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Analysis and Geometric Singularities

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Abstract. The Conference on "Analysis and Geometric Singularities" took place from June 27 to July 3, 2010 and had 53 participants. The organization of the meeting followed the well-established scheme, providing plenty of discussion time which was intensely used, especially by the young participants. The four survey talks were given by Gilles Carron, Jean-Michel Bismut, Ulrich Bunke and Xiaonan Ma.

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Introduction by the Organisers

The Conference on "Analysis and Geometric Singularities" took place from June 27 to July 3, 2010. The invitation to this meeting had a particularly positive response since only three invitees turned down the invitation with regret, all others agreed to come, bringing the number of participants to a total of 53. The organization of the meeting followed the already well-established scheme of five talks a day (except Wednesday), including one survey talk plus four talks on Wednesday; seven talks among the 24 were reserved for young participants. This provided plenty of discussion time which was intensely used, especially by the young participants, such that the workshop was characterized by an intense and vibrant atmosphere.

The four survey talks provided a certain structuring of the programme into four major areas of discussion as follows. The first topic (Gilles Carron, Monday) touched upon the analysis of heat kernels and resolvents on complete manifolds which were later also discussed on incomplete singular spaces and on spaces with various different regimes at infinity. The second topic (Jean-Michel Bismut, Tuesday) dealt with conceptual principles of index theory on quite general spaces and the role of a new class of hypoelliptic geometric operators in this field. Other talks dealt with the construction of the Chern character in singular situations, notably with noncommutative methods, or with equivariant index theory, including a proof of a conjecture of Vergne. The third major topic (Ulrich Bunke, Thursday) explained new topological constructions and applications, which was expanded by a higher signature theorem for Riemannian pseudomanifolds with the Witt property, and also various computations and applications of the analytic torsion, including also some number theory. The last main topic (Xiaonan Ma, Friday) concerned geometric applications of asymptotic spectral analysis, ranging from non-compact symplectic manifolds to the heat trace expansion on a Euclidian polygon.

Among the more stunning insights during this conference was the fact, corroborated in quite a few talks, that old problems of classical type or such as were intensely discussed when our series of workshops started in 1987, become accessible only now, apparently because the concepts and the techniques developed in the field have achieved a certain maturity, which raises high expectations for the future and which seems especially attractive for the young researchers who were present at the workshop.

Workshop: Analysis and Geometric Singularities

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Abstracts

Finding upper bounds for the Heat- & Green kernel

Gilles Carron

When (M^n, g) is a complete (non compact) Riemannian manifold, the Laplace operator is defined in local coordinates by

$$\Delta = -\frac{1}{\Theta}\sum_{i,j}\frac{\partial}{\partial x_i}\Theta g^{i,j}\frac{\partial}{\partial x_j},$$

where $\Theta = \sqrt{\det[g_{i,j}]}$.

The Laplace operator $\Delta : C_0^\infty(M) \to L^2(M)$ has a unique selfadjoint extension.

The heat kernel. The operator $e^{-t\Delta}$ has a smooth Schwartz kernel, the heat kernel $h_t(x,y)$: For $f \in C_0^\infty(M)$, we have that

$$f_t(x) := \left(e^{-t\Delta}f\right)(x) = \int_M h_t(x,y)f(y)dy$$

solves the evolution equation

$$\begin{cases} \frac{\partial}{\partial t}f_t + \Delta f_t = 0 & \text{on } (0,+\infty)\times M \\ f_0 = f \end{cases}$$

The Green kernel. If moreover, there is some $x, y \in M$ such that

$$\int_1^{+\infty} h_t(x,y)dt < \infty,$$

then for all $x \neq y$, we can define the *Green kernel*

$$G(x,y) = \int_0^{+\infty} h_t(x,y)dt.$$

And if $f \in C_0^\infty(M)$ then

$$u(x) := \int_M G(x,y)f(y)dy$$

solves the equation

$$\Delta u = f$$

with the extra property that $f \geq 0 \Rightarrow u \geq 0$.

In this survey talk, I introduced several ideas that lead to an Euclidean type upper bound on the heat kernel or on the Green kernel:

$$\forall x, y \in M, \forall t > 0, \quad h_t(x,y) \leq \frac{C}{t^{n/2}},$$

or

$$\forall x, y \in M, \quad G(x,y) \leq \frac{C}{d(x,y)^{n-2}}.$$

For a general account on this problem, there is a survey by T. Coulhon ([1]) and a book of A. Grigor'yan ([2]).

For instance, we explained the techniques of A. Grigor'yan and L. Saloff-Coste ([3]), and we showed how these results can be used to obtain a result of D. Joyce ([4]).

Theorem. *On a Q.A.L.E. manifold* $(M^{n>2}, g)$*, the above Euclidean type upper bound holds; in particular*

$$G(x,y) \leq \frac{C}{d(x,y)^{n-2}}.$$

Here a Quasi Asymptotically Locally Euclidean (Q.A.L.E.) manifold is a complete Riemannian manifold whose geometry at infinity is built upon a certain resolution of a quotient $\mathbb{R}^n/\Gamma$ where Γ is a finite subgroup of $SO(n)$.

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Equivariant index and Chern character

RICHARD B. MELROSE

(joint work with Pierre Albin and Frédéric Rochon)

Consider three forms of the index map in K-theory:

$$\text{ind} : K_{\mathrm{c}}(T^*(M/Y)) \longrightarrow K(Y) \tag{1}$$

$$\text{ind} : K_{G,\mathrm{c}}(T^*_G Z) \longrightarrow R_G \tag{2}$$

$$\text{ind} : K_{\mathrm{c}}(T^*Z/\sim) \longrightarrow \mathbb{Z}. \tag{3}$$

These maps correspond respectively to the families index theorem of Atiyah and Singer for fibrewise pseudodifferential operators for a smooth fibration of a compact manifold $M \longrightarrow Y$, to the equivariant index for invariant and transversally elliptic operators for the smooth action of a compact Lie group on a compact manifold Z and to the pseudodifferential extension of the numerical index map of Atiyah, Patodi and Singer for a compact manifold with boundary Z. The K-groups forming the domains are respectively the topological (compactly supported) K-theory of the fibrewise contangent bundle of the fibration, the G-equivariant K-theory of the (generally singular) space of fibre conormals of the group action and the compactly supported K-theory of the cotangent bundle of the manifold with boundary with the fibration of the boundary smashed to its base, the conormal line. The targets for the first two maps are the K-theory of the base and the ring of virtual representations of the group.

In this talk I had planned to try to present approaches to the analytic definitions of these maps, and the corresponding theorems – identifying them with topological push-forward maps – which I hoped would indicate how one can freely combine them to give an equivariant, families index of Atiyah-Patodi-Singer type. Unfortunately due to limitiations of time I will not be able to discuss the boundary case (in joint work with Frédéric Rochon) so will be content here with discussing an approach to a families equivariant index theorem. I will also discuss the closely related problem of deriving a Chern character formula, for the image of the index in an appropriate cohomology.

First consider the families index theorem. Here is a diagram of the construction of the analytic index which corresponds to push-forward under the map $\pi : T^*(M/Y) \longrightarrow Y$ which has fibres diffeomorphic to T^*Z, where Z is the model fibre of the fibration $M \longrightarrow Y$:

$$\begin{array}{ccc}
\bigcup_{m,E_+,E_-} \operatorname{iso}(S^*(M/Y); \pi^* E_+, \pi^* E_- \otimes N_m) & \longrightarrow & K_{\mathrm{c}}(T^*(M/Y)) \\
\uparrow \sigma_m & & \\
\bigcup_{E_\pm} \operatorname{Ell}^*(M/Y; E_+, E_-) & & \downarrow \operatorname{ind}_{\mathrm{a}} \\
\uparrow & & \\
\bigcup_{E_\pm} \operatorname{Ell}^*(M/Y; E_+, E_-)_{\mathrm{reg}} & \longrightarrow & K(Y)
\end{array}$$

The 'regular' elliptic families are those with null spaces of constant dimension. The 'model' for $K_{\mathrm{c}}(T^*(M/Y))$ is pairs of vector bundles over the base of the radial compactification, $\overline{T^*(M/Y)} \supset T^*(M/Y)$, of the fibrewise cotangent bundle with an identification (the symbol of the operator) between them over the boundary (the sphere at infinity).

There is a semiclassical version of the index based on a different model for K-theory. Namely for a compact manifold with boundary, X, the compactly-supported K-theory of the interior can be realized as equivalence classes of projection-valued maps (idempotents) $\gamma : X \longrightarrow M(N, \mathbb{C})$, $\gamma^2 = \gamma$, which are constant in Taylor series at the boundary $\gamma \equiv \gamma_0 \in M(N, \mathbb{C})$ at ∂X. Here $X = \overline{T^*(M/Y)} \supset T^*(M/Y)$. The algebra of semiclassical pseudodifferential operators has both a 'usual' and a semiclassical symbol map

$$\Psi^0(M/Y; \mathbb{C}^N) \xrightarrow{\sigma_{\mathrm{sl}}} \mathcal{C}^\infty(X; M(N, \mathbb{C}))$$

and this allows one to construct the diagram

$$\begin{array}{ccc}
\bigcup_N \left\{\gamma \in \mathcal{C}^\infty(X; M(N,\mathbb{C})),\ \gamma^2 = \gamma \equiv \gamma_0 \text{ at } \partial X,\ \gamma_0 \in M(N,\mathbb{C})\right\} & \longrightarrow & K_{\mathrm{c}}(T^*(M/Y)) \\
\uparrow \sigma_{\mathrm{sl}} & & \downarrow \mathrm{ind}_{\mathrm{sl}} \\
\bigcup_N \left\{\Gamma \in \Psi^0_{\mathrm{sl}}(M/Y; \mathbb{C}^N),\ \Gamma^2 = \Gamma,\ \sigma_{\mathrm{sl}}(\Gamma) \in M(N,\mathbb{C})\right\} & \longrightarrow & K(Y)
\end{array}$$

That

$$\mathrm{ind}_{\mathrm{a}} = \mathrm{ind}_{\mathrm{sl}} = \mathrm{ind}_{\mathrm{t}}$$

follows by showing that the two representations of the K-theory of the relative cotangent bundle discussed above are each the retraction of a larger model which can be directly quantized using similar constructions but of Toeplitz type. The right-hand inequality is then a reorganization of the proof given by Atiyah and Singer but in place of an axiomatic discussion one can simply follow the semiclassical quantization through the definition of the topological index by embedding. This approach to the index theorem is used in a twisted setting in [2] and a more detailed discussion can be found in [3].

The G-equivariant case arises from the smooth action of a compact group on a compact manifold Z, which gives a smooth homomorphism of the group into the diffeomorphisms of the manifold. The complexity of this setting arises from the variability of the isotropy groups, the subgroups which fix particular points. Again there is a semiclassical realization of the index map, now (2), analogous to that above. Indeed in on-going work with Pierre Albin a families equivariant index will be discussed. This arises from a fibration $M \longrightarrow Y$ where M and Y are compact G-manifolds and the fibration is equivariant. Then the families-equivariant index map induces the push-forward

$$K_{\mathrm{c},G}(T^*_G(M/Y)) \longrightarrow \hat{K}_G(Y)$$

into an appropriately completed equivariant K-theory of the base.

The resolution of group actions, to have fixed isotropy type, in recent work with Albin, allows the delocalized equivariant cohomology of Baum, Brylinski and MacPherson [1] to be extended to the non-Abelian case and to be given a deRham realization (on the resolution of the quotient). This in turn should allow explicit Chern character formulæ to be developed in this more general context – and ultimately in the (here unexplained) boundary case as well.

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The Witten deformation for singular complex curves

Ursula Ludwig

The Witten deformation is a method proposed in [10] by Witten which, given a smooth Morse function $f : M \to \mathbb{R}$ on a smooth compact Riemannian manifold M, leads to an analytical proof of the Morse inequalities. A rigorous account of the analytic proof of the Morse inequalities using semi-classical analysis has been done in [7].

In this note we will explain how to generalise the Witten deformation for a singular complex curve $X \subset \mathbb{P}^n(\mathbb{C})$ and a stratified Morse function $f : X \to \mathbb{R}$ in the sense of [6]. The results presented here were obtained in [8], [9].

Let us denote by Σ the singular set of X and by g be the metric on $X \setminus \Sigma$ induced from the Fubini-Study metric on the ambient projective space. Let us recall the local situation near a singular point $p \in \Sigma$ of multiplicity m_p. Let us first assume that X is unibranched at p. Then there exists an open neighbourhood $U_p \subset X$ of p, such that $(U_p \setminus \{p\}, g)$ is isometric to $\big(\mathrm{cone}(S^1_{m_p}), (1 + O(r^{1/m_p}))(dr^2 + r^2 d\varphi^2)\big)$, see [3]. Here $S^1_{m_p}$ denotes the circle of length $2\pi m_p$. Thus in particular the singular curve is a conformally conic Riemannian manifold in the sense of [4]. Moreover in local coordinates (r, φ) a stratified Morse function has the form

$$f(r,\varphi) = f(p) + r(a\cos\varphi + b\sin\varphi) + O(r^{1+\delta}),\ \delta > 0,\ (a,b) \in \mathbb{R}^2 \setminus \{0\} \tag{1}$$

near the singularity (see [8] for a detailed computation). If X is not unibranched near p, the arguments above can be applied to each branch separately.

In the rest of this note we will explain how to adapt Witten's method to the situation described above, state the main results and explain shortly the idea of proof.

The main principle in Morse theory is to give a relation between a "local datum" of the Morse function, namely the critical points, and a "global topological datum" of the space. For smooth manifolds the latter is the singular cohomology of the manifold. In the presence of singularities the topological invariant of interest is the so called intersection cohomology.

The intersection cohomology for a singular complex curve can be analytically expressed as the cohomology of the complex of L^2-forms: Let $(\Omega^*_0(X \setminus \Sigma), d)$ be the de Rham complex of differential forms acting on smooth forms with compact supports. For conformally conic manifolds the elliptic complex $(\Omega^*_0(X \setminus \Sigma), d)$ admits a unique extension to a Hilbert complex $(\mathcal{C}, d, \langle\ ,\ \rangle)$ in the Hilbert space of square integrable forms equipped with the L^2-metric (see [5], [4]). The L^2-*cohomology* of X, denoted by $H^i_{(2)}(X)$, is defined as the cohomology of this Hilbert complex. (We use the language of Hilbert complexes, as introduced in [2].)

Witten's idea for an analytic proof of the Morse inequalities on a smooth compact manifold consists in the deformation of the de Rham complex by means of a smooth Morse function (see [10], [7]). In the presence of singularities we deform the complex of L^2-forms instead of the de Rham complex. We use a stratified Morse function f for the deformation. In particular we deform the complex $(\Omega_0^*(X \setminus \Sigma), d)$ into

$$0 \to \Omega_0^0(X \setminus \Sigma) \xrightarrow{d_t} \dots \xrightarrow{d_t} \Omega_0^2(X \setminus \Sigma) \to 0,$$

where $d_t = e^{-ft} d e^{ft}$; here $t \in (0, \infty)$ is the deformation parameter. One can show that the deformed complex also admits a unique extension to a Hilbert complex, which is denoted by $(\mathcal{C}_t, d_t, \langle\ ,\ \rangle)$. The map $\omega \to e^{-tf}\omega$ yields an isomorphism of the two complexes $(\mathcal{C}, d, \langle\ ,\ \rangle)$ and $(\mathcal{C}_t, d_t, \langle\ ,\ \rangle)$. Therefore the cohomology of the deformed complex is also isomorphic to the L^2-cohomology of X, i.e. $H^i(\mathcal{C}_t, d_t, \langle\ ,\ \rangle) \simeq H^i_{(2)}(X)$. The Witten Laplacian Δ_t is defined as the Laplacian associated to the Hilbert complex $(\mathcal{C}_t, d_t, \langle\ ,\ \rangle)$. Note that in the presence of singularities $\Delta_t \mid_{\Omega_0^*(X \setminus \Sigma)}$ is not an essentially selfadjoint operator and therefore we have to specify the domain of the Witten Laplacian carefully. Hodge theory is still valid for the deformed complex, i.e.

$$\ker \Delta_t^{(i)} \simeq H^i(\mathcal{C}_t, d_t, \langle\ ,\ \rangle) \simeq H^i_{(2)}(X). \tag{2}$$

For a stratified Morse function f the restriction $f_{|X \setminus \Sigma}$ is a Morse function in the smooth sense and we denote by $c_i(f_{|X\setminus\Sigma})$ the number of critical points of $f_{|X\setminus\Sigma}$ of index i and by

$$c_i(f) := c_i(f_{|X\setminus\Sigma}), i = 0, 2; \quad c_1(f) := c_1(f_{|X\setminus\Sigma}) + \sum_{p\in\Sigma}(m_p - b_p),$$

where b_p is the number of analytic branches of X at p. The advantage of the deformed complex $(\mathcal{C}_t, d_t, \langle\ ,\ \rangle)$ compared to the initial complex $(\mathcal{C}, d, \langle\ ,\ \rangle)$ is that the spectrum of the Witten Laplacian has nice properties for large parameters t:

Theorem 1 (Spectral Gap Theorem)**.**

(1) Let X be a singular complex curve and let $f : X \to \mathbb{R}$ be a stratified Morse function. Then there exist constants $C_1, C_2, C_3 > 0$ and $t_0 > 0$ depending on X and f such that for any $t \geq t_0$,

$$\operatorname{spec}(\Delta_t) \cap (C_1 e^{-C_2 t}, C_3 t) = \emptyset.$$

(2) Let us denote by $(\mathcal{S}_t, d_t, \langle\ ,\ \rangle)$ the subcomplex of $(\mathcal{C}_t, d_t, \langle\ ,\ \rangle)$ generated by all eigenforms of the Witten Laplacian Δ_t to eigenvalues in $[0, 1]$. Then, for $t \geq t_0$,

$$\dim \mathcal{S}_t^i = c_i(f).$$

The following Morse inequalities follow from the spectral gap theorem and (2) by a standard argument

Corollary. *In the situation of Theorem 1, for all $0 \leq k < 2$*

$$\sum_{i=0}^{k}(-1)^{k-i} c_i(f) \geq \sum_{i=0}^{k}(-1)^{k-i} b_i^{(2)}(X), \quad \sum_{i=0}^{2}(-1)^i c_i(f) = \sum_{i=0}^{2}(-1)^i b_i^{(2)}(X),$$

where $b_i^{(2)}(X)$ denote the L^2-Betti numbers of X.

Note that from Corollary one can recover the Morse inequalities for intersection homology with middle perversity known already from stratified Morse theory [6].

The key step in the proof of the spectral gap theorem is the construction of a local model operator $\Delta_t^{p,\mathrm{loc}}$ for the Witten Laplacian near a singular point $p \in \Sigma$ of X. Using the local form (1) one can compute easily that formally

$$\Delta_t^{p,\mathrm{loc}} = \Delta^{p,\mathrm{loc}} + (a^2 + b^2)t^2. \tag{3}$$

Note however that $\mathrm{dom}(\Delta_t^{p,\mathrm{loc}}) \neq \mathrm{dom}(\Delta^{p,\mathrm{loc}})$ on forms of degree 1. The local spectral gap theorem can now be shown by an explicit computation. The forms in the kernel of $\Delta_t^{p,\mathrm{loc}}$ can be computed explicitly in terms of the modified Bessel functions. Thus one can see nicely that from the point of view of the analytic proof, the contribution of the singularity to the Morse inequalities is related to the "small eigenvalues of the transversal Laplacian". Once the local situation near the singularities of X is understood one can proceed as in the smooth case (see e.g. [1], Section 9) to complete the proof of the spectral gap theorem.

In the smooth situation we know from [10] and [7] that, if the gradient vector field satisfies the Morse-Smale transversality condition, the complex of eigenforms of Δ_t to small eigenvalues "converges", for $t \to \infty$, to the so called (geometric) Thom-Smale complex. This result can also be generalised to the singular situation treated here (see [9]).

Acknowledgement. I wish to thank J. M. Bismut for suggesting to work on the subject.

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The higher signature operator on Witt spaces

Pierre Albin
(joint work with Eric Leichtnam, Rafe Mazzeo, and Paolo Piazza)

1. Novikov conjecture

The signature theorem of Hirzebruch identifies the signature of a $4k$-dimensional closed manifold M with the evaluation of a particular cohomology class, the L-class of M, on its fundamental homology class, $[M]$. The signature is a homotopy invariant of M and, if M is simply connected, it is in a certain sense the unique homotopy invariant of M.

If M is not simply-connected there are other invariants of M, the higher signatures, and the Novikov conjecture is that these are homotopy invariant. If G is the fundamental group of M, and $EG \longrightarrow BG$ is the universal principal G-bundle, then there is a map $r : M \longrightarrow BG$ classifying the universal cover $\widetilde{M}$ of M. With this data each cohomology class $[c]$ of BG defines a higher signature: one pulls-back $[c]$ via r, takes the cup product with the cohomology L-class of M, and evaluates on the fundamental homology class of M,

$$\sigma([c]) = (r^*[c] \cup L(M), [M]).$$

The analytic approach to the Novikov conjecture consists in assembling these signatures together into a single object, the index of the 'higher signature operator,' defined as follows: If $C^*_r G$ is the reduced C^*-algebra of G (the closure of G within the bounded operators on $\ell^2(G)$), and $\mathcal{V} = \widetilde{M} \times_G C^*_r G$, then the higher signature operator $\widetilde{D}_{\text{sign}}$ is the signature operator of M twisted by $\mathcal{V}$.

The index of $\widetilde{D}_{\text{sign}}$ is an element of $K_*(C^*_r G)$ and has been shown to be equal to Mischenko's symmetric signature, which is known to be homotopy invariant. Novikov's conjecture would follow if we knew that the assembly map were injective, a possibility referred to as the 'strong Novikov conjecture'. The advantage is that the latter conjecture involves the group G, but not the manifold M, so one can think that the geometric part of the conjecture has been resolved. In the project being reported on we show that the analogue of the Novikov conjecture on stratified manifolds also 'reduces' to the strong Novikov conjecture.

2. Stratified spaces

A stratified space is a finite union of manifolds, known as the strata, together with (Thom-Mather) data specifying how the manifolds fit together. The highest dimensional strata is known as the 'regular part' and its complement is known as the 'singular part'. Each singular stratum S has a tubular neighborhood within the regular stratum that can be identified with a bundle over the S with fiber the cone over another, simpler, stratified space L known as the 'link' of the cone. Thus locally a stratified space can be described as a closed manifold, or the cone over a closed manifold, or the product of a cone over a closed manifold with a closed manifold, or the cone over the product of a cone with a closed manifold, etc.

Our approach, following unpublished work of Rafe Mazzeo with Richard Melrose, is to resolve a stratified space $\widehat{X}$ to a manifold with corners $\widetilde{X}$. The interior X of $\widetilde{X}$ is diffeomorphic to the regular part of $\widehat{X}$, and each of the boundary hypersurfaces of $\widetilde{X}$ corresponds, indeed fibers over, a singular stratum of $\widehat{X}$. The fibrations are compatible at the corners and form what is known as a 'resolution structure' or 'iterated fibration structure.' The Thom-Mather data of $\widehat{X}$ becomes the usual differential topological data of $\widetilde{X}$, e.g., the tubular neighborhoods correspond to collar neighborhoods of the boundary.

3. Iterated incomplete edge metrics and the signature operator

There is a natural class of metrics on X that reflects the stratified structure in that they degenerate conically as they approach a singular stratum. Thus if H is a boundary hypersurface of $\widetilde{X}$ with associated fibration $Z - H \longrightarrow Y$, and x is a coordinate measuring the distance to H, then the metric asymptotically takes the form

$$dr^2 + r^2 g_Z + g_Y$$

where g_Y is (the pull-back of) a metric on Y and g_Z is a metric of the same type on the simpler space Z. We refer to these metrics as 'incomplete iterated edge' (or *iie*) metrics, and to the associated signature operator as the *iie*-signature operator.

One convenient way of studying these objects is to replace the cotangent bundle of X, T^*X, with a suitable rescaling that better reflects the geometry. Namely, we define the iie-cotangent bundle, ${}^{iie}T^*X$, by specifying that its sections should be the covectors on X that have bounded pointwise length with respect to an iie-metric. One advantage of this approach is that an *iie*-metric defines a non-degenerate metric on ${}^{iie}T^*X$ over all of $\widetilde{X}$ and similarly the *iie*-signature operator induces a non-degenerate (uniformly elliptic) operator on sections of $\Lambda^*({}^{iie}T^*X) = {}^{iie}\Lambda^*X$, which we continue to refer to as the *iie*-signature operator.

From this perspective we carry out an inductive analysis of the *iie*-signature operator succored by the fact that its leading term at a boundary face (alternately at a singular stratum) involves the link of the cone only through its *iie*-signature operator. Indeed, near a boundary face H as above, the differential forms on X can be decomposed as

$$\Lambda^*X = (\Lambda^*Y \wedge \Lambda^*Z) \oplus dx \wedge (\Lambda^*Y \wedge \Lambda^*Z)$$
$${}^{iie}\Lambda^*X = (\Lambda^*Y \wedge x^{\mathbf{N}}\Lambda^*Z) \oplus dx \wedge (\Lambda^*Y \wedge x^{\mathbf{N}}\Lambda^*Z)$$

where $\mathbf{N}$ is the 'vertical number operator', i.e., the map given by multiplication by k when restricted to forms of vertical degree k, and the de Rham operator of X takes the form

$$(d_X + \delta_X) \sim \begin{pmatrix} \frac{1}{x}(d_Z + \delta_Z) + (d_Y + \delta_Y) & -\partial_x - \frac{1}{x}(f - \mathbf{N}) \\ \partial_x + \frac{1}{x}\mathbf{N} & -\frac{1}{x}(d_Z + \delta_Z) - (d_Y + \delta_Y) \end{pmatrix}.$$

We are able to define model operators at each boundary face and prove that these are invertible provided that the Witt condition holds, i.e., that the link is odd-dimensional or, if it is even dimensional, that its middle-degree, middle-perversity, intersection cohomology vanish. In this situation we put the inverses together to form a parametrix for $d_X + \delta_X$, hence for the signature operator, which leads to the following theorem:

Theorem (A.-Leichtnam-Mazzeo-Piazza) *Let $\widehat{X}$ be a Witt space with suitably scaled iie-metric.*

i) *As an unbounded operator on L^2 with core domain $\mathcal{C}_c^\infty$, the iie-signature operator D_{sign} has a unique closed extension and is essentially self-adjoint.*
ii) *The domain of this closed extension is compactly contained in L^2.*
iii) *D_{sign} is Fredholm.*
iv) *D_{sign} has discrete spectrum of finite multiplicity.*

Most of this was proven by Cheeger by different methods that in this context yield stronger results (e.g., an analysis of the associated heat kernel and its trace). Our methods however generalize easily to handle the higher signature operator described above.

Theorem (A.-Leichtnam-Mazzeo-Piazza) *Let $\widehat{X}$ be a Witt space with suitably scaled iie-metric and $\widetilde{D}_{\text{sign}}$ the signature operator with values in $\widetilde{M} \times_G C_r^* G$*

i) *$\widetilde{D}_{\text{sign}}$ has a unique closed extension and is essentially self-adjoint.*
ii) *$\widetilde{D}_{\text{sign}}$ defines a higher signature index class $\text{Ind}(\widetilde{D}_{\text{sign}}) \in K_*(C_r^* G)$ which is a Witt-bordism invariant and invariant under stratified homotopies.*
iii) *Push-forward by the classifying map $r : M \to BG$ followed by the assembly map sends the K-homology signature class $[D_{\text{sign}}] \in K_*(X)$ to the higher signature index class.*
iv) *The Novikov conjecture for stratified spaces 'reduces' to the strong Novikov conjecture.*

As in the case of closed manifolds, there is a topologically defined symmetric signature in $K_*(C_r^* G)$, due in this context to Banagl. We use a uniqueness result of Sullivan to show that Banagl's signature coincides with the higher signature index class (rationally). In contrast to the case of closed manifolds, the topologically defined class is not known to be a stratified homotopy invariant, though by our theorem it is a stratified homotopy invariant over the rationals.

Relative Connes-Chern character for manifolds with boundary

Matthias Lesch
(joint work with Henri Moscovici and Markus J. Pflaum)

Let M be a compact smooth m-dimensional manifold with boundary $\partial M \neq 0$. Assuming that M possesses a Spin^c structure, the fundamental class in the relative K-homology group $K_m(M, \partial M)$ can be realized analytically in terms of the Dirac operator D associated to the given Spin^c structure and to a Riemannian metric on M. This class is denoted by $[\mathsf{D}] \in K_m(M, \partial M)$.

The index map $\mathrm{Index}_{[\mathsf{D}]} : K^\bullet(M, \partial M) \to \mathbb{Z}$, defined by the pairing of $[\mathsf{D}]$ with the K-theory, can be expressed in cyclic cohomological terms by means of Connes' Chern character in K-homology [4]. Since

$$K_\bullet(M \setminus \partial M) \simeq KK_\bullet(\mathcal{J}^\infty(M, \partial M); \mathbb{C}),$$

one can compute the Connes-Chern character of $[\mathsf{D}]$ by restricting D, to the dense (and closed under holomorphic functional calculus) subalgebra $\mathcal{J}^\infty(M, \partial M) \subset C_0(M \setminus \partial M) = C\big(\{f \in C(M) \mid f_{|\partial M} = 0\}\big)$ of functions vanishing to infinite order at ∂M. The resulting periodic cyclic cocycle, which can be computed as in [4, Part I, §6], corresponds via the canonical isomorphism $HP^\bullet\big(\mathcal{J}^\infty(M, \partial M)\big) \simeq H_\bullet^{\mathrm{dR}}(M \setminus \partial M; \mathbb{C})$ [2] to the relative de Rham class of the $\hat{A}$-current associated to the Riemannian metric.

It is the purpose of the present project (for details see [7]) to find explicit representations for the Connes-Chern character of the fundamental relative K-homology class $[\mathsf{D}] \in K_\bullet(M, \partial M)$ that allow to retain geometric information about the boundary. A significant step in this direction has already been taken by Getzler [5], who used the setting of Melrose's b-calculus [8] to construct an entire version of the relative Connes-Chern character. Devised for the treatment of infinite-dimensional geometries, entire cyclic cohomology is less effective than periodic cyclic cohomology in the finite-dimensional case. To remedy this drawback, we give explicit cocycle realizations for the Connes-Chern character in the relative cyclic cohomology bicomplex associated to the pair of algebras $\big(C^\infty(M), C^\infty(\partial M)\big)$. This is achieved by adapting to the relative context the retraction procedure of [3], which converts the entire Connes-Chern character into the periodic one.

More concretely, we fix an exact b-metric g on M, and denote by D the corresponding b-Dirac operator. We define for each $t > 0$ and any $n \geq m = \dim M$, $n \equiv m \pmod 2$, a pair of cochains $\big({}^{\mathrm{b}}\mathrm{ch}_t^n(\mathsf{D}), \mathrm{ch}_t^{n+1}(\mathsf{D}_\partial)\big)$ over $\big(C^\infty(M), C^\infty(\partial M)\big)$, given by the following formulæ

$$\begin{aligned} {}^{\mathrm{b}}\mathrm{ch}_t^n(\mathsf{D}) &:= \sum_{j \geq 0} {}^{\mathrm{b}}\mathrm{Ch}^{n-2j}(t\mathsf{D}) + B\, {}^{\mathrm{b}}\mathrm{T\not{c}h}_t^{n+1}(\mathsf{D}) \\ \mathrm{ch}_t^{n+1}(\mathsf{D}_\partial) &:= \sum_{j \geq 0} \mathrm{Ch}^{n-2j+1}(t\mathsf{D}_\partial) + B\, \mathrm{T\not{c}h}_t^{n+2}(\mathsf{D}_\partial); \end{aligned}$$

here $\mathrm{Ch}^\bullet(\mathsf{D}_\partial)$ denote the components of the Jaffe-Lesniewski-Osterwalder realization [6] of the Connes-Chern character in entire cyclic cohomology, ${}^{\mathrm{b}}\mathrm{Ch}^\bullet(\mathsf{D})$ stand for their b-counterparts, and the components $\mathrm{T\not{c}h}_t^\bullet(\mathsf{D}_\partial)$, resp. ${}^{\mathrm{b}}\mathrm{T\not{c}h}_t^\bullet(\mathsf{D})$, are manufactured out of the canonical transgression formula as in [3].

Our main results about these cocycles are:

1.
$$(b + B)\big({}^{\mathrm{b}}\mathrm{ch}_t^n(\mathsf{D})\big) = \mathrm{ch}_t^{n+1}(\mathsf{D}_\partial) \circ i^*.$$

Hence $\big({}^{\mathrm{b}}\mathrm{ch}_t^n(\mathsf{D}), \mathrm{ch}_t^{n+1}(\mathsf{D}_\partial)\big)$ is a cocycle in the relative total (b, B)-complex of the pair of algebras $(C^\infty(M), C^\infty(\partial M))$. $i : \partial M \to M$ denotes the inclusion.

2. Its class in $HC^n(C^\infty(M), C^\infty(\partial M))$ is independent of $t > 0$ and its class in $HP^\bullet(C^\infty(M), C^\infty(\partial M))$ is independent of n.

3.

$$\lim_{t\searrow 0}\big({}^{\mathrm{b}}\mathrm{ch}_t^n(\mathsf{D}), \mathrm{ch}_t^{n-1}(\mathsf{D}_\partial)\big) = \Big(\int_{{}^{\mathrm{b}}M} \hat{A}({}^{\mathrm{b}}\nabla_g^2)\wedge -, \int_{\partial M} \hat{A}(\nabla_{g\partial}^2)\wedge -\Big),$$

thus $\big[{}^{\mathrm{b}}\mathrm{ch}_t^n(\mathsf{D}), \mathrm{ch}_t^{n+1}(\mathsf{D}_\partial)\big] \in HP^\bullet\big(C^\infty(M), C^\infty(\partial M)\big) \cong H_\bullet^{\mathrm{dR}}(M, \partial M)$ does represent the Chern character of $[\mathsf{D}] \in K_m(M, \partial M)$.

4. Under the assumption that $\ker \mathsf{D}_\partial = 0$, the pair of retracted cochains $\big({}^{\mathrm{b}}\widetilde{\mathrm{ch}}_t^n(\mathsf{D}), \mathrm{ch}_t^{n-1}(\mathsf{D}_\partial)\big)$ has a limit as $t \to \infty$. For n even, or equivalently M even-dimensional, the limit is

$$\begin{aligned} {}^{\mathrm{b}}\mathrm{ch}_\infty^n(\mathsf{D}) &= \sum_{j=0}^{n/2} \daleth^{2j}(\mathsf{D}) + B\,{}^{\mathrm{b}}\mathrm{T}\!\not{c}\mathrm{h}_\infty^{n+1}(\mathsf{D}), \\ \mathrm{ch}_\infty^{n+1}(\mathsf{D}_\partial) &= \qquad\qquad B\,\mathrm{T}\!\not{c}\mathrm{h}_\infty^{n+2}(\mathsf{D}_\partial), \end{aligned}$$

with the cochains $\daleth^\bullet(\mathsf{D})$, occurring only when $\ker \mathsf{D} \neq \{0\}$, given by

$$\daleth^{2j}(\mathsf{D})(a_0, \ldots, a_{2j}) = \mathrm{Str}\big(\varrho_H(a_0)\,\omega_H(a_1, a_2)\cdots\omega_H(a_{2j-1}, a_{2j})\big);$$

here H denotes the orthogonal projection onto $\ker \mathsf{D}$, and

$$\varrho_H(a) := HaH\,, \quad \omega_H(a,b) := \varrho_H(ab) - \varrho_H(a)\varrho_H(b), \text{ for all } a, b \in C^\infty(M).$$

The Pairing formula. The class $\big[{}^{\mathrm{b}}\mathrm{ch}_t^n(\mathsf{D}), \mathrm{ch}_t^{n+1}(\mathsf{D}_\partial)\big]$ pairs with relative K-theory classes in $K^0(M, \partial M)$. Such a class can be represented as a triple $[E, F, h]$, where E, F are vector bundles over M, which we will identify with projections $p_E, p_F \in \mathrm{Mat}_N(C^\infty(M))$, and $h : [0,1] \to \mathrm{Mat}_N(C^\infty(\partial M))$ is a smooth path of projections connecting their restrictions to the boundary E_∂ and F_∂.

We prove that the pairing between $[\mathsf{D}] \in K_0(M, \partial M)$ and $[E, F, h] \in K^0(M, \partial M)$ equals

$$\langle [\mathsf{D}], [E, F, h]\rangle = \mathrm{Ind}_{\mathrm{APS}}\,\mathsf{D}^F - \mathrm{Ind}_{\mathrm{APS}}\,\mathsf{D}^E + \mathrm{SF}(h, \mathsf{D}_\partial); \tag{1}$$

here $\mathrm{Ind}_{\mathrm{APS}}$ stands for the APS-index, and $\mathrm{SF}(h, \mathsf{D}_\partial)$ denotes a certain spectral flow associated to D_∂ and h.

The Chern character of $[E, F, h] \in K^0(M, \partial M)$ is represented by the relative cyclic homology cycle over the algebras $\big(C^\infty(M), C^\infty(\partial M)\big)$

$$\mathrm{ch}_\bullet\big([E, F, h]\big) = \Big(\mathrm{ch}_\bullet(F) - \mathrm{ch}_\bullet(E)\,,\, -\mathrm{T}\!\not{c}\mathrm{h}_\bullet(h)\Big),$$

where $\mathrm{ch}_\bullet$, resp. $\mathrm{T}\!\not{c}\mathrm{h}_\bullet$ denote the components of the standard Chern character in cyclic homology resp. of its canonical transgression. Since $\big[{}^{\mathrm{b}}\mathrm{ch}_t^n(\mathsf{D}), \mathrm{ch}_t^{n+1}(\mathsf{D}_\partial)\big]$

equals the Chern character of $[\mathsf{D}]$ we now obtain for *any* $t > 0$:

$$\begin{aligned}\big\langle [\mathsf{D}],\, [E,F,h]\big\rangle &= \big\langle \big({}^{\mathrm{b}}\widetilde{\mathrm{ch}}_t^{\,n}(\mathsf{D}), \mathrm{ch}_t^{n+1}(\mathsf{D}_\partial)\big),\, \mathrm{ch}_\bullet[E,F,h]\big\rangle \\ &= \big\langle \sum_{j\geq 0} {}^{\mathrm{b}}\mathrm{Ch}^{n-2j}(t\mathsf{D}) + B\ {}^{\mathrm{b}}\mathrm{T}\!\not{c}\mathrm{h}_t^{n+1}(\mathsf{D}),\, \mathrm{ch}_\bullet(F) - \mathrm{ch}_\bullet(E)\big\rangle \\ &\qquad - \big\langle \sum_{j\geq 0} \mathrm{Ch}^{n-2j+1}(t\mathsf{D}_\partial) + B\, \mathrm{T}\!\not{c}\mathrm{h}_t^{n+2}(\mathsf{D}_\partial),\, \mathrm{T}\!\not{c}\mathrm{h}_\bullet(h)\big\rangle.\end{aligned}$$

The situation is now reminiscent of the celebrated McKean-Singer formula. Indeed, letting $t \to 0$ yields the local form of the pairing formula

$$\int_{{}^{\mathrm{b}}M} \hat{A}({}^{\mathrm{b}}\nabla_g^2) \wedge \big(\mathrm{ch}_\bullet(F) - \mathrm{ch}_\bullet(E)\big) - \int_{\partial M} \hat{A}(\nabla_{g\partial}^2) \wedge \mathrm{T}\!\not{c}\mathrm{h}_\bullet(h).$$

Under the additional assumption that D_∂ is invertible the limit as $t \to \infty$ yields

$$\big\langle \sum_{0\leq k\leq \ell} \daleth^{2k}(\mathsf{D}) + B\, {}^{\mathrm{b}}\mathrm{T}\!\not{c}\mathrm{h}_\infty^{n+1}(\mathsf{D}), \mathrm{ch}_\bullet(F) - \mathrm{ch}_\bullet(E)\big\rangle - \big\langle B\, \mathrm{T}\!\not{c}\mathrm{h}_\infty^{n+2}(\mathsf{D}_\partial), \mathrm{T}\!\not{c}\mathrm{h}_\bullet(h)\big\rangle.$$

Comparing the limit as $t \to 0$ and as $t \to \infty$ gives, analogously to the Atiyah-Patodi-Singer Index Theorem, the following equation for all $n = 2\ell \geq m$

$$\begin{aligned}&\big\langle \sum_{0\leq k\leq \ell} \daleth^{2k}(\mathsf{D}),\, \mathrm{ch}_\bullet(F) - \mathrm{ch}_\bullet(E)\big\rangle \\ &= \int_M \hat{A}({}^{\mathrm{b}}\nabla_g^2) \wedge \big(\mathrm{ch}_\bullet(F) - \mathrm{ch}_\bullet(E)\big) - \int_{\partial M} \hat{A}(\nabla_{g\partial}^2) \wedge \mathrm{T}\!\not{c}\mathrm{h}_\bullet(h) \\ &\quad - \frac{\sqrt{\pi}}{2}\big\langle {}^{\mathrm{b}}\eta^{n+1}(\mathsf{D}), B\big(\mathrm{ch}_\bullet(F) - \mathrm{ch}_\bullet(E)\big)\big\rangle + \frac{\sqrt{\pi}}{2}\big\langle \eta^{n+2}(\mathsf{D}_\partial), B\, \mathrm{T}\!\not{c}\mathrm{h}_\bullet(h)\big\rangle.\end{aligned}$$

The left hand side is an interaction between the Chern character of $[E,F,h]$ and the *kernel* of the operator D while the right hand side is the sum of a local term and the pairing between a higher eta–cochain and $\mathrm{ch}_\bullet(E,F,h)$.

Invoking (1) and the APS Index Theorem (now with a metric on M which is non–degenerate and smooth up to the boundary) gives relations between higher eta pairings and eta invariants:

$$\begin{aligned}&\int_M \hat{A}(\nabla_g^2) \wedge \big(\mathrm{ch}_\bullet(F) - \mathrm{ch}_\bullet(E)\big) - \Big(\xi(\mathsf{D}_\partial^{F_\partial}) - \xi(\mathsf{D}_\partial^{E_\partial})\Big) + \mathrm{SF}(h, \mathsf{D}_\partial) \\ =&\big\langle \sum_{0\leq k\leq \ell} \daleth^{2k}(\mathsf{D}), \mathrm{ch}_\bullet(F) - \mathrm{ch}_\bullet(E)\big\rangle \\ &+ \frac{\sqrt{\pi}}{2}\big\langle {}^{\mathrm{b}}\eta^{n+1}(\mathsf{D}), B\big(\mathrm{ch}_\bullet(F) - \mathrm{ch}_\bullet(E)\big)\big\rangle - \frac{\sqrt{\pi}}{2}\big\langle \eta^{n+2}(\mathsf{D}_\partial), B\, \mathrm{T}\!\not{c}\mathrm{h}_\bullet(h)\big\rangle,\end{aligned}$$

where $\xi(\mathsf{D}_\partial^{E_\partial}) = \frac{1}{2}\Big(\eta(\mathsf{D}_\partial^{E_\partial}) + \dim\ker \mathsf{D}_\partial^{F_\partial}\Big)$.

Finally on $N = \partial M$ we obtain

$$\xi(\mathsf{D}_\partial^{F_\partial}) - \xi(\mathsf{D}_\partial^{E_\partial}) - \mathrm{SF}(h, \mathsf{D}_\partial) = \int_{\partial M} \hat{A}(\nabla_{g\partial}^2) \wedge \mathrm{T}\!\not{c}\mathrm{h}_\bullet(h)\,,$$

which is a generalization of APS index theorem for flat bundles.

Details can be found in [7].

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Index theorem and the hypoelliptic Laplacian

Jean-Michel Bismut

The purpose of the talk was to review various aspects of index theory, with an emphasis on the Gaussian aspect of the theory.

Various questions were reviewed in the talk:

- The algebraic de Rham complex $(\mathcal{A}(V), d)$ of a vector space V was described. Once a scalar product is fixed, the Bargmann isomorphism identifies the proper completion of $\mathcal{A}(V)$ and the Hilbert space of L_2 forms on V equipped with a Witten twist of the de Rham operator. The Bargmann isomorphism exchanges Dirac masses and Gaussian distributions.

  If X is a smooth manifold, its exterior algebra $\Lambda^{\cdot}(T^*X)$ has been of constant use in de Rham theory. If X is a Riemannian manifold, the symmetric algebra analogue of the operator $d - d^*$ acting on smooth forms is the generator of the geodesic flow ∇_Y.

- The McKean-Singer formula for the index of a Dirac operator is a Gaussian like multiplicative formula. I explained that the index formula for the index has exactly the same Gaussian character. This is obviously true for the Chern character form, less obvious for the form $\widehat{A}\left(TX, \nabla^{TX}\right)$, which is a generalised superconnection form associated with the Levi-Civita superconnection of the tangent bundle [1].

  The fact that both sides of the index formula have the same Gaussian character suggests that they are just one aspect of the same object. Such considerations play an important in the construction of the hypoelliptic Laplacian.

- The hypoelliptic Laplacian [2, 3] is a method to reconstruct the elliptic Dirac operator from its local index theory. The construction of the hypoelliptic Dirac operator [3] has been described in some detail. This operator is obtained as an easy perturbation of the Levi-Civita superconnection on the tangent bundle. The two sides of the index formula are now put on the same Gaussian footing.
- The scalar version of the hypoelliptic Laplacian is an operator acting on the total space of the tangent bundle of a Riemannian manifold. It is a weighted sum of the harmonic oscillator along the fibre, and of the generator of the geodesic flow. It deforms the classical Laplace-Beltrami operator on the base. The square of the distance is the natural action associated with the Laplace-Beltrami operator. Its analogue for the hypoelliptic Laplacian was described. It was emphasized that it does not defined a distance on the total space of the tangent bundle. In the case where the base manifold is an Euclidean vector space, uniform bounds were given for the action as the deformation parameter b tends to 0. When X is a symmetric space, the uniform bounds obtained in [4] for the hypoelliptic heat kernel were explained, in relation with the estimates on the action functional.

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Heat trace in a polyhedron

Leonid Friedlander

Let P be an n-dimensional Euclidean polyhedron, let Δ be the Dirichlet Laplacian in P, and let $h(t)$ be the heat trace of P, that is the trace of the operator $\exp t\Delta$. It is well known that, in the case of manifolds M with smooth boundary Γ, one has

$$h(t) \sim \sum_{k=0}^{\infty} a_k t^{(-n+k)/2}.$$

The first two coefficients can be computed easily: a_0 is proportional to the volume of M, and a_1 is proportional the $(n-1)$-dimensional volume of Γ; the proportionality coefficients equal $(4\pi)^{-n/2}$ and $-(1/4)(4\pi)^{-(n-1)/2}$, respectively. The coefficients that follow can be represented as sums of two terms: the first term is the integral over M of a function that depends on components of the curvature tensor and their derivatives, and the second term is an integral over Γ of a function that depends on the components of the curvature tensor, the second fundamental form, and their derivatives. In the case $n = 2$, one has

$$a_2 = \frac{1}{12\pi}\Big(\int_M K(x)\,dm(x) + \int_\Gamma \kappa(x)\,ds(x)\Big)$$

where K is the Gauss curvature and κ is the geodesic curvature of the boundary. By the Gauss–Bonnet theorem, $a_2 = \chi(M)/6$ where $\chi(M)$ is the Euler characteristic of M.

Computation of the coefficients in the heat trace expansion in a polyhedron is much more difficult. The $n = 2$ case is already highly non-trivial. The answer is that the contribution of a polygonal vertex with interior angle α to the heat trace asymptotics equals $(\pi^2 - \alpha^2)/(24\pi\alpha)$. This formula was derived in the PhD dissertation of Fedosov (see [3]). Fedosov studied the asymptotic expansion of the Riesz means, not of the heat trace; however a simple integral trasformation converts his results into results about the heat trace. The same formula was also derived by Ray (apparently unpublished; see [4]). A complete derivation of this formula can be found in [1] and [5]. When one goes from the previous dimension to the next one, one has to understand the constant term; each vertex contributes to it. The problem boils down to the analysis of the heat kernel in an infinite cone that is associated with a vertex. Such kind of analysis was done by Cheeger [2]. We give the answer in terms of a function that is associated with Brownian motion.

Let $B_{0,2}$ be the space of continuous functions $b(t)$ on $[0, 2]$ such that $b(0) = b(2) = 0$, and let $\mu_{0,2}$ be the conditional Wiener measure on $B_{0,2}$. For a given Brownian path $b(t)$, we define a function

$$\xi(r; b) = \frac{1}{2}\int_0^2 \frac{dt}{(r + b(t))^2} \quad .$$

This function is defined for $r > -\min\{b(t) : 0 \le t \le 2\}$, and it is decreasing. Let $r(\xi; b)$ be the inverse function, and let

$$\bar{r}(\xi) = \int r(\xi, b)\,d\mu_{0,2} \quad .$$

For a vertex x of a polyhedron, by C_x we denote the infinite cone associated with x, an ω_x is the intersection of C_x with the unit sphere centered at x. Let $\theta(t)$ be the heat trace in ω_x (with the Dirichlet boundary conditions), and let

$$p(t) = \exp\Big\{-\frac{(n-1)(n-3)}{4}t\Big\}\theta(t).$$

We consider the following function

$$J(\epsilon) = \int_\epsilon^\infty \bar{r}(\xi)p'(\xi)\,d\xi \quad .$$

It has a complete asymptotic expansion as $\epsilon \to 0$. Our main result is that, up to an explicitly computable expression, the contribution of the vertex x to the heat trace expansion in a polyhedron equals the constant term in the expansion of $J(\epsilon)$ as $\epsilon \to 0$.

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Radiation Field for Einstein Vacuum Equations

Fang Wang

I use the Radiation field theory to study the asymptotic behavior of solutions to Einstein Vacuum equations, which are close to Minkowski space-time $(\mathbb{R}^{1+n}_{t,x}, m)$: $m = -dt^2 + \sum_{i=1}^n (dx^i)^2$.

Radiation field. To study the asymptotic behavior of solutions to linear hyperbolic equations at null infinity, L. Hörmander used the *radiation field* introduced by F.G. Friedlander in [4] using the coordinates

$$\tau = t - |x|, \qquad \rho = |x|^{-1}, \qquad \theta = x/|x|,$$

for $|x|$ large. Consider the Cauchy problem for the wave equation as follows:

$$\Box_m u(t,x) = 0, \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \quad \text{where } u_0, u_1 \in C_c^\infty(\mathbb{R}^n).$$

By writing $u = \rho^{\frac{n-1}{2}} \tilde{u}$ near $\rho = 0$ and studying the equivalent equation

$$\left(\Box_{\tilde{m}} + \tfrac{(n-1)(n-3)}{4}\right)\tilde{u} = 0$$

with the conformal metric $\tilde{m} = \rho^2 m$ near $\rho = 0$, Friedlander showed that $\tilde{u}$ is smooth up to $\rho = 0$. The radiation field is the image of the map

$$\mathcal{R} : \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \ni (u_0, u_1) \longrightarrow \partial_\tau \tilde{u}|_{\rho=0} \in L^2(\mathbb{R}_\tau \times S^{n-1}_\theta), \tag{1}$$

which is an isometric isomorphism. Here $\mathcal{R}$ is essentially the *Møller wave operator* and also the free space translation representation of Lax and Phillips.

Compactification. Friedlander's idea can be interpreted geometrically by taking a suitable compactification of $\mathbb{R}^{1+n}$. Let X_0 be the radial compactification of $\mathbb{R}^{1+n}$ with boundary defining function of $\tilde{\rho}$ and X be X_0 blown up at a p-submanifold $\{|t| = |x|\} \cap \partial X_0$, resulting in a manifold with corners up to codimension 2 with 5 boundary hypersurfaces: the front face $S_1^{\pm}$, the top and bottom boundary hypersurfaces $S_2^{\pm}$ and the middle one S_0, with boundary defining function ρ_1, ρ_2 and ρ_0 respectively. Here $S_1^{\pm}$ are the compactification of null infinity, in the sense of Penrose, of Minkowski space-time. Let X^2 be the double of X across $S_1^{\pm}$; the conformal metric $\tilde{m}$ extends to a Lorentzian b-metric on X^2 with $S_1^{\pm}$ as characteristic surfaces; the vector fields generating the Lorentz group and translations span all the vector fields tangent to the boundary of X over $C^{\infty}(X)$, denoted by $\mathscr{V}_b(X)$. In this picture, the map (1) can be refined by applying the b-calculus due to Melrose [10] [11]:

$$\mathcal{R}_F : \rho_0^{\frac{n}{2}+\delta} H_b^{N+1}(\overline{\mathbb{R}^n}) \times \rho_0^{\frac{n}{2}+1+\delta} H_b^{N}(\overline{\mathbb{R}^n}) \ni (u_0, u_1)$$
$$\longrightarrow \tilde{u}|_{S_1} \in (\rho_0\rho_2)^{\frac{1}{2}+\delta}[H_b^{-\delta}(\overline{\mathbb{R}}; H^{m+1+\delta}(\mathbb{S}^{n-1})) \cap L^2(\mathbb{S}^{n-1}; H_b^{m+1}(\overline{\mathbb{R}}))].$$

which is an isomorphism for $\delta \in (-\frac{1}{2}, \frac{1}{2})$ and $n \geq 3$.

Einstein vacuum equations. The Einstein Vacuum equations on an $n+1$-dimensional manifold $\mathcal{M}^{1+n}$ for a Lorentzian metric g make the Ricci Curvature vanish:

$$R_{\mu\nu} = 0, \quad \forall \mu, \nu = 0, 1, ..., n. \tag{2}$$

The question of stability of Minkowski space-time concerns the Cauchy problem for (2): Given an n-dimensional manifold Σ_0 with a Riemannian metric g_0 and a symmetric two-tensor k_0, which satisfy the constraint equations

$$R_0 - [k_0]_i^j [k_0]_j^i + [k_0]_i^i [k_0]_j^j = 0, \quad \nabla^j [k_0]_{ij} - \nabla_i [k_0]_j^j = 0, \quad \forall i = 1, ..., n,$$

find a Lorentzian manifold $(\mathcal{M}^{1+n}, g)$ satisfying (2) and an embedding $\Sigma_0 \subset M$ such that g_0 is the restriction of g to Σ_0 and k_0 is the second fundamental form of (Σ_0, g_0) in $(\mathcal{M}^{1+n}, g)$. For the physical case of $n = 3$, in 1952 Choquet-Bruhat showed in [1] the local well-posedness of the Cauchy problem for the Einstein Vacuum equations with general smooth initial data and later in 1993 D. Christodoulou and S. Klainerman proved in [2] the global stability of Minkowski space-time for strongly asymptotically flat initial data with an asymptotic estimate of the gravitational field at null infinity.

The Einstein equations are invariant under diffeomorphisms. In [1] and her related work, Choquet-Bruhat broke this gauge invariance by working in *harmonic coordinates*:

$$\Box_g x^{\mu} = 0 \Leftrightarrow \Gamma_{\mu} = g^{\alpha\beta}\partial_{\alpha} g_{\mu\beta} - \tfrac{1}{2} g^{\alpha\beta} \partial_{\mu} g_{\alpha\beta} = 0, \quad \forall \mu = 0, 1, ..., n.$$

In terms of the harmonic gauge, the Einstein Vacuum equations reduce to a system of quasilinear wave equations

$$\Box_g g_{\mu\nu} = P(\partial_{\mu} g, \partial_{\nu} g) + Q_{\mu\nu}(\partial g, \partial g), \quad \forall \mu, \nu = 0, 1, ..., n, \tag{3}$$

where P and $Q_{\mu\nu}$ are quadratic forms with $Q_{\mu\nu}$ also satisfying the *null condition.* For $n \geq 4$, general results for quasilinear wave equations with small initial data ensures the existence of global solutions to (3); see [3], [5], [6] and [7]. For $n = 3$, in 2005, H. Lindblad and I. Rodnianski proved in [8] and [9] the global stability of Minkowski space-time with a decay estimate on $h = g - m$ at null infinity for general asymptotically flat initial data with $\Sigma_0 \simeq \mathbb{R}^n$ by introducing the *weak null condition.* They also showed in [8] that the harmonic gauge is stable by an argument of uniqueness of the wave equations for the components of the connection.

To study the asymptotic behavior of solutions to (3), it is equivalent to study the conformal transformation of it:

$$(4) \qquad \begin{aligned} (\Box_{\tilde{g}} + \tilde{\gamma})\tilde{h}_{\mu\nu} &= \rho_1^{\frac{n-5}{2}} (\rho_0\rho_2)^{\frac{n-1}{2}} \tilde{F}_{\mu\nu}(\tilde{h}, \tilde{\partial}\tilde{h}), \\ \text{where} \quad \tilde{g} = \tilde{\rho}^2 g,\ \tilde{h} &= \tilde{\rho}^{\frac{1-n}{2}},\ \tilde{\gamma} = -\tilde{\rho}^{\frac{n-1}{2}} \Box_{\tilde{g}} \tilde{\rho}^{\frac{1-n}{2}},\ \tilde{\partial} \in \mathscr{V}_b(X). \end{aligned}$$

Here $\tilde{\gamma}$ is an analytic function of $\tilde{h}$ with all coefficients smooth and uniformly bounded and $\tilde{F}_{\mu\nu}$ is a quadratic form in $(\tilde{h}, \tilde{\partial}\tilde{h})$ with all coefficients having the same property as $\tilde{\gamma}$.

Main result. Denote by $\mathcal{U}_\epsilon^{N,\delta}$ the space of (h^0, h^1) such that

$$\|(h^0, h^1)\|_{\rho_0^{\frac{n}{2}+\delta} H_b^{N+1}(\overline{\mathbb{R}^n}) \times \rho_0^{\frac{n}{2}+1+\delta} H_b^N(\overline{\mathbb{R}^n})} < \epsilon, \quad \Gamma_\mu|_{t=0} = \partial_t \Gamma_\mu|_{t=0} = 0,\ \mu = 0, ..., n,$$

with $(h^0, h^1) = (g - m|_{t=0}, \partial_t g|_{t=0})$. Here $\partial_t \Gamma_\mu|_{t=0} = 0$ are equations of (h^0, h^1) if combined with the Reduced Einstein equations (3).

Given Cauchy data $(h_0, h_1) \in \mathcal{U}_\epsilon^{N,\delta}$ with $n \geq 4$, $N \geq \frac{n}{2} + 6$, $\delta \in (-\frac{1}{2}, 0)$ and $\epsilon > 0$ small enough, then (4) has a global solution $\tilde{h}$ on X, which is $C^{0,\delta+\frac{1}{2}}$ up to $S_1^{\pm}$. Hence the radiation field $\tilde{h}|_{S_1}$ of h is well defined and satisfies

$$(5) \qquad \partial_\tau [\tilde{h}|_{S_1}]_{\mu\nu} \theta^\nu + \frac{1}{2} \theta_\mu \partial_\tau \mathrm{tr}_m [\tilde{h}|_{S_1}] = 0, \quad \mu = 0, ..., n.$$

Here we take the convention $\theta^0 = -\theta_0 = 1, \theta^i = \theta_i = x^i/|x|$. Denote by

$$\mathcal{W}_\epsilon^{N,\delta} = \Big\{ \tilde{h}|_{S_1} : \|\tilde{h}|_{S_1}\|_{(\rho_0\rho_2)^{\frac{1}{2}+\delta} [H_b^{-\delta}(\overline{\mathbb{R}}; H^{m+1+\delta}(\mathbb{S}^{n-1})) \cap L^2(\mathbb{S}^{n-1}; H_b^{m+1}(\overline{\mathbb{R}}))]} < \epsilon, \\ \tilde{h}|_{S_1} \text{ satisfies (5)} \Big\}.$$

Combining the linear theory and implicit function theorem, we have

Theorem 1. *For $n \geq 4$, $N \geq \frac{n}{2} + 6$, $\delta \in (-\frac{1}{2}, 0)$ and $\epsilon > 0$ small enough, the map*

$$\mathscr{R}_F : \mathcal{U}_\epsilon^{N,\delta} \ni (h^0, h^1) \longrightarrow \tilde{h}|_{S_1} \in \mathcal{W}_{C\epsilon}^{N,\delta}$$

is an isomorphism onto its image for some constant $C > 0$.

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Gluing semiclassical resolvent estimates, or the importance of being microlocal

András Vasy

(joint work with Kiril Datchev)

In this talk I give a method for gluing high energy or semiclassical resolvent estimates, i.e. to obtain global resolvent estimates when analogous estimates are known for local models. Such a method is useful because the estimates for the local models can be obtained using different techniques, which might not be easy to combine directly. The key point is the use of a microlocal understanding of the propagation of semiclassical singularities to patch the resolvents.

As an application, one can describe solutions of the wave equation modulo exponential decay when

(1) One has a model at infinity with good high energy resolvent estimates, such as asymptotically hyperbolic spaces, see the work of Melrose, Sá Barreto and the lecturer [7].
(2) One has 'mild' trapping in a compact set, such as normally hyperbolic trapped sets, see the recent work of Wunsch and Zworski [10].

This combination gives a *more robust* way of analyzing wave propagation on de Sitter-Schwarzschild space than done earlier in [7, 6], which relied on combining the first listed ingredient with high energy estimates for the cutoff resolvent (i.e. for the actual resolvent on the whole of de Sitter-Schwarzschild space, sandwiched between cutoffs, due to Bony and Häfner [1], who used one dimensional techniques) by the technique of Bruneau and Petkov [2].

We actually work with more general semiclassical resolvent estimates which can be motivated as follows. Let $\tilde{P} = \Delta_g + V$ on a Riemannian manifold (X, g) with a real potential V, and let $R(\lambda) = (\tilde{P} - \lambda)^{-1}$ be the resolvent of $\tilde{P}$ when $\operatorname{Im}\lambda > 0$, as well as its analytic continuation across the positive real axis $(\lambda_0, +\infty)$, λ_0 sufficiently large, when this exists. A contour deformation (in τ, the dual of t) argument shows that, as far as high energy behavior is considered, it suffices to obtain polynomial (in $\operatorname{Re}\tau$) estimates for $R(\tau^2)$, $|\operatorname{Im}\tau| < \Gamma'$, $\Gamma < \Gamma'$, in order to understand the solutions of the wave equation modulo exponential decay $e^{-\Gamma t}$, in spatially compact sets. With $h = |\operatorname{Re}\tau|^{-1}$, and rewriting τ, one is left to consider $h^{-2}(h^2(\Delta + V) - 1 - z)$, with $|\operatorname{Im} z| \leq Ch$ (and $\operatorname{Re} z$ is $\mathcal{O}(h^2)$).

The semiclassical principal symbol of the more general operator $P = h^2\Delta_g + V - 1$ is $p = |\xi|_g^2 + V(x) - 1$, which thus vanishes on the typically non-empty characteristic set $\Sigma_p = \{(x, \xi) : p(x, \xi) = 0\}$, so even though the standard principal symbol of P is elliptic, P is *not* elliptic in the semiclassical sense. From the local perspective, the best case scenario is if p is real principal type, i.e. the Hamilton vector field H_p does not vanish on Σ_p. This is analogous in the standard ps.d.o. world to (micro)hyperbolic equations, such as the wave equation, where one has the loss of one order of derivative relative to the elliptic case. Correspondingly, one may hope for estimates such as $\|(P - z)^{-1}\| \leq Ch^{-1}$, with the norm being as an operator acting on some weighted spaces. These indeed hold in asymptotically hyperbolic spaces, see [7], acting on optimally weighted spaces. In smaller than $\mathcal{O}(h)$ neighborhood of the real axis ($\operatorname{Im} z = 0$), such estimates hold if (X, g) is a non-trapping asymptotically Euclidean, or rather scattering, space, as proved by the lecturer and Zworski [9], as well as in more general geometries as shown by Cardoso and Vodev [3].

The semiclassical wave front set, WF_h, of a function u, measures microlocally, i.e. in T^*X, whether u rapidly decays in h relative to some space (here, L^2), see e.g. [5]. Then real principal type propagation of singularities is the following: Suppose that $u \in h^{-N}L^2$. Then $\mathrm{WF}_h(u) \setminus \mathrm{WF}_h(Pu)$ is a union of maximally extended nullbicharacteristics. Note that even if $Pu = 0$, this allows for $\mathrm{WF}_h(u)$ to be non-empty, much like solutions of the wave equations need not be smooth.

When one is considering a limit such as $R(z)$, with $\operatorname{Im} z \to 0$, for which one has an elliptic problem in $\operatorname{Im} z > 0$, one can sometimes get a one-sided estimate: if the *backward* bicharacteristic from (y, η) is disjoint from $\mathrm{WF}_h(Pu)$, and Pu is compactly supported, say, then $(y, \eta) \notin \mathrm{WF}_h(u)$. Thus, singularities propagate *forwards*. This holds, for instance, on asymptotically hyperbolic spaces, as follows from [7]. In other words, singularities do not appear 'out of nowhere' from $-\infty$ along bicharacteristics. The same holds for solutions of operators of the form $P - iW$, at least microlocally along bicharacteristics that reach $T^*W^{-1}(1)$ in finite time, where $W \in C^\infty(X_1'; [0, 1])$ has $W = 0$ on X_1 and $W = 1$ off a compact set, see the work of Nonnenmacher and Zworski [8]. In fact, complex absorbing potentials provide a convenient way of localizing problems to trapped sets, see e.g. [10, 4], so our gluing construction is expected to be very useful in applications.

To set up the gluing problem, suppose $\bar{X}$ is a compact manifold with boundary, X its interior, x a boundary defining function, (X, g) is complete, $P = h^2\Delta_g + V - 1$ is self-adjoint. Let $X_0 = \{x < 4\}$, $X_1 = \{x > 1\}$. The first serious assumption is that level sets of x are (null)bicharacteristically convex in the overlap $X_0 \cap X_1$, i.e. if γ is a nullbicharacteristic then $\dot{x}(\gamma(t)) = 0$ implies $\ddot{x}(\gamma(t)) < 0$. This states $x \circ \gamma$ can only have strict local maxima as critical points. It is this convexity that will assure that the iterative construction we give ends in finitely many (three) steps.

Next, we assume that there are manifolds X'_j, $j = 0, 1$, including X_j as open sets, with some not necessarily self-adjoint semiclassical Schrödinger operators P_j, such that $P_j|_{X_j} = P|_{X_j}$. We also assume that X_1 is bicharacteristically convex for P_1, i.e. that no (null)bicharacteristic of P_1 can leave X_1 and return there; this holds in most cases of interest. Assume also that the resolvents $R_j(z)$ extend analytically to some set $D \subset [-E, E] + i[-Ch, Ch]$, and, acting on certain weighted spaces, with weight non-vanishing in $X_0 \cap X_1$ for R_0 and in X_1 for R_1, satisfy polynomial bounds $\|R_j(z)\| \le a_j(h) \le h^{-N}$, for $0 < h \le h_0$ and some N.

The most important assumption is a microlocal one on the P_j. Suppose $q \in T^*X'_j$ is in the characteristic set of P_j, and let $\gamma_- : (-\infty, 0] \to T^*X'_j$ be the backward P_j-bicharacteristic from q. We say that the resolvent $R_j(z)$ is *semiclassically outgoing at q* if $u \in L^2_{\mathrm{comp}}(X_j)$ polynomially bounded, $\mathrm{WF}_h(u) \cap \gamma_- = \emptyset$ implies that $q \notin \mathrm{WF}_h(R_j(z)u)$, i.e. WF_h could only arise from the past of q. Our microlocal assumption is then that

(0-OG) $R_0(z)$ is semiclassically outgoing at all $q \in T^*(X_0 \cap X_1) \cap \Sigma_p$,
(1-OG) $R_1(z)$ is semiclassically outgoing at all $q \in T^*(X_0 \cap X_1) \cap \Sigma_p$ such that γ_- is disjoint from $T^*(X'_1 \setminus (X \setminus X_0))$, thus disjoint from any trapping in X_1.

Theorem 1. *There exists $h_0 \in (0, 1)$ such that for $h < h_0$, $R(z)$ continues analytically to D and obeys the bound $\|R(z)\| \le Ch^2 a_0^2 a_1$ there, with the norm taken in the same weighted space as for $R_0(z)$.*

In particular, when $a_0 = C/h$, we find that $R(z)$ obeys (up to constant factor) the same bound as $R_1(z)$, the model operator with infinity suppressed.

In order to prove the theorem, we construct a semiclassical parametrix. Let $\chi_1 \in C_0^\infty(X; [0, 1])$ be such that $\chi_1 = 1$ near $\{x \ge 3\}$ and $\operatorname{supp} \chi_1 \subset \{x > 2\}$ and let $\chi_0 = 1 - \chi_1$. Define a right parametrix for P by

$$F \equiv \chi_0(x-1)R_0(z)\chi_0(x) + \chi_1(x+1)R_1(z)\chi_1, \text{ so}$$
$$PF = \mathrm{Id} + [P, \chi_0(x-1)]R_0(z)\chi_0 + [P, \chi_1(x+1)]R_1(z)\chi_1 \equiv \mathrm{Id} + A_0 + A_1.$$

The error $A_0 + A_1$ is large, $\mathcal{O}(1)$, in h due to semiclassical propagation of singularities, but using an iteration argument we can replace it by a small error.

The key point is that by the *forward propagation of semiclassical singularities*, i.e. the outgoing assumptions on the resolvent, $\|A_0 A_1\|_{L^2 \to L^2} = \mathcal{O}(h^\infty)$. Indeed, for a pair of points to be in the wave front relation of the product, there must be a nullbicharacteristic of P going through three points in T^*X over $\operatorname{supp} \chi_1$,

$\operatorname{supp} d\chi_1(.+1)$ and $\operatorname{supp} d\chi_0(.-1)$ in this order, which is excluded by the convexity assumption. This implies that iterating the parametrix construction, i.e. solving away the A_0 error using R_1 and solving away the A_1 error using R_0, and repeating once more, the error is $\mathcal{O}(h^\infty)$.

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Analytic torsion of hyperbolic 3-manifolds

Werner Müller

The purpose of this talk is to show that analytic torsion can be used to detect torsion in the cohomology of locally symmetric spaces defined by arithmetic subgroups of semisimple Lie groups. We consider here only the case of hyperbolic 3-manifolds. However, the methods are expected to work also for higher dimensions. Furthermore, we consider only compact quotients. The goal, is of course, to extend the methods to the finite volume case.

We write the 3-dimensional hyperbolic space as $\mathbb{H}^3 = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$. Let $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ be a discrete torsion free co-compact subgroup. Then $X = \Gamma\backslash\mathbb{H}^3$ is a compact oriented hyperbolic 3-manifold. We are interested in co-compact arithmetic subgroups Γ. Such discrete groups are derived from a quaternion division algebra D over a imaginary quadratic number field F (see [5]).

For $m \in \mathbb{N}$ let $\rho_m\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(S^m(\mathbb{C}^2))$ be the m-th symmetric power of the standard representation of $\mathrm{SL}(2,\mathbb{C})$. Recall that this is the standard irreducible representation of $\mathrm{SL}(2,\mathbb{C})$ of dimension $m+1$ acting in the space of homogeneous polynomials $S^m(\mathbb{C}^2)$ of degree m. By restriction of ρ_m to Γ we obtain a representation of Γ which we continue to denote by ρ_m. This representation of Γ is unimodular and acyclic [3], which means that $|\det(\rho_m(\gamma))| = 1$ for all $\gamma \in \Gamma$ and

$H^*(X, E_m) = 0$, where $E_m \to X$ denotes the flat vector bundle over X attached to $\rho_m|_\Gamma$. Therefore the analytic torsion $T_X(\rho_m)$ and the Reidemeister $\tau_X(\rho_m)$ of X with respect to $\rho_m|_\Gamma$ are well defined and independent of any choice of a fibre metric in E_m which we need to choose in order to define $T_X(\rho_m)$. Moreover by [8] we have $T_X(\rho_m) = \tau_X(\rho_m)$. Our first main result is the following asymptotic formula for $\tau_X(\rho_m)$ as $m \to \infty$ (see [9]).

Theorem 1. *Let $X = \Gamma\backslash\mathbb{H}^3$ be a compact hyperbolic 3-manifold. Then we have*

$$-\log \tau_X(\rho_m) = \frac{\mathrm{vol}(X)}{2\pi} m^2 + O(m)$$

as $m \to \infty$.

We note that there is an analogous result in the holomorphic setting. In [2] Bismut and Vasserot studied the asymptotic behavior of the holomorphic Ray-Singer torsion for symmetric powers of a positive vector bundle.

An immediate consequence of Theorem 1 is that the set $\{\tau_X(\rho_m) \colon m \in \mathbb{N}\}$ of Reidemeister torsions determines the volume of the hyperbolic manifold X.

The proof of Theorem 1 is based on the study of the twisted Ruelle zeta function. We recall its definition. Let $\rho\colon \Gamma \to \mathrm{GL}(V)$ be a representation of Γ on a finite-dimensional complex vector space V. Given $\gamma \in \Gamma$, denote by $[\gamma]$ the Γ-conjugacy class of γ. For $\gamma \in \Gamma \setminus \{e\}$ let $\ell(\gamma)$ be the length of the unique closed geodesic that corresponds to $[\gamma]$. Then the twisted Ruelle zeta function is defined as

$$R(s;\rho) := \prod_{\substack{[\gamma]\neq e\\ \text{prime}}} \det\left(\mathrm{I} - \rho(\gamma) e^{-s\ell(\gamma)}\right). \tag{1}$$

The product runs over all nontrival primitive conjugacy classes. The infinite product converges in some half-plane $\mathrm{Re}(s) > c$ and admits a meromorphic extension to $\mathbb{C}$ [7, Sect. 7]. Let $R(s;\rho_m)$ denote the twisted Ruelle zeta function attached to $\rho_m|_\Gamma$. From the thesis of A. Wotzke [11] we obtain the following result. See also [9].

Theorem 2. *For each $m \in \mathbb{N}$ the Ruelle zeta function $R(s;\rho_m)$ is regular at $s = 0$ and its value at $s = 0$ satisfies*

$$|R(0;\rho_m)| = T_X(\rho_m)^2.$$

The corresponding result for unitary representations ρ of Γ was proved by Fried [6]. Using the equality of $T_X(\rho_m)$ and $\tau_X(\rho_m)$ [8], we get

$$|R(0;\rho_m)| = \tau_X(\rho_m)^2.$$

The proof of Theorem 1 is now obtained by the study of the asymptotic behavior of $|R(0;\rho_m)|$ as $m \to \infty$.

Next we apply Theorem 1 in the arithmetic setting. Let Γ be a co-compact arithmetic subgroup of $\mathrm{SL}(2,\mathbb{C})$ which is derived from a quaternion devison algebra D over an imaginary quadratic field [5]. We assume that Γ is torsion free. This can be achieved by choosing D appropriately or by passing to a subgroup of finite index. It follows from the construction of Γ that for all even $n \in \mathbb{N}$, there exists a lattice

$M_n \subset S^n(\mathbb{C}^2)$ which is Γ-stable with respect to the action given by $\rho_n = \mathrm{Sym}^n$. Let $\mathcal{M}_n$ be the associated local system of free $\mathbb{Z}$-modules over X. Denote by $H^*(X, \mathcal{M}_n)$ the cohomology of X with coefficients in $\mathcal{M}_n$ [10]. Then $H^*(X, \mathcal{M}_n \otimes \mathbb{R}) = 0$. Hence $H^*(X, \mathcal{M}_n)$ is a finite abelian group. Denote by $|H^p(X, \mathcal{M}_n)|$ the order of $H^p(X, \mathcal{M}_n)$. Then our second main result is the following theorem.

Theorem 3. *The alternating sum of* $\log |H^p(X, \mathcal{M}_{2k})|$ *is independent of the choice of a* Γ*-stable lattice* M_{2k} *in* $S^{2k}(\mathbb{C}^2)$ *and we have*

$$\sum_{p=1}^{3} (-1)^p \log |H^p(X, \mathcal{M}_{2k})| = \frac{2}{\pi} \operatorname{vol}(X) k^2 + O(k) \tag{2}$$

as $k \to \infty$.

An immediate consequence is the following corollary.

Corollary. *For any choice of lattices* M_{2k} *in* $S^{2k}(\mathbb{C}^2)$ *we have*

$$\liminf_k \frac{\log |H^2(X, \mathcal{M}_{2k})|}{k^2} \geq \frac{2}{\pi} \operatorname{vol}(X). \tag{3}$$

Actually, we expect that for $p = 1, 3$ we have $\log |H^p(X, \mathcal{M}_{2k})| = O(k)$ as $k \to \infty$. In other words, we pose the following

Conjecture. *For any choice of lattices* M_{2k} *in* $S^{2k}(\mathbb{C}^2)$ *we have*

$$\lim_{k \to \infty} \frac{\log |H^2(X, \mathcal{M}_{2k})|}{k^2} = \frac{2}{\pi} \operatorname{vol}(X).$$

The proof of Theorem 3 follows from Theorem 1 and the following result about Reidemeister torsion. Let $\rho \colon \Gamma \to \mathrm{GL}(V)$ be a finite-dimensional unimodular acyclic representation of Γ on a real vector space V. Assume that there exists a lattice $M \subset V$ which is Γ-stable. Let $\mathcal{M}$ be the associated local system of free $\mathbb{Z}$-modules over X. Then $H^*(X, \mathcal{M})$ is a finite abelian group and the Reidemeister torsion $\tau_X(\rho)$ associated to ρ satisfies

$$\tau_X(\rho) = \prod_{q=0}^{3} |H^q(X, \mathcal{M})|^{(-1)^{q+1}}.$$

This is a general algebraic fact, which was first observed by Cheeger [4, (1.4)]. As a simple example consider $d \in \mathbb{Z}$, $d \neq 0$. Let $A \colon \mathbb{R} \to \mathbb{R}$ be the multiplication by d. Then we have $|\mathbb{Z}/d\mathbb{Z}| = |d| = |\det A|$.

In [1] Bergeron and Venkatesh established results of similar nature but in a different aspect. They study the growth of the torsion in the cohomolgy for a fixed local system as the lattice varies in a decreasing sequence of congruence subgroups. Again the volume of the locally symmetric space appears as the main ingredient of the asymptotic formulas.

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Recent Advances on the Analytic Torsion of Singular Spaces

Boris Vertman

(joint work with Rafe Mazzeo and Werner Müller)

The analytic torsion has been introduced by Ray and Singer in [13] as the analytic counterpart to the combinatorially defined Reidemeister-Franz torsion in [11], [12] and [4]. Equality of both invariants has been proved independently by Cheeger [2] and Müller [9] for closed manifolds. The Cheeger-Müller Theorem asserts the topological nature of the Ray-Singer analytic torsion, which is a priori only a spectral invariant. In view of the general study of spaces with singularities, initiated by Cheeger in [3], we may ask ourselves the following question.

Question: *What topological information does the analytic torsion carry in presense of conical and edge singularities?*

We present here the recent advances on that question, based on the joint projects with Rafe Mazzeo [7] and with Werner Müller [10]. An earlier step forward on the question above has been done by the author in [16], where the analytic torsion of a bounded cone has been computed in terms of spectral and topogical data of the cross section. The computation uses the double-summation method developped by Spreafico in [14], [15], and a symmetry observation by Lesch [6].

Theorem 1. (Vertman, [16]) *Let $M^m = (0,1] \times N, g^M = dx^2 \oplus x^2 g^N$ be a bounded generalized cone over a closed oriented Riemannian manifold $(N^n, g^N), n = \dim N$.*

Denote the Euler characteristic of N by $\chi(N)$ and the Betti numbers by $b_k = \dim \mathcal{H}^k(N)$. Put $\alpha_k = (n-1)/2 - k$ and define

$$F_k := \{\nu \in \mathbb{R}^+ \mid \nu^2 = \eta + (k + 1/2 - n/2)^2, \eta \in \mathrm{Spec}\Delta_{k,ccl,N}\backslash\{0\}\},$$
$$\zeta_{k,N}(s) = \sum_{\nu \in F_k} \nu^{-s}, \quad \zeta_{k,N}(s,\alpha) := \sum_{\nu\in F_k} (\nu+\alpha)^{-s}, \quad Re(s) \gg 0.$$

Then the logarithm of the scalar analytic torsion of (M, g^M) is given by a sum of a topological, the torsion-like and the residual terms

$$\log T(M, g^M) = \mathrm{Top}(M) + \mathrm{Tors}(M, g^M) + \mathrm{Res}(M, g^M),$$

where the topological term is an algebraic combination of Betti numbers

$$\mathrm{Top}(M) = \tfrac{\log 2}{2}\chi(N) + \sum_{k=0}^{\frac{n}{2}-1} (-1)^k b_k \Big(\tfrac{1}{2}\log(n-2k+1) - \sum_{l=0}^{\frac{n}{2}-k-1} \log(2l+1)\Big), \ m \ odd,$$

$$\mathrm{Top}(M) = \sum_{k=0}^{(n-1)/2} \tfrac{(-1)^k}{2} b_k \log(n-2k+1), \ m \ even.$$

The torsion-like term is in fact the analytic torsion of (N, g^N) in even dimensions

$$\mathrm{Tors}(M, g^M) = \sum_{k=0}^{n/2-1} \frac{(-1)^k}{2}(\zeta'_{k,N}(0,\alpha_k) - \zeta'_{k,N}(0,-\alpha_k)), \ m \ odd,$$
$$\mathrm{Tors}(M, g^M) = -\frac{1}{2}\log T(N, g^N), \ m \ even.$$

The residual term is an intricate combination of residues of $\zeta_{k,N}(s)$

$$\mathrm{Res}(M, g^M) = \sum_{k=0}^{n/2-1} \tfrac{(-1)^k}{4} \sum_{r=1}^{n/2} \mathrm{Res}\zeta_{k,N}(2r) \sum_{b=0}^{2r} A_{r,b}(\alpha_k)\tfrac{\Gamma'(b+r)}{\Gamma(b+r)}, \ m \ odd,$$
$$\mathrm{Res}(M, g^M) = \sum_{k=0}^{\frac{(n-1)}{2}} \tfrac{(-1)^k}{4} \sum_{r=1}^{\frac{(n-1)}{2}} \mathrm{Res}\zeta_{k,N}(2r+1) \sum_{b=0}^{2r+1} B_{r,b}(\alpha_k)\tfrac{\Gamma'(b+r+\frac{1}{2})}{\Gamma(b+r+\frac{1}{2})}, \ m \ even,$$

where the coefficients $A_{r,b}(\alpha_k)$ and $B_{r,b}(\alpha_k)$ are determined by certain recursive formulas, associated to combinations of special functions.

In view of our general results in [16], de Melo, Hartmann and Spreafico subsequently evaluated in [5] and [8] the analytic torsion in the special case of a cone over the sphere S^n, equating in even dimensions the residual term $\mathrm{Res}(M, g^M)$ to the anomaly term of Brüning-Ma [1] by direct comparison. This approach however limits the result either to the special case of spheres or the general cross-section in lower (even) dimensions.

Nonetheless, one conjecturally expects $\mathrm{Res}(M, g^M)$ to equal the Brüning-Ma anomaly, coming from the non-product metric structure of the bounded cone at its regular boundary, and hence to vanish if the cone metric is smoothened to product away from the conical singularity. We answer this affirmatively with the following precise statement.

Theorem 2. (Müller-Vertman [10], see also [17]) *Under the notation of Theorem 1 we have the following identification of the residual term*

$$\mathrm{Res}(M, g^M) = A_{BM}(M, g^M) + \frac{\log\sqrt{\pi}}{2}\chi(N),$$

where $A_{BM}(M, g^M)$ *denotes the Brüning-Ma anomaly, coming from the non-product metric structure of the bounded cone* (M, g^M) *at its regular boundary, and the Euler characteristic* $\chi(N)$ *of the cross-section is zero for* $\dim N$ *odd.*

The statement is proved by considering a bounded generalized cone, with the conical singularity truncated off. Explicit computations relate its analytic torsion to the residual term $\mathrm{Res}(M, g^M)$. On the other hand, truncation of the conical singularity leads to a finite cylinder with a non-product metric, whose analytic torsion metric anomaly has been discussed by Brüning-Ma in [1].

It should be noted that our new result of Theorem 2, which had already appeared in the preprints [17], was subsequently announced by Hartmann-Spreafico in [5].

In view of the intricate structure of the analytic torsion in presense of conical singularities, it is a valid question whether one can at all hope to rediscover the Cheeger-Müller theorem in any singular configuration, with the analytic torsion being metric-independent and thus a topological invariant. The following result aims precisely at this issue in the setup of edge singularities.

Theorem 3. (Mazzeo-Vertman [7]) *Let* (M, g^M) *be an odd-dimensional compact Riemannian manifold with a simple edge singularity* B*. Let* $U = (0,1) \times Y$ *be an open neighborhood of* B*, where* Y *is the total space of a fibration* $\phi : Y \to B$ *with fibres* F*. The incomplete edge metric* g^M *takes the following form over* U

$$g^M\big|_U := dx^2 + x^2\kappa(x) + \phi^*\mathfrak{h}(x), \tag{1}$$

where $\mathfrak{h}(x)$ *is a smooth family of metrics on* B *and* $\kappa(x)$ *is a smooth family of bilinear forms restricting to Riemannian metrics on fibres* F*, such that* ϕ *is a Riemannian submersion for each* $x \in (0,1)$*. Then, the analytic torsion norm* $T(M, g^M)$ *is invariant under all deformations of* g^M *which fix* $\kappa(0)$ *and* $\mathfrak{h}(0)$*. If moreover* $\dim B$ *is odd, then* $T(M, g^M)$ *is invariant under all deformations of admissible edge metrics* g^M*, where* $\kappa(x)$ *and* $\mathfrak{h}(x)$ *are in fact functions of* x^2*.*

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Hodge theory for manifolds with fibered cusps

Jörn Müller

In this talk we present the main result of [2]; for details we refer to this article and the references given there.

Manifolds with fibered cusp metrics can be considered as a geometrical generalization of $\mathbb{Q}$-rank one locally symmetric spaces at "infinity" as well as of manifolds with cusps or cylindrical ends. In [1] methods from the ϕ-calculus developed by Melrose, Mazzeo, Vaillant and others have been used to find an identification of square integrable harmonic forms $\mathscr{H}^p_{(2)}(X)$ with a subspace of the middle perversity intersection cohomology. We want to take another approach and identify the de Rham cohomology of a manifold with fibered cusps with a space of harmonic forms; generally these forms will not be square integrable.

For a Riemannian submersion $M \to B$ with f-dimensional standard fiber F we equip $Z = \mathbb{R}^+ \times M$ with the Riemannian metric $g^Z = du^2 + \pi^* g^B + e^{-2u} g^{F_b}$, where g^B and g^{F_b} are the Riemannian metrics on B resp. the vertical tangent bundle TF. A Riemannian manifold X is called *manifold with fibered cusp metric*, if X is isometric to (Z, g^Z) outside a compact set. As long as the fibers are not points, X is a complete manifold with finite volume. The splitting $TM = \pi^* TB \oplus TF$ of the tangent bundle induces an isomorphism $\Omega^*(M) = \Omega^*(B, W)$, where W is the vector bundle whose fiber over $b \in B$ is $C^\infty(F_b, \Lambda(T^*F)|_{F_b})$. On W one can define a Hermitean metric and thus the fiber-wise "vertical" Laplacian. In this way we can define 'fiber-harmonic forms' over M, and from there, over Z. These turn out to play an crucial role in the spectral theory of the Laplacian on X.

For the analysis of the spectral theory it seems useful, if not necessary, to ensure that fiber-harmonic forms are an invariant subspace of Δ_Z. To that end we impose two obstructions on the submersion $M \to B$, namely (A) that the horizontal distribution is integrable and (B) that fiber-harmonic forms are an invariant subspace of the "horizontal Laplacian" $\Delta_{1,0}$. One important consequence is that the de Rham cohomology of M can be identified with $\Delta_{1,0}$-harmonic forms on B with values in harmonic sections in the fibers:

$$H^p(M) \cong \bigoplus_{r+s=p} \mathscr{H}^r(B, \mathscr{H}^s(F)).$$

First we investigate the spectral theory of the Hodge-Laplace operator on X. To that end we first examine the spectral theory of the non-compact end Z using the Friedrichs extension of the Laplacian on compactly supported forms. The conditions (A) and (B) introduced above allows the parametrix construction of the resolvent for the Laplacian known from the setting of manifolds with cusps (e.g. [4]) to be carried out in our more general fibered setting. The explicit knowledge of the resolvent kernel, and arguments from mathematical scattering theory show

Proposition 1.
The absolutely continuous part $L^2_{ac}\Omega^p(X)$ of $\operatorname{dom}\Delta_X$ is unitarily equivalent to the fiber-harmonic forms $\Pi_0 L^2\Omega^p(Z)$. Furthermore, eigensections orthogonal to fiber-harmonic forms ("cusp forms"), are square integrable.

In this sense, the spectral theory of Δ on X is determined by the spectral theory on Z.

The spectral resolution of $L^2_{ac}\Omega^p(X)$ is given by generalized eigenforms (GEs); these will be the main ingredient in our Hodge-type theorem.

Let $\phi \in \mathscr{H}^{p-k}(B, \mathscr{H}^k(F))$ and set $d_k = |f/2 - k|$. Then there is a unique GE $E(s, \phi) \in \Omega^p(X)$ such that

$$\Delta E(s, \phi) = s(2d_k - s)E(s, \phi),$$

$s \mapsto E(s, \phi)$ is meromorphic for $s \in U \subset \mathbb{C}$, $2d_k \in U$, and $E(s, \phi)$ satisfies a growth condition on the end Z. As a consequence the asymptotic expansion on the end Z of the fiber-harmonic part of E is given by

$$\Pi_0 E(s, \phi) = e^{(f/2-k-d_k+s)r}\phi + \sum_{l=0}^{f} e^{(f/2-l+d_l-s)r} T^{[l]}(s)(\phi) + G(s, \phi), \tag{1}$$

where $G(s, \phi) \in L^2\Omega^p(X)$ for $\operatorname{Re}(s) > d_k$ and $T^{[l]}(s)$ are linear operators $\mathscr{H}^*(M) \to \mathscr{H}^*(B, \mathscr{H}^l(F))$, which are meromorphic in s. In the context of mathematical scattering theory, the $T^{[l]}(s)$ are referred to as *scattering operators*.

In view of Hodge theory the spectral value $s = 2d_k$ is of particular interest, because if $E(\cdot, \phi)$ is regular there, it is harmonic and not square integrable. Then it remains to determine under which conditions $E(2d_k, \phi)$ is closed. For that, we employ functional equations that are derived from the asymptotic expansion (1).

The next task is to identify the poles of $s \mapsto E(s, \phi)$. In the theory of automorphic forms, information about the poles of generalized eigenforms can be read off from product formulas known as the Maaß–Selberg relations. In [2] we derive a similar formula for the inner product of GEs which are perpendicular to fiber-harmonic forms outside of a compact set. From that, we obtain that the order of a pole in $s = 2d_k$ coincides with the maximal order of a pole of the scattering operator $T(s)$ and is at most one. The residue $\widetilde{E}(\phi)$ at $2d_k$ is a closed L^2-harmonic form, thus a representative in both $H^p(X)$ and $H^p_{(2)}(X)$.

In this way to every $\phi \in \mathscr{H}^p(M)$ a so-called *singular value*, that is a closed harmonic form $\Xi(\phi) \in \Omega^p(X)$, can be associated. This idea for the classification of harmonic representatives of $H^p(X)$ by GEs goes back to G. Harder in the setting of locally symmetric spaces.

Now we can state our main result. Let $r : H^p(X) \to H^p(M)$ be the restriction map induced from the inclusion $M \subset X$.

Theorem 1.
Let $H^p_!(X) := \mathrm{im}(H^p_c(X) \to H^p(X))$ be the image of cohomology with compact support in the de Rham-cohomology. Let $H^p_{\mathrm{inf}}(X)$ be a complementary space to $H^p_!(X)$ in $H^p(X)$,

$$H^p(X) = H^p_!(X) \oplus H^p_{\mathrm{inf}}(X). \tag{2}$$

Let $R^p := \mathrm{im}(r : H^p(X) \to H^p(M))$. Then $\Xi(R^p)$ is isomorphic to $H^p_{\mathrm{inf}}(X)$ and $\Xi(H^p(M)) = \Xi(R^p)$.

Since all classes on the right hand side of (2) have unique harmonic representatives, this indeed is a “Hodge-type” theorem.

The image of the restriction map r is of independent interest, in fact, the proof of Theorem 1 relies heavily on its explicit description in terms of values and residues of the scattering operator (see [2]).

The detailed knowledge we have obtained about $H^p_{\mathrm{inf}}(X)$ also allows us to compute the signature of X, by showing that there are involutions τ on L^2-harmonic forms, which commute with the construction of $\widetilde{E}$,

$$\tau_X \widetilde{E}(\omega) = \widetilde{E}(\tau_Z \omega).$$

This leads to a direct proof of the identity L^2-sign$(X) = \mathrm{sign}(X_0, \partial X_0)$, which was already given in [5].

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Computations and Applications of η-invariants

Sebastian Goette

Riemannian metrics of positive sectional curvature on closed manifolds are a rare phenomenon, and sharp conditions for their existence are far from being understood. For this reason, one is still interested in finding new examples. Recently, Grove, Wilking and Ziller [10] found two families (P_k), (Q_k) of 7-manifolds and one exceptional space R, which possibly allow such metrics and contain new examples. In [9], Grove, Verdiani and Ziller constructed a positive sectional curvature metric on P_2 (note that $P_1 = S^7$); another construction is due to Dearricott [6]. Here, we determine the diffeomorphism types of the manifolds P_k using η-invariants and secondary characteristic classes.

The P_k are highly connected ($\pi_k(M) = 0$ for $k < \left[\frac{\dim M}{2}\right]$) with $\pi_3(P_k) \cong \mathbb{Z}/k\mathbb{Z} \cong H^4(P_k)$. By Crowley's work [2], their diffeomorphism type is determined by the Eells-Kuiper invariant $\mu(P_k) \in \mathbb{Q}/\mathbb{Z}$ and a quadratic form $q\colon H^4(P_k) \to \mathbb{Q}/Z$. In joint work [4] with Crowley, we refine $q(a)$ to an invariant $t(E)$ of a quaternionic line bundle E with $a = c_2(E) \in H^4(M)$, in analogy with the Kreck-Stolz invariants s_2 and s_3 for complex line bundles of [12].

In contrast to the complex case, a quaternionic line bundle is not uniquely determined by its characteristic class a, and not every integer class $a \in H^4(M)$ arises as $c_2(E)$ for a quaternionic line bundle $E \to M$. By definining and analysing $t(E)$ over suitable $4n-1$-manifolds, we can detect all quaternionic line bundles on S^7 and S^{11}. For higher n, we recover the Feder-Gitler conjecture [7] on the existence of quaternionic line bundle over $\mathbb{H}P^n$.

Coming back to P_k, the two invariants μ and t are classically defined on oriented spin manifolds N bounding P_k, but it is not clear how to construct N. On the other hands, both invariants can be expressed as linear combinations of η-invariants of certain Dirac operators and Cheeger-Chern-Simons correction terms on P_k itself. In order to determine the necessary η-invariants, we write the spaces P_k as Seifert fibrations with generic fibre S^3 over some base orbifold B_k as indicated in [10].

It has been shown by Bismut, Cheeger [1] and Dai [5] that the η-invariants of Dirac operators $D_{M,\varepsilon}$ on total spaces of fibre bundles converge in the adiabatic limit $\varepsilon \to 0$, if the kernels of the associated fibrewise Dirac operators D_X form a vector bundle. This result can be generalised to Seifert fibrations $M \to B$, so M is foliated with compact leaves such that the space of leaves forms an orbifold B. We assume that $H = \ker(D_X)$ is a vector orbibundle on B. Let ΛB be the inertia orbifold of B and let $\hat{A}_{\Lambda B}(TB, \nabla^{TB}) \in \Omega^\bullet(\Lambda B)$ denote the normalised equivariant $\hat{A}$-form as in Kawasaki's orbifold index theorem [11]. Let $\mathbb{A}$ denote Bismut's Levi-Civita superconnection, then we construct equivariant η-forms $\eta_{\Lambda B}(\mathbb{A}) \in \Omega^\bullet(\Lambda B)$ as in [8]. Let D_B^H denote the horizontal Dirac operator on $H = \ker(D_X) \to B$, and let $(\lambda_\nu(\varepsilon))_\nu$ denote the finite family of very small eigenvalues of $D_{M,\varepsilon}$.

Theorem 1 (cf. [1], [5])**.** *Let $p\colon M \to B$ be a Seifert fibration, and let $D_{M,\varepsilon}$ be an adiabatic family of Dirac operators over M such that $H = \ker(D_X)$ forms a* *vector orbibundle over B. With D_B^H and $\lambda_\nu(\varepsilon)$ as above and $\varepsilon > 0$ small, we have*

$$\lim_{\varepsilon\to 0} \eta(D_{M,\varepsilon}) = \int_{\Lambda B} \hat{A}_{\Lambda B}\big(TB, \nabla^{TB}\big)\, 2\eta_{\Lambda B}(\mathbb{A}) + \eta\big(D_B^H\big) + \sum_\nu \operatorname{sign}(\lambda_\nu(\varepsilon)) \,.$$

With this result and the equivariant η-forms computed in [8], one can compute the η-invariants in the definition of $\mu(P_k)$ and t_{P_k} for any single space P_k. In order to obtain a general formula for all members of the family (P_k), one notices that the contributions from the singular fibres are given by generalisations of Dedekind sums. Although we are not aware of an explicit general formula for these Dedekind sums, we can compute them inductively modulo integers by writing each P_k as a Seifert fibration in two different ways, see [10]. It should be noted that one can get rid of the integer ambiguity if one is able to exhibit a family of metrics of positive scalar curvature on each P_k that connects two adiabatic limit metrics stemming from the two Seifert fibration structures on P_k.

Theorem 2. *The Eells-Kuiper invariant of P_k is given by*

$$\mu(P_k) = -\frac{4k^3 - 7k + 3}{2^5 \cdot 3 \cdot 7} \quad \in \quad \mathbb{Q}/\mathbb{Z} \,. \tag{1}$$

Crowley's quadratic form q on $H^4(P_k) \cong \mathbb{Z}/k\mathbb{Z}$ is given by

$$q(\ell) = \frac{\ell(\ell - k)}{2k} \quad \in \quad \mathbb{Q}/\mathbb{Z} \,. \tag{2}$$

By comparing these values with the corresponding values for S^3-bundles over S^4 in [3] and [4], one can construct manifolds that are diffeomorphic to P_k.

Theorem 3. *Let $E_{k,k} \to S^4$ denote the principal S^3-bundle with Euler class $k \in H^4(S^4) \cong \mathbb{Z}$, and let Σ_7 denote the exotic seven sphere with $\mu(\Sigma_7) = \frac{1}{28}$. Then there exists an orientation preserving diffeomorphism*

$$P_k \cong E_{k,k} \# \Sigma_7^{\# \frac{k-k^3}{6}} \,.$$

In particular, P_k and $E_{k,k}$ are homeomorphic.

This result also implies that P_2 with reversed orientation is diffeomorphic to some S^3-bundle over S^4, and to $US^4 \# \Sigma_7$, where US^4 denotes the unit tangent bundle of S^4.

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Relative Index Pairing and Odd Index Theorem for Even Dimensional Manifolds

Zhizhang Xie

In this talk, we will prove an analogue for even dimensional manifolds of the Atiyah-Patodi-Singer twisted index theorem for trivialized flat bundles over odd dimensional closed manifolds [1, Proposition 6.2], and some related results. For notational simplicity, we will restrict the discussion mainly to spin manifolds. However all results can be straightforwardly extended to general manifolds. Unless we specify otherwise, we always fix the Riemannian metric for each manifold in this talk and use the associated Levi-Civita connection to define its characteristic classes.

To motivate the subject matter of this talk, we begin by recalling the APS twisted index theorem for odd dimensional closed manifolds in the following form, cf. [4, Corollary 7.9]. For $(p_s)_{0\leq s\leq 1}\in M_k(C^\infty(N))$, $s\in[0,1]$, a smooth path of projections over N,

$$\int_0^1 \frac{1}{2}\frac{d}{ds}\eta(p_sDp_s)ds = \int_N \hat{A}(N)\wedge \mathrm{Tch}_\bullet(p_s).$$

Here p_sDp_s is the Dirac operator twisted by p_s, $\eta(p_sDp_s)$ its η-invariant, $\hat{A}(N)$ the $\hat{A}$-genus form of N and $\mathrm{Tch}_\bullet(p_s)$ is the Chern-Simons transgression form of $(p_s)_{0\leq s\leq 1}$.

To prove our analogue for even dimensional closed manifolds, we shall replace a path of projections by a path of unitaries. The more interesting issue is what should replace the η-invariant appearing on the left hand side of the above formula. To answer this, let us first consider the case where the manifold in question bounds, that is, it is the boundary of some spin manifold. In this case, the η-invariant by Dai and Zhang [3, Definition 2.2] is the right candidate. Indeed, suppose the even dimensional manifold Y is the boundary of a spin manifold X and $(U_s)_{0\leq s\leq 1}$ is the restriction to Y of a smooth path of unitaries over X. Denote the η-invariant of Dai and Zhang by $\eta(Y, U_s)$ for each $s \in [0,1]$, then

$$\int_0^1 \frac{1}{2}\frac{d}{ds}\eta(Y,U_s)ds = \int_Y \hat{A}(Y)\wedge \mathrm{Tch}_\bullet(U_s). \tag{1}$$

When Y bounds, it follows from cobordism invariance of the index of Dirac operators that $\mathrm{Ind}(D^+) = 0$, where D^+ is the restriction of the Dirac operator over Y to the half of the spinor bundle according to its natural $\mathbb{Z}_2$-grading. The condition $\mathrm{Ind}(D^+) = 0$ is crucial for the definition of the η-invariant by Dai and Zhang, however is often not satisfied by an even dimensional closed spin manifolds in general. To cover the general case, we shall use another approach where we lift the data to $\mathbb{S}^1 \times Y$. The main ingredient of the method of proof is using an explicit formula of the cup product $K^1(\mathbb{S}^1)\otimes K^1(Y) \to K^0(\mathbb{S}^1\times Y)$, inspired by the Powers-Rieffel idempotent construction, cf. [6]. In fact, the formula given for the case when $Y = \mathbb{S}^1$ by Loring in [5] also works for all manifolds in general. Our analogue for even dimensional closed spin manifolds of the APS twisted index theorem is as follows.

Theorem (I). *Let Y be an even dimensional closed spin manifold and $(U_s)_{0\le s\le 1} \in U_k(C^\infty(Y))$ a smooth path of unitaries over Y. For $s \in [0,1]$, $e_s \in M_{2k}(C^\infty(\mathbb{S}^1\times Y))$ is a projection over $\mathbb{S}^1 \times Y$, which is the cup product of U_s with $e^{2\pi i\theta}$ a generator of $K^1(\mathbb{S}^1)$. Denote by $D_{\mathbb{S}^1\times Y}$ the Dirac operator over $\mathbb{S}^1\times Y$. Then*

$$\int_0^1 \frac{1}{2}\frac{d}{ds}\eta(e_sD_{\mathbb{S}^1\times Y}e_s)ds = \int_Y \hat{A}(Y)\wedge \mathrm{Tch}_\bullet(U_s). \tag{2}$$

A priori, the η-invariants in the formulas (1) and (2) appear to be different, we however will show that they are equal to each other modulo $\mathbb{Z}$ in the case where Y bounds.

Theorem (II). *Suppose Y is the boundary of an odd dimensional spin manifold. For $U \in U_k(C^\infty(Y))$ and e_U is the cup product of U with $e^{2\pi i\theta} \in K^1(\mathbb{S}^1)$, one has*

$$\eta(Y,U) = \eta(e_U D_{\mathbb{S}^1\times Y} e_U) \mod \mathbb{Z}.$$

The method of proof is based on a slight generalization of a theorem by Brüning and Lesch [2, Theorem 3.9]. In this sense, $\eta(e_U D_{\mathbb{S}^1\times Y} e_U)$ can be thought of as the extension to general even dimensional manifolds of the definition of the η-invariant by Dai and Zhang.

The same technique used above also allows us to prove the following analogue for odd dimensional manifolds with boundary of the relative index pairing formula by Lesch, Moscovici and Pflaum [4, Theorem 7.6].

Theorem (III). *For a relative K-cycle $[U,V,u_s] \in K^1(M,\partial M)$, that is, $U,V \in U_n(C^\infty(M))$ are two unitaries over M with $u_s \in U_n(C^\infty(\partial M))$, $s\in[0,1]$, a smooth path of unitaries over ∂M such that $u_0 = U|_{\partial M}$ and $u_1 = V|_{\partial M}$. If U and V are constant along the normal direction near the boundary, then*

$$\mathrm{Ind}_{[D]}([U,V,u_s]) = \mathrm{Ind}(T_V) - \mathrm{Ind}(T_U) + \mathrm{SF}\left(u_s^{-1}D_{[0,1]}u_s; P_0^{u_s}\right).$$

Here $D_{[0,1]}$ is the Dirac operator over $[0,1] \times \partial M$ and $\mathrm{SF}\left(u_s^{-1} D_{[0,1]} u_s; P_0^{u_s}\right)$ *is the spectral flow of the path of elliptic operators $u_s^{-1} D_{[0,1]} u_s$ with APS type boundary conditions $P_0^{u_s}$.*

This uses Dai and Zhang's Toeplitz index theorem for odd dimensional manifolds with boundary [3].

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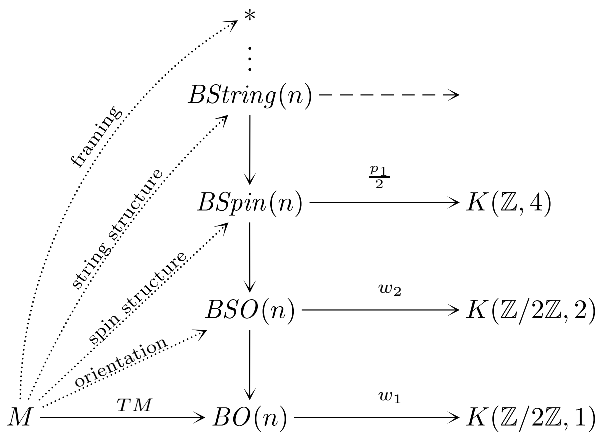
Seondary and ternary elliptic genera

Ulrich Bunke

(joint work with Niko Naumann)

In order to set up some notation we consider the first stages of the Postnikov tower of the classifying space $BO(n)$ for $n \geq 3$



Here M is an n-dimensional manifold, TM is the classifying map of its tangent bundle, and the dotted arows are labelled by the names of the geometric structures corresponding to the lifts of this classifying map. This sequence of classifying spaces gives rise to a sequence of Thom spectra

$$S \to \cdots \to MString \to MSpin \to MSO \to MO\ .$$

Their homotopy groups are only known for *MSpin*, *MSO* and *MO* by classical computations. Our main goal is to construct secondary and higher invariants which detect elements in the unknown bordism groups, in particular in $MString_*$ and the stable homotopy group S_*.

The treatment of secondary invariants is organized in the following steps.

(1) choice of a primary invariant
(2) investigation of locality and integrality
(3) construction of the secondary invariant
(4) intrisification
(5) calculation of the secondary invariant

Let us explain this in the example of Adam's e-invariant. The primary invariant is the KO-orientation $\alpha : MSpin \to KO$ of the *Spin*-bordism. We have a diagram

$$\begin{array}{ccccc} S_{4k} & \longrightarrow & & \longrightarrow & 0 \\ \downarrow & & & & \downarrow \\ MSpin_{4k} & \xrightarrow{\alpha} & KO_{4k} & \longrightarrow & \mathbb{Z} \\ & \searrow^{\epsilon_k \int_M \hat{A}(\nabla^{TM})} & & & \downarrow \\ & & & & \mathbb{R} \end{array}$$

The down-right arrow indicates the local formula in which $\alpha(M)$ is given as an integral of a characteristic form over the spin manifold M representing a class $[M] \in MSpin_{4k}$, where ϵ_k is 1 for even k and $\frac{1}{2}$ for odd k. This diagram explains locality and integrality of the primary invariant. This integral formula can be applied to a manifold with a framed boundary $\partial M = Z$, if we assume that the connection ∇^{TM} is compatible with the framing. It easily follows that

$$e([Z]) := \left[\epsilon_k \int_M \hat{A}(\nabla^{TM})\right] \in \mathbb{R}/\mathbb{Z}$$

only depends on the framed bordism class $[Z] \in S_{4k-1}$. This is the construction of the secondary invariant

$$e : S_{4k-1} \to \mathbb{R}/\mathbb{Z}$$

due to Atiyah-Patodi-Singer. They also give an intrinsic formula involving η-invariants. Its calculation is given by the sequence

$$\pi_{4k-1}(O) \xrightarrow{j} S_{4k-1} \xrightarrow{e} \mathbb{R}/\mathbb{Z}\ .$$

The image of the j-homomorphism is known and splits off as a summand, and the e-invariant is injective on this image and annihilates the complement.

Our main example is the secondary Witten genus associated to the primary invariant

$$R : MSpin \to KO[[q]]\ .$$

We consider the diagram

$$\begin{array}{ccccc} MString_{4k} & \xrightarrow[\sigma]{} & tmf_{4k} & \longrightarrow & \mathcal{M}^{\mathbb{Z}}_{2k} \\ \downarrow & & \downarrow & & \downarrow {\scriptstyle q\text{-expansion}} \\ MSpin_{4k} & \xrightarrow{R} & KO[[q]]_{4k} & \longrightarrow & \mathbb{Z}[[q]] \\ & \searrow^{\epsilon_k \int_M W(\nabla^{TM})} & & & \downarrow \\ & & & & \mathbb{R}[[q]] \end{array}$$

If M is a spin-manifold, then we form the power series of vector bundles

$$R(TM) := \prod_{n\geq 1}(1-q^n)^{8k} \bigotimes_{n\geq 1} Sym_{q^n}(TM \otimes \mathbb{C}) \ .$$

The map R is given by the index of a twisted Dirac operator

$$[M] \mapsto \mathtt{index}(D_M \otimes R(TM)) \ ,$$

the local formula involves the characteristic form

$$W(\nabla^{TM}) := \hat{A}(\nabla^{TM}) \wedge \mathbf{ch}(\nabla^{R(TM)}) \ ,$$

tmf is the spectrum of topological modular forms of Goerss-Hopkins and Lurie, σ is the Ando-Hopkins-Rezk orientation, and $\mathcal{M}^{\mathbb{Z}}_{2k}$ is the space of integral modular forms for $SL(2,\mathbb{Z})$ of weight $2k$. Integrality and modularity of the primary invariant is explained by the factorizations over KO and tmf.

We can write $W(TM)$ as a polynomial in the Pontragin classes with coefficients in the ring generated by the Eisenstein series:

$$W(TM) = \Phi(G_2, G_4, \dots)(p_1(TM), p_2(TM), \dots) \ .$$

We form

$$\tilde{\Phi} := \Phi \frac{1 - e^{G_2 p_1}}{p_1} \ .$$

We now consider a spin manifold M with boundary N which has a geometric string structure α as introduced recently by Waldorf. The geometric string structure gives rise to a form $H_\alpha \in \Omega^3(N)$ such that $2\, dH_\alpha = p_1(\nabla^{TN})$. We define the group

$$T_{2k} := \frac{\mathbb{R}[[q]]}{\mathbb{Z}[[q]] + \mathcal{M}_{2k}}$$

and consider the class

$$b^{an}([N]) := \left[2\epsilon_k \int_N H_\alpha \wedge \tilde{\Phi}(\nabla^{TN}) + \epsilon_k \eta(D_N \otimes (R(TN) \oplus \mathbb{R}))\right] \in T_{2k} \ .$$

The second term is the formal power series of degree-wise η-invariants of the Dirac operator twisted by a formal power series of bundles.

It is easy to see using the Atiyah-Patodi-Singer index theorem form manifolds with boundary that this gives rise to a bordism invariant

$$b^{an} : MString_{4k-1} \to T_{2k} \ .$$

The formula for b^{an} given above is already the intrinsic one.

The calculation of this invariant is explained by the following diagram

$$\begin{array}{ccc} A_{4k-1} & \longrightarrow & tmf_{4k-1} \\ \downarrow & & \downarrow {\scriptstyle b^{tmf}} \\ MString_{4k-1} & \xrightarrow{b^{an}} & T_{2k} \end{array}$$

Here $A_{4k-1} \subseteq MString_{4k-1}$ is the kernel of the map to $MSpin_{4k-1}$ (at the moment it is not known if this map is non-zero). The calculation of b^{an} is given in terms of the factorization over tmf_{4k-1} and a complete calculation of b^{tmf}. For all further details we refer to [1].

If we consider b^{an} as primary and proceed in a similar way, then we can construct a ternary invariant which detects pieces of S_{4k-2}. For details in a similar case we refer to [2].

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Extension of the resolvent of Laplacian on geometrically finite hyperbolic manifolds

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(joint work with Rafe Mazzeo)

The Laplacian Δ_X on a quotient $X := \Gamma\backslash\mathbb{H}^{n+1}$ of real hyperbolic space $\mathbb{H}^{n+1}$ by a geometrically finite hyperbolic group Γ of isometries (in the sense of Bowditch [2]) has essential spectrum $[n^2/4, \infty)$ and finitely many eigenvalues $\lambda_1, \dots, \lambda_N \in [0, n^2/4)$ and 0 is not an eigenvalue if $\mathrm{Vol}(X) = \infty$. The resolvent of Δ_X is defined by

$$R(s) := (\Delta_X - s(n-s))^{-1}$$

for $\mathrm{Re}(s) > n/2$ and $s(n-s) \neq \lambda_j$, as a family of bounded operators on $L^2(X)$. We shall assume that Γ has no torsion, or equivalently that Γ has no elliptic elements, so that X is smooth. The group is convex co-compact if $\mathrm{Vol}(X) = \infty$ and Γ has no parabolic elements. In this case, the manifold can be conformally compactified as a smooth manifold with boundary $\bar{X}$ and the resolvent $R(s)$ is shown to have a finite meromorphic extension to $s \in \mathbb{C}$ by Mazzeo-Melrose [10] and later by Guillopé-Zworski [7]. The case with parabolic elements is more complicated because of the presence, in general, of cusps. For geometrically finite hyperbolic surfaces, the cusps all have maximal rank and the extension of the resolvent has been proved by Guillopé-Zworski [8], essentially by using that this is satisfied on an exact cusp or funnel (elementary cyclic group acting on $\mathbb{H}^2$), and analytic Fredholm perturbation theory. In the 3-dimensional case, there can be cusps of rank 1, modeled on the quotient $\langle\gamma\rangle\backslash\mathbb{H}^3$ where γ is a parabolic transformation fixing ∞; it was proved by Froese-Hislop-Perry [4] that the resolvent has a meromorphic extension to $s \in \mathbb{C}$. Generalizations of this case to higher dimension have been studied later by Perry [12] and Guillarmou [5], these are cases where all the parabolic elements of Γ have a power which is a pure translation in the horosheres associated to their parabolic fixed point. The scattering theory on geometrically finite hyperbolic manifolds has been developped by Bunke-Olbrich [3], in particular they are able to show the meromorphic extension of the scattering operator acting on the quotient of the discontinuity set of the group by the group and defined using extension and restriction operators on the sphere S^n.

In the present talk (based on the preprint [6]), we explain how to prove the finite meromorphic extension of $R(s)$ to $s \in \mathbb{C}$ in weighted space, which implies the meromorphic extension of Poincarés series $P(s; m; m') := \sum_{\gamma\in\Gamma} e^{-sd(m,\gamma m')}$ for $m, m' \in \mathbb{H}^{n+1}$, of Eisenstein series and of scattering operator. The scattering operator is defined using the asymptotic profile of generalized eigenfunctions at infinity of X, it is an operator acting on $\Gamma\backslash\Omega(\Gamma)$ where $\Omega(\Gamma)$ is the discontinuity set of Γ (the set of points of S^n which are not in the limit set of Γ). Using arguments of Patterson [11], we recover as a corollary, a result recently proved by Roblin [13] (and known in certain cases by Lax-Philipps [9]), ie. the asymptotic as $R \to \infty$ of the number of lattice points of an orbit $\Gamma.m$ in hyperbolic ball of radius R, in terms of the Hausdorff dimension of the limit set. The proof of the meromorphic extension of $R(s)$ is based on a parametrix construction combined with a careful study of the resolvent of models $\Gamma_0\backslash\mathbb{H}^{n+1}$ where Γ_0 is an elementary parabolic group of isometries fixing ∞. These model quotients are warped products on $(0, \infty)\times\mathcal{F}$ where $\mathcal{F}$ is a flat bundle with basis a flat compact manifold of dimension k (where k is the rank of Γ_0) and fibers $\mathbb{R}^{n-k}$. We first use a spectral decomposition for the Euclidean Laplacian on such bundles $\mathcal{F}$ and use it to get a rather explicit expression of the resolvent $R_0(s)$ on $\Gamma_0\backslash\mathbb{H}^{n+1}$, and prove its meromorphic extension to $s \in \mathbb{C}$. This construction also has the advantage that one can get estimates on the norms of the extension and the number of poles of $R(s)$ in balls $|s| \leq R$. This is also a first step toward proving the meromorphic extension of Selberg zeta function for such cases, as well as the study of its zeros and poles (which is not known in this case by dynamical methods). Another interests of such geometries is that they appeared recently for number theoretic/seaving applications in the work of Bourgain-Gamburd-Sarnak [1].

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Large time limit and the L^2-local index theorem

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(joint work with Sebastian Goette, Thomas Schick)

The heat kernel of a Dirac operator interpolates between the local geometry of the manifold and a global invariant, the index of the Dirac operator. This fundamental property, combined with Quillen–Bismut's superconnection formalism, leads to the so called *local index theorems.*

For a family of Dirac operators on a smooth fibre bundle with compact fibres, Bismut's local index theorem [Bi] is a refinement at the level of differential forms of the cohomological Atiyah–Singer families index theorem. In the proof of the local index theorem it is crucial that the kernels of the fibrewise operators form a bundle, and that the fibres are compact.

When the fibres are not compact, the large time asymptotic of the heat operator is in general not convergent. For general Dirac operators, it is possible to prove the large time limit only assuming regularity conditions on the spectrum [HL].

We present here a new method to compute explicitely the large time limit in the L^2-setting of families of normal coverings for the signature operators, without assuming any regularity assumption.

Definition 1. Let $\underline{\pi}\colon \underline{M} \to B$ be a smooth fibre bundle with compact fibre. A *family of normal coverings* $(M,\Gamma) \to B$ *of* $\underline{\pi}$ consists of a bundle of discrete groups $\Gamma \to B$ and a covering $p\colon M \to \underline{M}$ such that for all $b \in B$ $p_b\colon M_b \to \underline{M}_b$ is a normal covering with group of covering transformations Γ_b.

We can restrict here to the simplest example, a normal covering of a fibre bundle, which also gives the local model of the definition above. Consider $\pi\colon M \to B$ be a smooth fibre bundle, with typical fibre Z^{2l} and let Γ be a discrete group acting on M fibrewise freely and properly discontinuosly such that $M/\Gamma \to B$ has compact fibres. Let g^{TZ} a Γ-invariant metric on the vertical tangent bundle TZ, and denote with $D^{sign} = (d^{Z_b} + d^{Z_b,*})_{b\in B}$ the family of signature operators, odd with respect to the chirality grading τ. Each D_b^{sign} is a Breuer–Fredholm operator affiliated to the semifinite von Neumann algebra $\mathcal{B}_\Gamma(L^2(Z_b, \Lambda T^* Z_b))$, and we denote with tr_Γ Atiyah's L^2-trace on it.

Fixing a horizontal Γ-invariant subbundle $T^H M$ s.t. $TM = T^H M \oplus TZ$, one constructs the Bismut superconnection $\mathbb{A}$ adapted to D^{sign} which satisfies

$$\mathrm{tr}_\Gamma\left(\tau e^{-\mathbb{A}_T^2}\right) - \mathrm{tr}_\Gamma\left(\tau e^{-\mathbb{A}_t^2}\right) = -d\int_t^T \mathrm{tr}_\Gamma\left(\tau \frac{d\mathbb{A}_s}{ds} e^{-\mathbb{A}_s^2}\right) ds$$

and $\lim_{t\to 0} \mathrm{tr}_\Gamma\left(\tau e^{-\mathbb{A}_t^2}\right) = \int_{M/B} L(M/B)$. On the other hand, at $t \to \infty$ the goal is to prove that $\lim_{t\to\infty} \mathrm{tr}_\Gamma\left(\tau e^{-\mathbb{A}_t^2}\right) = \mathrm{tr}_\Gamma\, \tau e^{-\nabla_0^2}$, where ∇_0 is the connection on the bundle $\mathrm{Ker}(d^Z + d^{Z,*}) \to B$ induced by the Gauss–Manin flat connection on the bundle of L^2-cohomology of the fibres, and to prove integrability at ∞ of the eta form.

Heitsch and Lazarov investigated the problem in the very general setting of a foliated manifold with Hausdorff graph, for a general Dirac type operator D: in [HL] they obtain the large time limit assuming that the spectral projections $\chi_{\{0\}}(D_b)$ and $\chi_{(0,\varepsilon)}(D_b)$ are transversally smooth (in this setting, smooth with respect to b), and that the fibrewise Novikov–Shubin invariants are bigger than three times the codimension of the foliation. While smoothness of $\chi_{\{0\}}(D_b^{sign})$ is fulfilled on Riemannian foliations (and, as particular case, in our L^2 setting) [GR, BH3], the other conditions are very restrictive and hard to verify.

Our contribution is a new method to prove the large time limit for the family of signature operators without assuming any regularity assumption. The same method, combined with estimates à la Cheeger–Gromov, leads to the definition of the L^2-eta form for the family D^{sign} under determinant class condition. We obtain the following

Theorem 2. *Let $(M, \Gamma) \to B$ be a family of normal coverings of $\underline{\pi}\colon \underline{M} \to B$. Let $\mathbb{A}$ denote the Bismut superconnection adapted to the family of signature operators. Then*

$$\lim_{t\to\infty} \mathrm{tr}_\Gamma(\tau e^{-\mathbb{A}_t^2}) = \mathrm{tr}_\Gamma(\tau e^{-\nabla_0^2})\ .$$

Theorem 3. *If the fibres Z_b are determinant-class, then $\mathrm{tr}_\Gamma\left(\tau \frac{d\mathbb{A}_t}{dt} e^{-\mathbb{A}_t^2}\right)$ is integrable on $[1, \infty)$, and the L^2-eta form $\hat{\eta}_\Gamma(D^{sign}) = \int_0^\infty \mathrm{tr}_\Gamma\left(\tau \frac{d\mathbb{A}_t}{dt} e^{-\mathbb{A}_t^2}\right)$ is well defined as a continuos form on B.*

As a consequence we get a (weak) L^2 local index theorem

Corollary 4. *Let c be a smooth chain in B. Then*

$$\int_c \int_{M/B} L(M/B) - \int_c \operatorname{tr}_\Gamma e^{-(\nabla^{sign})^2} = \int_{\partial c} \hat{\eta}_\Gamma \qquad \textit{if } \dim Z = \textit{even},$$

$$\int_c \int_{M/B} L(M/B) = \int_{\partial c} \eta_\Gamma \qquad \textit{if } \dim Z = \textit{odd}.$$

We present now the two fundamental ideas of the proofs. The first ingredient is the particular structure of the Bismut superconnection $\mathbb{A}$ adapted to D^{sign}. It is $\mathbb{A} = d^M + d^{M,*}$, where $d^{M,*}$ is the adjoint superconnection of the flat superconnection d^M [BL]. Therefore $-\mathbb{A}^2 = \mathbb{X}^2$, where $\mathbb{X} = d^{M,*} - d^M$. The operator $\mathbb{X}$ plays a fundamental role: being the difference of two superconnections, it does not contain transversal derivatives. Therefore when writing the Duhamel expansion of $e^{-\mathbb{A}_t^2} = e^{\mathbb{X}_t^2}$ we obtain the expression

$$(1) \quad e^{\mathbb{X}_t^2} = \sum_{n=0}^{m_B} \int_{\Delta^n} e^{s_0 t \hat{D}^2} (\sqrt{t} R_t \hat{D} + \sqrt{t} \hat{D} R_t + R_t^2) e^{s_1 t \hat{D}^2} \dots$$
$$\dots (\sqrt{t} R_t \hat{D} + \sqrt{t} \hat{D} R_t + R_t^2) e^{s_n t \hat{D}^2} d^n(s_0, \dots, s_n) \,.$$

where $\hat{D} = d^{Z,*} - d^Z$, and R_t does not contain transversal derivatives. This allows to group together functions of $\sqrt{t}\hat{D}$ and get better estimates than the approach used in [HL] for general Dirac type operators.

The second point is a new method to estimate the terms in (1) as $t \to \infty$. We split Δ^n in subsets such that on each of them we have "small" or "big" variables s_j, and use different kind of estimates accordingly.

To get the well definiteness of the eta form $\hat{\eta}_\Gamma$ we employ our estimates to obtain a generalization of the classical Cheeger–Gromov estimate [CG2]. Here we need to assume that the fibres are determinant-class. Yet, we conjecture that theorem 2 should be true also without this assumption. We obtain $\hat{\eta}_\Gamma$ as a continuous form on B, hence the local index theorem is in the weak form.

In parallel, our method gives an L^2 Bismut–Lott theorem, and allows to define the analytic L^2 torsion form under the same hypothesis.

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The resolvent trace of an elliptic cone operator

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(joint work with Juan B. Gil and Thomas Krainer)

This note discusses some aspects of the analysis leading to the proof of the main theorem in [10] (stated here as Theorem 1) on the structure of the asymptotics of the resolvent trace of a general elliptic cone operator as the spectral parameter tends to infinity, under suitable minimal growth assumptions on the principal symbols of the operator.

We deal with an elliptic cone differential operator

$$A : C_c^\infty(\mathring{\mathcal{M}}; E) \subset x^\gamma L_b^2(\mathcal{M}; E) \to x^\gamma L_b^2(\mathcal{M}; E) \tag{1}$$

of positive order m, an element of $x^{-m}\operatorname{Diff}_b^m(\mathcal{M}; E)$; $\mathcal{M}$ is a compact n-manifold with boundary, $E \to \mathcal{M}$ is a smooth complex Hermitian vector bundle, and L_b^2 is defined using some fixed smooth positive b-density $\mathfrak{m}$. As usual, x is a smooth defining function for $\mathcal{Y} = \partial\mathcal{M}$, positive in the interior of $\mathcal{M}$. The number $\gamma \in \mathbb{R}$ is arbitrary. Ellipticity means that $P = x^m A$ is a b-elliptic differential operator. Somewhat unnaturally (see [6]), we write here ${}^c\boldsymbol{\sigma}(A) = {}^b\boldsymbol{\sigma}(P)$ (see [16] for the notation ${}^b\boldsymbol{\sigma}(P)$ and the notion of b-ellipticity).

Details of the following setup can be found in [6]. Let $\mathcal{D}_{\min}$, resp. $\mathcal{D}_{\max}$, be the domains of the minimal, resp. maximal, extensions of the operator (1):

$$\mathcal{D}_{\max} = \{u \in x^\gamma L_b^2(\mathcal{M}; E) : Au \in x^\gamma L_b^2(\mathcal{M}; E)\}$$

is a Hilbert space with respect to the inner product $(u, v)_A = (Au, Av) + (u, v)$, $u, v \in \mathcal{D}_{\max}$, and $\mathcal{D}_{\min}$ is the closure of $C_c^\infty(\mathring{\mathcal{M}}; E)$ in $\mathcal{D}_{\max}$. From [13] we know that $\mathcal{D}_{\min}$ has finite codimension in $\mathcal{D}_{\max}$, hence every closed extension of (1) has as domain a subspace $\mathcal{D} \subset \mathcal{D}_{\max}$ of the form $\mathcal{D} = D + \mathcal{D}_{\min}$ where D is uniquely determined by the condition that D is orthogonal to $\mathcal{D}_{\min}$. We let $\mathcal{E}$ be the orthogonal complement of $\mathcal{D}_{\min}$ in $\mathcal{D}_{\max}$. Domains of closed extensions then correspond to the points of the various complex Grassmannian varieties associated with $\mathcal{E}$.

There is an operator

$$A_\wedge : C_c^\infty(\mathring{\mathcal{Y}}^\wedge;\wp^*E) \subset x_\wedge^\gamma L_b^2(\mathcal{Y}^\wedge;\wp^*E) \to x_\wedge^\gamma L_b^2(\mathcal{Y}^\wedge;\wp^*E) \tag{2}$$

canonically associated with A. Here $\wp : \mathcal{Y}^\wedge \to \mathcal{Y}$ is the inward pointing normal bundle of $\mathcal{Y}$ in $\mathcal{M}$ (the zero section is included), $x_\wedge = dx|_{\mathcal{Y}^\wedge}$ and the L^2 space is defined using the density $x_\wedge^{-1}dx_\wedge \otimes \mathfrak{m}_{\mathcal{Y}}$ where $\mathfrak{m} = x^{-1}dx \otimes \mathfrak{m}_{\mathcal{Y}}$ along $\mathcal{Y}$. The operator (2) has its own maximal and minimal domains $\mathcal{D}_{\wedge,\max}$ and $\mathcal{D}_{\wedge,\min}$. Lesch's result on the codimension of the latter in the former still holds. We let $\mathcal{E}_\wedge$ be the orthogonal complement of $\mathcal{D}_{\wedge,\min}$ in $\mathcal{D}_{\wedge,\max}$. There is a natural vector space isomorphism

$$\theta : \mathcal{E} \to \mathcal{E}_\wedge$$

which allows passage from domains of closed extensions of (1) to those of (2) and back, namely

$$\mathcal{D} = D + \mathcal{D}_{\min} \longleftrightarrow \mathcal{D}_\wedge = \theta(D) + \mathcal{D}_{\wedge,\min}. \tag{3}$$

Let Λ be a closed sector in $\mathbb{C}$. The main result of [7] asserts that if ${}^c\boldsymbol{\sigma}(A) - \lambda$ is invertible when $\lambda \in \Lambda$ and if in addition Λ is a sector of minimal growth for $A_{\wedge,\mathcal{D}_\wedge}$ ($A_\wedge$with domain $\mathcal{D}_\wedge$), then Λ is a sector of minimal growth for $A_{\mathcal{D}}$, where $\mathcal{D}$ and $\mathcal{D}_\wedge$ are related by (3). In [10] we show:

Theorem 1. Let Λ be a sector of minimal growth both for ${}^c\boldsymbol{\sigma}(A)$ and for $A_{\wedge,\mathcal{D}_\wedge}$. For any $\varphi \in C^\infty(M;\mathrm{End}(E))$ and $\ell \in \mathbb{N}$ with $m\ell > n$,

$$\mathrm{Tr}\big(\varphi(A_{\mathcal{D}} - \lambda)^{-\ell}\big) \sim \sum\nolimits_{j=0}^{n-1} \alpha_j \lambda^{\frac{n-\ell m - j}{m}} + \alpha_n \log(\lambda)\lambda^{-\ell} + s_{\mathcal{D}}(\lambda)$$

with coefficients $\alpha_j \in \mathbb{C}$ that are independent of the choice of $\mathcal{D}$, and

$$s_{\mathcal{D}}(\lambda) \sim \sum\nolimits_{j=0}^{\infty} r_j(\lambda^{i\mu_1},\dots,\lambda^{i\mu_N},\log\lambda)\lambda^{\nu_j/m} \text{ as } |\lambda| \to \infty, \tag{4}$$

where each r_j is a rational function in $N+1$ variables, $N \in \mathbb{N}_0$, with real numbers μ_k, $k = 1,\dots,N$, and $0 \geq \nu_j \searrow -\infty$ as $j \to \infty$. We have $r_j = p_j/q_j$ with $p_j,\ q_j \in \mathbb{C}[z_1,\dots,z_{N+1}]$ such that $q_j(\lambda^{i\mu_1},\dots,\lambda^{i\mu_N},\log\lambda)$ is uniformly bounded away from zero for large λ.

A number of references at the end of this note, needless to say incomplete, point to earlier related work by various other authors in special cases.

Let bg-res$(A_\wedge)$ be the set of $\lambda \in \mathbb{C}$ such that $A_\wedge - \lambda$ is injective on $\mathcal{D}_{\wedge,\min}$ and surjective on $\mathcal{D}_{\wedge,\max}$. For $\lambda \in$ bg-res$(A_\wedge)$ set $\mathcal{K}_\lambda = \ker(A_{\wedge,\mathcal{D}_{\wedge,\max}} - \lambda)$. Then $\lambda \in \mathrm{res}(A_{\wedge,\mathcal{D}_\wedge})$ iff $\lambda \in$ bg-res$(A_\wedge)$ and $\mathcal{K}_\lambda \oplus \mathcal{D}_\wedge = \mathcal{D}_{\wedge,\max}$. Let $\mathcal{R}_\lambda$ be the range of $A_\wedge - \lambda$ on $\mathcal{D}_{\wedge,\min}$. There exist

$$B_{\wedge,\min}(\lambda) : x_\wedge^\gamma L_b^2 \to \mathcal{D}_{\min} \text{ with kernel equal to } \mathcal{R}_\lambda^\perp$$

such that $B_{\wedge,\min}(\lambda)(A_\wedge - \lambda) = I$ on $\mathcal{D}_{\wedge,\min}$, and

$$B_{\wedge,\max}(\lambda) : x_\wedge^\gamma L_b^2 \to \mathcal{D}_{\max} \text{ with range equal to } \mathcal{K}_\lambda^\perp \cap \mathcal{D}_{\wedge,\max}$$

(the orthogonal in the space $x_\wedge^\gamma L_b^2$) such that $(A_\wedge - \lambda)B_{\wedge,\max}(\lambda) = I$ on $x_\wedge^\gamma L_b^2$. The resolvent of $A_{\wedge,\mathcal{D}_\wedge}$ is (see [6])

$$B_{\wedge,\mathcal{D}_\wedge}(\lambda) = B_{\wedge,\max}(\lambda) - \big[I - B_{\wedge,\min}(\lambda)(A_\wedge - \lambda)\big]\pi_{\wedge,\max}\pi_{\mathcal{K}_\lambda,\mathcal{D}_\wedge}\pi_{\wedge,\max}B_{\wedge,\max}(\lambda)$$

in which $\pi_{\mathcal{K}_\lambda,\mathcal{D}_\wedge} : \mathcal{D}_{\wedge,\max} \to \mathcal{D}_{\wedge,\max}$ is the projection on $\mathcal{K}_\lambda$ according to the decomposition $\mathcal{D}_{\wedge,\max} = \mathcal{K}_\lambda \oplus \mathcal{D}_\wedge$ (this holds when $\lambda \in \operatorname{res}(A_\wedge)$) and $\pi_{\wedge,\max}$ is the orthogonal projection on $\mathcal{E}_\wedge$. Altogether, $\pi_{\wedge,\max}\pi_{\mathcal{K}_\lambda,\mathcal{D}_\wedge}|_{\mathcal{E}_\wedge}$ is equal to the projection $\pi_{K_\lambda,D_\wedge} : \mathcal{E}_\wedge \to \mathcal{E}_\wedge$ on $K_\lambda = \pi_{\max}\mathcal{K}_\lambda$ according to $\mathcal{E}_\wedge = K_\lambda \oplus D_\wedge$.

The multiplicative group $\mathbb{R}_+$ acts canonically on $\mathcal{Y}^\wedge$. Define κ_ϱ on $C_c^\infty(\mathcal{Y}^\wedge; E)$ for $\varrho \in \mathbb{R}_+$ by $\kappa_\varrho(u)(\nu) = \varrho^{-\gamma}u(\varrho\nu)$. The operators κ_ϱ extend to give a strongly continuous unitary action of $\mathbb{R}_+$ on $x_\wedge^\gamma L_b^2(\mathcal{Y}^\wedge; \wp^*E)$. The operator $A_\wedge$ has the property $\kappa_\varrho^{-1}(A_\wedge - \varrho^m\lambda)\kappa_\varrho = \varrho^m(A_\wedge - \lambda)$, which in turn produces $\kappa_\varrho\mathcal{K}_\lambda = \mathcal{K}_{\varrho^m\lambda}$ as well as $\kappa_\varrho^{-1}B_{\wedge,\min}(\varrho^m\lambda)\kappa_\varrho = \varrho^{-m}B_{\wedge,\min}(\lambda)$ and the same formula with min replaced by max.

As a consequence, $B_{\wedge,\mathcal{D}_\wedge}(\varrho^m\lambda)$ is equal to

$$\varrho^{-m}\kappa_\varrho\big\{B_{\wedge,\max}(\lambda) - \big[I - B_{\wedge,\min}(\lambda)(A_\wedge - \lambda)\big]\pi_{K_\lambda,\kappa_\varrho^{-1}D_\wedge}\pi_{\wedge,\max}B_{\wedge,\max}(\lambda)\big\}\kappa_\varrho^{-1}.$$

This formula brings to the forefront the role played by the dynamical system $\varrho \mapsto \kappa_\varrho^{-1}D_\wedge$ in the Grassmannian $\operatorname{Gr}_k(\mathcal{E}_\wedge)$, $k = \dim D_\wedge$, especially the limiting sets, on the behavior of the resolvent. One can show that the ray through $\lambda \neq 0$ is a ray of minimal growth for $A_{\wedge,\mathcal{D}_\wedge}$ if and only if the set

$$\Omega^-(\mathcal{D}_\wedge) = \{D \in \operatorname{Gr}_k(\mathcal{E}_\wedge) : \exists\{\varrho_k\}_{k=1}^\infty,\ \varrho \to \infty,\ \lim \kappa_\varrho^{-1}D_\wedge = D\}$$

is disjoint from $\mathcal{V}_{K_\lambda} = \{D \in \operatorname{Gr}_k(\mathcal{E}_\wedge) : D \cap K_\lambda = 0\}$, see [8]. The hypotheses of Theorem 1 imply this is the case for λ in a closed arc in Λ. In [10] we elucidate the asymptotics of $\pi_{K_\lambda,\kappa_\varrho^{-1}D_\wedge}$ and use it to determine the asymptotics of the trace of the resolvent of $A_\mathcal{D}$.

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A glimpse of noncommutative curvature

Henri Moscovici

§1. The basic template for a space in noncommutative geometry is the notion of *p-summable spectral triple*, modeled on the Dirac operator. It consists of a (local Banach) $*$-algebra $\mathcal{A}$ represented in the Hilbert space $\mathfrak{H}$ by bounded operators, and a self-adjoint unbounded operator D with the resolvent of Schatten class $p \geq 1$, and having bounded commutators with any $a \in \mathcal{A}$. The noncommutative generalization of a "smooth space" is a spectral triple $(\mathcal{A}, \mathfrak{H}, D)$ which admits *residual zeta regularization*, in the sense that the zeta functions $\zeta_D(P, z) = \mathrm{Tr}(P\,|D|^{-z})$, $\mathrm{Re}\, z > p$, associated to analogues of pseudodifferential operators $P \in \Psi(\mathcal{A}, \mathfrak{H}, D)$ admit meromorphic extensions to $\mathbb{C}$. Assuming, for the sake of simplicity, that all poles are simple, the algebra $\Psi(\mathcal{A}, \mathfrak{H}, D)$ acquires a *residue trace functional* $⨍ P := \mathrm{Res}_{z=0}\, \zeta_D(P, z)$, which vanishes on trace class operators and therefore only depends on the "complete symbol" of P. The local index formula [5] expresses the K-homological Chern character of such a spectral triple in terms of a cocycle in the cyclic cohomology bicomplex $\{CC(\mathcal{A}), b, B\}$. Its components are linear combinations of terms of the form $⨍ a_0\, [D, a_1]^{(k_1)} \cdots [D, a_q]^{(k_q)}\, |D|^{-2|\mathbf{k}|-q}$, with $P^{(k)} = [D^2, \ldots, [D^2, P] \ldots]$ (k-th order commutator), and they play the role of the "Pontryagin forms" of the spectral triple. Actually, in the case of the Dirac operator on a closed spin manifold they do, indeed, coincide with the Pontryagin polynomials in the curvature of the underlying Riemannian metric.

§2. In order to shed more light on the elusive notion of "noncommutative curvature", it makes sense to look at the noncommutative 2-torus with irrational slope θ. Its standard spectral triple $(\mathcal{A}_\theta, \mathcal{H}, D)$ (cf. [4]) corresponds to the flat metric, but one can consider a conformal change by a positive element $k = e^{\frac{h}{2}}$, $h = h^*$ in its algebra of "smooth coordinates" $\mathcal{A}_\theta$. The analogue of the *scalar curvature* κ_h for the resulting twisted Laplacian is given by the equation $\tau(a\,\kappa_h) = \zeta_{k\triangle k}(a, 0)$, $\forall\, a \in \mathcal{A}_\theta$, where $\tau : A_\theta \to \mathbb{C}$ is the unique normalized trace on the (simple !) C^*-algebra A_θ. Using the pseudodifferential calculus developed in [3], it was shown in [7] that $\zeta_{k\triangle k}(0) := \zeta_{k\triangle k}(\mathrm{id}, 0)$ is independent of k and therefore a conformal invariant. Moreover, (computer aided) symbolic computations by A. Connes [1] give

[1]Private communication

the following expression for the scalar curvature:

$$\kappa_h = e^{-h}\Big(K(\log\Delta)(\triangle(h)) + \sum_{j=1}^{2} H(\nabla_{(1)},\nabla_{(2)})\big(\delta_j(h)\delta_j(h)\big)\Big);$$

here $\delta_1, \delta_2 \in \mathrm{Der}(\mathcal{A}_\theta)$ are the canonical derivations, $\triangle = \delta_1^2 + \delta_2^2$ is the Laplacian, while $\Delta(x) = e^{-h}xe^h$ is the modular operator, $\nabla_{(i)}$ signifies the action of $\log\Delta = -\mathrm{ad}\, h$ on the i-th factor of the product, and the functions K, H have the following expressions: $K(u) = \dfrac{e^u - e^{u/2}u - 1}{\left(-1+e^{u/2}\right)^3}$, and

$$H(s,t) = \left(1 + \mathrm{ch}\left(\tfrac{s+t}{2}\right)\right) \times$$
$$\times \frac{-t(s+t)\mathrm{ch}(s) + s(s+t)\mathrm{ch}(t) - (s-t)(s+t+\mathrm{sh}(s)+\mathrm{sh}(t)-\mathrm{sh}(s+t))}{s\,t\,(s+t)\mathrm{sh}\left(\frac{s}{2}\right)\mathrm{sh}\left(\frac{t}{2}\right)\mathrm{sh}\left(\frac{s+t}{2}\right)^2}.$$

This should be contrasted with the scalar curvature of the conformal metric on the ordinary torus T^2 with conformal factor e^h, $h \in C^\infty(\mathrm{T}^2, \mathbb{R})$, in which case $\mathrm{ad}\, h = 0$ and the above formidable expression simply becomes $\kappa_h^0 = e^{-h}K(0) = \frac{1}{3}e^{-h}\triangle h$.

§3. Let $(\mathcal{A}, \mathfrak{H}, D)$ be a p-summable spectral triple, and let $h = h^* \in \mathcal{A}$. Changing its "metric" by a conformal factor amounts to replacing D by $D_h = e^h\, D\, e^h$. The commutators $[D_h, a]$, $a \in \mathcal{A}$, are no longer bounded, unless h is central. However, inserting the automorphism $\sigma(a) = e^{2h}\, a\, e^{-2h}$ into the datum, one observes that the twisted commutators $[D_h, a]_\sigma := D_h\, a - \sigma_h(a)\, D_h$ are bounded. This gives a particular example of a *twisted spectral triple*, notion which was studied in [6] and also in [9].

For the purposes of this talk, we shall assume that $(\mathcal{A}, \mathfrak{H}, D)$ has *good pseudodifferential calculus*, in the sense that there exist asymptotic expansions of the form

$$\mathrm{Tr}\left(A\, e^{-tD_{sh}^2}\right) =_{t\searrow 0} \sum_{j=0}^{\infty} a_j(A,s)\, t^{\frac{j-N-p}{2}} + O(1),$$

for any operator A which is a (noncommutative) polynomial in D and a finite number of elements $a \in \mathcal{A}$. Moreover, we shall assume *good resolvent approximation*, property which ensures that these expansions can be differentiated in the term-by-term in $s \in [-1, 1]$.

Theorem (Conformal index à la Branson-Ørsted [1]) *For a spectral triple* $(\mathcal{A}, \mathfrak{H}, D)$ *with good pseudodifferential calculus and good resolvent approximation,* $\zeta_D(0)$ *is a conformal invariant.*

Proof. Relying on the above assumptions and notation, it can be easily shown that $\frac{d}{ds}a_j(\mathrm{id}, s) = 2(j-p)\, a_j(h, s)$, which implies $\frac{d}{ds}a_p(s) = 0$. □

Since by [3] the noncommutative tori have good pseudodifferential calculus, one obtains a non-computational proof of the corresponding result for the noncommutative 2-torus [7, Theorem 1.1], as well as for its generalization to a general translation invariant noncommutative metric structure (cf. [8]).

As another consequence, one immediately recovers the independence of dilaton field rescaling of the constant term in the Chamseddine-Connes spectral action (cf. [2]) for the noncommutative space of the standard model.

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Bergman kernel and geometric quantization

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(joint work with George Marinescu)

In the theory of quantization, one attempts to associate to a symplectic manifold (X, ω) a Hilbert space H and a mapping from the space of functions on X into the space of operators on H, and this in a canonical way. The mapping should give some reasonable relationship between the Poisson bracket on the function side and the commutator on the operator side. It is generally acknowledged that there is no canonical way to construct a quantization of X without making use of certain additional structures.

In the theory of the geometric quantization of Kostant and Souriau, (X, ω) is assumed to be prequantizable, that is, there exists a prequantum line bundle (L, h^L, ∇^L) on X (i.e., ω is the first Chern form of L associated with the Hermitian connection ∇^L). Given a compatible almost complex structure J and a Riemannian metric g^{TX}, we can define canonically a Dirac operator D^L acting on $\Omega^{0,\bullet}(X, L)$, the smooth $(0, \bullet)$-forms on X with coefficients in L.

Assume that X is compact. Following an observation by Bott, we take, as a quantization of X, $\mathrm{Ind}(D^L_+) = \mathrm{Ker}(D^L_+) - \mathrm{Coker}(D^L_+)$ of $D^L_+ := D^L|_{\Omega^{0,\mathrm{even}}}$, which is a formal difference of finite dimensional Hilbert spaces.

The virtual dimension of $\mathrm{Ind}(D^L_+)$, which can be computed by the Atiyah-Singer index theorem, does not depend on the choice of the connection and of the metric on L.

For $p \gg 1$, $\mathrm{Ind}(D^{L^p}_+) = \mathrm{Ker}(D^{L^p}_+)$ is an ordinary finite dimensional Hilbert space. The Bergman kernel is defined as the integral kernel $P_p(x, x')$ associated with the orthogonal projection P_p from $\Omega^{0,\bullet}(X, L^p)$ onto $\mathrm{Ker}(D^{L^p})$. We will show that when $p \to +\infty$, the Bergman kernel $P_p(x, x')$ has an asymptotic expansion whose coefficients contain interesting geometric informations about X and L. The kind of expansion obtained for the kernel $P_p(x, x')$ also characterizes the Berezin-Toeplitz operators. Their semi-classical limit provides a precise way to relate the classical and quantum observables.

If (X, ω, J) is a compact Kähler manifold and if L is holomorphic, then for $p \gg 1$, $\mathrm{Ker}(D^{L^p})$ is the space of holomorphic sections $H^0(X, L^p)$ of L^p on X. This leads to many applications of the asymptotic expansion of the Bergman kernel in Kähler geometry.

We refer the reader to our book with Marinescu [4] for a comprehensive study of the Bergman kernel and applications. This note is a short version of our survey [2].

In this survey talk, we start to explain our model situation: Let $L = \mathbb{C}$ be the trivial holomorphic line bundle on $\mathbb{C}^n$, but the metric on L is a non-trivial metric h^L: $|\mathbf{1}|_{h^L}(z) := e^{-\frac{1}{4}\sum_{j=1}^n a_j|z_j|^2} = \rho(Z)$ for $z \in \mathbb{C}^n$, with $a_j > 0$ for $j \in \{1, \dots, n\}$.

To introduce the model operator $\mathscr{L}$ we set:

$$b_i = -2\frac{\partial}{\partial z_i} + \frac{1}{2}a_i\overline{z}_i\,, \quad b_i^+ = 2\frac{\partial}{\partial \overline{z}_i} + \frac{1}{2}a_i z_i\,, \quad \mathscr{L} = \sum_i b_i\, b_i^+\,.$$

We can interpret the operator $\mathscr{L}$ in terms of complex geometry. Let $\overline{\partial}^{L*}$ be the adjoint of the Dolbeault operator $\overline{\partial}^L$ on (L, h^L) over $(\mathbb{C}^n, \frac{\sqrt{-1}}{2}\sum_j dz_j \wedge d\overline{z}_j)$. We have the isometry $\Omega^{0,\bullet}(\mathbb{C}^n, \mathbb{C}) \to \Omega^{0,\bullet}(\mathbb{C}^n, L)$ given by $\alpha \mapsto \rho^{-1}\alpha$. If $\square^L = \overline{\partial}^{L*}\overline{\partial}^L + \overline{\partial}^L\overline{\partial}^{L*}$ denotes the Kodaira Laplacian acting on $\Omega^{0,\bullet}(\mathbb{C}^n, L)$, then $\rho\,\square^L\rho^{-1}$: $\Omega^{0,\bullet}(\mathbb{C}^n,\ \mathbb{C}) \to \Omega^{0,\bullet}(\mathbb{C}^n, \mathbb{C})$ is given by $\frac{1}{2}\mathscr{L} + \sum_j a_j d\overline{z}^j \wedge i_{\frac{\partial}{\partial \overline{z}_j}}$, and its restriction on functions is $\frac{1}{2}\mathscr{L}$.

The operator $\mathscr{L}$ is the complex analogue of the harmonic oscillator, the operators b, b^+ are creation and annihilation operators respectively. Each eigenspace of $\mathscr{L}$ has infinite dimension, but we can still give an explicit description.

Theorem 1 (Ma-Marinescu [5, Th. 1.15], [4, Th. 4.1.20])**.** The spectrum of $\mathscr{L}$ on $L^2(\mathbb{R}^{2n})$ is given by

$$\mathrm{Spec}(\mathscr{L}) = \Big\{2\sum_{i=1}^n \alpha_i a_i\ :\ \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n\Big\}$$

and an orthogonal basis of the eigenspace of $\lambda \in \mathrm{Spec}(\mathscr{L})$ is given by

$$B_\lambda = \Big\{ b^\alpha \big(z^\beta \exp\big(-\tfrac{1}{4}\textstyle\sum_i a_i |z_i|^2\big)\big) \;:\; 2\sum_i \alpha_i a_i = \lambda, \text{ with } \alpha, \beta \in \mathbb{N}^n \Big\}$$

where $b^\alpha := b_1^{\alpha_1} \cdots b_n^{\alpha_n}$. Moreover, $\bigcup_\lambda \{B_\lambda : \lambda \in \mathrm{Spec}(\mathscr{L})\}$ forms a complete orthogonal basis of $L^2(\mathbb{R}^{2n})$.

Let $\mathscr{P}(Z, Z')$ be the smooth kernel of $\mathscr{P}$, which is the orthogonal projection from $(L^2(\mathbb{R}^{2n}), \|\cdot\|_{L^2})$ onto $\mathrm{Ker}(\mathscr{L})$, with respect to dZ'. Then $\mathscr{P}(Z, Z')$ is the classical *Bergman kernel* on $\mathbb{C}^n$ given by

$$\mathscr{P}(Z, Z') = \prod_{i=1}^{n} \frac{a_i}{2\pi} \; \exp\Big(-\frac{1}{4} \sum_i a_i \big(|z_i|^2 + |z_i'|^2 - 2 z_i \overline{z}_i' \big) \Big). \tag{1}$$

Then we explain for a positive line bundle L on a compact symplectic manifold, in which sense the Bergman kernel associated with $L^p := L^{\otimes p}$ can be approximated by the above model situation. In this way, we get a characterization of Toeplitz operator by using the asymptotic expansion of its kernel, and we can compute the coefficients of various expansions with the help of (1).

At the end of this talk, we review very briefly the "quantization commutes with reduction"–phenomenon for a compact Lie group action, and its relation to the Bergman kernel.

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Stochastic Completeness and Volume Growth

Christian Bär

Throughout the talk M denotes a connected but not necessarily geodesically complete Riemannian manifold of dimension n. We denote by Δ the Laplace-Beltrami operator acting on smooth compactly supported functions and also its Friedrichs extension to $L^2(M)$, the space of square-integrable functions.

Let $k \in C^\infty((0,\infty)\times M\times M)$ be the corresponding heat kernel. It is well-known that either $\int_M k(t,x,y)dy = 1$ for all $t > 0$ and all $x \in M$ or $\int_M k(t,x,y)dy < 1$ for all t and x. In the first case we call M *stochastically complete*. Examples for stochastically complete manifolds are closed manifolds, Euclidean space or hyperbolic space.

In the stochastically complete case fix $x \in M$ and construct a probability measure (the *Wiener measure*) on $\Omega_x(M) := \{\omega \in C^0([0,\infty), M) \,|\, \omega(0) = x\}$ as follows. For any $0 < t_1 < \cdots < t_m$ and any Borel sets $B_1, \ldots, B_m \subset M$ put $I_x(t_1, \ldots, t_m; B_1, \ldots, B_m) := \{\omega \in \Omega_x(M) \,|\, \omega(t_j) \in B_j\}$. Set

$$\mathbb{W}_x(I_x(\vec{t},\vec{B})) := \int\limits_{B_1\times\cdots\times B_m} k(t_1,x,y_1)k(t_2-t_1,y_1,y_2)\cdots k(t_m-t_{m-1},y_{m-1},y_m)d\vec{y}.$$

Here we used the abbreviations $\vec{t} = (t_1, \ldots, t_m)$, $\vec{B} = (B_1, \ldots, B_m)$ and $d\vec{y} = dy_1 \cdots dy_m$. In particular, for $m = 1$,

$$\mathbb{W}_x(I_x(t,B)) = \int_B k(t,x,y)\,dy.$$

One easily checks that the set of sets $\mathcal{J}_x := \{I_x(t_1, \ldots, t_m; B_1, \ldots, B_m)\}$ forms a semi-algebra. It requires some work to verify that $\mathbb{W}_x$ is countably additive on $\mathcal{J}_x$. The Caratheodory extension theorem now tells that $\mathbb{W}_x$ extends to a measure on the σ-algebra generated by $\mathcal{J}_x$. This σ-algebra is easily identified to be the Borel algebra of $\Omega_x(M)$ with respect to the compact-open topology.

Wiener measure being constructed we can define M to be *recurrent* if for all non-empty open subsets $U \subset M$

$$\mathbb{W}_x[\exists t_j \nearrow \infty : \omega(t_j) \in U] = 1.$$

Otherwise, we call M *transient*.

For $r > 0$ denote by $B(x,r) \subset M$ the closed ball of radius r which is centered at x. Put $V(x,y) := \mathrm{vol}_n(B(x,r))$ and $A(x,r) := \mathrm{vol}_{n-1}(\partial B(x,r))$.

Now we look at *spherically symmetric manifolds*. By this we mean $\mathbb{R}^n$ equipped with a metric which in polar coordinates takes the form

$$ds^2 = dr^2 + f(r)^2 d\sigma^2$$

where $d\sigma^2$ is the standard metric of S^{n-1} and $f \in C^\infty([0,\infty),\mathbb{R})$ with $f(r) > 0$ for $r > 0$, $f(0) = 0$, and $f'(0) = 1$. These manifolds are always geodesically complete. The most prominent examples are Euclidean space ($f(r) = r$) and hyperbolic space ($f(r) = \sinh(r)$). On a spherically symmetric manifold stochastic completeness and recurrence have nice characterizations in terms of volume growth.

Proposition 1. [4, Props. 3.1 and 3.2] Let M be a spherically symmetric manifold. Then

(i) M is stochastically complete if and only if

$$\int_1^\infty \frac{V(0,r)}{A(0,r)}\, dr = \infty.$$

(ii) M is recurrent if and only if

$$\int_1^\infty \frac{dr}{A(0,r)} = \infty.$$

Knowing this it is easy to construct geodesically complete but stochastically incomplete manifolds. E.g., choose $f(r) = r^{(\alpha-1)/(n-1)} \cdot \exp\left(\frac{r^\alpha}{n-1}\right)$ with $\alpha > 2$ for $r \geq 1$.

The beautiful survey article [4] contains a presentation of many different criteria for stochastic completeness and for recurrence. In particular, for a general geodesically complete Riemannian manifold M T. Lyons and D. Sullivan [5] and A. Grigoryan [2, 3] have shown independently that if for some $x \in M$ we have

$$\int_1^\infty \frac{dr}{A(x,r)} = \infty,$$

then M is recurrent.

Grigoryan asked the natural question whether the criterion for stochastic completeness of spherically symmetric manifolds is also sufficient for general manifolds [4, Problem 9]: Suppose that for some $x \in M$ we have

$$\int_1^\infty \frac{V(x,r)}{A(x,r)} dr = \infty. \tag{1}$$

Is M then stochastically complete? Somewhat to the surprise of the experts the answer turns out to be no. In [1] counterexamples are constructed. They are obtained as connected sums of two spherically symmetric manifolds M_1 and M_2. One chooses M_1 stochastically incomplete. This has the consequence that $M = M_1 \sharp M_2$ is stochastically incomplete as well, no matter what one chooses for M_2. Now the warping function of M_2 can be chosen such that (1) holds for M.

Conversely, one may also ask whether

$$\int_1^\infty \frac{V(x,r)}{A(x,r)} dr < \infty.$$

for some $x \in M$ implies that M is stochastically incomplete. It turns out that there are counterexamples as well [1].

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An Adiabatic Decomposition of the Hodge Cohomology of Manifolds Fibred over Graphs

Karsten Fritzsch

Let (X, g) be a smooth, closed and oriented Riemannian manifold and $\{Y_e\}$, $e \in E$ a finite set of mutually disjoint smooth, closed, oriented codimension 1 submanifolds such that on a tubular neighbourhood of each Y_e we have $g = dt^2 + g_{Y_e}$, t denoting the local coordinate normal to Y_e. Because of the product type of the metric, we may stretch cylindrical neighbourhoods Σ_e of the Y_e's to form a prolonged manifold $X(r)$. We try to answer the following

> **Question.** How does the space $\mathcal{H}^*(r)$ of harmonic (differential) forms on $X(r)$ behave for $r \to \infty$?

This question has been studied (for different objects) by many authors, including Atiyah-Patodi-Singer [1], Cappell-Lee-Miller [2], Grieser [4], Hassell-Mazzeo-Melrose [5], Mazzeo-Melrose [6] and Nicolaescu [7].

Placing cuts at the $Y_e \subset X(r)$ yields manifolds with boundary $X_v(r)$, $v \in V$ which may as well be stretched to manifolds with cylindrical ends $X_v(\infty)$. The way we placed the cuts is encoded by a map $\mathcal{X}$ and a graph $\mathcal{G} = (V, E)$:

$$\mathcal{X} : X \longrightarrow \mathcal{G} \quad , \quad x \to \begin{cases} v & , \quad x \in X_v(0) \\ e & , \quad x \in \Sigma_e \end{cases} . \tag{1}$$

Generalising Cappell-Lee-Miller's concept of "matching at infinity" and their splicing map $\mathcal{S}_r : \mathcal{W} \longrightarrow C^\infty\big(\Lambda T^* X(r)\big)$ to multiple cylinders, arguments of Cappell-Lee-Miller [2] and Nicolaescu [7] show that – for large r – splicing yields an isomorphism between the space of matching sets $\mathcal{W}$ (of extended harmonic forms) and the span of certain eigenforms of the Gauss-Bonnet operator $D(r)$:

Theorem 1 ([2])**.** Let $\varepsilon > 0$, $E(r)$ be the span of eigenforms of the Gauss-Bonnet operator $\mathcal{D}(r)$ on $X(r)$ corresponding to eigenvalues $|\mu| < r^{-(1+\varepsilon)}$ and Π_r denote orthogonal projection of $L^2\big(\Lambda T^* X(r)\big)$ onto $E(r)$. Then, there is $r_0 > 0$ such that for $r > r_0$

$$\Pi_r \circ \mathcal{S}_r : \mathcal{W} \longrightarrow E(r)$$

is an isomorphism.

From a more topological point of view, slightly enlarging the sets $\mathcal{X}^{-1}(\mathcal{G})$ gives an open cover of X. By abuse of notation we denote the cohomology of the corresponding Čech-de Rham complex by $H^*(\mathcal{X})$. The generalised Mayer-Vietoris sequence reduces to a short split exact sequence

$$0 \longrightarrow H^1(\mathcal{X}) \longrightarrow H^*(X(r)) \longrightarrow H^0(\mathcal{X}) \longrightarrow 0 \,, \tag{2}$$

which, by results similar to Atiyah-Singer-Patodi's [1], is isomorphic to an exact sequence related to a splitting of the space of matching sets:

$$0 \longrightarrow \mathcal{L}^{rel} \longrightarrow \mathcal{W} \longrightarrow \ker \mathcal{D}(\infty) \oplus \mathcal{L}^{abs} \longrightarrow 0 \,. \tag{3}$$

Here, $\mathcal{L}$ denotes the set of "limiting values" of matching sets and the superscripts *abs* and *rel* refer to absolute respectively relative boundary conditions at infinity.

Combining these approaches, we obtain equality of spaces $E(r) = \ker \mathcal{D}(r)$ and hence

Theorem 2 ([2], [3]). Let $\varepsilon > 0$ and r_0 suitable large. Then, for $r > r_0$

(1) There are no non-zero eigenvalues μ of $\mathcal{D}(r)$ satisfying $|\mu| < r^{-(1+\varepsilon)}$.
(2) The projected splicing map

$$\Pi_r \circ \mathcal{S}_r : \mathcal{W} \longrightarrow \ker \mathcal{D}(r)$$

is an isomorphism of matching sets of extended harmonic forms on the disjoint union of the $X_v(\infty)$'s and harmonic forms on $X(r)$.
(3) Any spliced form is exponentially close to a harmonic form, to be precise

$$\left\| \left(\Pi_r \circ \mathcal{S}_r - \mathcal{S}_r\right)(u) \right\|_{L^2} \leq e^{-cr} \left\| \mathcal{S}_r(u) \right\|_{L^2} \,,$$

where $c > 0$ depends on the geometry of the cross-sections Y_e only.

Remark. It is as well planned to relate exact sequences (2) and (3) respectively the corresponding splittings to the Hodge-Leray spectral sequence developed by Mazzeo-Melrose in [6]. This might enable us to identify subspaces of differential forms on $X(r)$ with the spaces $\mathcal{L}^{rel}$ and $\ker \mathcal{D}(r) \oplus \mathcal{L}^{abs}$, hence to understand the actual *behaviour* of $\mathcal{H}^*(r)$ as $r \to \infty$.

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Quantization on noncompact symplectic manifolds

Weiping Zhang
(joint work with Xiaonan Ma)

In this talk we report our joint work with Xiaonan Ma on the resolution of a conjecture due to Michèle Vergne [13] concerning the geometric quantization on noncompact symplectic manifolds.

To be more precise, let (M, ω) be a (not necessarily) compact symplectic manifold with symplectic form ω. We assume that (M, ω) is prequantizable, that is, there exists a complex line bundle L (called a prequantized line bundle) carrying a Hermitian metric h^L and a Hermitian connection ∇^L such that

$$\frac{\sqrt{-1}}{2\pi}\left(\nabla^L\right)^2 = \omega.$$

Let J be an almost complex structure on TM such that

$$g^{TM}(u, v) = \omega(u, Jv), \quad u,\ v \in TM$$

defines a J-invariant Riemannian metric on TM.

Let G be a compact connected Lie group with Lie algebra denoted by $\mathbf{g}$. We assume that the compact connected Lie group G acts on M and that this action lifts to an action on L. Moreover, we assume that G preserves g^{TM}, J, h^L and ∇^L.

For any $K \in \mathbf{g}$, let K^M be the vector field generated by K over M. Let $\mu : M \to \mathbf{g}^*$ be the moment map defined by the Kostant formula

$$2\pi\sqrt{-1}\mu(K) = \nabla^L_{K^M} - L_K,\ K \in \mathbf{g}.$$

Then μ verifies the Hamiltonian action condition that for any $K \in \mathbf{g}$,

$$d\mu(K) = i_{K^M}\omega.$$

From now on, we assume that the following fundamental assumption holds.

Fundamental Assumption. The moment map $\mu : M \to \mathbf{g}^*$ is proper, in the sense that the inverse image of a compact subset is compact.

Fix a maximal torus of G and let $\Lambda^*_+ \subset \mathbf{g}^*$ be the corresponding set of dominant weights of irreducible representations of G.

Take any $\gamma \in \Lambda^*_+$. If γ is a regular value of the moment map μ, then one can construct the Marsden-Weinstein symplectic reduction $(M_\gamma, \omega_\gamma)$, where $M_\gamma = \mu^{-1}(G\cdot\gamma)/G$ is a *compact* orbifold (since μ is proper). Moreover, the line bundle L (resp. the almost complex structure J) induces a prequantized line bundle L_γ (resp. an almost complex structure J_γ) over $(M_\gamma, \omega_\gamma)$. One can then construct the associated Spin^c-Dirac operator (twisted by L_γ), $D_+^{L_\gamma} : \Omega^{0,\mathrm{even}}(M_\gamma, L_\gamma) \to \Omega^{0,\mathrm{odd}}(M_\gamma, L_\gamma)$ on M_γ, of which the index

$$Q(L_\gamma) := \mathrm{Ind}\big(D_+^{L_\gamma}\big) := \dim \mathrm{Ker}\big(D_+^{L_\gamma}\big) - \dim \mathrm{Coker}\big(D_+^{L_\gamma}\big) \in \mathbb{Z},$$

is well-defined. If $\gamma \in \Lambda_+^*$ is not a regular value of μ, then by proceeding as in [8] (cf. [9, §7.4]), one still gets a well-defined quantization number $Q(L_\gamma)$ extending the above definition.

On the other hand, let $\mathbf{g}^*$ be equipped with an Ad_G-invariant metric. Then $\mathcal{H} = |\mu|^2$ is G-invariant. Let $X^{\mathcal{H}} = -J(d\mathcal{H})^*$ be the Hamiltonian vector field associated to $\mathcal{H}$.

Since μ is proper, for any $a > 0$, $M_a := \mathcal{H}^{-1}([0,a]) = \{x \in M : \mathcal{H}(x) \leq a\}$ is a compact subset of M. Recall that by Sard's theorem, the set of critical values of the function $\mathcal{H} : M \to \mathbb{R}$ is of measure zero.

For any regular value $a > 0$ of $\mathcal{H}$, it is clear that $X^{\mathcal{H}}$ is nowhere zero on $\partial M_a = \mathcal{H}^{-1}(a)$. Thus the triple $(M_a, X^{\mathcal{H}}, L)$ defines a transversally elliptic symbol

$$\sigma_{L,X^{\mathcal{H}}}^{M_a} = \sqrt{-1}\, c(\cdot + X^{\mathcal{H}}) \otimes \mathrm{Id}_L,$$

where $c(\cdot)$ is the Clifford action on $\Lambda(T^{*(0,1)}M)$, in the sense of Atiyah [1] and Paradan [9]. Let $\mathrm{Ind}(\sigma_{L,X^{\mathcal{H}}}^{M_a}) \in R[G]$ be the corresponding transversal index in the sense of [9].

For any $\gamma \in \Lambda_+^*$, let V_γ^G denote the corresponding irreducible representation of G, let $Q(L)_a^\gamma \in \mathbb{Z}$ denote the multiplicity of V_γ^G of $\mathrm{Ind}(\sigma_{L,X^{\mathcal{H}}}^{M_a}) \in R[G]$.

Theorem 1.

a) For any $\gamma \in \Lambda_+^*$, there exists $a_\gamma > 0$ such that $Q(L)_a^\gamma \in \mathbb{Z}$ does not depend on $a \geq a_\gamma$, with a a regular value of $\mathcal{H}$.

b) $Q(L)_a^{\gamma=0} \in \mathbb{Z}$ does not depend on $a > 0$, with a a regular value of $\mathcal{H}$.

According to Theorem 1, for any $\gamma \in \Lambda_+^*$, we have a well-defined integer $Q(L)_a^\gamma$ not depending on the large enough regular value $a > 0$. From now on we denote it by $Q(L)^\gamma$.

Theorem 2. For any $\gamma \in \Lambda_+^*$, the following identity holds,

$$Q(L)^\gamma = Q(L_\gamma).$$

Remark 3. If the zero set of $X^{\mathcal{H}}$ is compact, then Theorem 1 was already known, while Theorem 2 was conjectured by Michèle Vergne in [13, §4.3]. Thus Theorem 2 provides a solution of Vergne's conjecture even when the zero set of $X^{\mathcal{H}}$ is non-compact. If M is compact, then Theorem 2 reduces to the famous Guillemin-Sternberg geometric quantization conjecture [4] first proved in [7] and [8].

Outline of Proof. Our proof of Theorems 1 and 2 is analytic. We first interpret the transversal indices appearing in the context through the analytic indices of Atiyah-Patodi-Singer [2] type, by making use of a result of Braverman [3]. We then prove Theorems 1 and 2 by analyzing the corresponding APS type indices, by adapting the analytic methods developed in [11] and [12]. Extra difficulties appear in dealing with the $\gamma \neq 0$ case. For more details, see [5] and [6].

Remark 4. For an alternate proof of Theorems 1 and 2, see [10].

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