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Geometric Group Theory, Hyperbolic Dynamics and Symplectic Geometry

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Abstract. The main theme of the workshop is the interaction between the speedily developing fields of mathematics mentioned in the title. One of the purposes of the workshop is to highlight new exciting developments which are happening right now on the borderline between hyperbolic dynamics, geometric group theory and symplectic geometry.

Mathematics Subject Classification (2000): 20Fxx, 22E40, 22E46, 22F10, 37D05, 37D40, 37K65, 53C22, 53Dxx.

Introduction by the Organisers

Most of the talks of the workshop largely included different aspects related to the main themes of the conference. There were talks given by the well-known specialists as well as by young participants (recent PhD or postdoctoral students). The scientific atmosphere was very fruitful. Many interesting scientific discussions between participants certainly generated new ideas for the further research. The following report contains extended abstracts of the presented talks. By continuing our tradition we included an updated list of open important problems related to the workshop.

Workshop: Geometric Group Theory, Hyperbolic Dynamics and Symplectic Geometry

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Abstracts

Spectral Representations, Archimedean Solids, and finite Coxeter Groups

Norbert Peyerimhoff
(joint work with Ioannis Ivrissimtzis)

We consider finite, connected and simple (i.e., no loops and multiple edges) combinatorial graphs $G = (V, E)$, with a group $\Gamma \subset Aut(G)$ acting transitively on the set of vertices V. This induces the following equivalence relation on the set of (undirected) edges E: $\{v_1, w_1\} \sim_\Gamma \{v_2, w_2\}$ iff there exists a $\gamma \in \Gamma$ with $\{v_2, w_2\} = \{\gamma v_1, \gamma v_2\}$. We are particularly interested in the case when we have several equivalence classes $[e_1], \dots, [e_N]$ of edges. The multiplicity m_j of an equivalence class $[e_j]$ is defined to be the number of edges in this equivalence class meeting at an arbitrarily chosen vertex $v_0 \in V$. We associate positive weights $x_1, \dots, x_N > 0$ to the equivalence classes of edges and introduce the *simplex of weights*

$$\Delta_\Gamma := \{X = (x_1, \dots, x_N) \in (0, \infty)^N \mid \sum m_j x_j = 1\}.$$

Each point $X \in \Delta_\Gamma$ can be considered as a choice of transition probabilities for a Γ-equivariant random walk, and we consider the corresponding symmetric operator A_X, acting on the space of functions f on the vertices and given by

$$A_X f(v) = \sum_{w \sim v} x_{\{v,w\}} f(w).$$

Note that A_X can be interpreted as a doubly stochastic matrix and its second highest eigenvalue $\lambda_2 < 1$ can be related to the mixing rate of the corresponding random walk.

Assume the operator A_X has an eigenvalue $\lambda(X) \in (-1, 1)$ of multiplicity $k \geq 2$. The idea of a *spectral representation* is to use this eigenspace to construct a "geometric realisation" of the combinatorial graph G in Euclidean space $\mathbb{R}^k$, by using an orthonormal base $\phi_1, \dots, \phi_k$ of eigenfunctions of the eigenspace $E_{\lambda(X)}$ and defining the map

$$\Phi_X : V \to \mathbb{R}^k, \quad \Phi_X(v) = (\phi_1(v), \dots, \phi_k(v)).$$

Vertex transitivity forces the image of Φ_X to lie on a round sphere S^{k-1} and that pairs of vertex-pairs, belonging to equivalent edges, are mapped to pairs of Euclidean segments with the same Euclidean length. We call a spectral representation $\Phi_X : V \to \mathbb{R}^k$ *faithful* if Φ_X is injective, and *equilateral* if all images of edges have the same Euclidean lenghts (not only images of equivalent edges). We have the following general result:

Theorem 1. *Let $G = (V, E)$ be a finite, connected, simple graph. Let $\Gamma \subset Aut(G)$ be vertex transitive, $U \subset \Delta_\Gamma$ be an open set and $\lambda : U \to (-1, 1)$ be a smooth function such that $\lambda(X)$ is an eigenvalue of A_X with fixed multiplicity $k \geq 2$ for* *all $X \in U$. If $X_0 \in U$ is a* critical point *of λ, then the spectral representation $\Phi_X : V \to S^{k-1}$ is equialateral.*

Note that the above result holds for any eigenvalue and is not restricted to λ_2. Henceforth, we focus particularly on λ_2. The following result is straightforward and has a simple proof using the Rayleigh quotient:

Lemma 1. *Let $\Gamma \subset G$ be vertex transitive. Then $\lambda_2 : \Delta_\Gamma \to (-1, 1)$ is a convex function. If the graph G decomposes in multiple components by removing all edges of an equivalence class $[e_j]$, then $\lambda_2(X) \to 1$, as X approaches the face of Δ_Γ corresponding to $x_j = 0$.*

Lovász and Schrijver prove in [4] faithfulness of 3-dimensional spectral representations of λ_2 for 3-connected planar (not necessarily vertex transitive) planar graphs. Their considerations are related to Colin de Verdiére's graph invariant.

Next, we consider a special class of planar vertex transitive graphs: 1-skeletons of Archimedean solids. We focus on the 1-skeleton of the largest Archimedean solid, namely the *truncated icosidodecahedron* with vertex configuration $(4, 6, 10)$, which is bipartite and has 120 vertices. Our first result reads as follows (similar results holds for the Archimedean solid with vertex configuration $(4, 6, 8)$):

Theorem 2. *The G be the 1-skeleton of the truncated icosidodecahedron, $\Gamma = Aut(G)$. The simplex of weights is given by*

$$\Delta_\Gamma = \{(x, y, z) \mid x, y, z > 0, x + y + z = 1\},$$

where x, y, z are the weights on the edge-equivalence classes separating 4- and 6-gons, 4- and 10-gons, and 6- and 10-gons, respectively. Then $\lambda_2 : \Delta_\Gamma \to (0, 1)$ is analytic and strictly convex, $\dim E_{\lambda_2(X)} = 3$ for all $X \in \Delta_\Gamma$, and

- *$\lambda_2(X) \to 1$ as $X \to \partial\Delta_\Gamma$,*
- *$X_0 = \frac{1}{14+5\varphi}(6 + 2\varphi, 5, 3 + 3\varphi)$ is the unique minimum of λ_2, and*

$$\lambda(X_0) = \frac{10 + 7\varphi}{14 + 5\varphi},$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Moreover, $\Phi_{x_0} : V \to S^2$ is faithful and equilateral.

The proof of this result uses our previous results, the irreducible representations of $\Gamma = A_5 \times \mathbb{Z}_2$, and the fact that G is the Cayleygraph of the finite Coxeter group H_3.

Canonical Laplacians (with all weights being equal) for finite Coxeter graphs have been considered in earlier papers by [1], [2], and [3]. These sources studied the explicit value of λ_2 and its multiplicity. We consider a finite, irreducible Coxeter group

$$\Gamma = \langle s_1, \dots, s_k \mid (s_i s_j)^{m_{ij}} = e \rangle,$$

together with its geometric realisation $\Gamma \hookrightarrow O(k)$ as finite reflection group with simple roots $n_1, \dots, n_k$. Let $P_j = (-1)^{j-1} n_1 \times \cdots \times \widehat{n}_j \times \cdots \times n_k$ ($\widehat{n}_j$ means that this term is dropped) be the directions of the extremal rays of its conic fundamental domain $\mathcal{F} \subset \mathbb{R}^k$, and let $\mathcal{F}_0 = \mathcal{F} \cap S^{k-1}$. We construct a diffeomorphism $\Psi_\Delta : \mathcal{F}_0 \to \Delta_\Gamma$ (where Δ_Γ is the simplex of weights of the Cayleygraph $Cay(\Gamma, S = \{s_1, \ldots, s_k\}))$ and a map $\Psi_\lambda : \mathcal{F}_0 \to (0,1)$, such that the following holds:

Proposition 1. *Let $(\Gamma, S) \hookrightarrow O(k)$ be a finite, irreducible Coxeter group, and $\pi_j : \mathbb{R}^k \to \mathbb{R}$ be the canonical projection on the j-th coordinate. Then, for $P = [\alpha_1 P_1 + \cdots + \alpha_k P_k] \in \mathcal{F}_0$, the functions*

$$\phi_j(\gamma) = \pi_j(\gamma P)$$

form a system of linearly independent eigenfunctions of A_X on $Cay(\Gamma, S)$ to the eigenvalue $\lambda = \Psi_\lambda(P)$, where $X = \Psi_\Delta(P)$. Moreover,

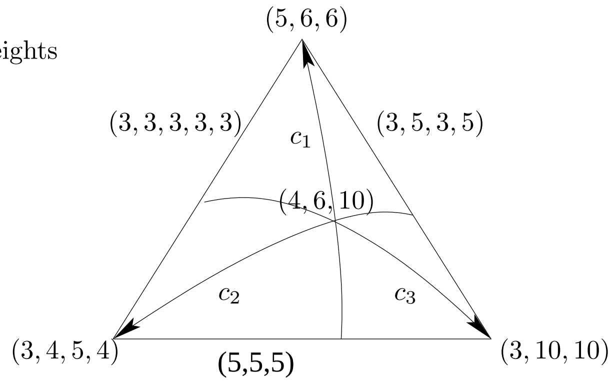
$$\Phi_X(\gamma) = (\phi_1(\gamma), \ldots, \phi_k(\gamma)) = \gamma P$$

is faithful and the Euclidean lenghts of the equivalence classes of edges are given by $\|P - \sigma_j(P)\| = 2\alpha_j \det(n_1, \ldots, n_k)$.

We also prove in the case $\Gamma = H_3$ that $\Psi_\Delta^{-1} \circ \Psi_\lambda = \lambda_2$, and conjecture that this is true for all finite, irreducible Coxeter groups.

Coming back to the 1-skeleton of the Archmidean Solid $(4, 6, 10)$, the diagram below illustrates the convergence behavior of spectral representations of λ_2, as we approach the boundary $\partial\Delta_\Gamma$. For weights in each of the three curves c_j, exactly two of the three equivalence classes of Euclidean edges have the same length (we can parametrise these curves explicitly). The intersection point of the curves is the global minimum X_0 of λ_2. If we choose a sequence of weights converging along one of the curves into a vertex of $\partial\Delta_\Gamma$, then the spectral representations converge to an equilateral representation of a simpler Archimedean solid with only two equivalence classes of edges. If we choose a sequence of weights converging towards the interior of an edge of $\partial\Delta_\Gamma$, the spectral representations converge to an equilateral representation of an Archimdedean or Platonic solid with only one equivalence class of edges. The solids are represented by their vertex configurations.

Simplex Δ_Γ of weights



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A brief survey of contact homology

Frédéric Bourgeois

Contact homology [1] is a powerful invariant for contact manifolds, with applications and ramifications in many directions. The aim of this talk is to give an overview of this technique and to illustrate its applications with some selected results.

Three versions of contact homology are described. All of these are obtained as the homology of a chain complex generated by closed orbits of a Reeb vector field for the contact structure, satisfying some index property (good orbits). The differentials of these complexes count suitable types of J-holomorphic spheres in the symplectization of the contact manifold.

First, the (full) contact homology is defined as the homology of a differential graded algebra with a unit, freely generated by the orbits, and the relevant holomorphic curves have one positive puncture and an arbitrary number of negative punctures. The resulting object is often too large for practical computations and applications.

Second, the cylindrical contact homology is obtained by replacing the algebra with the module generated by the orbits and by restricting to holomorphic cylinders with one positive and one negative punctures. For this object to be well-defined, the closed Reeb orbits need to satisfy some ad hoc conditions.

Third, the linearized contact homology is defined without the need of conditions as above, but with an augmentation of the differential graded algebra. This augmentation can be naturally obtained from a symplectic filling of the contact manifold.

The first application of contact homology is to distinguish contact structures. In this vein, one can prove the existence of infinitely many pairwise non diffeomorphic contact structures with the same classical invariants on manifolds such as S^{4k+1}, T^5 or $T^2 \times S^{2n-3}$.

Contact homology can also be used to study the Weinstein conjecture (stating that any Reeb field on a closed contact manifold has at least a periodic orbit) or more quantitative versions of this conjecture. Hofer, Wysocki and Zehnder showed that on the standard contact 3-sphere, there are either 2 or infinitely many simple closed Reeb orbits. In a joint result with Cieliebak and Ekholm, we showed that linearized contact homology can detect this alternative.

In the case of a unit cotangent bundle, Cieliebak and Latschev [2] proved that contact homology is isomorphic to the homology of the free loop space of the base modulo parametrization shift and constant loops. This is to be compared with an earlier result showing the isomorphism of symplectic homology of the corresponding cotangent bundle with the homology of the loop space of the base. The relation between contact and symplectic homology observed here is in fact a general phenomenon. In a joint work with Oancea, we proved that linearized contact homology is isomorphic to the (positive part of) S^1-equivariant symplectic homology and fits into a Gysin sequence with the ordinary symplectic homology.

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Models of 3-manifolds and hyperbolicity

Brian H. Bowditch

Let Σ be a closed surface of genus $g \geq 2$, and let Γ be the mapping class group of Σ. Let $\mathcal{T}$ be the Teichmüller space of Σ — the space of marked hyperbolic structures on Σ. This is homeomorphic to $\mathbf{R}^{6g-6}$. It admits a natural compactification to a ball, by adjoining the $(6g-7)$-sphere of projective laminations, $\partial\mathcal{T}$. This is the "Thurston compactification". Teichmüller space admits a number of natural metrics, notably the Teichmüller metric (a complete Finsler metric) and the Weil-Petersson metric (an incomplete geodesically convex negatively curved riemannian metric). The mapping class group acts properly discontinuously on $\mathcal{T}$, with quotient the moduli space — the space of unmarked hyperbolic structures. One aim is to understand the large scale geometry of $\mathcal{T}$ in these various metrics.

Let C be the set of homotopy classes of curves in Σ. (This is the vertex set of the curve complex.) Given $\alpha \in C$ and $\sigma \in \mathcal{T}$, we write $l_\sigma(\alpha)$ for the length of the geodesic realisation of α in Σ in the hyperbolic metric σ. Given $\eta > 0$, let

$$\mathrm{thin}(\mathcal{T}, \alpha) = \{\sigma \in \mathcal{T} \mid l_\sigma(\alpha) < \eta\}$$

.

$$\mathrm{thin}(\mathcal{T}) = \bigcup_{\alpha \in C} \mathrm{thin}(\mathcal{T}, \alpha)$$

and

$$\mathrm{thick}(\mathcal{T}) = \mathcal{T} \setminus \mathrm{thin}(\mathcal{T}).$$

We remark that, if η is chosen smaller than the Margulis constant, then given $A \subseteq C$, $\bigcap_{\alpha \in A} \mathrm{thin}(\mathcal{T}, \alpha) \neq \emptyset$ if and only if the elements of A are disjoint in Σ. (In this way, we can think of the curve complex as the nerve of the pieces of the thin part of Σ.) We also note that the quotent of $\mathrm{thick}(\mathcal{T})$ by the mapping class group is a compact subset of moduli space. As a result, any two invariant riemannian metrics on $\mathrm{thick}(\mathcal{T})$ are bilipschitz equivalent. (It also follows that $\mathrm{thick}(\mathcal{T})$ is quasi-isometric to the mapping class group.) We can therefore refer to lipschitz paths in $\mathrm{thick}(\mathcal{T})$ without specifically identifying a metric. To understand the large scale geometry of $\mathcal{T}$, one wants to understand the geometry of the thick part and how the various pieces of the thin part fit together.

Let $\pi : \mathbf{R} \longrightarrow \text{thick}(\mathcal{T})$ be a lipschitz path. We can associate to π a complete riemannian manifold, P, homeomorphic to $\Sigma \times \mathbf{R}$, where the structure on the "horizontal" fibre $\Sigma \times \{t\}$ corresponds to the structure $\pi(t)$ on Σ. This is well defined up to bilipschitz equivalence, so we can denote it by $P(\pi)$. Given another such path, π', we write $\pi \approx \pi'$ to mean that there is a bilipschitz reparameterisation, $s : \mathbf{R} \longrightarrow \mathbf{R}$, such that the distance between $\pi'(t)$ and $\pi(s(t))$ is bounded for all $t \in \mathbf{R}$. (In other words, the paths π and π' "fellow travel".) In this case, $P(\pi)$ and $P(\pi')$ are bilipschitz equivalent. It was shown independently by Mosher and myself that the universal cover, $\tilde{P}(\pi)$, is Gromov hyperbolic if and only if $\pi \approx \pi_0$ for some Teichmüller geodesic π_0. The "flaring condition" of Bestvina and Feighn gives a criterion for recognising when a manifold of the form $\tilde{P}(\pi)$ is Gromov hypebolic in terms of the uniform divergence of "vertical fibres".

An example of such a path, π, arises from a doubly degenerate hyperbolic 3-manifold, $M \cong \Sigma \times \mathbf{R}$, of positive injectivity radius. It follows from the work of Thurston and Bonahon that M is bilipschitz equivalent to a manifold of the form $P(\pi)$ for some proper bi-infinite path π. Moreover, π has well defined distinct limit points, λ^+ and λ^- in $\partial\mathcal{T}$, the "end invariants" of M. Note that $\tilde{P}(\pi)$ is Gromov hyperbolic, and so $\pi \approx \pi_0$ where π_0 is the Teichmüller geodesic with endpoints, λ^+ and λ^-. It follows that M is bilipschitz equivalent to the "model" $P(\pi_0)$. If M' is another such manifold with the same end invariants, then it follows that M and M' are bilipschitz equivalent to each other. It in turn follows from the theory of Ahlfors, Bers, Sullivan et al. that they must be isometric. This gives the ending lamination conjecture for such manifolds, as proven by Minsky.

The general case of the ending lamination conjecture, allowing for arbitrarily short curves, was proven by Brock, Canary and Minsky. In this case, short geodesics give rise to Margulis tubes in M, so one cannot associate a path in Teichmüller space in such a direct way. Minsky described a combinatorial model in this case. It turns out that one can also characterise the essential properties of such a model in terms of Gromov hyperbolicity, as follows. Suppose that P is a complete reimannian manifold diffeomorphic to $\Sigma \times \mathbf{R}$, and satifying:

(1) All sectional curvatures of P lie between two constants.

(2) Any lipschitz map of a circle into $\tilde{P}$ extends to a lipschitz map of a disc, and any lipschitz map of a 2-sphere extends to a lipschitz map of a ball.

(3) There is some $\eta > 0$ such that if $x \in P$ is contained in an essential curve of length at most η, then then P has all sectional curvatures at x equal to -1.

(4) $\tilde{P}$ is Gromov hyperbolic.

Then one can classify the ends of P as "geometrically finite" or "simply degenerate", similarly as in the work of Thurston/Bonahon for hyperbolic 3-manifolds. To a degenerate end we can associate an end invariant in $\partial\mathcal{T}$. If we assume that P is doubly degenerate, then it has two end invariants which must be distinct. If

P' is another such manifold, with the same pair of end invariants, the $\tilde{P}$ and $\tilde{P}'$ are equivariantly quasi-isometric.

Of the above conditions, (3) is somewhat unnatural — it is equivalent to asserting that the "thin part" of P is a disjoint union of constant curvature Margulis tubes. However, it would hold for many of the standard ways one might want to construct a model. Again, the hyperbolicity condition can be interpreted as a flaring condition on vertical fibres, though the statement becomes more technical. Note a hyperbolic 3-manifold automatically satisfies all the above, so it implies the ending lamination conjecture for product manifolds.

There are a number of ways one could construct potential models, for example, starting with Weil-Petersson geodesics. It would be interesting to know if these satisfy such a flaring condition, and thereby give rise to models of hyperbolic 3-manifolds.

Lagrangian submanifolds: their fundamental group and Lagrangian cobordism

Octav Cornea
(joint work with Paul Biran)

1. setting

Given a symplectic manifold (M^{2n}, ω), a Lagrangian submanifold $L \subset (M, \omega)$ is an n-dimensional submanifold so that $\omega|_M = 0$. All Lagrangians discussed here are assumed closed. Such a Lagrangian is called monotone if the two morphisms $\omega : \pi_2(M, L) \to \mathbb{R}$ and $\mu : \pi_2(M, L) \to \mathbb{Z}$, the first given by integrating the symplectic form ω and the second by the Maslov class, are proportional with a positive constant of proportionality. Whenever refering to a monotone Lagrangian we implicitly also assume that the minimal Maslov number

$$N_L = \min\{\mu(x) \mid x \in \pi_2(M, L),\ \omega(x) > 0\}$$

verifies $N_L \geq 2$. There are many examples of such Lagrangians: $\mathbb{R}P^n \subset \mathbb{C}P^n$, the Clifford torus $\mathbb{T}^n_{Cliff} \subset \mathbb{C}P^n$ and many others.

An important property of this class of Lagrangians is that Floer homology $HF(L, L)$ is defined (by early work of Oh [7]). More recently, various quantum structures of monotone Lagrangians have been discussed in [3] and [4].

2. Lagrangian cobordism

Assume that $L_i \subset (M, \omega)$, $1 \leq i \leq k$ are closed connected Lagrangian submanifolds and consider also a second such set of Lagrangian submanifolds, $L'_j \subset (M, \omega)$, $1 \leq j \leq h$.

Definition. We say that $(L_i)_{1 \leq i \leq k}$ is cobordant to $(L'_j)_{1 \leq j \leq h}$ if there exists a smooth, connected, cobordism $(V; \coprod_i L_i, \coprod_j L'_j)$ and a Lagrangian embedding

$$V \hookrightarrow (M \times T^*[0,1], \omega \oplus \omega_0)$$

so that

$$V|_{M\times[0,+\epsilon)} = \coprod_i L_i \times [0,\epsilon) \times \{i\} \ , \ V|_{M\times(1-\epsilon,1]} = \coprod_j L'_j \times (1-\epsilon,1] \times \{j\}$$

where $T^*[0,1] = [0,1] \times \mathbb{R}$ and $\epsilon > 0$ is very small.

It is useful to imagine a cobordism V as extended - trivially - to $T^*\mathbb{R} \supset T^*[0,1]$ and viewed as a non-compact manifold with k-cyclindrical ends to the left and h-cyclindrical ends to the right. A Lagrangian cobordism with $h = k = 1$ will be called an *elementary cobordism*.

Lagrangian cobordisms have been introduced, in a slightly different setting, by Arnold [1]. They have been studied by Audin [2], Eliashaberg [6] as well as by Chekanov [5]. The results obtained on this topic have been somewhat contrasting. On one side, the results of Eliashberg together with the calculations of Audin and combined with the Lagrangian surgery technique (see for instance Polterovich [8]) show that general Lagrangian cobordism is very flexible. The argument for this is roughly as follows: as shown by Eliashberg, if one considers the notion of immersed Lagrangian cobordism that corresponds to requiring V above to be only an immersion and not an embedding, then, by an application of the Gromov h-principle, classifying Lagrangians up to immersed cobordism is a purely algebraic topology question and is computable by classical homotopy theoretical techniques. At the same time, by surgery, any immersed cobordism between two embedded Lagrangians can be transformed in an embedded cobordism. On the other hand, using J-holomorphic techniques Chekanov's result shows a certain form of rigidity for monotone cobordisms. By definition, a cobordism V as above is monotone if V itself is a monotone Lagrangian. Chekanov's argument essentially shows that the number (mod 2) of J-holormophic disks passing through a point on any of the manifolds V_i or V'_j is the same, independently of i, j, J and the point in question, whenever $(V_i)_{1\leq i\leq k}$ is cobordant to $(V_j)_{1\leq j\leq h}$.

3. Results on cobordism

We list here a number of results of increasing degree of generality - we caution the reader that, at this time, this is still work in in progress.

Theorem 1. *Any monotone elementary Lagrangian cobordism $(V; L, L')$ is a quantum h-cobordism in the sense that for an appropriately defined relative quantum homology we have $QH(V,L) \cong QH(V,L') = 0$. In particular, there exists a ring isomorphism (depending on V):*

$$QH(L) \cong QH(L') \ .$$

In this result the coefficients are in $\mathbb{Z}_2$ in general and in $\mathbb{Z}$ if the cobordism V is oriented and carries a spin structure. The same convention is implicitly understood for the statements below.

It is useful to note at this time that there are examples of Lagrangian cobordisms as above that are not topological h-cobordisms (in the sense that singular homology $H(V, L) \neq 0$).

We also recall here that the quantum homology of a Lagrangian is the homology of the "pearl" chain complex $\mathcal{C}(L; f, \langle,\rangle, J)$ where $f : L \to \mathbb{R}$ is a Morse function on L, $\langle -, - \rangle$ is a generic metric on L and J is a generic almost complex structure on M. Its generators are the critical points of f and the differential counts configurations that combine Morse trajectories with J-holomorphic disks [3], [4].

Theorem 2. *Let $(V; (L_i)_{1\leq i\leq k}, L')$ be a monotone Lagrangian cobordism. Then for generic J and any Morse functions $f_i : L_i \to \mathbb{R}$ there are chain maps $\phi_i : \mathcal{C}(L_i; f_i, J) \to C_{i-1}$ where the chain complex C_{i-1} is the cone of the chain map ϕ_{i-1} for $i \geq 3$ and $C_1 = \mathcal{C}(L_1; f_1, J)$. Moreover, the chain complex $\mathcal{C}(L'; f, \langle,\rangle, J)$ is chain homotopy equivalent C_k.*

In other words, the chain homotopy type of L' can be recovered from that of the L_i's by an iterated cone-construction. A different and somewhat richer way to formulate this result is to say that the existence of the cobordism between L' and the family $(L_i)_{1\leq i\leq k}$ translates - inside the adequate Fukaya derived category - into the fact that the class of L' belongs to the subcategory spanned by the L_i's, $[L'] \in [L_1, \ldots, L_k]$ (the meaning of the Fukaya category used here appears in Seidel [9]).

There are two wide-reaching extensions of this result that are worth mentioning here. First, one can extend the theory to cobordisms in the total space of a Lefschetz fibration with basis $\mathbb{C}$ and with finitely many singular fibres. In this case the decomposition in Theorem 2 has to take into account also the vanishing cycles. In other words, in the Fukaya derived category language we have $[L'] = [L_1, \ldots, L_k, S_1, \ldots, S_r]$ where S_j's are the vanishing cycles of the fibration. Secondly, it is expected that most of the results here remain valid outside of the monotone category by using the appropriate algebraic formalism.

4. Application to the study of Fundamental groups of Lagrangians.

Not much is known in a systematic way concerning the following natural problem: *given a symplectic manifold (M, ω) what can be said about the class of groups G so that there exists a Lgrangian submanifold $L \subset (M, \omega)$ with $\pi_1(L) = G$.*

One way to approach this question is to analyze how the fundamental group of Lagrangians changes along Lagrangian cobordism. A first rigidity result in this direction is available.

Theorem 3. *Assume $(V; L, L')$ is a Lagrangian cobordism with L and L' connected. If $QH(L) \neq 0$, then the maps: $i : H_1(L; \mathbb{Z}_2) \to H_1(V; \mathbb{Z}_2)$ and $i' : H_1(L'; \mathbb{Z}_2) \to H_1(V; \mathbb{Z}_2)$ have the same image.*

When $\dim(L) = 2$ it is possible to say more: both i and i' are isomorphisms.

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Almost linear functionals on Lie algebras

Michael Entov

(joint work with Leonid Polterovich)

Let $\mathfrak{g}$ be a real Lie algebra. A function $\zeta : \mathfrak{g} \to \mathbb{R}$ is called a *Lie quasi-state* if its restriction to any abelian subalgebra is linear.

The interest to the notion of a Lie quasi-state is three-fold.

Lie quasi-states and quasi-morphisms on Lie groups: Recall that a *homogeneous quasi-morphism* on a group G is a function $\mu : G \to \mathbb{R}$ such that

- There exists $C > 0$ so that $|\mu(xy) - \mu(x) - \mu(y)| \leq C$ for all $x, y \in G$.
- $\mu(x^k) = k\mu(x)$ for all $k \in \mathbb{Z}$, $x \in G$.

It is known that restriction of any homogeneous quasi-morphism to an abelian subgroup is a genuine morphism, and that homogeneous quasi-morphisms are conjugation invariant. Therefore, given a continuous homogeneous quasi-morphism μ on a Lie group G, its pull-back to the Lie algebra $\mathfrak{g}$ by the exponential map,

$$\zeta : \mathfrak{g} \to \mathbb{R},\ a \mapsto \mu(\exp a)\ ,$$

is a continuous Ad_G-invariant Lie quasi-state.

Lie quasi-states and Gleason's theorem: Gleason's theorem [5] is one of the most famous and important results in the mathematical formalism of quantum mechanics. In the finite-dimensional setting the proof of Gleason's theorem yields the following result about Lie quasi-states.

Theorem 1 (Gleason)**.** *Any Lie quasi-state ζ on the Lie algebra $\mathfrak{u}(n)$, $n \geq 3$, which is bounded on a neighborhood of zero, is linear and has the form $\zeta(A) = tr(HA)$ for some $H \in \mathfrak{u}(n)$.*

Lie quasi-states in symplectic topology: As the third point of interest in Lie quasi-states, we note that such functionals on the infinite-dimensional Poisson-Lie algebra of Hamiltonian functions on a symplectic manifold appeared recently in symplectic topology and Hamiltonian dynamics before they were properly studied in the finite-dimensional setting. We refer to [1, 2, 3] for various aspects of this development.

The talk concerned the existence and uniqueness of (continuous) Lie quasi-states on various Lie algebras, covering the above-mentioned results as well as the following recent theorem of ours:

Theorem 2 ([4]). *Let $\mathfrak{g} = \mathfrak{sp}\,(2n, \mathbb{R})$, $n \geq 3$. Then the factor-space $\mathcal{Q}(\mathfrak{g})$ of the space of continuous Lie quasi-states on $\mathfrak{g}$ by $\mathfrak{g}^*$ (the dual space to $\mathfrak{g}$) is 1-dimensional.*

The generator ζ of $\mathcal{Q}(\mathfrak{g})$ looks as follows: its value on a matrix $B \in \mathfrak{sp}\,(2n, \mathbb{R})$ equals, roughly speaking, to the asymptotic Maslov index of the path e^{tB} as $t \to \infty$. The Lie quasi-state ζ comes from a continuous homogeneous quasi-morphism on the simply connected Lie group $G = \widetilde{Sp}\,(2n, \mathbb{R})$ and hence is Ad_G-invariant [4]. Moreover, the whole space of Ad_G-invariant Lie quasi-states on $\mathfrak{g} = \mathfrak{sp}\,(2n, \mathbb{R})$ is 1-dimensional and is generated by ζ (for any n) [4]. A similar result on Ad-invariant Lie quasi-states holds, in fact, for any Hermitian simple Lie algebra; Ad-invariant Lie quasi-states can be also completely described for compact Lie algebras – see [4]. The continuity assumption in Theorem 2 is essential:

Theorem 3 ([4]). *The space of (not necessarily continuous) Lie quasi-states on $\mathfrak{sp}\,(2n, \mathbb{R})$ which are bounded on a neighborhood of zero is infinite-dimensional for all $n \geq 1$.*

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Groups of interval exchange transformations

Koji Fujiwara

(joint work with François Dahmani, Vincent Guirardel)

Let $I = [0, 1)$ be an interval. Break it into finitely many subintervals, $[a, b)$, and rearrange those pieces to obtain the interval again. This is an *interval exchange transformation*. Let IET be the group of all interval exchange transformations of the interval. We want to know which groups can appear as subgroups of IET (cf. [1]). For example, J. Franks asked two years ago at Oberwolfach if one can find SO_3 in IET. We answer negatively.

Theorem 1. *If a subgroup of* IET *is isomorphic to a connected Lie group, then it is abelian.*

Next we show:

Theorem 2. *Every finitely generated subgroup of* IET *is the limit of a sequence of finite groups in the space of marked groups.*

Corollary 3. *Every finitely presented subgroup of* IET *is residually finite.*

Corollary 4. *Thompson's groups F (on the interval) and T (on a circle) are not subgroups of* IET.

We also find interesting subgroups:

Theorem 5. *There is a subgroup in* IET *generated by two elements which contains all finite groups. In particular this group is not linear.*

Theorem 6. *The lamplighter group L is a subgroup of* IET.

Recall that L is solvable and contains a free semigroup. It is still open if IET contains a free group of rank two (this is the other question Franks asked). We rather ask:

Question 1. *If $G <$ IET, then is G amenable ?*

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Geometry of convergence group actions

Victor Gerasimov
(joint work with Leonid Potyagailo)

1. A *star* is a tree which is a "cone over a nonempty finite set": it has one *central* vertex, and at least one *non-central*. Each non-central vertex is joined with the central vertex. Even when such a tree has just one edge we mark the central vertex.

A *star of groups* is a graph of groups being a star.

Theorem 1. *Let G be a group relatively hyperbolic with respect to a collection $\mathcal{P}$ of subgroups. Then G is the fundamental group of a star of groups whose set of non-central subgroups is $\mathcal{P}$ and the central group is finitely generated and relatively hyperbolic with respect to the collection of those edge group which are infinite.*

Remark. Without the hypothesis that the group is finitely generated some common definitions of relative hyperbolicity (r.h. for short) are not known to be equivalent [Hr08]. Furthermore, each of the definitions requires or implies that the group is countable. We adopt a definition that impose no restriction on the cardinality and for countable groups is equivalent to Gromov's definition. For example, a free product of two arbitrary group is r.h. with respect to the factors.

We call a group *relatively hyperbolic* [Ge09] if it possesses an action on a compactum T with at least 3 points such that the induced action on the space $\mathbf{\Theta}^3 T$ of subsets of cardinality 3 is properly discontinuous, the space $\mathbf{\Theta}^2 T/G$ is compact, and the limit set $\mathbf{\Lambda} G$ has at least 2 points. Remark: the countability of G is equivalent to the metrisability of T.

2. Let G be a finitely generated r.h. group with respect to $\mathcal{P}$. Denote by $\mathsf{GB}(G, \mathcal{P})$ the Gromov-Bowditch completion[Gr87],[Bo97] of G. This is a compactum containing G as a discrete dense open subset. The group G acts on $\mathsf{GB}(G, \mathcal{P})$ discontinuously on triples (i.e. with "convergence property") and cocompact on pairs. Denote by $\mathsf{St}_G \mathfrak{p}$ the stabilizer in G of a point $\mathfrak{p}$.

Denote by $\mathsf{Fl}(G)$ the Floyd's completion [Fl80] of G.

Proposition[Ge10]. *The identity map $G \to G$ extends to a continuous equivariant map $\varphi : \mathsf{Fl}(G) \to \mathsf{GB}(G, \mathcal{P})$ non-ramified over the conical points.*

Theorem 2. *For every parabolic point $\mathfrak{p} \in \mathsf{GB}(G, \mathcal{P})$ the set $\varphi^{-1}\mathfrak{p}$ is canonically homeomorphic to $\mathsf{Fl}(\mathsf{St}_G \mathfrak{p})$.*

This is the full generalization of Floyd's theorem [Fl80] for Kleinian groups. A weaker result was obtained in [GP09].

3. On the Cayley graph of a f.g. relatively hyperbolic group G along with the usual graph distance d one uses the geometry F of Farb's "conned-off" graph and the geometry GB of "Gromov-Bowditch space" when the distance between points x, y in an "horosphere" is changed from n to $\log(1+n)$. For every geodesic in the metrics F or GB one naturally defines its *lift* by joining any points x, y in an horosphere by a d-geodesic segment.

A *distortion function* is any non-decreasing function $\alpha : \mathbb{N}=\{0, 1, \dots\} \to \mathbb{R}_{\geqslant 0}$. Given two graphs Γ, Δ a map $f : \Delta^0 \to \Gamma^0$ between the sets of vertices is called α-distorted if
(1) $\mathsf{dist}(f(x), f(y))=n \implies \mathsf{dist}(x, y)\leqslant\alpha(n)$ and
(2) $\mathsf{dist}(f(x), f(y))\leqslant\alpha(1)$ if $\mathsf{dist}(x, y)=1$.

Lemma. *The lifts of geodesics in the geometries* F *and* GB *are* α*-distorted where* α *is a polynomial of degree two.*

For a Floyd scaling function f and a vertex v of a connected graph Γ denote by $\boldsymbol{\delta}_{v,f}$ the Floyd metric on Γ centered at v determined by f. Our main geometric tool is the following "generalized Karlsson lemma" (see [Ka03])

Let f be a scaling function and α be a distortion function such that $\sum_n \alpha_{2n} f_n\leqslant\infty$. Then, for every $\varepsilon>0$ there exists K such that for every connected graph Γ, a vertex v and an α-distorted path $\gamma : I \to \Gamma^0$, if $\mathsf{d}(v, \mathsf{Im}\gamma)>K$ then the $\boldsymbol{\delta}_{v,f}$-length of γ is less than ε.

We fix f and α from the above lemma.

Theorem 3 (compare with [BDM09]). *Let $\psi : H \to G$ be α-distorted map between f.g. groups. Suppose that G is r.h. modulo $\mathcal{P}$. Then H is r.h. modulo some collection $\mathcal{Q}$ (the case $\mathcal{Q}=\{H\}$ is allowed) and ψ maps each $Q\in\mathcal{Q}$ into a uniformly bounded neighborhood of a right coset gP of some $P\in\mathcal{P}$.*

4. Let a group G act on a compactum T discontinuously on triples. A subgroup H of G is *dynamically quasiconvex* if for every entourage $\mathbf{u}$ of T the set $\{g\in G : g{\cdot}\boldsymbol{\Lambda}H$ is not $\mathbf{u}$-small$\}/H$ is finite. D. Osin [Os06] conjectured that for r.h. groups the dynamical quasiconvexity is equivalent to the geometric relative quasiconvexity.

Theorem 4. *Osin's conjecture is true.*

5. As an application of the generalized Karlsson lemma we give simple proofs of Yaman theorem [Ya04] and Drutu-Sapir theorem "Morse property of r.h. groups" [DS05, theorem 1.12].

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No planar billiard has open set of quadrangular trajectories

Alexey Glutsyuk
(joint work with Yuri Kudryashov)

Consider the Dirichlet problem for the Laplace equation in a bounded domain of a Euclidean space. The asymptotic behavior of the number of its eigenvalues that are smaller than a given number N, as N tends to infinity, was studied by many mathematicians. In 1911 H. Weyl had found its first asymptotic term and stated a conjecture about the second asymptotic term. In 1980 the Weyl's conjecture about the second asymptotic term was proved by V.Ivrii modulo the following Ivrii's conjecture: in every billiard with piecewise-smooth boundary the set of periodic trajectories has Lebesgue measure zero. In 1989 M. Rychlik proved that in every planar billiard the set of triangular trajectories has measure zero. In 1994 Ya. Vorobets proved the same result in any dimension. We show that in any piecewise-smooth planar billiard (with sufficiently smooth pieces of boundary) the set of quadrangular orbits has measure zero.

Isomorphism problem for relative hyperbolic groups

Vincent Guirardel
(joint work with François Dahmani)

The isomorphism problem for a class of groups $\mathcal{C}$ asks for an algorithm that takes as input two presentations of groups G, G' in $\mathcal{C}$, and which decides whether G is isomorphic to G'. This is known to be unsolvable for the class of all finitely presented groups since the 50's [Ady55, Rab58]. In fact, the isomorphism problem is unsolvable for some very natural classes of groups, including the class of free-by-free groups (Miller [Mil71]), the class of [free abelian]-by-free groups (Zimmermann [Zim85]) or the class of finitely presented solvable groups of derived length 3 (Baumslag-Gildenhuys-Strebel [BGS85]).

On the positive side, the isomorphism problem is known to be decidable for the class of nilpotent groups and virtually polycyclic groups (Grunewald-Segal [GS80], Segal [Seg90]), and, following Sela, for the class of hyperbolic groups ([Sel95, DG08, DG10]), and toral relatively hyperbolic groups [DG08]. As a corollary, Dahmani and Groves give a solution to the isomorphism problem for fundamental groups of hyperbolic manifolds with finite volume [DG08].

In pinched variable negative curvature, the parabolic subgroups are virtually nilpotent instead of virtually abelian. Our initial motivation is to generalize this solution for fundamental groups of such manifolds, and more generally, to a class of relative hyperbolic groups with virtually nilpotent parabolic subgroups. However, one cannot rely on the same approach. Indeed, the solutions to the isomorphism problem for classes of hyperbolic and relative hyperbolic groups mentioned above fundamentally rely on a solution of the equations problem in these groups. But this problem is known to be unsolvable in the class of nilpotent groups [Rom79].

Instead, our strategy is to use Dehn filling theorems by Groves-Manning and Osin [GM08, Osi07] to produce sequences of canonical hyperbolic quotients of the given groups, and then to use our solutions of the isomorphism problem for hyperbolic groups with torsion to compare these Dehn fillings. The success of this approach might be surprising since there exists non-isomorphic groups having the same finite quotients, even among nilpotent groups.

For simplicity, we state our main result with only one parabolic subgroup, but the obvious generalization with several parabolic subgroups also holds.

Theorem 0.1. *Given two finite presentations of groups* $G_1 = \langle S_1 | R_1 \rangle$, $G_2 = \langle S_2 | R_2 \rangle$, *and two finite generating sets of subgroups* $P_1 < G_1$, $P_2 < G_2$ *such that*

- G_i *is a non-elementary relatively hyperbolic with parabolic group* P_i,
- G *does not split relative to* P_i *over an elementary subgroup,*
- P_i *is residually finite*

one can decide if there exists an isomorphism $f : G_1 \to G_2$ *sending* P_1 *to a conjugate of* P_2.

Here, an elementary subgroup of G_i is a group that is either virtually cyclic, or contained in a conjugate of P_i. The first assumption therefore asks that $G_i \neq P_i$, and G_i not virtually cyclic.

In many situations, building on work by Dahmani, one can find the parabolic subgroups. For instance, we get:

Theorem 0.2. *The isomorphism problem is solvable for the class of relative hyperbolic groups with virtually polycyclic parabolic groups, and which do not split over virtually polycyclic groups relative to their non virtually cyclic parabolic subgroups.*

In particular, the isomorphism problem is solvable for the class of fundamental groups of manifolds with pinched negative curvature and finite volume.

This also applies to parabolic groups in the class $\mathcal{C}$ of semi-direct products $F_r \ltimes \mathbb{Z}^n$ with $r, n \geq 2$. Since the isomorphism problem in $\mathcal{C}$ is unsolvable, the following corollary might be surprising.

Corollary 0.3. *The isomorphism problem is solvable for the class of non-elementary relative hyperbolic groups with parabolic groups in* $\mathcal{C}$ *that do not split over an elementary subgroup, relative to its parabolic subgroups.*

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Non-coherence of lattices

Michael Kapovich

Recall that a group Γ is called *coherent* if every finitely-generated subgroup of Γ is finitely-presented. Coherence is known for various families of groups. In particular, surface groups are coherent since every subgroup in a surface group is either free or has finite index. 3-manifold groups are coherent by a theorem of P. Scott [12]. In particular, lattices in $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$ are coherent.

This paper is motivated by the following

Conjecture 1. *Let G be a semisimple Lie group which is not locally isomorphic to $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$. Then every lattice in G is noncoherent.*

In the case of lattices in $O(n,1)$, this conjecture is due to Dani Wise. Conjecture 1 is true for all lattices containing direct product of two nonabelian free groups since the latter are incoherent. Therefore, it holds, for instance, for $SL(n,\mathbb{Z}), n \geq 4$. The case $n = 3$ is unknown (this problem is due to Serre, see the list of problems [13]).

Conjecture 1 is out of reach for general non-arithmetic lattices in $O(n,1)$ and $SU(n,1)$, since we do not understand the structure of such lattices. However, all known constructions of nonarithmetic lattices lead to noncoherent groups: See [7] for the case of Gromov–Piatetsky-Shapiro construction; the same argument proves noncoherence of nonarithmetic reflection lattices and non-arithmetic lattices obtained via Agol's [1] construction. In the case of lattices in $PU(n,1)$, all known nonarithmetic lattices contain fundamental groups of complex-hyperbolic surfaces which fiber over hyperbolic Riemann surfaces. Noncoherence of such groups is proven in [4].

In this talk (based on [5]) we discuss the case of arithmetic subgroups of rank 1 Lie groups. Conjecture 1 was proven in [7] for non-uniform arithmetic lattices in $O(n,1), n \geq 6$ (namely, it was proven that the noncoherent examples from [6] embed in such lattices). The proof of Conjecture 1 in the case of all arithmetic lattices of the *simplest type* appears as a combination of [7] and [2]. In particular, it covers the case of all non-uniform arithmetic lattices ($n \geq 4$) and all arithmetic lattices in $O(n,1)$ for n even, since they are of the simplest type. For odd $n \neq 3, 7$, there are also arithmetic lattices in $O(n,1)$ "quaternionic origin," while for $n = 7$ there is one more family of arithmetic groups associated with octonions. Lattices of "quaternionic origin" appear as groups commensurable to the integer groups of automorphisms of certain hermitian forms over division rings over number fields. One of the keys to the proof of noncoherence above is a virtual fibration theorem for certain classes of hyperbolic 3-manifolds. Our main result (Theorem 3) will be conditional to the existence of such fibrations:

Assumption 2. *We will assume that every arithmetic hyperbolic 3-manifold M of "quaternionic origin" admits a virtual fibration, i.e., M has a finite cover which fibers over the circle. (For general finite volume hyperbolic 3-manifolds, this is known as Thurston's Virtual Fibration Conjecture.)*

This assumption would follow if the recent work of on subgroup separability by Dani Wise [14] is correct: Wise's paper asserts Thurston's Virtual Fibration Conjecture for Haken hyperbolic 3-manifolds of finite volume. On the other hand, every arithmetic hyperbolic 3-manifold M of "quaternionic origin" is virtually Haken.

Our main results are:

Theorem 3. *Under the Assumption 2, Conjecture 1 holds for all arithmetic lattices of quaternionic type.*

Corollary 4. *Under the Assumption 2, Conjecture 1 holds for all arithmetic lattices in $O(n,1)$ for $n \neq 7$.*

Theorem 5. *Let $\Gamma < SU(2,1)$ be a uniform lattice (arithmetic or not) whose quotient $\mathbf{C}\mathbb{H}^2/\Gamma$ has positive first Betti number. Then Γ satisfies Conjecture 1.*

Corollary 6. *Let $\Gamma < SU(n,1)$ be a uniform lattice of the simplest type (also called* type 1 arithmetic lattices*, see* [9]*). Then Γ satisfies Conjecture 1.*

Proof. The lattice Γ always contains a (uniform) sublattice $\Lambda < SU(2,1)$ of the simplest type. By a theorem of Kazhdan [8], such Λ always contains a finite-index subgroup Λ' with infinite abelianization. Then, the result follows from Theorem 5. $\square$

Theorem 7. *Let X be either a quaternionic–hyperbolic spaces $\mathbf{H}\mathbb{H}^n$ or the octonionic–hyperbolic plane $\mathbf{O}\mathbb{H}^2$. Then every lattice $\Gamma < Isom(X)$ is noncoherent.*

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Entropies of covers of compact manifolds

François Ledrappier

Let $\pi : \overline{M} \to M$ be a regular Riemannian cover of a compact manifold: $\overline{M}$ is a Riemannian manifold and there is a discrete group G of isometries of $\overline{M}$ acting freely and such that the quotient $M = G\backslash\overline{M}$ is a compact manifold. The quotient metric makes M a compact Riemannian manifold. The following limits exist and do not depend on the reference point $x_0 \in \overline{M}$:

- the volume entropy $v := \lim_{R\to\infty} \dfrac{\ln \operatorname{vol}(B_{\overline{M}}(x_0,R))}{R}$,
- the stochastic entropy $h := \lim_{t\to\infty} -\dfrac{1}{t}\displaystyle\int_{\overline{M}} \ln \overline{p}(t,x_0,\overline{y})\overline{p}(t,x_0,\overline{y})d\overline{y}$,
- the linear drift $\ell := \lim_{t\to\infty} \displaystyle\int_{\overline{M}} \frac{1}{t} d_{\overline{M}}(x_0,\overline{y})\overline{p}(t,x_0,\overline{y})d\overline{y}$,

where we write $\overline{p}(t,\overline{x},\overline{y})$ for the heat kernel associated with the Laplacian Δ on $\overline{M}$. Our result is the following:

Theorem A.([**L2**]) *Let $\pi : \overline{M} \to M$ be a regular Riemannian cover of a compact manifold. With the above notation, we have:*

$$\ell^2 \ \leq \ h \ \leq \ v^2.$$

Recall that $h \leq \ell v$ ([**Gu**]). Theorem A then follows from

Theorem B.([**L2**]) *Let $\pi : \overline{M} \to M$ be a regular Riemannian cover of a compact manifold. With the above notations, we have:*

$$\ell^2 \ \leq \ h. \tag{1}$$

Moreover, either equality $\ell = v, h = v^2$ implies equality in (1).

Let l be the bottom of the spectrum of the Laplacian on $\overline{M}$:

$$l \ := \ \inf_{f\in C^2_K(\overline{M})} \frac{\int_{\overline{M}} \|\nabla f\|^2}{\int_{\overline{M}} f^2}.$$

Clearly (by considering C^2_K approximations to the functions $e^{-sd(x_o,.)}$ for $s > v/2$), we have $4l \leq v^2$. It can be shown that $4l \leq h$ ([**L1**], Proposition 3). Therefore,

Corollary 0.1. *Let $\pi : \overline{M} \to M$ be a regular Riemannian cover of a compact manifold. With the above notations, equality $4l = v^2$ implies equality in (1).*

In the case when $\overline{M}$ is the universal covering of a compact manifold with negative curvature, inequality (1) is due to V. Kaimanovich ([**K1**]). Moreover in that case, there is equality in (1) if, and only if, the manifold $\overline{M}$ is a symmetric space of negative curvature.

Our proof of (1) is based on the construction of a compact bundle space X_M over M which is laminated by spaces modeled on $\overline{M}$ and of a laminated Laplacian. In the case when M has negative curvature and $\overline{M}$ is the universal cover of M, the bundle space is the unit tangent bundle T^1M and the lamination on T^1M is the weak stable foliation of the geodesic flow.

We consider the *Busemann compactification* of the metric space $\overline{M}$: fix a point $x_0 \in \overline{M}$ and define, for $x \in \overline{M}$ the function $\xi_x(z)$ on $\overline{M}$ by:

$$\xi_x(z) \ = \ d(x,z) - d(x,x_0).$$

The assignment $x \mapsto \xi_x$ is continuous, one-to-one and takes values in a relatively compact set of functions for the topology of uniform convergence on compact subsets of $\overline{M}$. The Busemann compactification $\widehat{M}$ of $\overline{M}$ is the closure of $\overline{M}$ for that topology. The space $\widehat{M}$ is a compact separable space. The *Busemann boundary* $\partial\overline{M} := \widehat{M} \setminus \overline{M}$ is made of Lipschitz continuous functions ξ on $\overline{M}$ such that $\xi(x_0) = 0$. Elements of $\partial\overline{M}$ are called *horofunctions*. Observe that we may extend by continuity the action of G from $\overline{M}$ to $\widehat{M}$, in such a way that for ξ in $\widehat{M}$ and g in G,

$$g.\xi(z) \ = \ \xi(g^{-1}z) - \xi(g^{-1}(x_0)).$$

We define now the *horospheric suspension* X_M of M as the quotient of the space $\overline{M} \times \partial\overline{M}$ by the diagonal action of G. The projection onto the first component in $\overline{M} \times \partial\overline{M}$ factors into a projection from X_M to M so that the fibers are isometric to $\partial\overline{M}$. It is clear that the space X_M is metric compact.

To each point $\xi \in \partial\overline{M}$ is associated the projection W_ξ of $\overline{M} \times \{\xi\}$. As a subgroup of G, the stabilizer G_ξ of the point ξ acts discretely on $\overline{M}$ and the space W_ξ is homeomorphic to the quotient of $\overline{M}$ by G_ξ. We put on each W_ξ the smooth structure and the metric inherited from $\overline{M}$. The manifold W_ξ and its metric vary continuously on X_M. The collection of all $W_\xi, \xi \in \widehat{M}$ form a continuous lamination $\mathcal{W}_M$ with leaves which are manifolds locally modeled on $\overline{M}$. In particular, it makes sense to differentiate along the leaves of the lamination and we denote $\Delta^{\mathcal{W}}$ the laminated Laplace operator acting on functions which are smooth along the leaves of the lamination. A Borel measure on X_M is called *harmonic* if it satisfies, for all f for which it makes sense,

$$\int \Delta^{\mathcal{W}} f dm \;=\; 0.$$

By [**Ga**], there exist harmonic measures and the set of harmonic probability measures is a weak* compact set of measures on X_M. Moreover, if m is a harmonic measure and $\overline{m}$ is the G-invariant measure which extends m on $\overline{M} \times \partial\overline{M}$, then ([**Ga**]), there is a finite measure ν on $\partial\overline{M}$ and, for ν-almost every ξ, a positive harmonic function $k_\xi(x)$ with $k_\xi(x_0) = 1$ such that the measure m can be written as;

$$\overline{m} \;=\; k_\xi(x)(dx \times \nu(d\xi)).$$

The harmonic probability measure m is called ergodic if it is extremal among harmonic probability measures. In that case, for ν-almost every ξ, the following limits exist along almost every trajectory of the foliated Brownian motion:

- the linear drift of m $\ell(m) \;:=\; \lim_{t\to\infty} \frac{1}{t}\xi(\overline{X}_t)$.
- the transverse entropy $k(m) \;:=\; \lim_{t\to\infty} -\frac{1}{t}\ln k_\xi(\overline{X}_t)$.

The proof of Theorem B reduces to the three following results:

Proposition 0.2. *With the above notation, there exists an ergodic harmonic measure such that $\ell(m) = \ell$.*

Proposition 0.3. *For all ergodic harmonic measures m, we have $\ell^2(m) \leq k(m)$ with equality only if the harmonic functions k_ξ are such that $\nabla^{\mathcal{W}} \ln k_\xi = -\ell(m)\nabla^{\mathcal{W}}\xi$ m-almost everywhere.*

Proposition 0.4. *For all ergodic harmonic measures m, we have $k(m) \leq h$.*

The proof of Proposition 0.2 is an extension of the proof of the Furstenberg formula in [**KL**]. Kaimanovich ([**K1**]) proved Proposition 0.3 under the hypothesis that the horofunctions are of class C^2 by applying Itô's formula to the function ξ. In the general case, horofunctions are only uniformly 1-Lipschitz, but the integrated formulas of [**K1**] are still valid. See [**K2**] for Proposition 0.4.

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McCool groups

Gilbert Levitt

(joint work with Vincent Guirardel)

Let $\mathcal{C}$ be a finite set of conjugacy classes in a free group F_n. Let $Out_{\mathcal{C}}(F_n)$ be the pointwise stabilizer of $\mathcal{C}$ in $Out(F_n)$. If for instance $\mathcal{C}$ is the class of $[a,b][c,d]$ in $F(a,b,c,d)$, then $Out_{\mathcal{C}}(F_n)$ is a mapping class group. We call $Out_{\mathcal{C}}(F_n)$ a McCool group because of:

Theorem 1 (McCool)**.** *$Out_{\mathcal{C}}(F_n)$ is finitely presented.*

McCool's proof used peak reduction. Using JSJ theory and outer space, we can prove:

Theorem 2. *$Out_{\mathcal{C}}(F_n)$ is VFL: some finite index subgroup has a finite $K(\pi,1)$. This also holds if F_n is replaced by a torsion-free word-hyperbolic group G.*

Our motivation for studying McCool groups is the following result:

Theorem 3. *The stabilizer of a point in the boundary of outer space is built out of McCool groups (so is VFL).*

The goal of the talk was to present the basic techniques used in the proof of Theorem 2: JSJ theory (introduced by Rips-Sela) in the one-ended case (G is not a free product), outer space (introduced by Culler-Vogtmann) in the general case.

For a typical exemple, let G be the fundamental group of the space X obtained by gluing compact surfaces $\Sigma_1, \Sigma_2, \Sigma_3$, each with one boundary component, along their boundary. It is a torsion-free one-ended hyperbolic group.

By van Kampen's theorem, one can construct G by taking amalgamated free products. This is encoded in a graph of groups decomposition Γ of G, with a central vertex carrying $\mathbb{Z}$ and three other vertices carrying $\pi_1(\Sigma_i)$. Edge groups are $\mathbb{Z}$.

This decomposition (the cyclic JSJ decomposition of G) has two nice properties:

(1) every cyclic splitting of G (way of writing G as an amalgamated product over $\mathbb{Z}$, or an HNN extension) may be read from Γ: it comes from an edge of Γ or from a simple closed curve in a Σ_i.
(2) Γ is invariant under $Out(G)$: every automorphism of G comes from a homeomorphism of X.

In particular, automorphisms may come from permutations of the Σ_i's (if they have the same genus), a reflection (reversing orientation of each Σ_i), mapping classes of the Σ_i's. This leads to an exact sequence

$$1 \to \mathcal{T} \to Out^0(G) \to \prod_{i=1}^{3} MCG(\Sigma_i) \to 1$$

where $Out^0(G)$ is the finite index subgroup obtained by restricting to homeomorphisms mapping each Σ_i to itself in an orientation-preserving way, $MCG(\Sigma_i)$ is the mapping class group of the punctured surface, and $\mathcal{T}$ is generated by Dehn twists near the boundary of the Σ_i's ($\mathcal{T}$ is isomorphic to $\mathbb{Z}^2$ because the product of all three twists gives an inner automorphism). Finiteness properties of $Out(G)$ follow from this exact sequence.

To understand $Out_{\mathcal{C}}(G)$ in this example one has to use a different decomposition $\Gamma_{\mathcal{C}}$, which has properties similar to (1) and (2) above relative to $\mathcal{C}$. For instance, if $\mathcal{C}$ is represented by curves filling Σ_1, the graph of groups is the same for $\Gamma_{\mathcal{C}}$ as for Γ, but the vertex carrying $\pi_1(\Sigma_1)$ has become rigid: it has no cyclic splitting relative to $\mathcal{C}$, and no nontrivial element of $MCG(\Sigma_1)$ fixes $\mathcal{C}$. There is an exact sequence for $Out^0_{\mathcal{C}}(G)$ similar to the one above, but the quotient is now $\prod_{i=1}^{2} MCG(\Sigma_i)$.

For a general $Out_{\mathcal{C}}(G)$, with G one-ended (relative to $\mathcal{C}$), one uses the cyclic JSJ decomposition $\Gamma_{\mathcal{C}}$ of G relative to $\mathcal{C}$. Its vertex groups are $\mathbb{Z}$, surface groups, or rigid groups (having no further splitting), and there is an exact sequence as above with $\mathcal{T}$ free abelian and a product of mapping class groups as quotient. Understanding automorphisms of the rigid groups requires the Bestvina-Paulin method and Rips theory for actions on $\mathbb{R}$-trees.

Moving on to the infinitely-ended case (when G is a free product), Culler-Vogtmann proved that $Out(F_n)$ is VFL by constructing a space CV_n (outer space) with the following properties:

(1) $Out(F_n)$ acts on CV_n simplicially with finitely many orbits of simplices.
(2) Stabilizers are finite.
(3) CV_n is contractible.

These properties imply VFL by standard arguments. Outer space is constructed as a space of marked metric graphs, and the hard part is to prove contractibility.

To understand $Out_{\mathcal{C}}(G)$ for a general G (infinitely ended relative to $\mathcal{C}$), one uses an outer space relative to $\mathcal{C}$. The action of $Out_{\mathcal{C}}(G)$ satisfies the same properties as above, except for (2): stabilizers are not finite, but they are controlled by the one-ended case.

The Ehrenpreis and the surface subgroup conjectures

Vladimir Markovic
(joint work with Jeremy Kahn)

It is now well known that most 3-manifolds admit hyperbolic structure. Therefore in order to study topology of 3-manifolds it is necessary to understand topology of hyperbolic 3-manifolds. Machinery has been developed to study hyperbolic 3-manifolds that are Haken (a manifold is Haken if it contains an embedded incompressible surface). Although many hyperbolic 3-manifolds are not Haken it has been conjectured that every such manifold has a finite degree cover that is. There is an even stronger conjecture of Thurston that says that every such manifold has a cover that fibres over a circle. These two conjectures are known as the Virtual Haken conjecture and the Virtual fibering conjecture, and are among central problems in low dimensional topology.

A related well known problem is the Surface subgroup conjecture which asserts that every closed hyperbolic 3-manifold contains an immersed essential surface. Let $\mathbf{M} = \mathbb{H}^3/\mathcal{G}$ denote a closed hyperbolic three manifold. If such a surface exists then one may hope to find a finite cover $\mathbf{M}_1$ of $\mathbf{M}$ such that there exists a lift of this surface to $\mathbf{M}_1$ that is embedded, which implies that $\mathbf{M}_1$ is Haken. The most general version of the Surface subgroup conjecture is the following conjecture of Gromov.

Conjecture 1. *Given a word hyperbolic group G is there an injective homomorphism*

$$\rho : \pi_1(S) \to G,$$

where S is a closed surface of genus at least two.

In case when $\mathbf{M} = \mathbb{H}^3/\mathcal{G}$ is a closed hyperbolic three manifold, where $\mathcal{G}$ is the corresponding Kleinian group, Jeremy Kahn and I have announced the following theorem [2].

Theorem 1. *Let $\epsilon > 0$. Then there exists a Riemann surface $S_\epsilon = \mathbb{H}^2/F_\epsilon$ where F_ϵ is a Fuchsian group and a $(1+\epsilon)$-quasiconformal map $g : \partial\mathbb{H}^3 \to \partial\mathbb{H}^3$, such that the quasifuchsian group $g \circ F_\epsilon \circ g^{-1}$ is a subgroup of $\mathcal{G}$ (here we identify the hyperbolic plane $\mathbb{H}^2$ with an oriented geodesic plane in $\mathbb{H}^3$ and the circle $\partial\mathbb{H}^2$ with the corresponding circle on the sphere $\partial\mathbb{H}^3$).*

In particular we have the following.

Theorem 2. *Let $\mathbf{M}$ be a closed hyperbolic 3-manifold, $n \in \mathbb{N}$. Then one can find a closed surface S (of genus at least two) and an immersion $f : S \to \mathbf{M}$ such that the induced map between fundamental groups is injective.*

Let S and R be two finite type Riemann surfaces that are both either closed or both have at least one puncture. The well-known Ehrenpreis conjecture asserts that for a given $\epsilon > 0$ one can find covers S_1 and R_1, of S and R respectively, so that S_1 and R_1 are quasiconformally equivalent and the distance between them is less than ϵ. Since S_1 and R_1 are quasiconformally equivalent they belong to the same moduli space and the distance between them is measured in terms of a chosen metric on the Moduli space. There are two cases of this conjecture, the first is when S and R have punctures, and the second when they are both closed. The two natural metrics to be considered are the Teichmüller and the Weil-Petersson metrics.

In [1] Kahn-Markovic have announced the following result which settles this conjecture for punctured surfaces with respect to the Weil-Petersson metric.

Theorem 3. *Let S and R be two finite type Riemann surfaces that both have at least one puncture. Then given $\epsilon > 0$ one can find covers S_ϵ and R_ϵ of S and R respectively, so that S_ϵ and R_ϵ are quasiconformally equivalent and the Weil-Petersson distance between them is less than ϵ.*

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Finitely generated infinite simple groups of infinite square width and vanishing stable commutator length

Alexey Muranov

Definition. *Let G be a group. The* commutator length *of $g \in [G, G]$, denoted $\mathrm{cl}_G(g)$, is the minimal n such that there exist $x_1, \dots, x_n, y_1, \dots, y_n \in G$ such that $g = [x_1, y_1] \cdots [x_n, y_n]$. The* stable commutator length *$\overline{\mathrm{cl}}_G(g)$ is defined by*

$$\overline{\mathrm{cl}}_G(g) = \lim_{n\to\infty} \frac{\mathrm{cl}_G(g^n)}{n} = \inf_{n\in\mathbb{N}} \frac{\mathrm{cl}_G(g^n)}{n}.$$

The commutator width *of G is*

$$\mathrm{cw}(G) = \sup_{[G,G]} \mathrm{cl}_G\,.$$

The square length *of $g \in G^2 = \langle\, x^2 \mid x \in G\,\rangle$, denoted $\mathrm{sql}_G(g)$, is the minimal n such that there exist $x_1, \dots, x_n \in G$ such that $g = x_1^2 \cdots x_n^2$. The* square width *of G is*

$$\mathrm{sqw}(G) = \sup_{G^2} \mathrm{sql}_G\,.$$

Since every commutator is the product of 3 squares, finite commutator width implies finite square width.
Stable commutator length is related to the space of *homogeneous quasi-morphisms* on G, and hence to the kernel of the natural homomorphism $H_b^2(G, \mathbb{R}) \to H^2(G, \mathbb{R})$, which is isomorphic to the quotient of the space of all homogeneous quasi-morphisms $G \to \mathbb{R}$ by the subspace of all homomorphisms $G \to \mathbb{R}$, see Proposition 3.3.1(1) in [2] or Theorem 3.5 in [7].

Definition. *Let G be a group. A function $\phi\colon G \to \mathbb{R}$ is a* quasi-morphism *if the function $(x, y) \mapsto \phi(xy) - \phi(x) - \phi(y)$ is bounded on $G \times G$. A quasi-morphism is* homogeneous *if its restriction to every cyclic subgroup is a homomorphism to $(\mathbb{R}, +)$.*

Christophe Bavard [2, Proposition 3.4] proved that stable commutator length vanishes on the whole of the derived subgroup if and only if so do all homogeneous quasi-morphisms. Observe (or see Proposition 3.3.1(2) in [2]) that a homogeneous quasi-morphism vanishes on the derived subgroup only if it is a homomorphism. Therefore, the natural homomorphism $H_b^2(G, \mathbb{R}) \to H^2(G, \mathbb{R})$ is injective if and only if stable commutator length vanishes on $[G, G]$.

Until 1991, no simple group was known to have the commutator width greater than 1. For finite simple groups, it was shown in 2008 by Martin W. Liebeck, Eamonn A. O'Brien, Aner Shalev, and Pham Huu Tiep [8] that every element of every non-abelian finite simple group is a commutator (thus the long-standing conjecture of Oystein Ore [11] was proved). Jean Barge and Étienne Ghys [1, Theorem 4.3] showed that there are simple groups of symplectic diffeomorphisms of $\mathbb{R}^{2n}$ (kernels of *Calabi homomorphisms*) which possess nontrivial homogeneous quasi-morphisms, and thus their commutator width is infinite. Existence of *finitely generated* simple groups of commutator width greater than 1 was proved in [10]. Pierre-Emmanuel Caprace and Koji Fujiwara [4] recently shown that there are *finitely presented* simple groups for which the space of homogeneous quasi-morphisms is infinite-dimensional, and in particular whose commutator width is infinite. Those groups are the quotients of certain non-affine Kac-Moody lattices by the center; they were defined by Jaques Tits [12] and their simplicity was proved by P.-E. Caprace and Bertrand Rémy [5].

Commutator length in a group G is an example of a *conjugation-invariant norm* on the derived subgroup $[G, G]$.

Definition. *A* conjugation-invariant norm *on a group G is a function $\nu\colon G \to [0, \infty)$ which satisfies the following five axioms:*

(1) $\nu(g) = \nu(g^{-1})$ *for all* $g \in G$,
(2) $\nu(gh) \leq \nu(g) + \nu(h)$ *for all* $g, h \in G$,
(3) $\nu(g) = \nu(hgh^{-1})$ *for all* $g, h \in G$,
(4) $\nu(1) = 0$,
(5) $\nu(g) > 0$ *for all* $g \in G \setminus \{1\}$.

For brevity, conjugation-invariant norms shall be called simply norms.

Definition. *If ν is a norm on G, then its* stabilization *$\overline{\nu}$ is defined by*

$$\overline{\nu}(g) = \lim_{n\to\infty} \frac{\nu(g^n)}{n} = \inf_{n\in\mathbb{N}} \frac{\nu(g^n)}{n}, \quad g \in G.$$

A norm ν is stably unbounded *if $\overline{\nu}(g) > 0$ for some $g \in G$.*

(In general the stabilization of a norm is not a norm, as it has no reason to satisfy the axioms (2) and (5) of the definition.)

The following question was asked in [3]:

> Does there exist a group that does not admit a stably unbounded norm and yet admits a norm unbounded on some cyclic subgroup?

The main result of the author is the following.

Theorem ([9]). *There exists a torsion-free simple group G generated by 2 elements a and b such that*:

(1) *a^2 and b^2 freely generate a free subgroup H such that*
$$\lim_{n\to\infty} \mathrm{cl}_G(h^n) = \infty \quad \textit{for every } h \in H \setminus \{1\}$$
(*in particular*, $\mathrm{cw}(G) = \infty$),

(2) sql_G *is unbounded on* $H = \langle a^2, b^2\rangle$ (*in particular*, $\mathrm{sqw}(G) = \infty$),

(3) *G does not admit any stably unbounded conjugation-invariant norm* (*in particular*, $\overline{\mathrm{cl}}_G = 0$),

(4) *G is the direct limit of a sequence of hyperbolic groups with respect to a family of surjective homomorphisms*,

(5) *the cohomological and geometric dimensions of G are* 2,

(6) *G has decidable word and conjugacy problems.*

This theorem provides positive answer to the question of Burago-Ivanov-Polterovich [3] and also shows that stable commutator length can vanish on a simple group of infinite commutator width (and even of infinite square width). A recursive presentation of a desired group is found using small-cancellation methods.

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Contact topology on the space of light rays

Stefan Nemirovski
(joint work with Vladimir Chernov)

Let $\mathfrak{N}$ be the the space of light rays (that is to say, future pointing endless null geodesics considered up to an orientation preserving reparameterisation) of a globally hyperbolic spacetime X. Then $\mathfrak{N}$ is a co-oriented contact manifold contactomorphic to the spherical cotangent bundle ST^*M of a smooth spacelike Cauchy surface $M \subset X$. To each point $x \in X$, one associates a Legendrian sphere $\mathfrak{S}_x \subset \mathfrak{N}$ formed by the geodesics passing through x and calls it the *sky* of x.

It was suggested by Robert Low in a series of papers starting from his DPhil thesis in 1988 that under suitable assumptions on the topology of M, it should be possible to describe causal relations between points of X in terms of the (Legendrian) linking of their skies.

A basic observation is that if $x, y \in X$ are not causally related then their skies do not intersect and the Legendrian isotopy class of the link $\mathfrak{S}_x \sqcup \mathfrak{S}_y$ does not depend on the choice of x and y. Under the isomorphism $\mathfrak{N} \cong ST^*M$ this Legendrian linking class corresponds to the class of the link formed by any pair of distinct fibres of ST^*M. Thus, it seems natural to call a pair of skies *Legendrian linked* if they either intersect or form a Legendrian link that is not in that 'trivial' Legendrian isotopy class. The following result was conjectured by Natário and Tod [4] in the case when M is diffeomorphic to an open subset of $\mathbb{R}^3$.

Theorem 1. *Suppose that the universal cover of the Cauchy surface M of a globally hyperbolic spacetime X is non-compact. Then two points $x, y \in X$ are causally related if and only if their skies are Legendrian linked.*

For $(2+1)$-dimensional spacetimes, one has a stronger result in terms of usual (rather than Legendrian) linking conjectured by Low [3]. It is deduced from Theorem 1 and the classification of Legendrian cable links in $ST^*\mathbb{R}^2 \cong J^1(S^1)$ obtained by Ding and Geiges [2].

Theorem 2. *Suppose that M is a two-surface other than S^2 or $\mathbb{R}\mathrm{P}^2$. Then two points $x, y \in X$ are causally related if and only if their skies are smoothly linked.*

The proof of Theorem 1 is based on minimax (spectral) invariants of generating functions introduced by Viterbo [5] and on the following geometric characterisation of causal curves in terms of skies. Suppose that $x(t)$ is a past pointing curve in X, then the associated Legendrian isotopy of skies $\mathfrak{S}_{x(t)}$ is *non-negative* in the sense that it can be parameterised in such a way that the tangent vectors of the trajectories of individual points lie in the non-negative tangent half-spaces determined by the co-oriented contact structure on $\mathfrak{N}$.

It is not difficult to see that Theorems 1 and 2 are false for static spacetimes of the form $(M \times \mathbb{R}, \bar{g} \oplus -dt^2)$ such that the Riemannian manifold $(M, \bar{g})$ has the following 'Wiedersehen property': There exist a point $x \in M$ and a positive number $\ell > 0$ such that every unit-speed $\bar{g}$-geodesic starting from x returns back to x in time ℓ. In particular, the assumptions of Theorems 1 and 2 are sharp for spacetimes of dimension $2+1$ and $3+1$ because all surfaces and three-manifolds with compact universal cover are diffeomorphic to quotients of the standard round sphere by finite groups of isometries.

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Spectral partial quasistates and symplectic intersections

Yong-Geun Oh

(joint work with Fukaya, K., Ohta, H., Ono, K.)

In this lecture, I will first explain a deformation theory of Floer homology via ambient cycles (which we call *bulk deformations*) of symplectic manifold (M, ω) and extend construction of spectral invariants of Hamiltonian paths involving the bulk deformations. Then we explain how we can extend Entov and Polterovich's construction of spectral quasimorphisms of the Hamiltonian diffeomorphism group $Ham(M, \omega)$ and of spectral partial quasi-states of (M, ω) to a family thereof parameterized by the big quantum cohomology ring of (M, ω). We also apply these extended spectral invariants and Fukaya-Oh-Ohta-Ono's critical point theory of bulk-deformed potential functions of Lagrangian submanifolds. Finally using these machinery, we analyze the continuum family of non-displaceable Lagrangian tori in $S^2 \times S^2$ previously discovered by Fukaya-Oh-Ohta-Ono and prove that all of them cannot be separable from the product torus $S^1_{eq} \times S^1_{eq} \subset S^2 \times S^2$ where S^1_{eq} is the equator of S^2.

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On the flatness of Riemannian cylinders without conjugate points

Victor Bangert

(joint work with Patrick Emmerich)

In 1948 E. Hopf published the following celebrated result.

Theorem 1 ([5]). *A Riemannian metric without conjugate points on a two-dimensional torus is flat.*

This theorem and the method of its proof have attracted much interest ever since. Most importantly, by a completely different and beautiful proof, D. Burago and S. Ivanov [2] showed in 1994 that Theorem 1 also holds for the n-dimensional torus for all $n \geq 2$. E. Hopf's original method is short and elegant, and has proved useful also in other situations, see e.g. [1]. It depends on the Gauß-Bonnet theorem and the invariance of the Liouville measure under the geodesic flow via an integration over the unit tangent bundle. Thus it uses the compactness of the two-torus in an essential way. If one tries to generalize this method to a noncompact manifold, one will try to apply it to an appropriate sequence of compact sets exhausting the manifold. Since the compact sets have boundaries one will be confronted with boundary terms arising from the integration. These have to be controlled in the limit. In this way, K. Burns and G. Knieper proved

Theorem 2 ([3]). *Let g be a complete Riemannian metric without conjugate points on the cylinder $C = S^1 \times \mathbb{R}$. Assume that*

(i) *the Gaussian curvature of g is bounded below, and*
(ii) *there exists a constant L such that at every point $p \in C$ there exists a noncontractible loop of length at most L.*

Then g is flat.

Under the stronger assumption that g has no focal points, this had been proved by L. W. Green [4]. It is clear that some condition of the type of condition (ii) is necessary for the result to hold: There exist complete cylinders of negative Gaussian curvature (hence without conjugate points), e.g. surfaces of revolution in $\mathbb{R}^3$ generated by the graph of a positive function $f : \mathbb{R} \to \mathbb{R}$ with $f'' > 0$. In these examples at least one end of the cylinder "opens at least linearly", i.e. for points p in this end the length $l(p)$ of a shortest noncontractible loop at p grows at least linearly with the distance from p to a fixed point. In [6] H. Koehler showed that condition (ii) in Theorem 2 can be weakened to a condition that allows $l(p)$ to grow logarithmically with the distance to a fixed point.

The purpose of this paper is to prove versions of Theorem 2 where both conditions (i) and (ii) are considerably relaxed. Instead of the lower bound on the Gaussian curvature K, we only require K to be bounded below by $-t^\kappa$ for some $0 < \kappa < 1$, and instead of the logarithmic upper bound on l we allow l to grow at most like t^λ where $\lambda > 0$ and $\kappa + 2\lambda < 1$; here t denotes the distance to an arbitrarily fixed point $p_0 \in C$. The precise statement is:

Theorem 4. *Let g be a complete Riemannian metric without conjugate points on the cylinder $C = S^1 \times \mathbb{R}$. Assume that for some constants c, κ, λ in $\mathbb{R}_+$ with $\kappa + 2\lambda < 1$, and for all $p \in C$ we have that*

(i') *the Gaussian curvature K of g satisfies $K(p) \geq -c\,(\mathrm{d}(p, p_0) + 1)^\kappa$, and*

(ii') *the length $l(p)$ of the shortest noncontractible loop at p satisfies $l(p) \leq c\,(\mathrm{d}(p, p_0) + 1)^\lambda$.*

Then g is flat.

Here d denotes the Riemannian distance, and $p_0 \in C$ is an arbitrary point.

The improvement of Theorem 4 over Theorem 2 is made possible by a different choice of exhaustion of C. While in [3] the exhaustion is by compact subcylinders bounded by two geodesic loops, we use subcylinders bounded by horocycles. Here, we give a brief description of this exhaustion.

Given a *ray* γ in C, i.e. a minimal geodesic $\gamma : \mathbb{R}_+ \to C$, we consider its *Busemann function* $b_\gamma : C \to \mathbb{R}$ defined by

$$b_\gamma(p) := \lim_{t\to\infty} (\mathrm{d}(p, \gamma(t)) - t)$$

and its *horocyles*

$$h_t^\gamma := b_\gamma^{-1}\{-t\}.$$

The notation is chosen so that $\gamma(t) \in h_t^\gamma$. If C has no conjugate points and satisfies the condition

$$\liminf_{t\to\infty} \frac{1}{t} l(\gamma(t)) < 1, \tag{1}$$

then b_γ is a proper function and each of its horocyles h_t^γ is a closed curve winding once around the cylinder. Obviously, condition (ii') implies that (1) is satisfied for every ray γ. To define the exhaustion we choose two rays γ_1, γ_2 converging to the two ends of C. For sufficiently large t the horocyles $h_t^{\gamma_1}$ and $h_t^{\gamma_2}$ bound a compact subcylinder of C. For $t \to \infty$ these subcylinders form the exhaustion that we use in the proof of Theorem 4. The fact that horocycles of a ray are equidistant makes a crucial difference in the treatment of the boundary terms arising from the integration.

As a precursor to Theorem 4 we prove the following result in which we replace (i') and (ii') by a sublinear bound on the lengths of horocycles. In this case we can omit any bound on the curvature.

Theorem 3. *Let g be a complete Riemannian metric without conjugate points on the cylinder $C = S^1 \times \mathbb{R}$. Assume there exist two rays γ_1, $\gamma_2 : \mathbb{R}_+ \to C$ converging to the different ends of C such that, for $i \in \{1, 2\}$, the 1-dimensional Hausdorff measures of the corresponding horocycles $h_t^{\gamma_i}$ satisfy*

$$\lim_{t\to\infty} \frac{1}{t}\mathcal{H}^1(h_t^{\gamma_i}) = 0.$$

Then g is flat.

Remark. Surfaces of revolution with negative curvature provide nonflat examples of complete cylinders without conjugate points such that $\lim_{t\to\infty} \frac{1}{t}\mathcal{H}^1(h_t^{\gamma_i})$ is an arbitrarily small positive number. So, if one is willing to assume bounds on the lengths of horocycles, then Theorem 3 is close to being optimal.

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Quasi-morphisms defined by knot invariants

Michael Brandenbursky

Quasi-morphisms are known to be a helpful tool in the study of algebraic structure of non-Abelian groups. A quasi-morphism on a group G is a function $\varphi : G \to \mathbb{R}$ which satisfies the homomorphism equation up to a bounded error: there exists $D_\varphi > 0$ such that

$$|\varphi(gg') - \varphi(g) - \varphi(g')| \leq D_\varphi$$

for all $g, g' \in G$. A quasi-morphism φ is called *homogeneous* if $\varphi(g^m) = m\varphi(g)$ for all $g \in G$ and $m \in \mathbb{Z}$. Any quasi-morphism φ can be *homogenized*: setting $\widetilde{\varphi}(g) := \lim_{p\to+\infty} \varphi(g^p)/p$ we get a homogeneous (possibly trivial) quasi-morphism $\widetilde{\varphi}$.

Let $\mathcal{D}$ be the group of compactly supported area-preserving diffeomorphisms of a unit open two-dimensional disc in the Euclidean plane. The group $\mathcal{D}$ admits a unique (continuous, in the proper sense) homomorphism to the reals – the famous Calabi homomorphism (see e.g. [1], [4], [7]). At the same time $\mathcal{D}$ is known to admit many (linearly independent) homogeneous quasi-morphisms (see e.g. [2], [5], [8]). In this talk we explain an explicit geometric construction, due to Gambaudo and Ghys [8], which takes a homogeneous quasi-morphism $\widetilde{sign}_n$ on the pure braid group P_n on n strings and produces from it a quasi-morphism on $\mathcal{D}$. The quasi-morphism $sign_n$ on P_n is constructed, in turn, from the signature link invariant $sign$ of n-component links in $\mathbb{R}^3$ in the following way: close up a pure braid in a link in the standard way and take the value of the signature on that link. In [8] Gambaudo and Ghis proved that $sign_n$ is a quasi-morphism on P_n. By $\widetilde{sign}_n$ we denote it homogenization.

Denote by X_n the configuration space of ordered n-tuples of points in $\mathbb{D}^2$. First we fix a base point $\overline{z} = (z_1, \dots, z_n)$. Let $\overline{x} = (x_1, \dots, x_n)$ be any point in X_n. Take $g \in \mathcal{D}$ and any path g_t, $0 \le t \le 1$, in $\mathcal{D}$ between Id and g. Connect $\overline{z}$ to $\overline{x}$ by the straight line in $(\mathbb{D}^2)^n$, then act on $\overline{x}$ with the path g_t, and then connect $g(\overline{x})$ to $\overline{x}$ by the straight line in $(\mathbb{D}^2)^n$. The group $\mathcal{D}$ is path-connected and contractible [6]. Thus for almost every $\overline{x} \in X_n$ we get a loop $\gamma(g, \overline{x})$ in X_n with a homotopy type independent of the path g_t. Now identify P_n with $\pi_1(X_n, \overline{z})$. Thus $\gamma(g, \overline{x})$ is an element in P_n. Set

$$\widetilde{Sign}_{n,\mathbb{D}^2}(g) := \lim_{p\to\infty} \frac{1}{p} \int_{X_n} \widetilde{sign}_n(\gamma(g^p; \overline{x})) d\overline{x},$$

where $d\overline{x} = dx_1 \cdot \ldots \cdot dx_n$.

Gambaudo and Ghys proved that $\widetilde{Sign}_{n,\mathbb{D}^2} : \mathcal{D} \to \mathbb{R}$ is a well defined non-trivial homogeneous quasi-morphism for each $n > 1$.

There arises a natural

Question. *Which knot/link invariants may be used to define quasi-morphisms on P_n in a similar way?*

First we define some useful notions. Let I be a real-valued knot invariant. We define $\widehat{I} : P_n \to \mathbb{R}$ by setting $\widehat{I}(\alpha) := I(\widehat{\alpha\sigma})$. Here $\sigma = \sigma_1 \cdot \ldots \cdot \sigma_{n-1}$, where σ_i is a standard generator of the full braid group B_n. We denote by $g_4(K)$ the four-ball genus of a knot K and by $Conc(S^3)$ the concordance group of knots in S^3.

The following theorem gives a partial answer to the question above.

Theorem ([3]). *Suppose that a real-valued knot invariant I is a homomorphism $I : Conc(S^3) \to \mathbb{R}$, such that $|I(K)| \le cg_4(K)$, where c is some real positive constant independent of K. Then $\widehat{I}$ is a quasi-morphism on P_n.*

The definition of $\widehat{I}$ can be extended to B_n such that the theorem described above is true for B_n. Note that g_4 defines a norm on $Conc(S^3)$. The theorem above can be reformulated as follows: each element of $\mathrm{Hom}(Conc(S^3), \mathbb{R})$, which is Lipshitz with respect to this norm, defines a quasi-morphism on B_n.

Corollary ([3]). *The Rasmussen knot invariant s [11], which comes from a Khovanov-type homology, Oszvath-Szabo knot invariant τ [10], which comes from knot Floer homology, and the hermitian knot signatures induce quasi-morphisms on B_n.*

The invariants s and 2τ share similar properties and coincide on positive and alternating knots. It was conjectured by Rasmussen [11] that they are equal. This conjecture was disproved by Hedden and Ording [9]. In this work we prove the following

Theorem ([3]). *Let s and τ be the quasi-morphisms on B_n induced by Rasmussen and Oszvath-Szabo knot invariants. Then for every $\alpha \in B_n$ the following inequality holds $|s(\alpha) - 2\tau(\alpha)| \leq 2(n-1)$.*

Corollary ([3]). *The homogenization of s and 2τ are equal $\widetilde{s} = 2\widetilde{\tau}$.*

Recall that every knot K in $\mathbb{R}^3$ can be presented as a closure of some braid in B_n. The braid number of K is the minimal such n. It is denoted by $\mathrm{br}(K)$.

Corollary ([3]). *For every knot K the following inequality holds:*

$$|s(K) - 2\tau(K)| \leq 2(br(K) - 1).$$

In the remaining part we show that any quasi-morphism on P_n induces a quasi-morphism on $\mathcal{D}$.

Theorem ([3]). *Let φ be any quasi-morphism on P_n. Then*

$$\Phi(g) := \int\limits_{X_n} \varphi(\gamma(g;\overline{x}))d\overline{x},$$

is a quasi-morphism on $\mathcal{D}$.

We also discuss the computation of the homogeneous quasi-morphisms on $\mathcal{D}$, obtained by the construction above, on a diffeomorphism generated by a generic, **time-independent** (compactly supported) Hamiltonian H. We present the result of the computation in terms of the Reeb graph of H and the integral of the push-forward of H to the graph against a certain signed measure on the graph. This result helps us to proof the following asymptotic statement.

Theorem ([3]). *For each $g \in \mathcal{D}$ generated by an autonomous Hamiltonian*

$$\lim_{n\to\infty} \frac{\widetilde{Sign}_{n,\mathbb{D}^2}(g)}{\pi^{n-1}n(n-1)} = \mathcal{C}(g),$$

where $\mathcal{C}$ is the celebrated Calabi homomorphism.

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Heat kernels and counting in Cayley graphs

Anders Karlsson
(joint work with G. Chinta, J. Jorgenson)

1. Heat kernels on Cayley graphs

The heat kernel of a space with Laplacian is a fundamental object in its own right. To any group G generated by a finite set S there is an associated Cayley graph $X(G,S)$ and combinatorial Laplacian. We show that in a natural way the building blocks of the heat kernel $K_X(t,x) = K_X(t,e,x)$ of $X(G,S)$ are the functions

$$q^{-n/2}e^{-(q+1)t}I_n(2\sqrt{q}t),$$

where $q = 2|S| - 1$, $n \in \mathbb{Z}_{\geq 0}$, time $t \in \mathbb{R}_{\geq 0}$, and I_n the I-Bessel function of order n. This comes from a new expression for the heat kernel on free groups (quite different from the previous expression due to Chung-Yau [3]):

Proposition. *The heat kernel of the $(q+1)$-regular tree is given in the radial coordinate $r \geq 0$ as*

$$K(t,r) = q^{-r/2}e^{-(q+1)t}I_r(2\sqrt{q}t) - (q-1)\sum_{j=1}^{\infty} q^{-(r+2j)/2}e^{-(q+1)t}I_{r+2j}(2\sqrt{q}t),$$

where I denotes the I-Bessel function.

In view of a periodization procedure this leads to an expression on any Cayley graph:

Theorem. *The heat kernel on a Cayley graph $X(G,S)$ is*

$$K_X(t,x) = e^{-(q+1)t}\sum_{m=0}^{\infty} b_m(x)q^{-m/2}I_m(2\sqrt{q}t),$$

where I_m is the I-Bessel function of order m, $b_m(x) = c_m(x) - (q-1)(c_{m-2}(x) + c_{m-4}(x) + \ldots)$ and $c_m(x)$ is the number of geodesics from the identity to x of length $m \geq 0$.

In view of a combinatorial observation we obtain:

Corollary. *Let $N_e(m)$ denote the number of closed geodesics of length m in X at e. Then we have*

$$K_X(t,e) = K_{q+1}(t,e) + e^{-(q+1)t} \sum_{m=1}^{\infty} N_e(m) q^{-m/2} I_m(2\sqrt{q}t)$$

where K_{q+1} denotes the heat kernel of the $(q+1)$-regular tree.

Applications we have in mind include: asymptotics of number of spanning trees, counting of geodesics, Mahler measure computations. The general idea is to express the heat kernel in two ways, and use this equality using integral transforms and other manipulationsa to extract the desired information. In a sense it is the usage of a generalized Poisson summation principle.

2. Asymptotics of number of spanning trees

The number of spanning trees $\tau(G)$, called *the complexity*, of a finite graph G is an invariant which is of interest for several sciences: electrical networks, statistical physics, theoretical chemistry, etc. Via a well-known theorem the complexity equals the determinant of the combinatorial laplacian Δ_G divided by the number of vertices.

For compact Riemannian manifolds M there is an analogous invariant $h(M)$, *the height*, defined as minus the logarithm of the zeta regularized determinant of the Laplace-Beltrami operator, and which is of interest for quantum physics. The analogy of these two invariants has been commented on by Sarnak in [7].

In statistical physics it is of interest to study the asymptotics of the complexity, and other spectral invariants, for families of graphs. Important cases to study are various subgraphs of the standard lattice $\mathbb{Z}^d$. It is shown in [1] that in the asymptotics of the complexity of discrete tori, the height of an associated real torus appears as a constant.

Let Λ be an invertible $r \times r$ integer matrix. This matrix defines a lattice $\Lambda\mathbb{Z}^r$ in $\mathbb{R}^r$. Let the group quotient

$$DT(\Lambda) = \Lambda\mathbb{Z}^r \backslash \mathbb{Z}^r$$

be the associated *discrete torus.*

Theorem. *Given a sequence Λ_n of integer matrices converging normalized to $A \in SL_r(\mathbb{R})$ as $n \to \infty$. Then*

$$\log\det \Delta_{DT(\Lambda_n)} = c_r \det \Lambda_n + \frac{2}{r} \log\det \Lambda_n + \log\det \Delta_{A\mathbb{Z}^r \backslash \mathbb{R}^r} + o(1)$$

as $n \to \infty$, and where

$$c_r = \log 2r - \int_0^{\infty} e^{-2rt}(I_0(2t)^r - 1)\frac{dt}{t}.$$

Deninger and Lück informed us that also the constant c_r can be interpreted as the logarithm of a determinant of a Laplacian. The formula for this constant converges very fast making numerical computations easy. More generally one can make similar formulas for other Mahler measures of several variables in terems of Bessel integrals. Thanks to the fact that the first two terms in the asymptotics are universal in the sense that they only depend on Λ_n via $\det \Lambda_n$, the theorem gives a close connection between the complexity of certain graphs and the height of an associated manifold.

Conjecturally the height of $A\mathbb{Z}^r \backslash \mathbb{R}^r$ has a global minimum when $A\mathbb{Z}^r$ is the densest regular sphere packing. Extremal metrics for heights of tori has for example been studied in [2] and [8]. In these papers, the question is phrased as the study of the derivative of Epstein zeta functions at $s = 0$. From this theory we can for example deduce the following corollary from the theorem:

Corollary. *Given a sequence Λ_n of integer matrices with $\det \Lambda_n \to \infty$ that normalized stays in a compact subset of $SL_r(\mathbb{R})$. For all sufficiently large n we have that*

$$\tau(DT(\Lambda_n)) \leq \frac{(\det \Lambda_n)^{2/r-1}}{4\pi} \exp(c_r \det \Lambda_n + \gamma + 2/r),$$

where γ is Euler's constant and c_r is as in the theorem.

3. Counting of geodesics

A *geodesic* in a graph is a path without back-tracking. A *loop* is a closed path. A *closed geodesic* is a geodesic loop without tail, which means that no mather which starting point we take for the loop it remains withou bactrackings. Consider the numbers $a_n(x)$ defined by

$$e^{(q+1)t} K_X(t,x) = \sum_{n=0}^{\infty} a_n(x) \frac{t^n}{n!}$$

then it is well-known and simple to see that $a_n(x)$ is the number of paths from x_0 to x. We are after the counting of geodesics. The various counting functions can be extracted from the heat kernel expressions above, and formulas are obtained by a second heat kernel expression, such as

$$K_X(t,e) = \frac{1}{n} \sum_{j=0}^{n-1} e^{-\lambda_j t}$$

for a finite Cayley graph. We will employ the transform

$$Gf(u) = (1-u^{-2}) \int_0^{\infty} e^{-(u+1/u)t} e^{(q+1)t} f(t) dt.$$

to the heat kernel building block yielding

$$G(e^{-(q+1)t} q^{-k/2} I_k(2\sqrt{q}t))(u) = u^{k-1}$$

for $k \geq 0$ and $u > 0$. From our expressions in particular the Ihara determinantal formula [5] quickly comes out. Further applications are in progress.

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On the growth of quotients of Kleinian group

Marc Peigné

(joint work with Françoise Dal'Bo, Jean-Claude Picaud, Andrea Sambusetti)

1. Introduction and main results

We study the growth and divergence of quotients of Kleinian groups G i.e. discrete, torsionless groups of isometries of a Cartan-Hadamard manifold $\tilde{X}$ with pinched negative curvature. Namely, we give general criteria ensuring the divergence of a quotient group $\overline{G}$ of G and the "critical gap property" $\delta_{\overline{G}} < \delta_G$. As a corollary, we prove that every geometrically finite Kleinian group satisfying the parabolic gap condition (i.e. $\delta_P < \delta_G$ for every parabolic subgroup P of G) is "growth tight" for the distance on G induced by the Riemannian metric.

The notion of growth tightness was introduced by Grigorchuk and de la Harpe, relatively to word metrics of finitely generated groups. It was investigated with respect to more general distances by A. Sambusetti in the case of fundamental groups of hyperbolic surfaces, with some estimation of the gap between the different growth rates in terms of systolic lengths.

An important tool in this context is the *Poincaré series* $\mathcal{P}_G(s, \mathbf{x})$ of G, defined by

$$\mathcal{P}_G(s, \mathbf{x}) = \sum_{g \in G} e^{-sd(\mathbf{x}, g \cdot \mathbf{x})},$$

for $\mathbf{x} \in \tilde{X}$ and $s \in \mathbb{R}$. Its abscissa of convergence, called the *critical exponent of* G, does not depend on $\mathbf{x}$ and is equal to

$$\delta_G = \limsup_{R \to +\infty} \frac{1}{R} \ln \sharp\{g \in G / d(\mathbf{x}, g \cdot \mathbf{x}) \leq R\}.$$

One says that G is *divergent* when $\mathcal{P}_G(\delta_G, \mathbf{x}) = +\infty$; otherwise, it is convergent.

It is not easy to decide wether or not a Kleinian group G acting on X is convergent or divergent ; surprisingly, there exist only partial answers to this natural question. In particular, a result due to Sullivan states that geometrically finite groups in $\mathbb{H}^n$ are divergent. The fact that the parabolic groups are always divergent in constant curvature is a crucial fact in the proof ; this property fails in the variable curvature case where there may exist parabolic and geometrically finite groups of convergent type.

We now consider a proper normal subgroup N of G and study the action of $\bar{G} := N\backslash G$ on $\bar{X} := N\backslash\tilde{X}$. The metric on $\tilde{X}$ induce a distance $\bar{d}$ on $\bar{X}$ defined by

$$\forall \bar{\mathbf{x}} = N\mathbf{x}, \bar{\mathbf{y}} = N\mathbf{y} \in \bar{X} \quad \bar{d}(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = \inf_{\mathfrak{n} \in N} d(\mathbf{x}, \mathfrak{n} \cdot \mathbf{y})$$

and the elements of $\bar{G}$ are isometries of $(\bar{X}, \bar{d})$. We denote by

$$\mathcal{P}_{\bar{G}}(s, \bar{\mathbf{x}}) := \sum_{\bar{g} \in \bar{G}} e^{-s\bar{d}(\bar{\mathbf{x}}, \bar{g} \cdot \bar{\mathbf{x}})},$$

the Poincaré series of $\bar{G}$ and $\delta_{\bar{G}}$ its abscissa of convergence ; one clearly gets $\delta_{\bar{G}} \leq \delta_G$. We may state the

Theorem 1.1 (Growth tightness of geometrically finite groups)**.**
Let G be a geometrically finite Kleinian group of a Cartan-Hadamard manifold $\tilde{X}$ with pinched negative curvature. If G is convex-cocompact or $\delta_G > \delta_{\mathcal{P}}$ for any parabolic subgroup P of G, then G **is growth tight** *with respevt to the Riemannian metric, i.e. $\delta_{\bar{G}} < \delta_G$ for any proper quotient $\bar{G}$ of G.*

To prove this theorem, we will first establish the criteria :

Theorem 1.2. *Let $\tilde{X}$ be a complete, simply connected Riemannian manifold with pinched negative curvature, G a Kleinian group of $\tilde{X}$ and N a non trivial normal subgroup of G. If the group $G := N\backslash G$ is divergent, then $\delta_{\bar{G}} < \delta_G$.*

This result gives a new interest to the question of divergence/convergence of the group G, with respect to the induced distance $\bar{d}$ on $\bar{X}$. Theorem 1.1 will be thus a consequence of the divergence of the quotient group $\bar{G}$; this property holds in fact as soon as $\delta_{\bar{P}} < \delta_{\bar{G}}$ for any parabolic subgroup $P \subset G$, where $\bar{P} = (P \cap N)\backslash P$.

In the following section, we just propose a sketch of a new proof of the divergence of geometrically finite groups ; indeed, contrarily to previous works, we do not need the Patterson-Sullivan theory to obtain this result, the approach we developp is based on a purely geometrical point of view and a subadditivity type argument and may be easily applied when $\tilde{X}$ is replaced by a normal covering $\bar{X}$.

2. A subadditivity type argument for the annulus orbital function

Throughout this section, G is a geometrically finite group of isometries of $\tilde{X}$ such that $\delta_P < \delta_G$ for any parabolic subgroup P of G. We thus fix $\delta > 0$ such that $\delta_P < \delta < \delta_G$ and will use the following notations ; for any $\mathbf{x} \in \tilde{X}$ and any $R, \alpha \in \mathbb{R}^{*+}$, we denote by

- $\mathcal{A}_G(\mathbf{x}, \alpha, R)$ the "annulus " of width α defined by
$$\mathcal{A}_G(\mathbf{x}, \alpha, R) := \{\mathbf{y} \in \tilde{X} / R - \alpha < \bar{d}(\mathbf{x}, \mathbf{y}) \le R + \alpha\},$$
- $v_G(\mathbf{x}, \alpha, R) := \sharp\Big(\mathcal{A}_G(\mathbf{x}, \alpha, R) \cap G \cdot \mathbf{x}\Big)$,
- $w_G(\mathbf{x}, \alpha, R) := e^{-\delta R}\, v_G(\mathbf{x}, \alpha, R)$.

We have the

Proposition 2.1. *There exist $\alpha > 1$ and $C > 0$ such that, for any $A, B \ge 2\alpha$*

(1) *If G is convex-cocompact, then*

$$v_G(\mathbf{x}, 2\alpha, A + B) \le C \times v_G(\mathbf{x}, 2\alpha, A) \times v_G(\mathbf{x}, 2\alpha, B). \tag{1}$$

(2) *If G contains parabolic elements and , one gets*

$$w_G(\mathbf{x}, 2\alpha, A+B) \le C \times \Big(\sum_{0 \le n \le A+3\alpha+2} w_G(\mathbf{x}, 2\alpha, n)\Big) \times \Big(\sum_{0 \le n \le B+3\alpha+2} w_G(\mathbf{x}, 2\alpha, n)\Big). \tag{2}$$

We thus set $w_n := C w_G(\mathbf{x}, 2\alpha, n)$, $W_n := w_1 + \cdots + w_n$ and $\tilde{W}_n := 1 + W_1 + \ldots + W_n$ for all $n \ge 1$; by the above, for some $n_0 \ge 1$ one gets

$$\forall n, m \ge 1 \quad w_{n+m} \le W_{n+n_0} W_{m+n_0}.$$

It yields

$$\forall n, m \ge 1 \quad W_{n+m} = W_n + w_{n+1} + \cdots + w_{n+m} \le W_{n+n_0} \tilde{W}_{m+n_0}$$

and consequently

$$\forall n, m \ge 1 \quad \tilde{W}_{n+m} = \tilde{W}_n + W_{n+1} + \cdots + W_{n+m} \le \tilde{W}_{n+n_0} \tilde{W}_{m+n_0}.$$

Hence, the sequence $\left(\frac{\ln \tilde{W}_n}{n}\right)_n$ converges to some $L \ge 0$ (since $\tilde{W}_n \ge 1$) and one gets $\tilde{W}_n \ge e^{Ln}$. By the definition of the w_n, one gets $L = \delta_G - \delta > 0$ and one concludes using the following elementary lemma :

Lemma 2.2. *Let $(u_n)_{n \ge 1}$ be a sequence of positive numbers and set*

$$\forall n \ge 1 \qquad U_n := u_1 + \cdots + u_n.$$

Then, for any $s > 0$, the series $\sum_{n \ge 1} e^{-ns} u_n$ and $\sum_{n \ge 1} e^{-ns} U_n$ converge or diverge simultaneously. In particular, these series have the same critical exponent s_u and, when $s_u > 0$, they both diverge or converge for $s = s_u$.

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Statistical products of matrices

Mark Pollicott

(joint work with Richard Sharp)

1. Introduction

The basic theme of this talk was about how one can use that certain Fuchsian groups Γ are hyperbolic groups, to associate a dynamical model for the action $\Gamma \times \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$ on the boundary $\partial\mathbb{H}^2$ of Poincaré upper half space $\mathbb{H}^2$. In particular, we can then use this approach to recover results on the action $\Gamma \times \mathbb{H}^2 \to \mathbb{H}^2$.

In particular, since the group Γ is *strongly markov* (a consequence of being *Gromov hyperbolic*) elements in the group can be described in terms of a finite directed graph.

Lemma 1.1. *Let $\Gamma_0 \subset \Gamma$ be a symmetric finite set of generators. We can associate a finite directed graph $\mathcal{G}$ such that there is a bijection between the elements in $\Gamma - \{e\}$ and finite paths in Γ. (Moreover, the length of the path is equal to the word length of the corresponding element of the group).*

The ideas in this result seem to date back to work of Cannon (and perhaps in special cases of cocompact groups to earlier work of Hedlund and Morse). A very nice reference is the book of Ghys and de la Harpe [1].

Example 1.2 (Free group on two generators)**.** *For group $\Gamma = \langle a, b\rangle$ we let $\Gamma_0 = \{a^{\pm 1}, b^{\pm 1}\}$. Then Γ is described by a directed graph $\mathcal{G}$ with 4 vertices.*

Example 1.3 (Surface groups)**.** *For surface groups*

$$\Gamma = \langle a_1, \cdots, a_g, b_1, \cdots, b_g : \prod_i [a_i, b_i] = e\rangle$$

we let $\Gamma_0 = \{a_i^{\pm 1}, b_i^{\pm 1}\}$. The associated directed graphs $\mathcal{G}$ can be explicitly presented, although they can be quite large if g is large.

2. Shift spaces and boundaries

We can consider the space Σ of infinite (one-sided) paths in the graph $\mathcal{G}$. This is one particular definition of a subshift of finite type. As usual, we can then associate a shift map $\sigma : \Sigma \to \Sigma$ by starting the path one step further along its route. In the above examples, the shift $\sigma : \Sigma \to \Sigma$ is topologically mixing.

We can naturally identify sequences $\underline{i} = (i_n) \in \Sigma$ with points in the limit set, in the boundary of hyperbolic space, i.e., for $x \in \mathbb{H}^2$ we associate the sequence $g_{i_0}x, g_{i_0}g_{i_1}x, \ldots, g_{i_0}g_{i_1}\cdots g_{i_{n-1}}x \in \mathbb{H}^2$ which converges to a point in the boundary $\partial\mathbb{H}^2$ (with respect to the usual Euclidean metric). We recall that $\Gamma \subset PSL(2,\mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2)$ and thus we can interpret $g_{i_0}g_{i_1}\cdots g_{i_{n-1}}$ as a matrix product.

In order to consider statistical properties of this sequence it is convenient to consider shift invariant ergodic measures on Σ. For even stronger results, we want to consider a Gibbs measure μ associated to a Hölder continuous function (e.g., the Parry measure of maximal entropy on Σ or more generally a Markov measure on Σ; a measure corresponding to the Patterson-Sullivan measure on $\partial\mathbb{H}^2$, etc.). We can then associate the value $\lambda_\mu > 0$ characterized by

$$\lambda_\mu = \lim_{n\to+\infty} \frac{1}{n} d(x, g_{i_0}g_{i_1}\cdots g_{i_{n-1}}x) \text{ a.e.}(\mu)$$

which quantifies the average speed at which typical sequences move the point x.

3. Central Limit Theorem and Large Deviations

The following two theorems appear in [2] and were the main results in this talk.

Theorem 3.1 (Central Limit Theorem)**.** *There exists $\sigma > 0$ such that for any $y \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mu\left\{\underline{i} \in \Sigma : \frac{1}{\sqrt{n}}\left(d(x, g_{i_0}g_{i_1}\cdots g_{i_{n-1}}x) - n\lambda_\mu\right) \le y\right\} = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{y} e^{-t^2/2\sigma^2}\,dt.$$

It is also possible to prove stronger invariance principles, which imply the Central Limit Theorem, as well as deduce related results, such as the Law of the Iterated Logarithm.

The Central Limit Theorem also has a natural interpretation in terms of the action $\Gamma \times \partial\mathbb{H}^2 \to \partial\mathbb{H}^2$ on the boundary. This then has a natural extension to a Local Central Limit Theorem for the action on the boundary, although the method of proof is very different.

Finally, we also have results of the following type.

Theorem 3.2 (Large Deviations)**.** *Let $\epsilon > 0$. We have that*

$$\limsup_{n\to+\infty} \frac{1}{n}\log\mu\left\{\underline{i} \in \Sigma : \left|\frac{1}{n} d\left(x, g_{i_0}g_{i_1}\cdots g_{i_{n-1}}x\right) - \lambda_\mu\right| > \epsilon\right\} < 0$$

These results are somewhat similar in flavour to those for random walks or Brownian motion on $\mathbb{H}^2$, but again the methods of proof are very different.

4. Final Comments

Returning to the principle of encoding the boundary points by sequences in Σ, one can also use this method and standard methods involving Poincaré series to show the following simple result where Γ is a Fuchsian-Schottky group or a compact surface group acting on $\mathbb{H}^2$ (e.g., the earlier examples).

Proposition 4.1. *Let $x \in \mathbb{H}^2$. There exists $C > 0$ and $\delta > 0$ such that*

$$Card\{g \in \Gamma : d(gx, x) \leq T\} \sim Ce^{\delta T} \text{ as } T \to +\infty.$$

In particular, δ is the Hausdorff Dimension of the limit set.

With some modification this gives similar results for certain Kleinian groups, e.g., Apollonion circle packings.

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Spectral asymptotics on negatively curved surfaces and hyperbolic dynamics

Iosif Polterovich

(joint work with D. Jakobson and J. Toth)

Let M be a compact Riemannian manifold of dimension n without boundary. Let $0 < \lambda_1 \leq \lambda_2 \leq \dots$ be the eigenvalues of the Laplace–Beltrami operator on M. The eigenvalue counting function $N(\lambda) = \#\{\lambda_i < \lambda\}$ satisfies the following asymptotic formula called the *Weyl's law*:

$$N(\lambda) = \frac{\mathrm{Vol(M)}\,\lambda^{\frac{n}{2}}}{(4\pi)^{\frac{n}{2}}\,\Gamma(\frac{n}{2}+1)} + R(\lambda), \quad R(\lambda) = O(\lambda^{\frac{n-1}{2}}). \tag{1}$$

The upper bound on the remainder is sharp and attained on a round sphere.

Let us focus on the lower bounds for the error term. Under certain conditions on the manifold one can prove that

$$\frac{1}{\lambda}\int_0^\lambda |\,R(\mu)|\;d\,\mu >> \lambda^{\frac{n-2}{2}},$$

where $f(\lambda) >> g(\lambda)$ means that there exist constants C and λ_0 such that $f(\lambda) > C\,g(\lambda)$ for all $\lambda > \lambda_0$.

In dimension $n = 2$ this bound is quite weak. In particular, one may ask the following

Question: Is $\limsup_{\lambda\to\infty} |R(\lambda)| = \infty$ on any surface?

We apply methods of hyperbolic dynamics to give a positive answer to this question for negatively curved surfaces.

Theorem 1. [JPT] *Let M be a compact surface of negative curvature. Then for any $\delta > 0$*

$$R(\lambda) \neq o\left((\log \lambda)^{\frac{P(-H/2)}{h} - \delta}\right).$$

Here h is the topological entropy of the geodesic flow on M, P is the topological pressure and H is the Sinai–Ruelle–Bowen potential (see [JP, JPT]).

The exponent on the right hand is side is always positive: if the Gaussian curvature of M lies in the interval $[-K_1^2, -K_2^2]$, then

$$\frac{P(-H/2)}{h} \geq \frac{K_2}{2K_1}.$$

On surfaces of constant negative curvature, the inequality above becomes an equality and the exponent equals 1/2. In this case, Theorem 1 was proved independently by Randol [Ran] and Hejhal [Hej, section 17] more than thirty years ago using the Selberg zeta function techniques. Our approach, based on the Duistermaat–Guillemin wave trace formula [DG], hyperbolic dynamics and microlocal analysis, allows us to treat the variable curvature case as well.

We conclude by noting that Theorem 1 is in good agreement with a “folklore” conjecture stating that $R(\lambda) = o(\lambda^\epsilon)$ for any $\epsilon > 0$ on a generic (in particular, non–arithmetic) negatively curved surface (see [P, Conjecture 3]).

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From Symplectic Embeddings to Number Theory?

Felix Schlenk

(joint work with Dusa McDuff and Dorothee C. Müller)

In recent work with Dusa McDuff and Dorothee Müller on symplectic embeddings of 4-dimensional ellipsoids into balls and cubes, various algebraic, arithmetic and number theoretic notions and identities naturally arise. Among them are Diophantine equations and perfect solutions to them, Fibonacci numbers and Pell numbers, weight expansions and continued fractions, and lattice point counting problems.

In the standard symplectic vector space $\mathbb{R}^4$ with the symplectic form $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ consider, for $a, b > 0$, the closed ellipsoid

$$E(a,b) := \left\{ (z_1, z_2) \in \mathbb{C}^2 \equiv \mathbb{R}^4 \;\middle|\; \frac{|z_1|^2}{a} + \frac{|z_2|^2}{b} \leq 1 \right\}.$$

In particular, $B^4(a) := E(a,a)$ is the closed ball of radius $\sqrt{a}$. Given $a, b > 0$ and $c, d > 0$ we are looking for the smallest $\lambda > 0$ such that $E(a,b)$ symplectically embeds into $E(\lambda c, \lambda d)$. After scaling we can assume that $b = 1$ and $d = 1$.

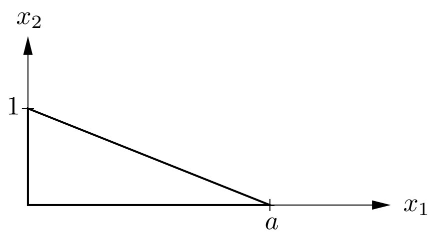
We first assume that $c = 1$, that is, we ask for the smallest λ such that

$$E(1,a) \overset{s}{\hookrightarrow} B^4(\lambda).$$

In other words, we want to find the function $c_B \colon [1, \infty) \to \mathbb{R}$,

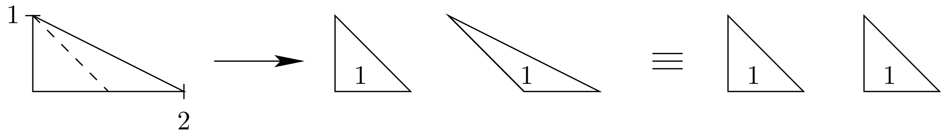
$$c_B(a) := \inf \left\{ \lambda \mid E(1,a) \overset{s}{\hookrightarrow} B^4(\lambda) \right\}.$$

Since the function $c_B(a)$ is continuous, we can assume throughout that a is rational. Using symplectic polar coordinates, we can think of the ellipsoid $E(1,a)$ as the triangle in the figure below.



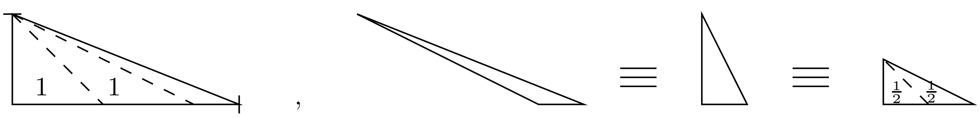
We decompose $E(1,a)$ into a finite collection of balls. Instead of giving the formal definition of this decomposition, [7], we explain it by examples:

(i) $E(1,2)$ decomposes into the disjoint union $B^4(1) \coprod B^4(1)$:



The **weight expansion** of 2 is $\boldsymbol{w}(2) = (1,1)$.

(ii) $E(1, 2\frac{1}{2})$ decomposes into the disjoint union $B^4(1) \coprod B^4(1) \coprod B^4(\frac{1}{2}) \coprod B^4(\frac{1}{2})$:



The weight expansion of $2\frac{1}{2}$ is $\boldsymbol{w}(\frac{5}{2}) = \left(1^{\times 2}, \frac{1}{2}^{\times 2}\right)$.

(iii) Similarly, the weight expansion of $4\frac{2}{3}$ is

$$\boldsymbol{w}(4\tfrac{2}{3}) = \left(1^{\times 4}, \tfrac{2}{3}^{\times 1}, \tfrac{1}{3}^{\times 2}\right).$$

In this way, to every ellipsoid $E(1,a)$ with $a \in \mathbb{Q}$ we associate the **weight expansion** $\boldsymbol{w}(a)$. The multiplicities of the weights in $\boldsymbol{w}(a)$ form the **continued fraction expansion** of a. For instance

$$4\tfrac{2}{3} \;=\; \mathbf{4} + \cfrac{1}{\mathbf{1} + \cfrac{1}{\mathbf{2}}}$$

Also note that

$$\boldsymbol{w}(a) \cdot \boldsymbol{w}(a) \;=\; a.$$

Denote by $\ell(a)$ the length of the weight expansion $\boldsymbol{w}(a)$. The weight expansion of a determines the disjoint union of balls $\coprod_{i=1}^{\ell(a)} B^4(w_i(a))$. If $E(1,a) \overset{s}{\hookrightarrow} B^4(\lambda)$, then, clearly, $\coprod_{i=1}^{\ell(a)} B^4(w_i(a)) \overset{s}{\hookrightarrow} B^4(\lambda)$. Dusa McDuff has shown in [5] by a geometric construction that the converse is true too:

Theorem (McDuff) *$E(1,a) \overset{s}{\hookrightarrow} B^4(\lambda)$ if and only if $\coprod_{i=1} \ell(a) B^4(w_i(a)) \overset{s}{\hookrightarrow} B^4(\lambda)$.*

Our problem $E(1,a) \overset{s}{\hookrightarrow} B^4(\lambda)$ is thus translated into a symplectic ball packing problem. This problem has a long history, starting with Gromov's work [2], followed by McDuff's and Polterovich's work [6], and culminating with Biran's work [1]. A further ingredient is the work by Li–Li [3, 4].

Theorem (Gromov, McDuff–Polterovich, Biran, Li–Li)

$$\coprod_{i=1}^{k} B^4(w_i) \overset{s}{\hookrightarrow} B^4(\lambda) \quad \textit{if and only if} \quad (1)\ \sum_{i=1}^{k} w_i^2 \;<\; \lambda^2 \ \textit{and}$$

(2) *for each solution $(d; m_1, \dots, m_k) \in \mathbb{N} \times \mathbb{N}^k$ of the* **Diophantine system**

$$(D_B) \qquad 3d - 1 \;=\; \sum_{i=1}^{k} m_i, \qquad d^2 + 1 \;=\; \sum_{i=1}^{k} m_i^2$$

it holds that

$$\frac{1}{d} \sum_i w_i m_i \;<\; \lambda.$$

The constraint (1) is the volume constraint, coming from the fact that symplectic embeddings preserve the volume. The constraint (2) comes from J-holomorphic curves. Define

$$\mu(d; \boldsymbol{m})(a) \;:=\; \frac{1}{d} \sum_i w_i(a)\, m_i \;<\; \lambda$$

and

$$c_B(a) \;:=\; \inf \left\{ \lambda \mid E(1,a) \overset{s}{\hookrightarrow} B^4(\lambda) \right\}.$$

In view of the above two theorems we have

$$c_B(a) \;=\; \max\Big\{\sqrt{a},\, \sup\big\{\mu(d;\boldsymbol{m})(a) \mid (d;\boldsymbol{m}) \text{ solves } (D_B)\big\}\Big\}.$$

Examples. a) Consider the problem $E(1,2)\overset{s}{\hookrightarrow}B^4(\lambda)$. By (1) we have $2<\lambda^2$. Since $(1;1,1)$ is a solution to (D_B), we have $1+1=2<\lambda$. Since $E(1,2)\overset{\mathrm{id}}{\hookrightarrow}B^4(2)$ we find that $c_B(2)=2$.

b) Consider the problem $E(1,5)\overset{s}{\hookrightarrow}B^4(\lambda)$. By (1) we have $5<\lambda^2$. Since $(2;1^{\times 5})$ is a solution to (D_B), we have $\frac{5}{2}<\lambda$. One checks that there is no stronger constraint. Thus $c_B(5)=\frac{5}{2}$.

c) For $a\geq 9$ we have $c_B=\sqrt{a}$, since then the volume constraint (1) is stronger than all constraints in (2). Indeed, since $w_i(a)\leq 1$ and by the first equation in (D_B),

$$\frac{1}{d}\sum m_i\, w_i(a) \;\leq\; \frac{1}{d}\sum m_i \;=\; \frac{1}{d}(3d-1) \;<\; 3 \;\leq\; \sqrt{a}.$$

We point out that it is impossible to find all solutions to (D_B). Namely, for $k\geq 9$, the system (D_B) has infinitely many solutions. Notice that by the Cauchy–Schwarz inequality and by the second equation in (D_B),

$$\mu(d;\boldsymbol{m})(a) \;:=\; \frac{1}{d}\sum m_i\, w_i(a) \;\leq\; \frac{1}{d}\,\|\boldsymbol{m}\|\,\|\boldsymbol{w}(a)\| \;=\; \sqrt{1+1/d^2}\sqrt{a}.$$

Therefore, solutions $(d;\boldsymbol{m})$ that give a constraint $>\sqrt{a}$ at a must have the vector $\boldsymbol{m}$ “essentially parallel” to $\boldsymbol{w}$. We say that a solution $(d;\boldsymbol{m})$ is a **perfect solution** if $\boldsymbol{m}$ is parallel to $\boldsymbol{w}(a)$ for some a.

We have worked out the function $c_B(a)$ on $[1,\infty)$ in [7]. We describe the answer for $a\in[1,\tau^4]$, where $\tau:=\frac{1+\sqrt{5}}{2}=[1;1,1,1,\dots]$ is the **golden ration**. Recall that the **Fibonacci numbers** f_n are recursively defined by

$$f_0=0, \quad f_1=1, \quad f_{n+1}=f_n+f_{n-1}.$$

Define $g_n=f_{2n-1}$. Then the sequence g_n starts with $1,2,5,13,34,\dots$. Abbreviate

$$a_n := \left(\frac{g_{n+1}}{g_n}\right)^2, \qquad b_n := \frac{g_{n+2}}{g_n}.$$

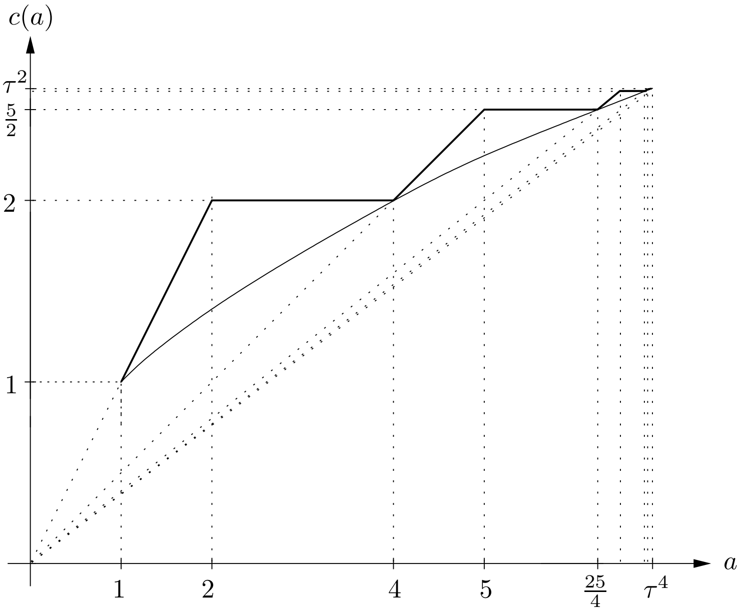
Then $a_0=1<b_0=2<a_1=4<b_1=5<a_2=\left(\frac{5}{2}\right)^2<b_2=\frac{13}{2}<\dots$, and, in general,

$$\cdots<a_n<b_n<a_{n+1}<\cdots\;\to\;\tau^4\approx 6.85.$$

Theorem 1 (McDuff–Schlenk) *The function $c_B(a)$ on $[1,\tau^4]$ is given by the “Fibonacci stairs”: For each $n\geq 0$, $c_B(a)=\frac{a}{\sqrt{a_n}}$ for $a\in[a_n,b_n]$, and c_B is constant with value $\sqrt{a_{n+1}}$ on the interval $[b_n,a_{n+1}]$.*

Define the rescaled weight expansions $\boldsymbol{W}(b_n):=g_n\boldsymbol{w}(b_n)$ and $\boldsymbol{W}(a_n):=g_n^2\boldsymbol{w}(a_n)$. The main step in the proof of Theorem 1 is to show that

$$E(b_n):=\big(g_{n+1};\boldsymbol{W}(b_n)\big) \qquad \text{and} \qquad E(a_n):=\big(g_ng_{n+1};\boldsymbol{W}(a_n),1\big)$$



are solutions to (D_B). Note that $E(b_n)$ is a perfect solution, while $E(a_n)$ is only almost perfect. It turns out that the solutions $E(b_n)$ are the only perfect solutions.

Assume next that $c = 2$, that is, we are looking for the smallest λ such that

$$E(1,a) \overset{s}{\hookrightarrow} E(\lambda, 2\lambda).$$

This problem is equivalent to the problem of embedding $E(1,a)$ into a polycube,

$$E(1,a) \overset{s}{\hookrightarrow} C^4(\lambda) := D^2(\lambda) \times D^2(\lambda).$$

This latter problem is, again, equivalent to the problem

$$\coprod B^4\big(w_i(a)\big) \overset{s}{\hookrightarrow} C^4(\lambda)$$

which is, this time, equivalent to (1) $\sum w_i^2(a) \,<\, 2\lambda^2$ and

(2) for each solution $(d,e;m_1,\dots,m_k) \in \mathbb{N} \times \mathbb{N}^k$ of the **Diophantine system**

$$(D_C) \qquad 2(d+e)-1 \,=\, \sum m_i, \qquad 2de+1 \,=\, \sum m_i^2,$$

it holds that

$$\frac{1}{d+e} \sum w_i(a)\, m_i \,<\, \lambda.$$

We thus have

$$c_{E(1,2)}(a) \,=\, c_{C^4(1)}(a) \,=\, \max\left\{ \sqrt{\frac{a}{2}},\, \sup\big\{\mu(d,e;\boldsymbol{m})(a) \mid (d,e;\boldsymbol{m}) \text{ solves } (D_C)\big\} \right\}.$$

It turns out that $\mu(d,e;\boldsymbol{m}) \geq \sqrt{\frac{a}{2}}$ only if $|d-e| \leq 1$.

The **Pell numbers** P_n are recursively defined by

$$P_0 = 0, \quad P_1 = 1, \quad P_{n+1} = 2P_n + P_{n-1}.$$

Define $x_n := P_{2n-1} + P_{2n}$ and $y_n := P_{2n-1}$, and

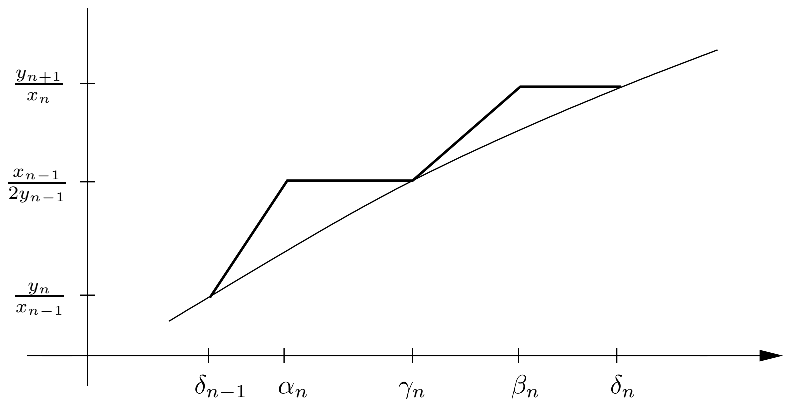
$$\alpha_n := \frac{x_n}{x_{n-1}}, \quad \beta_n := \frac{y_{n+1}}{y_n} \qquad \gamma_n := \frac{1}{2}\left(\frac{x_n}{y_n}\right)^2, \quad \delta_n := 2\left(\frac{y_{n+1}}{x_n}\right)^2.$$

Then

$$\cdots < \alpha_n < \gamma_n < \beta_n < \delta_n < \alpha_{n+1} < \cdots \to \sigma^2 \approx 5.83$$

where $\sigma = 1 + \sqrt{2} = [2; 2, 2, 2, \dots]$ is the **silver ratio**.

Theorem 2 (Müller [8]**)** *The function $c_{E(1,2)}(a) = c_C$ on $[1, \sigma^2]$ is given by the "Pell stairs": For each $n \geq 0$, the function c_C on $[\delta_{n-1}, \delta_n]$ forms a double step as described in the figure below.*



Again, the main step of the proof is to show that there are perfect solutions at α_n and β_n and almost perfect solutions at γ_n and δ_n.

For connections of our embedding problems to lattice point counting problems in triangles, and for curious quadratic identities for weight expansions we refer to [7].

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scl in free groups

Danny Calegari
(joint work with Alden Walker)

1. Abstract of talk

Let G be a group, and $[G,G]$ the commutator subgroup. Given $g \in [G,G]$ the *commutator length* of g, denoted $\mathrm{cl}(g)$, is the least number of commutators in G whose product is equal to g. The *stable commutator length*, denoted $\mathrm{scl}(g)$, is the lim inf of $\mathrm{cl}(g^n)/n$ as n goes to infinity.

A *quasimorphism* of G is a function $\phi : G \to \mathbb{R}$ for which there is a constant $D(\phi) \geq 0$ such that $|\phi(a) + \phi(b) - \phi(ab)| \leq D(\phi)$ for all $a, b \in G$. The least constant $D(\phi)$ with this property is called the *defect* of ϕ.

A quasimorphism is *homogeneous* if $\phi(a^n) = n\phi(a)$ for all integers n and all $a \in G$.

The Bavard Duality Theorem says that in any group G, there is an equality

$$\mathrm{scl}(g) = \sup_{\phi} \frac{\phi(g)}{2D(\phi)}$$

where the supremum is taken over all homogeneous quasimorphisms. See [1] for a more substantial introduction.

Bavard Duality is highly nonconstructive, and raises a fundamental question: given $g \in G$, to give an explicit construction of a homogeneous quasimorphism ϕ for which $\mathrm{scl}(g) = \phi(g)/2D(\phi)$.

In general, this question is much too hard to say anything reasonable about. However, for certain specific groups it might be able to say more.

We conjecture that for G a free group, for every $g \in [G,G]$ there is an extremal quasimorphism ϕ for g that arises “from symplectic geometry”. This is somewhat vague, but there are some intriguing families of examples where this is precise:

(1) Hyperbolic surfaces: G is realized as $\pi_1(S)$ where S is a hyperbolic surface with geodesic boundary. ϕ in this case is the *rotation quasimorphism*, a primitive for the hyperbolic area form.
(2) Taut sutured handlebodies: G is realized as $\pi_1(X)$ where X is the symplectization of a taut foliated handlebody. ϕ in this case is the rotation quasimorphism associated to a *universal circle* for the foliation.
(3) Siegel handlebodies: G is $\pi_1(X)$ where $X = \mathfrak{H}/\Gamma$, where $\mathfrak{H}$ is the Siegel upper half-space, and Γ is a free subgroup of $\mathrm{Sp}(2n, \mathbb{Z})$. ϕ in this case is the *symplectic rotation number*.

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Smooth rigidity at infinity of negatively curved manifolds

Chengbo Yue

Let M be a closed Riemannian manifold of negative curvature. Its universal cover $\widetilde{M}$ can be compactified by an ideal sphere at infinity. By Hadamard's Theorem, $\widetilde{M}$ is diffeomorphic to a round ball. The ideal sphere $\partial\widetilde{M}$, however, is in general only a rugged topological sphere.

Main Conjecture(Gromov[DG], Kanai[K1], et al.): *$\partial\widetilde{M}$ is C^2 if and only if M is locally symmetric.*

This has been solved in dimensional 2 by the work of Hurder-Katok[HK] and Ghys[Gh]. In higher dimension, after a long series of works by Kanai[K1,K2], Feres and Katok[FK1, FK2, F], Benoist-Foulon-Laborie[BFL] was able to prove the C^∞-case applying a general principle of Gromov[Gro] on rigid transformation groups.

This talk describes a solution to the higher dimensional optimal C^2-case. The key ingredient in our approach is the observation that if the ideal sphere is C^2, then a certain Busemann pairing cocycle must also be C^2 and its second order mixed partial derivatives recover the symplectic geometry of the geodesic flow completely. In fact, the Busemann pairing cocycle B, which can be thought of as the logarithm of a pseudo-distance between points at infinity, plays the role of some sort of potential for the transversal symplectic 2-form Ω of the geodesic flow:

$$\Omega = dd_y B.$$

This in turn implies that B satisfies a system of second order hyperbolic partial differential equations:

$$\frac{\partial^2 B}{\partial x_i \partial y_j} = a_{ij}.$$

To solve such equations, one needs to integrate twice: (1) We first integrate along a Lagrangian foliation in the guise of leaf-wise Poincaré Lemma, which shows that $\{\partial_{y_j} B\}$ define exactly the flat affine parameters along the Lagrangian leaves; (2) A further integration of $\{\partial_{y_j} B\}$ gives rise to the cross ratio, which in effect converts hyperbolic regularity to elliptic regularity. Using a bootstrap argument typical in elliptic regularity, one can promote C^2-regularity first to $C^{2+\alpha}(0<\alpha<1)$, then to C^∞, inductively.

As consequence, we prove:

Theorem[Y]: *If $\partial\widetilde{M}$ is C^2, then it is C^∞.*

Hence, combining with the C^∞-result of Benoist-Foulon-Laborie[BFL] and the entropy rigidity result of Besson-Courtois-Gallot[BCG], we obtain a complete solution to the main conjecture:

Theorem: *The ideal sphere $\partial\widetilde{M}$ of a negatively curved closed manifold M is C^2 if and only if M is locally symmetric.*

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Problem Session

A session for discussing interesting open questions was held at the workshop. It was organized and moderated by Anatole Katok and carefully transcripted by Lev Buhovsky and Wenyuan Yang.

Some questions were already solved during the workshop and have not been included here.

1. Questions by Danny Calegari

1. Let G be a hyperbolic group with a fixed generating set S.

Fact 1 (Calegari-Maher [2])**.** There exist positive constants C_1, C_2 such that the following holds.

Let scl_n denote the value of scl of a random element of word length n (in the generators S), conditioned to lie in $[G, G]$. Then the following is true:

$$\mathbf{P}(C_1 n/\log n < scl(g) < C_2 n/\log n) \to 1 \text{ as } n \to \infty$$

Conjecture 1. There exists a positive constant C such that the following holds: (with notation as above)

$$\mathrm{scl}_n \log n/nC \to 1 \text{ with respect to probability measure.}$$

Question 1. *What one can say about the constant C? What does the error term look like?*

2. Let G be a finitely generated group with a finite generating set S.

Fact 2 (Calegari-Maher [2])**.** Let G be a group where scl does not vanish identically, and let S be a symmetric generating set. Let scl_n denote the value of scl on a random walk of length n in the generators, conditions to lie in $[G, G]$.

Then the growth rate of scl_n is at least of order $\sqrt{n}$, and at most of order $n/\log n$.

Question 2. *Is there a finitely generated (finitely presented) group for which* scl_n *has growth intermediate between these two possibilities?*

3. Let F be a free non-abelian group.

Fact 3 (Calegari [1])**.** If $g, h \in F$ do not commute, then there is a homogeneous quasimorphism $\phi : F \to \mathbb{R}$ satisfying

$$\phi(g) + \phi(h) - \phi(gh) = D(\phi)$$

Question 3. *Which other groups G have the same property?*

4. Let G be a finitely generated subgroup of a free group F. Let $Out_G(F)$ denote the set of outer automorphisms of F that conjugate G to itself; i.e. for which $\phi(G) = tGt^{-1}$ for some $t \in F$.

Question 4. *Is $Out_G(F)$ finitely generated? finitely presented? VFL?*

2. Five questions from M. Entov and L. Polterivich

These questions arise from the joint work of M. Entov and L. Polterovich on Lie quasi-states.

Definition 1. A *Lie quasi-state* on a real Lie algebra $\mathfrak{g}$ is a functional $\zeta : \mathfrak{g} \to \mathbb{R}$ whose restriction to any abelian subalgebra is linear.

Fact:
If $\mathfrak{g} = Lie(G)$ for a Lie group G and $\mu : G \to \mathbb{R}$ is a continuous homogeneous quasi-morphism, then $\zeta := \mu \circ exp : \mathfrak{g} \to \mathbb{R}$ is a continuous Lie quasi-state, which is Ad -invariant.

Question 5. *Describe the space of continuous Lie quasi-states (modulo the linear functionals) on classical Lie algebras. The simplest open cases: $sl(4, \mathbb{R})$, $sp(4, \mathbb{R})$.*

Question 6. *Find a meaningful cohomology theory incorporating Lie quasi-states (similarly to the way the bounded cohomology of groups incorporates quasi-morphisms).*

Question 7. *Find dynamical/geometric constructions of Lie quasi-states that do* **not** *yield quasi-morphisms.*

Question 8. *Does any Ad -invariant Lie quasi-state come from a homogeneous quasi-morphism?*

Question 9. *Do there exist (continuous) functionals $\mu : G \to \mathbb{R}$ on Lie transformation groups which are* **not** *quasi-morphisms but such that the restriction of μ to any abelian subgroup is a homomorphism?*

3. Anna Erschler: Symmetric measures and entropy on groups

Let G be a finitely generated group.

Question 10. *Suppose there exist a finite subset V of G and a sequence of symmetric measures $\{\mu_i\}$ on G such that $Supp(\mu_i) \subset V$ and $< Supp(\mu_i) >= G$ for each i. If $\mu_i \to \mu$, does it follow that $h(\mu_i) \to h(\mu)$?*

Question 11. *Suppose that μ_1, μ_2 are two symmetric finitely-supported measures on G. If $< Supp(\mu_1) >= G$ and $< Supp(\mu_2) >= G$ and $h(\mu_1) > 0$, then is it true that $h(\mu_2) > 0$?*

4. Two questions by Gerhard Knieper

Definition 2. Let (M, g) be a complete Riemannian manifold. We call a geodesic flow $\phi^t : SM \to SM$ partially hyperbolic if there exists a continous splitting

$$T_vSM = E^s(v) \oplus E^u(v) \oplus E^c(v)$$

of the tangent bundle of SM into subbundles such that $\dim E^s(v) = \dim E^u(v) = k > 0$ and the following properties are fulfilled: there are constants $b \geq 1$, $\alpha > 0$ such that for all $\xi \in E^s(v)$

$$\|D\phi^t(v)\xi\| \leq b\|\xi\|e^{-\alpha t},\ t \geq 0, \quad \|D\phi^t(v)\xi\| \geq \frac{1}{b}\|\xi\|e^{-\alpha t},\ t \leq 0$$

and for all $\xi \in E^u(v)$

$$\|D\phi^t(v)\xi\| \geq \frac{1}{b}\|\xi\|e^{\alpha t},\ t \geq 0, \quad \|D\phi^t(v)\xi\| \leq b\|\xi\|e^{\alpha t},\ \ t \leq 0.$$

Furthermore, for all $\xi \in E^c(v)$ and $t \in \mathbb{R}$ we have

$$\|D\phi^t(v)\xi\| \leq \|\xi\| f(t).$$

where f is a positive function of subexponential growth. The norm is given by the Sasaki-metric

Question 12. *Let (M, g) be a compact Riemannian manifold and $\phi^t : SM \to SM$ a partially hyperbolic geodesic flow. Is it true that ϕ^t is Anosov?*

It is known by a theorem of Klingenberg [Kl] and Mané [Ma] that on compact manifolds Anosov geodesic flows have no conjugate points.

Question 13. *Let (M, g) be a compact Riemannian manifold and $\phi^t : SM \to SM$ a partially hyperbolic geodesic flow. Is it true that (M, g) has no conjugate points?*

In the case that the geodesic flow has no focal points, we answered question 1 affirmatively without assumption that M is compact [K]. However, we had to assume that f growth at most linearly.

5. A problem by Frederic LeRoux

Question 14. *Is it true that, for any smooth (C^∞) compact manifold M, every element of $\mathrm{Diff}_0(M)$ (the group of all smooth diffeomorphisms of M that are isotopic to the identity) can be expressed as the commutator $[a, b] = aba^{-1}b^{-1}$ of two elements a, b in $\mathrm{Diff}_0(M)$?*

This is a question about the *commutator width* of $\mathrm{Diff}_0(M)$. It is well-known that for any compact manifold M, the group $\mathrm{Diff}_0(M)$ is simple. Hence any element of $\mathrm{Diff}_0(M)$ is a product of a finite number of commutators. Given some element $g \in \mathrm{Diff}_0(M)$, the *commutator length* of g is the minimum number k such that g is the product of k commutators. The commutator width $cw(\mathrm{Diff}_0(M))$ is the maximum of the commutator length of elements of $\mathrm{Diff}_0(M)$. Hence the question is to determine whether $cw(\mathrm{Diff}_0(M)) = 1$ for every compact manifold.

Burago, Ivanov and Polterovich have recently shown that, for many manifolds M, $cw(\mathrm{Diff}_0(M)) < \infty$. This is true in particular for every sphere $\mathbb{S}^n$, and every compact manifold in dimension three (this has been generalized by Tsuboi to every odd-dimensional compact manifold). In the case of the circle, it is even known that $cw(\mathrm{Diff}_0(\mathbb{S}^1)) \leq 2$. But, to my knowledge, there is no manifold M for which the precise value of $cw(\mathrm{Diff}_0(M))$ is known.

6. A problem by Yong-Geun Oh

Consider the following groups:
The group $Homeo^\Omega(D^2, \partial D^2)$ of area preserving homeomorphisms of the standard 2-dimensional disc, that are identity near the boundary;
The group $Homeo_0^\Omega(S^2)$ of area preserving homeomorphisms of the standard 2-dimensional sphere, that are isotopic to the identity.

The following conjecture, originally posed by J. Mather, is still open:

Conjecture 2. Are the groups $Homeo^\Omega(D^2, \partial D^2)$, $Homeo_0^\Omega(S^2)$ simple?

A possible answer could come from the C^0 symplectic topology and continuous Hamiltonian dynamics. The groups of Hamiltonian homeomorphisms $Homeo(D, \partial D^2, \Omega)$ of the disc D^2 and $Homeo(S^2, \Omega)$ of the sphere S^2 are normal subgroups of $Homeo^\Omega(D^2, \partial D^2)$ and $Homeo_0^\Omega(S^2)$ respectively (we refer the reader to the paper "The group of Hamiltonian homeomorphisms and C^0 symplectic topology" by Y.-G. Oh and S. Müller, for the definitions of the group of Hamiltonian homeomorphisms, and more generally, for foundations of the C^0 symplectic topology). A positive answer to the following open question will provide a solution to the conjecture of J. Mather:
Are the groups $Homeo(D, \partial D^2, \Omega)$ and $Homeo(S^2, \Omega)$ proper subgroups of $Homeo^\Omega(D^2, \partial D^2)$ and $Homeo_0^\Omega(S^2)$ respectively?

7. Three questions by Mark Sapir

Let G be the group with following presentation

$$< x, y, t | txt^{-1} = xy, tyt^{-1} = yx > \cong < x, t | [[x, t], t] = x >.$$

Note that G is a hyperbolic group having cohomological dimension 2.

Question 15. *Is G linear?*

Conjecture: "no". The group is residually finite (Borisov and Sapir). That would be an easy example of a non-linear hyperbolic group.

Question 16. *Does G contain $\pi_1(S_g)$? Here S_g is the closed surface with genus $g \geq 2$.*

Conjecture: "no". That would solve a problem of Gromov.

Let F_n be the free group of rank $n \geq 2$, and ϕ be an irreducible proper injective endomorphism $F_n \to F_n$. Let G be the fundamental group of the mapping torus of ϕ.

Question 17. *Can G be embeddable into $SL_2(\mathbb{C})$?*

Conjecture: "no" is supported by computer experiments. By a result of D. Calegari and N. Dunfield, the irreducibility condition cannot be dropped.

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