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Cohomology of Finite Groups: Interactions and Applications

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Abstract. The cohomology of finite groups is an important tool in many subjects including representation theory and algebraic topology. This meeting was the third in a series that has emphasized the interactions of group cohomology with other areas.

Mathematics Subject Classification (2000): 20Jxx, 20Cxx, 55Nxx, 57Nxx.

Introduction by the Organisers

The workshop brought together mathematicians from several areas of algebra and topology. The common theme was the use and application of techniques from the cohomology theory of finite groups. There were 53 participants from eleven different countries at the meeting. This included several graduate students and postdocs. Three of the participants were invited speakers at the International Congress of Mathematicians in Hyderbad, India, a month after the Obwerwolfach workshop. Two spoke in the algebra section of the ICM and the third in the topology section.

This was the third Oberwolfach workshop in the series with the same title. The previous two had been in 2000 and 2005. The emphasis of this meeting was strongly on applications and interactions with subjects as diverse as homotopy theory, transformation groups, representation theory of finite groups and group schemes, triangulated categories and algebraic geometry, number theory and commutative algebra.

A few of the highlights included an introduction to generalized rank varieties and modules of constant Jordan type by Julia Pevtsova. Dave Benson showed us how the geometry of vector bundles can be used to obtain information on representation theory. Bernhard Hanke described a homotopy Euler characteristic that provides an upper bound on the free p-rank of symmetry of a finite complex. Ergün Yalçin connected fusion systems for finite groups with the study of group actions on products of spheres. Dave Hemmer told how topological and algebraic methods can be combined to prove that there are large gaps in the cohomology of Young modules over symmetric groups. Fred Cohen described an approach to understanding the homotopy of spaces of representations, in particular commuting elements in a Lie group. After a quick introduction into the geometry of tensor triangulated categories, Paul Balmer addressed the natural question of descent and presented some applications in the representation theory of groups. The Brauer construction from group representations has been adapted by Peter Symonds to study the fixed-point space of group action on varieties. Jesper Grodal proved a bound on the p-rank of the group of homotopy self-equivalences of a finite simply connected complex. There was some discussion of the classification of localizing subcategories of the stable module category of a finite group, by Benson, Henning Krause and Srikanth Iyengar.

The schedule allowed much time for lively discussions. There were 23 lectures, several by younger members of the group. Many of the participant joined the standard hike to St. Roman on Wednesday afternoon, in spite of somewhat unfavorable weather. On Thursday evening, the group was treated to a concert organized by Peter Webb, featuring excellent performances on piano, violin and flute by several of the conference participants. As usual, the friendly and stimulating atmosphere of Oberwolfach contributed to a lively and exciting conference.

Workshop: Cohomology of Finite Groups: Interactions and Applications

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Abstracts

Beyond the Zariski topology in tensor triangulated categories

Paul Balmer

A survey of *Tensor Triangular Geometry* is given in [1]. One of the major tools in this subject is the following. Given a tensor triangulated category K and a quasi-compact open $U \subset \mathrm{Spc}(K)$ of its spectrum, one can form $K(U)$ as the idempotent completion of the Verdier quotient of K by the thick subcategory $\{a \in K \,|\, \mathrm{supp}(a) \subset U\}$.

In algebraic geometry, for X a quasi-compact and quasi-separated scheme and $K = \mathrm{D}^{\mathrm{perf}}(X)$, one has $\mathrm{Spc}(K) \cong X$ and a result of Thomason says that $K(U) \cong \mathrm{D}^{\mathrm{perf}}(U)$ for every $U \subset X$. However, in modular representation theory, for G a finite group and k a field of characteristic p (dividing $|G|$) and for $K = \mathrm{stab}(kG)$, the spectrum $\mathrm{Spc}(K)$ is the projective support variety but for any proper non-empty open $U \subset \mathrm{Spc}(K)$ the category $K(U)$ is *never* a stable category of representations of a finite group.

Therefore, it would be interesting to find a purely tensor-triangular construction which would produce $\mathrm{stab}(kH)$ out of $\mathrm{stab}(kG)$ for $H < G$ for instance. Here is one:

Theorem. *Let $A = k(G/H)$ with obvious left G-action and define a multiplication $\mu : A \otimes_k A \to A$ by $\mu(\gamma \otimes \gamma') = \gamma$ if $\gamma = \gamma'$ and 0 else, for every $\gamma, \gamma' \in G/H$. Considering A as a ring object in $K(G) = \mathrm{D}(kG)$ (resp. in $K = \mathrm{stab}(kG)$, etc), one has an equivalence between the category of A-modules* in $K(G)$ *and the category $K(H)$, such that extension of scalars $(A \otimes -)$ coincides with restriction and such that forgetting the A-action coincides with (co)induction.*

It is then a legitimate question to ask, for a general triangulated category K, when is the category of A-modules in K triangulated. One answer (probably the best "general" answer) is that it holds when A is *separable*, meaning that $\mu : A \otimes A \to A$ has a section as A, A-bimodule. Some care has to be taken with higher octahedra and this can be found in [2].

Another legitimate question, inspired by the analogue situation in algebraic geometry, is the problem of descent. This is discussed in [3] and the bottom line is that descent holds as soon as the algebra A is faithful (i.e., $A \otimes -$ is a faithful functor). Note however that these two questions (triangulation and descent) make sense for modules over any monad, instead of a ring object, but that descent is slightly more complicated in that case, faithfulness not being sufficient.

When $H < G$ and $[G : H]$ is invertible in k, then descent holds for $A = k(G/H)$ as in the Theorem above. This will be further investigated in upcoming work.

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Modules over elementary abelian p-groups

David J. Benson

Let $E = \langle g_1, \ldots, g_r \cong (\mathbb{Z}/p)^r$ be a finite elementary abelian group, and let k be an algebraically closed field of characteristic p. Let

$$X_i = \log(g_i) = (g_i - 1) - \tfrac{1}{2}(g_i - 1)2 + \cdots - \tfrac{1}{p-1}(g_i - 1)^{p-1}.$$

If $\alpha = (\lambda_1, \ldots, \lambda_r) \in \mathbb{A}^r(k)$, we write

$$X_\alpha = \lambda_1 X_1 + \cdots + \lambda_r X_r.$$

These elements form a set of coset representatives for $J2(kE)$ in $J(kE)$.

Following Carlson, Friedlander and Pevtsova, we say that a finitely generated kE-module M has *constant Jordan type* (cJt) if for all $0 \neq \alpha \in \mathbb{A}^r(k)$, the action of X_α on M has the same Jordan canonical form. We write $[a_1, \ldots, a_t]$ if the Jordan canonical form has blocks of sizes $a_1, \ldots, a_t$ (with multiplicities). We abbreviate the notation, so that for example $[2, 13]$ means two Jordan blocks of length two and three of length one. We are often more interested in the *stable Jordan type*, where the Jordan blocks of length p are omitted from the notation. Then we talk of the *stable constant Jordan type* (scJt) of the module.

Theorem 1 (Friedlander, Pevtsova and Suslin)**.** *If M has constant Jordan type then the Jordan canonical forms of all elements of $J(kE) \setminus J2(kE)$ on M are the same.*

Dade's lemma states that if M of constant Jordan type $[p^t]$ for some t then M is free. Using this, a module of stable constant Jordan type $[1]$ or $[p-1]$ is automatically endotrivial, and then another theorem of Dade show that $M \cong \Omega^n k$ for some $n \in \mathbb{Z}$.

I proved the following in MSRI in 2008:

Theorem 2. *If $r \geq 2$ then there is no kE-module of scJt $[a]$ with $2 \leq a \leq p-2$.*

To get further, we introduce some algebraic vector bundles over projective space. Let $\mathbb{P}^{r-1} = \mathsf{Proj}\, k[Y_1, \ldots, Y_r]$, where $Y_1, \ldots, Y_r$ are the coordinate functions on $\mathbb{A}^r(k)$. Let $\mathcal{O}$ be the structure sheaf on $\mathbb{P}^{r-1}$.

If M is a finitely generated kE-module, let $\tilde{M} = M \otimes_k \mathcal{O}$, a trivial sheaf whose rank is the dimension of M. Following Friedlander and Pevtsova, we define an operator $\theta\colon M \to M(1)$ (and by abuse of notation $\theta\colon M(j) \to M(j+1)$ for $j \in \mathbb{Z}$) as follows:

$$\theta(m \otimes f) = \sum_i X_i m \otimes Y_i f.$$

Then $\theta^p = 0$, and for $1 \leq i \leq p$ we define

$$\mathcal{F}_i(M) = \frac{\mathrm{Image}(\theta^{i-1}) \cap \mathrm{Ker}(\theta)}{\mathrm{Image}(\theta^i) \cap \mathrm{Ker}(\theta)}.$$

Then M has cJt if and only if each $\mathcal{F}_i(M)$ is a vector bundle on $\mathbb{P}^{r-1}$, and the rank of this vector bundle is equal to the number of Jordan blocks of length i for each X_α. The following is proved in a joint paper with Pevtsova:

Theorem 3. *Given a vector bundle $\mathcal{F}$ on $\mathbb{P}^{r-1}$ of rank s, there exists a kE-module of scJt $[1^s]$ with*

(i) $\mathcal{F}_1(M) \cong \mathcal{F}$ *if $p = 2$, and*

(ii) $\mathcal{F}_1(M) \cong F^*(\mathcal{F})$ *if p is odd,*

where $F\colon \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$ is the Frobenius map.

The Frobenius map for p odd cannot be removed. To see this, we use Chern classes. Recall that the Chow group $A^*(\mathbb{P}^{r-1})$ is isomorphic to $\mathbb{Z}[h]/(h^r)$. Given a vector bundle $\mathcal{F}$ on $\mathbb{P}^{r-1}$, there is a *Chern polynomial*

$$c(\mathcal{F}) = 1 + c_1(\mathcal{F})h + \cdots + c_{r-1}(\mathcal{F})h^{r-1} \in A(\mathbb{P}^{r-1}).$$

The following is proved in the same paper with Pevtsova:

Theorem 4. *If $r \geq 2$ and M is a kE-module of scJt $[1^s]$ then*

$$p | c_i(\mathcal{F}_1(M)) \qquad \text{for} \quad 1 \leq i \leq p-2.$$

Similar Chern class techniques prove the following more recent theorem.

Theorem 5. *Let $r \leq 3$ and $p \geq 5$. If M has scJt $[a_1, \dots, a_t]$ with $\sum a_i \leq \min(r-1, p-2)$ then $a_1 = \cdots = a_t = 1$.*

This proves a weak form of a conjecture of Carlson, Friedlander and Pevtsova stating that if M has scJt $[2, 1^j]$ then $j \geq r-1$:

Corollary 6. *If M has scJt $[2, 1^j]$ and $p \geq j+4$ then $j \geq r-2$.*

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On the Cartan matrix of Mackey algebras

Serge Bouc

Let k be a commutative ring, and G be a finite group. It has been shown by Thévenaz and Webb ([3]) that the category $\mathsf{Mack}_k(G)$ of Mackey functors for G over k is equivalent to the category of modules over *the Mackey algebra* $\mu_k(G)$. Similarly, the subcategory $\mathsf{CoMack}_k(G)$ of cohomological Mackey functors for G over k is equivalent to the category of modules over a quotient $co\mu_k(G)$ of $\mu_k(G)$, called *the cohomological Mackey algebra*.

Many notions defined for kG-modules can be extented to Mackey functors and cohomological Mackey functors : for example, when H is a subgroup of G, there is a restriction functor $\mathsf{Mack}_k(G) \to \mathsf{Mack}_k(H)$, and an induction functor $\mathsf{Mack}_k(H) \to \mathsf{Mack}_k(G)$, which is both left and right adjoint to restriction. Both functors preserve cohomological Mackey functors.

The algebras $\mu_k(G)$ and $co\mu_k(G)$ share some nice properties with the group algebra kG : for example, they are free as k-modules, of finite rank independent on k. When k is a field of characteristic $p \geq 0$, Maschke's theorem holds, i.e. $\mu_k(G)$ and $co\mu_k(G)$ are semisimple if p does not divide the order of G. The Krull-Schmidt theorem also holds, the notions of relative projectivity, of vertex and source make sense for Mackey functors.

When $p > 0$, the subcategory $\mathsf{Mack}_k(G, \mathbf{1})$ of Mackey functors which are projective relative to p-subgroups of G is equivalent to the category of modules over a direct summand $\mu_k(G, \mathbf{1})$ of $\mu_k(G)$, called *the p-local Mackey algebra*. The cohomological Mackey algebra $co\mu_k(G)$ is a quotient of $\mu_k(G, \mathbf{1})$.

The algebras $\mu_k(G, \mathbf{1})$ and $co\mu_k(G)$ are by many other aspects similar to the group algebra kG : there is a good decomposition theory for modules from characteristic 0 to characteristic p, with a Cartan c, d, e triangle, and the corresponding Cartan matrices are symmetric. Also, the blocks of kG are in one to one correspondence with the blocks of $\mu_k(G, \mathbf{1})$ and the blocks of $co\mu_k(G)$.

There are also important differences between kG, $\mu_k(G, \mathbf{1})$, and $co\mu_k(G)$: in particular, projective Mackey functors need not be projective relative to the trivial subgroup. Moreover, the algebras $\mu_k(G, \mathbf{1})$, and $co\mu_k(G)$ are generally not self-injective.

One can also observe on small examples that the determinant of the Cartan matrix of $\mu_k(G, \mathbf{1})$ is generally not a power of p (e.g. for the group of order p, this matrix is equal to $\begin{pmatrix} 2 & 1 \\ 1 & p \end{pmatrix}$). The Cartan matrix of $co\mu_k(G)$ is often singular (e.g. when G is an elementary abelian p-group of rank 2). This raises the following natural questions :

- (Q1) What are the "exotic" prime factors of the determinant of the Cartan matrix $\mathsf{C}\big(\mu_k(G, \mathbf{1})\big)$ of $\mu_k(G, \mathbf{1})$?
- (Q2) When is the Cartan matrix $\mathsf{C}\big(co\mu_k(G)\big)$ of $co\mu_k(G)$ non singular ?
- (Q3) What is the rank of $\mathsf{C}\big(co\mu_k(G)\big)$?

The answer to Question (Q1) is the following explicit formula :

Theorem 1. *Let G be a finite group, and k be a (big enough) field of characteristic $p > 0$. Then the determinant of the Cartan matrix of the Mackey algebra $\mu_k(G, \mathbf{1})$ is equal to*

$$\det \mathsf{C}\big(\mu_k(G,\mathbf{1})\big) = \prod_{R\in[\mathcal{S}_p(G)]} \prod_{s\in[\overline{N}_G(R)_{p'}]} \Big(|C_{\overline{N}_G(R)}(s)|_p \sum_{x\in R/[<sR>,R]} \frac{1}{|x|} \Big) \ ,$$

where $[\mathcal{S}_p(G)]$ is a set of representatives of conjugacy classes of p-subgroups of G, where $[\overline{N}_G(R)_{p'}]$ is a set of representatives of p'-elements of $\overline{N}_G(R) = N_G(R)/R$, and $<sR>$ is the subgroup of G generated by sR.

Examples : • If G cyclic of order p^n, then

$$\det\ \mathsf{C}\big(\mu_k(G,\mathbf{1})\big) = p^{\binom{n}{2}} \prod_{i=1}^{n} \big(p + i(p-1)\big) \ .$$

• If G is an elementary abelian p group of rank 2, then

$$\det\ \mathsf{C}\big(\mu_k(G,\mathbf{1})\big) = p(2p-1)^{p+1}(p^2+p-1) \ .$$

Questions (Q2) and (Q3) are settled by the following :

Theorem 2. *With the same assumptions :*

(1) *The rank of $\mathsf{C}\big(co\mu_k(G)\big)$ is equal to the number of G-conjugacy classes of pairs (R, s), where R is a p-subgroup of G, and s is a p'-element of $\overline{N}_G(R)$, such that the group $<sR>$ is cyclic. In other words*

$$\mathrm{rk}\, \mathsf{C}\big(co\mu_k(G)\big) = \sum_{R\in[\mathcal{C}_p(G)]} |N_G(R)\backslash C_G(R)_{p'}|$$

where $[\mathcal{C}_p(G)]$ is a set of representatives of conjugacy classes of cyclic p-subgroups of G, and $N_G(R)\backslash C_G(R)_{p'}$ is the set of $N_G(R)$-conjugacy classes of p'-elements of $C_G(R)$.

(2) *The matrix $\mathsf{C}\big(co\mu_k(G)\big)$ is non singular if and only if G is p-nilpotent with cyclic Sylow subgroups, i.e. if $G = N \rtimes S$, where N is a p'-group, and S is a cyclic p-group. In this case, the determinant of $\mathsf{C}\big(co\mu_k(G)\big)$ is equal to $(p-1)^n p^m$, where n and m are explicitly computable natural integers.*

The proof of Theorem 1 and Theorem 2 relies on the following observations :

- (cf. [3]) The indecomposable modules for $\mu_k(G, \mathbf{1})$ and $co\mu_k(G)$ (up to isomorphism) are in one to one correspondence with the indecomposable p-permutation (i.e. trivial source) kG-modules (up to isomorphism). Thus, the Cartan matrices of $\mu_k(G, \mathbf{1})$ and $co\mu_k(G)$ can be indexed by the indecomposable p-permutation kG-modules.

  It also follows from [3] that the coefficient $\mathsf{C}^{coh}_{W,W'}$ of the Cartan matrix of $co\mu_k(G)$ indexed by W and W' is equal to

$$\mathsf{C}^{coh}_{W,W'} = \dim_k \operatorname*{Hom}_{kG}(W, W') \ .$$

On the other hand, it was shown in [1] that the coefficient $\mathsf{C}_{W,W'}$ of the Cartan matrix of $\mu_k(G,\mathbf{1})$ indexed by W and W' is equal to

$$\mathsf{C}_{W,W'} = \sum_{R\in[\mathcal{S}_p(G)]} \dim_k \operatorname*{Hom}_{k\overline{N}_G(R)}\big(W[R], W'[R]\big) \ ,$$

where $W[R]$ denotes the Brauer quotient of W at R.

In other words, both matrices $\mathsf{C}\big(\mu_k(G,\mathbf{1})\big)$ and $\mathsf{C}\big(co\mu_k(G)\big)$ are matrices of $\mathbb{Z}$-valued bilinear forms on the subgroup $pp_k(G)$ of the Green group $g_k(G)$ of kG-modules freely generated by the isomorphism classes of indecomposable p-permutation kG-modules.

- The group $pp_k(G)$ is actually a unital subring of $g_k(G)$, closed under k-duality. Each of the previous bilinear forms $\langle -\,,-\rangle$ is such that
$$\langle W, W'\rangle = \langle k, W^* \otimes_k W'\rangle \ .$$
- Let K be a (big enough) field of characteristic 0. The algebra $K\otimes_{\mathbb{Z}} pp_k(G)$ is a split semisimple K-algebra. In a joint work with Jacques Thévenaz (cf. [2]), we have stated explicit formulae for its primitive idempotents. These idempotents form an alternative K-basis of $K\otimes_{\mathbb{Z}} pp_k(G)$, in which the matrix of the bilinear form $\langle -,-\rangle$ is (essentially) diagonal : more precisely, the Cartan matrices C and C^{coh} can be factored as products of square matrices
$$\mathsf{C} = \mathsf{T}'\cdot\mathsf{D}\cdot{}^t\mathsf{T} \ \text{ and } \ \mathsf{C}^{coh} = \mathsf{T}'\cdot\mathsf{D}^{coh}\cdot{}^t\mathsf{T} \ ,$$
where D and D^{coh} are explicitly computable diagonal matrices, and T and T' are block triangular matrices, with blocks indexed by the set $[\mathcal{S}_p(G)]$. Moreover the diagonal block indexed by R can be read from the matrix of Brauer characters of the algebra $k\overline{N}_G(R)$. In particular T and T' are non singular, and the product det $\mathsf{T}'\cdot$ det T can be computed explicitly.

Theorem 2 can be generalized to blocks, as follows : if b is a block of kG, denote by $co\mu_k(b)$ the corresponding block algebra of $co\mu_k(G)$, under the above natural bijection. Then :

Theorem 3. *With the same assumptions, let b be a block of kG. Then :*

(1) *The rank of the Cartan matrix of $co\mu_k(b)$ is equal to*
$$\operatorname{rk}\mathsf{C}\big(co\mu_k(b)\big) = \sum_{(R,c)\in[\mathcal{C}_p(b)]} |N_G(R,c)\backslash\operatorname{Irr}\big(kC_G(R)c\big)| \ ,$$
where $[\mathcal{C}_p(b)]$ is a set of representatives of b-Brauer pairs (R,c) for which R is cyclic.

(2) *The Cartan matrix of the block $co\mu_k(b)$ is non singular if and only if b is a nilpotent block with cyclic defect groups.*

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Factorable groups and their homology

Carl-Friedrich Bödigheimer
(joint work with Balazs Visy)

Let G be a (discrete) group with a norm $N : G \to \mathbb{N}$, that is $N(a) = 0$ if and only if $a = 1$, $N(a^{-1}) = N(a)$, and $N(ab) \leq N(a) + N(b)$. We assume – and this is only a technical assumption – that the minimal value of N on non-trivial elements is 1. Call a pair (a, b) *geodesic* and write $a||b$ if $N(ab) = N(a) + N(b)$.

A group G with norm N is called *factorable*, if there is a function

$$\eta = (\bar{\eta}, \eta') : G \to G \times G, \qquad g \mapsto \eta(g) = (\bar{\eta}(g), \eta'(g)) = (\bar{g}, g')$$

such that for all $a, b \in G$ the following holds:

(F1) $a = \bar{a}a'$, (F2) $\bar{a}||a'$, (F3) $N(a') = 1$,

(F4.1) $a||b$ if and only if $a'||b$ and $\bar{a}||\overline{a'b}$,

(F4.2) if $a||b$, then $(ab)' = (a'b)'$ and $\overline{ab} = \bar{a}\,\overline{a'b}$.

Here g' is called *(right) prefix* and $\bar{g}$ is called *(left) remainder* of g.

Examples
(1) Any group G is factorable with respect to the trivial norm $N(g) = 1$ for $g \neq 1$; we simply set $g' := g$ and therefore $\bar{g} := 1$.
(2) For the free group $G = Fr(n)$ on generators $x_1, \ldots, x_n$ and N the word length norm we write $g = x_{i_1}^{e_1} \ldots x_{i_r}^{e_r}$ as a reduced word and set $g' := x_{i_r}$ if $e_r > 0$ and $g' := x_{i_r}^{-1}$ if $e_r < 0$.
(3) For the symmetric group $G = \mathfrak{S}_n$ we take N to be the word length norm with respect to all transpositions (so not the Coxeter length). For $g \neq 1$ let k be the largest non-fixed point, and set $g' = (k, g^{-1}(k))$.

There are many interesting examples, as the class of factorable groups is closed under direct, semi-direct and free products. For more examples see [4] and [2].

We filter the classifying space BG of G by $F_h BG = \bigcup |X|$, where the union is over cells $|X|$ with $X = [x_q| \ldots |x_1]$ (in inhomogeneous notation) with $N(x_q) + \ldots + N(x_1) \leq h$. Thus h is an upper bound for a kind of circumference of the simplex $|X|$.
The filtration induces a spectral sequence

$$E^0_{h,q} = H_q(F_h BG, F_{h-1}BG) \Rightarrow H_q(G)$$

for the homology of G. Our main results are the following two theorems.

Theorem 1
If G is factorable, then $H_*(F_h BG, F_{h-1}BG) = 0$ unless $* = h$.

Theorem 2
If G is factorable and has only finitely many elements of norm one, then $\mathbb{V}_h(G) := H_h(F_h BG, F_{h-1}BG)$ is isomorphic to the free abelian group generated by all $X = (x_h, \ldots, x_1)$ with $N(x_k) = 1$ for $k = 1, \ldots, h$ and $\eta(x_{k+1}, x_k) \neq (x_{k+1}, x_k)$ for $k = 1, \ldots, h-1$.

This leads to a new chain complex $\mathbb{V}_*(G)$ which computes the homology of G. There is a Mayer-Vietoris formula for free products $G = G_1 * G_2$ and a Künneth formula for direct products $G = G_1 \times G_2$. We mention that there is a similiar complex computing the homology with coefficients in a G-module. For more details see [4] and [3].

Application
Our main application is the computation of homology groups of moduli spaces of Riemann surfaces. (In fact, this was the origin of our investigations, namely the discovery of Theorem 1 for the symmetric groups.) For this application we consider families $G_\bullet$ of normed groups with functions $D_j : G_p \to G_{p-1}$ satisfying the simplicial identities and not increasing the norm. Taking of each classifying space BG_p the stratum $S_h(G_p) = F_h BG_p - F_{h-1}BG_p$ and gluing them with the functions D_j we obtan a space $S_h(G_\bullet)$. It turns out that for many families $G_\bullet$ these spaces decompose into interesting spaces.

For example, if $G_p = \mathfrak{S}_p$ is the family of symmetric groups, $S_h(\mathfrak{S}_\bullet)$ is homotopy equivalent to the disjoint union of all moduli spaces $\mathfrak{M}^m_{g,1}$ of moduli spaces of Riemann surfaces of genus g with one boundary curve and m punctures, where the union is over all pairs (g, m) such that $2g + m = h$. Other families are alternating groups or Coxeter groups of type B.

If in addition the groups G_p have compatible factorizations η_p, then the modules $\mathbb{V}(G_p)$ can be used to compute the homology of the spaces $S_h(G_\bullet)$. Again for the family of symmetric groups this was applied to compute homology groups of moduli spaces. This has been done in [1].

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On a small quotient of the big absolute Galois group

Sunil K. Chebolu
(joint work with Ido Efrat and Ján Mináč)

1. Introduction

This is a report of a talk given at the Oberwolfach workshop on "cohomology of finite groups: Interactions and applications." It is based on the paper "Quotients of absolute Galois groups which determine the entire Galois cohomology"; see arXiv:0905.1364. A main open problem in modern Galois theory is the characterization of the profinite groups which are realizable as absolute Galois groups of fields F. The non-trivial torsion in such groups is described by the Artin–Schreier theory from the late 1920's, namely, it consists solely of involutions. More refined information on the structure of absolute Galois groups is given by Galois cohomology, systematically developed starting the 1950's by Tate, Serre, and others. Yet, explicit examples of torsion-free profinite groups which are not absolute Galois groups are rare. In 1970, Milnor [8] introduced his K-ring functor $K_*^M(F)$, and pointed out close connections between this graded ring and the mod-2 Galois cohomology of the field. This connection, in a more general form, became known as the Bloch–Kato conjecture: it says that for all $r \geq 0$ and all m prime to char F, there is a canonical isomorphism $K_r^M(F)/m \to H^r(G_F, \mu_m^{\otimes r})$. The conjecture was proved for $r = 2$ by Merkurjev and Suslin [7], for r arbitrary and $m = 2$ by Voevodsky [13], and in general by Rost, Voevodsky, with a patch by Weibel ([14], [16], [15]).

We obtain new constrains on the group structure of absolute Galois groups of fields, using this isomorphism. We use these constrains to produce new examples of torsion-free profinite groups which are not absolute Galois groups. We also demonstrate that the maximal pro-p quotient of the absolute Galois group can be characterized in purely cohomological terms! The main object of the talk is a remarkable small quotient of the absolute Galois group, which, because of the above isomorphism, already carries a substantial information about the arithmetic of F.

2. Main theorems

Fix a prime number p and a p-power $q = p^d$, with $d \geq 1$. All fields which appear in this paper will be tacitly assumed to contain a primitive qth root of unity. Let F be such a field and let $G_F = \mathrm{Gal}(F_{\mathrm{sep}}/F)$ be its absolute Galois group, where F_{sep} is the separable closure of F. Let $H^*(G_F) = H^*(G_F, \mathbb{Z}/q)$ be the Galois cohomology ring with the trivial action of G_F on $\mathbb{Z}/q$. Our new constraints relate the descending q-central sequence $G_F^{(i)}$, $i = 1, 2, 3, \ldots$, of G_F with $H^*(G_F)$. Setting $G_F^{[i]} = G_F/G_F^{(i)}$, we show that the quotient $G_F^{[3]}$ determines $H^*(G_F)$, and vice versa. Specifically, we prove:

Theorem A. *The inflation map gives an isomorphism*

$$H^*(G_F^{[3]})_{\mathrm{dec}} \xrightarrow{\sim} H^*(G_F),$$

where $H^*(G_F^{[3]})_{\mathrm{dec}}$ *is the decomposable part of* $H^*(G_F^{[3]})$ *(i.e., its subring generated by degree* 1 *elements).*

We further establish the following result.

Theorem B. *Let* F_1*,* F_2 *be fields and let* $\pi\colon G_{F_1} \to G_{F_2}$ *be a (continuous) homomorphism. The following conditions are equivalent:*

(i) *the induced map* $\pi^*\colon H^*(G_{F_2}) \to H^*(G_{F_1})$ *is an isomorphism;*

(ii) *the induced map* $\pi^{[3]}\colon G_{F_1}^{[3]} \to G_{F_2}^{[3]}$ *is an isomorphism.*

Theorems A and B show that $G_F^{[3]}$ is a Galois-theoretic analog of the cohomology ring $H^*(G_F)$. Its structure is considerably simpler and more accessible than the full absolute Galois group G_F (cf. e.g., [2]). Yet, as shown in our theorems, these small and accessible quotients encode and control the entire cohomology ring.

In the case $q = 2$ the group $G_F^{[3]}$ has been extensively studied under the name "W-group", in particular in connection with quadratic forms [9], [1], [6]). In this special case, Theorem A was proved in [1, Th. 3.14]. It was further shown that then $G_F^{[3]}$ has great arithmetical significance: it encodes large parts of the arithmetical structure of F, such as its orderings, its Witt ring, and certain non-trivial valuations. Theorem A explains this surprising phenomena, as these arithmetical objects are known to be encoded in $H^*(G_F)$ (with the additional knowledge of the Kummer element of -1).

First links between these quotients and the Bloch–Kato conjecture, and its special case the Merkurjev–Suslin theorem, were already noticed in a joint work of Mináč and Spira and in work of Bogomolov.

Our approach is purely group-theoretic, and the main results above are in fact proved for arbitrary profinite groups which satisfy certain conditions on their cohomology. A key point is a rather general group-theoretic approach, partly inspired by [3], to the Milnor K-ring construction by means of quadratic hulls of graded algebras. The Rost–Voevodsky theorem on the bijectivity of the Galois symbol shows that these cohomological conditions are satisfied by absolute Galois groups as above. Using this we deduce aforementioned Theorems in their field-theoretic version.

3. Groups which are not maximal pro-p Galois groups

We now apply Theorem A to give examples of pro-p groups which cannot be realized as maximal pro-p Galois groups of fields (assumed as before to contain a root of unity of order p). The groups we construct are only a sample of the most simple and straightforward examples illustrating our theorems, and many other more complicated examples can be constructed along the same lines. Throughout this section $q = p$. We have the following immediate consequence of the analog of Theorem A for maximal pro-p Galois groups:

Proposition 3.1. *If G_1, G_2 are pro-p groups such that $G_1^{[3,p]} \cong G_2^{[3,p]}$ and that $H^*(G_1) \ncong H^*(G_2)$, then at most one of them can be realized as the maximal pro-p Galois group of a field.*

Corollary 3.2. *Let S be a free pro-p group and R a nontrivial closed normal subgroup of $S^{(3,p)}$. Then $G = S/R$ cannot occur as a maximal pro-p Galois group of a field.*

Example 3.3. Let S be a free pro-p group on 2 generators, and take $R = [S,[S,S]]$. By Corollary 3.2, $G = S/R$ is not realizable as $G_F(p)$ for a field F as above. Note that $G/[G,G] \cong S/[S,S] \cong \mathbb{Z}_p^2$ and $[G,G] = [S,S]/S^{(3,0)} \cong \mathbb{Z}_p$, so G is torsion-free.

Proposition 3.4. *Let G be a pro-p group such that $\dim_{\mathbb{F}_p} H^1(G) < \mathrm{cd}(G)$. When $p = 2$ assume also that G is torsion-free. Then G is not a maximal pro-p Galois group of a field as above.*

Example 3.5. Let K, L be finitely generated pro-p groups with $1 \leq n = \mathrm{cd}(K) < \infty$, $\mathrm{cd}(L) < \infty$, and $H^n(K)$ finite. Let $\pi\colon L \to \mathrm{Sym}_m$, $x \mapsto \pi_x$, be a homomorphism such that $\pi(L)$ is a transitive subgroup of Sym_m. Then L acts on K^m from the left by ${}^x(y_1, \ldots, y_m) = (y_{\pi_x(1)}, \ldots, y_{\pi_x(m)})$. Let $G = K^m \rtimes L$. It is generated by the generators of one copy of K and of L. Hence $\dim_{\mathbb{F}_p} H^1(G) = \dim_{\mathbb{F}_p} H^1(K) + \dim_{\mathbb{F}_p} H^1(L)$.

On the other hand, a routine inductive spectral sequence argument (cf. [10, Prop. 3.3.8]) shows that for every $i \geq 0$ one has

(1) $\mathrm{cd}(K^i) = in$;
(2) $H^{in}(K^i) = H^n(K, H^{(i-1)n}(K^{i-1}))$, with the trivial K-action, is finite.

Moreover, $\mathrm{cd}(G) = \mathrm{cd}(K^m) + \mathrm{cd}(L)$. For m sufficiently large we get $\dim_{\mathbb{F}_p} H^1(G) < mn + \mathrm{cd}(L) = \mathrm{cd}(G)$, so by Proposition 3.4, G is not a maximal pro-p Galois group as above. When K, L are torsion-free, so is G.

For instance, one can take K to be a free pro-p group $\neq 1$ on finitely many generators, and let $L = \mathbb{Z}_p$ act on the direct product of p^s copies of K via $\mathbb{Z}_p \to \mathbb{Z}/p^s$ by cyclicly permuting the coordinates.

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Commuting elements, simplicial spaces and filtrations of classifying spaces

Frederick R. Cohen

1. Introduction

This report is based on joint work with Alex Adem, Enrique Torres-Giese, and José Gomez.

Let G denote a topological group. The classifying space BG plays a central role in algebraic topology with important applications to bundle theory and cohomology of groups. In this paper a filtration of BG is introduced by using the descending central series of the free groups. A basic feature is that if F_n is the free group on n generators with Γ^q the q–th stage of its descending central series, then the spaces of homomorphisms $\mathrm{Hom}(F_n/\Gamma^q, G)$ can be assembled to form simplicial spaces with geometric realizations $B(q,G)$ which filter the usual classifying space BG. In other words there are inclusions

$$B(2,G) \subset B(3,G) \subset \cdots \subset B(q,G) \subset B(q+1,G) \subset \cdots \subset B(\infty,G) = BG$$

where each term is constructed from the simplicial spaces associated to terms in the descending central series of the free group. This naturally gives rise to a functorial construction on topological groups $G \mapsto B(q,G)$. In fact the construction provided here affords a principal G–bundle $E(q,G) \to B(q,G)$ which fits into a commutative diagram, namely there are natural morphisms of principal G-bundles

$$\begin{array}{ccccc} E(q,G) & \xrightarrow{e_q} & E(q+1,G) & \longrightarrow & EG \\ \big\downarrow p & & \big\downarrow p & & \big\downarrow \\ B(q,G) & \xrightarrow{b_q} & B(q+1,G) & \longrightarrow & BG \end{array}$$

and the maps e_q, b_q yield a natural filtration of subspaces for EG and BG.

Properties of these bundles are developed. Two basic cases arise where G is discrete, or connected Lie group. Some of the properties developed are listed next.

(1) Let

$$E_n(q,G) = G \times \mathrm{Hom}(F_n/\Gamma^q, G) \subset G^{n+1},$$

and define $d_i : E_n(q,G) \to E_{n-1}(q,G)$ for $0 \le i \le n$ and $s_j : E_n(q,G) \to E_{n+1}(q,G)$, for $0 \le j \le n$, given by

$$d_i(g_0,\ldots,g_n) = \begin{cases} (g_0,\ldots,g_i \cdot g_{i+1},\ldots,g_n) & 0 \le i < n \\ (g_0,\ldots,g_{n-1}) & i = n \end{cases}$$

and $s_j(g_0,\ldots,g_n) = (g_0,\ldots,g_i,e,g_{i+1},\ldots,g_n)$ for $0 \le j \le n$.

Similarly, let $B_n(q,G) = \mathrm{Hom}(F_n/\Gamma^q, G)$ with maps d_i and s_j defined in the same way, except that the first coordinate g_0 is omitted and the map d_0 takes the form $d_0(g_1,\ldots,g_n) = (g_2,\ldots,g_n)$.

The functions d_i and s_j satisfy the axioms for a simplicial space. The geometric realization of the simplicial space $\{E_n(q,G) \mid n \ge 0\}$ is denoted $E(q,G)$ while $B(q,G)$ denotes the geometric realization of the simplicial space $\{B_n(q,G) \mid n \ge 0\}$.

(2) Let F_jG^n be the subspace of G^n that consists of n-tuples with at least j coordinates equal to 1_G, define $S_n(j,q,G) = \mathrm{Hom}(F_n/\Gamma^q, G) \cap F_jG^n$. The spaces $\mathrm{Hom}(F_n/\Gamma^q, G)$, and $B(q,G)$ frequently admit associated stable decompositions. If G is a closed subgroup of $GL(n,\mathbb{R})$, then there are homotopy equivalences

$$\Sigma\,\mathrm{Hom}(F_n/\Gamma^q, G) \to \bigvee_{1 \le k \le n} \Sigma \bigvee^{\binom{n}{k}} \mathrm{Hom}(F_k/\Gamma^q, G)/S_k(q,G).$$

Furthermore, the natural filtration quotients

$$E_k^0(B(q,G)) = F_kB(q,G)/F_{k-1}B(q,G)$$

of the geometric realization $B(q,G)$ are stably homotopy equivalent to the summands $\mathrm{Hom}(F_k/\Gamma^q, G)/S_k(q,G)$.

One application is a formula for the number of homomorphisms from F_n/Γ^q to G in terms of the singular homology of the filtration quotients $E_k^0(B(q,G))$ and the stable decomposition of $\mathrm{Hom}(F_n/\Gamma^q, G)$.

(3) In case G is a discrete group, the fibration

$$E(q,G) \to B(q,G)$$

is the projection in a regular G-cover. Thus there is a natural representation in the automorphism group

$$G \to Out(\pi_1(E(q,G)).$$

Indeed, in case G is finite of odd order, the Feit-Thompson theorem that G is solvable is equivalent to the fact that the natural map

$$H_1(E(q,G)) \to H_1(B(q,G))$$

fails to be surjective for some q. The representation $G \to Out(\pi_1(E(q,G))$. is fundamental for the Serre exact sequence associated to the fibration $B(q,G) \to BG$, and for addressing the map $H_1(E(q,G)) \to H_1(B(q,G))$ (Caution: It is far from clear that the natural topology arising in this setting is useful.)

(4) The natural structure of

$$\mathrm{Hom}(F_n/\Gamma^q, G)/G^{\mathrm{ad}}$$

where G acts via the adjoint representation is developed. For example, assume that $G = U(m)$. Then the space $\mathrm{Hom}(F_n/\Gamma^2, G)/G^{\mathrm{ad}}$ is homeomorphic to the m-fold symmetric product of $(S^1)^n$. In case $G = Sp(m)$, the space $\mathrm{Hom}(F_n/\Gamma^2, G)/G^{\mathrm{ad}}$ is homeomorphic to the m-fold symmetric product of the orbifold quotient of $(S^1)^n$ by the diagonal action of $\mathbb{Z}/2\mathbb{Z}$ acting by complex conjugation.

(5) Let $\mathcal{N}_q(G)$ denote all of the subgroups of nilpotence class less than q in a finite group G. If G is a finite group, let $G(q) = \varinjlim_{A \in \mathcal{N}_q(G)} A$. Similarly, if G is a finite group, let $G(q) = \varinjlim_{A \in \mathcal{N}_q(G)} A$.

For example, $\mathcal{N}_2(G)$ is the family of abelian subgroups in G, hence $G(2)$ is the colimit of all abelian subgroups in G. The sequence of groups $G(2), G(3), G(4), \ldots$ captures information relevant to the subgroups of G of increasing nilpotence class. Their classifying spaces will play a role in the analysis of the spaces $B(q,G)$.

Let G be a finite group, then for any $q \geq 2$, there is a natural fibration $B(q,G) \to BG(q)$ with fiber a simply–connected finite dimensional complex K_q. A basic question is that of determining under what conditions the fibre K_q is contractible.

(6) A group G is defined to be transitively commutative (TC) if given elements $g, h, k \in G - Z(G)$, $[g, h] = 1 = [h, k]$, then $[g, k] = 1$. It follows that there exists a set of elements $a_1, \dots, a_k \in G - Z(G)$ be a set of representatives of their centralizers so that $G = \bigcup_{1 \le i \le k} C_G(a_i)$ and no smaller number of centralizers covers G. If G is a TC group with trivial center, then there is a homotopy equivalence $B(2, G) \to \bigvee_{1 \le i \le k} \left(\prod_{p || C_G(a_i)|} BP \right)$ where $P \in Syl_p(G)$.

Related, but less concrete results are obtained for more general finite groups.

(7) Related decompositions are given in case G is simply-connected, compact Lie.

(8) Torres-Giese has developed new features of these spaces by considering the probability that a chosen ordered set of q elements generates an abelian subgroup of G. He then establishes connections of this probability to the geometric properties of the spaces $B(2, G)$ and $E(2, G)$ by assembling these for all q.

Some of the papers posted on this subject are as follows.

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Stable cohomology of classical groups over finite fields with polynomial coefficients

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(joint work with Christine Vespa)

This is a report on the article [1].

Let $G \subset GL_N$ be a nice linear algebraic group and k a finite field of characteristic p. We are interested with the cohomology $H^*(G(k))$ of the *discrete* (finite) group $G(k)$ of points of G over k.

We will *not* use, in our approach, that $G(k)$ comes from an algebraic group, but we could make it to get some interesting results (see works of Cline-Parshall-Scott-van der Kallen, Friedlander-Suslin, Betley, or more recently Touzé). In fact our method applies in other situations.

With untwisted coefficients, we can do complete computation in characteristic different of p (Quillen, for GL_N), but the situation is very mysterious in characteristic p. We know also:

- Suppose that $G = G_n \subset G_{n+1} \subset \dots$ inserts in an infinite family of classical groups,the *stable cohomology* $H^*(G_\infty(k); \mathbb{F}_p) := \lim\limits_{i \in \mathbb{N}} H^*(G_i(k); \mathbb{F}_p)$, which has been shown to be trivial for $G_i = GL_i$ by Quillen; its result was adapted for other infinite families of classical algebraic groups.
- This cancellation gives also some informations on $H^*(G_n(k); \mathbb{F}_p)$ because of *stablity results* (due to Charney, van der Kallen and many other): in many cases ($G_i = GL_i, Sp_{2i}$...) the natural map $H^d(G_{n+1}(k)) \to H^d(G_n(k))$ is an isomorphism if $n > 2d$.

Our aim: to determine the stable cohomology in characteristic p with nice twisted coefficients.

It was done by Betley and Suslin for general linear groups (we will remind this soon) ; with C. Vespa, we dealt the case of orthogonal or symplectic groups.

One significant example of the computations that we got is the following:

Theorem 1 (Djament-Vespa). *Suppose that the prime p is odd and $q = p^d$. Then*

$$\lim_{n \in \mathbb{N}} H^*(Sp_{2n}(\mathbb{F}_q); \mathbb{F}_q[X_1, Y_1, \dots, X_n, Y_n])$$

is a polynomial algebra on generators $x_{m,s}$, for integers $m \geq 0$ and $s > 0$, of bidegree $(2mq^s, q^s + 1)$, where the first degree is the cohomological one and the second the internal one (coming from the graduation of the polynomial algebra).

1. Some reminders

Polynomial functors. Let $\mathcal{V}(k)$ denote the category of k-vector spaces and $\mathcal{V}^f(k)$ the full subcategory of finite-dimensional vector spaces. We denote by $\mathcal{F}(k) = \mathbf{Fct}(\mathcal{V}^f(k), \mathcal{V}(k))$ the category of functors from $\mathcal{V}^f(k)$ to $\mathcal{V}(k)$.

The (exact) endofunctor Δ, called *difference functor*, of $\mathcal{F}(k)$ is defined by

$$\Delta(F)(V) = \mathrm{Ker}\,(F(V \oplus k) \to F(V)).$$

A functor $F \in \mathcal{F}(k)$ in said *polynomial* of degree $< d$, where d is an integer, if $\Delta^d(F) = 0$.

Example. For all $n \in \mathbb{N}$, the functors n-th symmetric or exterior power, denoted respectively by S^n and Λ^n, are polynomial of degree n.

Stable cohomology of the general linear group with polynomial coefficients. For $F \in \mathcal{F}(k)$ and $n \in \mathbb{N}$, $F(k^n)$ is naturally a $GL_n(k)$-module. we form the natural projective system

$$\cdots \to H^*(GL_{n+1}(k); F(k^{n+1})) \to H^*(GL_n(k); F(k^n)) \to \dots$$

whose limit is called the *stable cohomology* of the general linear groups over k with coefficients in F and denoted by $H^*(GL_\infty(k); F_\infty)$.

More generally, supposing that F and G are two functors in $\mathcal{F}(k)$, we can endow the k-vector space $\mathrm{Hom}_k(F(k^n), G(k^n))$ with the diagonal action of $GL_n(k)$ and define similarly $H^*(GL_\infty(k); \mathrm{Hom}_k(F,G)_\infty)$.

Theorem (Betley). *For $F \in \mathcal{F}(k)$ a polynomial functor such that $F(0) = 0$,*

$$H^*(GL_\infty(k); F_\infty) = 0.$$

Theorem A (Betley, Suslin). *Let F and G be polynomial functors in $\mathcal{F}(k)$. Then the natural graded map*

$$\mathrm{Ext}^*_{\mathcal{F}(k)}(F,G) \to H^*(GL_\infty(k); \mathrm{Hom}_k(F,G)_\infty)$$

induced by the evaluation functors from $\mathcal{F}(k)$ to $GL_n(k)$-representations is an isomorphism.

2. Main result for symplectic groups

If F is a functor in $\mathcal{F}(k)$, we can define the stable cohomology of the symplectic groups over k with coefficients in F to be the limit of the projective system

$$\cdots \to H^*(Sp_{2n+2}(k); F(k^{2n+2})) \to H^*(Sp_{2n}(k); F(k^{2n})) \to \dots$$

(which stabilizes in each cohomological degree, as in the case of GL, if F is polynomial), that we denote by $H^*(Sp_\infty(k); F_\infty)$.

Remark. We can easily see that we do not get something more general with two functors.

The main theorem of our work is the following.

Theorem 2 (Djament-Vespa). *Let k be a finite field. If $F \in \mathcal{F}(k)$ is polynomial, there is a natural graded isomorphism*

$$H^*(Sp_\infty(k); F_\infty) \simeq \mathrm{Ext}^*_{\mathcal{F}(k)}(k[\Lambda^2], F).$$

In this statement, $k[E]$ denotes, where E is any set, the k-vector space (freely) generated by E.

We can compute these extension groups when the characteristic of k is odd and F is a nice polynomial functor (as a symmetric power, getting theorem 1)

by using fundamental computations made by Franjou-Friedlamder-Scorichenko-Suslin and some elementary facts around graded-exponential functors.

3. Structure of the proof of the main theorem

We introduce the following categories:

- $\mathcal{V}^f_{inj}$ the subcategory of injections in $\mathcal{V}^f(k)$;
- $\mathcal{V}^{deg}_{alt}$ the category of finite-dimensional symplectic k-vector spaces (possibly degenerate) with the linear *injections* preserving the alternating forms as morphisms ;
- $\mathcal{V}_{alt}$ the full subcategory of non-degenerate symplectic spaces.

The proof of theorem 2 decomposes as follows:

$$H^*(Sp_\infty(k); F_\infty) \underset{(1)}{\simeq} H^*(\mathcal{V}_{alt}; F) \underset{(2)}{\simeq} H^*(\mathcal{V}^{deg}_{alt}; F) \underset{(3)}{\simeq} \dots$$

$$\dots \mathrm{Ext}^*_{\mathbf{Fct}(\mathcal{V}^f_{inj}, \mathcal{V}(k))}(k[\Lambda^2], F) \underset{(4)}{\simeq} \mathrm{Ext}^*_{\mathcal{F}(k)}(k[\Lambda^2], F).$$

We now give some short indications for each step.
Isomorphism (1): we deduce this isomorphism from the general setting that we develop to deal with stable (co)homology with twisted coefficients.
Isomorphism (2): it is the hardest part of the proof, which use the fact that F is polynomial (to be able to apply some known cancellation results in functor cohomology) and that k is finite in a Mackey functor argument.
Isomorphism (3): it is a formal adjunction argument.
Isomorphism (4): it is a particular case of the a result, which is a key-step in Suslin's proof of theorem A.

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Cohomology of Hecke algebras over fields of characteristic zero

Karin Erdmann

(joint work with Dave Benson, Aram Mikaelian)

Let $\mathcal{H} = \mathcal{H}(n, q)$ be the Hecke algebra of the symmetric group $\mathfrak{S}_n$ over a field k of characteristic zero and where q is a primitive ℓth root of unity. This has generators $T_1, \dots, T_{n-1}$ satisfying braid relations together with the relations $(T_i + 1)(T_i - q) = 0$. We assume that $\ell \geq 2$. Write $n = \ell m + a$ where $0 \leq a < \ell$, and let $\mathcal{B} = \mathcal{H}(\lambda, q)$ where λ is the partition

$$\lambda = (\ell^m, 1^a).$$

Then $\mathcal{B}$ is a maximal ℓ-parabolic subalgebra of $\mathcal{H}$. It has been proved by Du [3] that every $\mathcal{H}$-module is relatively $\mathcal{B}$-projective. This suggests that $\mathcal{B}$ should play a role similar to that of the group algebra of a Sylow subgroup of a finite group.

The algebra $\mathcal{H}$ has a trivial module, so it has cohomology, that is

$$H^*(\mathcal{H}, k) := \mathrm{Ext}^*_{\mathcal{H}}(k, k),$$

and similarly we define cohomology for $\mathcal{B}$.

Here we relate the cohomology of $\mathcal{H}$ to that of $\mathcal{B}$. We prove an analogue of the result for group algebras which states that if G is a finite group and F is a field of characteristic p then $H^*(G,F)$ is isomorphic to the stable part of $H^*(P,F)$ where P is a Sylow p-subgroup of G (Cartan and Eilenberg [1], Theorem XII.10.1), and that if furthermore P is abelian then the stable elements are the invariants of the action of $N_G(P)/P$ (Swan [7], corrected in [8]).

The symmetric group $\mathfrak{S}_m$ acts naturally on $\mathcal{B}$ and on $H^*(\mathcal{B},k) = \mathrm{Ext}^*_{\mathcal{B}}(k,k)$, and we prove

Theorem. *The restriction map in cohomology induces an isomorphism*

$$H^*(\mathcal{H},k) \to H^*(\mathcal{B},k)^{\mathfrak{S}_m}.$$

We also calculate the invariants $H^*(\mathcal{B},k)^{\mathfrak{S}_m}$, by exploiting [6]. The answer is particularly interesting when $\ell = 2$ which is the only case where it is not graded commutative (the degree one elements do square to zero). We obtain presentations by generators and relations:

Theorem. (1) *If $\ell > 2$ then $H^*(\mathcal{B},k) = \Lambda(y_1,\dots,y_m) \otimes_k k[x_1,\dots,x_m]$ with $|y_i| = 2\ell - 3$ and $|x_i| = 2\ell - 2$. Defining a derivation d on $H^*(\mathcal{B},k)$ via $d(x_i) = y_i$, $d(y_i) = 0$, we have*

$$H^*(\mathcal{H},k) \cong H^*(\mathcal{B},k)^{\mathfrak{S}_m} = \Lambda(d\sigma_1,\dots,d\sigma_m) \otimes_k k[\sigma_1,\dots,\sigma_m]$$

where σ_i is the ith elementary symmetric polynomial in $x_1,\dots,x_m$.

(2) *If $\ell = 2$ then $H^*(\mathcal{B},k) = k\langle z_1,\dots,z_m\rangle/(z_iz_j + z_jz_i,\ i \neq j)$ with $|z_i| = 1$. Let v_i be the ith elementary symmetric function in $z_1^2,\dots,z_m^2$, so that $|v_i| = 2i$. There are elements $u_i \in H^{2i-1}(\mathcal{B},k)^{\mathfrak{S}_m}$ $(1 \le i \le m)$ satisfying*

$$u_i^2 = \sum_{l=0}^{i-1}(2l+1)v_{i-l-1}v_{i+l} \tag{0.1}$$

for $1 \le i \le m$ (for $i = 1$ this relation says that $u_1^2 = v_1$), and

$$u_iu_j + u_ju_i = 2\sum_{l=0}^{j-1}(i-j+2l+1)v_{j-l-1}v_{i+l} \tag{0.2}$$

for $1 \le j < i \le m$, where v_i is taken to be zero if $i > m$ and $v_0 = 1$. We have

$$H^*(\mathcal{H},k) \cong H^*(\mathcal{B},k)^{\mathfrak{S}_m} = k\langle u_1,\dots,u_m,v_2,\dots,v_m\rangle/(R)$$

where (R) is the following set of relations:

(1) $v_iv_j = v_jv_i \quad (1 \le i,j \le m)$
(2) $u_iv_j = v_ju_i \quad (1 \le i,j \le m)$
(3) *relation* (0.1) $(2 \le i \le m)$
(4) *relation* (0.2) $(1 \le j < i \le m)$

where, in the right hand side of relations (0.1) *and* (0.2)*, we take* v_1 *to be* u_1^2*.*

In both cases, $\ell > 2$ and $\ell = 2$, the following is the Poincaré series for the cohomology:

$$\sum_{i\geq 0} t^i \dim_k H^i(\mathcal{H}, k) = \frac{(1+t^{2(\ell-1)-1})(1+t^{4(\ell-1)-1})\dots(1+t^{2m(\ell-1)-1})}{(1-t^{2(\ell-1)})(1-t^{4(\ell-1)})\dots(1-t^{2m(\ell-1)})}.$$

We also deduce the analogous theorem for the Hecke algebras of types B_n and D_n for which the trivial module belongs to a component which is Morita equivalent to $\mathcal{H}(n,q)$. This holds for some parameters, by Morita classifications due to Dipper and James (types B_n, [2]), and to Pallikaros, and Hu (types D_n, [5], [4]).

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Using p-nilpotent operators to investigate extensions

Eric M. Friedlander

We suggest a somewhat different paradigm for the consideration of non-semisimple categories of representations of finite group schemes (and more general algebraic structures). Whereas it is often the case that one parametrizes or lists irreducible representations with respect to the action of semi-simple elements (such as elements of a maximal torus), we emphasize the role of the action of p-nilpotent elements in the study of indecomposable representations.

This perspective is employed in a recent preprint by the author [3] which is a refinement of work of Jon Carlson, Zongzhu Lin, and Daniel Nakano [1]. This talk presents some background, especially foundational work of Julia Pevtsova and the author, and ends with a brief discussion of results of [3].

The outline of this talk is as follows:

(1) Representation theory for algebraic groups $\mathfrak{G}$.
(2) π-points, giving a uniform treatment of certain aspects of the representation theory of $\mathfrak{G}(\mathbb{F}_p)$ and $\mathfrak{G}_{(r)}$.
(3) 1-parameter subgroups and the global p-nilpotent operator for infinitesimal group schemes.

(4) The 1-parameter subgroup ϕ_x of Gary Seitz associated to a p-nilpotent element $x \in \mathfrak{G}(k)$.
(5) Comparison of actions.
(6) Comparison of invariants.

We first set the stage: $\mathfrak{G}$ is a reductive algebraic group defined over $\mathbb{F}_p$, k is a field of characteristic $p \geq h(\mathfrak{G})$, and M is a rational $\mathfrak{G}$-module. In [1], a comparison of support varieties for the finite Chevalley group $\mathfrak{G}(\mathbb{F}_p)$ and the infinitesimal group scheme $\mathfrak{G}_{(1)}$ was given (see also [4]).

We briefly recall the definition of a π-point $\alpha_K : K[t]/t^p \to KG$ of a finite group scheme G and the scheme $\Pi(G)$ of equivalence classes of π-points as developed by Julia Pevtsova and the author in [5]. In particular, we recall the existence of a natural isomorphism $\operatorname{Proj} H^\bullet(G,k) \simeq \Pi(G)$ and the identification of the *support variety* of a kG-module as

$$\Pi(G)_M \; = \; \{[\alpha_K] : \alpha_K^*(M_K) \text{ is not free}\}.$$

We then consider a finite group scheme G which is infinitesimal of height $\leq r$ (such as the Frobenius kernel $\mathfrak{G}_{(r)}$ of a smooth algebraic group $\mathfrak{G}$). As shown by Andrei Suslin, Christopher Bendel, and the author in [9], [10], the functor on finitely generated, commutative k-algebras

$$A \; \mapsto \; \operatorname*{Hom}_{grpsch/A}(G_{a(r),A}, G_A)$$

is representable by an affine conical scheme $V(G)$, the scheme of 1-parameter subgroups of G. Moreover, there is a natural F-isomorphism

$$V(G) \; \simeq \; \operatorname{Spec} H^\bullet(G,k).$$

The existence of a universal 1-parameter subgroup for an infinitesimal group scheme G enabled Julia Pevtsova and the author in [6] to construct a *global p-nilpotent operator*

$$\Theta_G \; \in \; kG \otimes k[V(G)]$$

with the property that at any 1-parameter subroup $\mu \in V(G)$ the base change $\mu^*(\Theta_G) \in kG$ is given by

$$\mu^*(\Theta_G) \; = \; \mu_*(u_{r-1})$$

where $u_{r-1} \in k\mathbb{G}_{a(r)}$ is the k-linear dual of $t^{r-1} \in k[t]/t^r = k[\mathbb{G}_{a(r)}]$.

These background strands are tied together through the construction by Gary Seitz [8] (following work of Donna Testerman) of a canonically defined 1-parameter subgroup $\phi_x : \mathbb{G}_a \to \mathfrak{G}$ associated to a given p-unipotent element $x \in \mathfrak{G}(k)$. This enables the construction in [3] of a "canonical" map

$$\Pi(\mathfrak{G}(\mathbb{F}_p)) \; \to \; \Pi(\mathfrak{G}_{(r)})/\mathfrak{G}(\mathbb{F}_p),$$

where the action of $\mathfrak{G}(\mathbb{F}_p)$ on $\Pi(\mathfrak{G}_{(r)})$ is the adjoint action.

The work of Carlson, Lin, and Nakano [1] involved the comparison of support varieties for $\mathfrak{G}(\mathbb{F}_p)$ and $\mathfrak{G}_{(1)}$, a comparison necessarily limited because of the following phenomenon: if a rational $\mathfrak{G}$-module M is replaced by its Frobenius twist $M^{(1)}$, then its restriction to $\mathfrak{G}(\mathbb{F}_p)$ is unchanged whereas the restriction of $M^{(1)}$

to $\mathfrak{G}_{(1)}$ is trivial. In order to understand how large r must be in comparing $\mathfrak{G}(\mathbb{F}_p)$ with $\mathfrak{G}_{(r)}$, we introduce the p-nilpotent degree of a rational $\mathfrak{G}$-module.

With all of this background, we then mention how to compare the action of

$$x-1 \ \in kG(\mathbb{F}_p), \quad \phi_x \circ \sigma \in k\mathfrak{G}_{(r)}$$

on a rational $\mathfrak{G}$-module M, where $x \in \mathfrak{G}(\mathbb{F}_p)$ is p-unipotent and $\sigma : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(r)}$ sends t to $\sum_{i\geq 0} t^{p^i}$. In particular, these actions as elements of $End_k(M)$ have difference in the square of the maximal ideal.

This comparison enables the author to derive new results in [3], including the following (where r is required to satisfy the condition that p^r is at least the p-nilpotent degree of M).

- The image of $\Pi(\mathfrak{G}(\mathbb{F}_p))_M$ in $\Pi(\mathfrak{G}_{(r)})/\mathfrak{G}(\mathbb{F}_p)$ is described.
- The non-maximal support varieties $\Gamma^j(\mathfrak{G}(\mathbb{F}_p))_M$, $\Gamma^j(\mathfrak{G}_{(r)})_M$ (as recently defined by Julia Pevtsova and the author in [7]) are compared.
- For $q = p^d$, these results can be extended to $\mathfrak{G}(\mathbb{F}_q)$ using the Weil restriction $Res_{\mathbb{F}_q,\mathbb{F}_p}(\mathfrak{G})$ as in [2].

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Symmetry of spaces and subgroups of $B\mathrm{aut}(X)$

Jesper Grodal
(joint work with Bill Dwyer)

The goal of this talk is to report on some joint work in progress [2] on understanding the subgroup structure of the topological monoid of self-homotopy equivalences $\mathrm{aut}(X)$ of a finite simply connected CW–complex X. Here, by a subgroup we mean a map $BG \to B\mathrm{aut}(X)$ that satisfies one of several equivalent conditions that justifies calling it a "monomorphism". Subgroups in this sense correspond to faithful group actions on X, up to homotopy.

Perhaps the first indication that $B\mathrm{aut}(X)$ should have interesting group theoretic properties was discovered by Sullivan and Wilkerson, who showed that $\pi_0(\mathrm{aut}(X))$ is an arithmetic group [3]. It is natural to speculate in which ways this can be extended to a space-level statement.

The type of naïve questions we are interested in are the following:

(1) How many subgroups are there? (e.g., finite or infinite?)
(2) How "large" can a subgroup be?
(3) What does $B\mathrm{aut}(X)$ look like cohomologically?

Concerning (1), it was observed some time ago by J. Smith that the set of conjugacy classes of subgroups of, say, order p can be infinite, e.g., for $X = S^3 \vee S^3 \vee S^5$, a departure from what happens for arithmetic groups. However, it follows from the work of Grodal–Smith, that for $X = S^n$ the set of conjugacy classes of subgroups isomorphic to a fixed finite group G is indeed finite. Here we show that for any simply connected finite CW–complex the set of conjugacy classes of subgroups which correspond to free actions is also finite, and examine other cases where the same conclusion holds.

Concerning (2), one can define a homotopical version of the classical notion of the p–rank of symmetry as follows:

$$h\text{-}rk_p(X) = \max\{r \,|\, \exists \text{ mono } f\colon\ B(\mathbb{Z}/p)^r \to B\mathrm{aut}(X)\}$$

The homotopy p–rank of symmetry $h\text{-}rk_p(X)$ provides an upper bound for the free p–rank of symmetry, as well as the corresponding S^1–rank of symmetry. Earlier work of Grodal–Smith implies that $h\text{-}rk_2(S^n) = n+1$ and $h\text{-}rk_p(S^{2n-1}) = h\text{-}rk_p(S^{2n}) = n$, p odd, realized by the standard reflections in the coordinates.

One basic question one may ask is whether the rank is always finite. We answer this in the affirmative:

Theorem (Dwyer–Grodal)**.** *For any finite simply connected CW–complex X, $h\text{-}rk_p(X) < \infty$.*

Via Lannes' theory the statement implies that the transcendence degree of $H^*(B\mathrm{aut}_1(X);\mathbb{F}_p)$ is finite, where the subscript denotes the identity component, providing information regarding (3).

The current proof produces bounds on $h\text{-}rk_p(X)$ which are homotopic in nature, in particular they depend on information about the homotopy type of Postnikov sections of $B\mathrm{aut}(X)$. It seems reasonable to expect better and more algebraic bounds. This could provide analogs for faithful actions of various results and conjectures of Browder, Carlsson, and others on the free rank of symmetry of products of spheres and more general finite complexes [1].

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Homotopy Euler characteristic and the stable free rank of symmetry

Bernhard Hanke

1. Introduction

Let X be a finite connected complex. We define the *free toral rank* of X

$$\mathrm{rk}_T(X) := \max\{r \mid (S^1)^r \text{ acts freely on } X\}$$

and for any prime number p the *free p-rank*

$$\mathrm{rk}_p(X) := \max\{r \mid (\mathbb{Z}/p)^r \text{ acts freely on } X\}\,.$$

Both invariants measure the "amount of symmetry" of X in a certain sence. How are they related?

Obviously $\mathrm{rk}_p(X) \geq \mathrm{rk}_T(X)$ for all p. The reverse inequality is wrong in general, the circle $S^1 \subset \mathbb{C}$ with p copies of S^1 attached at the pth roots of unity providing a counterexample.

Now assume that the fundamental group of X is abelian and acts trivially on the higher homotopy groups and that the rational homotopy of X is a finitely generated $\mathbb{Q}$-vector space.

The *homotopy Euler characteristic* of X is defined as

$$\chi\pi(X) := \bigoplus_{n=1}^{\infty} (-1)^n \dim_{\mathbb{Q}}(\pi_n(X) \otimes \mathbb{Q})\,.$$

The following bound is classical, see [7].

Theorem 1.

$$\mathrm{rk}_T(X) \leq -\chi\pi(X)\,.$$

Our main result is an asymptotic upper bound for the free p-rank of X.

Theorem 2. *There is a $P \in \mathbb{N}$ so that for all $p \geq P$ we have*

$$\mathrm{rk}_p(X) \leq -\chi\pi(X)\,.$$

Corollary 1 ([6]). *Let $X = S^{n_1} \times \ldots \times S^{n_k}$ be a product of spheres. Then there is a P so that for all primes $p \geq P$ the free p-rank of X is equal to the number of odd spheres in X.*

This answers [1, Question 7.2] in the affirmative for large primes.

The following interesting question remains open: Is there a finite complex X with the above properties so that $\mathrm{rk}_p(X) > \mathrm{rk}_T(X)$ for infinitely many primes p?

We prove Theorem 2 by combining the tame homotopy approach from [6] with the methods in [7].

Set

$$d_h := \max\{n \in \mathbb{N} \mid \pi_n(X) \otimes \mathbb{Q} \neq 0\}$$

and

$$\overline{1} = 1\,, \quad \overline{t} = 3(t-1) \text{ for } t \geq 2\,.$$

Now choose

$$k \geq 2 \cdot \max\{d_h, \dim X\}$$

and $P \in \mathbb{N}$ so that

(1) $P > \overline{k}$.
(2) $\pi_t(X)$ and $H_t(X;\mathbb{Z})$ do not contain p-torsion for all $p \geq P$ and all $t \leq k$.
(3) $\dim \pi_t(X) \otimes \mathbb{Q} < P - 1$ for all all $t \leq k$.

Let $G = (\mathbb{Z}/p)^r$ act on X where $p \geq P$. We cite

Theorem 3. [6, Theorem 5.6] *There are commutative graded differential algebras (E^*, d_E) and (M^*, d_M) over $\mathbb{F}_p$ with the following properties:*

(1) *as graded algebras $E^* = \mathbb{F}_p[t_1, \ldots, t_r] \otimes \Lambda(s_1, \ldots, s_r) \otimes M^*$ where each t_i is of degree 2 and each s_i is of degree 1.*
(2) *the differential d_E is zero on $\mathbb{F}_p[t_1, \ldots, t_r] \otimes \Lambda(s_1, \ldots, s_r) \otimes 1$ and the map $E^* \to M^*$, $t_i, s_i \mapsto 0$, is a cochain map.*
(3) *M^* is free as a graded algebra with generators in degrees $1, \ldots, k$. As generators in degree $t \in \{1, \ldots, k\}$ we can take the elements of a basis of the free $\mathbb{Z}_{(p)}$-module $\pi_t(X) \otimes \mathbb{Z}_{(p)}$.*
(4) *the cohomology algebra of E^* is multiplicatively isomorphic to $H^*(X_G; \mathbb{F}_p)$ up to degree k.*

Here properties (2) and (3) for P are important: They make sure that G acts trivially on $\pi_t(X) \otimes \mathbb{Z}_{(p)}$ for all $p \geq P$ and $t \leq k$.

If $G = (S^1)^r$ acts on X then a small rational cochain model of X_G can be constructed by use of the Grivel-Halperin-Thomas theorem, cf. [3, Section 2.5]. Our Theorem 3 is an approximative analogue in the context of p-torus actions. It is based on the tame homotopy theory of Dwyer [5] and Cenkl and Porter [4].

We now assume that $G = (\mathbb{Z}/p)^r$ with $p \geq P$ acts freely on X. Then $H^*(X_G; \mathbb{F}_p)$ vanishes in degrees larger than $\dim X$ and hence $H^*(E; \mathbb{F}_p)$ vanishes in degrees between $\dim X + 1$ and $2 \dim X$ by the choice of k.

We denote the generators of M^* by $\tau_1, \ldots, \tau_\gamma$.

Let

$$F^* := \mathbb{F}_p[t_1, \ldots, t_r] \otimes M^*$$

be obtained from E^* by dividing out the ideal $(s_1, \ldots, s_r)$. The induced differential on F^* is denoted by d_F.

We deform d_F to another differential δ_F on F^* as follows: δ_F is a derivation and satisfies

$$\delta_F(\tau_j) = \pi(d_F(\tau_j))\,.$$

for $j = 1, \ldots, \gamma$ where the map $\pi : F^* \to F^*$ denotes the projection onto the even part of F^* given by evaluating the odd degree generators at 0. Then δ_F vanishes on the even degree generators of F^* and $\delta_F^2 = 0$.

Lemma 1. *The cohomology classes represented by* t_i, $i = 1, \dots, r$, *and* τ_j, $j = 1, \dots, \gamma$, $|\tau_j|$ *even, are nilpotent in* $H^*(F, \delta_F)$.

The proof is nontrivial and here will be omitted.

This lemma implies that $H^*(F, \delta_F)$ is a finite dimensional $\mathbb{F}_p$-vector space. Granted this fact the proof of Theorem 2 is finished as follows.

Let $\mathcal{E}$ be the set of even degree generators of F^* and $\mathcal{O}$ be the set of odd degree generators. We consider the ideal

$$I = \big(\delta_F(\mathcal{O})\big) \subset \mathbb{F}_p[\mathcal{E}]$$

contained in $\mathrm{im}(\delta_\mathrm{F})$ and obtain an inclusion

$$\mathbb{F}_p[\mathcal{E}]/I \subset H^*(F, \delta_F)\,.$$

Here we use the fact that the coboundaries in (F^*, δ_F) are contained in the ideal $I \cdot F^*$, whose intersection with $\mathbb{F}_p[\mathcal{E}]$ is equal to I (just apply the projection π).

We conclude that $\mathbb{F}_p[\mathcal{E}]/I$ is a finite dimensional $\mathbb{F}_p$-vector space. Since I is generated by homogenous elements of positive degree, it does not contain a unit of $\mathbb{F}_p[\mathcal{E}]$ and hence there is a minimal prime ideal $\mathfrak{p} \subset \mathbb{F}_p[\mathcal{E}]$ containing I. The quotient $\mathbb{F}_p[\mathcal{E}]/\mathfrak{p}$ is both a finite dimensional $\mathbb{F}_p$-vector space and an integral domain. Hence $\mathfrak{p} = (\mathcal{E})$ and consequently $\mathrm{height}(\mathfrak{p}) = |\mathcal{E}|$. By Krull's Principal Ideal Theorem the number of generators of I must be at least $|\mathcal{E}|$. From the definition of I we derive the inequality $|\mathcal{O}| \geq |\mathcal{E}|$. Subtracting the number $|\mathcal{E}| - r$ from both sides of this inequality, the assertion of Theorem 2 follows.

2. Construction of small $\mathbb{F}_p$-cochain models

The basic ingredients for the proof of Theorem 3 are as follows. Let X be a simplicial complex. For $q \in \mathbb{N}$ we define the ring $\mathbb{Q}_q := \mathbb{Z}[p^{-1} \mid p \leq q]$. There is a filtered commutative graded differential algebra

$$T^{*,0}(X) \subset T^{*,1}(X) \subset \ldots \subset T^{*,q}(X) \subset T^*(X)$$

where each $T^{*,q}(X)$ is a cochain complex of $\mathbb{Q}_q$-modules and $T^*(X)$ is a rational cochain algebra. The multiplication is filtration preserving in the sense that it defines maps

$$T^{n_1,q_1}(X) \times T^{n_2,q_2}(X) \to T^{n_1+n_2,q_1+q_2}(X)\,.$$

Elements in $T^*(X)$ are polynomial differential forms on the simplices of X which are compatible on intersections of simplices. The filtration is defined in such a way that the closed form $t^{q-1}dt$ appears in $T^{*,q}(X)$, but not in a lower filtration degree. Since q is invertible in $T^{*,q}(X)$, the cocycle $t^{q-1}dt$ is the coboundary of $\frac{1}{q}t^q$ in this group.

By integration of forms we get cochain maps

$$\int : T^{*,q}(X) \to C^*(X; \mathbb{Q}_q)\,,$$

which induce a multiplicative isomorphism (of filtered graded rings) in cohomology by the Cenkl-Porter theorem [4]. Since under multiplication of cochains in $T^{*,q}(X)$

more and more primes get inverted, nontrivial Steenrod reduced power operations do not appear.

Under appropriate homotopy theoretic assumption on X the filtered cochain algebra $T^*(X)$ can be replaced by a smaller filtered cochain algebra using a Postnikov decomposition of X and a *tame Hirsch lemma* [8], which says that passing one step up in the Postnikov decomposition of X corresponds to adjoining free polynomial generators of an appropriate filtration degree to the small filtered cochain algebra already constructed. The generators of the resulting filtered cochain algebra correspond to genererators of $\pi_*(X) \otimes \mathbb{Q}_q$ for various q.

Finally a small approximative unfiltered $\mathbb{F}_p$-cochain model of X is (for large enough p) obtained by tensoring an appropriate filtration level of this filtered cochain algebra with $\mathbb{F}_p$.

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Young modules for the symmetric group with large vanishing ranges in cohomology

David J. Hemmer

1. Introduction

In 1990 Benson, Carlson and Robinson proved the following theorem:

Theorem 1. [2, Theorem 2.4] *Given a finite group G, there exists a positive integer $r = r(G)$ such that for any commutative ring R of coefficients and any RG-module M, if the Tate cohomology $\hat{\mathrm{H}}^n(G, M) = 0$ for $r + 1$ consecutive values of n then $\hat{\mathrm{H}}^n(G, M) = 0$ for all n positive and negative.*

The proof gives an explicit construction of an r that works. Namely, suppose $\eta_1, \eta_2, \ldots, \eta_c$ are homogenous generators of the cohomology ring $\mathrm{H}^*(G, \mathbb{Z})$. Then one can choose $r = \sum_i (\deg \eta_i - 1)$.

Alternately, if the group G has a faithful complex representation of degree n, one can embed G in the unitary group $U(n)$. The cohomology of the classifying space $BU(n)$ is a polynomial ring in generators of degree $2, 4, 6, \dots 2n$, and one obtains then $r = (n-1)^2$, see the remark on p.205 of [1].

In neither case is the r one obtains expected to be best possible.

In recent joint work [3]with Cohen and Nakano we used homological methods and algebraic topology to compute, in many cases, Young module cohomology $\mathrm{H}^i(\Sigma_n, Y^\lambda)$ for the symmetric group Σ_n and a Young module Y^λ.

In our talk we discussed two related problems. The first is to compute the smallest degree $i \geq 0$ such that $\mathrm{H}^i(\Sigma_d, Y^\lambda)$ is nonzero. This computation led us to discover some Young modules that have nonzero cohomology, but nevertheless have very large vanishing ranges. Thus the second problem is an attempt to use Young modules to realize the "maximal gap", in the sense of the BCR result, and thus obtain lower bounds for the ideal value of $r(\Sigma_n)$. In this note we present these examples

2. Young modules with large vanishing ranges

The two methods discussed above for determining $r(\Sigma_n)$ give similar answers. Using generators of the cohomology ring, one gets roughly n^2. More straightforward is to use the second method. The faithful $n-1$ dimensional natural representation gives the best known upper bound known for $r(\Sigma_n)$ as $(n-1)^2$. So the question arises, is there a Σ_n-module M with vanishing cohomology for $(n-1)^2$ consecutive degrees (but not $(n-1)^2+1$.)

We prove there are Young modules in all characteristics that vanish for about $n^{3/2}$ consecutive degrees but are not identically zero. Thus one can say the best value for $r(\Sigma_n)$ lies somewhere between $n^{3/2}$ and n^2, as the following examples will demonstrate.

First observe that, since the Young modules Y^λ are self-dual, Tate duality lets us just consider $\mathrm{H}^i(\Sigma_d, Y^\lambda)$ for $i \geq 0$. Theorem 11.1.1 in [3] tells precisely which λ have the property that $\mathrm{H}^i(\Sigma_d, Y^\lambda) = 0$ for all $i \geq 0$. Clearly these are not relevant for determining the best possible r in Theorem 1.

First suppose we are working over a field k of characteristic $p = 2$. Let $\rho = (x, x-1, x-2, \dots, 2, 1) \vdash x(x+1)/2$. Let

$$\lambda = 2\rho = (2x, 2x-2, 2x-4, \dots, 4, 2) \vdash n = x(x+1).$$

Using the results in [3] we can show:

Theorem 2. *For $\lambda \vdash x(x+1)$ as above and $p = 2$, we have:*

$$\mathrm{H}^i(\Sigma_{x(x+1)}, Y^\lambda) \cong \begin{cases} 0 & -\frac{x^3-x}{6} < i < \frac{x^3-x}{6} \\ k & i = \pm\frac{x^3-x}{6}. \end{cases}$$

Furthermore, for any other $\tau \vdash n$, either $\mathrm{H}^\bullet(\Sigma_n, Y^\tau)$ vanishes identically, or $\mathrm{H}^j(\Sigma_n, Y^\tau) \neq 0$ for some $0 < j < \frac{x^3-x}{6}$. That is, our choice of λ gives the maximal possible vanishing range.

It follows from work in [3] that, in characteristic two, if $\mathrm{H}^i(\Sigma_d, Y^\tau) \neq 0$ then $\mathrm{H}^j(\Sigma_d, Y^\tau) \neq$ for all $j \geq i$, which is why we only need consider "initial" vanishing ranges in Theorem 2.

Notice in Theorem 2, the vanishing range is roughly $\frac{n^{3/2}}{3}$.

The case of odd characteristic is a little different, as the main results of [3] are also different in odd characteristic. Instead let ρ be as above but set

$$\mu = p(p-1)\rho \vdash n = \frac{p(p-1)x(x+1)}{2}. \tag{2.1}$$

Then the corresponding vanishing range for $\mathrm{H}^i(\Sigma_n, Y^\mu)$is approximately:

$$-\frac{2^{3/2}(p-1)^{1/2}}{p}n^{3/2} < i < \frac{2^{3/2}(p-1)^{1/2}}{p}n^{3/2}.$$

So in both cases we obtain the order of $n^{3/2}$ consecutive vanishing degrees, followed by a nonzero degree.

We are not aware of any other constructions in the literature where, for an arbitrary $d > 0$, one can construct a group G and a G-module M with $\mathrm{H}^i(G, M)$ vanishing for d consecutive i but not vanishing identically.

3. Remarks

We conclude with a few remarks and problems.

Remark 3. *The choice of n in the examples above is just for convenience. When n is not of the form $n = \frac{p(p-1)x(x+1)}{2}$, the maximal vanishing occurs for $\lambda \vdash n$ of similar shape and the vanishing range is of comparable size.*

Remark 4. *Those familiar with algebraic groups will recognize our μ in* (2.1) *as p times the Steinberg weight $(p-1)\rho$ for the general linear group. We have no good explanation for why the maximal vanishing range occurs at this weight.*

Remark 5. *A more general problem is, given $\lambda \vdash d$, find the first $i > 0$ where $\mathrm{H}^i(\Sigma_d, Y^\lambda) \neq 0$. In order to solve this one must consider the projective cover $P(L(\lambda))$ of the irreducible module $L(\lambda)$ of highest weight λ. Then determine a (the?) maximal μ such that $P(L(\lambda))$ has a nonzero μ weight space.*

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On blocks of finite groups whose defect groups are elementary abelian of order 8

Radha Kessar
(joint work with Shigeo Koshitani, Markus Linckelmann)

We prove the following result.

Theorem. *Let k be an algebraically closed field of characteristic 2, let G be a finite group and let b be a block of kG with an elementary abelian defect group P of order 8. Denote by c the block of $kN_G(P)$ corresponding to b. Then b and c have eight ordinary irreducible characters and there is an isotypy between b and c; in particular, Alperin's weight conjecture holds for all blocks of finite groups with an elementary abelian defect group P of order 8.*

For principal blocks, the theorem follows from work of Landrock [6] and Fong and Harris [5]. The theorem implies Alperin's weight conjecture for all blocks with a defect group of order at most 8. Indeed, the two remaining abelian cases C_8 and $C_2 \times C_4$ admit no automorphisms of odd order, hence arise as defect groups only of nilpotent blocks, and by work of Brauer [4] and Olsson [7], Alperin's weight conjecture is known in the case of the two non-abelian groups D_8 and Q_8 of order 8.

The proof of the theorem uses the classification of finite simple groups. Using a stable equivalence due to Rouquier we show in that Alperin's weight conjecture implies Broué's isotypy conjecture. By the work of Landrock already mentioned, Alperin's weight conjecture follows from the 'if' part of Brauer's height zero conjecture (which predicts that all characters in a block have height zero if and only if the defect groups are abelian). The 'if' part of this conjecture has been reduced to quasi-simple finite groups by Berger and Knörr [2]; we verify independently that this reduction works within the realm of blocks with an elementary abelian defect group of order at most 8 and certain fusion patterns. We finally prove Alperin's weight conjecture for blocks with an elementary abelian defect group of order 8 for quasi-simple groups making use of contributions of many authors, including the proof of Brauer's height zero conjecture for blocks in non-defining characteristic of quasi-simple finite groups of type A and ${}^2A$ by Blau and Ellers [3]. Assembling these parts yields the proof of the theorem in. It is noticeable how few blocks of quasi-simple groups have an elementary abelian defect group - and when they do, most of them are nilpotent.

Alperin announced the weight conjecture in [1]. At that time, the conjecture was known to hold for all blocks of finite groups with cyclic defect groups (by work of Brauer and Dade), dihedral, generalised quaternion, semidihedral defect groups (by work of Brauer and Olsson), and all defect groups admitting only the trivial fusion system (by the work of Broué and Puig on nilpotent blocks). Since then many authors have contributed to proving Alperin's weight conjecture for various classes of finite groups - such as finite p-solvable groups (Okuyama), finite groups of Lie type in defining characteristic (Cabanes), symmetric and general linear groups (Alperin, Fong, An) and some sporadic simple groups (An). It appears that the theorem is the first proof of a case of Alperin's weight conjecture for *all* blocks with a fixed defect group since this conjecture was announced more than two decades ago.

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Restriction to the center in group cohomology

Nicholas J. Kuhn

This is a survey of results from [K1, K2] concerning the restriction of the mod p cohomology of a finite group to its maximal central elementary abelian p–subgroup. This leads to group theoretic bounds on nilpotence in the cohomology ring.

1. A filtration of the center and the invariant $e(G)$.

Fixing a prime p, we let $H^*(G)$ be cohomology with trivial $\mathbb{F}_p$–coefficients of a finite group G. Let $C = C(G)$ be the maximal *central* elementary abelian p–subgroup of G, and let $c = c(G)$ denote its rank.

Note that C is an $\mathbb{F}_p[Out(G)]$–module. In fact, there is more structure: C is a *filtered* $\mathbb{F}_p[Out(G)]$–module, with the filtration determined by, and determining, the image of restriction, $\mathrm{Res} : H^*(G) \to H^*(C)$. This will be a sub–Hopf algebra of $H^*(C)$, due to the $H^*(C)$–comodule structure on $H^*(G)$ determined by the multiplication homomorphism $C \times G \to G$. This greatly limits the possibilities for the image of restriction.

For example, when $p = 2$, this image must have the form

$$\mathbb{F}_2[x_1^{2^{i_1}}, \ldots, x_c^{2^{i_c}}] \subseteq \mathbb{F}_2[x_1, \ldots, x_c] = H^*(C).$$

The filtration then determines an invariant $e(P)$, defined to be the top degree of the finite Hopf algebra $H^*(C) \otimes_{H^*(G)} \mathbb{F}_p$.

Continuing the example, $e(G)$ would equal $\sum_{j=1}^{c} (2^{i_j} - 1)$.

2. The filtration and the Lyndon–Serre spectral sequence

As discussed in detail in [K1, §§5,6], the filtration also equals the filtration defined by the kernels of the differentials off the 0–line in the spectral sequence associated to the extension

$$C \to G \to G/C.$$

The key point is that this is a spectral sequence of $H^*(C)$–comodules. This forces the only possible nonzero differentials off the 0–line to be induced by the 'long' differentials $d_2, d_3, d_5, d_9, \ldots$ when $p = 2$, and similarly when p is odd. These themselves tend to be determined by the extension class $d_2 : H^1(C) \to H^2(G/C)$ and the structure of $H^*(G/C)$ as an unstable algebra over the Steenrod algebra.

Example 2.1. With $p = 2$, let $W(r)$ denote the 2–Sylow subgroup of Σ_{2^r}. $C = C(W(r)) \simeq \mathbb{Z}/2$, so that $H^*(C) = \mathbb{F}_2[x]$. It is not too hard to prove by induction on r that the image of restriction is $\mathbb{F}_2[x^{2^{r-1}}]$, so that $e(W(r)) = 2^{r-1} - 1$. One concludes that in the spectral sequence associated to

$$\mathbb{Z}/2 \to W(r) \to W(r)/(\mathbb{Z}/2)$$

the following long differentials are all nonzero: $d_2, d_3, \ldots, d_{2^{r-2}+1}$.

3. Bounds on $e(P)$ when P is a p–group

We survey some upper bounds for $e(P)$ when P is a p–group. These are proved in our recent preprint [K2].

Using Chern classes of some easy-to-define representations, one can prove:

Proposition 3.1. *Let $A < P$ be an abelian subgroup of maximal order in a p–group P. Then $e(P) \leq c(P)(2|P|/|A| - 1)$.*

Using more general representations, we conjecture the following, in which $n(P)$ denotes the minimal dimension (over $\mathbb{C}$) of a faithful complex representation of P.

Conjecture 3.2. $e(P) \leq 2n(P) - c(P)$.

Let $e_{ab}(P) = c(P)(2|P|/|A| - 1)$, and $e_{rep} = 2n(P) - c(P)$. The proposition thus says that $e(P) \leq e_{ab}(P)$, and the conjecture asks if $e(P) \leq e_{rep}(P)$.

If Q is a subgroup of P, it is easy to see that $e_{rep}(Q) \leq e_{rep}(P)$. With a bit more work (using that P is a p–group), one can check that also $e_{ab}(Q) \leq e_{ab}(P)$.

This led us to the statement of the next theorem, which at first surprised us.

Theorem 3.3. *Let Q be a subgroup of a p–group P. Then $e(Q) \leq e(P)$.*

Used in conjunction with Example 2.1 and its odd prime analogue, one has

Corollary 3.4. *Suppose a p–group P acts faithfully on a set S with no fixed points. Then*

$$e(P) \leq \begin{cases} |S|/2 - |S/P| & \text{if } p = 2 \\ 2|S|/p - |S/P| & \text{if } p \text{ is odd.} \end{cases}$$

4. Application to nilpotence

Let $nil^{\mathcal{U}}(G)$ denote the nilpotence length of $H^*(G)$, viewed as an unstable module over the Steenrod algebra. Of many equivalent formulations, one is that $nil^{\mathcal{U}}(G)$ is the largest s such that $H^*(G)$ contains a nonzero submodule that is an s–fold suspension of an unstable module.

The work of Henn–Lannes–Schwartz [HLS] links this to algebraic nilpotence. For example, when $p = 2$, one has that

$$\mathrm{Rad}(G)^s = 0 \text{ if } s > nil^{\mathcal{U}}(G),$$

where $\mathrm{Rad}(G) \subset H^*(G)$ is the Jacobson radical.

To link $e(G)$ to $nil^{\mathcal{U}}(G)$, we need one more invariant of G defined via local cohomology. Let $\epsilon(G) = \min\{\epsilon \mid H_{\mathfrak{m}}^{c(G),-(c(G)+\epsilon)}(H^*(G)) \neq 0\}$ if $H^*(G)$ has depth $c(G)$, and ∞ otherwise. Symonds' proof [Sy] of the Regularity Conjecture [B2], combined with work of Greenlees [G], shows

Theorem 4.1. *$\epsilon(G) \geq 0$, and equals 0 iff every element of order p in G is central.*

With $V \leq G$ denoting an elementary abelian p-subgroup, I've shown [K1]

Theorem 4.2. *One has bounds:*

$$\max_{V \leq G}\{e(C_G(V)) \mid V \text{ is maximal}\} \leq nil^{\mathcal{U}}(G) \leq \max_{V \leq G}\{e(C_G(V)) - \epsilon(C_G(V))\}.$$

Armed with Theorem 3.3, one can conclude

Theorem 4.3. *If P is a finite p–group, $nil^{\mathcal{U}}(P) \leq e(P)$.*

Thus nilpotence in $H^*(P)$ is bounded by any of our various group theoretic bounds for $e(P)$. For example, the bound given by Proposition 3.1 implies

Corollary 4.4. *If a finite group G has p–Sylow subgroup P, then $nil^{\mathcal{U}}(G) \leq |P|/p$.*

5. Theorem 3.3 and invariant theory

When P is a p–group, the proof that $Q < P \Rightarrow e(Q) \leq e(P)$ goes by induction on the index, with the critical case being the following:

- Q is normal in P and $P/Q \simeq \mathbb{Z}/p$.
- $C(P) = C(Q)^{\mathbb{Z}/p}$.

One wishes to compare the image of $H^*(P) \to H^*(C(Q)^{\mathbb{Z}/p})$ to the image of $H^*(Q) \to H^*(C(Q))$. With $C = C(Q)$, one has a diagram

$$\begin{array}{ccc} H^*(P) & \longrightarrow & H^*(C^{\mathbb{Z}/p}) \\ \big\downarrow & & \big\uparrow \\ H^*(Q)^{\mathbb{Z}/p} & \longrightarrow & H^*(C)^{\mathbb{Z}/p}. \end{array}$$

With V equal to the dual of the filtered $\mathbb{Z}/p$–module C, one needs to understand a filtered version of the following invariant theory problem, where $S^*(V)$ denotes the polynomial algebra on an $\mathbb{F}_p[\mathbb{Z}/p]$–module V: describe the image of

$$S^*(V)^{\mathbb{Z}/p} \hookrightarrow S^*(V) \twoheadrightarrow S^*(V_{\mathbb{Z}/p}).$$

Though $S^*(V)^{\mathbb{Z}/p}$ is famously hard to compute, in [K2] we show the following:

Theorem 5.1. *If V is an $\mathbb{F}_p[\mathbb{Z}/p]$–module with no trivial direct summands, the image of $S^*(V)^{\mathbb{Z}/p} \hookrightarrow S^*(V) \twoheadrightarrow S^*(V_{\mathbb{Z}/p})$ is the subalgebra of pth powers of $V_{\mathbb{Z}/p}$.*

A filtered version of this theorem suffices to prove Theorem 3.3.

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Dyer-Lashof operations in Tate cohomology of finite groups

Martin Langer

Let G be a finite group, and let k be a field of prime characteristic $p > 0$. Since the group algebra kG is self-injective, the classes of injective and projective modules in the category $\mathfrak{mod}$-kG of (finitely generated) right kG-modules coincide, which allows us to form the stable module category $\underline{\mathfrak{mod}}$-$kG$ by dividing out morphisms which factor through projectives. The stable category $\underline{\mathfrak{mod}}$-$kG$ is triangulated with shift functor $\Sigma = \Omega^{-1}$, the triangles essentially coming from short exact sequences in $\mathfrak{mod}$-kG. The tensor product $\otimes$ on $\mathfrak{mod}$-kG, which is the tensor product $\otimes_k$ of the underlying k-vector spaces with diagonal G-action, descends to a symmetric monoidal product on $\underline{\mathfrak{mod}}$-$kG$ compatible with the triangulated structure. Let k also denote the trivial kG-module. Then $\underline{\mathrm{Hom}}(\Omega^n k, k) = \hat{H}^n(G; k) = \hat{H}^n(G)$ defines the Tate cohomology of G, which is a graded commutative algebra. In non-negative degrees it equals ordinary group cohomology. Recall that Tate cohomology can be computed by choosing a complete projective resolution

$$\begin{array}{ccccccccc} \cdots \leftarrow P_{-2} \leftarrow & P_{-1} & \longleftarrow & & P_0 & \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots \\ & & \nwarrow & & \swarrow & \\ & & & k & & \end{array}$$

and then taking cohomology of the complex $\mathrm{Hom}(P_*, k)$.

Theorem 4. *There is an E_∞-operad action on $\mathrm{Hom}(P_*, k)$. In the case $p = 2$ we therefore get operations of Kudo-Araki-Dyer-Lashof type $Q_i : \hat{H}^*(G) \to \hat{H}^{*-i}(G)$ which are natural with respect to injective group homomorphisms. If $x \in \hat{H}^*(G)$*

is of degree $-i$, *then* $Q_i(x) = x^2$; *if* $|x| > -i$ *then* $Q_i(x) = 0$. *The operations satisfy the Cartan formula, the Adem relations, and they extend the usual Steenrod operations on* $H^*(G) = H^*(BG)$ *in the sense that*

$$Q_i(x) = \begin{cases} \mathrm{Sq}^{-i}(x) & \text{for } i \le 0, \\ 0 & \text{for } i > 0. \end{cases}$$

The statement is also true for odd primes p, with the well-known adaptions. Let us define the total operation Q to be the formal sum $Q(x) = \sum_{i\in\mathbb{Z}} Q_i(x)$; then the Cartan formula says that $Q(xy) = Q(x)Q(y)$.

Example. Let $p = 2$, and let $G = \mathbb{Z}/2\mathbb{Z}$ be the cyclic group of order 2. We can easily compute all the operations using the statements of the theorem only. It is known that $\hat{H}^*(G) \cong k[s^{\pm 1}]$ for the unique non-zero class s of degree 1. We know that $Q(s) = s + s^2$, so that $1 = Q(1) = Q(s^{-1}s) = Q(s^{-1})(s + s^2)$. Using the fact that $Q(s^{-1}) = s^{-2} +$ (terms of degree less than -2), we obtain

$$Q(s^{-1}) = s^{-2} + s^{-3} + s^{-4} + \dots .$$

Remark. The operations must not be confused with the Steenrod operations on Tate cohomology defined by Benson and Greenlees in [1]; their operations raise the degree, whereas ours 'spread the degree from the origin'. Since the generalized Steenrod operations also satisfy the Cartan formula, the total Steenrod square $\mathrm{Sq} = \sum_{i\ge 0} \mathrm{Sq}^i$ also satisfies the formula $\mathrm{Sq}(s^{-1})(s + s^2) = 1$ in the preceding example. For degree reasons, it yields the second solution $\mathrm{Sq}(s^{-1}) = s^{-1} + 1 + s + s^2 + \dots$.

As an application of the operations, consider the following situation. Given a Tate cohomology class $\zeta \in \hat{H}^n(G)$ represented by some map $\zeta : \Omega^n k \to k$ in the stable module category $\mathfrak{mod}$-kG, define the module L_ζ to be the third term in an exact triangle $L_\zeta \to \Omega^n k \xrightarrow{\zeta} k$. A theorem of Carlson [2] says that if n is even and $p \ge 3$, then ζ annihilates the cohomology of L_ζ, that is, the map

$$\zeta \otimes \mathrm{id} : \Omega^n k \otimes L_\zeta \to L_\zeta$$

is stably zero.

The case $p = 2$ is more complicated, but the Dyer-Lashof operations can be used to determine whether a Tate cohomology class ζ annihilates the cohomology of L_ζ or not. More precisely, we have the following theorem:

Theorem 5. *Let* $p = 2$, *and let* $\zeta \in \hat{H}^n(G)$ *be a Tate cohomology class. Then multiplication by* ζ *vanishes on* L_ζ *if and only if* $Q_{-n+1}(\zeta) \in \hat{H}^{2n-1}(G)$ *is divisible by* ζ.

The operations have been computed for abelian groups, generalized quaternion groups and the dihedral group of order 8; see [3] for details. The new structure gives rise to open questions, including the following:

Open question. *Is the Tate cohomology ring* $\hat{H}^*(G)$ *finitely generated as an algebra over the Dyer-Lashof algebra?*

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On endotrivial modules

Nadia Mazza

(joint work with Jon Carlson, Jacques Thévenaz)

Abstract. The primary objective of this talk is to present a survey of the endotrivial modules, aimed at both: experts and non-experts. In the second part, we present new results concerned with the "exotic" endotrivial modules which arise for the generalised quaternion and semi-dihedral 2-groups in characteristic 2. The question is to determine whether these modules "lift" to endotrivial modules for arbitrary finite groups with a generalised quaternion or semi-dihedral Sylow 2-subgroup. We prove that in both cases, this question has an affirmative answer. Time permitting, we will discuss the key steps of the proofs, which use either cohomological methods (quaternion case), or Auslander-Reiten theory (semi-dihedral case).

Endotrivial modules: background. Throughout, let p be a prime, $k = \overline{k}$ a field of characteristic p, and G a finite group. Let M be a kG-module. Then M is *endotrivial* if $\mathrm{End}_k M \cong k \oplus$ (proj) as kG-modules. Equivalently, M is endotrivial if $M^* \otimes M \cong k \oplus$ (proj) , i.e. M is invertible in the stable module category of kG. For short, 'endotrivial' = 'e-t'. If M is e-t, then $M = M_0 \oplus$ (proj) , with M_0 indecomposable e-t. We call M_0 the *cap of* M. We define an equivalence relation on the class of e-t modules: for M, N e-t, set $M \sim N$ if they have isomorphic caps, i.e. if M and N are isomorphic in the stable module category of kG. The set of equivalence classes of e-t modules is the *group of endotrivial modules of* G and denoted $T(G)$. The composition law is induced by the tensor product over k with diagonal G-action : $[M] + [N] = [M \otimes N]$.

Non-trivial examples of e-t modules are the syzygies $\Omega^n(k)$ of k for all $n \in \mathbb{Z}$.

$T(G)$ is detected by restriction to all elementary abelian p-subgroups of G, that is, a kG-module M is e-t if and only if $\mathrm{Res}^G_E M$ is e-t for each elementary abelian p-subgroup E of G, and we have $T(E) = \langle [\Omega(k)] \rangle$. A key fact is that the product of the restriction maps to all the elementary abelian p-subgroups of G has finite kernel, which implies that $T(G)$ is a finitely generated abelian group. Hence, $T(G) = TT(G) \oplus TF(G)$, with $TT(G)$ the (finite) torsion subgroup of $T(G)$, and $TF(G)$ a torsion-free subgroup direct sum complement of $TT(G)$ in $T(G)$. The torsion-free rank of $T(G)$ only depends on the number of conjugacy classes of maximal elementary abelian subgroups of G of rank 2, and in case $TF(G)$ is cyclic, then $TF(G) = \langle [\Omega(k)] \rangle$.

The endotrivial modules for a finite p-group G have been determined by Carlson and Thévenaz, and the classification theorem for $T(G)$ in that case yields in particular that $TT(G) = 0$ unless :
(i) G is cyclic of order ≥ 3; then $TT(G) = T(G) = \langle [\Omega(k)] \rangle \cong \mathbb{Z}/2$;
(ii) G is generalised quaternion; then $TT(G) = T(G) = \langle [\Omega(k)], [M] \rangle \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$, where M is self-dual and $\dim M \equiv 1 + \frac{|P|}{2} \pmod{|P|}$;
(iii) G is semi-dihedral; then $TT(G) = \langle [L] \rangle \cong \mathbb{Z}/2$, where L is self-dual and $\dim L \equiv 1 + \frac{|P|}{2} \pmod{|P|}$.

For the passage from the case of a finite p-group to an arbitrary finite group G, say with Sylow p-subgroup P, the only useful tools are:
(i) An unpublished result by Dade states that if P is normal in G, then, any G-stable e-t kP-module extends to G. Thus the image of Res^G_P contains the set of fixed points $T(P)^{G/P}$ of $T(P)$, where $[M] \in T(P)^{G/P}$ means $M \cong {}^gM$ for all $g \in G$ and all $M \in [M]$. Moreover, $\ker\big(\mathrm{Res}^G_P : T(G) \to T(P)\big) = \langle\ [M] \mid \dim(M) = 1\ \rangle$.
(ii) By the Green correspondence, $\mathrm{Res}^G_{N_G(P)} : T(G) \to T(N_G(P))$ is injective.

Hence, for the entire group $T(G)$, since $T(N_G(P))$ is "known", it "suffices" to determine which indecomposable e-t $kN_G(P)$-modules have an e-t kG-Green correspondent. But this is a difficult question, which remains open.

An exotic turn.

Let G be a finite group and P a Sylow p-subgroup of G. We investigate the elements in the torsion subgroup of $T(G)$ when $TT(P) \neq 0$. There are 3 cases:
(i) $TT(P) = \langle [\Omega(k)] \rangle \cong \mathbb{Z}/2$ if P is cyclic of order ≥ 3;
(ii) $TT(P) = \langle [\Omega(k)], [M] \rangle \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$ if $p = 2$ and P is generalised quaternion;
(iii) $TT(P) = \langle [L] \rangle \cong \mathbb{Z}/2$ if $p = 2$ and P is semi-dihedral.

In particular, $\dim M\ ,\ \dim L\ \equiv 1 + \frac{|P|}{2} \pmod{|P|}$.

Definition. Suppose that $p = 2$. We call *exotic* an indecomposable e-t kG-module M with $\dim M \equiv 1 + \frac{|P|}{2} \pmod{|P|}$.

Observe that this congruence can only occur at the prime 2, and this motivates our choice of terminology.

Main Theorem [Carlson-M-Thévenaz, 2010]**.** *Suppose that $p = 2$ and let G be a finite group with semi-dihedral or generalised quaternion Sylow 2-subgroup. Write $K(G) = \ker\big(\mathrm{Res}^G_P : T(G) \to T(P)\big)$ for the subgroup of $T(G)$ spanned by the classes of all trivial source e-t kG-modules, as before. There is a split short exact sequence*

$$0 \longrightarrow K(G) \longrightarrow T(G) \xrightarrow{\mathrm{Res}^G_P} T(P) \longrightarrow 0$$

Moreover, if P is generalised quaternion and $H = C_G(Z(P))$, then

$$K(H) = \langle [M] \mid \dim M = 1 \rangle \quad \textit{and} \quad K(G) = \langle [\mathrm{Ind}^G_H M] \mid [M] \in K(H) \rangle.$$

About the proof. First, let G be a finite group with semi-dihedral Sylow 2-subgroup. K. Erdmann's investigation of algebras of semi-dihedral type shows that the stable Auslander-Reiten quiver of kG has a component $\mathbb{Z}D_\infty$. Hence, the *heart* $H_k = \mathrm{Rad}(R_k)/\operatorname{Soc}(R_k)$ of the projective cover R_k of k appears in an almost split sequence

$$0 \longrightarrow V \longrightarrow H_k \oplus \text{ (proj) } \longrightarrow U \longrightarrow 0$$

where $V^* \cong U \not\cong \Omega^{-1}(k)$. We prove that the modules U and V are e-t. Moreover, $\Omega(U)$ is exotic and self-dual. In fact, (proj) $= 0$, and U and V are uniserial.

Next, let G be a finite group with generalised quaternion Sylow 2-subgroup P with centre $Z = \langle z\rangle \cong C_2$, and without loss, we suppose that $Z \leq Z(G)$. Set $\overline{G} = G/Z$ and $\overline{P} = P/Z$. It is well-known that $\Omega^4(k) = k$ and that $\Omega^2(k)$ is self-dual. Let $M = \Omega^2(k)$ and $M_0 = \{\, m \in M \mid (z-1)m = 0 \,\}$. There is a filtration : $0 \subset (z-1)M \underbrace{\subset}_{\cong k} M_0 \underbrace{\subset}_{\cong (z-1)M} M$. Moreover, $(z-1)M$ and M_0 are $k\overline{G}$-modules. The *Deconstruction – Re-construction* method used in [1, 7, 8] works also in this case: as $k\overline{G}$-module, $(z-1)M = L_1 \oplus L_2$ with L_1, L_2 indecomposable and dimension congruent to $\frac{|P|}{4} \pmod{\frac{|P|}{2}}$.

Theorem [CMT]**.** *For $i = 1, 2$, there is an exotic e-t kG-module W_i such that $(z-1)W_i \cong L_i$ as $k\overline{G}$-modules and* $\dim W_i \equiv 1 + \frac{|P|}{2} \pmod{|P|}$.

Two side-results.

(i) It is known that a block algebra of quaternion type can have $1, 2$ or 3 simple modules. Using K. Erdmann's description of algebras of quaternion type, in the case of a block which can occur as principal 2-block of a group algebra with generalised quaternion Sylow 2-subgroup, with the same notation as in the above theorem, either W_i is uniserial, or $\Omega(W_i)$ is uniserial, or both.

(ii) The groups $\mathrm{SL}_2(q)$ for q odd have a generalised quaternion Sylow 2-subgroup P. Moreover, there are 3 non-isomorphic simple modules in the principal 2-block: the trivial module k and two non-isomorphic modules of dimension $\frac{q-1}{2}$. Using [11] again, we observe that there exist two non-isomorphic uniserial e-t modules of dimension q and composition length 3. More precisely,

- if $q \equiv 1 \pmod 4$, then these modules are exotic and self-dual;
- if $q \equiv 3 \pmod 4$, then their syzygies are exotic, not self-dual, and have dimension $1 + \frac{(q-1)|P|}{8}$.

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Invariants in modular representation theory

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(joint work with Eric Friedlander, Jon Carlson, and Dave Benson)

It is well known that the category of modular representations of a finite group scheme is almost always wild. Hence, there is no hope to classify indecomposable representations making them fit into a nice, concise picture. In this talk, we present several new constructions of invariants of modular representations with an idea that they will serve as navigational devices in this wild jungle of representations. Some of our invariants are defined locally and generalize the well-known notion of support variety. They allow for a finer distinction between modules than support varieties; in addition, they lead to distinguished families of modules characterized by vanishing of one of those invariants. For these distinguished classes of modules we define new global invariants which we prove to be algebraic vector bundles on the Proj of cohomology of our group scheme.

For the purposes of this talk, we shall concentrate the case of a restricted Lie algebra. Nonetheless, many of the results and new invariants we introduce exist for infinitesimal group schemes and sometimes more generally for any finite group scheme.

This is an outline of the talk:

(1) Local Jordan type and non-maximal rank varieties;
(2) Global p-nilpotent operator and vector bundles associated to a representation;
(3) π^r-points, constant Rad and Soc-type, and bundles on Grassmanians.

Let k be an algebraically closed field of characteristic $p > 0$, and let $\mathfrak{g}$ be a restricted Lie algebra defined over k. We denote by $[p] : \mathfrak{g} \to \mathfrak{g}$ the $[p]$-th power map that defines the restricted structure. A standard example of a restricted Lie algebra is gl_n over $\overline{\mathbb{F}}_p$.

Let $u(\mathfrak{g})$ be the restricted enveloping algebra of $\mathfrak{g}$, a finite dimensional cocommutative Lie algebra. We have an equivalence of categories:

$$\text{restricted } \mathfrak{g} - \text{mod} \ \sim \ u(\mathfrak{g}) - \text{mod}$$

Comment. Let $\mathfrak{g}_a = \operatorname{Lie} \mathbb{G}_a$ be a one-dimensional abelian Lie algebra with trivial $[p]$-restriction. Let $E = (\mathbb{Z}/p)^{\times n}$ be an elementary abelian p-group of rank n. Then

$$kE \simeq u(\mathfrak{g}_a^{\oplus n})$$

Hence, the theory that will be described applies to elementary abelian p-groups. We just need to think of them as abelian Lie algebras.

I. Local invariants. Let $\mathcal{N}^{[p]} = \{x \in \mathfrak{g} \,|\, x^{[p]} = 0\}$, the $[p]$-restricted nullcone of $\mathfrak{g}$. Recall that if $\mathfrak{g}$ is a classical Lie algebra and $h > p$ then $\mathcal{N}^{[p]} = \mathcal{N}$, the nullcone of $\mathfrak{g}$. The following theorem connects our work with cohomology. jpthm[Friedlander-Parshall, Andersen-Jantzen, Suslin-Friedlander-Bendel] There is an isomorphism of varieties

$$\operatorname{Spec} \mathrm{H}^\bullet(u(g), k) \simeq \mathcal{N}^{[p]}(\mathfrak{g})$$

where $\mathrm{H}^\bullet$ denotes the even dimensional cohomology unless $p = 2$. jpthm

Let M be a finite-dimensional (restricted) $\mathfrak{g}$-module. We study M "locally" on $\mathcal{N}^{[p]}$. Let $x \in \mathcal{N}^{[p]}$. The isomorphism class of the restriction of M to the abelian Lie algebra generated by x, $M \downarrow_{\langle x \rangle}$, is determined by the Jordan type of x considered as an operator on M. We denote this Jordan type by $\mathbf{Jtype(x, M)}$. For each number j, $1 \leq j \leq p-1$, we denote by $\operatorname{rk}(x^j, M)$ the rank of x^j as an operator on M. Since $x^{[p]} = 0$, the Jordan type $\operatorname{Jtype}(x, M)$ is completely determined by the sequence $\{\operatorname{rk}(x, M), \operatorname{rk}(x^2, M), \ldots, \operatorname{rk}(x^{p-1}, M), \dim M\}$. We generalize the notion of *support variety* to *non-maximal rank varieties*, geometric invariants defined "locally":

Definition 1. Let M be a finite-dimensional $\mathfrak{g}$-module.

(1) $\Gamma^j_{\mathfrak{g}}(M) = \{x \in \mathcal{N}^{[p]} \ : \ \operatorname{rk}(x^j, M) \text{ is not maximal } \} \cup \{0\}$, $1 \leq j \leq p-1$.
(2) $\Gamma_{\mathfrak{g}}(M) = \{x \in \mathcal{N}^{[p]} \ : \ \operatorname{Jtype}(x, M) \text{ is not maximal } \} \cup \{0\}$.

These sets are, indeed, closed subvarieties of $\mathcal{N}^{[p]}(\mathfrak{g})$. Unlike support varieties, they are always proper subvarieties. Using these invariants, we can define modules of constant Jordan type (compare to D. Benson's talk), and, more generally, modules of constant j-rank.

Definition 2. Let M be a finite-dimensional $\mathfrak{g}$-module.

(1) M is a module of constant j-rank if $\Gamma^j_{\mathfrak{g}}(M) = \{0\}$.
(2) M is a module of constant Jordan type if $\Gamma_{\mathfrak{g}}(M) = \{0\}$.

II. Global invariants. Let $k[\mathcal{N}^{[p]}]$ be the coordinate algebra of $\mathcal{N}^{[p]}$, and let $\{x_1, \ldots, x_n\}$ be a basis of $\mathfrak{g}$. Let $\{Y_1, \ldots, Y_n\}$ be the dual basis of $\mathfrak{g}^{\#}$. Denote by

$y_1, \dots, y_n$ the images of $Y_1, \dots, Y_n$ under the surjetive map $S^*(\mathfrak{g}^\#) \to k[\mathcal{N}^{[p]}]$. We define the *universal p-nilpotent element* $\Theta \in u(\mathfrak{g}) \otimes k[\mathcal{N}^{[p]}]$ via the explicit formula

$$\Theta = \sum x_i \otimes y_i.$$

For any $\mathfrak{g}$-module M, the element Θ induces the *global p-nilpotent operator*

$$\Theta_M : M \otimes k[\mathcal{N}^{[p]}] \to M \otimes k[\mathcal{N}^{[p]}]$$

defined via

$$\Theta_M : m \otimes f \mapsto \sum x_i(m) \otimes y_i f$$

This is a $k[\mathcal{N}^{[p]}]$-linear, homogeneous operator of degree one. If $\mathbb{P}(g) = \operatorname{Proj} k[\mathcal{N}^{[p]}]$ and $\mathcal{O} = \mathcal{O}_{\mathbb{P}(\mathfrak{g})}$, then Θ induces a map of $\mathcal{O}$-modules

$$\Theta_M : M \otimes \mathcal{O} \to M \otimes \mathcal{O}(1)$$

The operator Θ_M allows us to associate an algebraic vector bundle to a module of constant j-rank thanks to the following theorem.

Theorem 3 (Friedlander-P)**.** *Let M be a $\mathfrak{g}$-module of constant j-rank. Then*

(1) $\operatorname{Im} \Theta_M^j$ *and* $\operatorname{Ker} \Theta_M^j$ *are algebraic vector bundles on* $\mathbb{P}(\mathfrak{g})$.
(2) *For any point $\bar{x} \in \mathbb{P}(\mathfrak{g})$, the fiber of* $\operatorname{Im} \Theta_M^j$ *at the point $\bar{x}$ is isomorphic to* $\operatorname{im}\{x^j : M \to M\}$*, and similarly for kernel.*
(3) $\operatorname{rk} \operatorname{Im} \Theta_M^j = \operatorname{rk}(x^j, M)$*, and* $\operatorname{rk} \operatorname{Ker} \Theta_M^j = \dim \ker(x^j, M)$

Remark 1. This theorem is true for infinitesimal group schemes but we do not know the appropriate analogue for finite groups.

Various quotients of the bundles constructed above are again algebraic vector bundles. It turns out that using one such quotient we can recover all vector bundles on a projective space (up to a Frobenius twist) from modules of constant Jordan type for elementary abelian p-groups. Let M be a module of constant Jordan type. Then

$$\mathcal{F}_1(M) = \frac{\operatorname{Ker} \Theta_M}{\operatorname{Ker} \Theta_M \cap \operatorname{Im} \Theta_M}$$

is an algebraic vector bundle on $\mathbb{P}(\mathfrak{g})$.

Theorem 4 (Benson-P)**.** *Let E be an elementary abelian p-group of rank n.*

(1) *If $p = 2$ then for any algebraic vector bundle $\mathcal{F}$ on $\mathbb{P}_k^{n-1}$ there exists an E-module M of constant Jordan type such that $\mathcal{F}_1(M) \simeq \mathcal{F}$.*
(2) *If $p > 2$ then for any algebraic vector bundle $\mathcal{F}$ on $\mathbb{P}_k^{n-1}$ there exists an E-module M of constant Jordan type such that $\mathcal{F}_1(M) \simeq F^*(\mathcal{F})$ where $F : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ is the Frobenius map.*

Another quotient bundle relates endo-trivial modules which were discussed in detail in N. Mazza's talk to line bundles:

Theorem 5 (Frieldnader-P). *Let M be a module of constant Jordan type, and let*

$$\mathcal{H}^{[1]}(M) = \frac{\operatorname{Ker}\Theta_M}{\operatorname{Im}\Theta_M^{p-1}}.$$

Then $\mathcal{H}^{[1]}(M)$ is a line bundle on $\mathbb{P}(\mathfrak{g})$ if and only if M is an endo-trivial module.

III. Bundles on Grassmanians. By generalizing the notion of a π-point introduced in [6], [7], we can extend our bundle construction to produce bundles on Grassmanians.

Definition 6. Let E be an elementary abelian p-group of rank n, and let $r < n$.

(1) A π^r-point of kE is a flat map of algebras

$$\alpha : k[t_1, \ldots, t_r]/(t_1^p, \ldots, t_r^p) \to kE.$$

(2) Let α, β be π^r-points. We say that $\alpha \sim \beta$ if for any finite dimensional kE-module M, $\alpha^*(M)$ is free if and only if $\beta^*(M)$ is free.

Proposition 7 (Carlson-Friedlander-P). *Let $\Pi^r(E)$ be the set of equivalence classes of π^r-points. Then $\Pi^r(E) \simeq \mathrm{Grass}_{r,n}$, the Grassmanian of r-planes in n-space.*

Definition 8. An E-module M has *constant $r-\mathrm{Rad}$ type* if $\dim \mathrm{Rad}\alpha^*(M)$ is constant for all π^r-points $\alpha : k[t_1, \ldots, t_r]/(t_1^p, \ldots, t_r^p) \to kE$.

Similarly, M has *constant $r - \mathrm{Soc}$ type* if $\dim \mathrm{Soc}\,\alpha^*(M)$ is constant for all π^r-points α.

With a bit more work than in the case of π-points, one can define sheaves $\mathrm{Ker}\{\Theta, M\}$ and $\mathrm{Im}\{\Theta, M\}$. The following theorem asserts that this construction gives a way to obtain bundles on Grassmanians starting with modules of constant socle or radical type.

Theorem 9 (Carlson-Friedlander-P).

(1) *Let E be an elementary abelian p-group or rank n, and let M be a module of constant $r - \mathrm{Rad}$ type. Then $\mathrm{Im}\{\Theta, M\}$ is an algebraic vector bundle on $\mathrm{Grass}_{r,n}$.*

(2) *Let M be an E-module of constant $r - \mathrm{Soc}$ type. Then $\mathrm{Ker}\{\Theta, M\}$ is an algebraic vector bundle on $\mathrm{Grass}_{r,n}$.*

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Hopf rings, cohomology of symmetric groups, and related topics

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(joint work with Chad Giusti, Paolo Salvatore)

1. Outline

Though the mod-two homology of symmetric groups was computed exactly fifty years ago by Nakaoka [6], some of the cohomology structure is just being understood. The homology is well-understood both algebraically, as a polynomial algebra under the standard product (induced by the standard inclusion of a product of symmetric groups), and geometrically in terms of "orbital submanifolds," which are reminiscent of the submanifolds which represent homology of ordered configurations (in finite-dimensional Euclidean spaces) [2]. Our work started with a desire to understand the cohomology geometrically, starting with the hint that the cohomology of ordered configurations is Poincaré dual to linearly-defined submanifolds.

1.1. Skyline diagrams.

Definition 1.1. *Let $\gamma_{\ell,n} \in H^{n(2^\ell-1)}(B\mathcal{S}_{n2^\ell})$ be Poincaré dual to the subvariety of configurations which can be partitioned into n groups of 2^ℓ points, each of which share their first coordinates.*

One can also "mix" these conditions for example having six points which share ten coordinates while four points share one coordinate and two points share three coordinates. There is a convenient graphical notation we employ which can be viewed as generalizing Young diagrams (see below).

Definition 1.2. *Skyline diagrams are constructed as follows.*

- *Basic building blocks of width $2^{\ell-1}$ and height $2 - \frac{1}{2^{\ell-1}}$ or height zero.*
- *General building blocks of width $2^k 2^{\ell-1}$ and height $1 - \frac{1}{2^{\ell-1}}$.*
- *Blocks may be stacked on top of each other to form a column. Columns are placed next to each other to form a diagram.*
- *(Commutativity) Order of stacking and column placement doesn't matter.*
- *(Compatibility) Stacking blocks of different widths yields the zero diagram.*
- *(Divided powers) Repeating a column yields the zero diagram.*

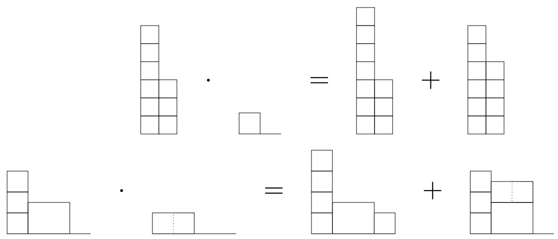


FIGURE 1. Some computations in $H^*(B\mathcal{S}_4)$ and $H^*(B\mathcal{S}_8)$, expressed by skyline diagrams.

Theorem 1.3 (Giusti, Salvatore, S- [4]). *Skyline diagrams of width n define an additive basis for the cohomology of the $2n$th symmetric group $H^*(B\mathcal{S}_{2n};\mathbb{F}_2)$, associating to a diagram the Poincaré dual of the variety defined by coordinate equalities translated from the diagram. The degree of an associated cohomology class is the area of the diagram.*

This basis is distinct from the linear dual to the monomial basis in the Dyer-Lashof algebra, as can be seen already for $H_6(B\mathcal{S}_4)$.

Question(s) 1.4.

- *Find Yoneda representatives of these cohomology classes.*
- *Understand the pairing between the Nakaoka monomial basis and the skyline basis.*
- *Identify/ develop the new(?) basis of the dual to the Steenrod algebra which corresponds to the skyline basis.*

Cup products can be computed by intersection theory (related to Schubert calculus). The coordinate equality conditions are compatible after making wise choices but one must allow for all possible ways the conditions can be "matched". Diagrammatically, this corresponds to matching and stacking of columns (up to automorphism).

From this additive basis and multiplication rule, we can with some work recover generator-and-relation presentations of Adem-Maginnis-Milgram, Feshbach and Hu'ng [1, 3, 5].

2. The transfer product and Hopf ring presentation

Definition 2.1 (Strickland-Turner [7]). *The standard multiplication maps $B\mathcal{S}_n \times B\mathcal{S}_m \to B\mathcal{S}_{n+m}$ are (homotopic to) covering maps. The associated transfer maps on cohomology defines an associative, commutative multiplication.*

We rediscovered and generalized this product for any collection of unordered configuration spaces.

Definition 2.2. *A Hopf ring is a ring object in the category of cocommutative coalgebras. Explicitly, a Hopf ring is a vector space V with two multiplications* *and one comultiplication* $(\odot, \cdot, \Delta)$ *such that each multiplication forms a bialgebra with the comultiplication and such that the three satisfy the distributivity relation*

$$\alpha \cdot (\beta \odot \gamma) = \sum_{\Delta\alpha = \sum a' \otimes a''} (a' \cdot \beta) \odot (a'' \cdot \gamma). \tag{2.1}$$

Because the transfer product does not have an antipode, the terminology should be "Hopf semi-ring." Honest Hopf rings occur in topology as the homology of infinite loop spaces which represent ring spectra.

Theorem 2.3 (Strickland-Turner [7]). *The generalised cohomology of* $\coprod_n B\mathcal{S}_n$ *with coefficients in a ring spectrum is a Hopf (semi-)ring, under transfer product, cup product and the standard coproduct.*

Theorem 2.4 (Giusti, Salvatore, S- [4]). *As a Hopf ring,* $H^*(\coprod_n B\mathcal{S}_n; \mathbb{F}_2)$ *is generated by classes* $\gamma_{\ell,n} \in H^{n(2^\ell-1)}(B\mathcal{S}_{n2^\ell})$*, along with unit classes on each component. The coproduct of* $\gamma_{\ell,n}$ *is given by*

$$\Delta\gamma_{\ell,n} = \sum_{i+j=n} \gamma_{\ell,i} \otimes \gamma_{\ell,j}.$$

Relations between transfer products of these generators are given by $\gamma_{\ell,n} \odot \gamma_{\ell,m} = \binom{n+m}{n}\gamma_{\ell,n+m}$*. Cup products of generators on different components are zero, and there are no other relations between cup products of generators.*

The skyline basis is the "Hopf monomial basis" for this Hopf ring presentation.

3. Related Hopf semi-ring structures

The transfer product is related (through the Atiyah-Segal completion theorem) to the induction product in representation theory, which sends V a representation of $\mathcal{S}_n$ and W of $\mathcal{S}_m$ to $\mathrm{Ind}_{\mathcal{S}_n \times \mathcal{S}_m}^{\mathcal{S}_{n+m}} V \otimes W$.

Theorem 3.1 (after Zelevinsky [8]). *The complex representation rings of symmetric groups* $\bigoplus_n \mathrm{Rep}_{\mathbb{C}}(\mathcal{S}_n)$ *form a Hopf ring under induction product, internal tensor product (zero for representations of different symmetric groups), and restriction coproduct. It is isomorphic to* $\mathbb{Z}[x_1, x_2, \ldots]$*, where* x_n *is the trivial representation of* $\mathcal{S}_n$*, with thus* ${x_n}^2 = x_n$ *and* $\Delta x_n = \sum_{n=i+j} x_i \otimes x_j$*.*

- The Hopf monomials are just the monomials under $\odot$, which correspond to permutation representations. Zelevinsky's presentation thus gives a reasonable handle on the ring structure of these representation rings, though is not good for understanding the standard bases of irreducibles.
- Verifying Hopf ring distributivity and that tensor product with restriction coproduct is a bialgebra are straightforward. Verifying that the induction product and restriction coproduct form a bialgebra is an application of double-cosets.

There has been a long and fruitful connection between group cohomology and invariant theory. The Hopf ring approach reflects this. Rings of symmetric invariants, for examples $\bigoplus_n \mathbf{k}[x_1, \ldots, x_n]^{\mathcal{S}_n}$, form Hopf rings. The, second "multiplicative" product just multiplies symmetric polynomials which share the same number of variables, or is zero if the number of variables differ. The coproduct is the standard "de-coupling" of two sets of variables followed by reindexing, so for example

$$\Delta_{2,1}({x_1}^2 x_2 x_3 + x_1 {x_2}^2 x_3 + x_1 x_2 {x_3}^2) = ({x_1}^2 x_2 + x_1 {x_2}^2) \otimes x_1 + x_1 x_2 \otimes {x_1}^2.$$

The first product $f \odot g$ reindexes the variables of g, multiplies that by f, and then symmetrizes with respect to $\mathcal{S}_{n+m}/\mathcal{S}_n \times \mathcal{S}_m$, as can be done with shuffles. A reasonable name for this product would be the shuffle product. For example

$$({x_1}^2 x_2 + x_1 {x_2}^2) \odot x_1 = ({x_1}^2 x_2 + x_1 {x_2}^2) x_3 + ({x_1}^2 x_3 + x_1 {x_3}^2) x_2 + ({x_2}^2 x_3 + x_2 {x_3}^2) x_1.$$

More generally, we consider $A(m) = \bigoplus_n (\mathbf{k}[x(1), \ldots, x(m)]^{\otimes n})^{\mathcal{S}_n}$ in which case the ring of symmetric functions are symmetric polynomials in m collections of variables.

Proposition 3.2. *The symmetric invariant Hopf ring $A(m)$ is the generated over $\mathbf{k}$ by unit elements and $\sigma(\ell)_n = x(\ell)_1 \cdot x(\ell)_2 \cdot \cdots \cdot x(\ell)_n$, for $1 \leq \ell \leq m$. The coproduct is given by*

$$\Delta \sigma(\ell)_n = \sum_{i+j=n} \sigma(\ell)_i \otimes \sigma(\ell)_j.$$

The $\odot$-products are given by $\sigma(\ell)_i \odot \sigma(\ell)_j = \binom{i+j}{i} \sigma(\ell)_{i+j}$, while $\odot$-products between classes with different ℓ are free.

In this presentation, the standard product is determined by Hopf ring distributivity, the fact that products of classes in different rings of invariants are zero, and the fact that the collection of $\sigma(\ell)_n$ for all ℓ with fixed n form a polynomial ring.

This Hopf ring presentation is simple, reflecting the fact that the basis of Hopf monomials is essentially that of symmetrized monomials. The individual component rings are determined by this structure, but are complex - understood through generators with recursively defined relations by Feshbach over $\mathbb{F}_2$ and still open(?) for other prime fields.

Let $V_n \cong (\mathbb{Z}/2)^n \subset B\mathcal{S}_{2^n}$ be the transitive elementary abelian 2-group embedded through acting on itself. Let $(V_n)^k \subset \mathcal{S}_{k2^n}$ be the product of such. Let D_n denote the nth Dickson algebra.

Theorem 3.3 ([4]). *The restriction maps $H^*(\coprod_k B\mathcal{S}_{k2^n}; \mathbb{F}_2) \to {D_n}^{\mathcal{S}}$ define a surjective map of Hopf rings.*

3.1. **Many questions.**

- Find the most general setting in which such Hopf (semi-)rings occur. Conjecturally, at least for "series groups with products and good intersections thereof," including symmetric groups, alternating groups, general linear

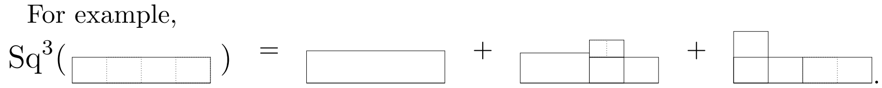
groups over finite fields, upper triangular matrices over finite fields, etc. in their cohomology, representation theory and invariant theory.
- Revisit and extend classical results such as on the cohomology of linear groups over finite fields.
- Find Hopf ring presentations for mysterious objects such as modular representations or generalized Dickson algebras and see if they can yield Poincaré series and the like.
- Develop a theory of commutative algebra of Hopf (semi-)rings, in particular those where the second product is a sum of component rings, and relate that commutative algebra to that of the component rings.

4. Steenrod action on cohomology

Since the transfer product is defined by stable maps, there are Cartan formulae for both transfer and cup products. Thus, Steenrod action on the skyline basis is determined (locally) by the action of the squares on the Hopf ring generators.

Theorem 4.1 ([4]). *A Steenrod square on $\gamma_{\ell,2^k}$ is represented by the sum of all diagrams which are of full width, with at most two boxes stacked one top of each other, and with the width of columns delineated by any of the vertical lines (of full height) at least ℓ.*

For example,

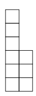


The proof is inductive, restricting cohomology through the product maps $B\mathcal{S}_{2^k} \times B\mathcal{S}_{2^k} \to B\mathcal{S}_{2^{k+1}}$. Note that there are no binomial coefficients (in contrast with Nishida relations). The cohomology of symmetric groups has long been known to have (sums of tensor powers) of Dickson algebras as an associated graded as an algebra over the Steenrod algebra. This formula shows that the difference with this associated graded is as large as possible.

Question(s) 4.2. *Better understand the A-module and A-algebra structure of the cohomology of $B\mathcal{S}_\infty$ (that is, of QS^0).*

5. Integral cohomology

The skyline basis for cohomology has canonical representatives in the Fox-Neuwirth cochain complex for cohomology of unordered configurations. The points in a configuration in $\mathcal{I}^\infty$ can be ordered by the dictionary order on their coordinates. Take the number of first coordinates which agree for each consecutive pair of points in this ordering, to associate a sequence of natural numbers to any configuration. The set of all configurations with the same sequence is a cell in the one-point compactification of this configuration space.

The codimension-zero cell of generic points is $[0, 0, \ldots, 0]$. The codimension of a cell is the sum of the entries which label it.  is represented by $[6, 0, 3]+[3, 0, 6]$. □

is represented by $[1,1,1]$. Boundary maps are combinatorial, though not as simple as one might expect (especially to determine signs). Preliminary calculations with the skyline basis representatives leads to the following.

Conjecture 5.1 (Bockstein splitting conjecture)**.** *There is an additive basis for the two-local cohomology of symmetric groups consisting of skyline basis classes which are integral cycles and (higher) Bocksteins on skyline basis classes.*

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Group Actions on Rings and Varieties and the Brauer Construction

Peter Symonds

As usual, k will be a field of characteristic p, G a finite group and M a kG-module of dimension n. We consider the action of G on $k[M] = S^*(M^*) = k[x_1, \ldots, x_n]$ and would like to understand $k[M]$ as a kG-module.

With Dikran Karagueuzian we proved a structure theorem for a p-group P, which describes the decomposition of $k[M]$ into indecomposable summands [4]. In this description, certain submodules are spread out by certain subsets of a set of invariant parameters. From this it is not hard to see that the Krull dimension of the part of vertex P is equal to the dimension of the fixed-point space M^P. Here the Krull dimension is equal to the maximum number of parameters that are used to spread out a piece, although it can also be given a more formal module-theoretic description.

We would like to have a more conceptual proof of this result and also one that extends to other rings.

Brauer Construction. Given a p-group P and a kP-module M we set $M^{[P]} = M^P/(\sum_{Q<P} \operatorname{tr}_Q^P M^Q)$.

If X is a P-set and M is the permutation module $k[X]$ then this has the property that $k[X]^{[P]} \cong k[X^P]$. Usually $k[X]$ is described as a vector space with basis the elements of X, but for our purposes it is better to think of it as the functions on X.

Now let V be a variety on which G acts and let $k[V]$ be its coordinate ring. Then $k[V/G] = k[V]^G$ and $k[V^G] = k[V]/(I_G k[V])$, the biggest quotient ring with trivial G-action. There are natural maps $V^G \to V \to V/G$.

Now write $S = k[V]$; the maps above become $S/(I_G S) \leftarrow S \leftarrow S^G$. For a p-group P this induces a map

$$S^{[P]} \to S/(I_P S).$$

Proposition 1. *This map is a purely inseparable isogeny (an F-isomorphism).*

This means that the kernel is nilpotent and also that for any $s \in S$ there is a power p^n such that s^{p^n} is in the image. As a consequence, the two rings have the same varieties as topological spaces i.e.

$$\operatorname{Spec}(S^{[P]}) \sim (\operatorname{Spec} S)^P.$$

The proof is by reducing modulo a maximal ideal of S^P to reduce to the case of a permutation module.

It is also possible to change the class of groups that one traces from, for example to just the trivial group.

$$\operatorname{Spec}(\hat{H}^0(G,S)) \sim \operatorname{Sing}(\operatorname{Spec} S).$$

For polynomial rings this last result is due to Feshbach [1]. Fleischmann [3] proved the general version for different classes of subgroups.

Application. Let F be a finite S-module with compatible group action. Observe that $F^{[G]}$ is finite over $S^{[G]}$ thus the krull dimension of $F^{[G]}$ is bounded by that of $S^{[G]}$.

Now assume that S is graded and let M be an indecomposable kG-module. We want to count the multiplicity of M as a summand of S^r. If the vertex of M is P then $M\downarrow_P$ has a summand N of vertex P. One basic result is that $\operatorname{End}_k(N)^{[P]} \neq 0$. Thus the multiplicity of M in $S^r \leq$ the multiplicity of N in $S^r\downarrow_P \leq \dim \operatorname{Hom}_k(M, S^r)^{[P]}$.

Now use $F = \operatorname{Hom}_k(M,S)$ and the observation above to obtain:

Proposition 2. *The Krull dimension of M in S is at most equal to* $\dim(\operatorname{Spec} S)^P$*, where P is the vertex of M.*

Proposition 1 generalizes easily to group actions on schemes. In fact it also generalizes to actions of unipotent finite group schemes. Here unipotent means that the only simple module is the trivial one, so this is a natural generalization of finite p-groups.

Our proof for finite p-groups was a formal consequence of the existence of a transfer with the usual good properties, indeed this is necessary even to be able to define the Brauer construction. Such a transfer was constructed by Feshbach [2].

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Sets with a Category Action

Peter Webb

1. $\mathcal{C}$-sets

Let $\mathcal{C}$ be a small category and Set the category of sets. We define a $\mathcal{C}$-set to be a functor $\Omega : \mathcal{C} \to \text{Set}$. Thus Ω is simply a diagram of sets, the diagram having the same shape as $\mathcal{C}$: for each object x of $\mathcal{C}$ there is specified a set $\Omega(x)$ and for each morphism $\alpha : x \to y$ there is a mapping of sets $\Omega(\alpha) : \Omega(x) \to \Omega(y)$. If $\mathcal{C}$ happens to be a group (a category with one object and morphism set G) then a $\mathcal{C}$-set is the same thing as a G-set, since the $\mathcal{C}$-set singles out a set and sends each morphism of $\mathcal{C}$ to a permutation of the set. We see that $\mathcal{C}$-sets form a category, the morphisms being natural transformations between the functors. Thus we have a notion of isomorphism of $\mathcal{C}$-sets.

Given two $\mathcal{C}$-sets Ω_1 and Ω_2 we define their *disjoint union* $\Omega_1 \sqcup \Omega_2$ to be the $\mathcal{C}$-set defined at each object x of $\mathcal{C}$ by $(\Omega_1 \sqcup \Omega_2)(x) := \Omega_1(x) \sqcup \Omega_2(x)$ with the expected definition of $\Omega_1 \sqcup \Omega_2$ on morphisms. Let us call a $\mathcal{C}$-set Ω a *single orbit $\mathcal{C}$-set* or *transitive* if it cannot be expressed properly as a disjoint union. A $\mathcal{C}$-set Ω may happen to be the disjoint union of two $\mathcal{C}$-sets, or not; if it can be broken up as a disjoint union we can ask if either of the factors is a disjoint union, and by repeating this we end up with a disjoint union of $\mathcal{C}$-sets each of which is transitive.

Proposition 1.1. *Every finite $\mathcal{C}$-set Ω has a unique decomposition*

$$\Omega = \Omega_1 \sqcup \Omega_2 \sqcup \cdots \sqcup \Omega_n$$

where each Ω_i is transitive. In the diagram

$$\Omega(\mathcal{C}) \xrightarrow{p} \varinjlim \Omega = \{1, \dots, n\}$$

we may take $\Omega_i(\mathcal{C}) = p^{-1}(i)$.

We present an example. Let $\mathcal{C}$ be the category

$$\mathcal{C} = \underset{x}{\bullet} \xrightarrow{\alpha} \underset{y}{\bullet}$$

which has two objects x and y, a single morphism α from x to y, and the identity morphisms at x and y. We readily see that the transitive (non-empty) $\mathcal{C}$-sets have the form

$$\Omega_n := \underline{n} \to \underline{1}, \quad n \geq 0$$

where $\underline{n} = \{1, \ldots, n\}$ is a set with n elements, the mapping between the two sets sending every element onto a single element. We see various things from this example, such as that a finite category may have infinitely non-isomorphic transitive sets, and also that transitive sets need not be generated by any single element.

We have available another operation on $\mathcal{C}$-sets, namely $\times$. Given two $\mathcal{C}$-sets Ω and Ψ we define $(\Omega \times \Psi)(x) = \Omega(x) \times \Psi(x)$, with the expected definition on morphisms of $\mathcal{C}$. In the above example we see that $\Omega_m \times \Omega_n \cong \Omega_{mn}$.

We are now ready to define the *Burnside ring* of the category $\mathcal{C}$ as

$$B(\mathcal{C}) := \text{Grothendieck group of finite } \mathcal{C}\text{-sets with respect to } \sqcup.$$

Thus $B(\mathcal{C})$ is the free abelian group with the (isomorphism classes of) transitive $\mathcal{C}$-sets as a basis. The multiplication on $B(\mathcal{C})$ is given by $\times$ on the basis elements. Note that this definition of the Burnside ring of a category appears to be quite different to the definitions given by Yoshida in [10] and May in [5].

As an example take $\mathcal{C}$ to be the category which we have seen before. From our calculations we have

$$\begin{aligned} B(\mathcal{C}) &= \mathbb{Z}\{\Omega_0, \Omega_1, \Omega_2, \ldots\} \\ &= \mathbb{Z}\mathbb{N}^{\times}_{\geq 0} \\ &\cong \mathbb{Z}\Omega_0 \oplus \mathbb{Z}\{\Omega_1 - \Omega_0, \Omega_2 - \Omega_0, \ldots\} \\ &\cong \mathbb{Z} \oplus \mathbb{Z}\mathbb{N}^{\times}_{>0} \end{aligned}$$

as rings, where $\mathbb{Z}\mathbb{N}^{\times}_{>0}$ (for example) denotes the monoid algebra over $\mathbb{Z}$ of the multiplicative monoid of non-zero natural numbers. This is the complete decomposition of $B(\mathcal{C})$ as a direct sum of rings. The ring $\mathbb{Z}\mathbb{N}^{\times}_{>0}$ is not finitely generated, and hence neither is $B(\mathcal{C})$.

We illustrate the kind of situation where these constructions may be applied. Quite regularly we consider diagrams of one thing or another, be it sets, or perhaps spaces. By a *space* we mean a simplicial set, in which case a diagram of spaces $\Omega : \mathcal{C} \to \text{Spaces}$ is the same thing as a simplicial $\mathcal{C}$-set. Given such Ω, in each dimension i the i-simplices Ω_i form a $\mathcal{C}$-set. We may form a Lefschetz invariant $\sum_{i\geq 0}(-1)^i\Omega_i$ and this is an element of the Burnside ring $B(\mathcal{C})$. It depends only on the $\mathcal{C}$-homotopy type of Ω. As an example of how this might arise, let G be a finite group and take $\mathcal{C}$ to be the orbit category of G with stablizers in some family of subgroups. Thus the objects of $\mathcal{C}$ are G-sets G/H with H in the specified family, and the morphisms are the equivariant maps. Given a G-space Δ we may obtain a $\mathcal{C}^{\text{op}}$-space $\hat{\Delta}$ by $\hat{\Delta}(G/H) = \Delta^H$, the fixed points, and hence we get a Lefschetz invariant $L(\hat{\Delta})$ in the Burnside ring of the opposite of the orbit category. This invariant carries more information than a similar invariant in the Burnside ring of

G considered in [6], [7, p. 358] and [4, Def. 1.6], since the latter invariant is the evaluation of $L(\hat{\Delta})$ at $G/1$.

2. Bisets for categories

Given categories $\mathcal{C}$ and $\mathcal{D}$ we define a $(\mathcal{C},\mathcal{D})$-*biset* to be the same thing as a $\mathcal{C}\times\mathcal{D}^{\mathrm{op}}$-set. Such a biset Ω is a functor $\mathcal{C}\times\mathcal{D}^{\mathrm{op}}\to \mathrm{Set}$, so given objects $x\in\mathcal{C}$ and $y\in\mathcal{D}$ and morphisms $\alpha: x\to x_1$ in $\mathcal{C}$ and $\beta: y_1\to y$ in $\mathcal{D}$, and an element $u\in\Omega(x,y)$ we get elements $\alpha u := \Omega(\alpha\times 1_y)(u)\in\Omega(x_1,y)$ and $u\beta := \Omega(1_x\times\beta)(u)\in\Omega(x,y_1)$. In this sense we have an action of $\mathcal{C}$ from the left and $\mathcal{D}$ from the right on Ω.

As an example, we define ${}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}$ to be the $(\mathcal{C},\mathcal{C})$-biset with ${}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}(x,y)=\mathrm{Hom}_{\mathcal{C}}(y,x)$, where we reverse the order of x and y because morphisms are composed on the left. In the case of a group this is the regular representation with the group acting by multiplication from the left and from the right.

Given a $(\mathcal{C},\mathcal{D})$-biset ${}_{\mathcal{C}}\Omega_{\mathcal{D}}$ and a $(\mathcal{D},\mathcal{E})$-biset ${}_{\mathcal{D}}\Psi_{\mathcal{E}}$ we construct a $(\mathcal{C},\mathcal{E})$-biset $\Omega\circ\Psi$ by the formula

$$\Omega\circ\Psi(x,z)=\bigsqcup_{y\in\mathcal{D}}\Omega(x,y)\times\Psi(y,z)/\sim$$

where $\sim$ is the equivalence relation generated by $(u\beta,v)\sim(u,\beta v)$ whenever $u\in\Omega(x,y_1)$, $v\in\Psi(y_2,z)$ and $\beta: y_2\to y_1$ in $\mathcal{D}$.

Proving the following result is a very good test of one's understanding of this construction:

Proposition 2.1. *The operation $\circ$ is an associative product, with identity the biset ${}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}$.*

We now define $A(\mathcal{C},\mathcal{D})$ to be the Grothendieck group of finite $(\mathcal{C},\mathcal{D})$-bisets with respect to $\sqcup$, thus extending the notion of the *double Burnside ring* for groups. If R is a commutative ring with 1 we put $A_R(\mathcal{C},\mathcal{D}) := R\otimes_{\mathbb{Z}} A(\mathcal{C},\mathcal{D})$ Using this construction we now define an analog $\mathbb{B}_{\mathrm{Cat}}$ of the *Burnside category* of [1] (see also [2] and [8], for example). The category $\mathbb{B}_{\mathrm{Cat}}$ has as objects all (finite) categories, with homomorphisms given by $\mathrm{Hom}_{\mathbb{B}_{\mathrm{Cat}}}(\mathcal{C},\mathcal{D}) = A_R(\mathcal{D},\mathcal{C})$. We define a *biset functor* over R to be an R-linear functor $\mathbb{B}_{\mathrm{Cat}}\to R$-mod. This notion evidently extends the usual notion of biset functors defined on groups, which are R-linear functors defined on the full subcategory $\mathbb{B}_{\mathrm{Group}}$ of $\mathbb{B}_{\mathrm{Cat}}$ whose objects are finite groups.

The Burnside ring functor $B_R(\mathcal{C}) := R\otimes_{\mathbb{Z}} B(\mathcal{C})$ is in fact an example of a biset functor defined on categories. Let $\mathbf{1}$ denote the category with one object and one morphism – in other words, the identity group. We see that if $\mathcal{C}$ is any category, $\mathcal{C}$-sets may be identified as the same thing as $(\mathcal{C},\mathbf{1})$-bisets, so that $B_R(\mathcal{C}) = A_R(\mathcal{C},\mathbf{1}) = \mathrm{Hom}_{\mathbb{B}_{\mathrm{Cat}}}(\mathbf{1},\mathcal{C})$. Thus B_R is a representable biset functor over R, and hence it is projective. It is indecomposable since its endomorphism ring is $\mathrm{End}(B_R)\cong A_R(\mathbf{1},\mathbf{1})\cong R$ by Yoneda's lemma (assuming R is indecomposable).

All this is similar to what happens with biset functors defined on groups, as described in [2], and the story continues. Supposing that the ring R we work over is a field or complete discrete valuation ring, for formal reasons the simple biset functors may be parametrized by pairs $(\mathcal{C}, V)$ consisting of a category $\mathcal{C}$ and a simple $\mathrm{End}_{\mathbb{B}_{\mathrm{Cat}}}(\mathcal{C})$-module V, subject to a certain equivalence relation described in a slightly different context in [9, Cor. 4.2]. Each simple functor $S^{\mathrm{Cat}}_{\mathcal{C},V}$ has a projective cover $P^{\mathrm{Cat}}_{\mathcal{C},V}$: an indecomposable projective with $S^{\mathrm{Cat}}_{\mathcal{C},V}$ as its unique simple quotient. Because the category of groups is a full subcategory of the category of small categories the relationship between functors defined on $\mathbb{B}_{\mathrm{Cat}}$ and $\mathbb{B}_{\mathrm{Group}}$ is similar to that of representations of an algebra Λ and of $e\Lambda e$ where $e \in \Lambda$ is idempotent. This kind of relationship was described by Green in [3] is described in a context close to the present one in sections 3 and 4 of [9]. Some of this relationship goes as follows.

Proposition 2.2. *Let S be a simple biset functor defined on categories. Then its restriction to groups is either zero or a simple functor and establishes a bijection $S^{\mathrm{Cat}}_{G,V} \leftrightarrow S^{\mathrm{Group}}_{G,V}$ between isomorphism types of simple biset functors defined on categories which are non-zero on groups, and simple biset functors defined on groups G. Furthermore $P^{\mathrm{Cat}}_{G,V} \downarrow^{\mathrm{Cat}}_{\mathrm{Group}} \cong P^{\mathrm{Group}}_{G,V}$, and $P^{\mathrm{Group}}_{G,V} \uparrow^{\mathrm{Cat}}_{\mathrm{Group}} \cong P^{\mathrm{Cat}}_{G,V}$ where $\uparrow^{\mathrm{Cat}}_{\mathrm{Group}}$ denotes the left adjoint to the restriction $\downarrow^{\mathrm{Cat}}_{\mathrm{Group}}$.*

Thus every simple biset functor defined on groups extends uniquely to a simple biset functor defined on categories, and the same holds for indecomposable projective biset functors. We see, when R is a field, that the Burnside ring functor B_R is in fact the indecomposable projective $P^{\mathrm{Cat}}_{\mathbf{1},R}$ with unique simple quotient $S^{\mathrm{Cat}}_{\mathbf{1},R}$.

We conclude by mentioning that the values of this simple functor may be identified in terms of a certain bilinear pairing between the Burnside ring of a category and of its opposite, generalizing a bilinear form introduced in [2]. The Burnside ring $B_R(\mathcal{C})$ has as basis the transitive $(\mathcal{C}, \mathbf{1})$-bisets ${}_{\mathcal{C}}\Omega_{\mathbf{1}}$, and $B_R(\mathcal{C}^{\mathrm{op}})$ has as basis the transitive $(\mathbf{1}, \mathcal{C})$-bisets ${}_{\mathbf{1}}\Psi_{\mathcal{C}}$. We define a bilinear map $\langle \ , \ \rangle : B_R(\mathcal{C}^{\mathrm{op}}) \times B_R(\mathcal{C}) \to R$ by $\langle {}_{\mathbf{1}}\Psi_{\mathcal{C}}, {}_{\mathcal{C}}\Omega_{\mathbf{1}} \rangle = |{}_{\mathbf{1}}\Psi_{\mathcal{C}} \circ {}_{\mathcal{C}}\Omega_{\mathbf{1}}|$, the size of this set.

Proposition 2.3. *If R is a field then the dimension of the simple biset functor $S_{\mathbf{1},R}$ is the rank of the above bilinear pairing.*

The observations here are just the start of a development of theory on which the author is currently working.

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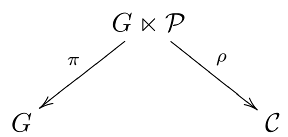
Subgroups as transporter categories and the transfer

Fei Xu

Let G be a finite group and $\mathcal{P}$ a finite G-poset. The Grothendieck construction on the G-poset $\mathcal{P}$ is a finite category and is written as $G \ltimes \mathcal{P}$. We may call such a category a transporter category as the usual transporter categories can be introduced in this way. Moreover, each subgroup H of G can be recovered as a transporter category $G \ltimes (G/H)$, for the G-poset G/H, up to a category equivalence. Therefore we may deem transporter categories as generalized subgroups for a fixed finite group. Topologically it is well known that $|G \ltimes \mathcal{P}| \simeq EG \times_G |\mathcal{P}|$, where $|\cdot|$ denotes the classifying space of a small category.

As two motivating examples, transporter categories were implicitly considered by Mark Ronan and Steve Smith in 1980s for constructing group modules, and later on played a key role in Bill Dwyer's work on homology decomposition of classifying spaces. Roughly speaking, one has a diagram of categories and functors



if $\mathcal{C}$ is a quotient category of $G \ltimes \mathcal{P}$. Here π is a canonical functor. (The quotient categories we have in mind are orbit categories, Brauer categories, Puig categories or even transporter categories themselves.) Dwyer used this diagram to establish connections among various homotopy colimits (e.g. classifying spaces), while Ronan and Smith constructed kG-modules via representations of $G \ltimes \mathcal{P}$ (equivalently, functors from $G \ltimes \mathcal{P}$ to $Vect_k$, where k is a field and $Vect_k$ is the category of finite-dimensional k-vector spaces). Applying classical tools in homological algebra, particularly the Kan extensions, we obtain two pairs of natural adjoint functors between kG-mod and $k\mathcal{C}$-mod, where $k\mathcal{C}$ is the (k-)category algebra of $\mathcal{C}$ in the sense of Peter Webb:

- $\mathrm{Hom}_{kG}(M, RK_\pi \mathrm{Res}_\rho N) \cong \mathrm{Hom}_{k\mathcal{C}}(LK_\rho \mathrm{Res}_\pi M, N)$;
- $\mathrm{Hom}_{kG}(LK_\pi \mathrm{Res}_\rho U, V) \cong \mathrm{Hom}_{k\mathcal{C}}(U, RK_\rho \mathrm{Res}_\pi V)$,

for $M, V \in kG$-mod and $N, U \in k\mathcal{C}$-mod. By direct calculation, these Kan extensions are

- $LK_\pi \cong \varinjlim_{\mathcal{P}}$, $RK_\pi \cong \varprojlim_{\mathcal{P}}$ (used by M. Ronan and S. Smith);
- $LK_\rho \cong \uparrow_{k(G\ltimes\mathcal{P})}^{k\mathcal{C}}$, $RK_\rho \cong \Uparrow_{k(G\ltimes\mathcal{P})}^{k\mathcal{C}}$, since ρ induces an algebra homomorphism $k(G \ltimes \mathcal{P}) \to k\mathcal{C}$ when $\mathcal{C}$ is a quotient category.

To some extent, comparing kG-mod with $k\mathcal{C}$-mod provides an extended context for local representation theory.

In group cohomology, one of the main themes is to compare cohomology of G with that of various subgroups H. The standard tools are the restriction and transfer. Based on our observation at the very beginning, we may replace subgroups H by transporter categories $G \ltimes \mathcal{P}$ when studying cohomology of G.

Theorem 1. *Suppose* $\mathrm{Res}_\pi : kG$*-mod* $\to k(G \ltimes \mathcal{P})$*-mod is the restriction along* π *and write* $\mathrm{Res}_\pi M = \underline{M}$ *for any* $M \in kG$*-mod. Then we have the following two maps, restriction and transfer,*

$$\mathrm{Ext}^*_{kG}(M, N) \xrightarrow{res} \mathrm{Ext}^*_{k(G\ltimes\mathcal{P})}(\underline{M}, \underline{N}) \xrightarrow{tr} \mathrm{Ext}^*_{kG}(M, N),$$

which compose to $\chi(\mathcal{P}; k) \cdot 1$*, multiplication by the Euler characteristic of (the order complex of)* $\mathcal{P}$*.*

This is a variation of the Becker-Gottlieb transfer for fibre bundles with compact fibres, reduced via Kan-Thurston Theorem to $|\mathcal{P}| \to EG \times_G |\mathcal{P}| \to BG = |G|$. Our construction/proof is entirely algebraic and similar to that for the classical situation where $G \ltimes \mathcal{P}$ "is" a subgroup (i.e. when $\mathcal{P} = G/H$). It uses Res_π and its left adjoint, the left Kan extensions (generalizing the induction), as well as certain standard categorical constructions.

From Venkov's proof of the finite generation of group cohomology ring, one knows the ordinary cohomology ring $\mathrm{Ext}^*_{k(G\ltimes\mathcal{P})}(\underline{k}, \underline{k}) \cong \mathrm{H}^*(EG \times_G |\mathcal{P}|, k)$ is finitely generated. Since every $k\mathcal{C}$-mod is a closed symmetric monoidal category with tensor identity $\underline{k}$, there exists a natural action, via cup product, of $\mathrm{Ext}^*_{k\mathcal{C}}(\underline{k}, \underline{k})$ on $\mathrm{Ext}^*_{k\mathcal{C}}(A, B)$ for $A, B \in k\mathcal{C}$-mod.

Theorem 2. *With the same notations as above,* $\mathrm{Ext}^*_{k(G\ltimes\mathcal{P})}(\underline{M}, B)$ *is finitely generated over* $\mathrm{Ext}^*_{k(G\ltimes\mathcal{P})}(\underline{k}, \underline{k})$*, where* $M \in kG$*-mod and* $B \in k(G \ltimes \mathcal{P})$*-mod.*

At present we are unable to replace $\underline{M}$ by arbitrary $A \in k(G \ltimes \mathcal{P})$-mod because the internal hom does not help.

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Fusion systems and constructing free actions on products of spheres

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(joint work with Özgün Ünlü)

Let G be a finite group. The rank of G, denoted by $\mathrm{rk}(G)$, is defined as the maximum s such that $(\mathbb{Z}/p)^s \leq G$ for some prime p. By a theorem of Heller [4], it is known that if G acts freely on a product of two spheres, then $\mathrm{rk}(G) \leq 2$. More recently, Adem-Smith [2] showed that every rank two p-group acts freely on a finite CW-complex homotopy equivalent to a product of two spheres. We prove that these actions can be constructed smoothly as well.

Theorem 1. *Every rank two p-group acts freely and smoothly on a product of two spheres.*

This theorem was proved for $p \geq 5$ by Adem-Davis-Ünlü [1] using fixity methods and for $p \geq 3$ by Ünlü in his PhD thesis using a method based on a theorem of Lück-Oliver [5] on constructions of equivariant vector bundles.

In our work [8], we also use the work of Lück-Oliver. An important ingredient in the Lück-Oliver construction is the existence of a finite group Γ satisfying certain properties. We find a systematic way of constructing this finite group using an intermediate group S. We consider a certain fusion system $\mathcal{F}$ on S and use a theorem of Park [6] which allows us to embed $\mathcal{F}$ into a fusion system $\mathcal{F}_S(\Gamma)$ of a finite group Γ. The group Γ is constructed as the automorphism group of a certain S-S-biset associated to $\mathcal{F}$. Existence of these bisets were proved by Broto-Levi-Oliver [3] for saturated fusion systems and they are called characteristic bisets. The fusion systems we consider are not saturated in general, so we use explicit constructions to show that these bisets exist.

Our methods also allow us to prove the following:

Theorem 2. *Let G be a finite group and p be a fixed prime. Suppose that G acts smoothly on a compact manifold M such that for every $x \in M$, the isotropy subgroup G_x is an elementary abelian p-group with rank $\leq k$. Then, G acts freely and smoothly on $M \times \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_k}$ for some positive integers $n_1, \dots, n_k$.*

An immediate corollary of this theorem is the following:

Corollary 3. *Let G be an (almost) extraspecial p-group of rank r. Then, G acts freely and smoothly on a product of r spheres.*

More generally, if G is a p-group such that every element of order p in the Frattini subgroup $\Phi(G) \leq G$ is central, then G acts freely and smoothly on a product of $\mathrm{rk}(G)$ many spheres. To see this, note that when G satisfies the above property, we can find representations $V_1, \ldots, V_t$, where $t = \mathrm{rk}(\Phi(G))$, such that G acts on

$$M = \mathbb{S}(V_1) \times \cdots \times \mathbb{S}(V_t)$$

in such a way that the induced action of the Frattini subgroup $\Phi(G)$ on M is free. So, the isotropy subgroups of G-action on M will be all elementary abelian with ranks $\leq \mathrm{rk}(G) - t$. Using Theorem 2, one obtains the desired free action.

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Stable degree zero Hochschild homology, Auslander-Reiten conjecture and tame blocks

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(joint work with Yuming Liu and Guodong Zhou)

Throughout let K be an algebraically closed field and let A and B be two finite dimensional K-algebras. In earlier work [9] we studied, in case the base field is perfect of characteristic $p > 0$, the invariance of Külshammer's ideals [3] of the centre of a symmetric algebra under derived equivalences. The definition of Külshammer ideals uses degree 0 Hochschild homology and cohomology and p-power maps on degree 0 Hochschild homology. A numerical version of Külshammer ideals, just involving certain linear subspaces does not need the symmetry of the algebras. Thankfully these objects are invariant under derived equivalences: If the derived categories $D^b(A)$ and $D^b(B)$ are equivalent as triangulated categories, then it is known that the Hochschild homology and the Hochschild cohomology of A and B are isomorphic:

$$D^b(A) \simeq D^b(B) \Rightarrow [HH^*(A) \simeq HH^*(B) \text{ and } HH_*(A) \simeq HH_*(B)]$$

Chang Chang Xi posed the question if a similar statement might hold for stable equivalences. Denote by $A-\underline{\text{mod}}$ the stable category and $\underline{\text{Hom}}_A(-,-)$ the morphisms in the stable category. For equivalences of stable categories much less is true than for derived equivalences. In particular the properties we need for Külshammer's ideal theory are not preserved.

Example. A striking example was given by Auslander and Reiten in 1973 [1]. Let

$$A := \begin{pmatrix} K & K \\ 0 & K \end{pmatrix},\ B := \begin{pmatrix} K & K & K \\ 0 & K & K \\ 0 & 0 & K \end{pmatrix} \Big/ \begin{pmatrix} 0 & 0 & K \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } C := K[X]/X^2.$$

We get

$$A \times A - \underline{\text{mod}} \simeq B - \underline{\text{mod}} \text{ and } A - \underline{\text{mod}} \simeq C - \underline{\text{mod}}.$$

The algebras $A \times A$ and B are hence stably equivalent and B is indecomposable whereas A is not. Neither A nor B nor C has any simple direct factor. Moreover, A is hereditary, whereas C is symmetric. The centre of $A \times A$ is 2-dimensional whereas the centre of B is 1-dimensional.

Much stronger is a special class of stable equivalences introduced by Broué.

Definition. (Broué [2]) Let K be a commutative ring, let A and B be two K-algebras and let $M \in A \otimes_K B^{op} - \text{mod}$ and let $N \in B \otimes_K A^{op} - \text{mod}$. Then (M, N) is said to induce a stable equivalence of Morita type if

- M as well as N are projective as A-modules and as B-modules.
- $M \otimes_B N \simeq A \oplus P$ as $A - A$-bimodules for a projective $A - A$-bimodule P
- $N \otimes_A M \simeq B \oplus Q$ as $B - B$-bimodules for a projective $B - B$-bimodule Q.

It should be noted that if A and B are derived equivalent symmetric algebras then A and B are stably equivalent of Morita type.

Liu shows [4] that if A and B are stably equivalent of Morita type and without any simple direct factor, then A is indecomposable if and only if B is indecomposable. A result of Xi [5] shows that if A and B are stably equivalent of Morita type, then the strictly positive degree Hochschild homology and cohomology is isomorphic: $HH^{>0}(A) \simeq HH^{>0}(B)$ and $HH_{>0}(A) \simeq HH_{>0}(B)$. Moreover, if A and B are stably equivalent of Morita type then Broué shows that

$$Z^{st}(A) := \underline{\text{End}}_{A \otimes_K A^{op}}(A) \simeq \underline{\text{End}}_{B \otimes_K B^{op}}(B) = Z^{st}(B).$$

A stable version of Hochschild homology was given by Eu for selfinjective algebras. However, a p-power map was missing there, and we search for an object available for general finite dimensional K-algebras admitting p-power maps. Let M be a $B - A$-bimodule, which is projective as B-module. Then the Hattori-Stallings trace tr_M induces a mapping on degree 0 Hochschild homology:

$$tr_M : A/[A, A] \longrightarrow B/[B, B].$$

Definition. [6] Let A be a finite dimensional K-algebra and let $\mathcal{P}$ be a set of representatives of the isomorphism classes of indecomposable projective A-modules. Let

$$HH_0^{st}(A) := \bigcap_{P \in \mathcal{P}} \ker(tr_P)$$

be the stable degree 0 Hochschild homology.

We obtain

Theorem. [6] *Let K be an algebraically closed field and let A and B be finite dimensional K-algebras without any simple direct factor and suppose that A and B are stably equivalent of Morita type.*

Then $HH_0^{st}(A) \simeq HH_0^{st}(B)$. Moreover, $HH_0(A) \simeq HH_0(B)$ if and only if A and B have the same number of isomorphism classes of non projective simple modules. Further, the dimension version of the Külshammer ideal theory coincide.

Observe that the statement that the number of non projective simple modules is preserved under stable equivalences is a conjecture of Auslander and Reiten.

The obstruction to the validity of the Auslander-Reiten conjecture appears as the rank of the Cartan matrices of the algebras, seen as matrices over K. In the case of symmetric algebras this rank equals the dimension of the projective centre, i.e. the dimension of the kernel of the natural map $Z(A) \longrightarrow Z^{st}(A)$.

This property, and the fact that by a result of Xi [5] the absolute value of the elementary divisors of the Cartan matrix are invariant under stable equivalences of Morita type, allows us to show

Theorem. [7] *Let K be an algebraically closed field and let A be an algebra of dihedral, (resp. semidihedral) (resp. quaternion) type in the sense of Erdmann. Then the Auslander-Reiten conjecture for stable equivalences of Morita type holds to be true for A. Moreover, if B is a finite dimensional K-algebra stably equivalent of Morita type to A, then B is an algebra of dihedral, (resp. semidihedral) (resp. quaternion) type.*

By the derived equivalence classification of these algebras, due to Holm, this gives an almost complete classification of these algebras up to stable equivalences of Morita type. For certain classes of algebras we can give a more distinctive statement distinguishing certain algebras with the same number of simples and of the same type (dihedral, resp semidihedral, resp quaternion).

Finally, for weakly symmetric tame algebras of domestic representation type and weakly symmetric algebras of polynomial growth representation type Bocian, Białkowski, Holm and Skowroński gave a classification up to derived equivalences.

We can show, using the results in [6],

Theorem. [8] *Two weakly symmetric algebras of domestic representation type are stably equivalent of Morita type if and only if they are derived equivalent.*

The classification up to stable equivalences of Morita type of weakly symmetric algebras of polynomial growth representation type coincides with the classification up to derived equivalences in the sense noted below.

Remark that in the classification up to derived equivalences a question could not be solved if a certain parameter in a socle relation of a specific quiver leads to two different derived equivalence classes. The same question is open also for stable equivalence of Morita type classification.

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