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Classical Algebraic Geometry

Organised by
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June 20th – June 26th, 2010

Abstract. Algebraic geometry studies properties of specific algebraic varieties, on the one hand, and moduli spaces of all varieties of fixed topological type on the other hand. Of special importance is the moduli space of curves, whose properties are subject of ongoing research. The rationality versus general type question of these and related spaces is of classical and also very modern interest with recent progress presented in the conference. Certain different birational models of the moduli space of curves and maps have an interpretation as moduli spaces of singular curves and maps. For specific varieties a wide range of questions was addressed, including extrinsic questions (syzygies, the k-secant lemma) and intrinsic ones (generalization of notions of positivity of line bundles, closure operations on ideals and sheaves).

Mathematics Subject Classification (2000): 14xx.

Introduction by the Organisers

The workshop *Classical Algebraic Geometry* held from June 20th to June 26th 2010 at the "Mathematisches Forschungsinstitut Oberwolfach" was organized by David Eisenbud (Berkeley), Frank-Olaf Schreyer (Saarbrücken), Ravi Vakil (Stanford) and Claire Voisin (Paris). It was very well attended with over 50 participants from the United States, Canada, United Kingdom, Italy, France, Poland and Germany. In total there were 18 one hour talks with a maximum of four talks a day and a session with short presentations allowing young participants to give 10 minute outlines of their current work. This schedule left plenty of room for many informal discussions and work in smaller groups.

The extended abstracts give a detailed account of the broad variety of topics of the meeting, many of them classical questions in algebraic geometry discussed with modern methods. Although it would be nice if we could mention all talks here, we should focus on a couple of highlights:

- The conference opened with Gavril Farkas' lecture on his major new work on Green's conjecture on syzygies of canonical curves. The previous major breakthrough was Voisin's proof earlier this decade that Green's conjecture holds for smooth curves lying on a K3 surface with Picard group of rank 1, and Aprodu's result (building on Voisin) that Green's conjecture holds for all curves satisfying the linear growth condition, thus turning Green's conjecture into a question in Brill-Noether theory. Farkas explained his recent proof that Green's conjecture in fact holds for every smooth curve lying on an *arbitrary* K3 surface.
- Daniel Erman, who received his Ph.D. shortly before arriving, reported on joint work with Melanie Wood (another recent Ph.D., and an AIM five-year-fellow). Their work deals with the space of degree n covers of varieties (or schemes), which they elegantly interpret in terms of the space (stack) of moduli of points. If $n \leq 5$, beautiful descriptions involving matrix presentations (or geometric variations thereof) lead to explicit useful results. (One prominent example is the celebrated work in number theory of Manjul Bhargava.) They begin to address the case $n = 6$, making clear what makes this case difficult. A special case are configurations arising from the Gulliksen-Negard complex, and they give nontrivial restrictions on the class of sextic covers which can arise from such a construction.
- The recent development of tropical geometry has had a significant impact on algebraic geometry, by turning many classical problems into piecewise linear problems that one can work with, either to calculate, or to prove theorems. Sam Payne reported on recent work with Brian Osserman developing intersection theory in tropical geometry, in such a way that the results will translate into classical algebraic geometry. This requires the systematic development of dimension theory and intersection theory on decidedly non-classical objects: schemes of finite type over non-Noetherian valuation rings of rank 1. Their general theory promises to subsume and extend earlier ad hoc methods, and will substantially change the subject.
- The concluding lecture of Joe Harris tied together a number of recent advances on the birational geometry of parameter spaces of curves. Different compactifications of the space of curves (both by themselves, and in projective space) have proved useful in a number of contexts. It has been recently observed (first by Hassett and Keel) that these different compactifications are often related in geometrically meaningful ways. For example, D. Chen, Coskun, and Crissman have shown that many of the spaces of rational curves in projective space arise in this way. Also, Viscardi has generalized recent work of Smyth and others on genus 1 curves to find a small compactification of genus 1 curves in projective space, which one

might hope is related to the recent "stable pairs" construction of Marian, Oprea, and Pandharipande.

The young participants' presentations by Michele Bolognesi, Dawei Chen, Christian Christensen, Thomas Dedieu, Florian Geiss, Andreas Höring, Grzegorz Kapustka, Paul Larsen and Margherita Lelli-Chiesa also covered a widespread variety of topics from moduli of curves, vector bundles, K3 surfaces to abelian varieties.

Workshop: Classical Algebraic Geometry

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Abstracts

The Green Conjecture for curves on $K3$ surfaces

Gavril Farkas
(joint work with Marian Aprodu)

Green's Conjecture on syzygies of canonical curves asserts that one can recognize existence of special linear series on an algebraic curve, by looking at the syzygies of its canonical embedding. Precisely, if C is a smooth algebraic curve of genus g, $K_{i,j}(C, K_C)$ denotes the (i,j)-th Koszul cohomology group of the canonical bundle K_C and $\mathrm{Cliff}(C)$ is the Clifford index of C, then M. Green predicted in 1984 that

$$K_{p,2}(C, K_C) = 0, \quad \text{for all } \ p < \mathrm{Cliff}(C). \tag{1}$$

In recent years, Voisin [V02], [V05] achieved a major breakthrough by showing that Green's Conjecture holds for smooth curves C lying on $K3$ surfaces S with $\mathrm{Pic}(S) = \mathbb{Z}\cdot C$. In particular, this establishes Green's Conjecture for general curves of every genus. Using Voisin's work, as well as a degenerate form of [HR98], it has been proved in [Ap05] that Green's Conjecture holds for any curve C of genus g of gonality $\mathrm{gon}(C) = k \leq (g+2)/2$, which satisfies the *linear growth condition*

$$\dim W^1_{k+n}(C) \leq n, \ \text{ for } 0 \leq n \leq g - 2k + 2. \tag{2}$$

Thus Green's Conjecture becomes a question in Brill-Noether theory. In particular, one can check that condition (2) holds for a general curve $[C] \in \mathcal{M}^1_{g,k}$ in any gonality stratum of $\mathcal{M}_g$, for all $2 \leq k \leq (g+2)/2$. Our main result is the following:

Theorem 1. *Green's Conjecture holds for every smooth curve C lying on an arbitrary $K3$ surface S.*

The proof of the statement does not cover, but it rather relies in an essential way on the most difficult case, that of curves of odd genus of maximal gonality. Precisely, when $g(C) = 2k - 3$ and $\mathrm{gon}(C) = k$, Theorem 1 as stated, is due to Voisin [V05] combined with results of Hirschowitz-Ramanan [HR98]. In the proof of Theorem 1, we distinguish two cases. When $\mathrm{Cliff}(C)$ is computed by a pencil (that is, $\mathrm{Cliff}(C) = \mathrm{gon}(C) - 2$), we use a parameter count for spaces of Lazarsfeld-Mukai bundles, in order to find a smooth curve $C' \in |C|$ in the same linear system as C, such that C' verifies condition (2). Since Koszul cohomology satisfies the Lefschetz hyperplane principle, one has that $K_{p,2}(C, K_C) \cong K_{p,2}(C', K_{C'})$. This proves Green's Conjecture for C.

When $\mathrm{Cliff}(C)$ is no longer computed by a pencil, it follows from [Kn09] that C is a *generalized ELMS example*, in the sense that there exist smooth curves $D, \Gamma \subset S$, with $\Gamma^2 = -2, \Gamma \cdot D = 1$ and $D^2 \geq 2$, such that $C \equiv 2D + \Gamma$ and $\mathrm{Cliff}(C) = \mathrm{Cliff}(\mathcal{O}_C(D)) = \mathrm{gon}(C) - 3$. This case requires a separate analysis since condition (2) is no longer satisfied.

Theorem 1 follows by combining results obtained by using the powerful techniques developed in [V02], [V05], with facts about the effective cone of divisors of $\overline{\mathcal{M}}_g$. Starting with any k-gonal curve $[C] \in \mathcal{M}_g$ satisfying condition (2), by identifying pairs of general points $x_i, y_i \in C$ for $i = 1, \ldots, g+3-2k$ one creates a stable curve

$$\Big[X := \frac{C}{x_1 \sim y_1, \ldots, x_{g+3-2k} \sim y_{g+3-2k}}\Big] \in \overline{\mathcal{M}}_{2g+3-2k}$$

having maximal gonality $g+3-k$. Since the class of the virtual failure locus of Green's Conjecture is a multiple of the Hurwitz divisor on $\overline{\mathcal{M}}_{2g+3-2k}$, cf. [HR98], Voisin's theorem can be extended to all irreducible stable curves of maximal gonality, in particular to X as well, and a posteriori to smooth curves sitting on $K3$ surfaces having arbitrary Picard lattice. On the other hand, showing that condition (2) is satisfied for a given curve C, is a question of pure Brill-Noether nature.

Theorem 1 has strong consequences on Koszul cohomology of $K3$ surfaces. It is known that for any globally generated line bundle L on a $K3$ surface S, the Clifford index of any smooth irreducible curve is constant, equal to, say c. Applying Theorem 1, Green's hyperplane section theorem, the duality theorem and finally the Green-Lazarsfeld nonvanishing theorem, we obtain a complete description of the distribution of zeros among the Koszul cohomology groups of S with values in L.

Theorem 2. *Suppose $L^2 = 2g-2 \geq 2$. The Koszul cohomology group $K_{p,q}(S, L)$ is nonzero if and only if one of the following cases occur:*

(1) $q = 0$ *and* $p = 0$*, or*
(2) $q = 1$, $1 \leq p \leq g-c-2$*, or*
(3) $q = 2$ *and* $c \leq p \leq g-1$*, or*
(4) $q = 3$ *and* $p = g-2$.

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Cones of higher-codimensional cycles

Olivier Debarre
(joint work with Lawrence Ein, Robert Lazarsfeld, Claire Voisin)

Introduction. Divisors on a smooth complex projective variety X have long been studied by looking at their classes in the finite-dimensional real vector space

$$N^1(X) := \{\mathbf{R}\text{-divisors on } X\}/\text{numerical equivalence}$$

and by studying the (closed convex) cones

$$\mathrm{Psef}^1(X) \subset N^1(X),$$

generated by classes of hypersurfaces in X, and

$$\mathrm{Nef}^1(X) \subset N^1(X),$$

generated by classes of nef divisors on X. There are inclusions

$$\mathrm{Nef}^1(X) \subset \mathrm{Psef}^1(X) \subset N^1(X).$$

We can make similar definitions in higher codimensions. For each $0 \leq k \leq n := \dim(X)$, define a finite-dimensional real vector space

$$N^k(X) := \{\text{codimension-}k \text{ algebraic } \mathbf{R}\text{-cycles on } X\}/\text{numerical equivalence}$$

and the closed convex cone $\mathrm{Psef}^k(X) \subset N^k(X)$ generated by classes of subvarieties of codimension k. There are perfect pairings $N^k(X) \times N^{n-k}(X) \to \mathbf{R}$ given by intersection, and we define

$$\mathrm{Nef}^k(X) \subset N^k(X)$$

to be the closed convex cone dual to $\mathrm{Psef}^{n-k}(X)$.

Our main result is that the nef cone is in general no longer contained in the pseudo-effective cone. We give an example with $k = 2$ and X is an abelian variety (for which $\mathrm{Psef}^k(X) \subset \mathrm{Nef}^k(X)$), answering negatively a question of Grothendieck ([2]).

Positive classes on abelian varieties. Let $X = (V/\text{lattice})$ be an abelian variety, where V is a complex vector space of dimension n. Since numerical and homological equivalence coincide on X, we have

$$\begin{aligned} N^k(X) \subset H^{k,k}(X) \cap H^{2k}(B, \mathbf{R}) &= \{\text{real } (k,k)\text{-forms on } V\} \\ &= \{\text{Hermitian forms on } \wedge^k V\} := N^k(V). \end{aligned}$$

If $\mathrm{Semi}^1(V) \subset N^1(V)$ is the cone of semipositive Hermitian forms,

$$\mathrm{Psef}^1(X) = \mathrm{Nef}^1(X) = \mathrm{Semi}^1(V) \cap N^1(X) := \mathrm{Semi}^1(X).$$

But for real (k,k)-forms, there are several distinct notions of positivity:

$$\begin{array}{rcl} \mathrm{Strong}^k(V) & = & \text{convex cone gen'd by } i^{k^2}\alpha\wedge\overline{\alpha},\ \alpha \text{ decomposable } k\text{-form on } V \\ \cap & & \\ \mathrm{Semi}^k(V) & = & \{\text{semipositive Hermitian forms}\} \\ & = & \text{convex cone gen'd by } i^{k^2}\alpha\wedge\overline{\alpha},\ \alpha\ k\text{-form on } V \\ \cap & & \\ \mathrm{Weak}^k(V) & = & \text{cone dual to } \mathrm{Strong}^{n-k}(V) \end{array}$$

The inclusions are strict for $2 \le k \le n-2$. By taking intersections with $N^k(X)$, we obtain a series of cones:

$$\mathrm{Psef}^k(X) \subset \mathrm{Strong}^k(X) \subset \mathrm{Semi}^k(X) \subset \mathrm{Weak}^k(X) \subset \mathrm{Nef}^k(X).$$

These cones are invariant under isogenies, so we may restrict ourselves to the principally polarized case.

Self-products of CM elliptic curves. Let E be a CM elliptic curve and let $X = E^n = (V/\text{lattice})$. We have $N^1(X) = N^1(V)$, and this implies, for all k,

$$\mathrm{Psef}^k(X) = \mathrm{Strong}^k(X) = \mathrm{Strong}^k(V) \subset \mathrm{Weak}^k(V) = \mathrm{Weak}^k(X) = \mathrm{Nef}^k(X).$$

For $2 \le k \le n-2$, the middle inclusion is strict. This gives a first example of nef cycles that are not pseudoeffective.

Self-product of an abelian variety. Let (A,θ) be a principally polarized abelian variety of dimension g. In $N^1(A\times A)$, we consider the classes $\theta_1 = p_1^*\theta$, $\theta_2 = p_2^*\theta$, and $\lambda = c_1(\text{Poincaré line bundle})$. We denote by $N^\bullet_{\mathrm{can}}(A\times A)$ the subalgebra of $N^\bullet(A\times A)$ generated by these classes. When (A,θ) is very general, we have, for $0 \le k \le g$ ([5], [4]),

$$N^k(A\times A) = N^k_{\mathrm{can}}(A\times A) \simeq \mathrm{Sym}^k N^1_{\mathrm{can}}(A\times A).$$

We define cones $\mathrm{Psef}^k_{\mathrm{can}}(A\times A), \ldots$ by taking intersections with the smaller vector space $N^k_{\mathrm{can}}(A\times A)$. The cones

$$\mathrm{Strong}^k_{\mathrm{can}}(A\times A) \subset \mathrm{Semi}^k_{\mathrm{can}}(A\times A) \subset \mathrm{Weak}^k_{\mathrm{can}}(A\times A)$$

are independent of (A,θ). Our main result is that for a very general principally polarized abelian surface (A,θ), we have

$$\mathrm{Psef}^2(A\times A) = \mathrm{Strong}^2(A\times A) = \mathrm{Semi}^2(A\times A) \subsetneq \mathrm{Weak}^2(A\times A) \subsetneq \mathrm{Nef}^2(A\times A).$$

Furthermore,

$$\mathrm{Psef}^2(A\times A) = \mathrm{Sym}^2\,\mathrm{Psef}^1(A\times A).$$

Moreover, if (A,θ) is a principally polarized abelian variety of dimension g,

$$\mathrm{Semi}^2_{\mathrm{can}}(A\times A) = \mathrm{Psef}^2_{\mathrm{can}}(A\times A) = \mathrm{Sym}^2\,\mathrm{Psef}^1_{\mathrm{can}}(A\times A).$$

Again, this gives examples of nef cycles that are not pseudoeffective: for A a very general abelian surface and $t \in \mathbf{R}$, we obtain for example:

$$4\theta_1\theta_2 + t\lambda^2 \quad \text{is} \quad \begin{cases} \text{semipositive} & \text{iff } t = 0; \\ \text{weakly positive} & \text{iff } |t| \leq 1; \\ \text{nef} & \text{iff } -1 \leq t \leq \frac{3}{2}. \end{cases}$$

The classes $4\theta_1\theta_2 + t\lambda^2$ and $2\theta_1^2 + 2\theta_2^2 - \lambda^2$ are weakly positive, hence nef, but their product is -8.

Questions and conjectures. We propose some questions and conjectures concerning possible characterizations of big classes, i.e., classes that lie in the interior of $\mathrm{Psef}^k(X)$, the intuition being that they should be those that "sit positively" in X, or that "move a lot."

Peternell's question ([3]): if $Y \subset X$ is a smooth subvariety with $N_{Y/X}$ ample, is $[Y]$ big? Voisin gave in [6] a counter-example involving a codimension-two subvariety Y that moves in a family covering X.

Voisin's conjecture: if $Y \subset X$ is "very moving", i.e., if, through a general point x of X, there is a deformation of Y that passes through x with general tangent space at x, is $[Y]$ big? Note that if $X = V/\Lambda$ is an abelian variety of dimension n that satisfies $\mathrm{Psef}^k(X) = \mathrm{Strong}^k(X)$, Voisin's conjecture holds for codimension-k subvarieties of X.

Ein and Lazarsfeld's conjecture: α is big if and only if

$\exists C > 0 \quad \exists m$ arbitrarily large

$\exists Z$ eff. cycle with class $m\alpha$ passing through at least $Cm^{n/k}$ v. gen'l points of X.

The implication $\Rightarrow$ is elementary, and the exponent $\frac{n}{k}$ appearing here is the largest that can occur. The statement has been verified in one nontrivial case, namely when $k = 2$ and $\mathrm{Pic}(X) = \mathbf{Z}$.

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Jacobians among abelian threefolds

Arnaud Beauville
(joint work with C. Ritzenthaler)

Let A be a principally polarized abelian variety of dimension 3 over a field k. If k is algebraically closed, A is the Jacobian variety of a curve C (or a product of Jacobians). If k is an arbitrary perfect field the situation is more subtle[1]: there is still a curve C defined over k, but either A is isomorphic to JC, or they become isomorphic only after a quadratic extension k' of k, uniquely determined by A (see [S]).

Now given A, how can we decide if it is a Jacobian, and more precisely determine the extension k'/k? For $k \subset \mathbb{C}$, an analytic solution has been given in [LRZ] in terms of modular forms. We propose a geometric approach, based on a classical construction of Recillas. We have to assume that A admits a rational theta divisor Θ, and a rational point $a \in A(k)$ outside a certain explicit divisor $\Sigma \subset A$; let us assume moreover, for simplicity, that Θ is symmetric, and $\mathrm{char}(k) \neq 2$. Then the curve $\Theta \cap (\Theta + a)$ is smooth, and the involution $z \mapsto a - z$ acts freely on that curve. The quotient X is a non-hyperelliptic genus 4 curve; its canonical model lies on a unique quadric $Q \subset \mathbb{P}^3$. Then the extension k' is $k(\sqrt{\mathrm{disc}(Q)})$. Another way of expressing this is to say that the character $\varepsilon_A : \mathrm{Gal}(\bar{k}/k) \to \{\pm 1\}$ associated to k'/k is the character ε_Q deduced from the action of $\mathrm{Gal}(\bar{k}/k)$ on the rulings of Q.

The proof has two steps. We consider first the case where A is a Jacobian, and prove that in that case the quadric Q is k-isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, so that both ε_A and ε_Q are trivial. In order to do that we use Recillas' construction to show that X has two distinct g^1_3 *defined over* k, giving two degree 3 maps f and $f' : X \to \mathbb{P}^1_k$; then the canonical map of X is the composition of $(f, f') : X \to \mathbb{P}^1 \times \mathbb{P}^1$ and the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, so that Q is k-isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Then we treat the case where A is *not* a Jacobian. We know that it becomes a Jacobian over k', so by the previous case the character ε_Q is trivial on the subgroup $\mathrm{Gal}(\bar{k}/k') = \mathrm{Ker}(\varepsilon_A)$ of $\mathrm{Gal}(\bar{k}/k)$. Thus ε_Q is either trivial or equal to ε_A; to rule out the first possibility it suffices to prove that the nontrivial automorphism σ of the extension k'/k exchanges the two g^1_3 of $Q_{k'}$. But from the explicit expression for f and f' obtained previously we see at once that they are exchanged by σ, hence the theorem.

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[1] We assume that A is indecomposable (that is, not isomorphic to a product) over $\bar{k}$.

The automorphisms group of $\overline{M}_{0,n}$

Massimiliano Mella
(joint work with Andrea Bruno)

The moduli space $M_{g,n}$ of smooth n-pointed curves of genus g, and its projective closure, the Deligne-Mumford compactification $\overline{M}_{g,n}$, is a classical object of study that reflects many of the properties of families of pointed curves. As a matter of fact, the study of its biregular geometry is of interest in itself and has become a central theme in various areas of mathematics.

Already for small n, the moduli spaces $\overline{M}_{0,n}$ are quite intricate objects deeply rooted in classical algebraic geometry. Under this perspective, Kapranov showed in [Ka] that $\overline{M}_{0,n}$ is identified with the closure of the subscheme of the Hilbert scheme parametrizing rational normal curves passing through n points in linearly general position in $\mathbb{P}^{n-2}$. Via this identification, given $n-1$ points in linearly general position in $\mathbb{P}^{n-3}$, $\overline{M}_{0,n}$ is isomorphic to an iterated blow-up of $\mathbb{P}^{n-3}$ at the strict transforms of all the linear spaces spanned by subsets of the points. In a natural way, then, base point free linear systems on $\overline{M}_{0,n}$ are identified with linear systems on $\mathbb{P}^{n-3}$ whose base locus is quite special and supported on so-called vital spaces, i.e., spans of subsets of the given points. Another feature of this picture is that all these vital spaces correspond to divisors in $\overline{M}_{0,n}$ which have a modular interpretation as products of $\overline{M}_{0,r}$ for $r<n$. In this interpretation, the modular forgetful maps $\phi_I:\overline{M}_{0,n}\to\overline{M}_{0,n-|I|}$, which forget points indexed by $I\subset\{1,\dots,n\}$, correspond, up to standard Cremona transformations, to linear projections from vital spaces. The aim of this talk is to study automorphisms of $\overline{M}_{0,n}$ with the aid of Kapranov's beautiful description.

It has been conjectured by William Fulton, that the only possible biregular automorphisms of $\overline{M}_{0,n}$ are the one associated to a permutation of the markings, as long as $n\geq 5$. Any such morphism has to permute the forgetful maps onto $\overline{M}_{0,n-1}$. This induces, on $\mathbb{P}^{n-3}$, special birational maps that switch lines through $n-1$ points in general position. On the other hand if one is able to prove that any automorphism has to permute forgetful maps this could lead to a proof that every automorphism is a permutation. From this point of view the main tool to study $Aut(\overline{M}_{0,n})$ is therefore the following theorem.

Theorem 1. *Let $f:\overline{M}_{0,n}\to\overline{M}_{0,r_1}\times\ldots\times\overline{M}_{0,r_h}$ be a dominant morphism with connected fibers. Then f is a forgetful map.*

The above theorem is an easy extension of the same statement with one factor only and the latter is obtained via an inductive argument starting from the case of a morphism with connected fibers onto $\mathbb{P}^1$.

Theorem 2. *Any dominant morphism with connected fibers $f:\overline{M}_{0,n}\to\overline{M}_{0,4}$ is a forgetful map.*

The idea of proof is as follows. Any morphism of this type produces a pencil of hypersurfaces on $\mathbb{P}^{n-3}$. The base locus of this pencil has severe geometric restrictions coming from Kapranov's construction. These are enough to prove that up to a standard Cremona transformations any such pencil is a pencil of hyperplanes.

This, together with some computation on birational endomorphisms of $\mathbb{P}^{n-3}$, is enough to completely describe the automorphisms group of $\overline{M}_{0,n}$.

Theorem 3. *Assume that* $n \geq 5$, *then* $\mathrm{Aut}(\overline{M}_{0,n}) = S_n$, *the symmetric group on* n *elements.*

This result has a natural counterpart in the Teichmüller-theoretic literature on the automorphisms of moduli spaces $M_{g,n}$ developed in a series of papers by Royden, Earle–Kra, and others, see [EK], but we do not see a straightforward way to go from one to the other.

In this talk I presented "modular" fiber type morphisms on $\overline{M}_{0,n}$ via the study of linear systems on $\mathbb{P}^{n-3}$ and applying whenever possible classical projective techniques. This program has been recently pursued also by Bolognesi, [Bo], in his description of birational models of $\overline{M}_{0,n}$, and Larsen, [La]. I think that other special morphism can be studied with similar techniques as fiber type morphisms of $\overline{M}_{0,n}$ onto either low dimensional varieties or low n or with linear general fiber.

I would like to thank Gavril Farkas for rising my attention on Kapranov's paper, [Ka], and the possibility to use projective techniques in the study of $\overline{M}_{0,n}$.

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Counter-examples to Prym-Torelli of high Clifford index

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(joint work with Elham Izadi)

1. INTRODUCTION

To any (non-trivial) étale double covering $\pi : \tilde{X} \to X$ of a smooth projective curve X of genus $g \geq 2$ one can associate a principally polarized abelian variety $P(\pi)$ of dimension $g-1$ in a canonical way, the *Prym variety* of π. This induces a morphism

$$pr_g : \mathcal{R}_g(2) \to \mathcal{A}_{g-1}$$

from the moduli space $\mathcal{R}_g(2)$ of (non-trivial) étale double coverings of curves of genus g to the moduli space $\mathcal{A}_{g-1}$ of principally polarized abelian varieties of dimension $g-1$, called the *Prym map.* It was shown independently by Friedman-Smith (1082), Kanev (1982), Welters (1985) and Debarre (1989), that pr_g is generically injective for $g \geq 7$. On the other hand, Mumford showed that the Prym variety of an étale double cover of a hyperelliptic curve is the product of the Jacobians of two other hyperelliptic curves. Via a dimension count, this implies that the Prym map has positive dimensional fibers on the locus of hyperelliptic Jacobians. Then Beauville remarked in 1977 that pr_g is not injective for $3 \leq g \leq 10$ at some non-hyperelliptic curves. In [1] Donagi gave a construction showing that pr_g is not injective at any étale double cover of a curve X admitting a map $X \to \mathbb{P}^1$ of degree 4 under some generality assumptions. Moreover, he conjectured (see [1, Conjecture 4.1] or [5, p. 253]) that pr_g is injective at any $\pi : \tilde{X} \to X$, whenever X does not admit a g^1_4. Verra showed in 2004 that pr_{10} is not injective at any étale double cover of a general plane sectic.

The curves which either admit a g^1_4 or are plane sextics (more precisely, admit a g^2_6) are exactly the curves of Clifford index ≤ 2. So one might ask whether pr_g is injective at $\pi : \tilde{X} \to X$ whenever X is of Clifford index ≥ 3. In [2] we showed that this is not the case. In Section 2 I will state the result and in Section 3 sketch a proof which is slightly different from the proof given in [2].

2. THE RESULT

Consider the following maps of smooth projective curves over $\mathbb{C}$

$$\tilde{X} \xrightarrow{f} X \xrightarrow{g} Y$$

where Y is of genus g_Y, g is a *simply* ramified cover of degree 4, X has genus g_X and $\tilde{X}$ is an étale double cover of X, not obtained by base change from a double cover of Y. Consider the cartesian diagram

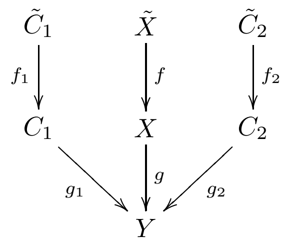
$$\begin{array}{ccc} \tilde{C} & \hookrightarrow & \tilde{X}^{(4)} \\ \downarrow & & \downarrow f^{(4)} \\ Y & \hookrightarrow & X^{(4)}. \end{array} \tag{1}$$

where Y embeds into the symmetric power $X^{(4)}$ via the map sending a point y of Y to the divisor obtained as the sum of its preimages in X. We assume that the monodromy group of the composed covering $\tilde{X} \to Y$ factors via the Weyl group $W(D_4)$. This is automatically the case if $Y = \mathbb{P}^1$. Then $\tilde{C}$ has 2 connected components $\tilde{C}_1$ and $\tilde{C}_2$, which moreover are smooth [3, Lemma 1.1].

The curve $\tilde{C}$ has an involution σ defined as follows. Let $\overline{z} := \overline{x}_1 + \ldots + \overline{x}_4$ be the sum of the points in a fiber of g, and, for each i, let x_i and x'_i be the two preimages of $\overline{x}_i$ in $\tilde{X}$. Then $z := x_1 + \ldots + x_4$ is a point of $\tilde{C}$ and

$$\sigma(z) = x'_1 + \ldots + x'_4.$$

Then σ respects $\tilde{C}_1$ and $\tilde{C}_2$ and defining $C_i := \tilde{C}_i/\sigma$, we get the following diagram



where g_1 and g_2 are also simply ramified 4-fold coverings and f_1 and f_2 étale double coverings with $g_{C_1} = g_{C_2} = g_X$. In particular the Prym varieties $P(f_1), P(f_2)$ and $P(f)$ are principally polarized of the same dimension. In the sequel we only work with one component of $\tilde{C}$ say $\tilde{C}_1$.

Consider the following correspondence

$$S = \{(z = x_1 + \cdots + x_4, x) \in \tilde{C}_1 \times \tilde{X} \mid x = x_i \text{ for some } i\}$$

with reduced scheme structure. Let S^t denote the transpose of S and $s : J\tilde{C}_1 \to J\tilde{X}$ the associated homomorphism of Jacobians. Our main result is

Theorem. *$s : J\tilde{C}_1 \to J\tilde{X}$ induces an isomorphism of principally polarized abelian varieties $(P(f_1), \Xi_1) \xrightarrow{\sim} (P(f), \Xi)$.*

It is easy to see that for a general choice of Y the coverings f_1 and f are non-isomorphic and, choosing $g(Y)$ sufficiently big, we get counter-examples to Donagi's conjeture of arbitrarily high Clifford index.

Donagi's tetragonal construction is the special case $Y = \mathbb{P}^1$. The proof outlined in the next section works also in this special case and is different from the existing proofs. These do not seem to generalize to the more general situation of the theorem.

3. SKETCH OF THE PROOF

Lemma. (1) $S^t \circ S(z) = 2z - 2\sigma(z) + 2Tr_{g_1 f_1}(z)$ *for any* $z \in \tilde{C}_1$;

$$S \circ S^t(x) = 2x - 2x' + 2Tr_{gf}(z) \textit{ for any } x \in \tilde{X};$$

(2) Let $A = \ker(Nm_{gf} : J\tilde{X} \to JY)^0$ and $A_1 = \ker(Nm_{g_1 f_1} : J\tilde{C}_1 \to JY)^0$. Then

$$s((g_1 f_1)^*(JY)) \subset (gf)^*(JY) \text{ and } s(A_1) \subset A,$$
$$s^t((gf)^*(JY)) \subset (g_1 f_1)^*(JY) \text{ and } s^t(A) \subset A_1.$$

Proof. Fix a general $y \in Y$. Then in the fibres over y the correspondences S, S^t, the involutions ' and σ and the trace correspondences are given by matrices. So here one checks the corresponding matrix equations. Using monodromy one extends the equations over all of Y. □

From this we immediately deduce the following proposition for the Prym varieties $P = P(f)$ and $P_1 = P(f_1)$.

Proposition 1. *The homomorphisms $s : J\tilde{C}_1 \to J\tilde{X}$ and $s^t : J\tilde{X} \to J\tilde{C}_1$ restrict to isogenies $s : P_1 \to P$ and $s^t : P \to P_1$ satisfying*

$$s^t s = 4 \cdot 1_{P_1} \quad \text{and} \quad ss^t = 4 \cdot 1_P.$$

Since the homomorphisms associated to transposed correspondences are of the same degree, we get as an immediate consequence

Corollary. $\deg(s : P_1 \to P) = \deg(s^t : P \to P_1) = 2^{2 \dim P_1} = \deg(2_{P_1})$.

The proof of the following proposition uses the explicit Hermitian forms of the polarizations Ξ on P and Ξ_1 on P_1 (see [2, Proposition 3.3]).

Proposition 2. *The isogeny $s : P_1 \to P$ factors via multiplication by 2 on P_1.*

Proof of the theorem. According to Proposition 2 the isogeny $s : P_1 \to P$ factors as $s = \psi \circ 2_{P_1}$ with an isogeny $\psi : P_1 \to P$. By the corollary, $\deg(s) = \deg(2_{P_1})$. Hence $\psi : P_1 \to P$ is an isomorphism.

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Lifting Tropical Intersections

Sam Payne
(joint work with Brian Osserman)

For any two subvarieties X and X' of a torus over a nonarchimedean field, the tropicalization of their intersection is contained in the intersection of their tropicalizations. Well-known examples show that this containment is sometimes strict. For instance, two distinct curves intersect in at most finitely many points, but the intersection of their tropicalizations may contain a segment. Motivated by the desire to apply tropical techniques in the study of classical intersection problems in algebraic geometry, we explore the problem of lifting tropical intersection points to algebraic intersection points. Our most basic result says that tropical intersection points lift whenever the tropical intersection has the expected dimension. In this respect, tropicalizing in the torus behaves like passing from the general fiber to the special fiber in a smooth scheme over a regular local ring.

Theorem 1. *Suppose* $\mathrm{Trop}(X)$ *and* $\mathrm{Trop}(X')$ *meet properly at* w. *Then* w *is contained in* $\mathrm{Trop}(X \cap X')$.

This generalizes an earlier lifting theorem for transverse tropical intersections [1] to the case where the tropicalizations meet properly, but not necessarily in the relative interiors of maximal faces.

For applications, one wants to study intersections inside an ambient subvariety Y of the torus, and to count the lifts of isolated points of proper intersections. One key to understanding intersections inside an ambient subvariety is the notion of a simple point, defined in terms of the multiplicities naturally assigned to facets in tropicalizations.

Definition 2. *A simple point of* $\mathrm{Trop}(Y)$ *is a point in the relative interior of a facet of multiplicity* 1.

Intersections of tropicalizations at simple points of $\mathrm{Trop}(Y)$ behave like intersections inside a torus of dimension equal to $\dim(Y)$.

Theorem 3. *Let* X *and* X' *be subvarieties of* Y. *If* $\mathrm{Trop}(X)$ *meets* $\mathrm{Trop}(X')$ *properly at a simple point* w *of* $\mathrm{Trop}(Y)$, *then* w *is contained in* $\mathrm{Trop}(X \cap X')$.

Furthermore, in the case of zero-dimensional intersections, the number of points of $X \cap X'$ whose tropicalization is w, counted with multiplicities, is exactly the tropical intersection multiplicity of X and X' at w, given by a local fan displacement rule in the facet of $\mathrm{Trop}(Y)$ containing w. For details, see [2].

The proofs of these lifting theorems are in two steps. First, we lift from the tropicalization to the initial degenerations at w. The main techniques used in this part are compactifications inside a suitable toric variety, the topology of the extended tropicalizations of these compactifications, and intersection theory in this toric variety. Next, we lift from the initial degeneration to the original variety, using naturally defined integral models over the valuation ring of our nonarchimedean field. The second step is accomplished with the following theorem which says, roughly speaking, that these integral models behave like schemes over DVRs, even when the valuation is not discrete.

Theorem 4. *Suppose the initial degnerations* X_w *and* X'_w *meet properly at a smooth point* x *of* Y_w. *Then* x *is contained in* $(X \cap X')_w$.

The proof of this result involves a systematic development of dimension theory and intersection theory in schemes of finite type over non-noetherian valuation rings of rank 1. Technical difficulties arise when the point w is not rational over the value group. In such cases, the associated integral models are not finite type over the valuation ring. These difficulties are overcome by extending to a valued field with larger value group, and applying a careful analysis of the behavior of initial degenerations with respect to extensions of valued fields.

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Rationality of the universal Jacobian of genus 5

Alessandro Verra

Let C be a smooth, irreducible complex projective curve of genus g and let $L \in Pic^d\ C$. Not so much seems to be known about the rationality of $Pic_{d,g}$ for very low values of g, say $g \leq 6$. This report summarizes the proof of the following new result:

Theorem 1. *The moduli space $Pic_{8,5}$ is rational.*

$Pic_{8,5}$ is of course biregular to the universal Picard group over the moduli space $\mathcal{M}_5$ of curves of genus 5. The proof of the theorem relies on the study of the multiplication map

$$\mu : H^0(L) \otimes H^0(\omega_C^{\otimes 2} \otimes L^{-1}) \to H^0(\omega_C^{\otimes 2})$$

and on a special K3 surface defined by μ. Let

$$\mathbf{P}^3 \times \mathbf{P}^3 := \mathbf{P}H^0(L)^* \times \mathbf{P}H^0(\omega_C^{\otimes 2} \otimes L^{-1})^*.$$

If (C, L) is sufficiently general then μ defines an embedding

$$C \subset \mathbf{P}^3 \times \mathbf{P}^3 \subset \mathbf{P}^{15},$$

where the right inclusion is the Segre embedding. In $\mathbf{P}^{15}$ consider the orthogonal space $\mathbf{P}Ker(\mu)^{\perp} \subset \mathbf{P}^{15}$ of $Ker(\mu)$. One can show that

Lemma 2. *For a general pair (C, L) the map μ is surjective and $\mathbf{P}Ker(\mu)^{\perp}$ is transversal to $\mathbf{P}^3 \times \mathbf{P}^3$.*

Since $dim\ Ker(\mu) = 4$, it follows that

$$S := \mathbf{P}(Ker\ \mu)^{\perp} \cap \mathbf{P}^3 \times \mathbf{P}^3$$

is a smooth surface. By adjunction formula S is a K3 surface. Let p_1, p_2 be the natural projection maps of $\mathbf{P}^2 \times \mathbf{P}^2$ and let $\mid H_i \mid := \mid \mathcal{O}_{\mathbf{P}^2}(1) \mid$, $i = 1, 2$. It is easy to see that $H_i^2 = 4$ and that the equation of $p_{i*}S$ is a 4×4 determinant of linear forms. Notice also that $\mathcal{O}_C(H_1 + H_2 - C)$ is ω_C. This implies a quite special property of S:

Proposition 3. *$H_1 + H_2 - C \sim B_1 + \cdots + B_6$, where B_i, $i = 1 \ldots 6$, is a conic embedded in S. Moreover one has*

$$Pic\ S \cong \mathbf{Z}[C] \oplus \mathbf{Z}[H_1] \oplus \mathbf{Z}[B_1] \oplus \cdots \oplus \mathbf{Z}[B_6]$$

if the pair (C, L) considered above is sufficently general.

The proof of the first statement follows from the standard exact sequence

$$0 \to \mathcal{O}_S(H_1 + H_2 - 2C) \to \mathcal{O}_S(H_1 + H_2 - C) \to \omega_C \to 0$$

and its associated long exact sequence. Fixing the lattice

$$\mathbb{L} := Pic\ S$$

and its basis $[C], [H_1], [B_1], \ldots, [B_6]$, we can consider the moduli space

$$\mathcal{K}$$

of pairs (S', f), where S' is a K3 surface and $f : Pic\ S' \to \mathbb{L}$ is an isometry. It turns out that $dim\ K = 12$. Furthermore there exists a projective bundle

$$\mathbf{P}_{\mathcal{K}} \to U,$$

U being a non empty open set of $\mathcal{K}$, with the following property: let x be the moduli point of (S', f) and let $[C'] = f^{-1}([C])$. Then the fibre of $\mathbf{P}_{\mathcal{K}}$ at x is the linear system $\mid C' \mid$. Let $[H_1'] = f^{-1}([H_1])$, we define the map

$$\alpha : \mathbf{P}_{\mathcal{K}} \to Pic_{8,5}$$

sending $C'' \in \mathbf{P}_{\mathcal{K}x} = \mid C' \mid$ to the moduli point of the pair $(C'', \mathcal{O}_{C''}(H_1'))$.

Theorem 4. *$\alpha : \mathbf{P}_{\mathcal{K}} \to Pic_{8,5}$ is a birational map.*

Since $\mathbf{P}_{\mathcal{K}} \cong \mathcal{K} \times \mathbf{P}^5$, the rationality of $Pic_{8,5}$ follows from the next theorem:

Theorem 5. *$\mathcal{K}$ is rational.*

The proof is related to nets of quadrics in $\mathbf{P}^5$ and even theta characteristics on genus 4 curves. To begin we fix a pair (S, f) defining a general point of $\mathcal{K}$. In particular S is constructed from a general pair (C, L) as above. Then $\mid C \mid$ defines a generically injective morphism $\phi : S \to \mathbf{P}^5$ and we have:

Proposition 6. *Let $\overline{S} = \phi(S)$, then:*

- *$\overline{S}$ is the complete intersection of three quadrics,*
- *Sing $\overline{S}$ consists of the six double points $\phi(B_i)$, $i = 1 \ldots 6$,*
- *each quadric Q containing $\overline{S}$ has rank ≥ 5.*

The special feature of $\overline{S}$ is reflected by the following properties of the net N of quadrics through $\overline{S}$.

Proposition 7. *Let $\Delta \subset N$ be the discriminant curve of the net, then Δ is a nodal, integral sextic with six nodes and they lie on a smooth conic.*

Fixing a conic $B \subset \mathbf{P}^2$ and coordinates $(x, y) = (x_1 : x_2 : x_3) \times (y_1 : \cdots : y_6)$ on $\mathbf{P}^2 \times \mathbf{P}^5$, we can assume that the six nodes $w_1, \ldots, w_6$ of Δ are in B and that the six singular points $o_1, \ldots, o_6$ of S are the fundamental points. It is easy to see that the net N is defined by a symmetric matrix $c(h_{ij})$, $(c \in \mathbf{C}^*)$, of linear forms h_{ij} in x such that: (1) if $i = j$, h_{ii} is zero; (2) if $i \neq j$, h_{ij} is an equation of the line joining w_i and w_j. This defines a natural $\mathbf{P}^{14}$-bundle $\mathbb{M}$ over B^6, endowed with a natural action of the group $G = Aut(B) \times Aut(\{o_1 \ldots o_6\})$ and such that $\mathbb{M}/G$ is birational to the moduli space of nets N as above. Finally the rationality of $\mathbb{M}/G$

is a consequence of a standard argument in invariant theory and of the rationality of $B^6/Aut(B)$.

The k-secant lemma

Christian Peskine
(joint work with Laurent Gruson)

Let $C \subset \mathbb{P}_3(\mathbb{C})$ be a smooth curve in the projective complex space . A general projection $C \to C_1 \subset \mathbb{P}_2(\mathbb{C})$ has only ordinary double points as singularities. In other words, the 3-secants to C, the tangents to C and the "tac-nod" (or stationary, or ramified) 2-secants to C do not fill up the space. This is the 3-secant lemma.

This result can easily be generalized to higher dimension in the following way. Let $X \subset \mathbb{P}_{2n+1}(\mathbb{C})$ be an n-dimensional smooth variety (note that the tangent spaces to X do not fill up $\mathbb{P}_{2n+1}(\mathbb{C})$). A general projection $X \to X_1 \subset \mathbb{P}_{2n}(\mathbb{C})$ has only a finite number of ordinary double points as singularities (a double point in X_1 is ordinary if it has 2 transverse branches) .

When the tangent spaces to $X \subset \mathbb{P}_N(\mathbb{C})$ do fill up $\mathbb{P}_N(\mathbb{C})$, new difficulties arise.

As an example, consider a smooth Veronese surface $X \subset \mathbb{P}_4(\mathbb{C})$ (yes, imbedded in $\mathbb{P}_4$), and a general projection $X \to X_1 \subset \mathbb{P}_3(\mathbb{C})$. The Steiner surface X_1 is well known. It has a reduced double curve X_2 consisting of 3 lines through a point x and not in a plane. The degree 1 scheme supported by x is the triple locus X_3 of X_1, as well as the triple locus of X_2. Furthermore, X_1 has 6 distinct pinch points (2 on each of the 3 lines of X_2). The degree 6 scheme T_2 supported by these 6 points is disjoint from X_3. Note that T_2 is also the image of the ramification divisor of the double cover of X_2. Obviously, X_3 is the singular locus of X_2.

We note the existence of disjoint smooth schemes X_3 (the triple locus) and T_2 (the first Boardmann locus) as well as the smoothness of X_2 outside X_3.

This last example can be generalized in two different ways.

The Veronese surface $X \subset \mathbb{P}_4(\mathbb{C})$ is the first Severi Variety; the next one is $X = \mathbb{P}_2 \times \mathbb{P}_2 \subset \mathbb{P}_7$ (yes, imbedded in $\mathbb{P}_7$). A general projection $X \to X_1 \subset \mathbb{P}_6$ has a reduced double surface X_2 consisting of 3 planes through a point x and generating $\mathbb{P}_6$. The degree 1 scheme supported by x is the triple locus X_3 of X_1, as well as the triple (as well as the singular) locus of X_2. Furthermore, X_1 carries a scheme T_2, formed by 3 distinct pinch conics (one on each of the 3 planes of X_2). We note that $T_2 \cap X_3 = \emptyset$ and that T_2 is also the image of the ramification divisor of the double cover of X_2.

Yes, your guess is right, the double locus X_2 of the general projection a Severi Variety is a configuration of 3 linear spaces through a point (the triple locus) and generating the total space.

The Palatini solid $X \subset \mathbb{P}_5(\mathbb{C})$ provides another interesting example of singularities. A general projection $X \to X_1 \subset \mathbb{P}_4$ has a reduced irreducible double surface X_2, a triple locus X_3 consisting of 4 lines (generating $\mathbb{P}_4$) through a point x. The degree 1 scheme supported by x is the quadruple locus X_4 of X_1. In this case, X_2 carries a pinch curve T_2. This curve intersects each of the triple lines of X_3 in six distinct points; it is not unimportant to note that the quadruple locus $X_4 = \{x\}$ is disjoint from T_2.

Note finally here that there is a second smooth Palatini variety. It has dimension 6 and is imbedded in $\mathbb{P}_9$. Not surprisingly, the triple locus of a general projection of this variety to $\mathbb{P}_8$ consists of 4 planes through a point (the 4-tuple locus) and generating $\mathbb{P}_8$.

Our original goal was to prove the following result. All examples given above are explained and described by this general projection theorem.

Theorem 1. (General Projection Theorem) Let $X \subset \mathbb{P}_{n+c}$ be a smooth dimension n variety and let $\pi : X \to X_1 \subset \mathbb{P}_{n+c-1}$ be a general projection.

1) For all $k > 0$, the scheme $X_k \subset X_1$ formed by points of multiplicity $\geq k$ of X_1 has dimension $N - 1 - k(c-1)$ (the empty set has all dimensions). The singular locus of X_k is X_{k+1} and the normalisation $\tilde{X_k}$ of X_k is smooth.

2) For $k = k_1 + ... + k_r$, with $k_i > 0$, the subscheme $X_{\{k_1,...,k_r\}} \subset X_k$ formed by points x such that $\pi^{-1}(x)$ contains r points $\{x_1, ..., x_r\}$ (possibly coinciding) with multiplicity $\geq k_i$ in x_i, has dimension $N - 1 - \sum(k_i c - 1) = N - 1 - kc + r$. The singular locus of $X_{\{k_1,...,k_r\}}$ is $X_{\{k_1,...,k_r,1\}}$ and its normalisation $\tilde{X}_{\{k_1,...,k_r\}}$ is smooth

Please note that 1) is a special case of 2), but it deserved to be stated by itself.

Be careful, if x_i and x_j do coincide, the multiplicity of $\pi^{-1}(x)$ at the point $x_i = x_j$ has to be $\geq (k_i + k_j)$.

As an interesting application (not straightforward) of this theorem we can prove the following vanishing result.

Corollary 2. If $X \subset \mathbb{P}_{n+2}$ is a smooth variety of dimension n, then

1) $H^p(J_X(q)) = 0$ for $p + q + 1 \leq n$ and $0 < p < N - 1$,

2) for $k < n$, the normalisation $\tilde{X_k}$ of the scheme X_k of points of multiplicity $\geq k$ of a general projection of X is smooth and connected of dimension $n + 1 - k$,

3) if the normalisation $\tilde{X_n}$ of the curve X_n of points of multiplicity $\geq n$ of a general projection of X is connected, then $H^1(J_X(n-1)) = 0$.

Note that 3) was well known to F. Zak (with a proof of a very different nature).

1) We recall here that the Veronese surface is the only surface whose general projection to $\mathbb{P}_3$ has a non irreducible double curve (this is Franchetta's theorem).

2) We recall also a conjecture (made by one of us in Trento twenty years ago) following which the Palatini threefold is the only smooth threefold whose general projection to $\mathbb{P}_4$ has a non irreducible triple curve.

3) Finally, we recall that for $n \geq 4$, the only known smooth varieties of dimension n in $\mathbb{P}_{n+2}$ are complete intersections.

With Fyodor Zak, we make the following conjecture (obviously related to the preceeding corollary).

Conjecture 3. If $X \subset \mathbb{P}_{n+c}$ is a smooth variety of dimension n, then

1) $H^p(J_X(q)) = 0$ for $p + (q+1)(c-1) \leq n$ and $p < N - 1$,

2) for $k(c-1) < n$, the normalisation $\tilde{X}_k$ of the scheme X_k of points of multiplicity $\geq k$ of a general projection of X is smooth and connected,

3) for $k(c-1) \leq n$, if the normalisation $\tilde{X}_k$ of the scheme X_k of points of multiplicity $\geq k$ of a general projection of X is connected, then $H^1(J_X(k-1)) = 0$.

Our "General Projection Theorem" is certainly easier to state and to prove in the language of Hilbert Schemes of aligned points. Indeed the smooth normalisation $\tilde{X}_k$ of X_k is the Hilbert scheme of aligned, finite, degree k subscheme of X. The normalisations $\tilde{X}_{\{k_1,\ldots,k_r\}}$ are smooth stratas of this Hilbert Scheme. They are described in the following theorem, of which the preceeding is clearly a consequence.

We denote by $G = G(1, n+c))$ the Grassmann variety of lines in $\mathbb{P}_{n+c}$ and by $\mathbb{Y} \subset G \times \mathbb{P}_{n+c}$ the incidence variety.

Theorem 4. (Aligned Hilbert Scheme Theorem) Let X be a smooth (equidimensional of) dimension n quasi-projective variety imbedded in $\mathbb{P}_{n+c}$.

1) Let $H_k(X)$ be the Hilbert scheme of aligned, finite, degree k subschemes of X. Consider the natural projective line bundle $H_k(X) \times_G \mathbb{Y}$ over H_k and the projection $\phi_k : H_k(X) \times_G \mathbb{Y} \to \mathbb{Y} \to \mathbb{P}_{n+c}$.

The general fiber of ϕ_k is smooth of dimension $N - 1 - k(c-1)$ (the empty set has all dimensions).

2) For $k = k_1 + \ldots + k_r$, with $k_i > 0$, let $H_{\{k_1,\ldots,k_r\}}(X) \subset H_k(X)$ be the Hilbert scheme of aligned, finite, degree k subschemes of X, with multiplicities k_i in points x_i (possibly coinciding). Consider the natural projective line bundle $H_{\{k_1,\ldots,k_r\}}(X) \times_G \mathbb{Y}$ over $H_{\{k_1,\ldots,k_r\}}(X)$ and the projection $\phi_{\{k_1,\ldots,k_r\}} : H_{\{k_1,\ldots,k_r\}}(X) \times_G \mathbb{Y} \to \mathbb{Y} \to \mathbb{P}_{n+c}$.

The general fiber of $\phi_{\{k_1,\ldots,k_r\}}$ is smooth of dimension $N - 1 - kc + r$.

Note once more that 1) is a special case of 2). In 2), when x_i and x_j do coincide, the multiplicity of a point $h \in H_{\{k_1,\ldots,k_r\}}(X)$ at $x_i = x_j$ has to be $(k_i + k_j)$.

As a special case of this theorem we recover Mather's celebrated result: higher polar varieties of a smooth variety are smooth of expected dimension.

Corollary 5. (Mather) Let $HB_k(X) \subset \mathbb{Y}$ be the Hilbert-Boardmann locus formed by couples (L, x) such that $L \cap X$ has multiplicity at least k at the point x.

Consider the natural projective line bundle $HB_{\{k\}}(X) \times_G \mathbb{Y}$ and the projection $\phi_{\{k\}} : H_{\{k\}}(X) \times_G \mathbb{Y} \to \mathbb{Y} \to \mathbb{P}_{n+c}$.

The general fiber of $\phi_{\{k\}}$ is smooth of dimension $n - (k-1)c$ (the empty set has all dimensions).

Ziv Ran's famous $(n+2)$-secant lemma and a recent generalisation of this lemma proved by R. Beheshti and D. Eisenbud can also be deduced from our theorem.

Corollary 6. (Ran) The $(n+2)$-secants to X fill up a variety of dimension at most $n+1$

More generally,

Corollary 7. (Beheshti-Eisenbud)

For $k \geq (n/s) + 2$, the k-secants to X fill up a variety of dimension at most $n + s$.

Asymptotic Syzygies of Higher Dimensional Varieties

Lawrence Ein

In the 80's Green and Lazarsfeld began a systematic study of the syzygies of smooth projective curves. On of the main driving problems in this area is the important conjecture of Green which predicts that the behavior of the syzygies of a canonical curve is determined by the Clifford index of the curve. See [G1] and [GL] for more details. In an important recent breakthrough, Voisin ([V1] and [V2]) proved that Green's conjecture is true for a generic curve of genus g. Combined with the result of Texidor Bigas [T], they show that Green's conjecture also holds for a general $p-$gonal curve.

It is natural to ask is whether we can generalize of these results and questions to higher dimensional varieties. In this report, we'll discuss my joint work with R. Lazarsfeld where we begin studying asymptotic behavior of the syzygies of higher dimensional varieties. First we'll recall some basic notations. Let $S = \mathbb{C}[x_0, x_1, \cdots, x_r]$ be the polynomial ring of $r+1$ variables. Suppose that G is a finitely generated graded S-module. We consider a minimal free resolution of G.

$$0 \longrightarrow E_s \longrightarrow \ldots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow G \longrightarrow 0$$

Let $\mathbb{C} = S/\mathfrak{m}$, where $\mathfrak{m}$ is the homogenous maximal ideal of S. By Nakayama's lemma, one observes that

$$\mathrm{Tor}_p(G, \mathbb{C}) = E_p \otimes \mathbb{C}.$$

Observe that $\mathrm{Tor}_p(G, \mathbb{C})$ is a graded vector space. Using the notations of Green, we set the Koszul group $K_{p,q}(G)$ to be the homogenous degree $p+q$ piece of $\mathrm{Tor}_p(G, \mathbb{C})$. Let X be a closed subvariety of dimension n in $\mathbb{P}^r$. In the following we assume that the restriction map gives an isomorphism between $H^0(\mathcal{O}_{\mathbb{P}^r}(1))$ and $H^0(\mathcal{O}_X(1))$. Let $\mathcal{F}$ be a coherent sheaf on X and F be its associated graded $S-$module. We will denote by

$$K_{p,q}(\mathcal{F}) \quad \text{for} \quad K_{p,q}(F).$$

We say that the pair $(X, \mathcal{O}_X(1))$ satisfies the property N_0, if $|\mathcal{O}_X(1)|$ gives a projectively normal embedding of X in $\mathbb{P}^r$. Note that this is equivalent to $K_{0,q}(\mathcal{O}_X) = 0$ for $q > 0$. For $p > 0$, inductively we say that $(X, \mathcal{O}_X(1))$ satisfies the property N_p, if it satisfies N_{p-1} and $K_{p,q}(\mathcal{O}_X) = 0$ for $q \geq 2$.

Let X be a smooth projective variety of dimension n and $\mathcal{O}_X(1)$ be a very ample line bundle on X and B be another line bundle on X. Choose $d >> 0$ with respect to B. We consider the coordinate ring

$$R = \oplus_{m=0}^{\infty} H^0(\mathcal{O}_X(md)).$$

We consider the S-module

$$N_{B,d} = \oplus_{m=0}^{\infty} H^0(\mathcal{O}_X(md) \otimes B).$$

We would like to investigate the behavior of the asymptotic behavior of the syzygy groups $K_{p,q}(\mathcal{O}_X(d); B) = \mathrm{Tor}_p(\mathbb{C}, N_{B.d})_{p+q}$ as $d \longrightarrow \infty$. Set $L = \mathcal{O}_X(d)$ and $r + 1 = h^0(\mathcal{O}_X(d))$. We would like to be able to predict the rough shape of the minimal resolution $\mathcal{O}_X$ and $N_{B,d}$ in $\mathbb{P}^r$ in some asymptotic sense. Consider the rank r vector bundle M_L, which is defined as the kernel of the natural surjective map from $H^0(L) \otimes \mathcal{O}_X \longrightarrow L$. It is well known that the Koszul groups can be computed by the cohomologies of the vector bundles of the form $\bigwedge^p M_L \otimes B \otimes L^{\otimes q}$. We recall that Ottaviani and Paoletti proved the following theorem for $\mathbb{P}^2$.

Theorem 1. [OP] $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ *for* $d \geq 3$ *satisfies* N_{3d-3} *but not* N_{3d-2}.

See also [EGHP] and [BCR] for different proofs. The following are two basic lemmas in dealing with the Koszul group of higher dimensional varieties.

Theorem 2. *Let L be a very ample line bundle on X. Suppose $L = \mathcal{O}_X(D+E)$, where D and E are nontrivial effective Cartier divisor on X. Then $h^0(\wedge^p M_L \otimes \mathcal{O}_X(D)) \neq 0$ if and only if $0 \leq p \leq h^0(\mathcal{O}_X(D)) - 1$.*

Proposition 3. *(Koszul duality) Let B be line bundle on X. Assume that d is sufficiently large and*

$$H^i(B \otimes \mathcal{O}(md)) = 0, \quad \textit{for all} \ \ m \in \mathbb{Z}.$$

Then

$$K_{p,q}(B)^* = K_{r-p-n,n+1-q}(K_X \otimes B^*).$$

The condition is equivalent to assuming that $N_{B,d}$ is a projectively Cohen Macaulay module on $\mathbb{P}^r$. So its minimal resolution will have the expected length. We also observe that if d is sufficiently large. It follows from Serre's vanishing theorem and duality B is $n+1$-regular with respect to $L = \mathcal{O}_X(d)$. If $h^n(B) = 0$, then B is even n-regular. So we see the Betti table of $M_{B,d}$ has only $n+1$ rows. First we now know the precise non-vanishing of $K_{p,q}(\mathcal{O}_{\mathbb{P}^2}(d); \mathcal{O}_{\mathbb{P}^2})$.

Theorem 4. *(Ein and Lazarsfeld) We set $r = h^0(\mathcal{O}_{\mathbb{P}^2}(d)) - 1$. For $d \geq 3$,*
(a) $K_{p,q}(\mathcal{O}_{\mathbb{P}^2}(d); \mathcal{O}_{\mathbb{P}^2}) = 0$ for $q \geq 3$. $K_{p,0}((\mathcal{O}_{\mathbb{P}^2}(d); \mathcal{O}_{\mathbb{P}^2}) \neq 0$ if any only if $p = 0$.
(b) $K_{p,1}(\mathcal{O}_{\mathbb{P}^2}(d); \mathcal{O}_{\mathbb{P}^2}) \neq 0$ if and only if $1 \leq p \leq h^0(\mathcal{O}_{\mathbb{P}^2}(d-1))$.
(c) $K_{p,2}(\mathcal{O}_{\mathbb{P}^2}(d)); \mathcal{O}_{\mathbb{P}^2}) \neq 0$, if and only if $r-2 \geq p \geq 3d-2$.

The following theorem due to myself and Lazarsfeld allows us to study the Koszul groups of $\mathbb{P}^n$ inductively.

Theorem 5. *$K_{p,q}(\mathcal{O}_{\mathbb{P}^{n-1}}(d); \mathcal{O}_{\mathbb{P}^{n-1}})$ is a direct summand of $K_{p,q}(\mathcal{O}_{\mathbb{P}^n}(d); \mathcal{O}_{\mathbb{P}^n})$.*

The above theorem can be used effectively to show inductively the non-vanishing of Koszul cohomology groups. We have also obtain similar precise non-vanishing results for the Koszul groups $K_{p,q}(\mathcal{O}_{\mathbb{P}^3}(d); \mathcal{O}_X)$. Assume $h^i(B) = 0$ for $i > 0$. One gets a similar non-vanishing theorem for $K_{p,q}(X, L; B)$ for a nonsingular projective threefold X.

Definition 6. *We fix an integer q. We set*

$$k_{\max,q}((X, \mathcal{O}_X(d)); \mathcal{O}_X) = \max\{p| \quad K_{p,q}(\mathcal{O}_X(d); \mathcal{O}_X) \neq 0\}.$$

and

$$k_{\min,q}((X, \mathcal{O}_X(d)); \mathcal{O}_X) = \min\{p| \quad K_{p,q}(\mathcal{O}_X(d); \mathcal{O}_X) \neq 0\}.$$

The above theorem and other partial results lead us to the following conjecture.

Conjecture 7. (Ein and Lazarsfeld) Let X be projective variety. Suppose $\mathcal{O}_X(1)$ is a very ample line bundle on X. Assume that $H^i(\mathcal{O}_X)$ for $1 \leq i < n$. For $d >> 0$, $K_{p,q}(\mathcal{O}_X(d); \mathcal{O}_X) = 0$, if $p < 0$ or $p \geq n$. Then

$$k_{\max,q}(\mathcal{O}_X(d); \mathcal{O}_X) = O(d^n)$$

and

$$k_{\min,q}(\mathcal{O}_X(d); \mathcal{O}_X)) \leq O(d^{n-1})$$

for $1 \leq q \leq n$.

We suspect the following precise picture may be true

$$k_{\min,q}(X, \mathcal{O}_X(d); \mathcal{O}_X) \leq O(d^{q-1}) \quad \text{for} \quad 1 \leq q \leq n.$$

As a consequence of our results with Lazarsfeld, we know the above conjecture is true for $X = \mathbb{P}^n$ or when X is a surface. We also expect that for almost every p, $K_{p,q}(\mathcal{O}_X)$ should be nonzero for $k_{\min,q} \leq p \leq k_{\max,q}$. We know this is the case for $X = \mathbb{P}^3$. One can also ask whether one can study the geometry of X and B from the syzygies.

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Sextic Covers

Daniel Erman
(joint work with Melanie Wood)

(This abstract refers to work in progress.)

We are interested in structure theorems for degree n covers of algebraic varieties, where by a degree n cover of Y we mean a map $f : X \to Y$ such that f is finite, flat, and degree n. Much previous work has produced strong structure theorems for degree n covers in all cases where $n \leq 5$ (see below). These results use matrix presentations for certain ideal sheaves to describe the structure of a finite cover of low degree. However, for larger values of n little is known in general. We focus on the case of sextic covers, as this is the first open case. A natural candidate for the sextic case arises from the Gulliksen-Negard complex; this approach is explored in [Ca01], where it is shown that this construction fails to provide a structure theorem for all sextic covers. We extend Casnati's work by precisely describing which sextic covers arise from the Gulliksen-Negard complex. Our hope is that this deeper understanding of why the Gulliksen-Negard complex fails to account for all sextic covers may point towards a more general construction for describing all sextic covers.

To provide a global view of families of degree n covers, we consider the moduli stack $\mathscr{B}_n$, where

$$\mathscr{B}_n(T) = \{\pi : X \to T \mid \pi \text{ is finite, flat and degree } n\}/\cong .$$

Not surprisingly, $\mathscr{B}_n$ is difficult to understand. However, for small values of n, [Wo09] constructs a birational model $\mathscr{X}_n$ of $\mathscr{B}_n$ with much nicer properties, including:

(i) There exists a proper, birational morphism $p : \mathscr{X}_n \to \mathscr{B}_n$.
(ii) $\mathscr{X}_n = [X/G]$ where X is a nice variety, and G is a nice group. (Generally $X = \mathbb{A}^N$ and G is a quotient of a product of general linear groups.)
(iii) $\mathscr{X}_n$ has a moduli interpretation.

Constructions of $\mathscr{X}_n$ are relatively well understood when $n \leq 5$, and are based on a huge variety of previous results [Wo09, Wo10, Bha04, Bha08, CaEk, Ca96, DeFa, GGS, HaMi, Mir, Par].

The existence of a model $\mathscr{X}_n$ with the above properties enables a wide array of applications. For instance, counting rational points of $\mathscr{X}_n$ for $n = 4, 5$ is related to Bhargava's asymptotic formulas for counting number fields of degree 4 and 5 [Bha04, Bha08]. More geometrically, the study of $\mathscr{X}_n(\mathbb{P}^1)$ is closely related to the study of the Hurwitz scheme of degree n covers of $\mathbb{P}^1$.

All known structure results for finite covers of low degree can essentially be related by the following observation:

> The structure of degree n covers is closely related to the study of matrix presentations of the ideals of Gorenstein, zero-dimensional, degree n subschemes of $\mathbb{P}^{n-2}$.

For instance, the construction of $\mathscr{X}_5$ is based on skew-symmetric 5×5 matrix presentations for ideals of 5 points in $\mathbb{P}^3$. The existence and uniqueness of such matrix presentations stems from the Buchsbaum-Eisenbud structure theorem for Gorenstein algebras of codimension 3.

From this perspective, it is not surprising that the sextic case is more difficult, as there is no good analogue of the Buchsbaum-Eisenbud structure theorem in codimension 4. However, the Gulliksen-Negard complex provides a known construction for producing matrix presentations of 6 points in $\mathbb{P}^4$. Over a field, this construction amounts to presenting the ideal of 6 points in $\mathbb{P}^4$ as the 2×2 minors of 3×3 matrix of linear forms. More generally, there is a similar construction for a sextic cover of an arbitrary base T by embedding into a $\mathbb{P}^4$-bundle and considering the ideal sheaf generated by the 2×2 minors of an appropriate map of vector bundles [Wey, (6.1.8)]. By working over an arbitrary base T, we can define a certain moduli stack $\mathscr{W}_6$ where $\mathscr{W}_6(T)$ parametrizes the sextic covers of T arising from a Gulliksen-Negard complex.

This construction has been previously used to produce interesting sextic covers [ABR, Ca01]. In addition, in [Ca01, Ex. 5.5], Casnati provides an example illustrating that the morphism $\mathscr{W}_6 \to \mathscr{B}_6$ is not surjective. Thus, the stack $\mathscr{W}_6$ fails to satisfy condition (i) from above.

Our goal is to understand this failure more precisely, with the hope of gaining new ideas for constructing a good birational model for $\mathscr{B}_6$. More specifically, we seek to understand precisely which sextic covers of a scheme T arise from $\mathscr{W}_6(T)$. We state our main result over an arbitrary field k.

Theorem 1. *Let k a field and let $\Gamma \subseteq \mathbb{P}^4_k$ be a Gorenstein 0 dimensional, degree 6 scheme in linearly general position. Then there is a bijection between:*

- *3×3 matrices M of linear forms on $\mathbb{P}^4_k$ such that the 2×2 minors of M generated the ideal of I_Γ, modulo $GL_3 \times GL_3$.*
- *Gale dual pairs of linear series (f_1, f_2) where $f_i : \Gamma \to \mathbb{P}^2$ and $f_i(\Gamma)$ lies on no conic.*

This result implies that the sextic covers of T which arise from a Gulliksen-Negard complex must embed into a $\mathbb{P}^2$-bundle over T. Since a generic sextic cover only embeds into a $\mathbb{P}^4$-bundle over T, this provides a nontrivial restriction on the class of sextic covers which can arise from a Gulliksen-Negard construction.

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Cohomology of generalized Kummer fourfolds and Lagrangian planes

Brendan Hassett

(joint work with Yuri Tschinkel)

Let X be an irreducible holomorphic symplectic variety, i.e., a smooth projective simply-connected manifold admitting a unique nondegenerate holomorphic two-form. Let $(,)$ denote the Beauville-Bogomolov form on the cohomology group $\mathrm{H}^2(X, \mathbb{Z})$, normalized so that it is integral and primitive. Duality gives a $\mathbb{Q}$-valued form on $\mathrm{H}_2(X, \mathbb{Z})$, also denoted $(,)$. When X is a K3 surface these coincide with the intersection form. In higher-dimensions, the form induces an inclusion

$$\mathrm{H}^2(X, \mathbb{Z}) \subset \mathrm{H}_2(X, \mathbb{Z}),$$

which allows us to extend $(,)$ to a $\mathbb{Q}$-valued quadratic form.

It is well known that a K3 surface contains a smoothly embedded $\mathbb{P}^1$ if and only if it admits a divisor ℓ such that $\ell \cdot \ell = -2$. We seek generalizations of this fact to higher-dimensional manifolds.

Suppose that X contains a Lagrangian projective space $\mathbb{P}^{\dim(X)/2}$; let $\ell \in \mathrm{H}_2(X,\mathbb{Z})$ denote the class of a line in $\mathbb{P}^{\dim(X)/2}$. Let λ denote the minimal positive multiple of ℓ contained in $\mathrm{H}^2(X,\mathbb{Z})$. Hodge theory [5, 6] shows that the deformations of X containing a deformation of the Lagrangian space coincide with the deformations of X for which $\lambda \in \mathrm{H}^2(X,\mathbb{Z})$ remains of type $(1,1)$. Infinitesimal Torelli implies this is a divisor on the deformation space, i.e.,

$$\lambda^{\perp} \subset \mathrm{H}^1(X,\Omega^1_X) \simeq \mathrm{H}^1(X,T_X).$$

Theorem 1. [3] *If X is deformation equivalent to the generalized Kummer manifold of dimension four and ℓ the class of a line on a Lagrangian plane in X then*

$$(\ell,\ell) = -3/2.$$

The main ingredients in our proof include an analysis of automorphisms of generalized Kummer varieties, their fixed-point loci, and the resulting 'tautological' absolute Hodge classes in middle cohomology. (Unlike the case of length-two Hilbert schemes, the middle cohomology is not generated by the second cohomology.) In particular, these tautological classes arise from explicit complex surfaces. We analyze the saturation of the lattice generated by these tautological classes in the middle cohomology, and integrality properties of the quadratic form associated with the cup product. Computing the orbit of a Lagrangian plane under the automorphism group, we obtain a precise characterization of the homology classes that may arise.

In collaboration with David Harvey, we have similar results for varieties deformation equivalent to Hilbert schemes of K3 surfaces. The four-dimensional case is addressed in [2].

Theorem 2. [1] *Let X be a six-dimensional variety, deformation equivalent to $S^{[3]}$ for S a K3 surface. Let $\mathbb{P}^3 \subset X$ be a smooth subvariety, $\ell \subset \mathbb{P}^3$ the class of a line. Then $(\ell,\ell) = -3$ and $2\ell \in \mathrm{H}^2(X,\mathbb{Z})$. Futhermore, we have*

$$\left[\mathbb{P}^3\right] = \frac{1}{12}\left(2\ell^3 + \ell^2 c_2(X)\right). \tag{1}$$

The main ingredients are Lehn and Sorger's description [4] of the cohomology ring of the Hilbert scheme of a K3 surface, and the resulting presentation of the ring of tautological classes. In addition, an analysis of integral points on an elliptic curve is required, to establish that certain indecomposable tautological classes do not appear in (1).

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Mutually intersecting planes in $\mathbb{P}^5$, Enriques surfaces, cubic 4-folds, and EPW-sextics

Igor Dolgachev
(joint work with Dimitri Markushevich)

Let $G(3,6)$ be the Grassmann variety of planes in $\mathbb{P}^5$ and S be a subset of $G(3,6)$ which consists of mutually intersecting planes. It is assumed that S is maximal with this property. In 1930 Ugo Morin [5] had classified all irreducible infinite sets with this property. In the same paper he acknowledged that the classification of finite sets presents essential difficulties. A simple counting constants argument shows that S consists of at least 10 planes. Let $\mathbb{X}_{10}$ be the closed subset of $G(3,6)^{10}$ which consists of maximal ordered 10-tuples $(\Lambda_1,\ldots,\Lambda_{10})$ of mutually intersecting planes. The tangent space of $\mathbb{X}_{10}$ at a point $\underline{\Lambda} = (\Lambda_1,\ldots,\Lambda_{10})$ is a linear subspace of the tangent space of $G(3,6)^{10}$ at $\underline{\Lambda}$ given by a system of 45 equations. So, the expected dimension of an irreducible component of $\mathbb{X}$ containing $\underline{\Lambda}$ is equal to 45. This is also confirmed by a counting constants argument. The problem is how to find 45-dimensional irreducible components of $\mathbb{X}$. We show that some of them arise from Fano's projective models of Enriques surfaces.

Let X be an Enriques surface over $\mathbb{C}$. Assume that X does not contain smooth rational curves. This is an open condition in the moduli space. It is known that X embeds in $\mathbb{P}^5$ by a complete linear system $|D|$ of degree $D^2 = 10$ (a Fano model) [2]. Also is known that the numerical class δ of D can be written uniquely in the form $\delta = \frac{1}{3}(f_1 + \ldots + f_{10})$, where $f_i^2 = 0$ and $f_i \cdot f_j = 1, i \neq j$. The lifts of f_i to $\mathrm{Pic}(X)$ can be represented by two elliptic curves F_i and F_{-i} such that $|2F_i| = |2F_{-i}|$ is an irreducible pencil of elliptic curves on X. The divisor class $3D$ can be written in 2^9-ways as a sum of the divisor classes of the curves F_i or $F_{-i}, i = 1,\ldots,10$. The image of each curve $F_{\pm i}$ in $\mathbb{P}^5$ under a map given the linear system $|D|$ spans a plane $\Lambda_{\pm i}$ and the fact that $F_i \cdot F_j = 1, i + j \neq 0$, implies that $\Lambda_i \cap \Lambda_j \neq \emptyset$ if $i + j \neq 0$. One can also show that $\Lambda_i \cap \Lambda_{-i} = \emptyset$. A choice of writing D as a sum of the $F_i's$ defines a 10-tuple of mutually intersecting planes $(\Lambda_1,\ldots,\Lambda_{10})$. We call it an Enriques 10-tuple of mutually intersecting planes.

Let $\mathcal{H}_{\mathrm{En}}$ be the Hilbert scheme of Fano models of Enriques surfaces and $\mathcal{H}'_{\mathrm{En}}$ be its Galois 2^9-cover corresponding to a choice of 10 curves F_i as above. Let $\mathcal{H}_0$ be an irreducible component of $\mathcal{H}'_{\mathrm{En}}$ (we conjecture that $\mathcal{H}'_{\mathrm{En}}$ is irreducible), then we show that the map from $\mathcal{H}_0$ to $\mathbb{X}_{10}$ is of degree 1 onto an irreducible 45-dimensional component of $\mathbb{X}_{10}$.

Let $\underline{\Lambda}_{10} = (\Lambda_1, \ldots, \Lambda_{10}) \in \mathbb{X}_{10}$. Assume that $\Lambda_i \cap \Lambda_j$ is a unique point q_{ij}, and, for each i, the points $q_{ij}, j \neq i$, lie on a unique cubic. Then, counting constants, we obtain that there exists a cubic hypersurface Φ containing all the ten planes. We show that the above condition is satisfied for any Enriques 10-tuple. In fact, the nine points in each plane form the set of base points of an Halphen pencil of elliptic curves of degree 6 with double points at the base points. Moreover, by choosing a special Enriques surface, we check that the cubic Φ is unique and nonsingular. Hence it remains unique and nonsingular for a general Enriques 10-tuple $\underline{\Lambda}$.

Let Φ be a smooth cubic hypersurface in $\mathbb{P}^5$ defined by a general Enriques surface in $\mathbb{P}^5$. The cohomology group $H^4(\Phi, \mathbb{Z})$ is a quadratic lattice with respect to the cup-product isomorphic to the odd unimodular lattice $I^{21,2} \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle 1 \rangle^{\oplus 3}$ of signature $(21, 2)$. Here U is the hyperbolic plane, E_8 is the root lattice of type E_8, and $\langle 1 \rangle$ denotes is the quadratic form x^2. Let $[h] \in H^2(\Phi, \mathbb{Z})$ be cohomology class of a hyperplane section h of Φ. Under the isomorphism $H^4(\Phi, \mathbb{Z}) \to I^{21,2}$, the class $[h]^2$ is mapped to an element $\eta \in \langle 1 \rangle^{\oplus 3}$ equal to the sum of generators. The orthogonal complement of η in $I^{21,2}$ is an even sublattice isomorphic to the lattice $U^2 \perp E_8^2 \perp B$, where B is the quadratic form $2x^2 + 2xy + 2y^2$. For any hyperbolic sublattice N of $I^{21,2}$ containing η we define an N-marking of a cubic 4-fold as an isometry $\phi : H^4(\Phi, \mathbb{Z}) \to L$ such that $\phi([h]^2) = \eta$ and $\phi(N) \subset H^{2,2}(\Phi) \cap H^4(\Phi, \mathbb{Z})$. The coarse moduli space of N-marked cubic 4-folds exists and is isomorphic to the quotient of a type IV hermitian symmetric domain $\mathcal{D}_r$ of dimension $r = 21 - \mathrm{rank}(N)$ modulo a certain finite index subgroup Γ_N of the orthogonal group of $N^\perp$. In this way the moduli space of smooth cubic 4-folds containing 10 mutually intersecting planes taken in some order becomes isomorphic to an open subset of D_{10}/Γ_N, where $N^\perp \cong \mathbb{E}(-2) \oplus C$, where $\mathbb{E} \cong U \oplus E_8$ and C is the quadratic form $-2x^2 + 6xy + 2y^2$. By using lattice-theoretical arguments this implies that a smooth cubic 4-fold contains at most 10 mutually intersecting planes.

Let $F(\Phi)$ be the Fano variety of lines on Φ. It is known that Φ carries a structure of an irreducible holomorphic symplectic 4-fold. The non-degenerate holomorphic 2-form is obtained from a unique (up to a multiplicative constant) 4-form on Φ of type $(3, 1)$ via the correspondence defined by the incidence variety $I = \{(x, \ell) \in \Phi \times F(\Phi) : x \in \ell\}$. In particular, we can consider the variety $F(\Phi)$, where Φ is a smooth cubic arising from an Enriques 10-tuple of planes. The variety $F(\Phi)$ contains 10 planes in its Plücker embedding. We do not know whether Φ is a rational variety, nor do we know that $F(\Phi)$ is isomorphic to the Hilbert scheme $K^{[2]}$ of any K3-surface.

There is another construction of an irreducible holomorphic symplectic 4-fold associated to an Enriques surface. Let $\mathbb{P}^5$ be the space of lines in a vector space V. Let A be a Lagrangian subspace of $\bigwedge^3 V$, where the latter is equipped with a nondegenerate symplectic bilinear form defined by the wedge product $\bigwedge^3 V \times \bigwedge^3 V \to \bigwedge^6 V \cong \mathbb{C}$. The locus of lines $\mathbb{C}v$ such that the image of the map $\bigwedge^2 V \to \bigwedge^3 V, \alpha \mapsto v \wedge \alpha$, intersects non-trivially A is either the whole space $\mathbb{P}^5$, or a hypersurface Y_A of degree 6 (maybe reducible). It depends only on the image of A in $\mathbb{P}(\bigwedge^3 V)$. The hypersurface Y_A was first considered in [3], and is called an EPW-sextic. Its singular locus in general is a surface Z of degree 40. It was shown by K. O'Grady that there exists a double cover X_A of Y_A ramified only over Z. It is an irreducible holomorphic symplectic 4-fold.

In our case 10-planes $\underline{\Lambda} = (\Lambda_1, \ldots, \Lambda_{10})$ arising from an Enriques surface, considered as decomposable 3-vectors in $\bigwedge^3 V$, span a Lagrangian subspace $A_{\underline{\Lambda}}$. We show that the corresponding EPW-sextic is singular along the union of 10 planes and a surface of degree 40 with the same Hilbert polynomial as the surface Z for a general A.

A special case of an Enriques surface is a Reye Congruence $R(W)$. It is the set of lines in $\mathbb{P}^3$ which contained in a subpencil L of a general web W of quadrics. Considered as a congruence of lines in $\mathbb{P}^3$ it has the type $(3,7)$. In particular, it embeds in $\mathbb{P}^5$ via the Plücker embedding of the Grassmannian $G(2,4)$. The subpencil L, considered as a line in W, is a bitangent to the Hessian surface $H(W)$ of W, the surface parameterizing singular quadrics in W. This defines a map from $R(W)$ to the surface of bitangents $\mathrm{Bit}(H(W))$ which coincides with the normalization map. The surface $\mathrm{Bit}(H(W))$ is a congruence of lines in $\mathbb{P}^3$ of type $(12,28)$. In particular, its degree in the Plücker embedding of $G(2,4) \subset \mathbb{P}^5$ is equal to 40. Its Hilbert polynomial is equal to the Hilbert polynomial of the singular surface Z of a general EPW-sextic. We show that $\mathrm{Bit}(H(W))$ is isomorphic to a projection of $R(W)$ to $\mathbb{P}^5$ when $R(W)$ embeds in $\mathbb{P}^{19}$ under the linear system $|2H|$, where H defines the Plücker embedding of $R(W)$.

We conjecture that the same is true for a general Fano model of an Enriques surface. The linear system $|2D|$ embeds it in $\mathbb{P}^{19}$ as a surface of degree 40, and a certain projection to $\mathbb{P}^5$ is a surface of degree 40 contained in the singular locus of the EPW-sextic.

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Cubics of dimension seven and the Cayley plane

Laurent Manivel
(joint work with Atanas Iliev)

Consider a smooth hypersurface X in the complex projective space $\mathbb{P}^{n+1}$. Suppose that X is Fano, i.e., that its degree d is at most $n+1$. Denote its index by $\iota = n + 2 - d$. Kuznetsov proved in [2]:

Theorem 1. *The derived category $D(X)$ of coherent sheaves on X has a semi-orthogonal decomposition*

$$D(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \dots, \mathcal{O}_X(\iota - 1)\rangle,$$

where the full subcategory $\mathcal{A}_X$ is a Calabi-Yau category of dimension $n - \frac{2\iota}{d}$.

(When d does not divide 2ι, this means that $\mathcal{A}_X$ has a Serre functor, a suitable power of which is a shift).

An interesting example is the case of cubic fourfolds ($d = 3, n = 4$), for which $\mathcal{A}_X$ is in general a *noncommutative K3 surface*. In certain cases there is a genuine K3 surface attached to the cubic fourfold X. This is for example the case when X is defined as the Pfaffian of a 6×6 skew-symmetric matrix of linear forms in 6 variables – otherwise said, as a linear section $X = \mathrm{Pf} \cap \mathbb{P}^5_X$ of the Pfaffian hypersurface $\mathrm{Pf} \subset \mathbb{P}(\wedge^2\mathbb{C}^6)^\vee$. On the dual side, $\mathbb{P}(\wedge^2\mathbb{C}^6)$ contains the Grassmannian $G(2, 6)$, whose intersection with $(\mathbb{P}^5_X)^\perp$ is a K3 surface S such that $\mathcal{A}_X \simeq D(S)$ [2]. When X deforms to a non Pfaffian cubic, $\mathcal{A}_X$ is still defined but S is no longer present.

The next case for which $\mathcal{A}_X$ has integral dimension is the case of cubic sevenfolds ($d = 3, n = 7$), for which $\mathcal{A}_X$ is a *noncommutative Calabi-Yau threefold*. A first important difference with the case of cubic fourfolds is that there can exist no genuine Calabi-Yau threefold Z such that $\mathcal{A}_X \simeq D(Z)$. One can nevertheless generalize the Pfaffian construction by replacing $G(2, 6)$ by the next Severi variety [4]. The *Cayley plane* $\mathbb{OP}^2$ is a sixteen dimensional complex projective variety, homogeneous under the exceptional group E_6, with an equivariant embedding inside $\mathbb{P}^{26}$. Its dual variety $C = (\mathbb{OP}^2)^*$ is a cubic hypersurface that we call the Cartan cubic (its equation was first written down by E. Cartan in terms of tritangent planes to a cubic surface).

Proposition 1. *A general seven dimensional cubic X can be represented as a linear section of the Cartan cubic, in finitely many ways.*

It would be interesting to know the number of such representations. If it is equal to one, the cubic sevenfold has a canonical form of a quite unexpected type. If it is bigger than one, the noncommutative Calabi-Yau threefold $\mathcal{A}_X$ must be very symmetric.

Indeed, each realization of X as a linear section of the Cartan cubic endows it with a special rank nine vector bundle E, defined as follows. Since C is also the secant variety to the dual Cayley plane $(\mathbb{OP}^2)^\vee$, each point $x \in X$ defines an entry locus $Q_x \subset (\mathbb{OP}^2)^\vee$, which turns out to be an eight-dimensional quadric. (As observed by Freudenthal and Tits in the fifties, these quadrics must be considered as projective lines over the octonions, and the family of these lines have many of the characteristic properties of a plane projective geometry.) The polar hyperplane to x with respect to Q_x inside its linear span, is by definition $\mathbb{P}(E_x)$ and this defines the bundle E on X.

Theorem 2. *The vector bundle E is arithmetically Cohen-Macaulay and infinitesimally rigid. Moreover E and $E(1)$ are spherical objects in $\mathcal{A}_X$.*

This means in particular that the cohomology of $End(E)$ is that of a three-dimensional sphere. Seidel and Thomas [6] showed how to associate to a spherical object a *spherical twist*, which is in our case a self-equivalence of the category $\mathcal{A}_X$. Thus every representation of X as a section of the Cartan cubic produces nontrivial self-equivalences of the corresponding noncommutative Calabi-Yau. Note that on the contrary, $D(X)$ itself has no interesting self-equivalence since X is Fano.

On the dual side, to a general $X = (\mathbb{OP}^2)^* \cap \mathbb{P}^8_X$ we can associate the orthogonal section $Y = \mathbb{OP}^2 \cap (\mathbb{P}^8_X)^\perp$ of the Cayley plane. This is a Fano manifold of dimension seven and index three.

Proposition 2. *The two varieties X and Y are birationally equivalent.*

More precisely one can associate a birationality to any general point of Y, and this birationality can be constructed in very simple terms from the plane projective geometry supported by the Cayley plane.

Now, the derived category of $\mathbb{OP}^2$ was described in [5] in terms of a special rank ten vector bundle S, and it follows that $S_Y, \mathcal{O}_Y, S_Y(1), \mathcal{O}_Y(1), S_Y(2), \mathcal{O}_Y(2)$ is an exceptional collection in $D(Y)$. Consider the semiorthogonal decomposition

$$D(Y) = \langle S_Y, \mathcal{O}_Y, S_Y(1), \mathcal{O}_Y(1), S_Y(2), \mathcal{O}_Y(2), \mathcal{A}_Y \rangle.$$

Conjecture. *$\mathcal{A}_X$ and $\mathcal{A}_Y$ are equivalent.*

This statement would be very similar to the Pfaffian-Grassmannian derived equivalence of Borisov and Caldararu [1], except that we deal with noncommutative Calabi-Yau's attached to our Fano varieties. It would also be a new instance of Kuznetsov's homological projective duality [3].

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Cones of Hilbert functions

Gregory G. Smith
(joint work with Mats Boij)

Studying the subvarieties of an given algebraic variety lies at the heart of classical algebraic geometry. For simplicity, consider $\mathbb{P}^n = \operatorname{Proj} S$ where $S := \Bbbk[x_0, \dots, x_n]$ and $\Bbbk$ is a field. The basic invariant used to classify projective subvarieties is the Hilbert function. For an $\mathbb{N}$-graded S-module $M = \bigoplus_{i \geq 0} M_i$ [e.g. if Y is a subvariety of $\mathbb{P}^n$, then we associate the $\mathbb{N}$-graded S-module $\bigoplus_{i \geq 0} H^0\big(\mathbb{P}^n, \mathcal{O}_Y(i)\big)$], the Hilbert function $h_M \colon \mathbb{N} \to \mathbb{N}$ is defined by $h_M(i) := \dim_{\Bbbk} M_i$. With this invariant, the classification problem for projective varieties breaks into two steps. Indeed, the two fundamental problems are:

1. (1) describe the set [or space] of all Hilbert functions for a collection of modules or subvarieties;
2. (2) describe the space of all modules with a fixed Hilbert function.

We will focus here on the first of these problems.

In 1927, F.S. Macaulay [3] provided a complete solution to the first problem for quotients of the polynomial ring S. To be more explicit, observe that, for two positive integers m and i, there exists a unique expression

$$m = \binom{k_i}{i} + \binom{k_{i-1}}{i-1} + \cdots + \binom{k_j}{j}$$

where integer numerators of these binomial coefficients satisfy the inequalities $k_i > k_{i-1} > \cdots > k_j \geq j \geq 1$. Using this expansion, we define

$$m^{\langle i \rangle} := \binom{k_i + 1}{i+1} + \binom{k_{i-1}+1}{i} + \cdots + \binom{k_j + 1}{j+1}.$$

Given this notation, Macaulay established that, for a function $h \colon \mathbb{N} \to \mathbb{N}$, the following are equivalent:

- (a) $h(0) = 1$ and $h(i+1) \leq \big(h(i)\big)^{\langle i \rangle}$ for $i \geq 1$;
- (b) there exists a homogeneous ideal I such that $h(i) = h_{S/I}(i)$ for all i.

From a modern perspective, the key insight arising from the proof of this theorem is the existence of the *lex-segment ideal*. This monomial ideal has extremal syzygies and plays a central role in establishing the connectedness of the Hilbert scheme. Nevertheless, this result has two notable weaknesses: the function $m \mapsto m^{\langle i \rangle}$ is cumbersome [e.g. does there exists a homogeneous ideal I such that the Hilbert polynomial $p_{S/I}(i)$ of S/I equals i^2?] and analogues of the lex-segment ideal fail to exist in many similar situations [e.g. when we replace $\mathbb{P}^n$ with another variety].

To attack these weaknesses, we examine a larger class of modules with an additional structure. The set of Hilbert functions naturally forms a semigroup: $h_{M \oplus N}(i) = h_M(i) + h_N(i)$. If we bound $a := \max\{i : h_M(i) \neq p_M(i)\}$, then this is an affine semigroup [i.e. a subsemigroup of $\mathbb{Z}^{a+1+n}$]. In this context, a modest variant of our first problem asks for a description of the convex hull of the set of Hilbert functions. By looking for the best linear approximation to the set of

Hilbert functions, we avoid both of the identified weakness of Macaulay's theorem. However, this approach will not yield a complete classification because this semigroup is typically not saturated — there exists lattice points in the convex hull that do not correspond to the Hilbert function of a module.

To describe this convex hull, it is convenient to choose a basis for the ambient rational vector space. Let $F_M(t) := \sum_{i\geq 0} h_M(i)t^i \in \mathbb{Q}[[t]]$ be the generating function associated with the Hilbert function h_M. Since the Hilbert function $h_M(i)$ agrees with the Hilbert polynomial $p_M(i)$ for all $i \gg 0$, it follows that the series $F_M(t)$ is a rational function of t and $\deg F_M = \max\{i : h_M(i) \neq p_M(i)\}$ [where the degree of a rational function is the degree of the numerator minus the degree of the denominator]. The forward difference operator Δ acts on $h\colon \mathbb{N} \to \mathbb{N}$ to produce the function $\Delta h\colon \mathbb{N} \to \mathbb{N}$ defined by $\Delta h(i) := h(i+1) - h(i)$. We write Δ^j for the j-fold composition of Δ with itself. Now, if M is a finitely generated $\mathbb{N}$-graded S-module, $\dim M = d$, and $a \geq \deg F_M$, then we have

$$F_M(t) = \sum_{i=0}^{a} h_M(i)t^i + t^a \sum_{j=0}^{d-1} \Delta^j h(a+1) \left(\frac{t}{1-t}\right)^{j+1} .$$

In particular, the Hilbert function $h_M\colon \mathbb{N} \to \mathbb{N}$ corresponds to the vector

$$\left[\, h_M(0)\ h_M(1)\ \cdots\ h_M(a)\ \Delta^0 h_M(a{+}1)\ \Delta^1 h_M(a{+}1)\ \cdots\ \Delta^{d-1} h_M(a{+}1)\ 0\ \cdots\ 0 \,\right]^{\mathsf{t}} \in \mathbb{Z}^{a+1+n} .$$

In this coordinate system, we can elegantly specify the facets for the cone of Hilbert functions.

Theorem. If M is an $\mathbb{N}$-graded S-module that is finitely generated in degree 0, $\dim M = d$, and $a \geq \deg F_M$, then the Hilbert function $h_M \in \mathbb{Z}^{a+1+d}$ lies in the rational simplicial convex cone defined by the half-spaces:

$$\frac{h_M(i)}{\binom{n+i}{i}} \geq \frac{h_M(i+1)}{\binom{n+i+1}{i+1}} \qquad \text{for } 0 \leq i \leq a,$$

$$\frac{\Delta^j h_M(a+1)}{\binom{n+a+1}{j+a+1}} \geq \frac{\Delta^{j+1} h_M(a+1)}{\binom{n+a+1}{j+a+2}} \qquad \text{for } 0 \leq j < d.$$

Informally, these two collections of inequalities can be thought of as Macaulay's theorem up to scaling and the Kruskal-Katona theorem up to scaling respectively. This cone is the surprisingly simple image, under the map from Betti diagrams to Hilbert functions, of the Boij-Söderberg simplicial fan; cf. [1], [2].

Dually, we can also describe the extremal rays of the cone of Hilbert functions. If $R(d,a)$ denotes the $\mathbb{N}$-graded S-module $S/\langle x_d, \ldots, x_n\rangle^{a+1+d}$, then we have

$$F_{R(d,a)}(t) = \sum_{i=0}^{a} \binom{n+i}{i} t^i + t^a \sum_{j=0}^{d-1} \binom{n+a+1}{j+a+1} \left(\frac{t}{1-t}\right)^{j+1} .$$

Theorem. If M is an $\mathbb{N}$-graded S-module that is finitely generated in degree 0, $\dim M = d$ and $a \geq \deg F_M$, then the Hilbert function h_M lies in the rational simplicial cone generated by $h_{R(d,a)}, h_{R(d-1,a)}, \ldots, h_{R(0,a)}, h_{R(0,a-1)}, \ldots, h_{R(0,0)}$.

It follows that every lattice point on an extremal ray of this cone corresponds to the Hilbert function of a module. Moreover, the lattice points on any face that correspond to the Hilbert function of a module form a finitely index subgroup of all the lattice points on the face.

We expect analogous results when $\mathbb{P}^n$ is replaced by a smooth projective toric variety. In particular, we give as part of an ongoing project a similar description for the cone of Hilbert functions over some multigraded polynomial rings.

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Partially positive line bundles

Burt Totaro

Ample line bundles are fundamental to algebraic geometry. The same notion of ampleness arises in many ways: geometric (some positive multiple gives a projective embedding), numerical (Nakai-Moishezon, Kleiman), or cohomological (Serre) [5, Chapter 1]. Over the complex numbers, ampleness of a line bundle is also equivalent to the existence of a metric with positive curvature (Kodaira).

Our goal is to study weaker notions of ampleness, and to prove some of the corresponding equivalences. The subject began with Andreotti-Grauert's theorem that on a compact complex manifold X of dimension n, a hermitian line bundle L whose curvature form has at least $n-q$ positive eigenvalues at every point has $H^i(X, E \otimes L^{\otimes m}) = 0$ for every $i > q$, every coherent sheaf E on X, and every m sufficiently large depending on E [1]. Call the latter property *naive q-ampleness* of L, for a given natural number q. Thus naive 0-ampleness is the usual notion of ampleness, while every line bundle is naively n-ample. Nothing is known about the converse to Andreotti-Grauert's theorem for $q > 0$, but we can still try to understand naive q-ampleness for projective varieties over any field.

Sommese gave a clear geometric characterization of naive q-ampleness when in addition L is semi-ample (that is, some positive multiple of L is spanned by its global sections). In that case, naive q-ampleness is equivalent to the condition that the morphism to projective space given by some multiple of L has fibers of dimension at most q [6]. That has been useful, but the condition of semi-ampleness is restrictive, and we do not want to assume it. For example, the line bundle $O(a, b)$ on $\mathbf{P}^1 \times \mathbf{P}^1$ is naively 1-ample exactly when at least one of a and b is positive, whereas semi-ampleness would require both a and b to be nonnegative.

Our main result applies to smooth projective varieties in characteristic zero. In that situation, we show that naive q-ampleness (which is defined using the vanishing of infinitely many cohomology groups) is equivalent to the vanishing of finitely many cohomology groups, a condition we call q-T-ampleness. This equivalence is an analogue of Serre's characterization of ampleness. Indeed, q-T-ampleness is an analogue of the geometric definition of ampleness by some power of L giving a projective embedding; the latter is also a "finite" condition, unlike the definition of naive q-ampleness. The definition of q-T-ampleness was suggested by Arapura's positive characteristic vanishing theorem [2]. The equivalence implies in particular that naive q-ampleness is Zariski open on families of varieties and line bundles in characteristic zero, which is not at all clear from the definition.

We also show that naive q-ampleness in characteristic zero is equivalent to DPS q-ampleness, a variant defined by Demailly-Peternell-Schneider [4]. It follows that naive q-ampleness defines an open cone (not necessarily convex) in the Néron-Severi vector space $N^1(X)$. After these results, it makes sense to say simply "q-ample" to mean any of these equivalent notions for line bundles in characteristic zero.

One natural open problem is to give a numerical characterization of q-ampleness, generalizing Kleiman's theorem in the case $q = 0$. Another is to characterize q-ampleness in terms of the asymptotic cohomological functions defined by Küronya, as de Fernex, Küronya, and Lazarsfeld did in the case $q = 0$ [3].

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Compact moduli for certain Kodaira fibrations

Sönke Rollenske

It is a general fact that moduli spaces of *nice* objects in algebraic geometry, say smooth varieties, are often non-compact. But usually there is a modular compactification where the boundary points correspond to related but more complicated objects.

Such a modular compactification has been known for the moduli space $\mathcal{M}_g$ of smooth curves of genus g for a long time and in [KSB88] Kollár and Shepherd-Barron made the first step towards the construction of a modular compactification $\overline{\mathfrak{M}}$ for the moduli space $\mathfrak{M}$ of surfaces of general type via so called stable surfaces; the boundary points arise from a stable reduction procedure. An overview over the technical issues arising in the construction can be found in [HC10].

But even 20 years later very few explicit descriptions of compact components of $\overline{\mathfrak{M}}$ have been published. The main idea in all approaches is to relate the component of the moduli space one wishes to study to some other moduli space, where a suitable compactification is known. Products of curves and surfaces isogenous to a product of curves have been treated by van Opstall [vO05, vO06] and reconsidered by Wenfei Liu [Liu10], and a recent paper of Alexeev and Pardini [AP09] studies Burniat and Campedelli surfaces relating them to hyperplane arrangements in (a blow-up of) $\mathbb{P}^2$.

In the talk we reported on progress in our programm to explicitly construct the stable surfaces in $\overline{\mathfrak{M}}$ that arise as stable degenerations of very simple Galois double Kodaira fibrations: let S be a compact complex surface of general type such that a finite group G acts on S and

- $S/G \cong C \times C$ for a smooth curve of genus at least 2,
- the quotient map $\psi : S \to C \times C$ is a ramified covering,
- there exist a set of automorphisms $\mathcal{S} \subset \mathrm{Aut}(C)$ such that the branch divisor is union of their graphs

$$B = \sum_{\sigma \in \mathcal{S}} \Gamma_\sigma \subset C \times C$$

and $\Gamma_\sigma \cap \Gamma_{\sigma'} = \emptyset$ for $\sigma \neq \sigma'$.

On the first glance these surfaces seem quite special but in joint work with Fabrizio Catanese we gave in [CR09] an effective method of construction and a precise description of their moduli space.

Theorem 1 ([CR09], Theorem 6.5)**.** *Let $\psi : S \to C \times C$ be a very simple Kodaira fibration, $\mathcal{S} \subset \mathrm{Aut}(C)$ as above. Let H be the subgroup of $\mathrm{Aut}(C)$ generated by $\mathcal{S}$. Then the connected component $\mathfrak{N}$ of the moduli space of surfaces of general type containing S contains only very simple Kodaira fibrations and, denoting by $\mathcal{M}_{g(C)}(H)$ the moduli space of curves with the given action of the group H, there is a map $\mathcal{M}_{g(C)}(H) \to \mathfrak{N}$ that is an isomorphim on geometric points.*

It is possible to work out an explicit description of the stable degenerations of these surfaces. They turn out to be ramified covers of explicitly given modifications of a product of stable curves, that is for every stable degeneration X of a very simple Galois Kodaira fibration there exists a stable curve C_0, a birational map $\pi : Y \to C_0 \times C_0$, which replaces some degenerate cusps by a projective line, and a ramified covering $\psi : X \to X/G = Y$; the branch divisor of ψ is the strict transform of the graph of some automorphisms of the stable curve C_0.

By standard deformation theoretic arguments, that however get quite involved where we deal with the degenerations, we can compare deformations of S and deformations of the pair (C, H).

Theorem 2. *In the same notation as in Theorem 1 assume that $S \to C \times C$ is a simple cyclic covering. Then the closure of $\mathfrak{N}$ in the moduli space of stable surfaces is a connected and irreducible component of $\overline{\mathfrak{M}}$. Its geometric points are in bijection to stable curves with the given automorphism group H.*

We certainly expect this to be true without the extra assumptions on S. More details can be found in [Rol09].

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Continuous closure of sheaves

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This is a preliminary report on my current work on certain closure operations on ideals and sheaves.

Definition 1. Let $I = (f_1, \dots, f_r) \subset \mathbb{C}[z_1, \dots, z_n]$ be an ideal. Following Brenner (see math.AC/0608611) a polynomial $g(z_1, \dots, z_n)$ is in the *continuous closure* of I iff there are continuous functions ϕ_i such that $g = \phi_1 f_1 + \cdots + \phi_r f_r$. These polynomials form an ideal $I^C \supset I$. For example

$$z_1^2 z_2^2 = \frac{\bar{z}_1 z_2^2}{|z_1|^2 + |z_2|^2} \cdot z_1^3 + \frac{\bar{z}_2 z_1^2}{|z_1|^2 + |z_2|^2} \cdot z_2^3$$

shows that $z_1^2 z_2^2 \in (z_1^3, z_2^3)^C \setminus (z_1^3, z_2^3)$.

The above definition is very natural, but it is not clear that it gives an algebraic notion (since $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ does not map continuous functions to continuous functions) or that it defines a sheaf in the Zariski topology (since a continuous function may grow faster than any polynomial).

The investigations have three principal aims:

(1) The first one is to give a purely algebraic construction of the continuous closure.

(2) Second, we intend to prove that one gets the same definition of I^C using various subclasses of continuous functions, for instance continuous semi-algebraic functions.

(3) The third goal is to show that taking continuous closure commutes with flat morphisms whose fibers are semi-normal. In particular, the continuous closure of a coherent ideal sheaf is again a coherent ideal sheaf (both in the Zariski and in the étale topologies) and it commutes with field extensions.

Instead of working with ideals, we work with maps of locally free sheaves $f : E \to F$. Thus an ideal sheaf $J = (f_1, \dots, f_r) \subset \mathcal{O}_X$ corresponds to the map $(f_1, \dots, f_r) : \mathcal{O}_X^r \to \mathcal{O}_X$.

One can simplify the local structure of J by taking a resolution $p : Y \to X$ such that $p^*J = \mathcal{O}_Y(-D)$ for some divisor D. Now we are asking:

Which global sections of p^*J are pull-backs of continuous sections of E?

It is pretty clear that most interesting aspects happen along the exceptional divisor D. That is, we should first study the question:

Which global sections of $(p^*J)|_D$ are pull-backs of continuous sections of $E|_Z$?

This allows one to use induction on the dimension, at the cost of having a more general set-up. This leads to the following definition.

Definition 2. A *descent problem* is a compound object

$$\mathbf{D} = \big(p : Y \to X, f : p^*E \to F\big)$$

consisting of a proper morphism of reduced schemes $p : Y \to X$, a locally free sheaf E on X, a locally free sheaf F on Y and a map $f : p^*E \to F$.

Fix a class of functions $\mathcal{C}$; for instance, if we work over $\mathbb{C}$, then a typical choice is $\mathcal{C}(X) = \big\{\text{continuous functions on } X(\mathbb{C})\big\}$. Let $\Gamma_{\mathcal{C}}$ denote global sections with coefficients in $\mathcal{C}$. Our aim is to understand the image of

$$f \circ p^* : \Gamma_{\mathcal{C}}(X, E) \to \Gamma_{\mathcal{C}}(Y, F)$$

and its intersection with $H^0(Y, F)$. If X is affine, $Y = X$, $F = \mathcal{O}_X$ and $I = \operatorname{im} f$ then the intersection turns out to be the continuous closure of I.

A general descent problem may be too far from the original set-up, so we need methods to simplify them. We identify some constructions that give new descent problems that are closely related to the original one. (At the moment, the following should be viewed as a preliminary choice. There is a tension between trying to use the smallest possible class that makes the theorems work or the largest possible class where all relevant constructions can be carried out.)

Definition 3 (Scions of descent problems)**.** Let $\mathbf{D} = \big(p : Y \to X, f : p^*E \to F\big)$ be a descent problem. A *scion* of $\mathbf{D}$ is any descent problem that can be obtained by a repeated application of

the following procedures.

(1) For a proper morphism $r : Y' \to Y$ set

$$r^*\mathbf{D} := \big(p \circ r : Y' \to X, r^*f : (p \circ r)^*E \to r^*F\big).$$

(2) Let $\mathbf{D}_i = \big(p_i : Y_i \to X, f_i : p_i^*E \to F_i\big)$ be descent problems. Take their disjoint union

$$\amalg_i \mathbf{D}_i := \big(\amalg_i p_i : \amalg_i Y_i \to X, (\amalg_i p_i)^*E \to \amalg_i F_i\big).$$

(3) Assume that p factors as $Y \stackrel{r}{\to} Z \stackrel{q}{\to} X$ where r is finite and flat. Then set
$$r_*\mathbf{D} := \bigl(q : Z \to X, r_* f : q^* E \to r_* F\bigr).$$

(4) Assume that f factors as $p^* E \twoheadrightarrow F' \hookrightarrow F$ where F' is locally free and set
$$\mathbf{D}' := \bigl(p : Y \to X, f' : p^* E \to F'\bigr).$$

Each scion remembers all of its forebears. That is, two scions are considered the "same" only if they have been constructed by identical sequences of procedures. This is quite important since the sheaf F_W on a scion $\mathbf{D}_W$ does depend on the whole sequence. The class of all scions of $\mathbf{D}$ is denoted by $\mathrm{Sci}(\mathbf{D})$.

Definition 4. Let $\mathbf{D} = \bigl(p : Y \to X, f : p^* E \to F\bigr)$ be a descent problem with scions
$$\mathrm{Sci}(\mathbf{D}) = \Bigl\{\bigl(p_i : Y_i \to X, f_i : p_i^* E \to F_i\bigr) : i \in I\Bigr\}.$$
A global section of F over $\mathrm{Sci}(\mathbf{D})$ is a collection of sections
$$\Phi := \bigl\{\phi_i \in H^0(Y_i, F_i) : i \in I\bigr\}$$
such that the ϕ_i commute with pull-backs for the operations (3.1–2) and with push-foward for the operations (3.3–4). All sections form a vector space $H^0\bigl(\mathrm{Sci}(\mathbf{D}), F\bigr)$.

One defines similarly $\Gamma_{\mathcal{C}}\bigl(\mathrm{Sci}(\mathbf{D}), F\bigr)$.

We call ϕ_i the restriction of Φ to Y_i. The most important of these restrictions is $\Phi|_Y$. Note that $\Phi|_Y$ uniquely determines Φ. Indeed, the constructions (3.1–2) automatically carry along ϕ and in (3.3–4) the natural maps $H^0(Z, r_* F) \to H^0(X, F)$ and $H^0(Y, F') \to H^0(Y, F)$ are injections.

Thus we usually think of $H^0\bigl(\mathrm{Sci}(\mathbf{D}), F\bigr)$ as a subspace of $H^0(Y, F)$.

Note also that every $\phi_X \in H^0(X, E)$ defines a global section of F over $\mathrm{Sci}(\mathbf{D})$ by setting $\phi_i := f_i(p_i^* \phi_X)$. Thus we have natural maps
$$\begin{array}{ccccc} H^0(X,E) & \to & H^0\bigl(\mathrm{Sci}(\mathbf{D}), F\bigr) & \hookrightarrow & H^0(Y,F) \quad \text{and} \\ \Gamma_{\mathcal{C}}(X,E) & \to & \Gamma_{\mathcal{C}}\bigl(\mathrm{Sci}(\mathbf{D}), F\bigr) & \hookrightarrow & \Gamma_{\mathcal{C}}(Y,F). \end{array}$$

The main results are the following.

Theorem 5. *Let $\mathbf{D} = \bigl(p : Y \to X, f : p^* E \to F\bigr)$ be a descent problem over $\mathbb{C}$ and $\mathcal{C}$ the sheaf of continuous functions. Then*
$$\Gamma_{\mathcal{C}}(X, E) \twoheadrightarrow \Gamma_{\mathcal{C}}\bigl(\mathrm{Sci}(\mathbf{D}), F\bigr) \quad \textit{is surjective.}$$

Theorem 6. *Let X be a seminormal affine scheme over $\mathbb{C}$ and $f : E \twoheadrightarrow J \subset \mathcal{O}_X$ a presentation of a coherent ideal sheaf. With $Y := X,\ F := \mathcal{O}_X$ and $p := \mathbf{1}_X$ we get a descent problem $\mathbf{D} = \bigl(p : Y \to X, f : p^* E \to F\bigr)$. Then*
$$J^C = H^0\bigl(\mathrm{Sci}(\mathbf{D}), F\bigr).$$

7 (Comments)**.** In general one would expect that solving equations of the form $g = \sum_i \phi_i f_i$ needs control of many terms of the Taylor expansions of g and of the ϕ_i. However, since the ϕ_i are only continuous, only the constant term of the Taylor expansion makes sense. The role of (3.3–4) is to access information about the higher order terms using the fact that $\Gamma_{\mathcal{C}}$ does not commute with push forward, not even for finite, flat morphisms.

As a simple example, consider $r : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^2$. Note that r is invariant under $\tau : z \mapsto -z$ and $r_*\mathcal{O}_{\mathbb{C}}$ is the direct sum of its τ-invariant part (generated by 1) and its τ-anti-invariant part (generated by z). If $h = a_0 + a_1 z + a_2 z^2 + \cdots$ then $r_* h$, as a section of $r_*\mathcal{O}_{\mathbb{C}}$ is

$$r_* h = \big(a_0 + a_2 z^2 + \cdots, a_1 + a_3 z^2 + \cdots\big). \tag{7.1}$$

Thus the 1^{st} order term a_1 becomes a 0^{th} order term. For continuous functions $\phi(z)$, the corresponding decomposition is

$$r_*\phi = \Big(\frac{\phi(z) + \phi(-z)}{2}, \frac{\phi(z) - \phi(-z)}{2}\Big). \tag{7.2}$$

Thus $r_*\phi$ is a continuous section of $r_*\mathcal{O}_{\mathbb{C}}$ only if $\frac{1}{2z}\big(\phi(z) - \phi(-z)\big)$ is continuous, that is, when the odd part of ϕ is differentiable at the origin. Thus $\phi \mapsto \frac{1}{2z}\big(\phi(z) - \phi(-z)\big)$ plays the role of the first derivative in the proof.

Birational Geometry of Parameter Spaces of Curves

Joe Harris

We propose to call a space whose points correspond naturally to isomorphism classes of varieties or schemes X of a given type a *moduli space*; we'll call a space whose points correspond naturally to subschemes $X \subset Z$ of a fixed scheme Z (not up to isomorphism) a *parameter space*. There is not always a clear line dividing the two—for example, the Kontsevich space parameterizing stable maps has elements of both—but it does reflect an important duality in how we view geometric objects. One of the fundamental ideas underlying much recent progress in the theory of curves, for example, is the fact that whenever we have a one-parameter family $\{C_t \subset \mathbb{P}^r\}_{t\in\Delta}$ of curves in projective space, with C_t smooth for $t \neq t_0$, we have two distinct notions of the "limit" $\lim_{t\to t_0} C_t$ of the curves C_t: the *flat limit*, which is a subscheme of $\mathbb{P}^r$ whose geometry can be pretty much arbitrarily messy; and the *stable limit*, which is the limit of the abstract curves C_t and has at worst nodes as singularities. (Other articles in this volume discuss alternative notions of stability, and correspondingly alternative definitions of the limit of the abstract curves C_t; as for the flat limit, we really don't have much of an alternative to that.)

That said, what should we take as the parameter space for curves of degree d and genus g in $\mathbb{P}^r$? There are principally three answers to this question: the *Chow variety*, the *Hilbert scheme* and the *Kontsevich space*. These agree on the common open subset $\mathcal{H}^\circ_{g,d,r}$ parametrizing smooth curves (at least if we ignore the scheme structure on these spaces), but give very different compactifications of $\mathcal{H}^\circ_{g,d,r}$. We start with a brief discussion of the properties of each. Actually, we'll pretty much ignore the Chow variety—in many ways, it has all the drawbacks of the Hilbert scheme and the Kontsevich space, and none of the virtues—and focus primarily on the other two.

To begin with, the Hilbert scheme $\mathcal{H} = \mathcal{H}_{g,r,d}(\mathbb{P}^r)$ is a parameter space for subschemes of $\mathbb{P}^r$ with Hilbert polynomial $p(m) = md - g + 1$; in the case of curves (one-dimensional subschemes) this means all subschemes with fixed degree and arithmetic genus. The Hilbert scheme has many good properties. For example, there is a useful cohomological description of its tangent spaces, and, beyond that, a deformation theory that in some cases can describe its local structure. And, of course, associated to a point on the Hilbert scheme is all the rich structure of a homogenous ideal in the ring $K[x_0, \ldots, x_n]$ and its resolution.

The Hilbert scheme, as a compactification of the space of smooth curves, has drawbacks that sometimes make it difficult to use:

(1) **It has extraneous components, often of differing dimensions.** We see this phenomenon already in the case of twisted cubics, above. Of course we could take just the closure in the Hilbert scheme of the locus of smooth curves, but we would lose some of the nice properties, like the description of the tangent space. Thus while it is relatively easy to describe the singular locus of $\mathcal{H}$, we don't know how to describe singular locus of $\mathcal{H}^\circ$ along the locus where it intersects other components.

In fact, we don't know for curves of higher degree how many such extraneous components there are, or their dimensions: for $r \geq 3$ and large d the Hilbert scheme of zero-dimensional subschemes of degree d in $\mathbb{P}^r$ will have an unknown number of extraneous components of unknown dimensions, and this creates even more extraneous components in the Hilbert schemes of curves.

(2) **No one knows what's in the closure of the locus of smooth curves**. If we do choose to deal with the closure of the locus of smooth curves rather than the whole Hilbert scheme—as it seems we must—we face another problem: Except in a few special cases, we can't tell if a given point in the Hilbert scheme is in this closure. That is, we don't know how to tell whether a given singular 1-dimensional scheme $C \subset \mathbb{P}^r$ is smoothable.

(3) **It has many singularities**. There are schemes that appear in the PGL_{r+1}-orbit of *every* subscheme of $\mathbb{P}^r$ with given Hilbert polynomial. At the corresponding points, the Hilbert scheme is necessarily horribly singular: its local geometry in some sense encodes the global geometry of the whole Hilbert scheme.

The Kontsevich space. These drawbacks often make it difficult to study the global geometry of the Hilbert scheme. An alternative is the *Kontsevich space.* The Kontsevich space $\overline{M}_{g,0}(\mathbb{P}^r, d)$ parametrizes what are called *stable maps* of degree d and genus g to $\mathbb{P}^r$. These are morphisms

$$f : C \to \mathbb{P}^r$$

with C a connected curve of arithmetic genus g having only nodes as singularities, such that the image $f_*[C]$ of the fundamental class of C is equal to d times the class of a line in $A_1(\mathbb{P}^r)$, and satisfying the one additional condition that the automorphism group of the map f—that it, automorphisms ϕ of C such that $f \circ \phi = f$—is finite.

As with the Hilbert scheme, there are difficulties in using the Kontsevich space:

(1) **It has extraneous components**. These arise in a completely different way from the extraneous components of the Hilbert scheme, but they're there. A typical example of an extraneous component of the Kontsevich space $\overline{M}_g(\mathbb{P}^r, d)$ would consist of maps $f : C \to \mathbb{P}^r$ in which C was the union of a rational curve $C_0 \cong \mathbb{P}^1$, mapping to a rational curve of degree d in $\mathbb{P}^r$, and C_1 an arbitrary curve of genus g meeting C_0 in one point and on which f was constant; if the curve C_1 does not itself admit a nondegenerate map of degree d to $\mathbb{P}^r$, this map can't be smoothed.

So, using the Konsevitch space rather than the Hilbert scheme doesn't solve this problem, but it does provide a frequently useful alternative: there are situations where the Kontsevich space has extraneous components and the Hilbert scheme not—like the case of plane cubics described above—and also situations where the reverse is true, such as the case of twisted cubics.

(2) **No one knows what's in the closure of the locus of smooth curves**. This, unfortunately, remains an issue with the Kontsevich space. Even in the case of the space $\overline{M}_g(\mathbb{P}^2, d)$ parametrizing plane curves, where it might be hoped that the Kontsevich space would provide a better compactification of the Severi variety than simply taking its closure in the space $\mathbb{P}^N$ of all plane curves of degree d, the fact that we don't know which stable maps are smoothable represents a real obstacle to its use.

(3) **It has points corresponding to highly singular schemes, and these tend to be in turn highly singular points of $\overline{M}_g(\mathbb{P}^r, d)$**. Still true; but in this respect, at least, it might be said that the Kontsevich space represents an improvement over the Hilbert scheme: even when the image $f(C)$ of a stable map $f : C \to \mathbb{P}^r$ is highly singular, the fact that the domain of the map is at worst nodal makes the deformation theory of the map relatively tractable.

Now, in the last ten years or so there has been a great deal of work on alternative compactifications of the moduli space M_g of smooth curves—moduli spaces of abstract curves that are allowed to have singularities other than nodes, such as cusps, tacnodes, etc. Recently, this has been done as well for parameter spaces, like the Kontsevich space. Dawei Chen, Izzet Coskun and Charley Crissman [1], for example, have investigated alternative birational models of the Kontsevich spaces of rational curves in $\mathbb{P}^r$ from the point of view of the minimal model program, and have found that many of the spaces arising in this way are again parameter spaces.

Another recent development is the work of Michael Viscardi [2], who considered modifications of the functor coarsely represented by the Kontsevich space. Basically, he constructs a parameter space $M_1^{[m]}(\mathbb{P}^r, d)$ for m-stable maps, where the domain curve is allowed to have an elliptic ℓ-fold point for any $\ell \leq m$. The remarkable fact is that this modification in effect gets rid of the extraneous components: he finds that for m sufficiently large, this space is irreducible.

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