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Progress in Surface Theory

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Abstract. The theory of surfaces is interpreted these days as a prototype of submanifold geometry and is characterized by the substantial application of PDE methods and methods from the theory of integrable systems, in addition to the more classical techniques from real and/or complex analysis. In addition, surfaces with singularities are studied intensively. In this workshop we brought together all the main strands of modern surface theory.

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Introduction by the Organisers

The workshop *Progress in Surface Theory*, organised by Uwe Abresch (Bochum), Josef Dorfmeister (München), and Masaaki Umehara (Osaka) was held May 2^{nd}–May 8^{th}, 2010. In recent years, studying surfaces in more general ambient spaces than space forms, as become an extremely active direction of research in surface theory, even though a complete understanding of all important surface classes in space forms for instance is still an issue. As a result, the gap between surface theory and submanifold geometry in general has closed considerably, despite the fact that studying typical questions about surfaces relies heavily on methods from a number of different fields. Crucial are in particular methods from the theory of elliptic differential equations and geometric measure theory, complex analytic methods, and integrable systems techniques. Moreover, it should be noted that surfaces/submanifolds with certain types of mild singularities have become much more mainstream.

The major goal of our workshop has been to continue bringing together the best experts in all these substantially different fields and to see the first results from our previous effort in 2007.

This workshop has been attended by 57 participants from ten different countries, the largest contingents coming from Germany, Japan, Spain, France, England and the US. We had participants of all ages, from the very young to the very senior people; we had specialists for all major directions in Surface Theory that are currently under active research, and we had some experts in neighboring fields.

As a result, our workshop provided a fairly complete picture of the recent developments in the field. This was of course particularly useful for our nine young participants. Given the size of the workshop, not even half of the participants had the opportunity to give a talk. In order to still facilitate everybody to present his/her latest results, we offered space to exhibit posters, an offer that has been particularly well received by our young participants and was well attended.

The official program consisted of 24 talks of 50 minutes each and 2 half hour talks. In addition, on Wednesday evening there was a more informal presentation concerning computer visualizations in differential geometry, and on Thursday evening there was an official poster session in order to stimulate discussions about the content of the various posters between their authors and other participants working on related topics. We had organized this poster session as a social gathering for the entire workshop, which worked out very well, both, socially and scientifically.

The free time in the afternoons and the evenings has been used extensively and intensively by the participants for discussions. In fact, it seems that the full schedule of this workshop has actually spawned a lot of new interaction among participants from different schools. This was strongly enhanced by the fact that most lectures were well prepared and well presented and provided a lot of useful background.

Several participants, both young and senior ones, commented to us that they had learned quite a bit from the talks and the discussions. Certainly, the great setting and the superb atmosphere, for which Oberwolfach is reknown around the world, making it a unique conference place, provided the environment that generated many fruitful discussions. Moreover, the young mathematicians also noted that the meeting had been an excellent opportunity to have direct contact with a broad international group of established mathematicians.

We feel that the meeting was exciting and highly sucessful, in particular in view of some startling new results and the extraordinary amount of scientific interaction among the participants.

Here is a brief summary of the mathematical contents: there were two *outstanding* new results that were presented for the first time to a larger community at our workshop.

R. Miyaoka has classified the isoparametric hypersurfaces $M^{12} \subset \mathbb{S}^{13}$ with six distinct principal curvatures of multiplicity 2. She proved that up to congruence there is only the known homogeneous family induced by the adjoint action of the exceptional Lie group G_2 on its Lie algebra. This result finishes off the classification of isoparametric hypersurfaces in spheres with six distinct principal curvatures. Isoparametric hypersurfaces had been introduced by E. Cartan around 1935, and their classification is known as a hard problem. Miyaoka's idea is to exploit the interplay between the two linear isospectral families obtained from the second fundamental forms of the focal manifolds. A non-trivial amount of cleverness is required to do the bookkeeping right and avoid an explosion of the number of cases to consider.

M. Kilian on the other hand used integrable system methods to study equivariant cmc tori in $\mathbb{S}^3$. He classified equivariant cmc tori of cohomogeneity 1 in terms of the spectral curves of flat tori with a double point on the real part. Using the maximum principle (at infinity) in order to analyse when self-intersections can appear or disappear on a given deformation of cmc tori, he even managed to decide which parts of the moduli space correspond to embedded or Alexandrov embedded cmc tori. It turns out that the Clifford torus is the only equivariant embedded minimal torus in $\mathbb{S}^3$, as conjectured by Lawson in 1971.

Moreover, the work of K. Große-Brauckmann and R. Kusner on coplanar cmc k-unduloids in $\mathbb{R}^3$ was received very well. Taking the Lawson transformation of the piece on one side of the symmetry plane and combining this map with the Hopf projection $\mathbb{S}^3 \to \mathbb{CP}^1$ that collapses the k boundary components to points, they assign to each coplanar k-unduloid a conformal developing map $\mathbb{D}^2 \to \mathbb{CP}^1$ with k branch points of infinity order on the boundary $\partial\mathbb{D}^2$. It is a hard PDE result that any such developing map can in fact be lifted (uniquely) w.r.t. the Hopf fibration $\mathbb{S}^3 \to \mathbb{CP}^1$ a to a map $\varphi\colon \mathbb{D}^2 \to \mathbb{S}^3$ that parametrizes a minimal surface which is the Lawson transformation of half a coplanar k-unduloid in $\mathbb{R}^3$. At this point it becomes feasible to investigate the moduli space of the relevant developing maps writing Hill's equation and suitable holomorphic quadratic differentials. In short, the theory of projective structures provides an explicit model for the moduli space of coplanar cmc k-unduloids in $\mathbb{R}^3$.

In the overall picture we observed three major lines of research. For one, quite a lot of work has been done on *geometric variational problems*. A. Ros presented a classification of embedded least area surfaces (mod 2) in flat 3-tori. B. Daniel studied the isoperimetric problem in $\mathsf{Solv}(3)$. Alexandrov's reflection principle implies that any isoperimetric domain Ω_v is actually a bi-graph w.r.t. two orthogonal foliations by totally-geodesic hyperbolic planes. This in fact shows that Ω_v is isotopic to a ball. However, for cmc surfaces in $\mathsf{Solv}(3)$ we do not know a holomorphic quadratic differential and thus we have no way of computing cmc spheres explicitely. The key difficulty is to prove that a (possibly immersed) cmc sphere Σ_H in $\mathsf{Solv}(3)$ with mean curvature H of index 1 is actually embedded and unique up to congruence. Somewhat related is M. Ritoré's work on stable area stationary surfaces in the Heisenberg group $\mathsf{Nil}(3)$ equipped with the standard Carnot metric. In this target space, all the relevant surfaces are known explicitely; here the difficulty is to cope with the subelliptic nature of the second variation formula.

U. Pinkall and E. Kuwert discussed geometric variational problems with an Euler-Lagrange equation of order 4. U. Pinkall described the singularities that appear in the rulings on closed elastic strips that are glued with one or more half twists. As usual he modelled the elastric strips by their center curve together with an additional transversal line field, and so he was dealing with complicated ODE systems and their first integrals. E. Kuwert on the other hand considered the infimum β_p^n of Willmore functional $\mathcal{W}(\Sigma_p)$ on smooth immersed surfaces $\Sigma_p \looparrowright \mathbb{R}^n$ of genus p. It is by now standard that $4\pi < \cdots \leq \beta_p^n \leq \cdots \leq \beta_p^4 \leq \beta_p^3 < 8\pi$ for any $p \geq 1$. Kuwert proved using an appropriate monotonicity formula of L. Simon that $\lim_{p\to\infty} \beta_p^n = 8\pi$ for any $n \geq 3$.

F. Pacard on the other hand started out with the problem of minimizing the first Dirchlet eigenvalue $\lambda_1(\Omega)$ over all domains Ω with precribed volume $v > 0$ in some Riemannian manifold M^n. It is standard to study the regularity properties of the boundary of such extremal domains using blow-up techniques. In fact blow-up leads to noncompact flat manifolds $\tilde{M}^n$ with boundary that comes with a positive harmonic function u that has zero Dirichlet boundary data and constant non-zero Neumann boundary data. The surprising result is that there are more such exceptional flat surfaces $\tilde{M}^n$ than half spaces and the complements of balls in $\mathbb{R}^n$.

There were two more analysis oriented lectures, the talk by M. Koiso about bifurcation and stability criteria for cmc surfaces and the survey by L. Hauswirth on the theory of minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. It seems remarkable that in $\mathbb{H}^2 \times \mathbb{R}$ there are both, complete minimal surfaces of parabolic and of hyperbolic type. The parabolic objects typically come from Jenkins-Serrin constructions and are pretty rigid at infinity, whereas the hyperbolic objects are quite flexible at infinity.

The second major theme has been *Weierstrass representations and integrable systems methods.* F. Pedit explained how to describe cmc k-noids with asymptotic Delaunay ends, which are not necessarily coplanar, in terms of connections $d + \xi$ on the loop group of $\mathsf{Sl}_2(\mathbb{C})$. The key problem with this approach so far is that in case $k > 3$ — because of the existence of accessory parameters — it has not been possible to decide whether $d + \xi$ is indeed unitarizable when given a concrete choice for all the parameters in the construction. Moreover, S. Kobayashi worked out the integrable systems approach for cmc surfaces in the hyperbolic space $\mathbb{H}^3$. The key problem that he had to handle is the fact that the (harmonic) Gauss map of such a surface is naturally a map into the unit tangent bundle of $\mathbb{H}^3$ which can be identified as the 4-symmetric space $\mathsf{Sl}_2(\mathbb{C})/\mathsf{SU}(1)$. Handling the loop group approach in this case requires a modifie technique.

I. Khemar on the other hand started out from the m-th integrable elliptic system associated to a given k-symmetric space and investigated which geometric objects are described by such a system. Depending on the relative magnitudes of k and m, he distinguished a primitive case where the integrable system decribes J-holomorphic curves, a minimal determined case where the system describes maps that are horizontally holomorphic and vertically harmonic, and a maximal determined case where the system describes so-called stringy harmonic maps.

Finally I. Taimanov talked about transformations of surfaces and their applications to spectral theory. He pointed out that the classical Laplace, Darboux, Moutard, and Bianchi-Bäcklund transformations of non-linear differential equations had historically been developed in the context of surface theory. With this in mind, he could extend the Moutard transformation to a transformation of the Novikov-Veselov equation, which is a 2-dimensional generalization of the KdV equation. Applying this technique, he was able to construct explicite solutions to the Novikov-Veselov equation with fast decaying Cauchy data at $t = 0$ that blow up in finite time T^*.

The classical transformations in surface theory and their impact on integrable systems theory have also been discussed in the lectures of F. Burstall on Ω-surfaces and K. Leschke on Darboux transforms and simple factor dressing.

The third topic common to several talks has been *surfaces with singularities.* In his talk on the duality of wave front sets and its applications, K. Yamada investigated hypersurfaces N^n in some Riemannian manifold M^{n+1} that effectively come as projections of immersed Legendre submanifolds $\hat{N}^n$ in the unit tangent bundle $UM^{n+1} \to M^{n+1}$. This allows N^n itself to have a certain class of mild singularities like swallowtails. This is the framework that has been employed by S. Fujimori when studying maximal surfaces with singularities in the Minkowski space $\mathbb{R}^{2,1}$.

In his lecture on the geometric Cauchy problem, P. Mira studied isolated singular points on $H = 1$ surfaces in $\mathbb{H}^3$, intrinsically flat surfaces in $\mathbb{H}^3$ or $\mathbb{S}^3$, and surfaces with constant positive curvature in $\mathbb{R}^3$. Note that the adherence points of the unit normal field define a Jordan curve in the unit sphere in the tangent space of the target space at the singular point. It is this Jordan curve that provides the Cauchy data necessary for reconstructing the surface near the singular point in a way that resembles the solution of the Björling problem. Based on these results, A. Martinéz has classified the complete flat surfaces in $\mathbb{H}^3$ with one end and one or two isolated singularities.

The remainig talks have been *special topics* that cannot be naturally subsumed into one of the previous threads. For instance, F. Martin has given a nice survey on the Calabi-Yau problem, which made it very clear how wild complete minimal surfaces can get when admitting non-proper surfaces. In her talk Ch. Breiner explained how to use Colding-Minicozzi theory in order to study complete cmc surfaces with finite topology and one end. She proved that such a minimal surface is either flat or $\mathcal{C}^0$-asymptotic to a helicoid.

On the other end of the spectrum, M. Guest gave a talk about tt^*-geometry explaining how the integrable systems methods developed in the theory of cmc surfaces relate to the observations of Cecotti and Vafa about holomorphic data in quantum field theory like quantum cohomolgy rings or Landau-Ginzberg potentials, which predict the existence of harmonic maps with surprisingly good global properties.

In his talk about Noether's theorem, conserved quantities, minimal and cmc surfaces, P. Romon explained how to obtain the flux and the torque of a cmc surface in $\mathbb{R}^3$ from Noether's theorem. He then approached the question whether the existence of the generalized Hopf differential for cmc surfaces in homogeneous 3-manifolds $M^3_{\kappa,\tau}$ may also be a consequence of Noether's theorem. After all, each point in such an $M^3_{\kappa,\tau}$ has a 1-dimensional isotropy group. Unfortunately, there is no conclusive answer yet.

This particular workshop included an evening session on *visualization*. H. Karcher and R. Palais presented the Virtual Math Museum (http://vmm.math.uci.edu/3D-XplorMath/), a fairly mature piece of software that can draw and animate quite a number of beautiful examples from curve theory, surface theory, integrable systems, and many other mathematical subjects. A key feature of the Virtual Math Museum is that it comes with well-writen elementary documentation on the concepts of visulization and on the mathematical background of the various galleries. Unfortunately, the Virtual Math Museum contains only few cmc surfaces. This is largely due to the subtleties of the numerics for generating fairly general classes of cmc surfaces.

Workshop: Progress in Surface Theory

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Abstracts

Minimal Surfaces with One End

Christine Breiner
(joint work with Jacob Bernstein)

The conformal type and asymptotic geometry of complete, embedded, minimal surfaces with finite topology (in $\mathbb{R}^3$) and two or more ends is well understood. In this talk we address recent results concerning such surfaces when they have one end. Work by Colding-Minicozzi on the structure of compact, embedded minimal surfaces with connected boundary paved the way for Meeks and Rosenberg's proof of the uniqueness of the helicoid as well as an understanding of the conformal type of these surfaces in the complete case. In particular, we outline the proof of the following theorem, which is proven in [2].

Theorem 1. *Let $\Sigma \in \mathbb{R}^3$ be a complete, embedded minimal surface with finite topology and one end. Then Σ is conformally a once punctured, compact Riemann surface. Moreover, if the surface is not flat, then it is C^0 asymptotic to some helicoid.*

The proof of this result draws heavily on the fundamental work of Colding and Minicozzi on the geometric structure of embedded minimal surfaces in $\mathbb{R}^3$ [3, 4, 5, 6, 7]. Assuming only mild conditions on the boundaries, they give a description of the geometric structure of essentially all embedded minimal surfaces with finite genus. From this structure, they deduce a number of important consequences. We highlight, in particular, two results of Colding-Minicozzi. The first concerns embedded minimal disks with large curvature at their center, which are thus known to be not graphical on a sufficiently large scale.

Theorem 2. *Let $0 \in \Sigma \subset B_R \subset \mathbb{R}^3$ be an embedded minimal disk with $\partial\Sigma \subset \partial B_R$. Then there exist $C, \Omega > 1$ such that:*
if $|A|^2(0) > CR^{-2}$ then the component of $B_{R/\Omega} \cap \Sigma$ containing 0 is the union of two multi-valued graphs that spiral together.

"Multi-valued graph" should be thought of as looking roughly like one half of the helicoid. Another important result is the "one sided curvature estimate". One can think of it as a local version of the strong half-space theorem. Precisely, it is as follows:

Theorem 3. *There exists $\epsilon > 0$ such that if $\Sigma \subset B_{2R} \cap \{x_3 > 0\} \subset \mathbb{R}^3$ is an embedded minimal disk with $\partial\Sigma \subset \partial B_{2R}$, then every component Σ' of $\Sigma \cap B_R$ that intersects $B_{\epsilon R}$ has*

$$\sup_{\Sigma'} |A|^2 \leq R^{-2}.$$

Colding and Minicozzi's work is also an essential ingredient in understanding minimal surfaces with infinite total curvature, i.e., complete surfaces with one end. Prior to their work, the study of these surfaces required very strong assumptions on the conformal structure and behavior of the Gauss map at the end. For example, Hauswirth, Perez and Romon [9] consider $E \subset \mathbb{R}^3$, a complete embedded minimal annulus with one compact boundary component and one end with infinite total curvature. They assume, in addition, that E is conformal to a punctured disk, the Weierstrass data (g, dh) has the property that dg/g and dh extend across the puncture, and the flux over the boundary of E has zero vertical component. They then deduce more precise information about the asymptotic geometry of E. Indeed, their result immediately gives the asymptotic behavior of the Σ of interest in Theorem 1, once the conditions on the Weierstrass data are shown.

By using Colding and Minicozzi's work, in particular the compactness result of [6], Meeks and Rosenberg were able to remove such strong assumptions for disks. Indeed, in [10], they resolve the question of the uniqueness of the helicoid.

Theorem 4. *Let Σ be a complete, embedded minimal disk in $\mathbb{R}^3$. Then Σ is the plane or the helicoid.*

Let us now recall the argument of [1], where we provide an alternative proof to the uniqueness of the helicoid. There it is shown that any complete, non-flat, properly embedded minimal disk can be decomposed into two regions: one a region of strict spiraling, i.e. the union of two strictly spiraling multi-valued graphs over the $x_3 = 0$ plane (after a rotation of $\mathbb{R}^3$), and the other a neighborhood of the region where the graphs are joined and where the normal has small vertical component. By strictly spiraling, we mean that each sheet of the graph meets any (appropriately centered) cylinder with axis parallel to the x_3-axis in a curve along which x_3 strictly increases (or decreases). This follows from existence results for multi-valued minimal graphs in embedded disks found in [4] and an approximation result for such minimal graphs from [8]. The strict spiraling is then used to see that $\nabla_\Sigma x_3 \neq 0$ everywhere on the surface; thus, the Gauss map is not vertical and the holomorphic map $z = x_3 + ix_3^*$ is a holomorphic coordinate. By looking at the log of the stereographic projection of the Gauss map, the strict spiraling is used to show that z is actually a proper map and thus, conformally, the surface is the plane. Finally, this gives strong rigidity for the Weierstrass data, implying the surface is a helicoid.

For Σ as in Theorem 1, as there is finite genus and only one end, the topology of Σ lies in a ball in $\mathbb{R}^3$, and so, by the maximum principle, all components of the intersection of Σ with a ball disjoint from the genus are disks. Hence, outside of a large ball, one may use the local results of [3, 4, 5, 6] about embedded minimal disks. For Σ non-simply connected , the presence of non-zero genus complicates matters. Nevertheless, the global structure will follow from the far reaching description of embedded minimal surfaces given by Colding and Minicozzi in [7]. In particular, as Σ has one end, globally it looks like a helicoid. We first prove a sharper description of the global structure; indeed, one may generalize the decomposition of [1] as:

Theorem 5. *There exist $\epsilon_0 > 0$ and a decomposition of Σ into disjoint subsets $\mathcal{R}_A$, $\mathcal{R}_S$, and $\mathcal{R}_G$ such that:*

(1) *$\mathcal{R}_G$ is compact, connected, has connected boundary and $\Sigma\backslash\mathcal{R}_G$ has genus 0;*
(2) *after a rotation of $\mathbb{R}^3$, $\mathcal{R}_S$ can be written as the union of two (oppositely oriented) strictly spiraling multi-valued graphs Σ^1 and Σ^2;*
(3) *in $\mathcal{R}_A$, $|\nabla_\Sigma x_3| \geq \epsilon_0$.*

Remark 1. *We say Σ^i $(i = 1, 2)$ is a multi-valued graph if Σ^i is the graph, Γ_{u^i}, of a single function u^i with $u^i_\theta \neq 0$.*

To prove this decomposition, we first find the region of strict spiraling, $\mathcal{R}_S$. The strict spiraling controls the asymptotic behavior of level sets of x_3 which, as x_3 is harmonic on Σ, gives information about x_3 in all of Σ.

By Stokes' Theorem, x_3^* (the harmonic conjugate of x_3) exists on Γ and thus there is a well defined holomorphic map $z : \Gamma \to \mathbb{C}$ given by $z = x_3 + ix_3^*$. Using Theorem 5 and a Rado type theorem, we deduce that z is a holomorphic coordinate on Γ. We claim that z is actually a proper map and so Γ is conformally a punctured disk. This can be shown by studying the Gauss map. On Γ, the stereographic projection of the Gauss map, g, is a holomorphic map that avoids the origin. Moreover, the minimality of Σ and the strict spiraling in $\mathcal{R}_S$ imply that the winding number of g around the inner boundary of Γ is zero. Hence, by monodromy there exists a holomorphic map $f : \Gamma \to \mathbb{C}$ with $g = e^f$. Then the strict spiraling in $\mathcal{R}_S$ imposes strong control on f which is sufficient to show that z is proper. Further, once we establish Γ is conformally a punctured disk, the properties of the level sets of f imply that it extends meromorphically over the puncture with a simple pole. This gives Theorem 1.

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Ω-surfaces

Francis Burstall
(joint work with Udo Hertrich-Jeromin, Wayne Rossman)

Ω-surfaces are an incarnation of isothermic surfaces in Lie sphere geometry and, as such, inherit a rich integrable structure [1, 2, 3, 5, 12]. In particular, they possess a spectral deformation (they are precisely the Lie applicable surfaces); Ribaucour transforms [9, §90] and an integrable discretisation. In this note, I shall sketch the definitions and discuss some interesting examples.

Isothermic surfaces in quadrics. Let Q be a real, smooth quadric. Thus $Q = \mathbb{P}(\mathcal{L}) \subset \mathbb{P}(\mathbb{R}^{p+1,q+1})$ is the projective light-cone of a vector space with indefinite inner product. The quadric Q is a $(p+q)$-manifold with a conformal structure of signature (p,q) which may be viewed as the conformal compactification of $\mathbb{R}^{p,q}$. The orthogonal group $\mathrm{O}(p+1,q+1)$ acts of Q by conformal diffeomorphisms and, in this way, is a finite cover of the conformal diffeomorphism group of Q.

Now let $f : \Sigma \to Q$ be an immersion. According to Darboux [6], f is *isothermic* if it admits conformal coordinates which diagonalise the second fundamental form. More invariantly, this is equivalent to the existence of a holomorphic[1] quadratic differential q which commutes with the second fundamental form. It follows that f has flat normal bundle.

An alternative and manifestly conformally invariant formulation of this condition is due to Burstall and Pinkall [3] (see also [11, para. 5.3.19]). For this, view f as a null line-subbundle of the trivial $\mathbb{R}^{p+1,q+1}$-bundle over Σ. Then f is isothermic if and only if there is a non-zero *closed* 1-form η with values in $\mathfrak{o}(p+1,q+1)$ such that

$$\eta f = 0, \qquad \eta f^{\perp} \subset \Omega^1(f).$$

The two approaches are related by

$$q(X,Y)\sigma = \eta_X \mathrm{d}_Y \sigma,$$

for any $\sigma \in \Gamma f$, but the latter formulation continues to make sense when the metric induced by f degenerates or even when f fails to immerse.

We readily see that the pencil of connections $\mathrm{d} + t\eta$, $t \in \mathbb{R}$, are all flat and this gives an efficient route into the integrable systems theory of isothermic surfaces.

Mobius and Lie sphere geometry of S^3. As is well known, when $(p,q) = (n,0)$, Q is conformally diffeomorphic to the n-sphere [11, para. 1.1.1]. Indeed, fix $t_1 \in \mathbb{R}^{n+1,1}$ with $(t_1,t_1) = -1$ to get a conformal diffeomorphism $x \mapsto \langle x + t_1 \rangle$ between the unit sphere in $t_1^{\perp}$ and Q.

When $(p,q) = (3,1)$, Q can be (non-canonically) identified with the space of oriented 2-spheres in S^3 together with the zero-radius spheres (the points of S^3). For this, we choose a *point-sphere complex* $Q_0 \subset Q$ to represent the points of S^3. Thus this should be a hyperplane section of Q of signature $(3,0)$, or, equivalently,

[1] When the induced metric has indefinite signature, one must adjust one's notion of holomorphic!

$Q_0 = \mathbb{P}(\mathcal{L} \cap t_0^\perp)$ for a fixed unit timelike $t_0 \in \mathbb{R}^{4,2}$. Now any $q \in Q \setminus Q_0$ can be uniquely written as the span of $x + t_0$ with x unit spacelike in $t_0^\perp$ and the corresponding 2-sphere is $Q_0 \cap x^\perp$. The same sphere with the opposite orientation is represented by $\langle -x + t_0 \rangle$.

In this picture, $q, q' \in Q$ are orthogonal if and only if the corresponding spheres have oriented contact. It follows that the lines in Q correspond to oriented contact elements of Q_0 so that the space Z of lines in Q is diffeomorphic to the bundle of oriented hyperplanes in T^*Q_0—a 5-dimensional contact manifold.

We can now do surface theory in this setting: a surface $f : \Sigma \to Q_0$ has a contact lift which is a Legendre map $E : \Sigma \to Z$ and, conversely, any Legendre map E gives rise to a map $f = E \cap Q_0 : \Sigma \to Q_0$ of which it is the contact lift wherever f immerses.

A section s of the contact lift E is now a family of 2-spheres with $s(p)$ tangent to f at $p \in \Sigma$. In classical language s is a 2-sphere congruence enveloped by f.

With all this understood, we can, following Demoulin [7, 8], make the following definition:

Definition 1. *$f : \Sigma \to Q_0$ is an* Ω-surface *if it is enveloped by an isothermic sphere congruence $s : \Sigma \to Q$.*

In fact, in this situation, Demoulin proves that we have to do with *two* isothermic sphere congruences:

Theorem 1 ([7])**.** *An Ω-surface f is enveloped by two isothermic sphere congruences $s, \hat{s}$ which are harmonically separated by the curvature spheres of f.*

Examples. Demoulin identifies three large classes of Ω-surfaces:

(1) Isothermic surfaces $f : \Sigma \to Q_0 = S^3$ are Ω: take $s = f$ and then $\hat{s}$ is the central sphere congruence of f.
(2) L-isothermic surfaces $f : \Sigma \to \mathbb{R}^3$, that is, surfaces which admit curvature line coordinates that are conformal with respect to the third fundamental form, are Ω: take for s the congruence of tangent planes.
(3) Guichard surfaces, which can be characterised as the orthogonal surfaces to curved flats in the space of circles, are Ω.

The first two of these classes are characterised by the demand that the sphere congruence s lie in some hyperplane section of Q: in the first case, the hyperplane section is Q_0 and, in the second, that defined by a lightlike vector. It is therefore interesting to consider the case where both s and $\hat{s}$ lie in hyperplane sections.

For this, suppose that the hyperplane sections are defined by $q, \hat{q} \in \mathbb{R}^{4,2}$ so that $(s, q) = (\hat{s}, \hat{q}) = 0$. If we take q timelike, we can take the corresponding hyperplane section to be Q_0 and then the condition on the central sphere congruence $\hat{s}$ amounts to the demand that f have constant mean curvature in a 3-dimensional space-form.

Again, take $q \in Q$ so that s has contact with a fixed 2-sphere which we may view as the sphere at infinity of hyperbolic 3-space H^3. Now, Q_0 can be chosen so that $f : \Sigma \to H^3$ is a *linear Weingarten surface of Bryant type* [10], that is, the mean and Gauss curvatures H and K of f satisfy:

$$\alpha K + 2(\alpha - 1)(1 - H) = 0,$$

for a constant α depending on $\hat{q}$. Thus:

Theorem 2. *A surface $f : \Sigma \to H^3$ is linear Weingarten of Bryant type if and only if it is Ω with one isothermic sphere congruence in oriented contact with a fixed sphere while the other lies in a hyperplane section disjoint from that fixed sphere.*

The geometry of the situation varies with α: in all cases, by construction, s is the tangent horosphere to f while:

$\alpha > 1$: $\hat{s}$ cuts the infinity sphere at constant angle. In particular, when $\alpha = 2$, $\hat{s}$ is the congruence of totally geodesic spheres tangent to f.

$\alpha = 1$: Here $\hat{q}$ is also lightlike and represents the infinity sphere with the opposite orientation so that $\hat{s}$ is the other tangent horosphere. This is the case $K = 0$ [4].

$\alpha < 1$: Here $\hat{s}$ does not intersect the infinity sphere and cuts the normal geodesic to f for the second time at distance $\ln(1 - \alpha)$. In particular, when $\alpha = 0$, $\hat{s} = f$ is a congruence of zero-radius spheres and $H \equiv 1$.

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Constant mean curvature spheres in the Lie group $\mathbf{Sol_3}$

Benoît Daniel
(joint work with Pablo Mira)

Hopf's theorem states round spheres are the unique (immersed) constant mean curvature (CMC) spheres in Euclidean 3-space $\mathbb{R}^3$. The proof relies on the existence of a holomorphic quadratic differential, the so-called Hopf differential. This theorem can be generalized immediately to hyperbolic 3-space $\mathbb{H}^3$ and the round sphere $\mathbb{S}^3$

Recently, Hopf's theorem was extended by Abresch and Rosenberg [1, 2] to all simply connected homogeneous Riemannian 3-manifolds with a 4-dimensional isometry group, i.e., $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$, the Heisenberg group Nil_3, the universal cover of $\mathrm{PSL}_2(\mathbb{R})$ and Berger spheres: any immersed CMC sphere in any of these manifolds is a rotational CMC sphere. To do this, they proved the existence of a holomorphic quadratic differential, which is a linear combination of the usual Hopf differential and of a term coming from a certain ambient Killing field. Once here, the proof is similar to Hopf's: such a differential must vanish on a sphere, and this implies that the sphere is rotational.

Our purpose is to investigate the Hopf problem in the Lie group Sol_3 endowed with a left-invariant Riemannian metric, which is the only Thurston geometry where this problem remains open.

Several difficulties arise. The first is that Sol_3 has an isometry group only of dimension 3, and has no rotations. Hence, there are no known explicit CMC spheres, since, contrarily to the case of the abovementioned 3-manifolds, we cannot reduce the problem of finding CMC spheres to solving an ordinary differential equation. Moreover, even the existence of a CMC H sphere for a specific value $H \in \mathbb{R}$ of the mean curvature needs to be settled. Another difficulty is that the Abresch-Rosenberg differential does not exist in Sol_3.

On the other hand, in Sol_3 there exist two foliations by totally geodesic surfaces. Any of these surfaces is the invariant set of an orientation-reversing isometry of Sol_3, which permits Alexandrov reflection. This is enough to prove that a compact embedded CMC surface in Sol_3 is topologically a sphere.

The main results we prove are the following.

Theorem 1 ([4]). *For every $H > 1/\sqrt{3}$ there exists a unique (immersed) CMC H sphere in Sol_3 up to ambient translations. Moreover this sphere is embedded.*

Theorem 2 ([4]). *Let $H > 0$ such that there exists some (immersed) CMC H sphere Σ_H in Sol_3 satisfying one of the following properties, where actually $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$:*

- (a) *it is a solution to the isoperimetric problem in Sol_3,*
- (b) *it is a weakly stable surface,*
- (c) *it has index one,*
- (d) *its Gauss map is a (global) diffeomorphism into $\mathbb{S}^2$.*

Then Σ_H is embedded and it is the unique (immersed) CMC H sphere in Sol_3 *up to ambient translations.*

Let us remark than we can deduce from Alexandrov reflection and results of Pittet [6] that the infimum of the set of $H > 0$ such that there exists a CMC H sphere satisfying (a) is 0. One deduces Theorem 1 from Theorem 2 by proving that for every $H > 1/\sqrt{3}$ there exists a CMC H sphere satisfying (c).

To prove uniqueness of CMC H spheres, the main idea will be to ensure the existence of a quadratic differential that satisfies the *Cauchy-Riemann inequality* (a property weaker than holomorphicity introduced by Alencar, do Carmo and Tribuzy [3]) for all CMC H immersions. It seems very difficult and maybe impossible to obtain such a differential *explicitly*. We are able to prove the existence of this differential provided there exists a CMC H sphere whose Gauss map G is a (global) diffeomorphism (our differential will be defined using this G).

The next step is to study the existence of CMC spheres whose Gauss maps are diffeomorphisms. We first prove that the Gauss map of an isoperimetric sphere, and more generally of an index one CMC sphere, is a diffeomorphism. For this purpose we use a nodal domain argument. We also prove that a CMC sphere whose Gauss map is a diffeomorphism is embedded.

Then we deform an isoperimetric sphere with large mean curvature by the implicit function theorem. More generally we prove than we can deform index one CMC spheres, and that the property of having index one is preserved by this deformation. In this way we prove that there exists an index one CMC H sphere for every $H > 1/\sqrt{3}$. To do this we need a bound on the second fundamental form and a bound on the diameter of the spheres. This diameter estimate is the consequence of a theorem by Rosenberg [7] and relies on a stability argument; however this estimate only holds for $H > 1/\sqrt{3}$. This will complete the proof.

To conclude, let us mention that very recently Meeks [5] proved height estimates for CMC H spheres that are valid for *all* values of the mean curvature H (to do this he used the fact that the infimum of the mean curvatures of isoperimetric spheres is 0). Together with the previous arguments, this shows that Theorem 1 can be improved by replacing $H > 1/\sqrt{3}$ by $H > 0$. Hence this completes the study of the Hopf problem in Sol_3. A natural related problem remains open in Sol_3: is is true that all CMC spheres are isoperimetric?

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Maximal surfaces with singularities in the Lorentz-Minkowski 3-space

Shoichi Fujimori

Let $\mathbb{L}^3$ be the Lorentz-Minkowski 3-space with the metric $\langle\ ,\ \rangle = dx_1^2 + dx_2^2 - dx_3^2$. A spacelike immersion $X : M \to \mathbb{L}^3$ is said to be *maximal* if X has vanishing mean curvature. Although a complete maximal immersion in $\mathbb{L}^3$ is necessarily a spacelike plane [1], complete maximal surfaces with certain kinds of singularities have given rise to an interesting theory (see for instance [2, 3, 10]). Like minimal surfaces in $\mathbb{R}^3$, maximal surfaces admit the following Weierstrass-type representation: Let M be a Riemann surface and (g, φ) a pair of a meromorphic function and a holomorphic 1-form on M so that $(|g|^{-1} + |g|)^2|\varphi|^2$ gives a positive definite metric on M. Then

$$X = \mathrm{Re} \int \left(\frac{i}{2}\left(\frac{1}{g} - g\right), \frac{1}{2}\left(\frac{1}{g} + g\right), 1 \right) \varphi : M \to \mathbb{L}^3$$

defines a maximal surface with singularities [8]. g is called the *Gauss map* and the pair (g, φ) are called the *Weierstrass data* of X. The induced metric is given by $ds^2 = (|g|^{-1} - |g|)^2|\varphi|^2/4$ and the singular set is given by $\{p \in M\,;\,|g(p)| = 1\}$. If the nonsingular set $\{p \in M\,;\,|g(p)| \neq 1\}$ is dense in M, X is called a *maxface* [10].

In a joint work with Kentaro Saji, Masaaki Umehara and Kotaro Yamada [6], we have the following classification result for singularities of maxfaces:

Theorem 1 ([6]). *Generic singularities of maxfaces are cuspidal edges, swallowtails and cuspidal cross caps.*

A maxface $X : M \to \mathbb{L}^3$ is said to be *complete* if there exist a compact set C and a symmetric (0,2)-tensor T on M such that T vanishes on $M - C$ and $ds^2 + T$ is a complete Riemannian metric [10].

Proposition 1 ([10]). *Let $X : M \to \mathbb{L}^3$ be a complete maxface with the Weierstrass data (g, φ). Then there exist a compact Riemann surface $\overline{M}$ and finite number of points $p_1, \ldots, p_n \in \overline{M}$ so that M is biholomorphic to $\overline{M} - \{p_1, \ldots, p_n\}$. Moreover, the Weierstrass data g and φ extend meromorphically to $\overline{M}$.*

By definition, the genus of X is the genus of $\overline{M}$. The removed points $p_1, \ldots, p_n \in \overline{M}$ correspond to the *ends* of X.

Theorem 2 (Osserman-type inequality [10]). *Let $X : \overline{M} - \{p_1, \ldots, p_n\} \to \mathbb{L}^3$ be a complete maxface with meromorphic Gauss map g. Then*

$$2 \deg g \geq -\chi(\overline{M}) + 2n, \tag{1}$$

where $\chi(\overline{M})$ denotes the Euler characteristic of $\overline{M}$. Moreover, the equality holds if and only if X is an embedding around any end of M.

In [7] by Kim and Yang, it was shown that, although the only complete minimal surfaces in $\mathbb{R}^3$ with two embedded ends are catenoids (Schoen [9]), there does exists a complete maxface of genus 1 with two embedded ends in $\mathbb{L}^3$. We shall call this example the *Kim-Yang troidal maxface.* Until now, the only known examples of complete maxfaces with embedded ends were the spacelike plane, the Lorentzian catenoid and the Kim-Yang toroidal maxface. Also, until now the only known complete positive-genus maxfaces were the Lorentzian Chen-Gackstatter surface and the Kim-Yang toroidal maxface. In a joint work with Wayne Rossman, Masaaki Umehara, Kotaro Yamada and Seong-Deog Yang [5], we construct complete maxfaces with two ends and arbitrary genus, which are embedded if the genus is equal to 1.

Theorem 3 ([5])**.** *There exists a family of complete maxfaces X_k for $k = 1, 2, 3, \dots$ with two ends, and of genus k if k is odd and genus $k/2$ if k is even. Moreover, X_1 and X_2 have embedded ends. Furthermore, the number of swallowtails and the number of cuspidal cross caps are both equal to $4(k+1)$ if k is odd and $2(k+1)$ if k is even. In particular, X_1 and X_2 are both of genus one, but are not congruent.*

We note that X_1 is the Kim-Yang toroidal maxface, but the X_k are new examples for all $k \geq 2$.

Until now, all known examples of complete maxfaces are orientable. In a joint work with Francisco J. López [4], we investigate the geometry and topology of complete nonorientable maxfaces.

Let M' be a *nonorientable Riemann surface*, that is to say, a nonorientable surface endowed with an atlas whose transition maps are holomorphic or antiholomorphic. Let $\pi : M \to M'$ be the orientable conformal double cover of M'. A conformal map $X' : M' \to \mathbb{L}^3$ is said to be a *nonorientable maxface* if the composition $X = X' \circ \pi : M \longrightarrow \mathbb{L}^3$ is a maxface. In addition, X' is said to be *complete* if X is complete.

Let $X' : M' \to \mathbb{L}^3$ be a nonorientable maxface, and let $I : M \to M$ denote the antiholomorphic order two deck transformation associated to the orientable double cover $\pi : M \to M'$. Since $X \circ I = X$, we see that

$$g \circ I = \frac{1}{\bar{g}} \qquad \text{and} \qquad I^*(\varphi) = \bar{\varphi}. \tag{2}$$

As a consequence, I leaves invariant the singular set $\{p \in M \ ; \ |g(p)| = 1\}$. Conversely, if (g, φ) are the Weierstrass data of an orientable maxface $X : M \to \mathbb{L}^3$ and I is an antiholomorphic involution without fixed points in M satisfying (2), then the unique map $X' : M' = M/\langle I \rangle \to \mathbb{L}^3$ satisfying $X = X' \circ \pi$ is a nonorientable maxface. We call (M, I, g, φ) as the Weierstrass data of $X' : M' \to \mathbb{L}^3$. Assume that $X' : M' = M/\langle I \rangle \to \mathbb{L}^3$ is complete. Then I extends conformally to the compactification $\overline{M}$ of M and $M = \overline{M} - \{q_1, \dots, q_m, I(q_1), \dots, I(q_m)\}$, where $q_1, \dots, q_m \in \overline{M}$. Consequently, $M' = \overline{M}' - \{\pi(q_1), \dots, \pi(q_m)\}$, where $\overline{M}' = \overline{M}/\langle I \rangle$ is a compact nonorientable conformal surface of genus $2 - \chi(\overline{M}') = 2 - (1/2)\chi(\overline{M})$. By definition, the genus of X' is the genus of M'.

Let $X' : M' \to \mathbb{L}^3$ be a complete nonorientable maxface with Weierstrass data (M, I, g, φ), and label as $\pi : M \to M'$ as the orientable double cover of M'. Denote by $A : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ the complex conjugation $A(z) = 1/\bar{z}$, and consider the projection $p_0 : \mathbb{C} \cup \{\infty\} \to (\mathbb{C} \cup \{\infty\})/\langle A \rangle$. Then the unique conformal map $\hat{g} : M' \to (\mathbb{C} \cup \{\infty\})/\langle A \rangle$ satisfying $\hat{g} \circ \pi = p_0 \circ g$ is called the *Gauss map* of X'.

By Proposition 1, if X' is complete then $\hat{g}$ extends conformally to the compatification $\overline{M}'$ of M'. Moreover, $\hat{g}$ has the same degree as $g : \overline{M} \to \mathbb{C} \cup \{\infty\}$. Hence the inequality (1) becomes

$$\deg \hat{g} \geq -\chi(\overline{M}') + 2m, \tag{3}$$

where m is the number of ends of M'.

In [4], we first give the following topological congruence formulae for nonorientable maxfaces:

Theorem 4 ([4])**.** *Let $X' : M' \to \mathbb{L}^3$ be a conformal complete nonorientable maxface with Gauss map $\hat{g}$, then*

(i) $\deg \hat{g}$ *is even and greater than or equal to* 4*.*
(ii) *If in addition X' has embedded ends, then $\chi(\overline{M}')$ is even.*

We next produce the first known examples of complete nonorientable maxfaces. To be more precise, we describe the moduli space of complete maxfaces with the topology of a Möbius strip and Gauss map of degree four, and construct two complete one-ended Klein bottles, named KB_1 and KB_2, with Gauss map of degree four as well. Both KB_1 and KB_2 contain the x_1- and x_2-axes, and therefore their symmetry group contains four elements. Finally, we have the following characterization theorem:

Theorem 5 ([4])**.** *KB_1 and KB_2 are the unique complete maxfaces with the topology of a one-ended Klein bottle, Gauss map of degree four and have at least four symmetries.*

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tt*-geometry

Martin A. Guest

tt*-geometry (a generalization of special geometry) provides a new source of potentially interesting examples in differential geometry. This theory arose in the study of quantum field theories in physics, in the 1990's, in the work of Cecotti and Vafa. The connection with harmonic maps was noticed by Dubrovin.

The main observation of Cecotti and Vafa (see [1]) can be summarized as follows: physics predicts the existence of certain solutions of the harmonic map equation with surprisingly good "global" properties. These solutions correspond to natural holomorphic data, such as quantum cohomology rings or Landau-Ginzburg potentials. In retrospect, this is an example of the DPW (Generalized Weierstrass) representation. However, in the DPW representation, global properties are hard to predict, and this makes the examples of Cecotti and Vafa particularly interesting for differential geometers.

Roughly speaking, a tt*-structure on a complex manifold U of dimension r is a pluriharmonic map $f : U \to GL_N(\mathbb{R})/O_N$, for some N.

Example 1. U = special Kähler manifold. Here one obtains a holomorphic map $U \to Sp_{2r}\mathbb{R}/U_r$. It can be shown that this corresponds to a holomorphic map $F : U \to \mathbb{C}$ with the property that *the imaginary part of the matrix* $(\partial^2 F/\partial z_i \partial z_j)$ *is positive definite.* From the viewpoint of tt*-geometry, the positive definiteness condition is the only nontrivial aspect of this example. It can be interpreted as saying that a certain (Iwasawa) matrix factorization can be done.

Example 2. U = the base space of a variation of polarized Hodge structures. Here again there is a nontrivial positive definiteness condition, part of the Riemann-Hodge bilinear relations. One obtains a pluriharmonic map of "twistor type", i.e. the result of composing a superhorizontal holomorphic map with a twistor fibration.

The above pluriharmonic maps are rather elementary in comparison to the example that we shall concentrate on in this talk, namely the case of quantum cohomology (or more generally "Frobenius manifolds with real structure").

The quantum cohomology of a manifold M gives immediately a "normalized DPW potential" $\eta = \frac{1}{\lambda}\sum \eta_i dz_i$, where η_i is the matrix of quantum multiplication by the i-th basis vector of $H^*(M;\mathbb{C})$. (Here η_i depends on the z_j, the quantum parameters, which are usually called q_j.) From a choice of real form of the appropriate loop group, after performing an Iwasawa factorization in the usual way, one obtains a pluriharmonic map into $GL_N(\mathbb{R})/O_N$, where $N = \dim_{\mathbb{C}} H^*(M;\mathbb{C})$.

We shall discuss the loop groups involved here in some detail. The positive definiteness condition (which is conjectured to hold, at least for certain kinds of manifolds M) is that this Iwasawa factorization can indeed be done. It is nontrivial because the group $GL_N(\mathbb{R})$ is noncompact; the condition amounts to saying that the "extended solution" remains within a single Iwasawa cell. In the theory of harmonic maps, very few examples of this behaviour are known (other than the well known cases of Examples 1 and 2 above, where the harmonic maps are "isotropic", in particular of finite uniton number).

The first example is the quantum cohomology of $\mathbb{C}P^1$. The harmonic map equations can be identified with the Gauss-Codazzi equation for (the metrics of) spacelike CMC surfaces in $\mathbb{R}^{2,1}$, which in turn can be identified with the elliptic sinh-Gordon equation $\Delta w = \sinh w$. The quantum cohomology of $\mathbb{C}P^1$ provides a solution to these equations, and thus corresponds to a surface (a surface whose properties have yet to be fully investigated, incidentally). The solution in this case is radially invariant, a consequence of the grading property of quantum multiplication. Thus we have a solution of the radially invariant sinh Gordon equation, and it is well known that this is a case of the third Painlevé equation, an o.d.e. whose solutions have been exhaustively studied. Cecotti and Vafa pointed out that the quantum cohomology of $\mathbb{C}P^1$ gives a very special solution, namely the "boundary case" of a 1-parameter family of solutions discovered by McCoy, Tracy, and Wu, all of which are smooth on the interval $(0, \infty)$. (In contrast "most" solutions have infinitely many singularities in this interval.) This is a manifestation of the positive definiteness property referred to earlier. A surface-theoretic treatment of this example was given in [2].

We shall mention some other examples where the conjecture can be verified (joint work with Chang-Shou Lin). However, in view of the disproportionate effort required so far, the problem remains a challenge to differential geometers.

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Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$

Laurent Hauswirth

Examples of minimal surfaces in product spaces $\mathbb{M}^2 \times \mathbb{R}$ ($\mathbb{M}^2$ a riemannian surface) are given by surfaces invariant by one parameter groups of isometries (see the paper of Sa Earp and E. Toubiana [7] and [8]). This way constitute first examples of a theory.

In $\mathbb{H}^2 \times \mathbb{R}$ ($\mathbb{H}^2$ the hyperbolic plane), there is a family of vertical catenoid, rotationally invariant around a vertical geodesic. A one parameter family of helicoid are given by a screw motion invariant surface foliated by horizontal geodesic.

This examples has hyperbolic conformal type. To construct parabolic examples, we need to use different techniques.

In $\mathbb{R}^3$, graphs $X = (x, y, h(x, y))$ are an important class of examples. They satisfy the quasilinear equation

$$(1 + h_y^2)h_{xx} - 2h_x h_y h_{xy} + (1 + h_x^2)h_{yy} = 0.$$

In $\mathbb{H}^2 \times \mathbb{R}$, the equation is modified by the metric of the model used for the hyperbolic plane $\mathbb{H}^2 = (D, \sigma^2(u)|du|^2)$. We have the minimal surface equation:

$$\operatorname{div}_\sigma \left(\frac{\nabla_\sigma h}{\sqrt{1 + |\nabla_\sigma h|_\sigma^2}} \right) = 0 = (1 + \sigma^{-2}h_y^2)h_{xx} - 2\sigma^{-2}h_x h_y + (1 + \sigma^{-2}h_x^2)h_{yy}.$$

To solve the Dirichlet problem on a domain $P \subset \mathbb{H}^2$ of this equation, the geometry of P has to satisfy Jenkin-Serrin's conditions. We describe below how to understand this conditions when the domain P is ideal in $\mathbb{H}^2$.

Let Γ be an ideal polygon of $\mathbb{H}^2$ which bound P a polygonal domain. All the vertices of Γ are at infinity of $\mathbb{H}^2$ and Γ has an even number of sides A_1, B_1, A_2, B_2,... ,A_k, B_k, ordered by traversing Γ clockwise. At each vertices a_i, we consider a horocycle H_i with $H_i \cap H_j = \emptyset$, which bound a horodisk F_i. Each A_i(resp B_i) meets exactly two horocycles. Denote by $\tilde{A}_i$ (resp $\tilde{B}_i$), the compact arc of A_i(resp B_i) which is the part of A_i outside the two horodisks. We define by $|A_i|$ the distance between the two horodisks i.e. the length of $|\tilde{A}_i|$. Define $\tilde{B}_i$ and $|B_i|$ in the same way.

Now we can consider $a(\Gamma) = \sum_{i=1}^k |A_i|$ and $b(\Gamma) = \sum_{i=1}^k |B_i|$. We observe that $a(\Gamma) - b(\Gamma)$ does not depend on the choice of the horocycle H_i at a_i. Horocycle with same points at infinity are equidistant. Keeping in mind this data, we can state the following theorem analogous to a result of Jenkin and Serrin in $\mathbb{R}^3$ and prove in $\mathbb{H}^2 \times \mathbb{R}$:

Theorem 1. *(Collin, Rosenberg* [1]*, Nelli, Rosenberg* [5]*) There is a solution to the minimal surface equation in the polygonal domain P, equal to $+\infty$ on A_i and $-\infty$ on B_j if and only if the following condition are satisfied*

1- $a(\Gamma) = b(\Gamma)$,

2- For each inscribed polygon P in Γ, $P \neq \Gamma$, and for some choice of horocycles at the vertices one has

$$2a(P) < |P| \text{ and } 2b(P) < |P|.$$

One can use now the Gauss-Bonnet theorem to prove that graphs on ideal polygonal domain are of finite total curvature when the number of edges are finite. We say that this graphs are of Scherk type. Of course, this existence theorem is a generalization of a more general Dirichlet problem with finite data at the boundary. In this case we ask to the domain P to be convex along the side where the boundary data is finite in view to admit solutions.

To study the theory of finite total curvature surfaces, it is useful to parametrize the surface in conformal parameter. We consider $X = (F, h) : \Sigma \to \mathbb{H} \times \mathbb{R}$ a confomal immersion where $F : \Sigma \to \mathbb{H}^2$ is the vertical projection on $\mathbb{H}^2 = \mathbb{H}^2 \times (0)$, and $h : \Sigma \longrightarrow \mathbb{R}$ the horizontal projection on R. Then F is a harmonic map and h is a real harmonic function. The harmonic map equation in the complex coordinate $u = u_1 + iu_2$ of D is

$$F_{z\bar{z}} + 2(\log \sigma \circ F)_u F_z F_{\bar{z}} = 0$$

where $2(\log \sigma \circ F)_u = 2\bar{F}(1 - |F|^2)^{-1}$. In the theory of *harmonic maps* the holomorphic *quadratic Hopf differential* associated to F is

$$Q(F) = (\sigma \circ F)^2 F_z \bar{F}_z (dz)^2 := \varphi(z)(dz)^2$$

The function φ depends on z, whereas $Q(F)$ does not. Since we consider conformal immersions, we have $(h_z)^2(dz)^2 = -Q(F)$. Then the zeroes of Q are double and we can define η as the holomorphic one form $\eta = \pm 2i\sqrt{Q}$. The sign is chosen so that:

$$h = \operatorname{Re} \int \eta$$

Then we define the function ω on Σ by $n_3 = \tanh \omega$ (n_3 is the third coordinate of the normal). By identification we have

$$\omega = \frac{1}{2} \ln \frac{|F_z|}{|F_{\bar{z}}|}$$

The induced metric is $ds^2 = 4\cosh^2 \omega |Q|$. We remark that the zeroes of Q correspond to the poles of ω so that the immersion is well defined. Moreover the zeroes of Q are points of Σ, where the tangent plane is horizontal.

It is a well known fact that harmonic mappings satisfy the Böchner formula:

$$\triangle_0 \omega = 2\sinh(2\omega)|Q|$$

where $\triangle_0$ denote the laplacian in the euclidean metric $|dz|^2$. This equation is the *sinh*-Gordon equation. To construct minimal surfaces it suffices to obtain the harmonic map F.

T.Y. Wan derive from Q, an harmonic map F. First he solve the *sinh*-Gordon equation and obtain a solution ω. This equation is the Gauss equation of a $H = 1/2$ surface in the Minkovski space $M^{2,1}$. By Gauss-Codazzi equation he obtain F as the Gauss map of a constant mean curvature surface $H = 1/2$ in $M^{2,1}$. Then there is an important dictionnary between the study of harmonic maps in $\mathbb{H}^2$ and minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. In particular P. Collin and H. Rosenberg use such Jenkin-Serrin's type theorem with Dirichlet finite data and isometries to construct an entire minimal graph on $\mathbb{H}^2$ which has a conformal type $\mathbb{C}$. Then after considering the horizontal part of the immersion they obtain an example discarting a conjecture of R. Schoen:

Theorem 2. *(Collin, Rosenberg [1]) There exists harmonic diffeomorphism from $\mathbb{C}$ to $\mathbb{H}^2$.*

We can study the theory of finite total curvature immersion in the following theorem:

Theorem 3. *(Hauswirth, Rosenberg* [4]*) Let X be a complete minimal immersion of Σ in $\mathbb{H} \times \mathbb{R}$ with finite total curvature. Then*

a) Σ is conformally $\overline{M} - \{p_1 ...., p_n\}$, a Riemann surface punctured in a finite number of points.

b) Q is holomorphic on M and extends meromorphically to each puncture. If we parameterize each puncture p_i by the exterior of a disk of radius R_0, and if $Q(z) = z^{2m_i}(dz)^2$ at p_i then $m_i \geq -1$.

c) The third coordinate of the unit normal vector $n_3 \to 0$ uniformly at each puncture.

d)The total curvature is a multiple of 2π:

$$\int K dA = 2\pi \left(2 - 2g - 2k - \sum_{i=1}^{n} m_i \right).$$

e) Every finite total curvature end is asymptotic to a Scherk's type graph on a polygonal domain with finite number of edges when $m \geq 1$. The end is asymptotique to a vertical flat plane when $m = 0$

Very recently J. Pyo [6] has construct n-noids with finite total curvature. Each ends are asymptotic to vertical flat ends. He solve a Jenkin-Serrin's type problem on a domain of $\mathbb{H}^2$. He consider an hyperbolic quad with two opposite vertices in $\mathbb{H}^2$ and two vertices at the infinity of $\mathbb{H}^2$. Then he obtain a graph bounded by two vertical geodesic line. He use a theorem of existence of conjugate minimal surface

Theorem 4. *(Hauswirth, Sa Earp, Toubiana* [3]*, Daniel* [2]*) Let $\Omega \subset \mathbb{C}$ be a simply connected open set and consider two conformal minimal immersions $X, X^* : \Omega \to \mathbb{H}^2 \times \mathbb{R}$ which are isometric each other. Assume $Q = Q^*$. Then X and X^* differ from an isometry of $\mathbb{H}^2 \times \mathbb{R}$. In summary $\{\omega, Q\}$ define uniquely (up to an isometry) the minimal surface. If $Q^* = e^{2i\theta} Q$ then X and X^* are θ-associate.*

and a Krust's type theorem:

Theorem 5. *(Hauswirth, Sa Earp, Toubiana* [3]*) If we consider a minimal graph $X(\Omega)$ on a convex domain $F(\Omega)$ in $\mathbb{H}^2$, then the associate surface $X^\theta(\Omega)$ is a graph on a domain Ω' of $\mathbb{H}^2$.*

Finally J. Pyo using this results, produce a minimal embedded annulus asymptotic to two vertical flat ends. This annulus has parabolic conformal type and finite total curvature. He generalize his construction to construct n-noids.

In conclusion we can observe that the theory of minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ will split into two different properties following their conformal type:

-A hyperbolic theory with examples construct by one parameter invariant isometries (but not always). This theory will admit a huge number of examples and the rigidity at infinity will be depends on the closure of the surface in the closure of $\mathbb{H}^2 \times \mathbb{R}$. The asymptotic behavior will depends in Fatou's type theorem on harmonic maps.

-A parabolic theory, where objects will satisfy some rigidity properties at infinity. The behavior will be very close to the theory of properly embedded minimal surfaces in $\mathbb{R}^3$.

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Evening Demo of the Visualization Program 3D-XplorMath

Hermann Karcher

(joint work with R.S.Palais, U.Pinkall)

In the demo we emphasized deformation sequences that explain mathematical properties of the objects shown. Three-dimensional objects are shown in anaglyph (red/green) stereo.

Polyhedra: Deformation of the octahedron to the icosahedron inside a cube. Deformation of a Platonic solid to its snub-polyhedron.

Planar Curves: For curves with a mechanical generation the moving plane with its momentary center of rotation is shown while the curve is drawn. The segment from the center of rotation to the current curve point is orthogonal to the curve tangent.

Spherical Curves: Spherical ellipses and spherical rolling curves allow the "same" constructions (including tangents and evolutes) as their simpler planar versions.

Holomorphic Functions: These are visualized by showing a domain grid and its image grid. Derivatives are shown as linear approximations which move with the cursor point. Elliptic functions on rhombic tori are shown on the Riemann sphere. The image shows the torus in two ways: (a) The parameter quadrilaterals are scaled down fundamental domains and (b) the cross ratio of the four branch points on the sphere is a coordinate on the moduli space of tori.

Surfaces: Inverse images of closed spherical curves under the Hopf projection are flat tori in $\mathbb{S}^3$. They can be parametrized so that parameter quadrilaterals are scaled fundamental domains. Their stereographic projections into $\mathbb{R}^3$ are shown. A conformal 180° rotation around a Hopf fibre is an anti-involution of the torus with a connected fixed point set, a property that is characteristic for rhombic tori. The family of minimal surfaces that for the first time showed "minimal surfaces connected by a handle" was shown.

Fractals: A family of continuous curves with increasing Hausdorff dimensions joining a segment to the Hilbert square filling curve was shown. To the Feigenbaum Tree we added a demo that illustrates, for each parameter value, the invariant measure of the iteration map.

Space Curves: For torus knots and for closed curves of constant curvature (with non-constant torsion) we showed: Osculating circles and evolute, the principal curvature vector function in the normal plane (planes identified by parallel translation along the curve). To turn this into a Frenet integration demo we first integated a given angular velocity vector function $\vec{\omega}(t)$ to a rotational motion ($\vec{x}\,'(t) = \vec{\omega}(t) \times \vec{x}(t)$). Next, $\vec{\omega}(t)$ is given in the moving frame. Finally, the rotational motion is obtained by the Frenet equation from the principal curvature vector function and the first vector of the moving frame is integrated to the curve (which is thus determined by the given principal curvature vector function).

Rotating Solid Body: The Euler equations determine a spherical curve (in fact, a spherical ellipse), the so called Euler Polhode. If one takes the Polhode as the given angular velocity vector function $\vec{\omega}(t)$ in the moving frame then the integrated rotational motion is the motion of a solid body (in its center of mass system). In the laboratory system the angular velocity $\vec{\omega}(t)$ is a curve in a plane orthogonal to the constant angular momentum vector, this curve is called Herpolhode, its plane is called the invariant plane.

Point Clouds: R.S. Palais explains the use of point clouds in applications, for example laser scans of objects determine point clouds which approximate the surface of the object. Such point clouds are used in 3D-XplorMath to visualize implicit surfaces. They can also serve to emphasize (and explain) contours.

jReality: U. Pinkall illustrates how mathematical objects can be fed into jReality to show them in some scene. The scene may include gravity. The observer can move around in the scene and can manipulate the objects. In this case the objects were fed from 3D-XplorMath into jReality.

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home: `http://3D-XplorMath.org`

museum: `http://VirtualMathMuseum.org`

jReality: `http://www.math.tu-berlin.de/geometrie/lab/misc.shtml\#vmm`

Integrable elliptic systems in homogeneous geometries

Idrisse Khemar

1. Introduction

In this talk, we study all the elliptic integrable systems, in the sense of C.L. Terng [12]. That is to say the family of all the m-th elliptic integrable systems associated to a k-symmetric space $N = G/G_0$. Here $m \in \mathbb{N}$ and $k \in \mathbb{N}^*$ are integers. For example, it is known that the first elliptic integrable system associated to a symmetric space (resp. to a Lie group) is the equation for harmonic maps into this symmetric space (resp. this Lie group). Indeed it is well known that this harmonic maps equation can be written as a zero curvature equation:

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0, \quad \forall \lambda \in \mathbb{C}^*,$$

where $\alpha_\lambda = \lambda^{-1}\alpha_1' + \alpha_0 + \lambda\alpha_1''$ is a 1-form on a Riemann surface L taking values in the Lie algebra $\mathfrak{g}$. This 1-form α_λ is obtained as follows. Let $f\colon L \to N = G/G_0$ be a map from the Riemann surface L into the symmetric space G/G_0. Then let $F\colon L \to G$ be a lift of f, and consider $\alpha = F^{-1}.dF$ its Maurer-Cartan form. Then decompose α according to the symmetric decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of $\mathfrak{g}$: $\alpha = \alpha_0 + \alpha_1$. Finally, we define

$$\alpha_\lambda := \lambda^{-1}\alpha_1' + \alpha_0 + \lambda\alpha_1'', \quad \forall \lambda \in \mathbb{C}^*.$$

Then the zero curvature equation for this α_λ, for all $\lambda \in \mathbb{C}^*$, is equivalent to the harmonic maps equation for $f\colon L \to N = G/G_0$, and is by definition the first elliptic integrable system associated to the symmetric space G/G_0. Thus the methods of integrable system theory apply to give generalised Weierstrass representations, algebro-geometric solutions, spectral deformations and so on. In particular, we can apply the DPW method [4] to obtain a generalised Weierstrass representation. More precisely, we have a Maurer-Cartan equation in some loop Lie algebra

$$\Lambda\mathfrak{g}_\tau = \{\xi\colon S^1 \to \mathfrak{g} | \xi(-\lambda) = \tau(\xi(\lambda))\},$$

then we can integrate it in the corresponding loop group and finally apply some factorizations theorems in loop groups to obtain a generalised Weierstrass representation: this is the DPW method. Moreover, these methods of integrable system theory hold for all the systems written in the forms of a zero curvature equation for some

$$\alpha_\lambda = \lambda^{-m}\hat{\alpha}_{-m} + \cdots + \hat{\alpha}_0 + \cdots + \lambda^m\hat{\alpha}_m.$$

Namely, these method apply to construct the solutions of all the m-th elliptic integrable systems. So it is natural to ask what is the geometric interpretation of these systems. Do they correspond to some generalisations of harmonic maps? This is the problem that we solve in the work [9] presented in this talk: to describe the geometry behind this family of integrable systems whose we know how to construct (at least locally) all the solutions.

2. The m-th elliptic integrable system associated to a k-symmetric space.

Let $\mathfrak{g}$ be a real Lie algebra and $\tau\colon \mathfrak{g} \to \mathfrak{g}$ be an automorphism of order k, $\tau^k = \mathrm{Id}$. Then we have the following eigenspace decomposition:

$$\mathfrak{g}^{\mathbb{C}} = \bigoplus_{j \in \mathbb{Z}/k\mathbb{Z}} \mathfrak{g}_j^{\mathbb{C}}, \qquad [\mathfrak{g}_j^{\mathbb{C}}, \mathfrak{g}_l^{\mathbb{C}}] \subset \mathfrak{g}_{j+l}^{\mathbb{C}}$$

where $\mathfrak{g}_j^{\mathbb{C}}$ is the $e^{2ij\pi/k}$-eigenspace of τ.

Definition 1 (C.-L. Terng)**.** *Let L be a Riemann surface. The m-th $(\mathfrak{g}, \tau)$-system on L is the equation for $(u_0, \ldots, u_m)$, $(1,0)$-type 1-form on L with values in $\prod_{j=0}^m \mathfrak{g}_{-j}^{\mathbb{C}}$:*

$$\alpha_\lambda = \sum_{j=0}^{m} \lambda^{-j} u_j + \lambda^j \bar{u}_j = \sum_{j=-m}^{m} \lambda^j \hat{\alpha}_j \tag{1}$$

satisfies the zero curvature equation:

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0, \quad \forall \lambda \in \mathbb{C}^*. \tag{2}$$

Let us set $m_k = [(k+1)/2]$. Then the general problem splits into three cases: the primitive case ($1 \leq m < m_k$), the determined case ($m_k \leq m \leq k-1$) and the underdetermined case ($m > k-1$). The lowest determined ($m = m_k$) is called minimal determined and the highest one ($m = k-1$) is called maximal determined.

3. Geometric interpretation.

• The primitive system has an interpretation in terms of J-holomorphic curves, $f\colon L \to G/G_0$, with respect to a canonical almost complex structure J, in the odd case (i.e. the order k of the target space G/G_0 is odd). Moreover it has an interpretation in terms of F-holomorphic maps with respect to a canonical F-structure (i.e $F^3 + F = 0$) in the even case (i.e. the order k is even).

• In the minimal determined case, we have an interpretation in terms of horizontally holomorphic and vertically harmonic maps $f\colon L \to G/G_0$.

• In the maximal determined case, we prove that we have an interpretation in terms of *stringy harmonic maps*. In a good geometric context these maps are exactly the solutions of the Euler Lagrange equation of a sigma model with a Wess-Zumino term.

A map $f\colon L \to (N, J, \nabla)$ from a Riemann surface into an almost complex manifold (N, J) endowed with a linear connection ∇ is stringy harmonic if

$$-\tau_g(f) + (J \cdot T)_g(f) = 0.$$

Here $\tau_g(f)$ is the tension field of f w.r.t. ∇, g is an Hermitian metric on L, T is the torsion of ∇ and $J \cdot T = -JT(J\cdot, J\cdot)$.

We also define, in the same way, stringy harmonic maps into a f-manifold (N, F). In this geometric interpretation, we use linear metric connections with totally

skew-symmetric torsion, preserving some geometric structures. For example, in the odd case (the order k of symmetries of the target is odd), the underlying geometry is the one of almost hermitian $\mathcal{G}_1$-manifolds (in the sense of the Gray-Hervella classification of almost hermitian manifolds).
In the even case, we construct a new class of metric f-manifols (which generalizes the class of almost contact metric manifolds obtained in [5]). We also obtained a supersymmetric interpretation of stringy harmonicity: F-stringy harmonicity can be viewed as a supersymmetric extention of the J-stringy harmonicity [6, 10].

• Finally, we show that any underdetermined system is equivalent to a determined system associated to new $\tilde{k}$-symmetric space of the form $G^p/G^{\tilde{\tau}}$, where $\tilde{\tau}\colon G^p \to G^p$ is an automorphism obtained by composing the initial automorphism τ with some permutation in the product G^p.

• Moreover we also prove that we have a geometric interpretation in terms of twistors.

Let us quote some related subjects and works:
− Linear metric connections with totally skew-symmetric torsion recently became a subject of interest in theoretical and mathematical physics.
− The target space of supersymmetric sigma models with Wess-Zumino term carries a geometry of a metric connection with skew-symmetric torsion [11].
− Works of T. Friedrich, S. Ivanov, I. Agricola. [1, 2, 5, 3]

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Flows of equivariant constant mean curvature tori in the 3-sphere

Martin Kilian
(joint work with Martin Ulrich Schmidt, Nicholas Schmitt)

A compact minimal surface in the 3-sphere $\mathbb{S}^3$ is rigid, and cannot be deformed smoothly without breaking the minimality or the topology, but is it possible to smoothly deform it through CMC surfaces while preserving its topology? We affirm this in the case of CMC tori, and prove that any (generic) CMC torus in $\mathbb{S}^3$ can be deformed smoothly through CMC tori. To obtain an insight into the structure of the moduli space we explore equivariant CMC tori in $\mathbb{S}^3$. These are the tori whose harmonic Gauss map [9] is $\mathbb{R}$-equivariant [2, 10]. An equivariant CMC torus is thus either flat or some member of an associated family of a Delaunay surface [3], and it turns out that the moduli of equivariant CMC tori in $\mathbb{S}^3$ is an infinite connected graph whose edges are parameterized by the mean curvature.

Harmonic maps come in families [12, 8, 11] and the holomorphic dependence on this additional (spectral) parameter make it possible to obtain some deep global results. Amongst such harmonic maps there is a dense subset consisting of harmonic maps of finite type [1, 5, 7]. To such a harmonic map of finite type there corresponds an associated algebraic curve, called the spectral curve. The crucial fact that makes it possible to adapt the Whitham deformation technique [4] is that all CMC tori in $\mathbb{S}^3$ are of finite type [5, 7]. The spectral curve of a CMC torus is a double cover of the Riemann sphere with $2g + 2$ many branch points. The non-negative integer g is called the spectral genus of the curve. Every spectral curve of a CMC torus has two branch points which lie at specific points on the Riemann sphere. These two branch points remain fixed during the deformation, while the other $2g$ branch points may move around. The closing conditions involve a choice of two double points on the real part of the spectral curve: we call these the sym points. The mean curvature is the cotangent of the angle between the two sym points. Now consider the Clifford torus, which has the simplest possible spectral curve since its spectral genus is $g = 0$. During the deformation of the Clifford torus only the sym points move on $\mathbb{S}^1$, and it turns out that the angle between the sym points is strictly monotonic. Thus the Clifford torus lies in a smooth $\mathbb{R}$-family of flat embedded CMC tori parameterized by the mean curvature.

In the deformation family of the Clifford torus there is a $\mathbb{Z}$-family of embedded flat tori which allow a bifurcation into cohomogeneity one rotational embedded CMC tori. Such a bifurcation is possible when in addition to the two sym points there is a further double point on the real part of the spectral curve. By opening up this additional double point and moving the resulting two branch points off the real part, the spectral curve becomes a double cover of the Riemann sphere branched now at four points: It has spectral genus $g = 1$ and is the spectral curve of a Delaunay surface. The corresponding CMC torus is a truncation of a Delaunay surface in $\mathbb{S}^3$. We show that at the end of the flow the new branch points pair-wise coalesce with the two fixed branch points. Hence in the limit the coalescing pairs of branch points disappear, and the limit curve is an unbranched double cover of

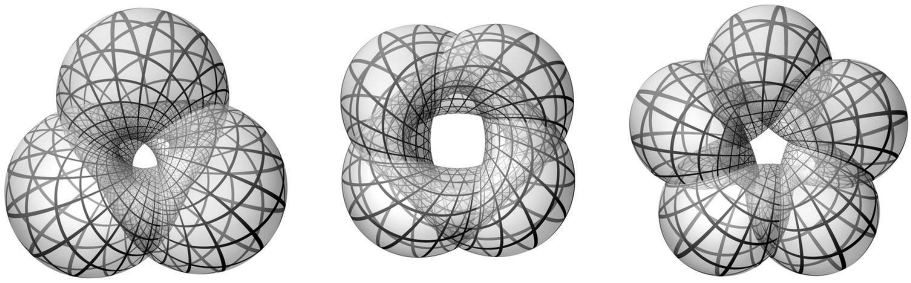

FIGURE 1. The simplest types of equivariant non-rectangular CMC tori.

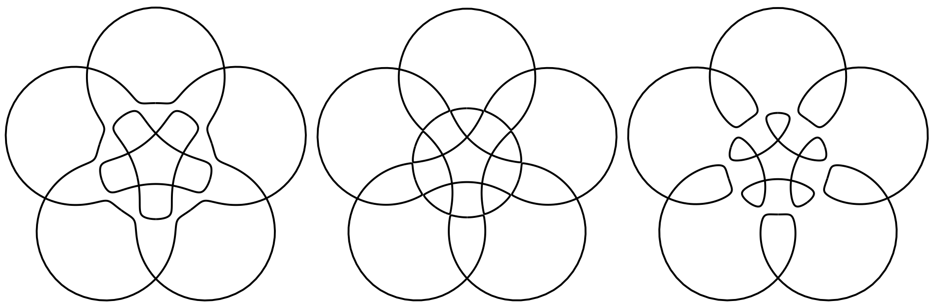

FIGURE 2. Profile curves of the five lobed non-rectangular CMC torus family as the torus flows through its axis. The turning number of the inner profile curve jumps from 2 to 4. Figure 1 shows a 5-lobed torus in the family of which these are cross-sections.

the sphere: the spectral curve of a bouquet of spheres. In the rotational case our deformation corresponds to pinching the neck of a Delaunay surface, starting at a flat torus and continuing through to a bouquet of spheres. Thus the connected component of the Clifford torus is an infinite comb: The spine ($g = 0$) consists of embedded flat CMC tori parameterized by the mean curvature, and each tooth ($g = 1$) of embedded Delaunay tori ends in a bouquet of spheres. By considering covers of the Clifford torus the moduli of rotational tori is a $\mathbb{Z}^2$–family of combs. It turns out that each bouquet of spheres occurs exactly twice in this moduli, so that we may glue the two families together there. Hence the moduli of rotational CMC tori in $\mathbb{S}^3$ is an infinite graph, whose edges are parameterized by the mean curvature.

A similar picture emerges in the non-rotational case. In each isogeny class there is a sequence of $g = 0$ tori that can be deformed into $g = 1$ tori. In the non-rotational case a $g = 1$ deformation family stays away from bouquets of spheres, and we prove that every $g = 1$ deformation family begins and ends at a $g = 0$ torus. The above results combined give that every deformation family of cohomogeneity one CMC tori ends at a cohomogeneity two CMC torus. The classification of cohomogeneity one CMC tori is thus reduced to that of spectral curves of flat tori with a double point on the real part; this initial data is classified and interpreted geometrically. We classify the equivariant minimal tori, as well as the embedded and Alexandrov embedded equivariant CMC tori, and give an independent proof that the Clifford torus is the only minimal embedded equivariant torus in the 3-sphere [6]. We also show that the spectral curve of an equivariant CMC torus has no double points off the real part. This implies that there can not be a Bianchi-Bäcklund transform of an equivariant CMC torus into a CMC torus. Our results carry over to analogous statements about constrained equivariant Willmore tori in $\mathbb{R}^3$ of spectral genus $g \leq 1$.

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Constant mean curvature surfaces in $\mathbb{H}^3$ by integrable system methods

Shimpei Kobayashi

(joint work with Josef F. Dorfmeister, Jun-ichi Inoguchi)

Constant mean curvature surfaces in $\mathbb{R}^3$ have been studied intensively last forty years. The harmonicity of *Gauß maps*, which are the unit normals to surfaces, is an important ingredient for theory. In fact, for the minimal (resp. non minimal constant mean curvature) surfaces case, the Gauß maps are conformal harmonic (resp. non-conformal harmonic) maps into the unit sphere S^2, which is obviously a symmetric space. Moreover, for harmonic maps from surfaces into a symmetric space, it is known that the integrable system methods can be applied efficiently, and thus we have the generalized Weierstraß representation for non minimal constant mean curvature surfaces [2].

On the one hand, for surfaces in $\mathbb{H}^3$, several natural Gauß maps can be defined, see the following diagram. In this talk, we use the *Gauß maps* as the maps from surfaces into the unit tangent sphere bundle of $\mathbb{H}^3$, $U\mathbb{H}^3$. Such maps are defined by the pair of an immersion and its unit normal. Then a surface in $\mathbb{H}^3$ has constant mean curvature if and only if the Gauß map is harmonic [3]. It is known that $U\mathbb{H}^3$ is *not* a symmetric space, but a 4-symmetric space. Then for harmonic maps into a 4-symmetric space, the integrable system methods cannot be applied efficiently, and thus we do not have the generalized Weierstraß representation in general. On the contrary, the primitive maps, which is the special kind of harmonic map, see Definition 1, into 4-symmetric space can have the generalized Weierstraß representation.

In this talk, using the Lawson correspondence between surfaces in space forms, we discuss how the generalized Weierstraß representation for constant mean curvature surfaces in $\mathbb{H}^3$ with $|H| \neq 1$ can be established. We also discuss analytic extensions of constant mean curvature surfaces in $\mathbb{H}^3$ with mean curvature $|H| < 1$ crossing the ideal boundary of $\mathbb{H}^3$. This talk is based on the paper [1].

We first consider the *Lawson correspondence.*

Proposition 1 (Lawson correspondence)**.** *Let $f : \mathbb{D} \to M^3(c)$ be a simply connected surface of constant mean curvature H. Take a pair $(\tilde{c}, \tilde{H})$ of real numbers such that $H^2+c = \tilde{H}^2+\tilde{c}$. Then there exists a conformal immersion $\tilde{f} : \mathbb{D} \to M^3(\tilde{c})$ with constant mean curvature $\tilde{H}$ whose induced metric is the original metric of $(\mathbb{D}, f)$.*

Remark 1. *In particular, there exist the correspondences between*

- *Constant mean curvature surfaces with H in $\mathbb{R}^3$.*
- *Constant mean curvature surfaces with $\pm\sqrt{H^2+1}$ in $\mathbb{H}^3(-1)$.*

and

- *Constant mean curvature surfaces with $|H| < 1$ in $\mathbb{H}^3(-1)$.*
- *Minimal surfaces in $\mathbb{H}^3(\tilde{c})$, $-1 \leqq \tilde{c} = -1 + H^2 < 0$.*

Let M be a Riemann surface and $f : M \to \mathbb{H}^3$ a conformal constant mean curvature surface. Moreover, let n be the unit normal to f. Then $F = (f, n)$ is the map into $U\mathbb{H}^3$, which is called the *Gauß map.* $U\mathbb{H}^3$ can be represented as $SL_2\mathbb{C}/U(1)$, which is a 4-symmetric space. Since $\langle df, n\rangle = 0$, F satisfies *Legendre* property.

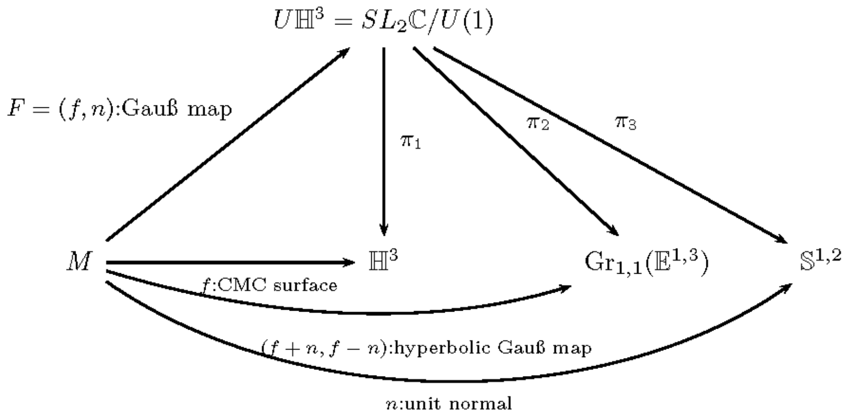
Theorem 1 (T. Ishihara, [3])**.** *A conformal immersion $f : M \to \mathbb{H}^3$ has constant mean curvature if and only if its Gauß map is harmonic with respect to the Killing metric of $SL_2\mathbb{C}$.*

Definition 1. *A map F into $k(>2)$-symmetric space G/K is called primitive if $dF(\frac{\partial}{\partial z}) \subset \mathfrak{g}_1$, where $\mathfrak{g} = \mathrm{Lie}\, G = \sum_{k \in \mathbb{Z}_k} \mathfrak{g}_k$ is the eigen space decomposition of $\mathfrak{g}$.*

Remark 2. *The primitive map is a harmonic map.*

Then minimal surfaces in $\mathbb{H}^3(c)$ can be characterized as follows:

Theorem 2. *A conformal immersion $f : M \to \mathbb{H}^3(c)$, $-1 \leqq c < 0$ is minimal if and only if its Gauß map is primitive map with respect to to the Killing metric of $SL_2\mathbb{C}$.*



$$\begin{aligned} U\mathbb{H}^3 &= SL_2\mathbb{C}/U(1) : \text{4-symmetric space} \\ \mathbb{H}^3 &= SL_2\mathbb{C}/SU(2) : \text{Symmetric space,} \\ \mathrm{Gr}_{1,1}(\mathbb{E}^{1,3}) &= SL_2\mathbb{C}/\mathrm{diag} : \text{Symmetric space,} \\ \mathbb{S}^{1,2} &= SL_2\mathbb{C}/SU(1,1) : \text{Symmetric space.} \end{aligned}$$

From Proposition 1, Theorem 1 and Theorem 2, the Gauß maps for constant mean curvature surfaces in $\mathbb{H}^3$ with $|H| \neq 1$ can be interpreted as

(1) Harmonic maps into S^2 for $|H| > 1$.
(2) Primitive maps into $U\mathbb{H}^3$ for $|H| < 1$.

Using these interpretations, we obtain the generalized Weierstraß representation. The detailed discussion is given in [1].

It is known that constant mean curvature surfaces in $\mathbb{H}^3$ with $|H| < 1$ cannot be compact. The generalized Weierstraß representation naturally gives the analytic extension of constant mean curvature surfaces in $\mathbb{H}^3$ with $|H| < 1$ acrossing the ideal boundary $\partial\mathbb{H}^3$ of $\mathbb{H}^3$, which is induced by the crossing the *small* cells for the Iwasawa decomposition of the loop group, see Figure 1 and [1].

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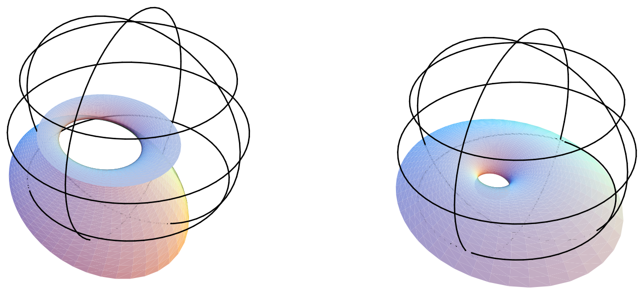

FIGURE 1. A portion of a surface of revolution with $H = \tanh(0.3)$ and and a portion of a minimal surface of revolution (right). Surfaces are shown in the Poincaré ball model and the outside of the Poincaré ball model.

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Stability and bifurcation for surfaces with constant mean curvature and their generalizations

MIYUKI KOISO

1. INTRODUCTION

In the study of a variational problem, it is natural to ask whether each critical point is stable (that is, the second variation of the considered energy functional is nonnegative) or not. It is important also to determine the geometric properties of stable solutions. For example, how about the curvatures of the solutions? Do they have the same symmetry as the original variational problem? It is also important to study the structure of the set of solutions. In this talk, as one of the steps to investigate these problems, we study stability and bifurcation for solutions of variational problems for (hyper)surfaces with constraint.

First let us give a simple but interesting example. Let C be the union of two coaxial circles C_1, C_2 of the same radius in $\mathbf{R}^3$. Consider critical points of area with volume constraint among surfaces bounded by C. For certain volumes V, Delaunay surfaces (CMC surfaces of revolution) with volume V bounded by C are stable, while, for large V, Delaunay surfaces with volume V bounded by C are unstable. This suggests the possibility of the existence of a bifurcation of solutions from a one-parameter family of Delaunay surfaces, and the existence of stable solutions with less symmetry. This is interesting in contrast to the fact that only spherical caps and disks are stable disk-type CMC surfaces bounded by a circle (Alías-Lopez-Palmer, 1999).

In this talk we consider mainly CMC surfaces whose boundary values are prescribed. We give sufficient conditions on a one-parameter family of solutions so that there exists a bifurcation of solutions. Moreover, we give a criterion for solutions in this bifurcation branch to be stable.

These results are applied to various variational problems.

2. Main results and idea of proofs

Let $X : \Sigma = \Sigma^2 \to \mathbf{R}^3$ be a $C^{3+\alpha}$ $(0 < \alpha < 1)$ immersion with constant mean curvature. Denote by $\nu : \Sigma \to S^2 \subset \mathbf{R}^3$ the Gauss map of X. For a volume-preserving variation X_t of X, the second variation of the area is

$$\partial^2 A = -\int_\Sigma \varphi L[\varphi]\, d\Sigma =: I(\varphi), \quad \varphi := \left\langle \left.\frac{\partial X_t}{\partial t}\right|_{t=0},\ \nu \right\rangle,$$

where $L[\varphi] = 2\,\partial H = \Delta\varphi + \|d\nu\|^2\varphi$. Hence, X is stable if and only if $I(\varphi) \geq 0$ for all $\varphi \in C_0^{3+\alpha}(\Sigma)$ which satisfy $\int_\Sigma \varphi\, d\Sigma = 0$.

Now consider the eigenvalue problem:

$$(*) \qquad L[\varphi] = -\lambda\varphi, \qquad \varphi|_{\partial\Sigma} = 0, \qquad \varphi \in H_0^1(\Sigma) - \{0\}.$$

Denote by λ_n the n'th eigenvalues of $(*)$. Set $E := \{e \in C_0^{2+\alpha}(\Sigma) \mid L[e] = 0\}$.

Theorem 1 (Existence and uniqueness of CMC deformation. Koiso [4]). *Let $X : \Sigma \to \mathbf{R}^3$ be a CMC immersion. Assume either the following (i) or (ii) holds.*

(i) $E = \{0\}$. *(ii)* $\dim E = 1$ *and* $\int_\Sigma e\, d\Sigma \neq 0$ *for all* $e \in E - \{0\}$.

Then, in a small neighborhood of X, there exists a unique (up to diffeomorphisms of Σ) one-parameter family $\{X_t\}$ ($X_t : \Sigma \to \mathbf{R}^3$, $X_0 = X$) of CMC immersions with the same boundary values as X.

Therefore, there is no bifurcation in this case, and bifurcation may occur only in the case where $\lambda_k = 0$ for some $k \geq 2$.

On the other hand, we have a criterion for the stability of CMC surfaces. For one-parameter family $\{X_t\}$ of immersions, denote by $H(t)$ and $V(t)$, the mean curvature and the volume of X_t, respectively.

Theorem 2 (Criterion for Stability. Koiso [4]). *Let X be a CMC immersion.*

(I) If $\lambda_1 \geq 0$, then X is stable.

(II) Assume $\lambda_1 < 0 \leq \lambda_2$. If there is a deformation X_t of X such that $H'(0) =$ constant $\neq 0$, *then the following (i) and (ii) hold.*

(i) If $H'(0)V'(0) \geq 0$, then X is stable.

(ii) If $H'(0)V'(0) < 0$, then X is unstable.

If there is no such deformation, then X is unstable.

(III) If $\lambda_2 < 0$, then X is unstable.

Therefore, in order to study the stability of CMC surfaces in a bifurcation branch, we need to study only the case where $\lambda_2 = 0$.

Theorem 3 (Stability of bifurcation branch)**.** *Assume we have one-parameter family $X_t = X + \varphi_t \nu$, ($t \in I = (-\epsilon, \epsilon) \subset \mathbf{R}$), of CMC $C^{3+\alpha}$ immersions with $X = X_0$, which satisfy the following (i)-(iii).*

(i) $V'(0) > 0$ and $H'(0) > 0$.

(ii) $\lambda_2(X_0) = 0$, and $\dfrac{d}{dt}\lambda_2(X_t)|_{t=t_0} > 0$. (resp. $\dfrac{d}{dt}\lambda_2(X_t)|_{t=t_0} < 0$.)

(iii) $E = \{ae\,;\, a \in \mathbf{R}\}$.

Then there exists an open interval $\hat{I}$ ($0 \in \hat{I} \subset \mathbf{R}$) and C^1 functions $\psi : \hat{I} \to C_0^{3+\alpha}(\Sigma)$, $\hat{V} : \hat{I} \to \mathbf{R}$, such that $\psi(0) = 0$, and $X(s) := X + (se + s\psi(s))\nu$ is a CMC immersion with volume $\hat{V}(s)$. In a small neighborhood X, CMC immersions with the same boundary values as X consists of $\{X_t\,;\, t \in I\}$ and $\{X(s)\,;\, s \in \hat{I}\}$. If $\hat{V}(s) > V(0)$ ($s \neq 0$) (resp. $\hat{V}(s) < V(0)$ ($s \neq 0$)), $X(s)$ are stable. If $\hat{V}(s) < V(0)$ ($s \neq 0$) (resp. $\hat{V}(s) > V(0)$ ($s \neq 0$)), they are unstable.

In order to prove Theorem 3, we need the following lemma.

Lemma 1. *Assume we have one-parameter family $X_t = X + \varphi_t \nu$, ($t \in I = (-\epsilon, \epsilon) \subset \mathbf{R}$), of CMC $C^{3+\alpha}$ immersions with $X = X_0$, which satisfy the following (i)-(iii).*

(i) $H(X_t) = t$. (ii) $\lambda_k(X_0) = 0$, $\dfrac{d}{dt}\lambda_k(X_t)|_{t=t_0} > 0$. (iii) $E = \{ae; a \in \mathbf{R}\}$.

Then there exist an open interval $\hat{I}$ ($0 \in \hat{I} \subset \mathbf{R}$) and C^1 functions $\varphi : \hat{I} \to C_0^{3+\alpha}(\Sigma)$ and $\hat{H} : \hat{I} \to \mathbf{R}$, such that $\hat{H}(0) = t_0$, $\varphi(0) = 0$, and $Y(\sigma) := X + (\sigma e + \sigma\varphi(\sigma))\nu$ is a CMC immersion with mean curvature $\hat{H}(\sigma)$. Moreover, in a small neighborhood of X, CMC immersions with the same boundary values as X consists of $\{X_t\,;\, t \in I\}$ and $\{Y(\sigma)\,;\, \sigma \in \hat{I}\}$. Moreover, the zero eigenvalue of X corresponds to small eigenvalues $\lambda(t) := \lambda(X_t)$ of X_t and $\lambda(Y(\sigma))$ of $Y(\sigma)$. $\lambda(Y(\sigma))$ and $-\sigma\hat{H}'(\sigma)\lambda'(t_0)$ have the same zeroes and, where $\hat{H}'(\sigma) \neq 0$, the same sign.

The first half of Lemma 1 is proved by applying a general result on bifurcation by Crandall-Rabinowitz [2]. This idea was originally used by Patnaik [5]. He obtained a similar result to the first half of Lemma 1, where he used the volume instead of the mean curvature. On the other hand, the second half of Lemma 1 is proved by applying a general result on bifurcation by Crandall-Rabinowitz [3].

Theorem 3 is proved by using Theorem 2, Lemma 1, and a corresponding result of Lemma 1 with volume instead of mean curvature.

Remark 1. The variation vector field of $X(s)$ (at $s=0$) in Theorem 3 is e, and $\int_\Sigma e\, d\Sigma = 0$. This implies that, $X(s)$ does not have the same symmetry as X_t.

3. Applications and generalizations

The method developed in the previous section is applied to various examples:

(I) The example which was mentioned in the first section: a bifurcation at a certain part of a nodoid with $\lambda_2 = 0$ occurs, where the rotationally symmetry breaks.

(II) A free boundary problem for CMC hypersurfaces between two parallel hyperplanes in $\mathbf{R}^{n+1}$ (cf. Pedrosa-Ritore [6]). We have a bifurcation from a one-parameter family of cylinders to produce a half period of an unduloid-type solutions. Symmetry with respect to a hyperplane breaks. The stability of the unduloid depends on the dimension.

(III) We can apply our method to more general variational problems: Free or fixed boundary problem for surfaces with constant anisotropic mean curvature, which are critical points of an anisotropic surface energy with volume constraint (cf. Arroyo-Koiso-Palmer[1]).

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Moduli spaces of complex projective structures and CMC surfaces

Rob Kusner

(dedicated to my father, David Kusner, 1932-2010)

Complete embedded constant mean curvatures (CMC) surfaces in $\mathbf{R}^3$ are highly transcendental objects [8, 3, 12, 10, 13] whose moduli spaces are understood in only a few special cases [1, 14, 9, 2, 6]. This talk (reporting on joint work [5] with Karsten Grosse-Brauckmann, Nick Korevaar and John Sullivan) discusses a surprising connection between CMC surfaces and complex projective structures, allowing us to make the former just a bit more explicit.

The well-known (see [4]) correspondence between projective structures and holomorphic quadratic differentials $q(z)\,dz^2$ via the holomorphic Hill equation

$$(\star) \qquad\qquad u_{zz} + q(z)\,u = 0,$$

guides our work: if $u_1(z), u_2(z)$ is a basis of solutions to $(\star)$, then their ratio $\frac{u_1}{u_2}$ is the developing map for a projective structure whose Schwarzian is $q(z)$, where z is a local coordinate belonging to some background projective structure. This makes the moduli space of projective structures over a fixed Riemann surface into a complex affine space modeled on the vector space of quadratic differentials.

In case of coplanar k-unduloids, CMC surfaces of genus 0 with k ends [7], the underlying Riemann surface is $\mathbf{C}$ with global coordinate z (unique up to $z \to az+b$) belonging to its standard projective structure, and $q(z)$ is a polynomial of degree $k-2$ normalized to be monic with root-sum zero. For example, the unduloids all have $q(z) = 1$ and an exponential developing map, while all triunduloids have $q(z) = z$ and developing map given by a ratio of Airy functions. For $k \geq 4$, it is not practical to solve the Hill equation $(\star)$ explicitly, so instead we perform a careful asymptotic analysis.

Each of the k ends corresponds to an asymptotic half-space in the flat metric given by $|q(z)|\,|dz|^2 \sim |dw|^2$. We use the flat half-space coordinate w to rewrite $(\star)$ as an $O(\frac{1}{w^2})$ perturbation of the constant coefficient equation. This allows us to analyze growing and decaying solutions on each half-space and show that the ratio of two independent global solutions to $(\star)$ is the developing map of a k-point projective structure: an equivalence class of the k-point spherical metrics previously used [7] to classify coplanar k-unduloids, except now two k-point metrics are equivalent if they differ by a fractional linear map (rather than an isometry) of $\mathbf{S}^2$.

Since the quotient space of fractional linear maps by isometries is a 3-ball, the moduli space of all k-point metrics – or equivalently, of all coplanar k-unduloids – is homeomorphic to the product of this ball with the space of k-point projective structures, and thus to $\mathbf{R}^{2k-3} = \mathbf{B}^3 \times \mathbf{C}^{k-3}$, where the second factor comes from realizing k-point structures by (affine) space of normalized polynomials of degree $k-2$. We already knew [6, 7] the topology of these moduli spaces for the cases $k = 3, 4$, but we had suspected that for $k \geq 5$ these spaces were not even simply connected, and thus it came as quite a surprise that they were contractible!

An interesting question we continue to explore is how this description for CMC moduli space compares with others, such as that coming from spherical metrics or, more speculatively, from the holomorphic potentials methods stemming from [3]. And although not discussed in this talk, one hopes these ideas may also be applied to give a more explicit description of minimal surfaces [11] in $\mathbf{S}^3$ which are cousins of CMC surfaces in $\mathbf{R}^3$.

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The large genus limit of the infimum of the Willmore energy

Ernst Kuwert
(joint work with Yuxiang Li, Reiner Schätzle)

The Willmore energy of an immersed surface $\Sigma \hookrightarrow \mathbb{R}^n$ with mean curvature vector $\vec{H}$ and induced area measure μ is given by

$$\mathcal{W}(\Sigma) = \frac{1}{4}\int_\Sigma |\vec{H}|^2 \, d\mu.$$

Let $\mathcal{C}(n,p)$ be the class of oriented, closed (i.e. compact without boundary), smoothly immersed surfaces Σ with genus $(\Sigma) = p$, and put

$$\beta_p^n = \inf\{\mathcal{W}(\Sigma) | f \in \mathcal{C}(n,p)\}. \tag{1}$$

It is well-known that $\mathcal{W}(\Sigma) \geq 4\pi$ for any closed immersed surface, with equality only for round spheres [10]. In [8] L. Simon proved the existence of smooth minimizers in $\mathcal{C}(n,p)$ under the Douglas-type condition

$$\beta_p^n < 4\pi + \min\Big\{\sum_{i=1}^r (\beta_{p_i}^n - 4\pi) : 1 \leq p_i < p, \, \sum_{i=1}^r p_i = p\Big\} =: \tilde{\beta}_p^n. \tag{2}$$

In particular he obtained the existence for $p = 1$. The inequality (2) was proved later in [1], so β_p^n is attained for all n, p and $\beta_p^n > 4\pi$ for $p \geq 1$. By conformal invariance the area of a minimal surface in $\mathbb{S}^3$ equals the Willmore energy of the surface in $\mathbb{R}^3$ obtained by stereographic projection [9], which leads to an upper bound for β_p^n. Namely, Pinkall [2] and independently Kusner [3, 4] observed that the minimal surfaces $\xi_{p,1}$ in $\mathbb{S}^3$ described by Lawson in [6] have area less than 8π. In summary we know that

$$4\pi < \beta_p^n < 8\pi \quad \text{for } p \geq 1. \tag{3}$$

An important consequence of the upper bound is that minimizers are automatically embedded, due to an inequality of Li and Yau [7]. It was conjectured that the β_p^n might be monotonically increasing in p, see [2, p. 446], and that the projected $\xi_{p,1}$ could in fact be minimizers for their genus [4, p. 318 and p. 344]. For large p these surfaces look like two spheres connected by minimal handles, see [10, p. 293] for $p = 5$, in particular their Willmore energy converges to 8π as $p \to \infty$ [3]. Here we prove the following.

Theorem. *Let β_p^n be the infimum of the Willmore energy among oriented, closed surfaces of genus p immersed into $\mathbb{R}^n$. Then*

$$\lim_{p\to\infty} \beta_p^n = 8\pi. \tag{4}$$

We would like to thank Tom Ilmanen for asking the question addressed in this paper when one of us gave a talk in Zürich.

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Darboux transforms and simple factor dressing

Katrin Leschke

(joint work with Francis E. Burstall, Josef Dorfmeister, Áurea Quintino)

Introduction. By the Ruh–Vilms Theorem [15] a constant mean curvature surface $f : M \to \mathbb{R}^3$ of a Riemann surface M into 3–space is characterized by the harmonicity of its Gauss map $N : M \to S^2$. This fact allows to use integrable system methods for constant mean curvature surfaces: one can introduce the spectral parameter and thus obtains the associated $\mathbb{C}_*$–family d_λ of flat connections. This family is unitary on the unit circle where it describes the associated family of harmonic maps N_λ of N, and thus, with the Sym–Bobenko formula [1], the associated family of constant mean curvature surface $f_\lambda : \tilde{M} \to \mathbb{R}^3$ on the universal cover $\tilde{M}$ of M. Moreover, the dressing operation [17, 16] on harmonic maps gives rise to new constant mean curvature surfaces.

On the other hand, in the case when $M = T^2$ is a torus, the holonomy representation of d_λ is abelian, and thus gives rise to the spectral curve of the harmonic torus $N : T^2 \to S^2$, see [9]. More generally, a spectral curve for a conformal torus $f : T^2 \to S^4$ has been defined recently and points on the spectral curve have the geometric interpreation [2] as (generalized) Darboux transforms of f. In this presentation, we explain an extension of results of [8, 3, 10]: a generic Darboux transform is indeed given by simple factor dressing, and vice versa.

Other surface classes are also linked to harmonicity, e.g., Hamiltonian stationary Lagrangians $f : M \to \mathbb{C}^2$, or Willmore surfaces $f : M \to S^4$, and thus allow an associated family of flat connections; however to construct new surfaces, a Sym–Bobenko formula is still needed. Our result extends to these cases and gives a generalized notion of simple factor dressing [14, 11, 12, 13].

Dressing. We will interpret the Euclidean 3-space as the imaginary quaternions, and identify $\mathbb{C}^2 = (\mathbb{H}, I)$ where the complex structure I is given by right multiplication by i. In particular, a smooth map $N : M \to S^2 \subset \operatorname{Im}\mathbb{H}$ gives by

$$J\varphi = N\varphi \quad \text{for} \quad \varphi \in \underline{\mathbb{H}}\,,$$

a complex structure $J \in \operatorname{End}(\underline{\mathbb{H}})$ on the trivial $\mathbb{H}$ bundle, that is a quaternionic linear endomorphism with $J^2 = -1$. In terms of the Hopf field, that is the $(1,0)$–part

$$A = \frac{1}{4}(*dJ + J(dJ))$$

of the connection form $\frac{1}{2}J(dJ)$, the harmonicity of N reads as the condition that $d * A = 0$. This allows to introduce the spectral parameter $\lambda \in \mathbb{C}_*$, and we obtain the well–known fact that N is harmonic if and only if the family of connections

$$d_\lambda = d + (\lambda - 1)A^{(1,0)} + (\lambda^{-1} - 1)A^{(0,1)}$$

on $\underline{\mathbb{C}}^2$ are flat for all $\lambda \in \mathbb{C}_*$ where $A^{(1,0)}$ and $A^{(0,1)}$ denote the $(1,0)$ and $(0,1)$–parts of the Hopf field A with respect to the complex structure I. New harmonic maps can be constructed by gauging the family of flat connections:

Theorem 1 (Dressing, [17, 16, 4]). *Let $N : M \to S^2$ be harmonic and $r_\lambda : \tilde{M} \to \mathrm{GL}(2, \mathbb{C})$ be smooth on the universal cover $\tilde{M}$ of M (satsifying a reality condition and normalisation) with*

(1) *$\lambda \to r_\lambda$ is meromorphic on $\mathbb{CP}^1$ and holomorphic at $0, \infty$ and*
(2) *$\lambda \to \hat{d}_\lambda = r_\lambda \cdot d_\lambda$ is holomorphic on $\mathbb{C}_*$.*

Then $\hat{d}_\lambda$ is the associated family of flat connections of a harmonic map $\hat{N} : \tilde{M} \to S^2$, a so–called dressing *of N.*

In particular, for fixed $\mu \in \mathbb{C}_*$ and d_μ–parallel line bundle M_μ over $\tilde{M}$, the map

$$r_\lambda = \pi_\mu \frac{1 - \bar{\mu}^{-1}}{1 - \mu} \frac{\lambda - \mu}{\lambda - \bar{\mu}^{-1}} + \pi_\mu^\perp$$

(where $\pi_\mu, \pi_\mu^\perp$ are the projections onto M_μ and $M_\mu^\perp$ respectively) satisfies the condition above and gives a simple factor dressing *of N.*

By the Sym–Bobenko formula this theorem gives (simple factor) dressing of constant mean curvature surfaces. To obtain a simple factor dressing on a torus $M = T^2$ rather than on the universal cover, one has to find a d_μ–parallel bundle M_μ over T^2. This amounts to finding eigenvectors of the holonomy of d_μ, and thus gives the link to the spectral curve of a harmonic torus.

Darboux transformation. In [2] a Darboux transformation on conformal tori in the 4–sphere is defined. This transformation extends the classical Darboux transformation [7]: two conformal immersions $f, \hat{f} : M \to \mathbb{R}^3$ from a Riemann surface M into 3–space are called a *classical Darboux pair* if there exists a sphere congruence enveloping both f and $\hat{f}$. To construct classical Darboux transforms in the case when $f : M \to \mathbb{R}^3$ has constant mean curvature $H = 1$, one solves the Riccati equation [5]

$$d\hat{T} = -df + \hat{T}(dg\rho)\hat{T}, \quad \rho \in \mathbb{R}\,, \tag{1}$$

where $g = f + N$ is the parallel constant mean curvature surface. Then $\hat{f} = f + \hat{T}$ is a classical Darboux transform of f. Moreover $\hat{f}$ has constant mean curvature if and only if

$$(\hat{T} - N)^2 = \rho^{-1} - 1\,. \tag{2}$$

To generalize this transformation choose for fixed $\mu \in \mathbb{C}_*, \mu \neq 1$, a parallel section φ (on the univeral cover) of d_μ. Putting $\rho = \frac{2-\mu-\mu^{-1}}{4}, \nu = \frac{i(\mu^{-1}-\mu)}{4} \in \mathbb{C}$ and $T = -N\varphi\rho\varphi^{-1} + \varphi\nu\varphi^{-1}$, one obtains a solution of the Riccati type equation

$$dT = -dg\hat{\rho} + TdfT$$

with $(T\hat{\rho}^{-1} + N)^2 = \hat{\rho}^{-1} - 1$ where $\hat{\rho} = \varphi\rho\varphi^{-1} : \tilde{M} \to \mathbb{H}_*$ is now a smooth function. In fact, $\hat{f} = f + T^{-1}$ is [6], up to translation, a constant mean curvature surface in $\mathbb{R}^3$ for $\mu \in \mathbb{C}_*, \mu \neq 1$. We call $\hat{f}$ a *μ–Darboux transform* of f. Note that for $\rho \in \mathbb{R}$ the above two equations are equivalent to (1) and (2) with $\hat{T} = T^{-1}$, so that we obtain [6] the classical Darboux transforms with constant mean curvature exactly as the special case when $\mu \in \mathbb{R}_* \cup S^1$. Note that for $\rho, \nu \notin \mathbb{R}$, the Darboux transform depends again on the choice of the line bundle $M_\mu = \varphi\mathbb{C}$ over $\tilde{M}$. Indeed:

Theorem 2. *A μ–Darboux transform of the Gauss map N of a constant mean curvature surface $f : M \to \mathbb{R}^3$ is a simple factor dressing of $-N$, and vice versa.*

Note that though our Riccati equations are expressed in terms of the constant mean curvature surface (and its parallel surface) both equations can be reformulated in terms of the harmonic map N, and we obtain a Darboux transformation on harmonic maps $N : M \to S^2$. From [13] we thus obtain an analogue result for Hamiltonian stationary Lagrangian surfaces. Moreover, this Darboux transformation can be extended to a transformation on the conformal Gauss map of a Willmore surface [5, 12], and one obtains an analogue for Willmore surfaces [11].

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Calabi-Yau domains in Riemannian three manifolds

Francisco Martín
(joint work with William H. Meeks III)

The Calabi-Yau problem for minimal surfaces has generated a remarkable amount of mathematical literature in recent years, possibly because of the diversity of interesting techniques developed for its study. The problem dates back to the sixties, when E. Calabi conjectured that there were no complete bounded minimal surfaces in $\mathbb{R}^3$. In 1996, Nadirashvili [14] provided the first counterexample to the conjecture.

The problem was revisited several times by S.-T. Yau [15, 16] and he proposed new questions related with this conjecture. Among all them, the most interesting one is the *embedded* Calabi-Yau problem, that is, the existence (or non-existence) of complete embedded minimal surfaces in a bounded domain $\mathcal{D}$ of $\mathbb{R}^3$.

We would like to point out the dichotomy between the immersed case and the embedded one. On one hand, we have a lot of existence theorems for immersed minimal surfaces in bounded domains of $\mathbb{R}^3$, even under the assumption of interesting geometric and topological properties for the surface and the domain $\mathcal{D}$ [1, 2, 3, 10, 11, 12].

On the other hand, if we impose to the surface the hypothesis of embeddedness we only have non-existence results:

Theorem 1 (Colding, Minicozzi, [5]). *A complete embedded minimal surface with* **finite topology** *in $\mathbb{R}^3$ must be proper in $\mathbb{R}^3$.*

Theorem 2 (Meeks, Perez, Ros, [13]). *If M is a complete embedded minimal surface in $\mathbb{R}^3$ with* **finite genus** *and a* **countable number of ends***, then M is proper in $\mathbb{R}^3$.*

These theorems mean that if we were looking for an example of a complete embedded minimal surface in a bounded domain of $\mathbb{R}^3$, then we should seek it among surfaces with infinite genus or with an uncountable number of ends. If in addition the surface is non-orientable, then we have an important obstruction as shows this theorem:

Theorem 3 (Ferrer, Martín, Meeks, [6]). *If M is a nonorientable surface and has an infinite number of nonorientable ends, then M cannot properly embed in any smooth bounded domain of $\mathbb{R}^3$.*

Taking into account all the above information, it makes sense to conjecture the following:

Conjecture 1 (Martín-Meeks-Nadirashvili; Meeks-Perez-Ros). *Let M be an open surface.*

(1) *There exists a complete proper minimal embedding of M in* ***some*** *smooth bounded domain $\mathcal{D} \subset \mathbb{R}^3$ iff the* ***number of nonorientable ends is finite*** *and* ***every end of M has infinite genus.***

(2) *There exists a complete proper minimal embedding of M in* ***every*** *smooth bounded domain $\mathcal{D} \subset \mathbb{R}^3$ iff M is orientable with every end having infinite genus.*

If the domain is non-smooth then we have:

Conjecture 2 (Ferrer, Martín, Meeks)**.** *Let $\mathcal{D}_\infty$ be the bounded domain in $\mathbb{R}^3$ described in the figure, which is* **smooth except at one point***. A necessary and sufficient condition for an open surface M to admit a complete, proper minimal embedding in $\mathcal{D}_\infty$ is that every end of M has* **infinite genus***.*

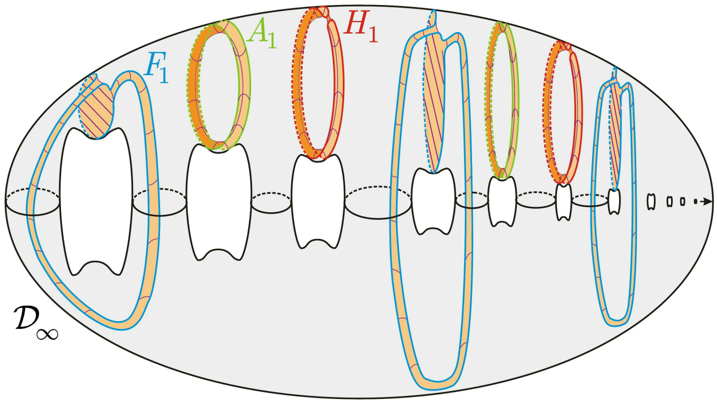

Figure 1. The domain $\mathcal{D}_\infty$

Recently, Ferrer, Martín and Meeks [6] have given a first approach to the proof of the embedded Calabi-Yau problem by demonstrating that for every smooth bounded domain $\mathcal{D} \subset \mathbb{R}^3$ and for every open surface M, there exists a complete proper minimal immersion $f\colon M \to \mathcal{D}$; furthermore, in [6], they proved that such an immersion $f\colon M \to \mathcal{D}$ can be constructed so that for any two distinct ends E_1, E_2 of M, the limit sets $L(E_1)$, $L(E_2)$ in $\partial\mathcal{D}$ are disjoint compact sets.

In contrast to the above existence results, in [8] we prove the existence of non-smooth bounded domains $\mathcal{D}$ in $\mathbb{R}^3$, and more generally, domains $\mathcal{D}$ inside any Riemannian three-manifold, for which some open surface M can not be properly immersed into $\mathcal{D}$ as a complete surface with bounded mean curvature. In this case, we will say that $\mathcal{D}$ is a **Calabi-Yau domain** for M. The result described in the next theorem generalizes the main theorem of Martín, Meeks and Nadirashvili in [9] which demonstrates the existence of nonsmooth bounded domains in $\mathbb{R}^3$ which do not admit any complete, properly immersed minimal surfaces with compact boundary (possibly empty) and at least one annular end.

Theorem 4. *Let $\overline{W}$ be a smooth compact Riemannian three-manifold with nonempty boundary and let $W = \mathrm{Int}(\overline{W})$. There exists a properly embedded one-manifold $\Delta \subset W$ whose path components are smooth simple closed curves, such that $\mathcal{D} =$*

$W - \Delta$ is a Calabi-Yau domain for any surface with compact boundary (possibly empty) and at least one annular end. In particular, $\mathcal{D}$ does not admit any complete, noncompact, properly immersed surfaces of finite topology, compact boundary and constant mean curvature.

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Complete Flat Surfaces with Isolated Singularities in $\mathbb{H}^3$

Antonio Martínez

(joint work with Armando V. Corro, Francisco Milán)

The theory of flat surfaces in $\mathbb{H}^3$ has undergone an important development in the last few years. The starting point of this renewed interest has been the discovery in [4] that flat surfaces in $\mathbb{H}^3$ admit a Weierstrass representation formula in terms of meromorphic data, like the classical one for minimal surfaces in $\mathbb{R}^3$. This has generated a great interest in such class of surfaces, even though the only complete examples are the horospheres and hyperbolic cylinders (see [11]).

The last mentioned lack of complete examples has motivated an important advance in the problem of studying the singularities in these surfaces. Questions such as their generic behaviour or the existence of complete examples with singularities have been solved thanks to the works [8], [7] and [10].

Contrarily to the minimal case, flat surfaces in $\mathbb{H}^3$ can have isolated singularities around which the surface might be regularly embedded. Geometrically, isolated singularities correspond to points where the Gauss map has not well defined limit. Locally, this kind of singularities have been classified in [5], where is proved that the class of flat surfaces that have $p \in \mathbb{H}^3$ as an embedded isolated singularity admits a one-to-one correspondence with the class of analytic regular convex Jordan curves in the $2-$sphere. But there are many interesting problems in this theory that remain unsolved. For example, we can quote the existence of compact or complete examples with a finite number of isolated singularities. In this sense and up to now, the only known example of complete flat surface with isolated singularities is the revolution one (also call *the half hourglass*) which is a graph over a horosphere with only one point removed.

We start with some information about how flat surfaces in $\mathbb{H}^3$ can be represented by holomorphic data and use it in order to study the global behaviour of complete embedded flat surfaces with a finite number of isolated singularities. To be precise, we prove

Proposition 1. *Let $\psi : \Sigma \longrightarrow \mathbb{H}^3$ be a complete flat immersion with $\psi(\mathcal{F})$ as set of isolated singularities. Then there is a compact Riemannian surface $\overline{\Sigma}$, n disjoint discs $\mathcal{D}_1, \cdots, \mathcal{D}_n \subset \overline{\Sigma}$ and finitely many points $q_1, \cdots, q_m \in \overline{\Sigma} \setminus \mathcal{D}$, where $\mathcal{D} = \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_n$ such that $\Sigma \setminus \mathcal{F}$ endowed with the conformal structure induced by the second fundamental form has the conformal type of $\overline{\Sigma} \setminus \{\{q_1, \cdots, q_m\} \cup \mathcal{D}\}$*

The points $q_1, \cdots, q_m$ are called the ends of ψ.

Theorem 1. *If $\psi : \Sigma \longrightarrow \mathbb{H}^3$ is a complete flat embedding with $\psi(\mathcal{F})$ as set of isolated singularities, then ψ is globally convex.*

Corollary 1. *Every complete flat embedding $\psi : \Sigma \longrightarrow \mathbb{H}^3$ with a finite number of isolated singularities and only one end is a graph over a finitely punctured horosphere.*

We shall spend a important part of the talk to the construction of examples of complete embedded surfaces with only one end an either one or two isolated singularities (see Figigure 1 and Figure 2). In the case of two isolated singularities, which we called canonical examples, the construction relies on the conformal representation of flat surfaces in $\mathbb{H}^3$ and the existence of conformal equivalences between a one punctured annulus and a horizontal slit domain in $\mathbb{C}$.

Finally, we give the following characterizations results:

Theorem 2. *The revolution examples are the unique complete flat embeddings in $\mathbb{H}^3$ with only one isolated singularity and one end.*

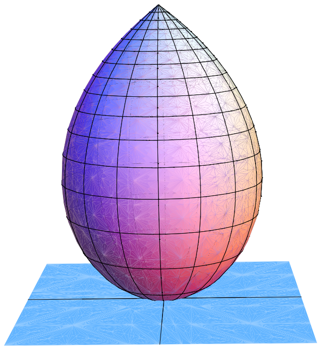

FIGURE 1. Complete flat surface with only one isolated singularity

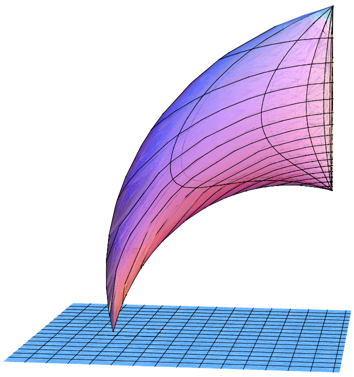

FIGURE 2. Complete flat surface with two isolated singularities.

Theorem 3. *Each complete flat embedding in $\mathbb{H}^3$ with only two isolated singularities and one end must be congruent to one of the canonical examples.*

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The geometric Cauchy problem in surface theory

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(joint work with Jose A. Gálvez)

Let $(\bar{M}^3, \langle , \rangle)$ denote a Riemannian 3-manifold, and $\mathcal{A}$ denote a class of surfaces immersed in $\bar{M}^3$ having an underlying second order PDE (for instance, CMC surfaces in the ambient manifold $\bar{M}^3$). Let $\beta(s)$ denote a regular curve in $\bar{M}^3$, and $V(s)$ be a vector field along $\beta(s)$ in $\bar{M}^3$ which is orthogonal to the curve. That is, $V(s) \in T_{\beta(s)}\bar{M}^3$ and $\langle \beta', V \rangle = 0$ for every s.

In these conditions, the *geometric Cauchy problem* for the class $\mathcal{A}$ asks for the existence and uniqueness of a surface Σ belonging to the class $\mathcal{A}$ such that $\beta(s)$ is a regular curve on Σ and the unit normal of Σ at $\beta(s)$ is given by $V(s)$.

It is clear that this is a geometric formulation of the usual Cauchy problem for second order PDEs. Thus, in many cases the problem will have a unique solution, just as an application of the Cauchy-Kowalevsky theorem. Here, we are interested in the construction of such a solution by some formula, and in the geometric global applications to surface theory of this resolution.

This geometric Cauchy problem is a general formulation of the classical *Björling problem* for minimal surfaces in $\mathbb{R}^3$, posed by E.G. Björling in 1844 and solved by H.A. Schwarz in 1890. In the specific setting of minimal surfaces in $\mathbb{R}^3$, the information provided by solving such a Björling type problem is, in general and with some exceptions, not specially relevant to the theory. Indeed, minimal surfaces in $\mathbb{R}^3$ are very explicit, and can be globally studied using stronger techniques. However, when one moves to some less explicit theories, the solution to the geometric Cauchy problem provides relevant non-trivial information about them, as we will explain.

In what follows, I will present some results obtained in collaboration with J.A. Gálvez regarding the solution to the geometric Cauchy problem in several surface theories. These theories are: $H = 1$ surfaces in $\mathbb{H}^3$, flat surfaces in $\mathbb{H}^3$, flat surfaces in $\mathbb{S}^3$ (jointly also with J.A. Aledo) and surfaces of constant positive curvature in $\mathbb{R}^3$ (jointly also with L. Hauswirth).

1. The geometric Cauchy problem in $\mathbb{H}^3$ and $\mathbb{S}^3$

1.1. **The Cauchy problem for $H = 1$ surfaces in $\mathbb{H}^3$.** The class of $H = 1$ surfaces (H is the mean curvature) in $\mathbb{H}^3$ has many interesting properties. One of them is that the hyperbolic Gauss map is a conformal map into the Riemann sphere. Another one is that they admit a Weierstrass-type representation, obtained by R.L Bryant [2]. Yet another one is that its metric is governed by the Liouville equation $\Delta u + e^u = 0$.

In the work [5] we solved the geometric Cauchy problem for $H = 1$ surfaces in $\mathbb{H}^3$. Specifically, we obtained a formula describing the unique solution to that problem, in terms of the solution to the Cauchy problem for Liouville's equation. We also investigated the solution to this analytic problem, and found several global consequences. For instance, we gave a rough classification of complete $H = 1$ cylinders in $\mathbb{H}^3$ having finite dual total curvature, in terms of initial data given by vector trigonometric polynomials. Another one was the reconstruction in explicit coordinates of any $H = 1$ surface in $\mathbb{H}^3$ from the knowledge of one of its planar geodesics. Also, in [6] we used these results for the Liouville equation in order to solve a Neumann problem in the half-plane for that equation.

1.2. **The Cauchy problem for flat surfaces in $\mathbb{H}^3$.** Flat surfaces in $\mathbb{H}^3$ share many similarities with $H = 1$ surfaces. Indeed, with respect to their extrinsic conformal structure (i.e. the conformal structure induced by the second fundamental form, which is definite), the positive and negative hyperbolic Gauss maps are conformal into the Riemann sphere, and so the surface has a quite explicit Weierstrass-type representation (see [7]).

In [4], we gave an explicit formula solving the geometric Cauchy problem for flat surfaces in $\mathbb{H}^3$, and we used that formula for studying singularities of these surfaces. It must be said that any complete flat surface in $\mathbb{H}^3$ is a horosphere or a hyperbolic cylinder; thus, it is natural to allow the presence of singularities for theses surfaces, and to investigate how the nature of singularities determines the global geometry of the flat surface.

In this line, we gave a classification of all flat surfaces in $\mathbb{H}^3$ that are regularly embedded around an isolated singularity. The way to do this was to solve an adequate *singular* Cauchy problem, and to analyze in terms of the initial data to such problem when the surface is regular or embedded around the singularity.

1.3. **The Cauchy problem for flat surfaces in $\mathbb{S}^3$.** The geometry of flat surfaces in $\mathbb{S}^3$ is governed by the homogeneous wave equation $\omega_{uv} = 0$, which is hyperbolic. As a consequence, both existence and uniqueness of the solution to the geometric Cauchy problem for flat surfaces in $\mathbb{S}^3$ fail when we meet an asymptotic direction on the surface.

In [1], we solved jointly with J.A. Aledo the geometric Cauchy problem for flat surfaces in $\mathbb{S}^3$. Specifically, we proved existence and uniqueness in the non-characteristic case, and we gave a necessary and sufficient condition for the initial data of the problem in order to ensure existence and uniqueness in a restricted sense, in the characteristic case.

As an application of this resolution, we classified the space of flat surfaces in $\mathbb{S}^3$ with the topology of a Möbius strip. It must be pointed out that these surfaces cannot be complete or real analytic, and that one always hits a characteristic direction when traveling around a non-null homotopic curve on it.

2. Surfaces of constant positive curvature in $\mathbb{R}^3$

In [3], we studied jointly with L. Hauswirth the class of surfaces of constant curvature $K > 0$ ($K = 1$ w.l.o.g.) in $\mathbb{R}^3$ with isolated singularities. In the local case, we proved that such an isolated singularity is *extendable* in an adequate sense if and only if the mean curvature of the surface is bounded around the singularity. For the case of non-extendable singularities, we proved:

Theorem. Let $\alpha : \mathbb{S}^1 \to \mathbb{S}^2$ denote a closed, real analytic, locally convex curve with admissible cusps in $\mathbb{S}^2$. Then it can be realized as the limit unit normal of a unique K-surface in $\mathbb{R}^3$ having a non-extendable isolated singularity of finite area at the origin.

Conversely, any non-extendable isolated singularity of finite area of a K-surface in $\mathbb{R}^3$ is constructed like this. Moreover:

i) From an intrinsic point of view, all these singularities are conical.
ii) The curve α is a regular convex Jordan curve in $\mathbb{S}^2$ if and only if the surface is embedded around the singularity.

As a consequence, there exists a correspondence between the space of embedded isolated singularities of K-surfaces in $\mathbb{R}^3$ and the class of regular, real analytic convex Jordan curves in $\mathbb{S}^2$.

From a global point of view, we define a *peaked sphere* in $\mathbb{R}^3$ to be a closed convex K-surface in $\mathbb{R}^3$ that is everywhere regular except for a finite number of points. These points will then be embedded isolated singularities. A peaked sphere with 0 singularities is a round sphere. There are no peaked spheres with exactly 1 singularity, and a peaked sphere with exactly 2 singularities is rotational. For the case of $n > 2$ singularities, we can combine results by Alexandrov, Pogorelov, Troyanov and Luo-Tian to obtain:

Theorem. For $n > 2$, the space of peaked spheres in $\mathbb{R}^3$ with $n > 2$ singularities is, up to ambient isometries, a $3n-6$-dimensional family. This family is described in terms of the conical angles at the singularities, and the intrisic conformal structure.

By our local analysis, we know that the extrinsic conformal structure of a peaked sphere is that of a circular domain. This provides several applications to the Neumann problem for harmonic diffeomorphisms into $\mathbb{S}^2$, or to a free boundary problem for CMC surfaces in $\mathbb{R}^3$ with boundary on a finite collection of spheres.

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Homogeneity of isoparametric hypersurfaces with six principal curvatures

Reiko Miyaoka

An isoparametric hypersurface M in S^{n+1} is a hypersurface with constant principal curvatures. When M has $g = 6$ principal curvatures $\lambda_1 > \cdots > \lambda_6$, the multiplicity m is common and takes values 1 or 2 [1]. We give a solution to the long-standing classification problem : [13] Isoparametric hypersurfaces in S^{13} with $(g, m) = (6, 2)$ are homogeneous and are given by the adjoint G_2 orbits in its Lie algebra. The geometry of G_2 orbits is given in [12]. The strategy of the proof is the same as the new geometric proof [11] of Dorfmeister-Neher's theorem for $m = 1$ [7], but the case $m = 2$ is overwhelmingly difficult. At every stage of the proof, a non-generic matter occurs, which we cannot avoid since what we have to prove is the homogeneity *starting without any homogeneity.* The classification is by no means a matter of calculation, but is done through some new ideas and a deep investigation of the geometry behind.

Here, the so-called Condition A plays a crucial role. This means that *the shape operators of a focal submanifold have the kernel independent of normal directions.*

Proposition 1. [10], [13] *When $g = 6$, an isoparametric hypersurface is homogeneous if and only if Condition A is satisfied.*

In fact, thanks to the small multiplicity when $g = 6$, Condition A is sufficient to calculate all the structure coefficients explicitly using the Gauss equation and a symmetry of M. Then these turn out to coincide with those of the G_2 orbits, and applying the rigidity theorem of hypersurfaces, we obtain the theorem.

Note that in the case $g = 4$, Condition A does not necessarily imply the homogeneity [4],[6],[14].

To show Condition A is the most difficult part of the proof. Singular matters have to be excluded via, for instance, the analyticity, the geometric properties, as well as the Fourier expansion of an S^1 parameter family of shape operators.

Now we give a brief summary of the argument. Decompose $T_pM = D_1 \oplus \cdots \oplus D_6$ into curvature distributions D_j of λ_j with $\dim D_j = m$. The focal map $f_i : M \to M_i$ makes each leaf $L_i(p)$ of D_i shrink to a focal point $\bar{p}$, and the image M_i is a regular submanifold of dimension $5m$, called the focal submanifold. We mainly work on $M_+ = M_6$, and put $f = f_6$. Then $TM_+ = \oplus_{i=1}^{5} f_* D_i$ and $T^\perp M_+ = \mathbb{R}\eta \oplus f_* D_6$ follows, where $\eta = f_*\xi$ for a unit normal vector ξ of M. Now, let $m = 2$. We denote an orthonormal basis of D_i by $e_i, e_{\bar{i}}$, $i = 1, \ldots, 6$. We use $\underline{i}$ for $i, \bar{i}$. The leaf $L_6(p)$ is identified with the (unit) normal sphere at $\bar{p} \in M_+$, and we have an orthonormal basis $\eta, \zeta = e_6$ and $\bar{\zeta} = e_{\bar{6}}$ of $T^\perp M_+$. We denote the shape operator of M_+ by $B_{\boldsymbol{n}}$ for $\boldsymbol{n} \in T^\perp M_+$.

Lemma 1. *$B_{\boldsymbol{n}}$ is isospectral, i.e., has eigenvalues $\pm\mathbb{S}^2 3, \pm 1/\mathbb{S}^2 3, 0$ with multiplicity 2, where the eigendirections* depend *on $\boldsymbol{n}$.*

Thus $L(t) = \cos t B_\eta + \sin t B_\zeta$ is isospectral, and with respect to the orthonormal basis $(e_1, e_{\bar{1}}, \ldots, e_5, e_{\bar{5}}) \in D_1 \oplus \cdots \oplus D_5$, we can express

$$B_\eta = \begin{pmatrix} \mathbb{S}^2 3 I & & & & \\ & \frac{1}{\mathbb{S}^2 3} I & & & \\ & & 0 & & \\ & & & -\frac{1}{\mathbb{S}^2 3} I & \\ & & & & -\mathbb{S}^2 3 I \end{pmatrix}, B_\zeta = \begin{pmatrix} 0 & B_{12} & B_{13} & B_{14} & B_{15} \\ & 0 & B_{23} & B_{24} & B_{25} \\ & & 0 & B_{34} & B_{35} \\ & & & 0 & B_{45} \\ & & & & 0 \end{pmatrix},$$

where each block is a 2 by 2 matrix, and B_ζ is symmetric. Obviously, Condition A holds if and only if $\ker B_\zeta = D_3$, i.e., $B_{i3} = 0 = B_{3j}$. Here, B_{ij} is given by

$$(1) \qquad B_{ij} = \frac{1}{\sin\theta_6(\lambda_i - \lambda_6)} \begin{pmatrix} \Lambda_{i6}^{j} & \Lambda_{i6}^{\bar{j}} \\ \Lambda_{\bar{i}6}^{j} & \Lambda_{\bar{i}6}^{\bar{j}} \end{pmatrix} = {}^t B_{ji}, \quad \Lambda_{\alpha\beta}^{\gamma} = \langle \nabla_{e_\alpha} e_\beta, e_\gamma \rangle.$$

Using a symmetry of M, we can see that Condition A implies $\Lambda_{\underline{36}}^{\gamma} = 0 = \Lambda_{\underline{14}}^{\gamma} = \Lambda_{\underline{25}}^{\gamma}$.

Lemma 2. *Put $E = \mathrm{span}_t\{\ker L(t)\} \subset TM_+$, and $d = \dim E$. Suppose Condition A fails. Then each $L(t)$ maps E into $E^\perp$, and $3 \le d \le 6$ follows.*

The lemma is proved by showing

$$E = \mathrm{span}\{e_{\underline{3}}, \nabla_{e_6} e_{\underline{3}}, \nabla_{e_6}^2 e_{\underline{3}}, \ldots\},\ E^\perp = \mathrm{span}\{\nabla_{e_{\underline{3}}} e_6, \nabla_{e_6}\nabla_{e_{\underline{3}}} e_6, \nabla_{e_6}^2 \nabla_{e_{\underline{3}}} e_6 \ldots\}$$

where $e_6 = \zeta$, and $\underline{3}$ stands for 3 and $\bar{3}$. Therefore, the behavior of $e_3(t), e_{\bar{3}}(t)$ along the geodesic c in the direction $\zeta = e_6$ is important. A vector filed $v(t)$ along c is called *even* when $v(t+\pi) = v(t)$, and *odd* when $v(t+\pi) = -v(t)$.

Lemma 3. *We can choose $e_3(t), e_{\bar{3}}(t)$ as even vectors. In this case, all $\nabla_{e_6}^k e_{\underline{3}}(t)$ are even, and all $\nabla_{e_6}^k \nabla_{e_{\underline{3}}} e_6(t)$ are odd (possibly vanish somewhere).*

The latter follows from $L(t+\pi) = -L(t)$ and Lemma 2. Using the fact that *a subspace of TM_+ cannot have a continuous moving frame including odd number of odd vectors*, we obtain the following proposition, after some further argument.

Proposition 2. *Only $d = 6$ is possible, and we have an orthonormal basis of $E \oplus E^\perp$ explicitly in terms of $e_i, e_{\bar{i}}$, $1 \le i \le 5$.*

We can express $L(t) = \begin{pmatrix} 0 & R(t) \\ {}^tR(t) & S(t) \end{pmatrix}$ w.r.t. this basis, while $L(t) = U(t)L(0)^tU(t)$ holds for $U(t) \in O(10)$ by the isospectrality. Then by a careful investigation of the coefficient matrices of the Fourier expansion of $L(t)$, $U(t) = \mathrm{diag}\big(U_1(t) \quad U_2(t)\big) \in O(6) \times O(4)$ follows. Note that this never follows from the above shape of $L(t)$.

Lemma 4. *$T(t) = {}^tR(t)R(t)$ and $S(t)$ are isospectral 4×4 matrices.*

Thus 10×10 isospectral matrices $L(t)$ is reduced into two 4×4 isospectral matrices, which is easier to handle, still we need Mathematica, and evenness and oddness argument to obtain:

Proposition 3. *$S(t) = 0$.*

At this stage, applying a similar argument to another focal submanifold M_-, we would obtain $M = S^6 \times S^6$, a contradiction. Therefore Condition A is satisfied, and the homogeneity follows from Proposition 1.

Classification of isoparametric hypersurfaces in S^n up to now

g	1	2	3	4*(3 exceptions)	6
M	S^{n-1}	$S^k \times S^{n-k-1}$	$C_{\mathbb{F}}$	homogeneous or of OT-FKM type	$SO(4)$-orbits or G_2-orbits

For $g \le 3$, É. Cartan showed they are all homogeneous [2]. Here, $C_{\mathbb{F}}$ is the Cartan hypersurface, namely, a tube over the standard $P^2\mathbb{F}$ in S^n for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathcal{C}$.
For $g = 4^*$, Ozeki-Takeuchi [14], and then Ferus-Karcher-Münzner [8] constructed infinitely many non-homogeneous examples using the representation of the Clifford algebras. Cecil-Chi-Jensen [3], and independently, Immervoll [9] showed that when $g = 4$, except for $(m_1, m_2) = (3,4), (4,5), (7,8), (6,9)$, they are either homogeneous or of OT-FKM type. Recently, Q.S. Chi [5] shows that Condition A holds at a point in the case $(m_1, m_2) = (3,4)$ and they are also of OT-FKM type.

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On Lagrangian submanifolds in complex hyperquadrics obtained from isoparametric hypersurfaces

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(joint work with Ma, Hui, Tsinghua University, Beijing, P. R. China)

A smooth immersion φ of a smooth manifold L into a $2n$-dimensional symplectic manifold (M, ω) is called a *Lagrangian immersion* if $\dim L = n$ and $\varphi^*\omega = 0$. A submanifold satisfying only the second condition is called an *isotropic* submanifold. It is an interesting and important problem in differential geometry to study Lagrangian submanifolds in specific Kähler manifolds and to discuss their relationship with other geometries.

The notion of Hamiltonian minimality and Hamitonian stability of Lagrangian submanifolds in Kähler manifolds was introduced and investigated first by Y. G. Oh (1990). It is the study of Lagrangian submanifolds from the viewpoint of minimal submanifold theory in Riemannian geometry and geometric variational problems. A Lagrangian immersion $\varphi : L \to M$ of a compact smooth manifold L into a Kähler manifold M is called *Hamiltonian minimal* if it has extremal volume under every Hamiltonian deformation of φ. A Hamiltonian minimal Lagrangian immersion φ is called *Hamiltonian stable* (shortly, *H-stable*) if the second variation of the volume is nonnegative under every Hamiltonian deformation of φ. Moreover we call it *strictly Hamiltonian stable* if φ satisfies the following two conditions :

(1) φ is H-stable.
(2) The null-space of the second variations is exactly the span of the normal projections of holomorphic Killing vector fields of the ambient (simply connected) Kähler manifold M ("*Hamiltonian rigid* ").

In [5] we discussed Lagrangian submanifolds in complex hyperquadrics

$$Q_n(\mathbf{C}) \cong \widetilde{\mathrm{G}r}_2(\mathbf{R}^{n+2}) \cong SO(n+2)/SO(2) \times SO(n),$$

which are obtained as Gauss images of isoparametric hypersurfaces in spheres.

Let $N^n \subset S^{n+1}(1) \subset \mathbf{R}^{n+2}$ be an oriented hypersurface immersed in the unit standard sphere. Denote by $\mathbf{x}$ its position vector of points p of N^n and by $\mathbf{n}$ the unit normal vector field of N^n in $S^{n+1}(1)$. Its "Gauss map" is defined as

$$\mathcal{G} : N^n \ni p \longmapsto \mathbf{x}(p) \wedge \mathbf{n}(p) \cong [\mathbf{x}(p) + \sqrt{-1}\mathbf{n}(p)] \in \widetilde{Gr}_2(\mathbf{R}^{n+2}) \cong Q_n(\mathbf{C}).$$

Then we know that $\mathcal{G} : N^n \longrightarrow Q_n(\mathbf{C})$ is always a Lagrangian immersion. Moreover we can observe that a small deformation of N^n corresponds to a small Hamiltonian deformation of $\mathcal{G}$.

Suppose that $N^n \subset S^{n+1}(1)$ is a compact oriented embedded hypersurface with constant principal curvatures, i. e. "isoparametric hypersurface". By the Münzner's famous result [7], the number g of distinct principal curvatures must be $g = 1, 2, 3, 4, 6$ and their multiplicities $m_1 = m_3 = \cdots \leq m_2 = m_4 = \cdots$. By the mean curvature form formula of B. Palmer, we see that its Gauss map $\mathcal{G} : N^n \longrightarrow Q_n(\mathbf{C})$ is a *minimal* Lagrangian immersion. Moreover we can observe that the "Gauss image" of $\mathcal{G} : N^n \longrightarrow Q_n(\mathbf{C})$ is a compact embedded minimal Lagrangian submanifold $L^n = \mathcal{G}(N^n)(\cong N^n/\mathbf{Z}_g) \subset Q_n(\mathbf{C})$ and it is a compact monotone and cyclic embedded Lagrangian submanifold with minimal Maslov number $\Sigma_L = 2n/g = m_1 + m_2$.

We can show that N^n is homogeneous if and only if L^n is homogeneous ([5]). Due to Hsiang-Lawson [4] and Takagi-Takahashi [14], we know that all *homogeneous* isoparametric hypersurfaces $N^n \subset S^{n+1}(1)$ can be obtained as principal orbits of compact Riemannian symmetric pairs (U, K) of rank 2.

g	Type	(U, K)	$\dim N$	m_1, m_2	K/K_0
1	$S^1\times$ BDII	$(S^1 \times SO(n+2), SO(n+1))$ $(n \geq 1)$	n	n	S^n
2	BDII $\times$BDII	$(SO(p+2) \times SO(n+2-p),$ $SO(p+1) \times SO(n+1-p))$ $(1 \leq p \leq n-1)$	n	$p, n-p$	$S^p \times S^{n-p}$
3	AI_2	$(SU(3), SO(3))$	3	$1, 1$	$\frac{SO(3)}{\mathbf{Z}_2+\mathbf{Z}_2}$
3	$\mathfrak{a}_2$	$(SU(3) \times SU(3), SU(3))$	6	$2, 2$	$\frac{SU(3)}{T^2}$
3	AII_2	$(SU(6), Sp(3))$	12	$4, 4$	$\frac{Sp(3)}{Sp(1)^3}$
3	EIV	(E_6, F_4)	24	$8, 8$	$\frac{F_4}{Spin(8)}$
4	$\mathfrak{b}_2$	$(SO(5) \times SO(5), SO(5))$	8	$2, 2$	$\frac{SO(5)}{T^2}$
4	AIII_2	$(SU(m+2), S(U(2) \times U(m)))$ $(m \geq 2)$	$4m-2$	$2,$ $2m-3$	$\frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(m-2))}$
4	BDI_2	$(SO(m+2), SO(2) \times SO(m))$ $(m \geq 3)$	$2m-2$	$1,$ $m-2$	$\frac{SO(2)\times SO(m)}{\mathbf{Z}_2\times SO(m-2)}$
4	CII_2	$(Sp(m+2), Sp(2) \times Sp(m))$ $(m \geq 2)$	$8m-2$	$4,$ $4m-5$	$\frac{Sp(2)\times Sp(m)}{Sp(1)^2\times Sp(m-2)}$
4	DIII_2	$(SO(10), U(5))$	18	$4, 5$	$\frac{U(5)}{SU(2)\times SU(2)\times T^1}$
4	EIII	$(E_6, U(1) \cdot Spin(10))$	30	$6, 9$	$\frac{U(1)\cdot Spin(10)}{S^1\cdot SU(4)}$
6	$\mathfrak{g}_2$	$(G_2 \times G_2, G_2)$	12	$2, 2$	$\frac{G_2}{T^2}$
6	G	$(G_2, SO(4))$	6	$1, 1$	$\frac{SO(4)}{\mathbf{Z}_2+\mathbf{Z}_2}$

By using these results in isoparametric hypersurface theory, we can classify all compact homogeneous Lagrangian submanifolds in complex hyperquadrics ([5]).

Let N^n be a compact isoparametric hypersurface embedded in $S^{n+1}(1)$. Palmer [13] showed that its Gauss map $\mathcal{G} : N^n \longrightarrow Q_n(\mathbf{C})$ is Hamiltonian stable if and only if $N^n = S^n \subset S^{n+1}(1)$ $(g = 1)$.

Question. *Hamiltonian stability of its Gauss image $\mathcal{G}(N^n) \subset Q_n(\mathbf{C})$?*

$g = 1$: strictly Hamiltonian stable, more strongly, stable minimal and homologically volume minimizing. In fact, it is a calibrated submanifold of $Q_n(\mathbf{C})$.

$g = 2$: $N^n = S^{m_1} \times S^{m_2}$ Clifford hypersurface $(n = m_1 + m_2, 1 \leq m_1 \leq m_2)$ and $\mathcal{G}(N^n) = Q_{m_1+1,m_2+1}(\mathbf{R}) = (S^{m_1} \times S^{m_2})/\mathbf{Z}_2 \subset Q_n(\mathbf{C})$. (1) If $m_2 - m_1 \geq 3$, then $\mathcal{G}(N^n) \subset Q_n(\mathbf{C})$ is NOT H-stable. (2) If $m_2 - m_1 = 2$, then it is H-stable but not strictly H-stable (not Hamiltonian rigid). If $m_2 - m_1 = 0$ or 1, then it is strictly H-stable.

$g = 3$: $L = \mathcal{G}(N^n) \subset Q_n(\mathbf{C})$ is strictly H-stable ([5]).

We have already reported partial results in cases of $g = 4, 6$ and homogeneous N^n in [10], [11]. Recently we have obtained the final result on Hamiltonian stability of the Gauss images of all homogeneous isoparametric hypersurfaces as follows :

Theorem ([6])**.** *Suppose that (U, K) is not of type* EIII*. Then (1) $L = \mathcal{G}(N)$ is not H-stable if and only if $m_2 - m_1 \geq 3$, (2) $L = \mathcal{G}(N)$ is H-stable but not strictly H-stable (not Hamiltonian rigid) if and only if $m_2 - m_1 = 2$, (3) $L = \mathcal{G}(N)$ is strictly H-stable if and only if $m_2 - m_1 < 2$. Moreover if (U, K) is of type* EIII*, then $m_2 - m_1 = 9 - 6 = 3$ but $L = \mathcal{G}(N)$ is strictly H-stable.*

It is a further problem to investigate the Hamiltonian stability and other properties of the Gauss images of compact non-homogeneous isoparametric hypersurfaces (OT-FKM type).

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Positive harmonic functions with 0 Dirichlet and constant Neumann data

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(joint work with L. Hauswirth and F. Hélein)

Given (M, g), a m-dimensional Riemannian manifold, and Ω, a smooth bounded domain in M, we denote by $\lambda_1(\Omega)$ the first eigenvalue of the Laplace-Beltrami operator under 0 Dirichlet boundary condition. Critical points of the functional $\Omega \longmapsto \lambda_1(\Omega)$ under the volume constraint $\mathrm{Vol}(\Omega) = \alpha$ (where $\alpha \in (0, \mathrm{Vol}(M))$ is fixed) are called *extremal domains*. Smooth *extremal domains* are characterized by the property that the eigenfunctions associated to the first eigenvalue of the Laplace-Beltrami operator have constant Neumann boundary data [1]. In other words, a smooth domain is extremal if and only if there exists a positive function u_1 and a constant λ_1 such that

$$\Delta_g u_1 + \lambda_1 \, u_1 = 0 \, ,$$

in Ω with $u_1 = 0$ and $\nabla_n u_1 = \text{constant}$ on $\partial\Omega$, where n denotes the inward unit normal vector to $\partial\Omega$. The theory of *extremal domains* is very reminiscent of the theory of constant mean curvature surfaces or hypersurfaces. To give some credit to this assertion, let us recall that, in the early 1970's, J. Serrin has proved that the only compact, smooth, extremal domains in Euclidean space are round balls [4], paralleling the well known result of Alexandrov asserting that round spheres are the only (embedded) compact constant mean curvature hypersurfaces in Euclidean space. More recently, F. Pacard and P. Sicbaldi have proved the existence of extremal domains close to small geodesic balls centered at critical points of the scalar curvature function [3], paralleling an earlier result of R. Ye which provides constant mean curvature topological spheres (with high mean curvature) close to small geodesic spheres centered at nondegenerate critical points of the scalar curvature function [5]. We propose the following :

Definition 1. *A smooth domain $\Omega \subset \mathbb{R}^m$ is said to be an* exceptional domain *if it supports positive harmonic functions having* 0 *Dirichlet boundary data and constant (nonzero) Neumann boundary data. Any such harmonic function is called a* roof *function.*

Exceptional domains arise as limits under scaling of sequences of *extremal domains* in the same way minimal surfaces arise as limits under scaling of sequences of constant mean curvature surfaces. More generally, we propose the :

Definition 2. *A m-dimensional flat Riemannian manifold M is said to be* exceptional *if it supports positive harmonic functions having* 0 *Dirichlet boundary data and constant (nonzero) Neumann boundary data. Any such harmonic function is called a* roof *function.*

We report here some recent results on the study of *exceptional* flat surfaces. Detailed statements and proofs can be found in [2].

1. Examples of *exceptional domain* in $\mathbb{R}^2$

The following are obviousl examples of *exceptional domains* in $\mathbb{R}^m$: The half space $\{x = (x_1, \ldots, x_m) \in \mathbb{R}^m \, : \, x_1 > 0\}$ is an *exceptional domain* in $\mathbb{R}^m$ with *roof* function $u(x) = x_1$. The complement of a ball of radius 1 in $\mathbb{R}^m$ is an *exceptional domain* with *roof* function u defined by $u(x) := \log |x|$, when $m = 2$ and $u(x) := 1-|x|^{2-m}$, when $m \geq 3$. The product $\Omega \times \mathbb{R}^k$ is an *exceptional domain* in $\mathbb{R}^m$ provided $\Omega \subset \mathbb{R}^{m-k}$ is an *exceptional domain* in $\mathbb{R}^{m-k}$. Finally, bserve that the property of being an *exceptional domain* is preserved under the action of the group of similarities of $\mathbb{R}^m$.

In dimension $m = 2$, there exists (up to a similarity) at least another *exceptional domain.* To describe this domain, it is be convenient to identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. We claim that the domain

$$\Omega := \left\{ w \in \mathbb{C} \, : \, |\,\mathrm{Im}\ w| < \tfrac{\pi}{2} + \cosh(\mathrm{Re}\ w) \right\} , \tag{1}$$

is an *exceptional domain.* To prove the claim, first observe that $F(z) := z + \sinh z$ is a conformal diffeomorphism from the infinite strip $S := \left\{ z \in \mathbb{C} \, : \, \mathrm{Im}\ z \in (-\frac{\pi}{2}, \frac{\pi}{2}) \right\}$ into Ω, next, one checks that the real valued function u defined on Ω by the identity $u(F(z)) = \mathrm{Re}\ \cosh z$ is harmonic and positive in Ω, vanishes and a direct computation shows that it has constant Neumann boundary data on $\partial\Omega$.

We suspect that this example generalises to any dimension $m \geq 3$, namely that there exists *exceptional* domains in $\mathbb{R}^m$ which are invariant under the action of rotations about a given axis, for all $m \geq 3$. In dimension $m = 2$, it is tempting to conjecture that (up to similarity) the only *exceptional domains* are the half spaces, the complement of a ball and the above example.

2. A representation formula for exceptional flat surfaces

Let M be a simply connected *exceptional flat surface* with smooth (nonempty) boundary ∂M. We assume that $F : (M, g) \longrightarrow (\mathbb{C}, g_{\mathbb{C}})$ (where $g_{\mathbb{C}}$ is the canonical Euclidean metric on $\mathbb{C}$) is an holomorphic, orientation preserving isometric immersion which induces a smooth immersion of ∂M. In particular $\|dF\|_g = 1$ in $M \cup \partial M$. We define the holomorphic $(1,0)$-form

$$\Phi := dF = \partial_z F \, dz \,,$$

Observe that Φ does not vanish and admits a smooth extension to $M \cup \partial M$.

We let $u : M \longrightarrow \mathbb{R}^+$ be a *roof* function on M which is normalized so that $\|\nabla u\|_g = 1$ on ∂M. We consider the harmonic conjugate function $v : M \longrightarrow \mathbb{R}$ (which is uniquely defined up to some additive constant) which is the solution of $\partial_z (u - i\, v) = 0$. And we set

$$U := u + i\, v \,.$$

Observe that U is a holomorphic function from M into $\mathbb{C}$. The property that u takes positive values in M and vanishes on ∂M can be translated into the fact that U maps M to

$$\mathbb{C}^+ := \{ w \in \mathbb{C} \, : \, \mathrm{Re}\; w > 0 \} \,,$$

and ∂M to $i\,\mathbb{R}$. Since $\Phi \neq 0$ on $\tilde{M}$ there exists a unique holomorphic function h on $\tilde{M}$ such that

$$dU = \partial_z U \, dz = h \, \Phi \,.$$

We deduce from the fact that u vanishes on $\partial \tilde{M}$ and has constant Neumann data normalized to be equal to -1 that $\|\partial_z U\|_g = 1$ on $\partial \tilde{M}$. In particular $\|\Phi\|_g = \|dF\|_g = 1 = \|dU\|_g$ on $\partial \tilde{M}$. Clearly, this is equivalent to the fact that $|h| = 1$ on ∂M. Therefore, we end up with the following data : (i) An oriented simply connected complex surface $\tilde{M}$ with smooth boundary ∂M. (ii) A holomorphic function U, defined on $\tilde{M}$, which takes values in $\mathbb{C}^+$ and which maps ∂M into $i\,\mathbb{R}$. (iii) A holomorphic function h, defined on $\tilde{M}$, such that $|h| = 1$ on ∂M and for which the 1-form Φ defined by $\Phi := \frac{1}{h}\, dU$ does not vanish on $\tilde{M}$.

Conversely, given a set of such data, we can define the map $F : M \longrightarrow \mathbb{C}$ by integrating $dF = \Phi$. Thanks to (iii), this map is an immersion and its image is an immersed *exceptional flat surface* with *roof* function given by $u = \mathrm{Re}\; U$.

For example, given an integer $n \in \mathbb{N} \setminus \{0\}$ and choosing the Riemann surface to be $D = \{ z \in \mathbb{C} \, : \, |z| < 1 \}$, we define on D the holomorphic functions

$$h(z) = z^{n-1} \qquad \text{and} \qquad U(z) := \frac{1 + z^n}{1 - z^n} \,.$$

Then, the 1-form Φ is given by

$$\Phi(z) := \frac{2n}{(1 - z^n)^2} \, dz \,.$$

Observe that both U and Φ have singularities at the n-th roots of unity. The function F is then obtained by integrating Φ and the *roof* function u is then defined by $u = \mathrm{Re}\, U$.

When $n = 1$, we can take

$$F(z) = \frac{1 + z}{1 - z} \,.$$

In this case, we simply have $F(D) = \mathbb{C}^+$ and we recover the fact that the half plane is an *exceptional domain*.

When $n = 2$, we can take

$$F(z) = \frac{2z}{1-z^2} + \log\left(\frac{z+1}{z-1}\right).$$

and the *exceptional flat surface* we find can be isometrically embedded in $\mathbb{C}$. In fact, $F(D)$ corresponds (up to some similarity) to the domain Ω which has been defined in (1).

Finally, when $n \geq 3$ the *exceptional flat surface* we find cannot be isometrically embedded in $\mathbb{C}$ anymore.

In [2] we obtain a general representation formula for *exceptional flat surfaces* whose immersion in $\mathbb{C}$ which have finitely many *regular ends* and are locally finite coverings of $\mathbb{C}$ (we refer to [2] for precise definitions). In particular we show that the representation of these *exceptional flat surfaces* is of the form of the above examples.

Finally, we prove the following Bernstein type result for 2-dimensional exceptional domains :

Proposition 1. [2] *Assume that Ω is a 2-dimensional* exceptional domain *which is conformal to $\mathbb{C}^+$ and let u be a* roof *function on Ω. We further assume that $\partial_x u > 0$ in Ω, then Ω is a half plane.*

As a Corollary, we can prove that there is no *exceptional domain* contained in a wedge

$$\Omega \subset \{z \in \mathbb{C} \,:\, \mathrm{Re}\, z \geq \kappa\, |\, \mathrm{Im}\, z|\},$$

for some $\kappa > 0$ [2].

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Description of constant mean curvature k-noids with Delaunay ends

Franz Pedit

(joint work with Josef Dorfmeister and Nicholas Schmitt)

The study of the global properties of constant mean curvature (CMC) surfaces in $\mathbb{R}^3$ began in the mid 20th century with two results of a rather different flavor. Using the fact that on a compact surface of genus zero there are no nontrivial holomorphic differentials, Hopf showed that any compact genus zero *immersed*

CMC surface had to be a round sphere (and thus embedded). On the other hand, from the maximums principle for the Laplacian Alexandrov deduced that any compact *embedded* CMC surface (of arbitrary genus) is reflection symmetric with respect to planes dissecting it, and thus had to be a round sphere. By now there is a rather complete description of all (necessarily immersed) CMC tori in terms of integrable systems [9], [1]. Using nonlinear elliptic analysis, we also have existence of (necessarily immersed) compact CMC surfaces of any genus [6], but one is far from understanding the full moduli space. For the study of complete non-compact surfaces the behavior of ends is crucial. It has been shown [7] that a properly embedded CMC annulus is smoothly asymptotic to a Delaunay unduloid (and thus of the conformal type of a punctured disk).

This suggests that a natural class of surfaces to consider are complete immersed CMC surfaces of genus g with k ends asymptotic to Delaunay surfaces (unduloids and immersed nodoids). We call this space $\mathcal{M}_{g,k}$ and the aim is to give a description of this space. Hopf's result states that $\mathcal{M}_{0,0}$ consists of the round spheres and from the balancing formula for the ends we know that $\mathcal{M}_{g,1}$ is empty. For the subset $\mathcal{M}^*_{g,k}$ of embedded surfaces Alexandrov's result says that $\mathcal{M}^*_{g,0}$ is empty for $g \geq 1$. There is a complete description [3] of $\mathcal{M}^*_{0,3}$ in terms of triangles on the 2-sphere (and, more generally, in terms of k-gons [4] for the coplanar surfaces in $\mathcal{M}^*_{0,k}$). As in the compact case, there are constructions of examples in $\mathcal{M}^*_{g,k}$ by gluing methods [8], [5].

This note is concerned with the space $\mathcal{M}_{0,k}$ when $k \geq 2$. We will use the description of CMC surfaces in terms of loop group valued holomorphic connections (their Weierstrass data) and characterize those connections for $\mathcal{M}_{0,k}$. For a subset $B \subset \mathbb{C}$ we denote by $\mathbf{Sl}_2(B)$ the (loop) group of real analytic maps $g\colon B \to \mathbf{Sl}_2(\mathbb{C})$. If B is symmetric with respect to $\lambda \mapsto 1/\bar{\lambda}$, we let $\mathbf{SU}_2(B)$ be the subgroup for which g is special unitary along the unit circle $S^1 \subset \mathbb{C}$. Let $f\colon M \to \mathbb{R}^3$ be a surface in $\mathcal{M}_{0,k}$ then $M = S^2 \setminus \{p_1, \dots, p_k\}$ and it is known [2] that there is a holomorphic $\mathbf{Sl}_2(O^\times)$-connection

$$d + \xi\,, \qquad \xi = \lambda^{-1}\xi_{-1} + \xi_0 + \lambda\xi_1 + \dots$$

with holomorphic 1-forms $\xi_i \in H^0(K)$ on M which can be unitarized by an $\mathbf{Sl}_2(O)$-gauge to an $\mathbf{SU}_2(\mathbb{C}^\times)$-connection. Here $O^\times \subset \mathbb{C}$ is a small punctured disk around the origin. The $\mathbf{SU}_2(\mathbb{C}^\times)$-connection ∇^λ describes the associated family of CMC immersions to f and the original surface is obtained at $\lambda = 1$. As an example, the Delaunay surfaces in $\mathcal{M}_{0,2}$ are described by $M = S^2 \setminus \{0, \infty\}$ and

$$\xi_D = \begin{pmatrix} 0 & a\lambda + b\lambda^{-1} \\ a\lambda^{-1} + b\lambda & 0 \end{pmatrix} \frac{dz}{z}\,.$$

Here $a + b = 1/2$ and the neck size radius ν satisfies $4ab = \nu(1-\nu)$. The cylinder has $\nu = 1/2$, the embedded unduloids have $0 < \nu < 1/2$ and the immersed nodoids have $\nu < 0$ (where the mean curvature is assumed to be 1). In this example the holomorphic connection $d + \xi_D$ extends meromorphically with simple poles to S^2.

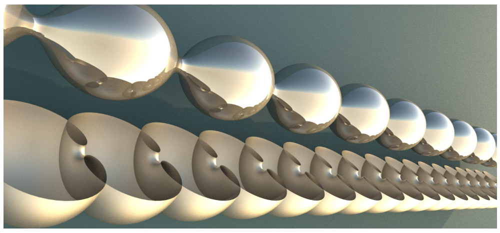

Figure 1. Delaunay unduloid and nodoid.

Even though the CMC equation is nonlinear, we expect superposition in the linearized picture of holomorphic Weierstrass data, that is, the connections describing the surfaces in $\mathcal{M}_{0,k}$ should be sums of their asymptotic Delaunays ξ_{D_i}.

Theorem. *Let $M = S^2 \setminus \{p_1, \ldots, p_k\}$ and let $f\colon M \to \mathbb{R}^3$ be a CMC surface in $\mathcal{M}_{0,k}$ with asymptotic Delaunay ends $\xi_{D_1}, \ldots, \xi_{D_k}$. Then f is described by a unitarizable Fuchsian connection $d + \xi$ where*

$$\xi = \sum_{i=1}^{k} \frac{A_i(\lambda)}{z - p_i} dz + \sum_{j=1}^{k-2} \frac{B_j(\lambda)}{z - q_j} dz \, .$$

The residues A_i of the geometric poles p_i have the same eigenvalues than the asymptotic Delaunay ends D_i which are determined by their asymptotic necksizes. The residues B_j at the (movable) apparent poles have eigenvalues $\pm 1/2$ (apparent means, that the pole can locally be gauged away by an $\mathbf{Sl}_2(O)$-gauge). The connection contains $2k - 6$ accessory parameters over the ring $\mathbb{C}(O)$ of holomorphic functions near the origin.

When passing from the connection to the CMC surface there is a choice of initial condition (responsible for the *dressing action*) which in our setting is an element of $\mathbf{Sl}_2(O)$. It can be shown that any such dressing on the above connections is rational, that is, a finite product of simple factor dressings. On the CMC surface this has the effect of *adding Bubbletons*. This leaves us with the following picture:

(1) $\mathcal{M}_{0,2}$ consists of the Delaunay surfaces with finitely many Bubbletons. This extends the description of $\mathcal{M}^*_{0,2}$ in [7].
(2) $\mathcal{M}_{0,3}$ consist of the 3-noids with finitely many Bubbletons added. There are no accessory parameters and the unitarizability of the Fuchsian connections is characterized by the spherical triangle inequalities on the asymptotic neck sizes. This gives a complete description of the space $\mathcal{M}_{0,3}$ extending the description of $\mathcal{M}^*_{0,3}$ in [3].
(3) For $k \geq 4$ the unitarizability condition of the Fuchsian connections is known only in special cases (e.g., platonic symmetric k-noids [10]).

The basic idea of the proof is very much in the spirit of the Riemann-Hilbert problem for loop group valued monodromies. We start with a holomorphic connection $d + \xi$ on M describing the CMC surface f. The assumption of Delaunay

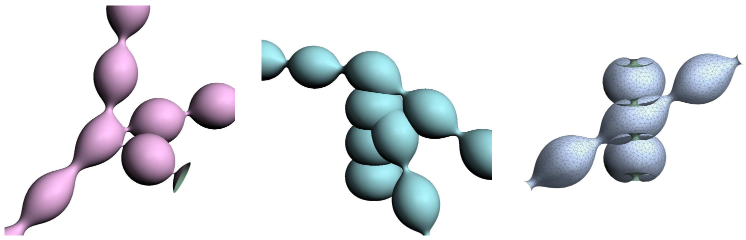

FIGURE 2. CMC 4-noids.

ends implies that the monodromy of this connection around the pole p_i has the same eigenvalues than the asymptotic Delaunay connection $d + \xi_{D_i}$. This implies that there is a local $\mathbf{Sl}_2(O)$-gauge g on a punctured disk around p_i of $d + \xi$ to a connection $d + \frac{D_i}{z - p_i} dz + O(z - p_i)$. We factorize this gauge into $g = g_+ w g_-$, with the $\mathbf{Sl}_2(O)$-valued map g_+ extending holomorphically into the puncture p_i, the map g_- extending holomorphically to the complement of the disk around p_i and w a rational map (in z) of specific form into $\mathbf{Sl}_2(O)$. This provides us with a new connection $d + \xi$ which has simple poles at the p_i and a pole of order $k - 2$ at ∞ accumulated from the rational maps w. Finally we separate the order $k - 2$ pole at ∞ to the simple poles at q_j. By construction the residues A_i at the simple poles p_i have the same eigenvalues than D_i. Since ∞ is not an end of the CMC surface f the poles at q_j are apparent with residue eigenvalues $\pm 1/2$.

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Elastic strips – Does nature do cut and paste?

Ulrich Pinkall
(joint work with David Chubelaschvili)

Let $\gamma : [0, L] \to \mathbb{R}^3$ be an arclength parametrized curve with Frenet frame T, N, B. Assume that there is a smooth function $\lambda : [0, L] \to \mathbb{R}$ such that the curvature and torsion of γ satisfy

$$\tau = \lambda\kappa. \tag{1}$$

Then $f : [0, L] \times [-\epsilon, \epsilon] \to \mathbb{R}^3$ defined by

$$f(s, t) = \gamma(s) + t(B(s) + \lambda T(s)) \tag{2}$$

parametrizes a developable strip of width 2ϵ. Here $\lambda = \tan\alpha$ if α denotes the angle between the curve and the normal to the rulings of the ruled surface f. The bending energy

$$E = \int H^2 dA \tag{3}$$

can be expressed as

$$E = \int_0^L \kappa^2(1 + \lambda^2)^2 \, \frac{\log(1 + \epsilon\lambda') - \log(1 - \epsilon\lambda')}{\lambda'} ds. \tag{4}$$

Starostin and van der Heijden computed in 2007 the Euler-Lagrange equations using computer algebra and carried out a numerical study of energy-minimizing Moebius bands. In the limit $\epsilon \to 0$ they discovered that the curvature κ jumps from +1 to -1 at the inflection point and the torsion τ is continuous with value 1. Thus the angle between curve and rulings jumps from 45° to -45° and it seems that energy minimizing Moebius bands are only C^1. In the limit $\epsilon \to 0$ the energy becomes

$$E = \int_0^L \kappa^2(1 + \lambda^2)^2 ds. \tag{5}$$

The Euler-Lagrange equations for E were computed in 2005 by Hagan and subsequently corrected by Rominger and Chubelaschwili. Since E has to be minimized among strips with fixed length they contain a Langrange multiplier μ:

$$\begin{aligned}
0 &= (\kappa'(1 + \lambda^2)^2 + 2\kappa(1 + \lambda^2)\lambda\lambda')' + \frac{\kappa}{2}(\kappa^2(1 + \lambda^2)^2 - \mu) \\
&+ \lambda\kappa(\kappa^2(1 + \lambda^2)^2\lambda + (\frac{\kappa'}{\kappa}(1 + \lambda^2)2\lambda))' + (1 + \lambda^2)2\lambda)'') \\
0 &= (\kappa^2(1 + \lambda^2)^2\lambda + (\frac{\kappa'}{\kappa}(1 + \lambda^2)2\lambda)' + ((1 + \lambda^2)2\lambda)'')' \\
&+ \kappa\lambda(\kappa'(1 + \lambda^2)^2 + 2\kappa(1 + \lambda^2)\lambda\lambda').
\end{aligned} \tag{6}$$

A more elegant formulation relies on conservation laws:

Theorem 1. *For a curve γ with Frenet frame* **T**,**N**,**B** *define*

$$
\begin{aligned}
\mathbf{b} &= \frac{1}{2}(\kappa^2(1+\lambda^2)^2+\mu)\,\mathbf{T} + (\kappa'(1+\lambda^2)^2+2\kappa(1+\lambda^2)\lambda\lambda')\,\mathbf{N} \\
&- (\kappa^2(1+\lambda^2)^2\lambda + (\frac{\kappa'}{\kappa}(1+\lambda^2)2\lambda)' + ((1+\lambda^2)2\lambda)'')\,\mathbf{B} \qquad (7)\\
\mathbf{a} &= 2\,\kappa\,\lambda\,(1+\lambda^2)\,\mathbf{T} + \frac{1}{\kappa}\,(2\,\kappa\,\lambda\,(1+\lambda^2))'\,\mathbf{N} \\
&+ \kappa\,(1+\lambda^2)\,(1-\lambda^2)\,\mathbf{B} - \mathbf{b}\times\gamma.
\end{aligned}
$$

Then the strip defined by γ is elastic if and only if the force vector **b** *is constant. Moreover, for any elastic strip the* torque vector **a** *is constant.*

The force **b** and the torque **a** have to be applied to the end point of the strip to keep it in equilibrium. An elastic strip is called *force-free* if the force vector **b** vanishes. For force-free elastic strips the bending energy is critical even if the end point of γ is allowed to move, only the frame at the end point is held fixed. Since we do not impose any condition on end point of γ we essentially have a variational problem for the tangent image **T** as a parametrized spherical curve. For a force-free elastic strip the Lagrange multiplier μ is positive and we can normalize it to 1 by scaling the strip. We are looking then for critical points of the energy

$$\tilde{E} = 1/2\int_0^L (\kappa^2(1+\lambda^2)^2+1)ds. \qquad (8)$$

For a force-free elastic strip κ does not vanish and the tangent image **T** is therefore a regular curve in S^2 with curvature λ:

$$
\begin{aligned}
\mathbf{T}' &= \quad\; +\kappa\,\mathbf{N} \\
\mathbf{N}' &= -\kappa\mathbf{T} \qquad\quad + \kappa\,\lambda\,\mathbf{B} \qquad (9)\\
\mathbf{T}' &= \quad\; -\kappa\,\lambda\,\mathbf{N}
\end{aligned}
$$

The strip γ can be reconstructed using an arclength parameter $\tilde{s}$ of **T** as

$$\gamma(\tilde{s}) = \int_0^{\tilde{s}} \frac{1}{\kappa}\,\mathbf{T}\,d\tilde{s}. \qquad (10)$$

We can then formulate a variational problem for the tangent image:

Theorem 2. *Let* $\mathbf{T}\colon [0,\tilde{L}] \to S^2$ *be an arclength parametrized curve with curvature λ. Then among all strips $\gamma : [0,\tilde{L}] \to \mathbb{R}^3$ with tangent image* **T** *the one given by*

$$\gamma(\tilde{s}) = \int_0^{\tilde{s}} (1+\lambda^2)\,\mathbf{T}\,d\tilde{s}$$

has minimal Sadowski functional

$$\tilde{E} = \int_0^{\tilde{L}} (1+\lambda^2)\,d\tilde{s}.$$

γ has curvature

$$\kappa = 1/(1+\lambda)^2. \tag{11}$$

Corollary 1. *The tangent images of force-free elastic strips are elastic curves in S^2, in fact critical points of*

$$\int_0^{\tilde{L}} (1+\lambda^2)\, d\tilde{s}. \tag{12}$$

Conversely, for any such spherical curve $\mathbf{T}$ the space curve

$$\gamma = \int (1+\lambda^2)\, \mathbf{T}\, d\tilde{s} \tag{13}$$

defines a force-free elastic strip.

All possible adapted frames (lifted to S^3) define the *frame cylinder*

$$F : [0, L] \times S^1 \to S^3 \tag{14}$$

of a space curve γ. F is the preimage of the tangent image $\mathbf{T}$ under the Hopf map $S^3 \to S^2$.

Corollary 2. *The frame cylinder of a force-free elastic strip is Willmore in S^3.*

Concerning the surprising non-smoothness of some energy minimizers we have the following result:

Theorem 3. *At any point $\gamma(s)$ of an elastic strip the following are equivalent:*

(1) *The rulings make an angle of $45°$ with γ.*
(2) *The curvature of the tangent image satisfies $\lambda(s) = 1$.*
(3) *A gluing construction is possible where the curvature κ is discontinuous but nevertheless we still have an elastic strip in the sense of balanced force* $\mathbf{b}$ *and torque* $\mathbf{a}$.

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Stable area-stationary surfaces in the sub-Riemannian Heisenberg group $\mathbb{H}^1$

Manuel Ritoré
(joint work with Ana Hurtado and César Rosales)

The sub-Riemannian Heisenberg group $\mathbb{H}^1$ with its Carnot-Carathéodory distance is a Hausdorff-Gromov limit of Riemannian Nil_3 manifolds. Sequences of Riemannian minimal surfaces may converge to critical points of the sub-Riemannian area in $\mathbb{H}^1$, thus providing a motivation to consider this variational problem, see [2].

Area-stationary surfaces of class C^2 in $\mathbb{H}^1$ are well understood. It is known [4], [17], that, outside the singular set of points where the tangent plane is horizontal, such surfaces are ruled by characteristic horizontal segments. Using the description of the singular set given by Cheng, Hwang, Malchiodi and Yang [4], and a general first variation formula of the area which allows to move the singular set [16], Ritoré and Rosales [17] proved that a C^2 surface Σ immersed in $\mathbb{H}^1$ is area-stationary if and only if its mean curvature is zero and the characteristic segments in Σ meet orthogonally the singular curves (when they exist). A similar result was independently obtained for area-minimizing t-graphs (Euclidean graphs over $t = 0$) by Cheng, Hwang, and Yang [5]. The classification of C^2 complete, connected, orientable, area-stationary surfaces with non-empty singular set was achieved in [17]: the only examples are, up to congruence, non-vertical Euclidean planes, the hyperbolic paraboloid $t = xy$, and the classical left-handed minimal helicoids in $\mathbb{R}^3$. Though some results for complete area-stationary surfaces with empty singular set have been obtained, [16, Thm. 5.4], [3], [17, Prop. 6.16], a detailed description seems far from being established. This provides a motivation to classify the second order minima of the area in $\mathbb{H}^1$.

As in the Euclidean case, we define a *stable area-stationary* surface in $\mathbb{H}^1$ as a C^2 area-stationary surface with non-negative second derivative of the area under compactly supported variations. These surfaces have been considered in connection with some Bernstein type problems in $\mathbb{H}^1$. A classification of C^2 entire solutions of the minimal surface equation for t-graphs in $\mathbb{H}^1$ was given in [4]. This classification was used in [17] to show that the only complete area-stationary t-graphs are Euclidean non-vertical planes or those congruent to the hyperbolic paraboloid $t = xy$. All these surfaces are area-minimizing by a calibration argument [17]. In [1] and [7] the Bernstein problem for *intrinsic graphs* (Riemannian graphs over vertical planes [11]) in $\mathbb{H}^1$ was considered. Examples of C^2 complete area-stationary intrinsic graphs different from vertical Euclidean planes, which are not area-minimizing, were found in [7]. So a natural question is to study complete area-minimizing intrinsic graphs. Barone, Serra Cassano and Vittone gave in [1] a classification of complete C^2 area-stationary intrinsic graphs. Then they computed the second variation formula of the area to conclude that the only stable ones are the Euclidean vertical planes, which are area-minimizing by a calibration argument. In [9], it was proven that C^2 complete stable area-stationary Euclidean graphs with empty singular set must be vertical planes.

The following natural step was to consider complete stable surfaces in $\mathbb{H}^1$. In this talk we shall give an overview of the proof of the recent result

Theorem ([12, Thm. 6.1]). *The only complete, connected, orientable, stable area-stationary surfaces in $\mathbb{H}^1$ of class C^2 are the Euclidean planes and the surfaces congruent to the hyperbolic paraboloid $t = xy$.*

This result provides a classification of complete C^2 orientable area-minimizing surfaces in $\mathbb{H}^1$. In a related paper [8], Danielli, Garofalo, Nhieu and Pauls have proven that the only complete, embedded, connected, stable area-stationary surfaces of class C^2 without singular points are vertical planes.

The main tool in the proof of this result is a general second variation formula of the sub-Riemannian area that allows *to move the singular set.* Previous formulae for variations *supported in the regular set* appeared in several contexts: in [4] for C^3 surfaces in a 3-dimensional pseudo-hermitian manifold; in [1], for intrinsic graphs of class C^2; in [6], for C^2 variations by Euclidean straight lines of a C^2 surface.

Once we have the second variation formula we proceed into two steps. First we prove that a C^2 complete, connected, oriented, stable area-stationary surface Σ with empty singular set must be a vertical plane [12, Thm. 4.7]. The second derivative of the area for a compactly supported normal variation uN is given by the index form $\mathcal{Q}(u) = -\int_\Sigma u\,\mathcal{L}(u)$, where $\mathcal{L}$ is a hypoelliptic operator on Σ. Then we choose the function $u = |N_h|$, where N is the Riemannian unit normal to Σ for the usual left invariant Riemannian metric g on $\mathbb{H}^1$, N_h is the horizontal projection of N. We see that this function u satisfies $\mathcal{L}(u) \geq 0$, and the inequality is strict in pieces of Σ which are not contained inside Euclidean vertical planes. Multiplying by a suitable cutoff function φ we get a compacty supported function so that $\mathcal{Q}(\varphi\,|N_h|) < 0$ still holds. Observe that the function $|N_h|$ is associated to the variation of Σ by parallel surfaces with respect to the Carnot-Carathéodory distance. Hence, our choice of test function is somewhat similar to the one in the Euclidean case, where the equivalent test function is obtained from $u \equiv 1$, [10]. It must be remarked that $\varphi\,|N_h|$ was already used as a test function in [1], [7] and [9].

In the second step of the proof we consider a surface Σ with non-empty singular set. From the classification in [17], we conclude that Σ must be either a non-vertical plane, or congruent to the hyperbolic paraboloid, or to a left-handed helicoid. Non-vertical planes and the hyperboic paraboloid are area-minimizing t-graphs by a calibration argument [17]. To prove the instability of the left-handed helicoids we use our general second variation formula. We must remark that variations with compact support in the regular set of the left-handed helicoids have non-negative second derivative of area.

Examples of area-minimizing surfaces in $\mathbb{H}^1$ with low Euclidean regularity have been obtained in [5], [14], [15] and [13]. Hence our result is optimal in the class of C^2 area-stationary surfaces.

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Noether theorem, conserved quantities, minimal and CMC surfaces

Pascal Romon

The *flux* (a.k.a. *force*) has been frequently used in minimal and constant mean curvature (CMC) surface theory for a long time, where it plays a key role in classification of ends, moduli space theory, gluing and as an all-purpose tool (see [6, 8, 10] for more specific examples). It is actually one of two known "conserved quantities", i.e. cohomology classes that can be defined on any minimal (or CMC) surface, the second being the *torque*. In $\mathbb{R}^3$ they are classically given for a minimal immersion f by

$$\mathrm{Flux}([\gamma]) = \int_\gamma \star \mathrm{d}f, \quad \mathrm{Torque}([\gamma]) = \int_\gamma f \times (\star \mathrm{d}f)$$

where $\star$ denotes the Hodge star operator. Their existence is a consequence of Noether theorem, which links infinitesimal symmetries to cohomology classes, applied here to ambient space isometries, e.g. translations and rotations in $\mathbb{R}^3$ respectively. We will explain here briefly how to formulate this theorem in any ambient three dimensional space, with homogeneous spaces in mind, but also how and why its use goes beyond simple isometries. This raises the question of understanding the nature of the Hopf or the Abresch–Rosenberg differentials. An interpretation will be given in the (simpler) case of the harmonic map PDE.

We will henceforth use the language of exterior differential geometry. Our goal is the study of a functional $\mathcal{F}$ defined by the integral of a lagrangian form Λ, which involves (some) first order derivatives. We define the manifold $\mathcal{C}$ of contact elements as the grassmannian of 2-planes over M, or equivalently the unit sphere bundle $\pi : UM \to M$. It is endowed with a contact form θ^0 defined thus: Let e_0 be an element of $\mathcal{C}$, i.e. a unit vector, and $v \in T\mathcal{C}$, then $\theta^0(v) = \langle e_0, \mathrm{d}\pi(v)\rangle$ where $\langle \cdot, \cdot \rangle$ denotes the metric on TM. We may as well work on the frame bundle FM as a bundle over UM. A point in FM will be denoted by the frame (e_0, e_1, e_2), and $(\theta^0, \theta^1, \theta^2)$ will be the associated (unit) coframe. Then the θ^0 on $\mathcal{C}$ lifts to the θ^0 on FM, so that we will abuse notations and denote them identically. For any immersion $f : \Sigma \to M$ denote by N the legendrian lift to $\mathcal{C}$ w.r.t. the contact structure θ^0. Then N is a legendrian lift if and only if N is a normal vector to the tangent bundle.

For the area functional define the following lagrangian $\Lambda = \theta^1 \wedge \theta^2$. For the CMC-$H$ surface equation, we need to add a Lagrange multiplier, and define the volume constraint by means of a volume vector field Ξ (i.e. such that $\operatorname{div} \Xi = 1$):

$$\Lambda = \theta^1 \wedge \theta^2 + 2H\Xi \lrcorner (\theta^0 \wedge \theta^1 \wedge \theta^2).$$

Then the Euler–Lagrange operator is the 2-form Ψ such that $\mathrm{d}\Lambda = \theta^0 \wedge \Psi$, and

$$\Psi = -\omega_0^2 \wedge \theta^1 - \omega_1^0 \wedge \theta^2 + 2H\theta^1 \wedge \theta^2,$$

where the ω_j^i are the structure 1-forms on the frame bundle. Check that Ψ vanishes on legendrian lifts if and only if the mean curvature is H.

Let v be a vector field on $\mathcal{C}$. It is called a *variational symmetry* if (i) v leaves the contact structure invariant: $\mathcal{L}_v\theta^0 \equiv 0$ and (ii) v preserves the lagrangian: $\mathcal{L}_v\Lambda \equiv 0$ ([1]). Preserving the lagrangian implies preserving the set of solutions to the Euler–Lagrange PDE, i.e. CMC-H surfaces. Then the Noether form is

$$\varphi = v \lrcorner \Lambda$$

and

$$\mathrm{d}\varphi = \mathcal{L}_v\Lambda - v \lrcorner \mathrm{d}\Lambda = \mathcal{L}_v\Lambda - \theta^0(v)\Psi + \theta^0 \wedge (v \lrcorner \Psi) \equiv 0$$

meaning that φ is closed whenever evaluated on a CMC-H surface. Noether theorem is actually much stronger. First, it can be stated with higher order derivatives (including infinite order). Second, under suitable hypotheses, it states

[1] All congruences are modulo the differential ideal generated by $\theta^0, \mathrm{d}\theta^0, \Psi$. These quantities will vanish when evaluated along a solution of the PDE.

a bijection between variational symmetries and closed one-forms, which is the last point of this talk. In the case at hand the Noether 1-form is

$$\varphi = \langle v, \star \mathrm{d}f - 2H\Xi \times \mathrm{d}f \rangle \,.$$

Obviously riemannian isometries of M can be lifted to $\mathcal{C}$ and they preserve the area. They will also preserve the volume part of the lagrangian, up to a divergence term (in which case they are called divergence symmetries). Translations (or their equivalent in homogeneous spaces) yield the flux form, while rotation(s) yield the torque. Explicit cases and computations will be given in [4].

The converse statement in Noether theorem raises an interesting question when considering the Hopf differential, or its generalization [1] by Abresch–Rosenberg to the $E(\kappa, \tau)$ homogeneous spaces. Indeed the Hopf differential is defined on a CMC-H surface in $\mathbb{R}^3$ as the $(2,0)$-part Q of the second fundamental form. In local conformal coordinates we may write $Q = q\mathrm{d}z^2$. Its major properties are that Q vanishes exactly at umbilic points and that Q is holomorphic (a consequence of the Codazzi equation when H is constant). Away from umbilic points (and up to sign), we can may take the square root $\varphi = \sqrt{Q} = \sqrt{q}\mathrm{d}z$. Holomorphicity of Q is equivalent to closedness of φ. Since φ is complex-valued we consider two conjugate 1-forms. Then there must exist two variational symmetries corresponding to the real and imaginary parts of φ. The same holds for the Abresch–Rosenberg form. Yet these symmetries cannot be simple ambient space isometries, since they depend on higher order derivatives. That requires a higher order contact (or more precisely multi-contact) setting. The question is then: *to which higher order variational symmetry do these 1-forms correspond to ?* An answer to this question would pave the way for a generalization of the Hopf–Abresch–Rosenberg approach, which in particular allows to classify spheres .

Let us finish with a much simpler example where the solution is known, though I haven't seen it written in these words: the harmonic map equation from a surface (we shall take $\mathbb{R}^2$ with a metric) to a riemannian manifold M^m. The setup will then be the manifold of 1-jets of maps $f : \mathbb{R}^2 \to M$ (see [3] for details).

Consider the following complex-valued quadratic form

$$Q = (f^*\mathrm{d}s^2)^{(2,0)} = \langle f_z, f_z \rangle \,\mathrm{d}z^2$$

sometimes also called the Hopf differential. Q is holomorphic whenever f is harmonic, and vanishes at points where f is conformal. Away from those points, Q admits a square root φ, whose real and imaginary parts φ_R, φ_I are closed 1-forms, which correspond to variational symmetries v_R, v_I.

The holomorphicity of Q has certainly been known for a long time, and so must be its link with Noether theorem. Indeed a conformal change of variable on the source surface preserves harmonicity so we can apply Noether theorem to conformal Killing fields. The corresponding conservation laws state simply the holomorphicity of Q (or some holomorphic function of Q). See [2, 9] for a first explicit (modern) mention of Noether theorem (at least in its ambient isometries version) and also [5] for a proof of $\bar{\partial}Q = 0$ using the stress-energy tensor, which is a reformulation of Noether theorem.

However, to understand exactly which two fields corresponds to Q itself, one needs to take appropriate coordinates, using the fact that Q is holomorphic. Indeed, away from conformal points, write $Q = \mathrm{d}w^2$, and by letting v act only on the source, we obtain $v_R = -\partial/\partial y$ and $v_I = \partial/\partial x$ where $w = x + iy$. In other words, the holomorphicity of Q says exactly that constant motion along the eigenlines of the stress-energy tensor are variational symmetries.

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Area minimizing surfaces in flat 3-manifolds

Antonio Ros

A properly embedded surface S in a complete Riemannian 3-manifold M is *area-minimizing mod* 2 if any compact region in S has least area among homologous (homology with $\mathbb{Z}_2$ coefficients) compact surfaces (orientable or nonorientable) with the same boundary. If M is closed, each nonzero homology class in $H_2(M, \mathbb{Z}_2)$ admits a compact area minimizing surface which is a smooth minimal surface without singularities, see [6].

Another important and related kind of minimal surfaces are the stable ones. These surfaces have nonnegative variation formula for every compactly supported infinitesimal deformation. Do Carmo and Peng [3], Fischer-Colbrie and Schoen [4] and Pogorelov [7] proved that complete two-sided stable minimal surfaces in $\mathbb{R}^3$ are planes. Ros [8] showed that the one-sided case cannot occur. If M is a flat 3-manifold, then complete stable two-sided surfaces are always flat but the behaviour of one-sided stable surfaces is quite different. For instance, certain nonorientable quotients of the classic Schwarz P and D periodic minimal surfaces are stable in

Nickname	$\mathcal{G}$	$H_2(M, \mathbb{Z}_2)$	Area minimizing conjecture
Torus	$P1$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	
Di	$P2_1$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	
Tri	$P3_1$	$\mathbb{Z}_2$	rPD
Tetra	$P4_1$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	tP tD CLP
Hexa	$P6_1$	$\mathbb{Z}_2$	H
Didi	$P2_12_12_1$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	

TABLE 1. Compact orientable flat 3-manifolds $M = \mathbb{R}^3/\mathcal{G}$. Space groups $\mathcal{G}$ and homology groups in the second and third column, respectively. The last column indicates conjectured nonplanar area minimizing surfaces (mod 2) in these spaces.

their ambient 3-tori, see Ross [10], and Ros [8] showed that nonflat stable compact minimal surfaces in 3-tori are Klein bottles with a handle.
The following is a basic and natural problem in classical minimal surface theory:

> Classify properly embedded area minimizing surfaces (mod 2) in complete flat 3-manifolds.

Nonflat area minimizing surfaces in flat 3-manifolds are necessarily one-sided. In [8], Ros obtained the following result for the simplest quotients of $\mathbb{R}^3$.

Theorem 1. *The Helicoid and the doubly periodic Scherk minimal surfaces of total curvature -2π are the only properly embedded nonflat area minimizing surfaces (mod 2) in quotients of $\mathbb{R}^3$ by one or two linearly independent translations, respectively.*

In these notes we will discuss some recent progresses about the above problem when the ambient space is compact and orientable. The first one is the following.

Theorem 2. [9] *Area minimizing surfaces (mod 2) in flat 3-tori are planar 2-tori.*

Note that from the result of Ross [10] there exist nonflat stable minimal surfaces in some 3-tori which are strict local minima of the area in their homology classes. One of the key steps in the proof that global minima are necessarily planar uses that if the moduli space of 3-tori which admit nonflat area minimizing surfaces were noncompact, then Theorem 1 allows us to control the limit surfaces of this moduli space. Now we introduce compact orientable flat 3-manifolds and some of the most significant classical periodic minimal surfaces.
Compact orientable flat 3-manifolds $M = \mathbb{R}^3/\mathcal{G}$ are listed in Table 1 following the *Cosm* notation, [2], in the first column, and the International Crystallographic notation for the space group $\mathcal{G}$, [1]:

In the 3-*torus*, the group $\mathcal{G} = P1$ is generated by three independent translations.

The *Di Cosm* is the quotient of $T^2 \times \mathbb{R}$, T^2 being a flat 2-tori, by the screw motion of order two $(x, y, z) \mapsto (-x, -y, z + a)$, $a > 0$.

The same holds for the *Tri, Tetra* and *Hexa Cosms*. These spaces are the quotient of $T^2 \times \mathbb{R}$ by a screw motion of vertical axis and order 3, 4 and 6, respectively. Note that in this case T^2 is either a square or an hexagonal 2-torus.

The *Didi Cosm* is the quotient of a general orthorhombic 3-torus $\mathbb{R}^3/P1$, where $P1$ is generated by $(a, 0, 0), (0, b, 0)$ and $(0, 0, c)$, $a, b, c > 0$, by the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ given by

$$(x, y, z),\ \ (-x+a/2, -y, z+c/2),\ \ (-x, y+b/2, -z+c/2),\ \ (x+a/2, -y+b/2, -z).$$

It has pairwise disjoint 2-fold screw axes parallel to the three coordinate axes.

Periodic minimal surfaces appearing in the last column of Table 1 where constructed by Schwarz [5] and are the natural candidates to nonflat area minimizing surfaces in compact orientable flat 3-manifolds. They are described in the following paragraphs.

1. *Catenoid-like surfaces.* These are the periodic minimal surfaces tP, H and rPD which give nonorientable minimal surfaces of total curvature -4π in the *Tetra, Hexa* and *Tri Cosms*, respectively. The *tetragonal P* surface, tP, is a deformation of the classical cubic Schwarz P minimal surface. It is generated by a Catenoidal piece spanned by two horizontal squares related by a vertical translation. If we change the squares by equilateral triangles we get the H surface in the *Hexa Cosm* and if we rotate one of these triangles by 60 degrees around the common axis one obtains the rPD minimal surfaces, a rhombohedral family of surfaces in the *Tri Cosm*, which for suitable values of the hight parameter produce the cubic P and D Schwarz minimal surfaces. All the above surfaces are homologous (mod 2) to the horizontal 2-torus in the corresponding *Cosm* and, if the height of the manifold is small enough, then the area of the Catenoid-like surface is smaller than the one of the horizontal 2-torus. We conjecture that, depending of the height parameter, either the horizontal planar surface or the Catenoid-like surface minimize the area in their homology classes.

2. *Discoidal surfaces.* In the *Tetra Cosm*, the two remaining nonzero homology classes do not admit embedded planar representatives and so the area minimizing surfaces in this case are nonflat. We conjecture that these area minimizing surfaces are CLP (Crossed layers of planes) and tD (tetragonal Diamond), which area nonorientable minimal surfaces of total curvature -2π constructed as follows: In a tetragonal box (with vertical 4-fold axis) consider the embedded hexagonal contours given by six edges of the box (two in the top and bottom faces and two opposite vertical edges). There are two possibilities according to whether the contour is centrally symmetric or it admits a horizontal mirror plane. The discoidal minimal surfaces obtained by solving the Plateau Problem for these contours generate the tD and CLP surfaces, respectively. We also have shown that

Theorem 3. [9] *Area minimizing surfaces (mod 2) in the Di and Didi Cosms are planar.*

An important ingredient in the proof of this theorem is the fact that there is a two sheeted covering from a 3-torus onto *Di* and that *Didi* admits three 2 : 1 coverings by the *Di Cosm*. For the other compact flat 3-manifolds the problem remains open.

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Transformations of surfaces and their applications to spectral theory

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(joint work with Sergey P. Tsarev)

Many transformations which maps solutions of certain nonlinear equations into solutions of equations of the same type were developed in the 19th century in the framework of surface theory. These are so-called Laplace, Moutard, and Bianchi–Backlund transformations. Although the latter transformation originates in surface theory two other transformations were introduced just for constructing explicit solutions of equations of certain types and in particular cases the Laplace transformation and the one-dimensional reduction of the Moutard transformation (so-called the Darboux transformation) can be found already in articles by Euler who applied them to analytical problems.

In particular, in late 1760s Euler showed that a general solution to the equation

$$u_{xy} = \frac{k(1-k)}{(x-y)^2} u$$

the form

$$u(x,y) = (x-y)^k \frac{\partial^{2k-2}}{\partial x^{k-1} \partial y^{k-1}} \left(\frac{f(x)+g(y)}{x-y} \right).$$

For $k = 1$ this formula reduces to the d'Alembert formula.

In his original article Moutard mentioned that by using his transformation one can rederive the Euler formula. Only later many applications of the Moutard transformation to surface theory were found by Bianchi, Demoulin, Guichard and others.

Moutard dealed with the equation

$$f_{xy} + U(x,y)f = 0$$

where U is a real-valued scalar potential. In surface theory this equation was named by him and it is known that every vector-valued solution $F(x,y)$ (with values in $\mathbb{R}^3$) to the Moutard equation gives rise to the Gauss map

$$n = \frac{F}{|F|}$$

of a negatively-curved surface in $\mathbb{R}^3$ with asymptotic coordinates x and y.

Let us expose the Moutard transformation in the form when x and y are complex conjugate parameters: $x = z, y = \bar{z}$. We may do that because the transformation is given by formal analytical expressions in terms of these parameters. For such a choice of parameters the Moutard equation takes the form of a two-dimensional Schrödinger equation

$$(\partial\bar{\partial} + U)\varphi = 0. \tag{1}$$

Given a solution ω to this equation, the Moutard transformation is defined by the following formulas:

$$U \longrightarrow \widetilde{U} = U + 2\partial\bar{\partial}\log\omega,$$

$$\varphi \to \theta = \frac{i}{\omega}\int(\varphi\partial\omega - \omega\partial\varphi)dz - (\varphi\bar{\partial}\omega - \omega\bar{\partial}\varphi)d\bar{z}$$

and it maps every solution φ of (1) to a solution θ of another equation of the same type:

$$(\partial\bar{\partial} + \widetilde{U})\theta = 0.$$

We remark that θ is defined modulo multiples of $\frac{1}{\omega}$ due to the integration constant.

The one-dimensional reduction of this transformation was later derived by Darboux and now it is called the Darboux transformation. It corresponds to the case when U depends on one variable x: $U = U(x)$, and ω takes the form $\omega = f(x)\exp(\mathrm{const}\cdot y)$. Therewith it gives a transformation of one-dimensional Schrödinger operators $H = -\frac{d^2}{dx^2} + U$ and of all their formal eigenfunctions, i.e. solutions ψ to $H = E\psi, E \in \mathbb{C}$, preserving E. This transformation was rediscovered may times and was widely used in the spectral theory (Dirac, Schrödinger, Crum and others). In particular, it was applied for constructing N-soliton potentials, their degenerations of the form $\frac{N(N+1)}{x^2}$ and N-soliton solutions to the Korteweg–de Vries (KdV) equation.

In [1] we applied the Moutard transformation to the spectral theory of two-dimensional Schrödinger operators. In fact,

- we constructed smooth two-dimensional potentials U which are sufficiently fast decaying for $H = \partial\bar{\partial} + U$ to have good scattering theory and for which H has a kernel in $L_2(\mathbb{R}^2)$.

These are the first examples of such potentials and it is known that in the one-dimensional situation such examples do not exist. In fact, our examples are rational functions which decay as r^8 whenever the kernel at least contains a two-dimensional subspace consisting of functions which decay as r^{-3} where $r = \sqrt{x^2 + y^2}$.

The KdV equation has a two-dimensional generalization, the Novikov–Veselov (NV) equation:

$$U_t = \partial^3 U + \bar{\partial}^3 U + 3\partial(UV) + 3\bar{\partial}(\bar{V}U) = 0,$$

$$\bar{\partial} V = \partial U$$

which is a compatibility condition for the system

$$(\partial\bar{\partial} + U)\psi = 0,$$

$$\partial_t \psi = (\partial^3 + \bar{\partial}^3 + 3V\partial + 3\bar{V}\bar{\partial})\psi.$$

One can show that the Moutard transformation can be extended to the transformation which respects this evolution and therefore constructs from solutions of the NV equation new solutions to it.

In [1] by using the extended Moutard transformation

- we constructed solutions to the NV equation with fast decaying ($\sim r^{-3}$) Cauchy data at $t = 0$ which blows up at some critical time $T_* > 0$.

Our solutions are given by explicit rational functions and thus we expose the scenario of blowing up.

In both constructions we use a double iteration of the Moutard transformation which is defined by a pair of holomorphic functions ω_1 and ω_2. The first iteration is defined by ω_1 and the second iteration is defined by the image of ω_2 under the first transformation. We do that to obtain smooth solutions because just one iteration always either gives singular potentials due to the term $\partial\bar{\partial}\log\omega$.

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Duality of wave fronts and applications

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(joint work with Kentaro Saji)

A differentiable map $f : M^n \to N^{n+1}$ of a differentiable n-manifold M^n into a Riemannian $(n+1)$-manifold N^{n+1} is called a *wave front* or a *front* if for each $p \in M^n$, there exists a unit normal vector field $\nu\colon U_p \to T_1N^n$ as a smooth immersion of a neighborhood $U_p \subset M^n$ of p into the unit tangent bundle T_1N^{n+1} of N^{n+1}. This condition is equivalent to that f is obtained as the projection of a Legendrian immersion $L\colon M^n \to P(T^*N^{n+1})$, where $P(T^*N^{n+1})$ is the projectified cotangent bundle with the canonical contact structure.

For a wave front $f\colon M^n \to N^{n+1}$, one can define the *limiting tangent bundle* $\mathcal{E}_f$ over M^n whose fiber at $p \in M^n$ is the orthogonal complement in $T_{f(p)}N^{n+1}$ of the unit normal vector ν_p at p. Furthermore, we have a bundle homomorphism $\varphi_f := df\colon TM^n \to \mathcal{E}_f$. A point $p \in M_n$ is a *singular point* of f if $(\varphi_f)_p\colon T_pM^n \to (\mathcal{E}_f)_p$ is not bijective.

As an abstract setting of the situation above, the notion of *coherent tangent bundle* is introduced in [5, 6]. In fat, a coherent tangent bundle is a 5-tuple $(M^n, \mathcal{E}, \varphi, \langle\ ,\ \rangle, D)$ of a differentiable n-manifold M^n, a vector bundle $\mathcal{E}$ of rank n over M^n, a bundle homomorphism $\varphi\colon M^n \to \mathcal{E}$, a metric $\langle\ ,\ \rangle$ and a metric connection D of $\mathcal{E}$ which satisfy the compatibility condition

$$D_X\varphi(Y) - D_Y\varphi(X) - \varphi([X,Y]) = 0$$

for any vector fields X and Y on M^n.

In particular, let $f\colon M^n \to N^{n+1}(c)$ be a front defined on an orientable manifold M^n into an $(n+1)$-dimensional space form $N^{n+1}(c)$ of constant curvature c, and assume that f is co-orientable, that is, there exists a globally defined unit normal vector field ν. Then, in addition to $\varphi_f = df$, one can take another bundle homomorphism $\psi_f := d\nu\colon TM^n \to \mathcal{E}_f$ which gives a structure of coherent tangent bundle because of the Codazzi equation. Thus there exist two structure of coherent tangent bundles for each orientable and co-orientable wave front in a space form, which is considered as the *duality* of wave fronts. A singular point of ν, that is a point at which $\varphi_f = d\nu$ degenerates, is called an *inflection point* of f. Abstract setting representing this duality, a notion of *front bundles*, is introduced in [8].

From now on, we restrict our attention to the case of $n = 2$. It is known that generic singularities of 2-dimensional wave fronts are cuspidal edges (A_2-singular points) and swallowtails (A_3-singular points). The criteria in [3] for cuspidal edges and swallowtails allow us to define the notions of A_2 and A_3-singular points for coherent tangent bundles. Moreover, the notion of *singular curvature* (denoted by κ_s) as a function of the set of cuspidal edges (A_2-singular points) are defined for 2-dimensional coherent tangent bundles as well as for 2-dimensional wave fronts [5, 6].

Let $f\colon M^2 \to N^3$ be a co-orientable wave front defined on a compact oriented 2-manifold M^2 into an oriented Riemannian 3-manifold N^3. For simplicity, we assume that the set $\Sigma_f \subset M^2$ of singular points of f consists of cuspidal edges and swallowtails. Then the following two Gauss-Bonnet type formulas hold [5, 6]:

$$2\pi\chi(M^2) = \int_{M^2} K\, dA + 2\int_{\Sigma_f} \kappa_s\, ds, \tag{1}$$

$$\chi(\mathcal{E}_f) = \chi(M_+) - \chi(M_-) + 2(S_+ - S_-), \tag{2}$$

where $\chi(\mathcal{E}_f)$ is the Euler characteristic of the limiting tangent bundle $\mathcal{E}_f$, K is the Gaussian curvature of f, dA is the area element, ds is the line element of the singular set (singular curve), and S_+ (resp. S_-) is the number of positive (resp. negative) swallowtails, for details, see [5, 6]. The formula (2) is stated in [4] (without proof) and two formulas are proved in [2]. (In [6], the singular curvature appeared implicitly as the measure $\kappa_s\, ds$).

The formulas (1) and (2) also hold for orientable and co-orientable coherent tangent bundles on 2-manifolds [6]. An advantage of this abstract setting is the following: Consider a front $f\colon M^2 \to N^3(c)$ in a space form $N^3(c)$ (or more generally, a front bundle). Then we have two coherent tangent bundles, one is derived from $\varphi_f = df$ and the other comes from $\psi_f = d\nu$. Hence we have two pairs of Gauss-Bonnet type formulas. Combining these four, one can obtain several topological properties of wave fronts and their Gauss maps.

For example, let $f\colon M^2 \to \mathbb{R}^3$ be a wave front of an oriented 2-manifold. Assume that the singular points of the Gauss map consists of folds and cusps. Then we have the Bleecker-Wilson formula ([1, 8]): $2\chi(\{p \in M^2\,;\, K(p) < 0\}) = S_+^\nu - S_-^\nu$, where K is the Gaussian curvature, and S_+^ν (resp. S_-^ν) denotes the number of positive (resp. negative) cusps. For details, see [8].

Further applications are given in [8, 7].

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Spectral Curves for Constant Mean Curvature Tori in $\mathbb{R}^3$

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(joint work with Katrin Leschke, Franz Pedit)

Spectral curve constructions describing harmonic maps of genus one surfaces stem from the classical S^1 family of associated surfaces; extending the circle parameter to $\mu \in \mathbb{C}_*$ yields a family of flat connections ∇^μ. The (simultaneous) eigenlines of the holonomy of ∇^μ around any non-trivial $\gamma_p \in \pi_1(T^2, p)$ define the eigenline spectral curve Σ_e, and for each p we obtain a line bundle on Σ, yielding a linear flow in the Jacobian of Σ_e. This algebro–geometric description of harmonic maps has proved useful both in answering moduli-space questions and in producing bounds for geometrically interesting quantities such as energy. For conformal immersions $f : T^2 \to S^4 \cong \mathbb{H}P^1$ one can alternatively define the multiplier spectral curve. This consists of holonomies realised by $\mathbb{H}$–holomorphic sections of V/L, where $V = T^2 \times \mathbb{H}^2$ and L is the pull-back under f of the tautological $\mathbb{H}$-line bundle on $\mathbb{H}P^1$. It is natural to ask what the relationship is between these two curves, and what geometric information they each encode? Holomorphic sections of V/L determine Darboux transforms, which are a generalisation of the classical Darboux transform, obtained by relaxing the usual condition that there be a sphere congruence mutually tangent to both surfaces to allow "half–touching". Those holomorphic sections which are additionally parallel with respect to a ∇^μ are called μ–Darboux transforms. We find that the μ–Darboux transforms of any constant mean curvature surface again have constant mean curvature and that they are classical Darboux transforms only for certain values of μ.

We find that the eigenline and multiplier curves are not isomorphic, but have the same normalisation. The multiplier curve is always singular whereas the eigenline curve is generically smooth. Geometrically, the space of closed μ–Darboux transforms of f is given by the quotient of $\Sigma_e \setminus \{\mu^{-1}(0,\infty)\}$ by an antiholomorphic involution together with finitely many complex projective lines. Analogously, the space of Darboux transforms of f is given by the quotient of the multiplier spectral curve by an antiholomorphic involution together with most countably many complex and quaternionic projective spaces. Furthermore, if $f : T^2 \to \mathbb{R}^3$ be a constant mean curvature immersion then there are natural maps

$$\begin{tikzcd} & \mathbb{CP}^3 \arrow[d, "\pi"] \\ T^2 \times \Sigma_e \arrow[ur, "\widehat{\mathcal{E}}"] \arrow[r, "\hat{\mathcal{E}}_{\mathbb{H}}"'] & \mathbb{H}P^1 \end{tikzcd}$$

such that

(i) For each $x \in \Sigma_e^\circ$, $\hat{f}^x = \pi\widehat{\mathcal{E}}(\cdot, x)$ is a μ–Darboux transform of f.
(ii) We recover f as the limit of μ–Darboux transforms for $\mu \to 0, \infty$.
(iii) For $p \in T^2$ the eigenline curve is algebraically mapped into $\mathbb{CP}^3$ by $\hat{\mathcal{E}}(p, \cdot)$. Thus we obtain a smooth T^2–family of algebraic curves $\Sigma_e \to \mathbb{CP}^3$.

Isometric immersions of the Hyperbolic plane into the Hyperbolic space

Atsufumi Honda

We shall deal with isometric immersions of the hyperbolic plane $\boldsymbol{H}^2$ into $\boldsymbol{H}^3$. The theorem of Portnoy [1] says that *an isometric immersion of* $\boldsymbol{H}^2$ *into* $\boldsymbol{H}^3$ *is developable*. Here, a surface in $\boldsymbol{H}^3$ is *developable* if *extrinsic flat*, namely the extrinsic curvature (= the product of principal curvatures = Gauss curvature +1) of the surface vanishes identically, and *ruled*, that is, the surface is the trace of 1-parameter family of geodesics.

A representation by null curves. Now, we focus on the property that a ruled surface is the trace of 1-parameter family of geodesics. That is, a ruled surface corresponds to a curve in *the space of oriented geodesics*. It is well known that the space of oriented geodesics $\mathcal{L}(\boldsymbol{H}^3)$ has natural 2-dimensional indefinite Kähler structure $(\mathcal{L}(\boldsymbol{H}^3), G, J)$, and $(\mathcal{L}(\boldsymbol{H}^3), J)$ is biholomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \bar{\Delta}$, where $\bar{\Delta}$ is reflected diagonal.

Theorem I. *A developable surface gives a null curve in* $\mathcal{L}(\boldsymbol{H}^3)$*. Conversely, a null curve*

$$\alpha = (\mu_1, \mu_2) \in \mathbb{P}^1 \times \mathbb{P}^1 \setminus \hat{\Delta} \cong \mathcal{L}(\boldsymbol{H}^3)$$

with $\dfrac{\dot{\mu}_1 \dot{\bar{\mu}}_2}{(1+\mu_1\bar{\mu}_2)^2} \leq 0$ *gives a developable surface in* $\boldsymbol{H}^3$.

Here, a curve α in $(\mathcal{L}(\boldsymbol{H}^3), G)$ is *null*, if $G(\dot{\alpha}, \dot{\alpha}) = 0$.Theorem I says that we have the representation formula of isometric immersion of $\boldsymbol{H}^2$ into $\boldsymbol{H}^3$, in terms of null curve.

Ideal cones. As for the inequality in the Theorem I, we get that the equality holds if and only if one side end of the developable surface is asymptotic to a point, that is *ideal cone*. We shall investigate the properties of the ideal cones.

For a complete developable surface, let t be the arc length parameter of the geodesic line included in the set of non-umbilic points. Then the non-zero principal curvature λ is proportional to $e^{\pm t}$ or $1/\cosh t$ on the geodesic line. So, we call the developable surface *exponential type* when the surface has the non-zero principal curvature λ is proportional to $e^{\pm t}$ on the set of non-umbilic points.

Theorem II. *The deveopable surface of exponential type is assymptotic to a point in the ideal boundary, in the direction where the non-zero principal curvature diverge. That is, it is cone whose vertex is on the ideal boundary. Conversely, a ruled surface which is 1-parameter family of geodesics whose one side end is same point is deveopable surface of exponential type.*

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The value distribution of the Gauss map of improper affine spheres

Yu Kawakami
(joint work with Daisuke Nakajo)

The study of improper affine spheres in the affine three-space $\mathbf{R}^3$ has made significant advances. Recently, Martínez [3] introduced the notion of improper affine map, that is, a class of (locally strongly convex) improper affine spheres with some admissible singularities. Afterward, Nakajo [4], Umehara and Yamada [5] showed that an improper affine map in $\mathbf{R}^3$ is a front. So we call this class improper affine front in this report. Moreover, Martínez [3] gave a representation formula for improper affine fronts in terms of two holomorphic functions and defined the Lagrangian Gauss map for this class. We give the best possible upper bounds for the number of exceptional values of the Lagrangian Gauss map of complete (in the sense of [2, 3]) and weakly complete (in the sense of [5]) improper affine fronts in $\mathbf{R}^3$.

Theorem 1 ([1]). *Let $\psi\colon \Sigma = \overline{\Sigma}_\gamma \backslash \{p_1, \ldots, p_k\} \to \mathbf{R}^3$ be a complete improper affine front defined on a closed Riemann surface $\overline{\Sigma}_\gamma$ of genus γ with k points removed, and $\nu\colon \Sigma \to \mathbf{C} \cup \{\infty\}$ be the Lagrangian Gauss map of ψ. Suppose that ν is nonconstant, and d is the degree of ν considered as a map on $\overline{\Sigma}_\gamma$. If we denote by D_ν the number of exceptional values of ν, then we have*

$$D_\nu \leq 2 + \frac{2}{R}, \qquad \frac{1}{R} = \frac{\gamma - 1 + k/2}{d} < \frac{1}{2}\,.$$

In particular, ν can omit at most two values. The number "two" is sharp.

Theorem 2 ([1]). *Let $\psi\colon \Sigma \to \mathbf{R}^3$ be a weakly complete improper affine front and $\nu\colon \Sigma \to \mathbf{C} \cup \{\infty\}$ be the Lagrangian Gauss map of ψ. Suppose that ν is nonconstant. If we denote by D_ν the number of exceptional values of ν, then we have*

$$D_\nu \leq 3\,.$$

In particular, ν can omit at most three values. The number "three" is sharp.

As an application of Theorem 2, we obtain a brief proof of the well-known result that any affine complete improper affine sphere must be an elliptic paraboloid.

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A generalization of Unduloid and Nodoid

Katsuei Kenmotsu

Theorem(Delaunay). For a surface of revolution M in the Euclidean three space with constant mean curvature h, M is periodic if and only if $h \neq 0$.

We extend this to the case of non-constant mean curvature $H(s)$ in [1], [2] for two-dimension and in [3], [4] for higher dimension. We proved

Theorem. M is periodic with period L if and only if $H(s)$ is periodic with period L, and moreover it satisfies the conditions (1) and (2) below :

$$2\int_0^L H(s)ds = 2\pi m, \text{ where } m \text{ is an integer,} \tag{1}$$

$$\int_0^L \cos(2\int_0^u H(u)du)ds = \int_0^L \sin(2\int_0^u H(u)du)ds = 0. \tag{2}$$

The next question is: how to get a periodic function satisfying the conditions (1) and (2) above ? The answer is: for the curvature $k(s)$ of any planar smooth closed curve Γ, $k(s)/2$ satisfies the conditions (1) and (2).

When Γ is a circle, we have the usual unduloid and nodoid:

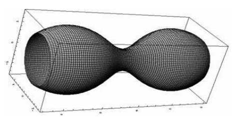

Figure 1. unduloid

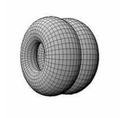

Figure 2. nodoid

When Γ is an ellipse, we have a generalized unduloid and nodoid:

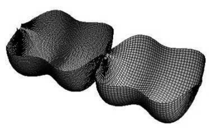

Figure 3. elliptic unduloid

Figure 4. elliptic nodoid

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Singularity of the asymptotic completion of developable Möbius strips

Kosuke Naokawa

Generic singular points of developable (zero Gaussian curvature and ruled) surfaces in Euclidean three-space $\mathbb{R}^3$ consist of cuspidal edges, swallowtails and cuspidal cross caps, and the most generic ones are cuspidal edges (cf.[2, Proposition 2.16]). We gave sharp lower bounds of the number of singular points other than cuspidal edges on the asymptotic completion of developable Möbius strips. Let

$$F(s,u) = \gamma(s) + u\xi(s) \qquad (|u| < \epsilon)$$

be an immersed developable Möbius strip in $\mathbb{R}^3$, where $\epsilon > 0$, $\gamma(s)$ is a generating curve and $\xi(s)$ is a ruling vector field of F. Then, F is called a *rectifying Möbius strip* if the generating curve γ is a closed geodesic. Moreover, the smooth map

$$\tilde{F}(s,u) = \gamma(s) + u\xi(s) \qquad (u \in \mathbb{R})$$

is called the *asymptotic completion* of the immersed strip F. We obtain the following assertions:

Proposition 1. *The asymptotic completion of a developable Möbius strip has at least one singular points other than cuspidal edges.*

Moreover, there exists a developable Möbius strip with only one non-cuspidal-edge singular point on the asymptotic completion.

Theorem 1. *The asymptotic completion of a rectifying Möbius strip has at least three singular points other than cuspidal edges.*

To prove these assertions, we use the criterion for cuspidal edges given in [1]. We also found an example having exactly three non-cuspidal-edge singular points.

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A new approach to the existence of harmonic maps

Toshiaki Omori

Theorem 1. *Let (M,g), (N,h) be closed Riemannian manifolds and $\{u_\varepsilon\}_{\varepsilon>0}$ be a sequence of smooth critical points $u_\varepsilon : (M,g) \to (N,h)$ of*

$$\mathbb{E}_\varepsilon(u) = \int_M \frac{e^{\varepsilon|\nabla u|^2} - 1}{\varepsilon} d\mu_g$$

with uniformly bounded energy. If the sectional curvature of (N,h) is nonpositive, then a subsequence $\{u_{\varepsilon'}\}_{\varepsilon'\to 0}$ converges to a harmonic map $u : (M,g) \to (N,h)$:

$$u_{\varepsilon'} \to u \quad (\varepsilon' \to 0) \quad \textit{in } C^\infty(M,N).$$

Critical points of $\mathbb{E}_\varepsilon$ were first proposed by Professor J. Eells and are called *exponentially harmonic maps*. The rapid growth of the functional $\mathbb{E}_\varepsilon$ gives an expectation that its minima have high regularity. In fact, Duc [1] and Naito [3] proved that any homotopy class $\mathcal{H} \in [M, N]$ admits a smooth critical point of $\mathbb{E}_\varepsilon$. The above theorem, together with this observation, yields the following corollary.

Corollary 1 ([2]). *Let (M, g) and (N, h) be closed Riemannian manifolds. If the sectional curvature of (N, h) is nonpositive, then any homotopy class $\mathcal{H} \in [M, N]$ admits a harmonic map $u : (M, g) \to (N, h)$.*

If one tries to remove the curvature assumption, a blow-up phenomenon may occur. In case $\dim M = 2$, Sacks-Uhlenbeck [4] considered a blow-up phenomenon for a sequence (as $\alpha \to 1$) of critical points of the functional

$$E_\alpha(u) = \int_M (1 + |\nabla u|^2)^\alpha d\mu_g \quad (\alpha > 1),$$

so-called *α-harmonic maps*. In [4], it was verified that there only exist finitely many points, outside which the sequence uniformly converges to a harmonic map.

The corresponding result to [4] for the functional $\mathbb{E}_\varepsilon$ is as follows.

Theorem 2. *Let (M, g) be a Riemann closed surface and (N, h) be a general closed Riemannian manifold, and $\{u_\varepsilon\}_{\varepsilon>0}$ be as in Theorem 1. Then there exist a finite singular set $\{p_1, \ldots, p_k\} \subseteq M$, outside of which a subsequence $\{u_{\varepsilon'}\}_{\varepsilon' \to 0}$ uniformly converges to a harmonic map $u : (M, g) \to (N, h)$:*

$$u_{\varepsilon'} \to u \quad (\varepsilon' \to 0) \quad \text{in } C^\infty_{\mathrm{loc}}(M \setminus \{p_1, \ldots, p_k\}, N).$$

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