_1^*) \to \mathrm{Hom}_R(N^*, M^*) \to 0$ is exact.

We denote the kernels by

$$0 \to K_0 \stackrel{c}{\to} N_0 \stackrel{a}{\to} M \quad \text{and} \quad 0 \to K_1 \stackrel{d}{\to} N_1^* \stackrel{b}{\to} M^*$$

and define the right mutation of M at N to be $\mu_N(M) := N \oplus K_0$ and the left mutation of M at N to be $\nu_N(M) := N \oplus K_1^*$.

We denote $[N]$ to be the two-sided ideal of $\Lambda := \mathrm{End}_R(M)$ consisting of morphisms $M \to M$ which factor through a member of $\mathrm{add}N$, and we set $\Lambda_N := \Lambda/[N]$. One of our main theorems is the following:

Theorem 5. *With notation as above if* $\dim_k \Lambda_N < \infty$*, then*
(1) $\mathrm{End}_R(M)$ *and* $\mathrm{End}_R(\mu_N(M))$ *are derived equivalent.*
(2) $\mathrm{End}_R(M)$ *and* $\mathrm{End}_R(\nu_N(M))$ *are derived equivalent.*

There are many other statements involving mutation including a study of the $\dim_k \Lambda_N = \infty$ case and also conditions for when $\mu_N(M) = \nu_N(M)$. See [1] for more details. Note however that the homological algebra always splits into two depending on whether $\dim \Lambda_N$ is finite or not. This, together with many geometrical examples, suggests the following:

Conjecture 6. *Let* R *be a complete normal Gorenstein 3-fold and suppose there exists* $Y \xrightarrow{f} X = \mathrm{Spec} R$*, a projective birational morphism such that* $Rf_* \mathcal{O}_Y = \mathcal{O}_X$ *and every fibre has dimension* ≤ 1*. Denote* $\Lambda := \mathrm{End}_Y(\mathcal{O}_Y \oplus (\oplus_{i \in I} \mathcal{V}_i))$ *where* $\mathcal{O}_Y \oplus (\oplus_{i \in I} \mathcal{V}_i)$ *is the tilting bundle constructed in* [2]*. Let* e_i *be the idempotent corresponding to* $\mathcal{V}_i$ *(which in turn corresponds to a curve* E_i *in* Y*) and denote* $\Lambda_i := \Lambda / \Lambda(1 - e_i)\Lambda$*. Then*

$$E_i \text{ contracts to a point without contracting a divisor} \iff \dim_k \Lambda_i < \infty.$$

A positive answer to the above conjecture would have both theoretical and computational consequences in the running of the minimal model program in dimension three.

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Quiver Grassmannians and their Euler characteristics

Andrei Zelevinsky

The aim of this talk is to advertise a very interesting class of algebraic varieties called *quiver Grassmannians.* They are defined as follows. Let Q be a quiver on vertices $\{1, \ldots, n\}$. A Q*-representation* is a family $M = (M_i, \varphi_a)$, where each M_i is a finite dimensional $\mathbb{C}$-vector space attached to a vertex i, and each $\varphi_a : M_j \to M_i$ is a linear map attached to an arrow $a : j \to i$. The *dimension vector* of M is the integer vector $\mathbf{dim}\, M = (\dim M_1, \ldots, \dim M_n)$. A *subrepresentation* of M is an n-tuple of subspaces $N_i \subseteq M_i$ such that $\varphi_a(N_j) \subseteq N_i$ for any arrow $a : j \to i$. With all this terminology in place, for every integer vector $\mathbf{e} = (e_1, \ldots, e_n)$, the quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ is defined as the variety of subrepresentations of M with the dimension vector $\mathbf{e}$. As a special case, for a one-vertex quiver we get an ordinary Grassmannian.

Any quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ is Zariski closed in the product of ordinary Grassmannians $\prod_i \mathrm{Gr}_{e_i}(M_i)$, hence is a projective algebraic variety (not necessarily irreducible or smooth). Not much is known about their properties. Motivated by applications to the theory of cluster algebras we focus on the problem of computing the Euler characteristic $\chi(\mathrm{Gr}_{\mathbf{e}}(M))$.

For a given quiver representation M, we assemble all the integers $\chi(\mathrm{Gr}_{\mathbf{e}}(M))$ into the generating polynomial $F_M \in \mathbb{Z}[u_1, \ldots, u_n]$ (*F-polynomial*) given by

$$F_M(u_1, \ldots, u_n) = \sum_{\mathbf{e}} \chi(\mathrm{Gr}_{\mathbf{e}}(M)) u_1^{e_1} \cdots u_n^{e_n} \ .$$

It is not hard to show that $F_{M \oplus N} = F_M F_N$, hence the study of arbitrary F-polynomials F_M reduces to the case of M indecomposable.

Example 1. For a one-vertex quiver, i.e., for the ordinary Grassmannians, we have $F_{\mathbb{C}^m} = (F_{\mathbb{C}})^m = (1+u)^m$, hence $\chi(\mathrm{Gr}_e(\mathbb{C}^m)) = \binom{m}{e}$.

Example 2. Let Q be the *Kronecker quiver* with two vertices and two arrows from 1 to 2. There are three kinds of indecomposable Q-representations: preprojectives, preinjectives, and regular ones. More precisely, for every $m \geq 1$, there is a unique (up to an isomorphism) preprojective indecomposable Q-representation $M^{\mathrm{pr}}(m)$ of the dimension vector $(m-1, m)$, a unique preinjective indecomposable $M^{\mathrm{inj}}(m)$ of the dimension vector $(m, m-1)$, and a family of regular indecomposables $M_\lambda^{\mathrm{reg}}(m)$(parameterized by $\lambda \in \mathbb{P}^1$) of the dimension vector (m, m). As shown in [1], the Euler characteristics of the corresponding quiver Grassmannians are given as follows:

$$\chi(\mathrm{Gr}_{\mathbf{e}}(M^{\mathrm{pr}}(m))) = \binom{m-e_1}{e_2-e_1}\binom{e_2-1}{e_1},$$
$$\chi(\mathrm{Gr}_{\mathbf{e}}(M^{\mathrm{inj}}(m))) = \binom{m-e_2}{e_1-e_2}\binom{e_1-1}{e_2},$$
$$\chi(\mathrm{Gr}_{\mathbf{e}}(M_\lambda^{\mathrm{reg}}(m))) = \binom{m-e_1}{e_2-e_1}\binom{e_2}{e_1} \ .$$

Example 3. Let Q be a Dynkin quiver, i.e., an orientation of a simply-laced Dynkin diagram; thus, its every connected component is of one of the ADE types. The indecomposable Q-representations are determined by their dimension vectors. Identifying $\mathbb{Z}^n$ with the root lattice of the corresponding root system, these dimension vectors get identified with the positive roots. Let $M(\alpha)$ denote the indecomposable representation with the dimension vector identified with a positive root α. A unified (type-independent) "determinantal" formula for $F_{M(\alpha)}$ was given in [6]. To state it we need some preparation.

First, different orientations of a given Dynkin diagram are in a bijection with the different Coxeter elements in the Weyl group W: following [6], to a Coxeter element $c = s_{i_1} \cdots s_{i_n}$ (the product of all simple reflections taken in some order) we associate an orientation with an edge between i_ℓ and i_k oriented from i_ℓ to i_k whenever $k < \ell$.

Now let G be the simply-connected semisimple complex algebraic group associated with our Dynkin diagram. Each weight γ belonging to the W-orbit of some fundamental weight ω_i gives rise to a regular function $\Delta_{\gamma,\gamma}$ on G (a *principal generalized minor*) defined as follows: $\Delta_{\gamma,\gamma}(x)$ is the diagonal matrix entry of $x \in G$ associated with the one-dimensional weight subspace of weight γ in the fundamental representation V_{ω_i}. Recall also the one-parameter subgroups $x_i(u) = \exp(ue_i)$ and $y_i(u) = \exp(uf_i)$ in G associated with the Chevalley generators e_i and f_i of the Lie algebra of G.

With this notation in place we have the following result essentially proved in [6]:

$$F_{M(\alpha)}(u_1, \ldots, u_n) = \Delta_{\gamma,\gamma}(y_{i_1}(1) \cdots y_{i_n}(1) x_{i_n}(u_{i_n}) \cdots x_{i_1}(u_{i_1})) \ ,$$

where γ is uniquely determined from the equation $c^{-1}\gamma - \gamma = \alpha$.

If a quiver Q is *acyclic* (i.e., has no oriented cycles), the quiver Grassmannians have the following properties:

- If M is a *general* representation of a given dimension vector then all its quiver Grassmannians are smooth. In particular, this is true if M is *rigid*, i.e., $\mathrm{Ext}^1(M, M) = 0$ (as explained in [3, Proposition 3.5], this follows from the results in [5]).
- If M is indecomposable and rigid then $\chi(\mathrm{Gr}_{\mathbf{e}}(M)) \geq 0$ for all $\mathbf{e}$.

The following example shows that the rigidity condition is essential for the positivity of the Euler characteristic.

Example 4. Let Q be the generalized Kronecker quiver with two vertices and four arrows from 1 to 2. As shown in [3, Example 3.6], if M is a general representation of dimension vector $(3, 4)$, and $\mathbf{e} = (1, 3)$ then $\mathrm{Gr}_{\mathbf{e}}(M)$ is isomorphic to a smooth projective curve of degree 4 in $\mathbb{P}^2$, hence $\chi(\mathrm{Gr}_{\mathbf{e}}(M)) = -4$.

With applications to the theory of cluster algebras in mind, we will assume from now on that every quiver Q under consideration has no loops or oriented 2-cycles. If in addition a quiver Q is acyclic, the quiver Grassmannians most important for these applications are those in rigid indecomposable Q-representations. In [3] it was suggested how to extend this class of representations to the case of not necessarily acyclic quivers. Here is a brief account of this approach.

First, we restrict our attention to Q-representations satisfying relations imposed by a generic *potential* S on Q. Roughly speaking, S is a generic linear combination (possibly infinite) of cyclic paths in Q, viewed as an element of the completed path algebra $\widehat{\mathbb{C}Q}$ (see [2] for a detailed setup). By a (Q, S)-representation we mean a Q-representation annihilated by sufficiently long paths in $\mathbb{C}Q$ and by all *cyclic derivatives* of S.

Second, following [4, 2], we consider *decorated* (Q, S)-representations: such a representation is a pair $\mathcal{M} = (M, V)$, where M is a (Q, S)-representation as above, and $V = (V_i)$ is a collection of finite-dimensional vector spaces attached to the vertices (with no maps attached).

Mutations of quivers with potentials and their representations at arbitrary vertices were introduced and studied in [2]. As shown in [3], a natural class of (Q, S)-representations generalizing rigid indecomposable representations of acyclic quivers consists of (Q, S)-representations obtained by mutations from *negative simple* representations (those having some V_i equal to $\mathbb{C}$, and the rest of the spaces M_j and V_j equal to 0). One of the main constructions in [3] is that of an integer-valued function $E(\mathcal{M})$ on the set of (Q, S)-representations, which is invariant under mutations and vanishes on negative simple representations. An open question is whether the condition $E(\mathcal{M}) = 0$ for an indecomposable (Q, S)-representation $\mathcal{M}$ implies that $\mathcal{M}$ can be obtained by mutations from a negative simple representation. Regardless of the answer to this question, quiver Grassmannians in the indecomposable (Q, S)-representations with $E(\mathcal{M}) = 0$ deserve further study.

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Geometric Approaches to Growth of Algebras

Efim Zelmanov

Let A be an algebra over a field F. Suppose that A is generated by a finite dimensional subspace V. Then A is a union of the ascending chain of finite dimensional subspaces $V(n) = \operatorname{span}\{v_1, \ldots, v_k \mid v_i \in V, k \leq n\}$. By the growth function we mean

$$g(n, V) := \dim V(n).$$

Let W be a finite dimensional subspace of A. Following M. Gromov we call

$$b(W) = (VW + W)/W$$

the V-boundary of W. Let's adopt the following terminology : $\dim W$ will be conveniently called the volume of W or the area of W.

The Isoperimetric Problem. Among all subspaces of volume n find the one with the minimal area of the surface $b(W)$.

Let $I(n, V)$ denote this minimal area. The function $I(n, V)$ is called the isoperimetric profile of A.

Although the functions $g(n,V), I(n,V)$ depend on a choice of V, their asymptotics do not. Let f,g be two functions on $\mathbb{N}$ with positive real values. We say that f is asymptotically smaller than g if there exists a constant $C \in \mathbb{N}$ such that $f(n) \leq Cg(Cn)$ for all n. The functions f,g are asymptotically equivalent if each of them is asymptotically smaller than the other one.

If $A =< V >=< V' >$ then $g(n,V)$ is asymptotically equivalent to $g(n,V')$ and $I(n,V)$ is asymptotically equivalent to $I(n,V')$. Thus we will speak simply of $g(n)$ and $I(n)$.

For groups the isoperimetric profile provides basically the same information as the probablity of return function for random walks. It would be interesting to find some analog of random walks for algebras (probably, involving products of random matrices ?)

Also for groups (and cancellation semigroups) Coulhon and Saloff-Coste proved an important inequality that relates $g(n)$ and $I(n)$: if g' is the inverse function of g, then we have

Theorem 1. *$I(n)$ is asymptotically greater than $n/g'(n)$.*

In many cases, for example, for polycyclic groups, I is asymptotically equivalent to $n/g'(n)$.

Gromov asked if the Coulhon Saloff-Coste inequality holds for domains. M. D'Adderio constructed a counterexample: a finitely generated Noetherian domain of GKdim 3 and the isoperimetric profile $n^(1/2)$.

Definition (Elek, Gromov)**.** We say that the algebra A is amenable if $I(n)$ is strictly asymptotically smaller than n.

G. Elek proved that an amenable domain satisfies Ore condition but a subdomain of an amenable domain is not necessarily amenable. From the results of D'Adderio it follows though that an Ore subdomain is amenable. For an amenable algebra A we consider the Folner function $F(n)$ which is the minimal dimension of a subspace W such that $\dim b(W)/\dim W < 1/n$. M. D'Adderio formulated a gap conjecture for Folner functions that is related to Artin and Zhang conjectures on gaps in low transcendence degrees. Surprisingly, in some important particular case this gap conjecture can be interpreted as Shannon enthropy inequality.

As above, let $A =< V >$, V is finite dimensional and let M be a left module over A. Let a be a positive number.

Definition. We say that M is an a-expander if the isoperimetric profile of M is greater than an, $I(n,M) > an$.

In other words, for an arbitrary finite dimensional subspace W of M we have $\dim(W + VW) > (1+a)\dim W$. In particular it means that the module M is not amenable.

We are primarily interested in infinite families of a-expanders with a fixed.

Definition. An algebra A has property τ if

(1) the family of irreducible finite dimensional A-modules is an expander family for some $a > 0$,

(2) The intersections of kernels of these representations is (0).

The motivation for this notion comes from Wigderzon's question about dimension expanders.

Let $chF = 0, K = Z/pZ, SL'(n, K[t])$ is the congruence group of matrices over polynomials. A. Lubotzky and I showed that the group algebra $F[SL'(n, K[t])$, $n > 2$, has property τ. In particular, it answers Wigderson's question in characteristic zero. Recently an answer for modular fields was obtained by J. Bourgaine. As shown by G. Margulis expansion properties come from Kazhdan property T.

We can define property T for Lie algebras and even for arbitrary associative bialgebras over reals. Practically all good algebras (affine Kac-Moody, root graded algebras etc) have property T. But even in that case only those representations have the expansion property, that are unitarizable. In particular we get :

Theorem 2. *Let L be a compact finite dimensional simple Lie algebra. Then the universal enveloping algebra $U(L)$ has property τ.*

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Growth of pointed Hopf algebras

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(joint work with D. G. Wang and G. Zhuang)

A seminal result of Gromov states that a finitely generated group G has polynomial growth, or equivalently, the associated group algebra has finite Gelfand-Kirillov dimension, if and only if G has a nilpotent subgroup of finite index [Gr]. Group algebras form a special class of cocommutative Hopf algebras. A natural question for us is to look for a necessary and sufficient condition for an affine (i.e., finitely generated) Hopf algebra H to have finite GK-dimension (short for Gelfand-Kirillov dimension). Let k be a base field, and assume that, for similicity, k is algebraically closed of characteriztic zero. It is clear that an affine commutative Hopf algebra over k has a finite GK-dimension which equals to its Krull dimension. If H is cocommutative, by a classification result [Mo, Corollary 5.6.4 and Theorem 5.6.5], it is isomorphic to a smash product $U(\mathfrak{g})\#kG$ for some group G and some Lie algebra $\mathfrak{g}$. Consequently, $\text{GKdim}H = \text{GKdim}kG + \dim \mathfrak{g}$. This question is also answered for several classes of noncommutative and noncocommutative Hopf algebras, including quantum groups $U_q(\mathfrak{g})$ and $\mathcal{O}_q(G)$, see [BG2]. We take an initial step to attack this question for general noncommutative and noncocommutative Hopf algebras.

Quite a few classes of Hopf algebras of finite GK-dimension have been studied by several authors, for example, Andruskiewitsch-Angiono [AA], Andruskiewitsch-Schneider, [AS1, AS2], Brown [Br1, Br2], Brown-Goodearl [BG1, BG2], Brown-Zhang [BZ], Goodearl-Zhang [GZ], Lu-Wu-Zhang [LWZ], Wu-Zhang [WuZ1, WuZ2] and Zhuang [Zhu], during the last few years. But the classification of such Hopf algebras is far from complete. We provide three lower bounds of GK-dimension in terms of certain invariants of skew primitive elements. These estimations should be useful for studying pointed Hopf algebras of low GK-dimension. It seems that the GK-dimension of H is closely related to some combinatorial data which we will define soon.

Let H be a Hopf algebra over k. A nonzero element $y \in H$ is called $(1, g)$-*primitive* (or generally skew primitive) if $\Delta(y) = y \otimes 1 + g \otimes y$ and such a g is called the *weight* of y and denoted by $\mu(y)$. Let $G(H)$ denote the set of group-like elements in H and let $C_0 = kG(H)$. It is clear that $\mu(y) \in G(H)$. Here is the first lower bound theorem.

Theorem 1 (First lower bound theorem). *Let $\{y_i\}_{i=1}^{w}$ be a set of skew primitive elements such that*

(1) *$\{y_i\}_{i=1}^{w}$ are linear independent in H/C_0.*
(2) *for all $i \leq j$, $y_i\mu(y_j) = \lambda_{ij}\mu(y_j)y_i$ for some $\lambda_{ij} \in k^{\times}$,*
(3) *for each i, λ_{ii} is either 1 or not a root of unity.*

Then $GKdimH \geq GKdimC_0 + w$.

In general λ_{ij} in condition (b) may not exist. Let W denote the set of weights $\mu(y)$ for all skew primitive elements $y \notin C_0$ and let $W_{\sqrt{}}$ be the subset consisting of weights $\mu(y)$ for all y such that y^n is also a skew primitive for some $n > 1$. (Note that in this paper the term "skew primitive" means "$(1, g)$-primitive"). For any subset $\Phi \subset G(H)$, the subgroup of $G(H)$ generated by Φ is denoted by $\langle\Phi\rangle$. Here is the second lower bound theorem.

Theorem 2 (Second lower bound theorem). *Suppose $\langle W \setminus W_{\sqrt{}}\rangle$ is abelian. Then*

$$GKdimH \geq GKdimC_0 + \#(W \setminus W_{\sqrt{}}).$$

There are examples such that $W = W_{\sqrt{}}$ and $\mathrm{GKdim}H = \mathrm{GKdim}C_0$, but $\#(W_{\sqrt{}})$ is arbitrarily large. Therefore $W_{\sqrt{}}$ has to be removed from W when we estimate the GK-dimension of H. Let y be a skew primitive element. If

$$\mu(y)^{-1}y\mu(y) - cy = \tau(\mu(y) - 1)$$

for some $c \in k^{\times}$ and $\tau \in k$, then c is called the *commutator* of y (with its weight) and denoted by $\gamma(y)$. Define Γ to be the set of $\gamma(y)$ for all skew primitive elements $y \notin C_0$ such that $\gamma(y)$ exists and let $\Gamma_{\sqrt{}}$ be the subset of Γ consisting of those $\gamma(y)$ which are roots of unity but not 1. If $\gamma(y)$ exists, the pair $(\mu(y), \gamma(y))$ is denoted by $\omega(y)$. Define Ω to be the set of $\omega(y)$ for all skew primitive elements $y \notin C_0$ such that $\omega(y)$ exists and let $\Omega_{\sqrt{}}$ be the subset of Ω consisting of those $\omega(y)$ in which $\gamma(y)$ is a root of unity but not 1. Theorem 2 can be improved a little: suppose $\langle W \setminus W_{\sqrt{}}\rangle$ is abelian. Then

$$\mathrm{GKdim}H \geq \mathrm{GKdim}C_0 + \#(\Omega \setminus \Omega_{\sqrt{}}).$$

Let y be a skew primitive element with $g = \mu(y)$. Let T_g be the conjugation by g, namely, $T_g : a \to g^{-1}ag$. A scalar c is called a commutator of y of level n if

$$(T_g - cId_H)^n(y) \in C_0.$$

In this case we also write $\gamma(y) = c$. Let Z denote the space spanned by all skew primitive elements and let $Y_{\sqrt{}}$ denote the subspace of Z spanned by those y with commutator of finite level and with $\gamma(y)$ being a root of unity but not 1. Here is the third lower bound theorem.

Theorem 3 (Third lower bound theorem)**.** *Suppose $\langle W \setminus W_{\sqrt{}} \rangle$ is abelian. Then*

$$GKdimH \geq GKdimC_0 + \dim Z/(C_0 + Y_{\sqrt{}}).$$

There are examples such that $Z = Y_{\sqrt{}} + C_0$ and GKdim$H$ = GKdimC_0, but $\dim Y_{\sqrt{}}$ is arbitrarily large. Therefore it is sensible to consider the quotient space $Z/(C_0+Y_{\sqrt{}})$ in the above theorem. This is analogous to removing $W_{\sqrt{}}$ in Theorem 1.

The hypothesis that $\langle W \setminus W_{\sqrt{}} \rangle$ being abelian could be superfluous, but it is not clear to us how to deal with the non-abelian case. When $\langle W \rangle$ is non-abelian, a better lower bound could be obtained by replacing $\#(W \setminus W_{\sqrt{}})$ in Theorem 2 by GKdim$k\langle W \rangle$. If $\langle W \setminus W_{\sqrt{}} \rangle$ is nilpotent-by-finite (otherwise GKdim$H = \infty$), results for Hopf algebras analogous to the group case (see Gromov's theorem [KL, Theorem 11.1] and to Bass' theorem [KL, Theorem 11.14]) would be interesting.

These lower bounds should be improved. It is expected that better estimates can be achieved once finer invariants of skew primitive elements are introduced. There are further connections between the growth of Hopf algebras and the W and other invariants defined by skew primitive elements.

Proposition 4. *Suppose $\langle W \setminus W_{\sqrt{}} \rangle$ is abelian. If rank $\langle \Gamma \rangle >$ rank $\langle W \setminus W_{\sqrt{}} \rangle = 1$, then H has exponential growth.*

Note that rank $\langle \Gamma \setminus \Gamma_{\sqrt{}} \rangle$ = rank $\langle \Gamma \rangle$ since elements in $\Gamma_{\sqrt{}}$ has finite order. The rank of $\langle W \rangle$ and $\langle \Gamma \rangle$ should be related when GKdim$H$ is finite. The condition of H having finite GK-dimension should be related to some restrictive condition in terms of W and Γ.

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