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Low-Dimensional Topology and Number Theory

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August 15th – August 21st, 2010

Abstract. The workshop on Low-Dimensional Topology and Number Theory brought together researchers in these areas with the intent of exploring the many tantalizing connections between Low-Dimensional Topology and Number Theory. Some of the most actively discussed topics were the appearances of modularity in quantum invariants and mutual relations between hyperbolic volume, K-theory, and asymptotics of quantum invariants.

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Introduction by the Organisers

The workshop *Low-Dimensional Topology and Number Theory*, organized by Paul E. Gunnells (Amherst), Walter Neumann (New York), Adam S. Sikora, (Buffalo), Don Zagier (Bonn/Paris) was held August 15th – August 21st, 2010. This meeting brought together 45 researchers from around the world, at different stages of their careers – from graduate students to some of the most established scientific leaders in their areas. The participants represented diverse backgrounds from low-dimensional topology, number theory, and quantum physics.

The workshop consisted of 28 talks ranging from 30 to 50 minutes intertwined with informal discussions. As suggested by its name, the workshop was devoted to the connections between Low-Dimensional Topology and Number Theory. Some of the most actively discussed topics are elaborated below.

0.1. **Arithmetic Topology.** Starting from the late 1960's, B. Mazur and others observed the existence of a curious analogy between knots and prime numbers and, more generally, between knots in 3-dimensional manifolds and prime ideals in algebraic number fields. For example, the spectrum of the ring of algebraic integers in any number field has étale cohomological dimension 3 (modulo higher 2-torsion). Moreover, the étale cohomology groups of such spectra satisfy Artin-Verdier duality, which is reminiscent of the Poincare duality satisfied by closed, oriented, 3-dimensional manifolds. Furthermore, there are numerous analogies between algebraic class field theory and the theory of abelian covers of 3-manifolds. As an example we mention that the universal abelian cover surprisingly yields complete intersections on both the number-theoretic (De Smit–Lenstra) and topological/geometric (Neumann–Wahl) sides. However, there is no good understanding of the underlying fundamental reasons for the above analogies, and of the full extent to which these analogies hold.

0.2. **Arithmetic of Hyperbolic Manifolds.** A hyperbolic 3-manifold M is an orientable 3-dimensional manifold that is locally modeled on hyperbolic three-space $\mathbf{H}^3$. Its fundamental group is a torsion-free discrete subgroup $\Gamma \subset PSL_2(\mathbf{C})$ defined uniquely (up to conjugation) by the requirement $M = \Gamma\backslash\mathbf{H}^3$. According to the geometrization program of Thurston, which has been the guiding force in 3-manifold topology research since the 1980's, hyperbolic 3-manifolds constitute the largest and least understood class of 3-manifolds. Such manifolds have been studied from a variety of perspectives, including geometric group theory, knot theory, analysis, and mathematical physics.

Recently, number theory has played a particularly crucial role in the study of hyperbolic 3-manifolds. The connection arises when one assumes that M has finite volume. In this case, Mostow rigidity asserts that the matrices representing $\Gamma \subset SL_2(\mathbf{C})$ can be taken to have entries in a number field. A consequence of this connection is that techniques from algebraic number theory now play a central role in the experimental investigation of hyperbolic 3-manifolds, to the extent that the premier software for performing computations with hyperbolic 3-manifolds (SNAP) incorporates the premier software for experimental algebraic number theory (GP-PARI)! Furthermore, via this connection low-dimensional topology and number theory inform each other in surprising ways. For example, volumes of hyperbolic manifolds can be expressed in terms of special values of Dedekind zeta functions (Borel).

0.3. **Connections with algebraic K-theory.** To a hyperbolic 3-manifolds one can associate a canonical element in algebraic K-theory, more specifically in $K_3(\bar{\mathbb{Q}})$. Thanks to work of Neumann–Yang and more recent work of Zickert, this element can be completely computed from any triangulation of the 3-manifold. Many invariants are related to, or even determined by, this class, the most prominent being the hyperbolic volume, which is given by the Bloch–Wigner dilogarithm function. This leads to a variety of connections with number theory. Some of these have to do with the asymptotics and modularity discussed below. Another connection is the complex of questions connected with the Mahler measure. This is an invariant of polynomials that appears in surprisingly diverse areas of mathematics, for example, as the entropy of certain dynamical systems and as the value of higher regulators (Beilinson, Boyd, Rodriguez Villegas), and the theory of heights and Lehmer's conjecture. But more surprisingly, they appear in the theory of hyperbolic 3-manifolds: the Mahler measures of polynomials appearing naturally in 3-dimensional topology (such as A-polynomials, Alexander polynomials, and Jones polynomials) give interesting examples from the perspective of arithmetic.

0.4. **Quantum topology, asymptotics, modular forms.** Further connections between quantum topology and number theory appear in the context of quantum invariants of 3-manifolds. These invariants are typically functions that are defined only at roots of unity, and one can ask whether they extend to holomorphic functions on the complex disk with interesting arithmetical properties. A first result in this direction was found by R. Lawrence and D. Zagier, who showed that for certain 3-manifolds (e.g. the Poincaré homology sphere) the quantum invariants at roots of unity are limiting values of Eichler integrals of certain modular forms. More recently Zagier has found experimental evidence of modularity properties of a new type for the Kashaev invariants of knots. The outstanding open problem concerning Kashaev invariants, the so-called Volume Conjecture (which relates the value at $e^{2\pi i/N}$ of the Nth Kashaev invariant of a knot to its hyperbolic volume), also has a beautiful arithmetic refinement in which the Kashaev invariants have asymptotic expansions around all roots of unity that have algebraic coefficients (in the trace field of the knot) and are related by modular transformations. Testing these conjectures is a challenging computational problem that now finally seems in reach.

0.5. **Multiple zeta values.** Although originally defined analytically, Kontsevich's universal finite type invariant of links can be calculated algebraically, by an application of the Drinfeld associator, a certain formal infinite power series on two non-commuting variables. It turns out that the coefficients of this series are given by special values of the Riemann zeta function and the multiple zeta functions of Euler and Zagier. By choosing different diagrams representing the same link L, one obtains different expressions for the Kontsevich invariant of L. Equating these expressions then yields non-trivial identities for the special values of these zeta functions. This topic was not lectured on at the meeting; nevertheless, it remains of considerable interest from the point of view of both number theory and topology.

Workshop: Low-Dimensional Topology and Number Theory

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Abstracts

Increasing the number of fibered faces of arithmetic hyperbolic 3-manifolds

Nathan M. Dunfield
(joint work with Dinakar Ramakrishnan)

Summary: We exhibit a closed hyperbolic 3-manifold which satisfies a very strong form of Thurston's Virtual Fibration Conjecture. In particular, this manifold has finite covers which fiber over the circle in arbitrarily many ways. More precisely, it has a tower of finite covers where the number of fibered faces of the Thurston norm ball goes to infinity, in fact faster than any power of the logarithm of the degree of the cover, and we give a more precise quantitative lower bound. The example manifold M is arithmetic, and the proof uses detailed number-theoretic information, at the level of the Hecke eigenvalues, to drive a geometric argument based on Fried's dynamical characterization of the fibered faces. The origin of the basic fibration $M \to S^1$ is the modular elliptic curve $E = X_0(49)$, which admits multiplication by the ring of integers of $\mathbb{Q}[\sqrt{-7}]$. We first base change the holomorphic differential on E to a cusp form on GL(2) over $K = \mathbb{Q}[\sqrt{-3}]$, and then transfer over to a quaternion algebra D/K ramified only at the primes above 7; the fundamental group of M is a quotient of the principal congruence subgroup of $\mathcal{O}_D^*$ of level 7. To analyze the topological properties of M, we use a new practical method for computing the Thurston norm, which is of independent interest.

Details: The most mysterious variant of the circle of questions surrounding Waldhausen's Virtual Haken Conjecture is:

0.1. **Virtual Fibration Conjecture** (Thurston). *If M is a finite-volume hyperbolic 3-manifold, then M has a finite cover which fibers over the circle, i.e. is a surface bundle over the circle.*

This is a very natural question, equivalent to asking whether $\pi_1(M)$ contains a *geometrically infinite* surface group. However, compared to the other forms of the Virtual Haken Conjecture, there are relatively few non-trivial examples where it is known to hold. Moreover, there are indications that fibering over the circle is, in suitable senses, a rare property compared, for example, to simply having non-trivial first cohomology.

Despite this, we show here that certain manifolds satisfy Conjecture 0.1 in a very strong way, in that they have finite covers which fiber over the circle in many distinct ways. For a 3-manifold M, the set of classes in $H^1(M;\mathbb{Z})$ which can be represented by fibrations over the circle are organized by the Thurston norm on $H^1(M;\mathbb{R})$. The unit ball in this norm is a finite polytope where certain top-dimensional faces, called the fibered faces, correspond to those cohomology classes coming from fibrations. The number of fibered faces thus measures the number of fundamentally different ways that M can fiber over the circle. We will sometimes abusively refer to these faces of the Thurston norm ball as "the fibered faces of M".

If $N \to M$ is a finite covering map, the induced map $H^1(M;\mathbb{R}) \to H^1(N;\mathbb{R})$ takes each fibered face of M to one in N; if $H^1(N;\mathbb{R})$ is strictly larger than $H^1(M;\mathbb{R})$, then it may (but need not) have additional fibered faces. The qualitative form of our main result is:

0.2. **Theorem.** *There exists a closed hyperbolic 3-manifold M which has a sequence of finite covers M_n so that the number of fibered faces of the Thurston norm ball of M_n goes to infinity.*

Moreover, we prove a quantitative refinement of this result (Theorem 0.4) which bounds from below the number of fibered faces of M_n in terms of the degree of the cover.

The example manifold M of Theorem 0.2 is arithmetic, and the proof uses detailed number-theoretic information about it, at the level of the Hecke eigenvalues, to drive a geometric argument based on Fried's dynamical characterization of the fibered faces. To state the geometric part of the theorem, we need to introduce the Hecke operators. Suppose M is a closed hyperbolic 3-manifold, and we have a pair of finite covering maps $p, q\colon N \to M$; when M is arithmetic there are many such pairs of covering maps because the commensurator of $\pi_1(M)$ in $\mathrm{Isom}(\mathcal{H}^3)$ is very large. The associated *Hecke operator* is the endomorphism of $H^1(M)$ defined by $T_{p,q} = q_* \circ p^*$, where $q_*\colon H^1(N) \to H^1(M)$ is the transfer map. The simplest form of our main geometric lemma is the following:

0.3. **Lemma.** *Let M be a closed hyperbolic 3-manifold, and $p, q\colon N \to M$ a pair of finite covering maps. If $T_{p,q}(\omega) = 0$ for some $\omega \in H^1(M;\mathbb{Z})$ coming from a fibration over the circle, then $p^*(\omega)$ and $q^*(\omega)$ lie in distinct fibered faces.*

We give a manifold with an infinite tower of covers to which Lemma 0.3 applies at each step, thus proving Theorem 0.2. In fact, we show the following refined quantitative version:

0.4. **Theorem.** *There is a closed hyperbolic 3-manifold M of arithmetic type, with an infinite family of finite covers $\{M_n\}$ of degree d_n, where the number ν_n of fibered faces of M_n satisfies*

$$\nu_n \geqslant \exp\left(0.3\frac{\log d_n}{\log\log d_n}\right) \quad \text{as } d_n \to \infty.$$

In particular, for any $t < 1$, there is a constant $c_t > 0$ such that

$$\nu_n \geqslant c_t e^{(\log d_n)^t}.$$

Note that this bound for ν_n is slower than any positive power of d_n, but is faster than any (positive) power of $\log d_n$. For context, the Betti number $b_1(M_n) = \dim H^1(M_n;\mathbb{R})$ is bounded above by (a constant times) the degree d_n, and bounded below, for any $\varepsilon > 0$, by (a constant times) $d_n^{1/2-\varepsilon}$ for n large enough (relative to ε), while ν_n is at least as large as (a constant times) $e^{(\log d_n)^{0.99}}$.

Generators for the S-integral points of algebraic groups

Ted Chinburg
(joint work with Matt Stover)

Let L be a number field and let M_L (resp. $M_{L,f}$) be the set of all (resp. all finite) places of L. Suppose S is a finite subset of $M_{L,f}$. The ring $O_{L,S}$ of S-integers of L is the ring of elements $x \in L$ such that $|x|_v \leqslant 1$ for all $v \in M_{L,f} - S$. Let G be a linear algebraic group over L. In this talk I discussed two results concerning generators for the S-integral points $G(O_{L,S})$ of G relative to an embedding of G into GL_m for some integer m. The first result concerns the existence of a set of generators of small height once S is sufficiently large. The second result has to do with generating $G(O_{L,S})$ when S is empty by Fuchsian subgroups.

Define a height function $H_L : L \to \mathbb{R}$ by $H(0) = 0$ and

$$H_L(x) = \prod_{v \in M_L} |x|_v$$

if $0 \neq x \in L$. We can extend this to a height function $H_G : G(L) \to \mathbb{R}$ by letting

$$H_G(g) = \max\{H_L(g_{i,j}) : 1 \leqslant i,j \leqslant m\}$$

if $g \in G(L)$ is sent to the matrix $(g_{i,j})_{i,j=1}^m \in GL_m(L)$ via the chosen embedding of G into GL_m. Define

$$m_S = \max(\{N(v) : v \in S\} \cup \{1\})$$

where $N(v)$ is the order of the residue field of v. Define d_L to be the discriminant of L.

Lenstra proved the following remarkable result in [2, Thm. 6.2] concerning the multiplicative group $G = \mathbb{G}_m = \mathrm{GL}_1$. Note that $\mathbb{G}_m(O_{L,S})$ is the group $O^*_{L,S}$ of S-units of L.

0.5. **Theorem.** (Lenstra) *Let $c = (2/\pi)^{r_2(L)}$ where $r_2(L)$ is the number of complex places of L. If S contains all $v \in M_{L,f}$ such that $N(v) \leqslant c\sqrt{|d_L|}$ then $\mathbb{G}_m(O_{L,S}) = O^*_{L,S}$ is generated by elements x for which $H_L(x) = H_{\mathbb{G}_m}(x) \leqslant c m_S \sqrt{|d_L|}$.*

For example, suppose $S = \{v \in M_{L,f} : N(v) \leqslant c\sqrt{|d_L|}\}$ is chosen to be the smallest set for which the hypotheses of the Theorem are satisfied. Then $O^*_{L,S}$ is generated by elements of height bounded by $c^2|d_L|$. (Exercise: Show $c\sqrt{|d_L|} \geqslant 1$.)

By contrast, when S is empty, one does not expect to always be able to generate $O^*_{L,\varnothing} = O^*_L$ by elements of height bounded by a fixed polynomial in $|d_L|$ as L ranges over the set of all fields of a given degree larger than 1. For example, one expects that the set T of real quadratic fields L having class number 1 is infinite. Suppose $L \in T$ and that $\epsilon_L > 1$ is a fundamental unit in O^*_L. The Brauer-Siegel theorem implies that for $\delta > 0$ there is a constant $c(\delta) > 0$ independent of L such that

$$H_L(\epsilon_L) = \epsilon_L > c(\delta)\ \exp(|d_L|^{1/2-\delta}).$$

Our first result generalizes Theorem 0.5 to groups G over L defined by the multiplicative group B^* of a quaternion division algebra B over L. Let D be a maximal O_L-order in B, and define $D_S = O_{L,S} \otimes_{O_L} D$. We are interested in generating the unit group D_S^* by elements of small height.

There is a quadratic extension F/L and an algebra isomorphism $F \otimes_L B = \mathrm{Mat}_2(F)$ which takes $1 \otimes D$ into $\mathrm{Mat}_2(O_F)$. The natural height to use in this context is $H_B : B^* \to \mathbb{R}$ defined by setting $H_B(0) = 0$ and by letting

$$H_B(g) = (\max\{H_F(g_{i,j}) : 1 \leqslant i, j \leqslant 2\})^{1/2}$$

if $g \in B^*$ is sent by the above algebra embedding to the matrix $(g_{i,j})_{i,j=1}^2 \in \mathrm{Mat}_2(F)$. The role of $|d_L|$ in Lenstra's theorem is played by

$$d_D = \mathrm{vol}(B_{\mathbb{R}}/D)$$

where D is embedded as a lattice in

$$B_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Q}} B = \prod_{v \in M_L - M_{L,f}} B_v$$

and the completion $B_v = \mathbb{Q}_v \otimes_{\mathbb{Q}} B$ of B at an archimedean place v is given the standard Haar measure defined in [3, §7.5].

0.6. **Theorem.** *There are constants $a, a' > 0$ which depend only the signature of L for which the following is true. If S contains all $v \in M_{L,f}$ such that $N(v) \leqslant a\sqrt{d_D}$ then D_S^* is generated by elements x for which $H_B(x) \leqslant a' m_S \sqrt{d_D}$.*

The problem of optimizing the constants a and a' in this Theorem raises interesting questions in the geometry of numbers.

The proof of Theorem 0.6 follows the method used by Lenstra in [2]. One defines an adelic group $\mathcal{G}$ containing D_S^* and uses Minkowski arguments to produce an explicit bounded fundamental domain $\mathcal{F}$ such that $\mathcal{G}$ is the product set $\mathcal{F} D_S^*$. From such an $\mathcal{F}$ and a well chosen set of pro-generators $\mathcal{P}$ for $\mathcal{G}$ one can produce a bounded set of generators for D_S^* (compare [2, Lemmas 6.3, 6.4]). In fact, can also find a presentation for D_S^* using these generators, $\mathcal{F}$ and some well-known results in combinatorial group theory (see [4, Appendix to Chapter 3]).

Our second result concerns the case in which S is empty, so that $D_{L,\varnothing}^* = D^*$.

0.7. **Theorem.** *Suppose L is a real quadratic field and that the quaternion division algebra B over L is isomorphic to $L \otimes_{\mathbb{Q}} A$ for a quaternion algebra A over $\mathbb{Q}$ which splits at infinity. Let B^1 be the multiplicative group of elements of B of reduced norm 1. Then for all maximal O_L-orders D of B, the group $D^* \cap B^1$ has a subgroup of finite index which is generated by two Fuchsian arithmetic subgroups of $D^* \cap B^1$.*

The proof of this result is effective. Since B splits over the two infinite places of L, there is a natural embedding $B^1 \to \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ which gives an action of B^1 on the product $\mathcal{H} \times \mathcal{H}$ of two copies of the upper half plane. The fact that B is isomorphic to $L \otimes_{\mathbb{Q}} A$ implies that such an isomorphism exists for infinitely many non-isomorphic quaternion algebras A over $\mathbb{Q}$. By a subgroup separability argument and the fact that the commensurator of D^* in B^* is known to be all of B^*, one can show that there is a subgroup Γ of finite index in $D^* \cap B^1$ as well as Fuchsian arithmetic subgroups $\Gamma_1, \Gamma_2 \subset D^* \cap B^1$ defined via quaternion subalgebras over $\mathbb{Q}$ for which the following is true. There are embeddings $\pi_i : \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ for $i = 1, 2$ such that $Y_i = \Gamma_i \backslash \pi_i(\mathcal{H})$ is a smooth compact curve inside the smooth compact complex surface

$$X = \Gamma \backslash (\mathcal{H} \times \mathcal{H})$$

and Y_1 intersects Y_2 at a unique point with transverse intersection. One then shows using intersection theory that $Y = Y_1 \cup Y_2$ has ample normal bundle in X and that Y is a retract of a complex open neighborhood U of Y in X. A result of Nori [1] then implies that the image of $\pi_1(Y) \to \pi_1(X)$ has finite index. This together with Van Kampen's theorem implies Theorem 0.7.

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The probability of presenting a finite group

Nigel Boston

The motivation for this work comes from three problems. In each case let G be a finite nontrivial group and g a positive integer.

Problem 1 is from number theory and asks whether there exists a G-extension of $\mathbf{Q}$ ramified at g primes (including the infinite prime) and if so, how many. Problem 2 is from topology and asks if for a random 3-manifold of Heegard genus g there exists a G-cover and how many. Problem 3 is from group theory and asks if a random g-generator g-relator group has a G-quotient and how many.

Problems 2 and 3 were answered by Dunfield and Thurston [4] when G is simple. We generalize their work to general G by using probabilistic zeta functions, we generalize problem 2 to a number-theoretical setting, and then we use that to get answers to problem 1.

A partial answer to Problem 1 is my conjecture with Markin [3] that says that a G-extension of $\mathbf{Q}$ ramified at g primes exists if and only if $g \geqslant d(G^{ab})$, where d denotes the minimal number of generators (taken to be at least 1). This is somewhat surprising for e.g. perfect groups but Jones and Roberts [5] has supporting data in that case. The general conjecture combines their conjecture with conjectures made elsewhere, e.g. in the p-group case, and also gives a conjectural positive density for such G-extensions.

A new approach to Problem 3 is given in recent joint work with Lucchini. To produce a measure we switch to profinite groups. Letting F_g denote the free profinite group on g generators and $X_g(G)$ denote the set of its closed normal subgroups with quotient G, we want the measure of the set $\Omega_{g,s}(G)$, consisting of ordered s-tuples of elements of F_g that present a group that has a G-quotient. This set is indeed measurable, with measure $1 - P(F_g, X_g(G), s)$, where $P(F_g, X_g(G), s) = \sum \mu(H)/[F_g : H]^s$, the sum being over the lattice of all intersections of groups in $X_g(G)$ with μ the Moebius function of that lattice.

In particular, if G is simple, nonabelian, letting $n = |X_g(G)|$, this yields

$$\sum_0^n \binom{n}{i}(-1)^i/|G|^{is} = (1 - |G|^{-s})^n$$

as shown by Dunfield and Thurston [4].

As another example, if G is cyclic of prime order p and $g = 2$, the probability that the randomly s-related group has a G-quotient is $(1 - 1/p^s)(1 - p/p^s)$. This implies that the probability that the group is perfect is $1/(\zeta(s)\zeta(s-1))$.

Most work has been for the case of p-groups. For example, in [1], it is shown that if $d(G) = g$ and $r(G) = r$, then the probability that a g-generator r-relator presentation gives G is $\phi_p(g)\phi_p(r)p^{gr-g(g+1)/2-r(r+1)/2}|G|^{g-r}/|\mathrm{Aut}(G)|$, where $\phi_p(n) = (p^n - 1)(p^{n-1} - 1)...(p - 1)$. This can be used to show that with probability 21/64 a 3-generator 1-relator pro-2 group is isomorphic to $< x, y, z | x^y = x^3 z^2 >$.

If S is a finite set of g primes, all $1 \pmod p$, then G_S, the Galois group over $\mathbf{Q}$ of the maximal p-extension unramified outside S, is a g-generator g-relator pro-p group. For simplicity suppose p is odd. With Ellenberg [2] I conjecture the probability that, as S varies, G_S is isomorphic to a given g-generator g-relator p-group G. Among p-groups with the same $G^{ab} \cong \mathbf{Z}/p^{n_1} \times ... \times \mathbf{Z}/p^{n_g}$, this should be $A(G)/|\mathrm{Aut}(G)|$, where $A(G) = |\{(c_1, ..., c_g) \in k(G)^g | < c_1, ..., c_g >= G, c_i^{p^{n_i}+1} = c_i\}|$.

This has been proven in many cases with plenty of computational evidence besides. For each G that it is proven, we obtain a positive density result as regards Problem 1, for all rank-preserving quotients of G.

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Volume bounds for generalized twisted torus links

Abhijit Champanerkar

(joint work with David Futer, Ilya Kofman, Walter Neumann, Jessica Purcell)

Recently, there has been interest in relating the volume of a hyperbolic knot and link to other link properties. Lackenby related volumes to number of twist regions for alternating links [6], and this relationship was extended to larger classes of links that satisfy a certain threshhold of complexity, such as a high amount of symmetry or twisting [4]. To better understand volumes in general, it seems natural to investigate properties of knots and links that are "simple".

Twisted torus knots and links are obtained by twisting a subset of strands of a closed torus braid. These knots are geometrically simple by several different measures of geometric complexity. Dean [3] showed that they often admit small Seifert fibered and Lens space surgeries. It was discovered in [1, 2] that twisted torus knots dominate the census of "simplest hyperbolic knots," those whose complements can be triangulated with seven or fewer ideal tetrahedra.

In this paper we investigate the geometry of twisted torus links and closely related generalizations. We determine upper bounds on their volumes in terms of their description parameters. In particular,

0.8. **Theorem.** *Let $T(p,q,r,s)$ be a twisted torus link. Then*

$$\begin{aligned} \mathrm{Vol}(T(p,q,r,s)) &\leqslant 10v_3 && \textit{if } r=2,\\ \mathrm{Vol}(T(p,q,r,s)) &\leqslant 2v_3(r+5) && \textit{if } s \bmod r = 0 \;\&\; r\geqslant 3\\ \mathrm{Vol}(T(p,q,r,s)) &\leqslant 4v_3(r^2-2r+4) && \textit{if } s \bmod r \neq 0 \;\&\; r\geqslant 3. \end{aligned}$$

For twisting on two strands we exhibit families where our bounds are sharp.

0.9. **Theorem.** *Choose any sequence $(p_N, q_N) \to (\infty,\infty)$, such that* $\gcd(p_N,q_N) = 1$*. Then the twisted torus knots $T(p_N,q_N,2,2N)$ have volume approaching $10v_3$ as $N\to\infty$.*

We also exhibit a family of twisted torus knots with volumes approaching infinity.

0.10. **Theorem.** *For any number V, there exists a twisted torus knot whose complement has volume at least V.*

The noteworthy feature of Theorem 0.8 is that the upper bound only depends on the parameter r.

0.11. **Corollary.** *The twisted torus knots $T(p,q,2,s)$, $p,q>2$, $s>0$, have arbitrarily large braid index and volume bounded by $10v_3$.*

The reverse of this Corollary is also true: for example, closed 3–braids can have unbounded hyperbolic volume. See [5]. One consequence of this is that there is no direct relationship between the braid index and volume of a link.

We also prove similar volume bounds for generalizations of twisted torus links.

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Zeros of Twisted Alexander Polynomials

Daniel S. Silver

(joint work with Susan G. Williams)

The classical Alexander polynomial $\Delta(t)$ was the result of J.W. Alexander's efforts in the 1920's to compute the order of the torsion subgroup of H_1M_r, where M_r is the r-fold cyclic cover of a knot k.

In 1990, X.S. Lin [4] defined a twisted generalization of $\Delta(t)$ by incorporating information from representations ρ of the knot group G. Improvements were subsequently given by M. Wada [7], P. Kirk and C. Livingston [3] and others.

The commutator subgroup K of G admits many more representations, most of which do not extend over G. We show how periodic representations of K can be used to extend Lin's invariant.

Alexander recognized that $\Delta(t)$ is essentially a group invariant. Wada's invariants also are group invariants. We formulate our results for general finitely presented groups.

Let G be a finitely presented group with an epimorphism $\epsilon : G \to \mathbb{Z} = \langle t \rangle$. The kernel K has a presentation of the form

$$\langle a_n u, b_\nu, \ldots, c_\nu \mid r_\nu, \ldots, s_\nu \rangle,$$

where ν ranges over the integers. The words $r_{\nu+j}, \ldots, s_{\nu+j}$ are obtained from $r_\nu, \ldots, s_\nu$, respectively, by adding j to the subscripts of all generators that occur.

The abelianization K^{ab} is a finitely generated module over $\mathbb{Z}[t^{\pm 1}]$ with $t \cdot u_\nu = u_{\nu+1}$, for all generators u_ν and extending linearly. The Alexander polynomial $\Delta(t)$ is the module order; that is, the greatest common divisor of the maximal minors of a presentation matrix for the module. It is well defined up to multiplication by $\pm t^i$, and it depends on (G, ϵ) only up to an obvious equivalence relation.

Given G and ϵ, choose $x \in G$ such that $\epsilon(x) = t$. We consider representations $\rho : K \to S_N$, where S_N is the symmetric group on $\{1, \ldots, N\}$. Then K acts on $\{1, \ldots, N\}$ on the right via ρ. We assume that the action is transitive.

We define a self-mapping σ of $\mathrm{Hom}(K, S_N)$ by

$$(\sigma\rho)(u) = \rho(xux^{-1}) \ \forall u \in K.$$

A representation ρ is *periodic* (with *period* r) if $\sigma^r \rho = \rho$.

0.12. **Remark.** *The pair (*$\mathrm{Hom}(K, S_N), \rho$*) has the structure of a shift of finite type, a dynamical system with excellent properties. Representations correspond to bi-infinite paths in a single finite directed graph. Periodic representations correspond to cycles. See* [6] *for details.*

0.13. **Definition.** *The Crowell group K_ρ of K with respect to ρ has generator set $\{1, \ldots, N\} \times K$, which we will write as $\{{}^i u \mid i \in [1, N], u \in K\}$, and relations ${}^i(uv) = ({}^i u)({}^{i\rho(u)}v)$, for all $u, v \in K, 1 \leqslant i \leqslant N$.*

It is easily seen that K_ρ has presentation

$$\langle {}^i a_\nu, {}^i b_\nu, \ldots, {}^i c_\nu \mid {}^i r_\nu, \ldots, {}^i s_\nu \rangle.$$

Here i ranges over $\{1, \ldots, N\}$, while ν ranges over $\mathbb{Z}$. Each relation can be expressed in terms of generators using ${}^i(uv) = ({}^i u)({}^{iu}v)$.

Henceforth we will assume that ρ is periodic with period r. Then K_ρ^{ab} is a finitely generated module over $\mathbb{Z}[s^{\pm 1}]$, where $s = t^r$, and

$$s \cdot {}^i u_\nu = {}^i u_{\nu+r},$$

for all ${}^i u_\nu \in K_\rho$.

0.14. **Definition.** *The Alexander-Lin twisted polynomial $D_{\rho,r}(s)$ is the module order of K_ρ^{ab}.*

The polynomial $D_{\rho,r}(s)$ is well defined up to multiplication by $\pm s^i$. It is an invariant of G, ϵ, x, ρ up to a suitably defined equivalence.

If $f(t) = c_0 \prod(t - \zeta_j)$ is a polynomial with complex coefficients, we define $f^{(r)}(s) = c_0^r \prod(s - \zeta_j^r)$.

0.15. **Proposition.**
- *If ρ is trivial, then $D_{\rho,1}(s) = \Delta(s)$;*
- *$\Delta^{(r)}(s)$ is a factor of $D_{\rho,r}(s)$;*
- *If ρ extends over G, then $D_{\rho,r}(s) = (s-1)^{N-1}\Delta_\rho^{(r)}(s)$, where $\Delta_\rho(t)$ is the ρ-twisted Alexander polynomial defined in* [3].

It is not difficult to show that K_ρ is finitely generated if and only if K is. In this case, $D_{\rho,r}(s)$ is monic with degree equal to $N \dim K_\rho^{\mathrm{ab}} \otimes Q$.

0.16. **Theorem.** *If K is finitely generated, then*

$$\max\{|z| \mid D_{\rho,r}(z) = 0\}^{1/r} \leqslant \mathrm{GR}(\mu),$$

where $\mu : K \to K$ is the automorphism $\mu(u) = xux^{-1}$ and $\mathrm{GR}(\mu)$ is the exponential growth rate of μ (see [1]*).*

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Higher Mahler measure as a Massey product in Deligne Cohomology

Matilde N. Lalín

0.17. **Definition.** *Given a non-zero polynomial $P(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$ and a positive integer k, the k-higher Mahler measure of P is defined by*

$$\mathrm{m}_k(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log^k |P(x_1 \dots, x_n)| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n},$$

or, equivalently, by

$$\mathrm{m}_k(P) := \int_0^1 \dots \int_0^1 \log^k |P(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})| d\theta_1 \dots d\theta_n.$$

In particular, notice that for $k = 1$ we obtain the "classical" (logarithmic) Mahler measure $\mathrm{m}_1(P) = \mathrm{m}(P)$ and $\mathrm{m}_0(P) = 1$.

The simplest example of higher Mahler measure is with the polynomial $P = 1 - x$. In that case, it was computed that $\mathrm{m}_2(1-x) = \frac{\pi^2}{12}$, $\mathrm{m}_3(1-x) = -\frac{3\zeta(3)}{2}$, etc (a general formula for $\mathrm{m}_k(x-1)$ is known [3]). Observe that the Mahler measure of cyclotomic polynomials is zero while these examples show that the situation is different for higher Mahler measures. In fact, if $\phi_n(x)$ denotes the n-th cyclotomic polynomial and φ is the Euler's function, then

$$\mathrm{m}_2(\phi_n(x)) = \frac{\pi^2}{12} \frac{\varphi(n) 2^{r(n)}}{n} = \frac{\pi^2}{12} \prod_{\substack{p|n \\ p \text{ prime}}} 2(2 - p^{-1}),$$

where $r(n)$ denotes the number of distinct prime divisors of n. Thus, m_2 for cyclotomic polynomials consists of rational multiples of $\frac{\pi^2}{12}$. Moreover, it is always greater or equal than this number. More can be said in this respect. Recall that Lehmer [5] asked the following question: *Given $\epsilon > 0$, can we find a polynomial $P(x) \in \mathbb{Z}[x]$ such that $0 < \mathrm{m}(P) < \epsilon$?*

This question is still open. The smallest known measure greater than 0 is that of a polynomial that he found in his 1933 paper:

$$\mathrm{m}(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) \cong 0.1623576120\ldots.$$

We have found a negative answer for Lehmer's question in the case of even higher measures.

0.18. **Theorem.** [4] *If $P(x) \in \mathbb{Z}[x]$, different from a monomial. Then for any $h \geqslant 1$,*

$$\mathrm{m}_{2h}(P) \geqslant \begin{cases} \left(\frac{\pi^2}{12}\right)^h, & \text{if } P(x) \text{ is reciprocal,} \\ \left(\frac{\pi^2}{48}\right)^h, & \text{if } P(x) \text{ is non-reciprocal.} \end{cases}$$

For odd degree, consider the following family of examples.

$$\mathrm{m}_3\left(\frac{x^n-1}{x-1}\right) = \frac{3}{2}\zeta(3)\left(\frac{-2+3n-n^3}{n^2}\right) + \frac{3\pi}{2}\sum_{\substack{j=1 \\ n\nmid j}}^{\infty} \frac{\cot\left(\pi\frac{j}{n}\right)}{j^2}.$$

In particular,

$$\mathrm{m}_3\left(x^2+x+1\right) = -\frac{10}{3}\zeta(3) + \frac{\sqrt{3}\pi}{2}L(2,\chi_{-3}),$$

$$\mathrm{m}_3\left(x^3+x^2+x+1\right) = -\frac{81}{16}\zeta(3) + \frac{3\pi}{2}L(2,\chi_{-4}).$$

These examples show that the nature of values of m_3 of cyclotomic polynomials is very different from the values of m_2 of cyclotomic polynomials.

We can use the above to give an idea of Lehmer's question for odd higher measure.

0.19. **Theorem.** *Let $P_n(x) = \frac{x^n-1}{x-1}$. For $h \geqslant 0$ fixed,*

$$\lim_{n\to\infty} \mathrm{m}_{2h+1}(P_n) = 0.$$

For $h \geqslant 1$, the sequence $\mathrm{m}_{2h+1}(P_n)$ is nonconstant and we obtain in this way a positive answer for Lehmer's question for m_{2h+1}.

Several other examples have been computed in single and multi-variable polynomials in [3], [4].

The relation with Massey product of Deligne Cohomology. We would like to interpret higher Mahler measures in terms of evaluations of regulators.

Deninger [2] interpreted the Mahler measure as a Deligne period of a mixed motive. Explicitly, one can write (under certain circumstances)

$$\mathrm{m}(P) = -\frac{1}{(2\pi i)^{n-1}}\int_A \eta(x_1,\ldots,x_n),$$

where

$$A = \{P(x_1,\ldots,x_n)=0\} \cup \{|x_1| = \cdots = |x_{n-1}| = 1, |x_n| \geqslant 1\},$$

and $\eta(x_1,\dots,x_n)$ is certain $\mathbb{R}(n-1)$-form in the variety Z defined by $\{P(x_1,\dots,x_n)=0\}$. In cohomological language,

$$\mathrm{m}(P) = -\left\langle r_{\mathcal{D}}\{x_1,\dots,x_n\}, [A]\otimes(2\pi i)^{1-n}\right\rangle$$

where $r_{\mathcal{D}}$ is a regulator from the motivic cohomology to the Deligne cohomology, so that

$$r_{\mathcal{D}}\{x_1,\dots,x_n\} = \log|x_1|\cup\cdots\cup\log|x_n|$$

is a cup product of n classes in the Deligne cohomology $H^1_{\mathcal{D}}(Z,\mathbb{R}(1))$ and the pairing is the natural one (Poincaré Duality). In this way, many special values of zeta and L functions that appear in computations of Mahler measures can be interpreted as special values of regulators.

Deninger suggested that we consider the analogous problem of interpreting the higher Mahler measure in terms of a Massey product in Deligne cohomology instead of the cup product.

Let X be a smooth variety over $\mathbb{C}$. We have the following

$$H^1_{\mathcal{D}}(X,\mathbb{R}(1)) = \{\varphi\in C^\infty(X)\,|\,d\varphi=\Re(\omega),\omega\in\Omega_{\log}(X)\}.$$

There is a similar concrete description for $H^j_{\mathcal{D}}(X,\mathbb{R}(j))$.

The cup product

$$\cup : H^i_{\mathcal{D}}(X,\mathbb{R}(i))\times H^j_{\mathcal{D}}(X,\mathbb{R}(j))\to H^{i+j}_{\mathcal{D}}(X,\mathbb{R}(i+j))$$

can be described explicitly in the same language.

Deninger [1] found an analogous description for the triple Massey product in Deligne cohomology and Wenger [6] showed a similar formula for the quadruple Massey product $\langle\varphi_1,\varphi_2,\varphi_3,\varphi_4\rangle$.

Consider the integral

$$\mathrm{m}_3(1-x) = \frac{1}{2\pi i}\int_{\mathbb{T}^1}\log^3|1-x|\frac{dx}{x} = -\frac{3\zeta(3)}{2}.$$

Let us take $\varphi_1=\varphi_2=\varphi_4=\log|1-x|$ and $\varphi_3=\log|x|$. All of these are elements in the Deligne cohomology. Thus

$$\langle\varphi_1,\varphi_2,\varphi_3,\varphi_4\rangle = -3diD_4(1-x)+\frac{i}{2}d(D_2(x)\log^2|1-x|),$$

where

$$D_k(x) := \pi_{k-1}\left(\sum_{h=1}^k\frac{(-\log|x|)^{k-h}}{(k-h)!}\mathrm{Li}_h(x)\right)$$

corresponds to a modified version of the polylogarithm.

Notice that the integral of $\langle\varphi_1,\varphi_2,\varphi_3,\varphi_4\rangle$ in the unit circle is zero. On the other hand, we also have

$$\langle\varphi_1,\varphi_2,\varphi_3,\varphi_4\rangle = -3diD_4(1-x)+\frac{1}{2}\log^2|1-x|\log|x|di\arg(1-x)$$
$$-\frac{1}{2}\log^3|1-x|di\arg x + iD_2(x)\log|1-x|d\log|1-x|.$$

Thus, the term that we wish to integrate appears as a piece of the Massey product. After a somewhat lengthly computation, we obtain

$$\frac{1}{2\pi i}\int_{\mathbb{T}^1}\log^3|1-x|\frac{dx}{x}=\frac{1}{\pi}\int_{\mathbb{T}^1}D_2(x)\log|1-x|d\log|1-x|=-\frac{3\zeta(3)}{2}.$$

It remains to understand this computation in the context of Deligne cohomology.

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The classical and quantum geometry of the $(-2, 3, n)$ pretzel knots

Stavros Garoufalidis

The talk will discuss the classical and quantum $\mathfrak{sl}_2$ character variety of the 1-parameter family of the $K_n = (-2, 3, 3 + 2n)$ pretzel knots for integer values of n. This family was chosen because the pairs $(K_n, -K_{-n})$ (where $-K$ denotes the mirror of K) are *geometrically similar*. The latter means that they are scissors congruent, and in particular they have equal volume. When $n = 0, 1, -1$, the knot K_n is a torus knot, thus not-hyperbolic. When $n = 2$, $K_{-2} = 5_2$ and $K_2 = (-2, 3, 7)$ pretzel knot.

The talk focuses on the following aspects:

(a) Computation of the A-polynomial of K_n.
(b) Computation of the colored Jones polynomial of K_n.
(c) Computation of the recursion relation for the colored Jones polynomial of K_n.
(d) Computation of the Kashaev invariant of K_n.

For part (a), in joint work with T. Mattman [GM] we compute the A-polynomial $A_n(M, L)$ of K_n by giving an explicit linear fourth order linear recursion relation with coefficients in $\mathbb{Z}[M, L]$. We also show that $A_n(M, L)$ and $A_{-n}(M, L^{-1})$ are related by an explicit $\mathrm{GL}(2, \mathbb{Z})$ transformation $\gamma_0^n\gamma_1$ for all n, where $\gamma_0, \gamma_1 \in \mathrm{GL}(2, \mathbb{Z})$, answering a question raised by Mattman and the author. The proofs of [GM] use three ingredients:

- A computation of the Newton polygon of $A_n(M, L)$ using the exceptional Dehn fillings of K_n and Culler-Shalen theory, done in [Ma].

- The fact that $A_n(M, L)$ divides a polynomial $P_n(M, L)$ which satisfies a constant coefficient linear fourth order recursion relation, done in [TY].
- The fact that the Newton polygons of $P_n(M, L)$ and $A_n(M, L)$ coincide, done in [GM].

For part (b), in [Ga2] the author uses fusion, and the identification of K_n with some 2-fusion knots to give an explicit double sum formula for the colored Jones polynomial $J_{n,N}(q)$ of K_n colored by the N-dimensional irreducible representation of $\mathfrak{sl}_2$.

For part (c), in [GK] Koutschan and the author use a table of values of $J_{n,N}(q)$, linear algebra, and a modular arithmetic computation to *guess* an explicit formula for the recursion relation of $J_{n,N}(q)$ with respect to N when $-4 \leqslant n \leqslant 4$. This recursion relation is the so-called *non-commutative A-polynomial* of A_n; see [Ga2]. Although our formulas for the non-commutative A-polynomial are experimental, they confirm the AJ-Conjecture. Moreover, for $n = -3, 3$ they provide examples of knots with reducible A-polynomial which confirms the AJ-Conjecture and uses all components of the A-polynomial. Needless to say, the non-commutative A-polynomial of K_4 is huge in size.

For part (d), D. Zagier and the author observed that once the non-commutative A-polynomial of a knot is known, then one can compute the N-th Kashaev invariant of the knot in $O(N)$ steps. This requires a finite number of differentiations of the recursion relation. This simple principle is applied to compute and compare the asymptotics of the Kashaev invariant of $-K_2$ and K_2. In joint work with D. Zagier [GZ], the N-th Kashaev invariants of $-K_{-2}$ and K_2 were computed to high accuracy for $N = 3000$ (and for $N = 1000$ computed exactly) and it was found that the asymptotics of the Kashaev invariant of K_2 are completely independent from those of $-K_{-2}$ answering in negative a short exact sequence dream of some authors.

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The extended Bloch group and representations of 3-manifold groups

Christian Zickert

Summary of results

This report summarizes joint work in progress with Stavros Garoufalidis and Dylan Thurston on representations of 3-manifold groups into $\mathrm{SL}(n,\mathbb{C})$. We show that such representations can be determined explicitly using a parametrization inspired by coordinates on higher Teichmüller spaces due to Fock and Goncharov [2]. We here restrict our attention to the parabolic representations, i.e. the representations taking peripheral subgroups to a unipotent subgroup. The parabolic representations are more computationally tractable, and have a well defined Chern-Simons invariant.

The extended Bloch group $\widehat{\mathcal{B}}(\mathbb{C})$ is a $\mathbb{Q}/\mathbb{Z}$-extension of the classical Bloch group. It was introduced by Walter Neumann [5], who used it to give an explicit formula for the universal Cheeger-Chern-Simons class $\hat{c}_2\colon H_3(\mathrm{PSL}(2,\mathbb{C}))\to\mathbb{C}/\pi^2\mathbb{Z}$. There are two variants of the extended Bloch group. One is isomorphic to $H_3(\mathrm{SL}(2,\mathbb{C}))$ and the other is isomorphic to $H_3(\mathrm{PSL}(2,\mathbb{C}))$. We denote them by $\widehat{\mathcal{B}}(\mathbb{C})_{\mathrm{SL}}$ and $\widehat{\mathcal{B}}(\mathbb{C})_{\mathrm{PSL}}$, respectively. There are regulator maps

$$R\colon \widehat{\mathcal{B}}(\mathbb{C})_{\mathrm{SL}}\to\mathbb{C}/4\pi^2\mathbb{Z},\quad R\colon \widehat{\mathcal{B}}(\mathbb{C})_{\mathrm{PSL}}\to\mathbb{C}/\pi^2\mathbb{Z}.$$

These are defined using a Rogers dilogarithm [5, 3], and agree with the Cheeger-Chern-Simons class under the identifications with $H_3(\mathrm{SL}(2,\mathbb{C}))$ and $H_3(\mathrm{PSL}(2,\mathbb{C}))$.

In the following, M denotes the interior of a compact 3-manifold.

0.20. **Theorem.** *A parabolic representation $\rho\colon\pi_1(M)\to\mathrm{SL}(n,\mathbb{C})$ determines an explicitly computable fundamental class $[\rho]$ in $\widehat{\mathcal{B}}(\mathbb{C})_{\mathrm{SL}}$, such that*

$$R([\rho]) = i(\mathrm{Vol}(\rho)+i\,\mathrm{CS}(\rho))\in\mathbb{C}/4\pi^2\mathbb{Z}.$$

0.21. **Remark.** *If F is a number field, a parabolic representation $\rho\colon\pi_1(M)\to\mathrm{SL}(n,F)$ has a fundamental class in the algebraic version of the extended Bloch group defined in* [1]. *This group is shown in* [1] *to be isomorphic to the algebraic K-group $K_3^{\mathrm{ind}}(F)$.*

The existence of parabolic representations in $\mathrm{SL}(2,\mathbb{C})$ is rare even for hyperbolic manifolds. Although the geometric representation always lifts, no lift is parabolic (any lift takes the longitude to a matrix with trace -2). The simplest knot complement admitting parabolic $\mathrm{SL}(2,\mathbb{C})$-representations is the 9_{34} knot complement. For n even, we therefore instead consider representations into the group $\mathrm{SL}(n,\mathbb{C})/\pm I$.

0.22. **Theorem.** *If n is even, a parabolic representation $\rho\colon\pi_1(M)\to\mathrm{SL}(n,\mathbb{C})/\pm I$ determines an explicitly computable fundamental class $[\rho]$ in $\widehat{\mathcal{B}}(\mathbb{C})_{\mathrm{PSL}}$, such that*

$$R([\rho]) = i(\mathrm{Vol}(\rho)+i\,\mathrm{CS}(\rho)\in\mathbb{C}/\pi^2\mathbb{Z}.$$

0.23. **Remark.** *Theorems 0.20 and 0.22 were proved for $n=2$ in* [6].

Let ϕ_n denote the unique irreducible representation $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(n,\mathbb{C})$. Note that $\phi_n\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(n,\mathbb{C})$ descends to a representation
$\mathrm{PSL}(2,\mathbb{C}) \to \mathrm{SL}(n,\mathbb{C})/(-1)^{n+1}I$.

0.24. **Theorem.** *Let M be a complete, oriented, hyperbolic 3-manifold of finite volume, and let $\rho = \phi_n \circ \rho_{geo}$, where ρ_{geo} is the geometric representation. We have $[\rho] = \binom{n+1}{3}[\rho_{geo}]$. In particular,* $\mathrm{Vol}(\rho) = \binom{n+1}{3}\mathrm{Vol}(M)$.

0.25. **Remark.** *The volume is defined for any representation, but only the parabolic representations have a Chern-Simons invariant. Also, only the parabolic representations define elements in $\widehat{\mathcal{B}}(\mathbb{C})$.*

0.26. **Conjecture.** *Let M be a complete, oriented, hyperbolic 3-manifold of finite volume and let $\rho\colon \pi_1(M) \to \mathrm{SL}(n,\mathbb{C})/(-1)^{n+1}I$ be a representation. We have*

$$\mathrm{Vol}(\rho) \leqslant \binom{n+1}{3}\mathrm{Vol}(M),$$

with equality if and only if $\rho = \phi_n \circ \rho_{geo}$.

Computer implementation

Our formula for the volume and Chern-Simons invariant has been implemented by Matthias Goerner, a graduate student at Berkeley. His program uses the computer algebra system MAGMA [4] to determine all parabolic representations. When there are finitely many representations (which is usually the case), the program computes for each representation the element in the extended Bloch group, and uses the regulator to compute the volume and Chern-Simons invariant.

Exotic representations.

0.27. **Definition.** *Let M be a hyperbolic 3-manifold. A representation $\pi_1(M) \to \mathrm{SL}(n,\mathbb{C})/(-1)^{n+1}I$ is* exotic *if it is not Galois conjugate to $\phi_n \circ \rho_{geo}$. The volume of an exotic representation (if non-zero) is an* exotic volume.

Examples. Using Matthias' program, we have found an abundance of exotic volumes that are equal to the volume of other manifolds. We give a few examples: The 7_7 knot complement has an exotic $\mathrm{PSL}(2,\mathbb{C})$-volume equal to the volume of the 5_2 knot complement (this phenomenon seems to be very common); The 7_2 knot complement has lots of exotic $\mathrm{SL}(3,\mathbb{C})$-volumes. Among them occur the (distinct) volumes of the following manifolds: $m120$, $s462$, the 11_{162} knot complement, and twice the volume of $m220$; The figure 8 knot has (distinct) exotic $\mathrm{SL}(4,\mathbb{C})/\pm I$-volumes equal to the volume of $m261$, $m045$, and twice the volume of $m234$.

0.28. **Conjecture.** *Every exotic volume is an integral linear combination of (Galois conjugate) volumes of hyperbolic manifolds.*

This conjecture would follow from a conjecture by Walter Neumann stating that all elements in the Bloch group are integral linear combinations of Bloch elements of hyperbolic manifolds.

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On mapping class group representations coming from Integral TQFT

Gregor Masbaum

The Witten-Reshetikhin-Turaev TQFT-invariants of 3-manifolds give rise to finite-dimensional representations of mapping class groups of surfaces. Our joint work with P. Gilmer shows that (at least) for the SO(3)-theory at an odd prime, these representations preserve a natural lattice defined over a ring of cyclotomic integers. We call this an Integral TQFT and my talk discussed some properties and applications of it. For example, it allows one to approximate the mapping class group representation at a fixed prime by representations into finite groups. More about this can be found in the extended abstract [1] of my talk at the previous Oberwolfach meeting *Invariants in Low-dimensional Topology* (May 4-10, 2008) (see also the papers and preprints on my home page `www.math.jussieu.fr/∼masbaum`).

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The universal deformation and the associated homological invariants for hyperbolic knots

Masanori Morishita

(joint work with Yuji Terashima)

0. Introduction. This is the joint work with Yuji Terashima about deformation theory for knot group representations and its applications. The motivation and ideas are coming from the analogy with Galois deformation theory due to B. Mazur ([Ma]). So, before describing our results, let us recall some basic analogies between knot theory and number theory ([Mo]):

Knot $K : S^1 = K(\mathbb{Z}, 1) \hookrightarrow \mathbb{R}^3$	Prime $\mathrm{Spec}(\mathbb{F}_p) \hookrightarrow \mathrm{Spec}(\mathbb{Z})$
Tubular n.b.d V_K $\partial V_K = V_K \backslash K$	p-adic integers $\mathrm{Spec}(\mathbb{Z}_p)$ $\mathrm{Spec}(\mathbb{Q}_p) = \mathrm{Spec}(\mathbb{Z}_p) \backslash \mathrm{Spec}(\mathbb{F}_p)$
Knot group $G_K = \pi_1(X_K),\ X_K = \mathbb{R}^3 \backslash \mathrm{int}(V_K)$	Prime group $G_p = \pi_1^{\text{ét}}(X_p),\ X_p = \mathrm{Spec}(\mathbb{Z}[1/p])$
Peripheral group $D_K = \mathrm{Im}(\pi_1(\partial V_K) \to G_K)$	Decomposition group $D_p = \mathrm{Im}(\pi_1^{\text{ét}}(\mathrm{Spec}(\mathbb{Q}_p)) \to G_p)$
$G_K^{\mathrm{ab}} = \mathrm{Gal}(X_K^\infty / X_K)$ X_K^∞ = infinite cyclic cover of X_K	$G_p^{\mathrm{ab}} = \mathrm{Gal}(X_p^\infty / X_p)$ X_p^∞ = cyclotomic p-cover of X_p $= \mathrm{Spec}(\mathbb{Z}[\sqrt[p^\infty]{1}, 1/p])$
Alexander polynomial $\Delta(H_1(X_K, \chi_K))$ χ_K = 1-dim. tautological repr. of G_K	Iwasawa polynomial $\Delta(H_1((X_p)_{\text{ét}}, \chi_p))$ χ_p = 1-dim. tautological repr. of G_p

Based on these analogies, there are close parallels between Alexander-Fox theory and Iwasawa theory. From the viewpoint of group representations, Alexander-Fox and Iwasawa theory may be regarded as "$GL(1)$-theory". To push the analogies further, let us consider a "$GL(2)$-generalization" of Iwasawa theory, initiated by H. Hida and Mazur. Hida ([H1]) produced the big Galois representation

$$\boldsymbol{\rho}_p : G_p \longrightarrow GL_2(\mathbb{Z}_p[[T]])$$

such that the specializations $T = (1+p)^k - 1$ give rise to a family of p-ordinary representations $\rho_p^{(k)} : G_p \to GL_2(\mathbb{Z}_p)$ parametrized by $k \in \mathbb{Z}_p$ and for $k = 2, 3, \ldots$, $\rho_p^{(k)}$ is Deligne's representation associated to a p-ordinary cusp form of weight k. Here the residual representations $\rho_p^{(k)}$ mod p, say $\overline{\rho}_p$, are independent of k and $\boldsymbol{\rho}_p$ is the universal p-ordinary modular deformation of $\overline{\rho}_p$ in the sense of Mazur. Attached to the adjoint representation $Ad\boldsymbol{\rho}_p$, the adjoint Selmer group $\mathrm{Sel}(Ad\boldsymbol{\rho}_p)$ is defined. The Pontryagin dual $\mathrm{Sel}(Ad\boldsymbol{\rho}_p)^\vee$ turns out to be a finitely generated, torsion $\mathbb{Z}_p[[T]]$-module and the order ideal $L_p(T) = \Delta(\mathrm{Sel}(Ad\boldsymbol{\rho}_p)^\vee)$ provides the algebraic $GL(2)$ adjoint p-adic L-function ([H2, Ch.1]). So our natural question is then:

Problem. Can we produce analogues of $\boldsymbol{\rho}_p$ and $L_p(T)$ in knot theory ?

We will give an affirmative answer to this problem in Sections 1 and 2, based on the following analogy between hyperbolic geometry and Hida-Mazur theory, which may be regarded as an "$SL(2)$-generalization" of the analogy between Alexander-Fox theory and Iwasawa theory:

hyperbolic structure and the associated holonomy representation, and its deformation	p-adic ordinary modular form and the associated Galois representation, and its deformation

We note that the above analogy can be extended, including Seiberg-Witten theory, to the following intriguing analogies:

Hida-Mazur theory (p-adic gauge theory)	Hyperbolic geometry (3-dim. gauge theory)	Seiberg-Witten theory (4-dim. gauge theory)
Deformation space $\mathfrak{X}_p$ parameterizing p-adic ordinary modular forms f	Deformation space $\mathfrak{X}_K$ parameterizing hyperbolic structures z	Moduli space of vacua $\mathfrak{U}$ parameterizing elliptic curves $y^2 = (x^2-1)(x-u)$ $(u \in \mathfrak{U})$
$\rho_f\|_{D_p} \simeq \begin{pmatrix} \chi_{f,1} & * \\ 0 & \chi_{f,2} \end{pmatrix}$ monodromy function on $\mathfrak{X}_p$ $x_p(f) = \chi_{f,1}(\tau)$ Frobenius function on $\mathfrak{X}_p$ $y_p(f) = \chi_{f,2}(\sigma)$	$\rho_z\|_{D_K} \simeq \begin{pmatrix} \chi_z & * \\ 0 & \chi_z^{-1} \end{pmatrix}$ meridian function on $\mathfrak{X}_K$ $x_K(z) = \chi_z(\mu)$ longitude function on $\mathfrak{X}_K$ $y_K(z) = \chi_z(\lambda)$	$\omega = \frac{u-x}{y}dx$ electric charge $a = \int_\mu \omega$ magnetic charge $a_D = \int_\lambda \omega$
$\mathfrak{L}$-invariant dy_p/dx_p	modulus of ∂V_K dy_K/dx_K	gauge coupling da_D/da
Hida potential Φ_p $d\Phi_p/dx_p = y_p$	Neumann-Zagier potential Φ_K $d\Phi_K/dx_K = y_K$	Seiberg-Witten prepotential F $dF/da = a_D$

1. The universal deformation of a holonomy representation. Fix a field F of characteristic $\neq 2$ and a complete discrete valuation ring $\mathcal{O}$ with residue field F. For a representation $\overline{\rho} : G \to SL_2(F)$ of a finitely generated group G, a representation $\rho : G \to SL_2(A)$ is called a *deformation* of $\overline{\rho}$ if A is a complete local $\mathcal{O}$-algebra with residue field $A/\mathfrak{m}_A = F$ and ρ mod $\mathfrak{m}_A = \overline{\rho}$. A deformation $\boldsymbol{\rho} : G \to SL_2(\mathbf{A})$ is called the *universal deformation* of $\overline{\rho}$ if for any deformation $\rho : G \to SL_2(A)$ of $\overline{\rho}$ there exists a unique local $\mathcal{O}$-algebra homomorphism $\psi : \mathbf{A} \to A$ such that $\psi \circ \boldsymbol{\rho}$ is strictly equivalent to ρ.

Theorem 1. *Suppose that $\overline{\rho} : G \to \mathrm{SL}_2(F)$ is an absolutely irreducible representation. Then there exists the universal deformation $\boldsymbol{\rho} : G \to SL_2(\mathbf{A})$ of $\overline{\rho}$.*

For the proof we make use of the method of pseudo-representations ([Ta]).

Now let $K \subset S^3$ be a hyperbolic knot and let $\overline{\rho}_K : G_K \to SL_2(\mathbb{C})$ be an $SL_2(\mathbb{C})$-lift of the holonomy representation associated to the complete hyperbolic structure on $S^3 \backslash K$. The following theorem may be regarded as an analogue of Hida's universal deformation mentioned in Introduction.

Theorem 2. *The universal deformation of $\overline{\rho}_K$ is given by*

$$\boldsymbol{\rho}_K : G_K \longrightarrow SL_2(\mathcal{O}[[T]]),$$

where $\mathcal{O} = \mathbb{C}[[h]]$, the ring of formal power series over $\mathbb{C}$.

For the proof we use Luo's theorem ([L]) to relate pseudo-representations with characters, and Thurston's theorem on the character variety ([Th]).

Remark. For example, if $G_K = \langle x, y \,|\, wx = yw \rangle$ $(w = x^{-1}yxy^{-1}x^{-1})$ is the figure eight knot group, $\overline{\rho}_K$ is given by $\overline{\rho}_K(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\overline{\rho}_K(x) = \begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix}$ $(\zeta = (1+\sqrt{-3})/2)$. The universal deformation $\boldsymbol{\rho}_K$ of $\overline{\rho}_K$ induces all deformation ρ of $\overline{\rho}_K$ e.g., $\rho(x) = \begin{pmatrix} 1+h & 1 \\ 0 & 1+h \end{pmatrix}$ and $\overline{\rho}_K(x) = \begin{pmatrix} 1+h & 0 \\ f(h) & 1+h \end{pmatrix}$ with $f(h) \in \mathbb{C}[[h]]$.

3. Adjoint Selmer module. Let $K \subset S^3$ be a hyperbolic knot and let $\boldsymbol{\rho}_K : G_K \to SL_2(\mathcal{O}[[T]])$ be the universal deformation of the holonomy representation as in Theorem 2. Consider the adjoint representation $Ad\boldsymbol{\rho}_K : G_K \to End(sl_2(\mathcal{O}[[T]]))$ defined by

$$(Ad\boldsymbol{\rho}_K(g))(X) := \boldsymbol{\rho}_K(g)X\boldsymbol{\rho}_K(g)^{-1}$$

for $X \in sl_2(\mathcal{O}[[T]])$. Let $C := C_*(X_K, Ad\boldsymbol{\rho}_K)$ and let $C' :=$ the subcomplex of C generated by $[\mu]$ (μ being a meridian) and $[\partial X_K]$. Set $C'' := C/C'$ and define the *adjoint Selmer module* by the $\mathcal{O}[[T]]$-module $H_1(C'')$.

Theorem 3. *The adjoint Selmer module $H_1(C'')$ is a finitely generated, torsion $\mathcal{O}[[T]]$-module.*

For the proof we use Porti's theorem on the behavior of twisted torsions with coefficients $Ad\rho$ on the character variety ([P, Ch.4]) and Nakayama's lemma. By Theorem 3, we can introduce the *adjoint L-function* $L_K(T)$ of K as (a generator of) the order ideal $\Delta(H_1(C''))$.

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The colored Jones polynomial of zero volume links

Roland van der Veen

In this note we study how the unnormalized colored Jones polynomial or quantum sl_2 invariant of a link changes under the operation of cabling. We work with a banded link or ribbon link L so that every component is an embedded annulus.

Given a diagram D of a banded link inside an annulus we can construct a satellite of L by embedding D into a component L_i of L. The (r, s)-cabling operation is the special case where we take D to be the closure of the (r, s)-torus braid $B_s^r = (\sigma_1 \cdots \sigma_{s-1})^r$, where $r \in \mathbb{Z}, s \in \mathbb{N}$. To turn B_s^r into a banded tangle we use the blackboard framing and add a positive curl to every overpassing arc, see the figure below. The banded link obtained by (r, s)-cabling the component L_i of a banded link L will be denoted by $L_{i;s}^r$, we will also call it the $(i; r, s)$-cabling of L.

In order to state our cabling formula we need to introduce the following generalizations of the trinomial (not multinomial) coefficients defined in [1]. For a vector $\mathbf{N} = (n_0, \ldots, n_{g-1})$ define $\binom{g}{w}_{\mathbf{N}}$ to be the coefficient of x^w in the expansion of the product $\prod_{k=0}^{g-1}(x^{\frac{N_k-1}{2}} + x^{\frac{N_k-1}{2}-1} + \ldots + x^{-\frac{N_k-1}{2}})$.

0.29. **Theorem.** *Let $g = \gcd(r, s)$, $p = s/g$ and $\mathbf{N} = (N_0, \cdots, N_{g-1})$. The unnormalized colored Jones polynomial of the zero framed $(i; r, s)$-cabling of a banded link L with c components can be expressed as follows:*

$$J_{M_1,\ldots,M_{i-1},\mathbf{N},\ldots,M_c}(L_{i;s}^r)(q) =$$

$$\prod_{j=1}^{g} q^{-\frac{rs}{g^2}\frac{N_j^2-1}{4}} \sum_{w=-\frac{|\mathbf{N}|-g}{2}}^{\frac{|\mathbf{N}|-g}{2}} \binom{g}{w}_{\mathbf{N}} q^{\frac{rw(wp+1)}{g}} J_{M_1,\ldots,M_{i-1},2wp+1,\ldots,M_c}(L)(q)$$

In the statement of the theorem we have used the notation $|\mathbf{N}| = N_0 + \ldots + N_{g-1}$ and the convention that $J_{M_1,\ldots,M_{i-1},-j,\ldots,M_c}(L)(q) = -J_{M_1,\ldots,M_{i-1},j,\ldots,M_c}(L)(q)$. In the case where $g = 1$ and L is the unknot the above cabling formula agrees with Morton's formula for the (r, s)-torus knot [6], where his variables are related to ours as $s^2 = q, m = r, p = s$. The case of a $(r, 2)$-cabling is also known [8]. In all other cases our formula seems to be new.

Our main motivation for proving such a formula is to verify the volume conjecture [4],[7] in the cases where cabling is involved. The volume conjecture states that the normalized colored Jones polynomial $J'(L)$ of a link L determines the simplicial volume of the link complement as follows:

$$\lim_{N\to\infty} \frac{2\pi}{N} |J'_{N,N,\ldots,N}(L)(e^{\frac{2\pi i}{N}})| = \mathrm{Vol}(\mathbb{S}^3 - L)$$

As a corollary to our cabling formula obtain a weak form of the volume conjecture for all knots and links whose complement has zero simplicial volume. This is because it is shown in [2] that all such links can be obtained from the unknot by repeated cabling and connected sum. Using the cabling formula we can therefore in principle write down the colored Jones polynomial of any such link and observe that it cannot grow exponentially fast.

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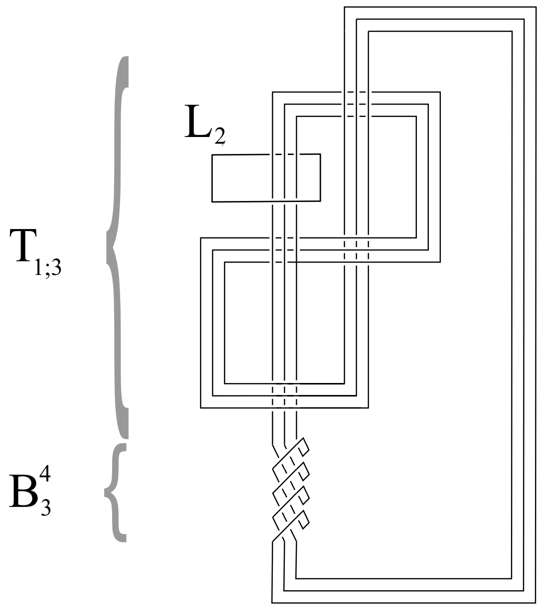

FIGURE 1. We have drawn the link $L^4_{1;3}$, the $(1;3,4)$-cabling of $L = (L_1, L_2)$, where L_1 is the figure eight knot and L_2 is an unknot. We have indicated the torus braid B^4_3 and the opened tangle $T_{1;3}$ mentioned in section 2.

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On Character Varieties of Certain Families of 3-Manifolds

Kathleen Petersen
(joint work with Melissa Macasieb, Ronald van Luijk)

Given a finitely generated group Γ, the set of all representations $\Gamma \to \mathrm{SL}_2(\mathbb{C})$ naturally carries the structure of an algebraic set. So does the set of characters of these representations. The union of the components containing characters of non-abelian representations is called the $\mathrm{SL}_2(\mathbb{C})$-character variety of Γ and is

$$X(\Gamma) = \{\chi_\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})\}$$

where $\chi_\rho : \Gamma \to \mathbb{C}$ is defined by $\chi_\rho(\gamma) = \mathrm{tr}(\rho(\gamma))$ for all $\gamma \in \Gamma$.

Over the last few decades, the $\mathrm{SL}_2(\mathbb{C})$-character variety of the fundamental groups of hyperbolic 3-manifolds has proven to be an effective tool in understanding their topology (see [2], [4], [5]). The same can be said for their $\mathrm{PSL}_2(\mathbb{C})$-character variety, but in general it is difficult to find even the simplest invariants of these varieties, such as the number of irreducible components.

If M is a hyperbolic knot complement, then M is isomorphic to a quotient of hyperbolic 3-space $\mathbb{H}^3$ by a discrete group. By Mostow-Prasad rigidity there is then a discrete faithful representation $\overline{\rho}_0 \colon \Gamma \hookrightarrow \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}_2(\mathbb{C})$ that is unique up to conjugation, defining an action of Γ on $\mathbb{H}^3$ whose quotient $\mathbb{H}^3/\Gamma$ is isomorphic with M. Moreover, the representation $\overline{\rho}_0$ can be lifted to a discrete faithful representation $\Gamma \hookrightarrow \mathrm{SL}_2(\mathbb{C})$. By work of Thurston [7], the character of such a lift, ρ_0, is contained in a unique component of $X(\Gamma)$, which has dimension 1. This component is called the canonical component and is denoted by $X_0(\Gamma)$.

Two-bridge knots are knots with projections admitting exactly two maxima and two minima. Following work of Burde [3] one can obtain a recursive formula for the $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{PSL}_2(\mathbb{C})$ character varieties of two-bridge knots, depending on two-bridge knots with fewer crossings.

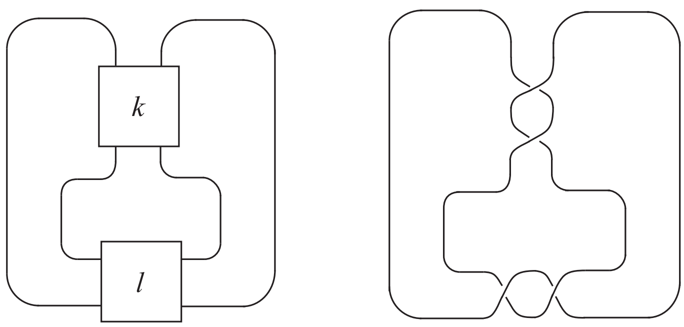

Figure 1. The knot $J(k, l)$ and the figure-eight knot $J(2, -2)$.

The knots $J(k, l)$ as described in Figure 1 are a sub-family of two-bridge knots. The integers k and l in the diagram denotes the number of half twists in the labeled boxes; positive numbers correspond to right-handed twists and negative numbers correspond to left-handed twists. Note that $J(k,l)$ is a knot if and only if kl is even; otherwise it is a two-component link. The subfamilies of knots $J(\pm 2, l)$, with $l \in \mathbb{Z}$, consist of all twist knots, containing the figure-eight knot $J(2,-2)$ and the trefoil $J(2,2)$. The complement of the knot $J(k,l)$ is hyperbolic if and only if $|k|, |l| \geqslant 2$ and $J(k,l)$ is not the trefoil.

For any nonzero integers k and l with kl even, let $M(k,l)$ denote the complement $S^3 \backslash J(k,l)$ and let $X(k,l)$ and $Y(k,l)$ denote the $\mathrm{SL}_2(\mathbb{C})$- and $\mathrm{PSL}_2(\mathbb{C})$-character variety of the fundamental group $\pi_1(M(k,l))$. Both varieties are curves and $X(k,l)$ is a two-fold branched cover of $Y(k,l)$ [1, Thm. 1].

Let j be any integer. Define the j^{th} Fibonacci polynomial inductively by $f_{j+1} - uf_j + f_{j-1} = 0$ with $f_0 = 0$ and $f_1 = 1$. (Note that if $u = s + s^{-1}$, then $f_j = (s^j - s^{-j})(s - s^{-1})$.) For each integer j we define $\Phi_{2j} = f_j$ and $\Phi_{2j-1} = f_j - f_{j-1}$. Furthermore, for each integer k we set $\Psi_k = \Phi_{k+1} - \Phi_{k-1}$.

0.30. **Theorem** (Macasieb-P-van Luijk). *Let k, l be any integers with l even. The variety $Y(k,l)$ is isomorphic to the subvariety $C(k,l)$ of $\mathbb{A}^2(r,y)$ defined by*

$$C(k,l)\colon\ f_n(t)\big(\Phi_{-k}(r)\Phi_{k-1}(r)(y-r)-1\big) + f_{n-1}(t) = 0,$$

with $t = \Phi_{-k}(r)\Psi_k(r)(y-r) + 2$ and $n = l/2$. The variety $X(k,l)$ is isomorphic to the double cover of $C(k,l)$ defined in $\mathbb{A}^2(r,x)$ by $y = x^2 - 2$.

This 'natural' model, which directly comes from the presentation of the fundamental groups is not smooth. However, these equations allow one to write the A-polynomial of these knots as an explicit resultant.

0.31. **Theorem** (Macasieb-P-van Luijk). *Let $D(k,l)$ be the variety in $\mathbb{P}^1_{\mathbb{Q}}(r) \times \mathbb{P}^1_{\mathbb{Q}}(t)$ that is the projective closure of the affine variety given by*

$$\Phi_{k+1}(r)\Phi_{l-1}(t) = \Phi_{k-1}(r)\Phi_{l+1}(t).$$

Then $C(k,l)$ and $D(k,l)$ are isomorphic.

In the case where $k \neq l$, $D(k,l)$ is smooth and irreducible, and $D(k,k)$ has exactly two irreducible components. From this, we compute the genus of every component of the character varieties associated to these knots.

0.32. **Theorem** (Macasieb-P-van Luijk). *Let k, l be any integers with kl even, $|k|, |l| \geqslant 2$, and $k \neq l$. Let $m = \lfloor \frac{1}{2}|k| \rfloor$ and $n = \lfloor \frac{1}{2}|l| \rfloor$.*

(1) *The curve $Y(k,l)$ is irreducible. It has geometric genus*

$$(m-1)(n-1)$$

and is hyperelliptic if and only if $|k| \leqslant 5$ or $|l| \leqslant 5$.

(2) *If $|l| > 2$ is even, then the curve $Y(l,l)$ has two components. The component $Y_0(l,l)$ has genus 0. The other component has genus $(n-2)^2$ and is hyperelliptic if and only if $|l| \leqslant 6$.*

0.33. **Theorem** (Macasieb-P-van Luijk). *Let k, l be any integers with kl even, $|k|, |l| \geqslant 2$, and $k \neq l$. Let $m = \lfloor \frac{1}{2}|k| \rfloor$ and $n = \lfloor \frac{1}{2}|l| \rfloor$.*

(1) *The curve $X(k,l)$ is irreducible. It has geometric genus*

$$3mn - m - n - b$$

with

$$b = \begin{cases} (-1)^{k+l} & \textit{if } (-1)^{k+l}kl > 0, \\ 0 & \textit{otherwise.} \end{cases}$$

(2) *If $|l| > 2$ is even, then the curve $X(l,l)$ has two components, namely $X_0(l,l)$ of genus $n-1$ and an other component of genus $3n^2 - 7n + 5$.*

This is the first time such results have been found for an infinite family of knots. In particular it shows that the genus of the canonical components of both the $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{PSL}_2(\mathbb{C})$ character varieties of knot complements can be arbitrarily large, which was not known before.

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Codes, arithmetic and 3-manifolds

Matthias Kreck

(joint work with Volker Puppe)

Codes in this talk are binary linear codes $C \subset \mathbb{Z}/2^n$. Moreover we are only interested in self dual codes, which means that $\sum_i x_i, y_i = 0 \; mod \; 2$ for all x , $y \in C$. These codes are interesting for several reasons, one is the close relation to arithmetic.

Here **arithmetic** stands for integral, unimodular lattices $L \subseteq \mathbb{R}^n$. They are not understood and interesting for many reasons. There are not so many constructions of such lattices. We define a construction, which we call **box product** which assigns to an odd number of **special lattices** $L_i \subseteq \mathbb{R}^{n_j}$ a new such lattice $L_1 \boxtimes \cdots \boxtimes L_{2r+1} \subseteq \mathbb{R}^{n_1} \otimes \cdots \otimes \mathbb{R}^{n_{2r+1}}$. Here a lattice is called **special** (ad hoc notation), if $\sqrt{2}L \subseteq \mathbb{Z}^n$. The simplest special lattice is $\mathbb{L} = \frac{1}{\sqrt{2}}\{(x,y) \in \mathbb{Z}^2 \mid x = y \; mod \; 2\}$. The box product is interesting, since for example

$$\mathbb{L} \boxtimes \mathbb{L} \boxtimes \mathbb{L} = E_8, \text{the } E_8 - \text{lattice or}$$

$$(\mathbb{L} \perp \mathbb{L}) \boxtimes \mathbb{L} \boxtimes \mathbb{L} = E_{16}, \text{the } E_{16} - \text{lattice}$$

One can give an explicit formula for the box-product but this does not explain its nature which comes from topology through binary self dual codes $C \subseteq (\mathbb{Z}/_2)^n$. To such a code one associates a special unimodular lattice $L(C) := \frac{1}{\sqrt{2}}\{x \in \mathbb{Z}^n \mid x \ mod\ 2 \in C\}$. The box product for lattices comes from a box product for codes. This in term comes from topology. Let (M, τ) be an odd-dim closed manifold with involution τ such that the number of fixed points is finite. We associate a code $C(M, \tau)$, which by Poincaré Lefschetz duality is self dual. One way to define $C(M, \tau)$ is to remove small open equivariant discs around each fixed point, divide by the involution and consider the kernel of the induced map in $\mathbb{Z}/2$-homology in degree k, where $\dim M = 2k + 1$. The boundary is a disjoint union of real projective spaces and so we obtain a code in $\mathbb{Z}/2^n$, where n is the number of fixed points. The background for the construction of the box product is

0.34. **Theorem.** *All self dual codes C are of the form $C(M, \tau)$ for some 3-manifolds M.*

Now the box product of codes appears as follows. Let C_i be an odd number of self dual codes and (M_i, τ_i) be involutions on 3-manifolds with $C(M_i, \tau_i) = C_i$. Then the box product of the codes C_i is $C((M_1 \times M_2 \times \ldots, \tau_1 \times \tau_2 \times \ldots)$. As mentioned above there is a direct construction of the box product on codes reflecting filtrations in Serre spectral sequences. The box product on special lattices is the product induced by the box product on codes.

At the end I listed some obvious questions which hopefully are solved and presented in future Oberwolfach meetings. For example:
1.) Find the Leech manifold, an explicit manifold with involution whose code is the Golay code (from which by some slightly different construction one obtains the Leech lattice).
2.) Find a formula for the theta series of the lattice associated to a manifold with involution via the associated code.

Torus links and scissor equivalence for Euclidean rectangles

Sebastian Baader

Torus links $T(p, q)$ bound pieces of smooth algebraic curves in the 4-ball. The Euler characteristic of these pieces of curves equals $-pq + p + q$ and is known to be maximal, due to the Thom conjecture [1], [3]. The maximal Euler characteristic can easily be extended to a distance function on links. Let K, $L \subset S^3$ be two oriented links. We define the cobordism distance $d_\chi(K, L)$ as the absolute value of the maximal Euler characteristic among all smooth oriented cobordisms between the links K, L. We will estimate the cobordism distance for pairs of torus links.

0.35. **Theorem.** *The following inequalities hold for all* $a, b, c, d \in \mathbb{N}$*:*

$$A \leqslant d_\chi(T(a,b), T(c,d)) \leqslant A + 2(a+b+c+d),$$

where $A = |(a-1)(b-1) - (c-1)(d-1)|$.

Surprisingly, the lower bound is sharp for all pairs of torus links of type $T(ab, c)$, $T(a, bc)$. This is illustrated in Figure 1 for the two triples $(a, b, c) = (3, 2, 7)$ and $(a, b, c) = (2, 3, 7)$ on the left and right, respectively. By looking at a certain family

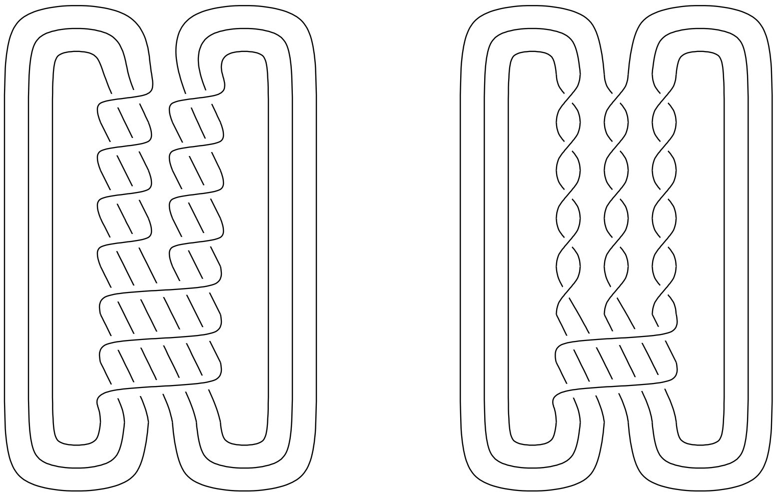

FIGURE 1

of examples, one can see that the upper bound is optimal, up to a constant factor 4 in the linear term.

Let us quickly mention an application concerning the stable 4-genus. The stable 4-genus of knots was recently defined by Livingston [2] and defines a semi-norm on the knot concordance group. Theorem 1 implies that the restriction of this semi-norm to the span of pairs of torus knots has extremely flat unit balls.

The proof of Theorem 1 makes use of elementary cobordisms and suggests to reduce the determination of the cobordism distance of torus links to a combinatorial problem on rectangles. More precisely, we may introduce three kinds of operations on Euclidean rectangles with integer side lengths:

(1) cutting a rectangle into two rectangles of integer side lengths,

(2) gluing two rectangles along sides of coinciding length,

(3) deleting or creating a rectangle.

These operations should be weighted as follows: cutting or gluing a side of length l costs l units, deleting or creating a rectangle of size $a \times b$ costs $(a-1)(b-1)$ units. Let us define $E(a \times b, c \times d)$ as the minimal costs for transforming a rectangle of size $a \times b$ into a rectangle of size $c \times d$, using the operations (1)-(3). We conclude with two questions.

0.36. **Question.** *What is the precise value of $E(a \times b, c \times d)$?*

0.37. **Question.** *What is the precise value of $d_\chi(T(a,b), T(c,d))$?*

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Cell decompositions of Siegel modular spaces

Anton Mellit

We propose to construct cell decompositions of Siegel modular spaces with the help of Morse theory. Thus the first step is to choose a function on a Siegel modular space and describe its set of critical points. We begin by introducing notations.

Let $V = \mathbb{R}^n$, $\Lambda = \mathbb{Z}^n \subset V$. $\operatorname{Sym} V^*$ denotes the space of quadratic forms on V, $\operatorname{Sym}_{>0} V^*$ the cone of positive definite forms. Let W be a symplectic integral-valued form on Λ of determinant 1. In general $H \in \operatorname{Sym}_{>0} V^*$ defines a lattice. The lattice is called symplectic if $J := W^{-1}H$ satisfies $J^2 = -1$ (we identify forms with their matrices). Denote the space of symplectic lattices by $\mathfrak{H}$.

Let $h(H)$ be defined as

$$h(H) = \min_{x \in \Lambda \setminus \{0\}} x^t H x.$$

The vectors for which the minimum is attained are called *minimal*.

Function h is the function whose critical points we are going to study. Similarly to [1], one can show that H is critical implies H is a *symplectic eutactic lattice*, i.e. there are numbers $\lambda_i > 0$ and minimal vectors x_i such that for each M from the tangent space to $\mathfrak{H}$ at H

$$\sum \lambda_i x_i^t M x_i = 0.$$

Eutaxy condition can be formulated conveniently as follows:

Proposition. *A symplectic lattice H is eutactic if and only if there is D, a positive linear combination of matrices of the form xx^t for minimal vectors x, such that $JDJ^t = D$.*

Contrary to what happens with ordinary lattices, we have two essentially different cases:

(1) If D is degenerate, our symplectic lattice is isogenious to a product of two symplectic lattices.

(2) If D is non-degenerate, H can be determined from D as $H = |WD|D^{-1}$ (to define $|\cdot|$ of a matrix we use the standard approach from spectral theory to evaluate a function of a matrix.) In this case we call H a non-degenerate eutactic lattice.

We restrict our attention only to the second case. Let $f : \mathrm{Sym}_{>0} V \to \mathfrak{H}$ be defined by $f(D) = |WD|D^{-1}$. This is a smooth map and the following very interesting property holds:

Lemma. *For each $N \in \mathrm{Sym}\, V$ and $D \in \mathrm{Sym}_{>0} V$ one has $((\nabla_N f)(D), N) \leqslant 0$, with equality if and only if $(\nabla_N f)(D) = 0$.*

Thus, f satisfies a multidimensional analogue of monotonicity property, which, I believe, should give strong topological implications. In particular, we have two applications of the lemma. The first one is theoretical.

Theorem. *Given a configuration of minimal vectors there is at most one non-degenerate symplectic eutactic form with this set of minimal vectors.*

The practical application allows us to find eutactic forms numerically by an iterative method similar to the Newton method.

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Two-dimensional quantum field theory, modular functions and lattice paths

Werner Nahm

Rational conformally invariant quantum field theories in two dimensions typically have integrable massive perturbations with calculable scattering matrix. If the latter has the form $\exp i\phi_{ab}(\theta)$, $a, b = 1, \ldots, r$, $r =$ number of particle types, one obtains partition functions of the form:

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \frac{q^{Q(n)}}{(q)_{n_1}\cdots(q)_{n_r}} \qquad Q(n) = \frac{1}{2}n^t A n + Bn + C$$

where $n = (n_1, \ldots, n_r)$ and $(q)_n = (1-q)\cdots(1-q^n)$ and

$$-2\pi A_{ab} = \phi_{ab}(+\infty) - \phi_{ab}(-\infty)$$

The simplest example is the free fermion theory with n-point functions

$$\langle \psi(z_1)\ldots\psi(z_n)\rangle = \begin{cases} 0 & \text{for } n \text{ odd} \\ Pf(\mu) & \text{for } n \text{ even} \end{cases}$$

$$\mu_{ij} = \begin{cases} 0 & \text{for } i = j \\ \frac{1}{z_i - z_j} & \text{for } i \neq j \end{cases}$$

This theory has two deformations of the type mentioned above, with partition functions

$$\sum \frac{q^{\frac{1}{2}n^2}}{(q)_n} = \prod_{k=1}^{\infty}(1+q^{k-\frac{1}{2}}) \quad \text{(Euler)}$$

$$\sum \frac{q^{n^t \mathcal{C}(E_8)^{-1} n}}{(q)_{n_1}\cdots(q)_{n_8}} = \frac{1}{2}\left(\prod_{k=1}^{\infty}(1+q^{k-\frac{1}{2}}) + \prod_{k=1}^{\infty}(1-q^{k-\frac{1}{2}})\right) \quad \text{(Warnaar, Pearce)}$$

The matrices A yield elements of the extended Bloch group $\hat{B}(\mathbb{C}) \cong K_3(\mathbb{C})$ (Zickert). The case $A = 1$ (Euler) yields a generator of $K_3(\mathbb{Q}) \cong \mathbb{Z}/48$ by $x^A = 1 - x \Rightarrow x = \frac{1}{2}$, $L_{\text{Rogers}}(\frac{1}{2}) = \frac{\pi^2}{12} = \frac{1}{48}4\pi^2$.

Modularity of the partition functions requires $q_i = 0$ for $i > 0$ in the expansion $\sim \tilde{q}^C \sum a_i \tau^i$ around $\tau = 0$, $(q = e^{2\pi i \tau})$. The dual form Q' with $A' = A^{-1}$, $B' = A^{-1}B$, $C' = -C - \frac{r}{24} + \frac{1}{2}BA^{-1}B$ yields $a'_i = (-)^i a_i$, as can be shown by comparing two methods to compute the expansion. Vanishing of the a_i, $i > 0$ might be related to supersymmetric gauge theory. Expansion in $\tilde{q} = \exp(-2\pi i/\tau)$ by a Laplace transform so far yields an alternative form of the partition function in terms of lattice paths. In particular

$$\sum_{k=0}^{\infty} \sum_{n \in P(k)} (-)^k q^{E_k(n)} = \prod_{k=1}^{\infty}(1+q^{k-\frac{1}{2}})$$

with $P(k) = \{(n_1, \ldots, n_k) \in \mathbb{N}^k | n_i \neq n_{i+1} \text{ for } i = 1, \ldots, k-1\}$

$$E_k = \binom{n_1}{2} + \sum_{i=1}^{k-1} \frac{1}{2}(n_i - n_{i+1})^2 + \frac{n_k^2}{2} \quad \text{for } k = 1, 2, \ldots$$

$P(0) = \{\varnothing\}, E_0(\varnothing) = 0$.

Geodesics, volumes and Lehmer's conjecture

Mikhail Belolipetsky

We first recall a well known relation between Lehmer's conjecture (also known as "Lehmer's question") and the Short Geodesic conjecture for arithmetic 2- and 3-orbifolds. Here we follow the exposition in [6, Chapter 12].

Let $P(x)$ be an irreducible monic polynomial with integer coefficients of degree n, and let $\theta_1, \ldots, \theta_n$ denote its roots. The *Mahler measure* of P is defined by

$$M(P) = \prod_{i=1}^{n} \max(1, |\theta_i|).$$

If $P(x)$ is a cyclotomic polynomial then its Mahler measure is equal to 1. Now Lehmer's conjecture says that the measures of all other $P(x)$ are separated from 1 by an absolute positive constant which is called Lehmer's number:

Lehmer's Conjecture. *There exists $m > 1$ such that $M(P) \geqslant m$ for all non-cyclotomic P.*

On the other hand we have the following geometric conjecture.

Short Geodesic Conjecture. *There is a universal positive lower bound for the length of geodesics of arithmetic hyperbolic* 2*- and* 3*-orbifolds.*

To draw the relation between the two conjectures we can argue as follows. Consider a hyperbolic element $\gamma \in \mathrm{PSL}(2,\mathbb{R})$ or $\mathrm{PSL}(2,\mathbb{C})$. We can define its trace and γ being *hyperbolic* means that $\mathrm{tr}(\gamma) = u + u^{-1}$ with $|u| > 1$ (in case of $\mathrm{PSL}(2,\mathbb{C})$ such elements are often called loxodromic but we will not use this terminology). If P is the minimal polynomial of u, then the displacement of γ is given by

$$\ell_0(\gamma) = 2\log M(P) \quad \text{or} \quad \log M(P)$$

for 2 and 3 dimensional cases, respectively. It can be shown that if γ is an element of an arithmetic subgroup then u is an algebraic integer and, moreover, its minimal polynomial is not cyclotomic. This leads to a relation between the two conjectures:

Corollary 1. (1) *Lehmer's conjecture implies Short Geodesic conjecture.*
(2) *Short Geodesic conjecture implies a special case of Lehmer's conjecture, namely, Lehmer's conjecture for Salem numbers.*

Our next goal is to consider the quantitative side of this relation.

The *systole* of a Riemannian manifold M, denoted by $\mathrm{Syst}_1(M)$, is the length of a closed geodesic of the shortest length in M. This notion can be also generalized to Riemannian orbifolds. It is clear from the definition that systole is a geometric invariant of a manifold (or orbifold). By Borel's theorem there exists only finitely many arithmetic hyperbolic 2- or 3-orbifolds of bounded volume [5]. Therefore, if the Short Geodesic conjecture is false then there should exist a sequence of orbifolds M_i such that

$$\mathrm{Syst}_1(M_i) \to 0 \quad \text{and} \quad \mathrm{Vol}(M_i) \to \infty.$$

Using our current knowledge of Mahler measures of polynomials and some recent results about volumes of arithmetically defined orbifolds we can say more here.

Let γ be an element of the group uniformising M_i with the smallest positive displacement and P_i is the minimal polynomial associated to γ as above so that $\mathrm{Syst}_1(M_i) = 2\log M(P_i)$ or $\log M(P_i)$ depending on the dimension.

First, let us recall a well known Dobrowolski's bound for the Mahler measure:

$$\log M(P_i) \geqslant C_1 \left(\frac{\log\log d_i}{\log d_i}\right)^3, \tag{1}$$

where d_i is the degree of the polynomial P_i and $C_1 > 0$ is an explicit constant. To my knowledge, Dobrowolski's result is still the best of its kind except possibly for the value of constant C_1. We refer to an excellent survey article by C. Smyth [7] for more about this topic.

Now we claim that the field of definition k_i of the arithmetic orbifold M_i satisfies

$$\deg(k_i) \geqslant \frac{1}{2} d_i. \tag{2}$$

This indeed follows from the results discussed in [6, Chapter 12].

Finally, we can bring facts together using an important inequality which relates the volume of an arithmetic hyperbolic orbifold with its field of definition:

$$\deg(k_i) \leqslant C_2 \log \mathrm{Vol}(M_i) + C_3. \tag{3}$$

This result was first proved by Chinburg and Friedman, in a form stated here it can be found in [2]. We note that the constants in inequalities (1)–(3) can be computed explicitly.

For sufficiently large x, $\frac{\log x}{x}$ is a monotonically decreasing function hence for sufficiently small $\mathrm{Syst}_1(M_i)$ we get

$$\mathrm{Syst}_1(M_i) \geqslant C_1 \left(\frac{\log\log\log \mathrm{Vol}(M_i)^C}{\log\log \mathrm{Vol}(M_i)^C} \right)^3. \tag{4}$$

This is a *very slowly decreasing function*! It shows that if the Short Geodesic conjecture is false, it has to be violated by a sequence of orbifolds with extremely fast growing volumes. We also get the following immediate corollary.

Corollary 2. *When* $\mathrm{Syst}_1(M_i) \to 0$*, the arithmetic orbifolds* M_i *are non-commensurable and defined over fields of degree going to* ∞*.*

Let us now consider the general, not necessarily arithmetic, hyperbolic n-manifolds and orbifolds. For dimensions $n \geqslant 4$, in a straight contrast with $n = 2$ and 3, we then have an analogue of Borel's theorem cited above. This result, proved by H. C. Wang in [8], implies that for $n \geqslant 4$ there exist only finitely many hyperbolic n-orbifolds of bounded volume. Hence again if we would like to have a sequence of higher dimensional hyperbolic manifolds or orbifolds M_i with $\mathrm{Syst}_1(M_i) \to 0$, we would necessarily have $\mathrm{Vol}(M_i) \to \infty$. In line with the previous discussion and known rigidity properties of higher dimensional hyperbolic manifolds we come to the questions of whether such sequences do exist at all and if yes, then what can we say about the isosystolic properties of corresponding manifolds. The first question was answered for $n = 4$ in a remarkable short paper by I. Agol [1]. In a joint work with S. A. Thomson we modify and extend Agol's argument which allows us to construct hyperbolic n-manifolds with short systole uniformly for all dimensions n and also to answer the second question about their isosystolic properties [3]. Our main results can be summarized as follows.

Theorem 1.

(A) *For every* $n \geqslant 2$ *and any* $\epsilon > 0$*, there exist compact* n*-dimensional hyperbolic manifolds* M *with* $\mathrm{Syst}_1(M) < \epsilon$*.*

(B) *For every* $n \geqslant 3$ *there exists a positive constant* C_n *(which depends only on* n*), such that the systole length and volume of the manifolds obtained in the proof of part (A) satisfy* $\mathrm{Vol}(M) \geqslant C_n/\mathrm{Syst}_1(M)^{n-2}$*.*

Concerning part (B), we can show that it is possible to achieve that $\mathrm{Vol}(M)$ grows exactly like a polynomial in $\mathrm{Syst}_1(M)$, therefore, Theorem 1(B) captures the growth rate of the volume in our construction. This growth can be compared with inequality (4) which says that if a similar phenomenon can occur in arithmetic setting, the volume would have to grow much faster. It is unknown if for $n \geqslant 4$

there exist hyperbolic n-manifolds M with $\mathrm{Syst}_1(M) \to 0$ and $\mathrm{Vol}(M)$ growing slower than a polynomial in $1/\mathrm{Syst}_1(M)$. Let us also remark that an alternative proof of part (A) of Theorem 1 can be given using the original Agol's construction combined with a recent work of Bergeron, Hugland and Wise [4].

To conclude the comparison with the low dimensional case and Corollary 2 stated above, here we have

Corollary 3. *When* $\mathrm{Syst}_1(M) < \epsilon$*, the manifolds* M *from Theorem 1 are non-arithmetic and all but finitely many of them are non-commensurable to each other.*

For the proof of this corollary, its relation to Lehmer's conjecture and some other results we refer to Section 5 of [3].

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Real Places and Surface Bundles

Jonah Sinick

A finite volume orientable *hyperbolic 3-manifold* has the form $M = \mathbb{H}^3/\Gamma$ where $\mathbb{H}^3$ is hyperbolic 3-space and Γ is a discrete, torsion-free, finite covolume subgroup of $\mathrm{Isom}^+(\mathbb{H}^3) = PSL_2(\mathbb{C})$. Such manifolds occupy a prominent role in the study of 3-manifolds in general [Thu1]. By Mostow-Prasad rigidity, the topology of such an M determines Γ uniquely up to conjugation and the *trace field* $\mathbb{Q}(\mathrm{tr}(\Gamma)) = \mathbb{Q}(\{\mathrm{tr}(\gamma) : \gamma \in \Gamma\})$ is a *finite* extension of $\mathbb{Q}$ which is a topological invariant of M. For a justification of these facts and more, see Chapter 3 of [MR]. Since all hyperbolic 3-manifolds that we consider are finite volume and orientable, we now drop these modifiers.

The trace field of a hyperbolic 3-manifold is not an abstract number field, but a concrete subfield of the complex numbers, that is, a pair (K, σ) where K is an abstract field and $\sigma : K \to \mathbb{C}$ is nonzero ring homomorphism. A short argument shows that $\sigma(K) \not\subset \mathbb{R}$ since otherwise Γ could not have finite covolume [MR]. However, it can happen that there is some other ring homomorphism $\sigma' : K \to \mathbb{C}$

that $\sigma'(K) \subset \mathbb{R}$. If this is so then we say that K has a *real place.* The condition that K has a real place is the same as the condition that a minimal polynomial for K over $\mathbb{Q}$ has a real root.

A natural question is

0.38. **Question.** *Is every pair (K, σ) with $\sigma(K) \not\subset \mathbb{R}$ the trace field of some hyperbolic 3-manifold?*

W. Neumann has conjectured that the answer is yes, but the question is still open. An adjacent question is whether there are restrictions on the trace fields that arise from natural families of hyperbolic 3-manifolds. One of the few known results of this type is D. Calegari's intriguing theorem [Cal]. Calegari's theorem is

0.39. **Theorem.** *The trace field of a hyperbolic once punctured torus bundle has no real places.*

Here we consider whether Calegari's result can be extended to hyperbolic surface bundles with other fibers. We show that many surfaces occur as fibers of hyperbolic surface bundles with trace field having real place.

We now review hyperbolic surface bundles. Let $S = S(g, p)$ be the orientable, connected surface of genus g with p punctures. Let $\psi : S \to S$ be an orientation preserving homeomorphism. The *mapping torus* of the pair (S, ψ) is $M = S \times [0, 1]/\sim$, where $\sim$ identifies $S \times \{0\}$ with $S \times \{1\}$ via ψ. A 3-manifold is called a *surface bundle over* S^1 if it arises from this construction. If ψ and ψ' differ by a homeomorphism isotopic to the identity map, then the associated mapping cylinders are homeomorphic. Hence M depends only on the class that ψ represents in the *mapping class group* $\mathrm{Mod}(S) := \mathrm{Homeo}^+(S)/\mathrm{Homeo}_0(S)$; here $\mathrm{Homeo}^+(S)$ is the group of orientation preserving homeomorphisms of S and $\mathrm{Homeo}_0(S)$ is the group of homeomorphisms of S that are isotopic to the identity.

W. Thurston showed that a mapping torus M is a hyperbolic 3-manifold if and only if ψ is *pseudo-Anosov* [Thu3]. J. Maher showed that if the Euler characteristic $\chi(S) = 2 - 2g - p < 0$ and $(g, p) \neq (0, 3)$, then a "generic" mapping class $\psi \in \mathrm{Mod}(S)$ is pseudo-Anosov, so that "most" surface bundles over S^1 are hyperbolic [Mah]. One motivation for studying hyperbolic surface bundles over S^1 is that Thurston has conjectured that every hyperbolic 3-manifold is a finite quotient of such a bundle.

In this paper we refer to the unique surface with p punctures and $\chi(S) = \chi$ as the *surface of type* $(-\chi, p)$, and a 3-manifold that can be realized as a surface bundle fiber of type $(-\chi, p)$ as a *surface bundle of type* $(-\chi, p)$. We remark that the type of a surface bundle is not in general uniquely determined, since a given manifold can fiber over S^1 in multiple ways.

In this language, Theorem 0.39 is that the trace field of a hyperbolic surface bundle of type $(1, 1)$ has no real places. Calegari's proof does not readily generalize to surface bundles of other types. Our goal is to substantiate

0.40. **Conjecture.** *For each pair $(-\chi, p)$ with $-\chi > 1$, there exists a hyperbolic mapping torus of type $(-\chi, p)$ with trace field having a real place.*

Those surfaces with $\chi \geqslant 0$ and $(-\chi, p) = (1, 3)$ and are excluded since there are no hyperbolic surface bundles with these as fibers, while the surface of type $(1, 1)$ is excluded by Theorem 0.39. Throughout the remainder of the paper we restrict ourselves to surfaces with $-\chi > 1$.

We were led to Conjecture 0.3 by utilizing existing software to collect data which led to the following.

0.41. **Observation.** *For each of the following pairs $(-\chi, p)$, there exists a hyperbolic mapping torus of that type with trace field having a real place:*

$-\chi$	p	$-\chi$	p	$-\chi$	p	$-\chi$	p
2	2, 4	7	1, 3, 5	12	0	18	0, 2
3	1, 3, 5	8	0, 2, 4	14	0, 2	19	1
4	0, 2, 4, 6	9	1, 3	15	1	20	0, 2
5	0, 1, 3, 5, 7	10	0, 2, 4	16	0, 2	21	1, 3
6	0, 2, 4	11	1, 3	17	1	22	0, 2

We were not able to prove our conjecture in general, but were able to prove it for several infinite families of surfaces.

0.42. **Theorem.** *Conjecture 0.3 is true for mapping tori of type $(-\chi, p)$ if*

a) $p \geqslant 4$ *and* $2|(-\chi)$
b) $p = 0$
c) $5p \leqslant -\chi$

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On the compact arithmetic hyperbolic 5-orbifold of smallest volume

Vincent Emery
(joint work with Mikhail V. Belolipetsky, Ruth Kellerhals)

Let $\mathbb{H}^n$ be the hyperbolic n-space and $\mathrm{Isom}^+(\mathbb{H}^n)$ its group of orientation-preserving isometries. The group of spinors $\mathcal{G} = \mathrm{Spin}(n,1)$ is a double covering of $\mathrm{Isom}^+(\mathbb{H}^n)$. Every (complete) orientable hyperbolic n-orbifold can be written as a quotient $\mathbb{H}^n/\Gamma$, where Γ is a discrete subgroup of $\mathcal{G}$.

In a recent work with M. Belolipetsky, we determined for each odd dimension $n \geqslant 5$ the unique compact orientable hyperbolic arithmetic n-orbifold of smallest volume [1, 3]. Let us denote this orbifold by $\mathbb{H}^n/\Gamma_n$, with $\Gamma_n \subset \mathcal{G}$. Our result completes previous results by Siegel ($n = 2$), Chinburg and Friedman ($n = 3$), and Belolipetsky ($n \geqslant 4$ even).

The discrete subgroup $\Gamma_n \subset \mathcal{G}$ is given as a normalizer $\Gamma_n = N_G(\Lambda)$, where Λ is a *principal* arithmetic subgroup of $\mathcal{G}$. More precisely, Λ can be written as

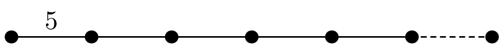
$$\Lambda \quad = \quad G(k) \cap \prod_{v \in V_f(k)} P_v,$$

where

- G is a simply connected algebraic group defined over the number field $k = \mathbb{Q}(\sqrt{5})$ and such that $G(k \otimes \mathbb{R}) \cong \mathcal{G} \times \mathrm{Spin}(n+1)$;
- for each finite place $v \in V_f(k)$, P_v is a parahoric subgroup of $G(k_v)$.

The covolume of Λ in $\mathcal{G}$ can be computed using Prasad's volume formula [4]. Using some methods developed in [2], the index $[\Gamma_n : \Lambda]$ can also be computed. Altogether this yields the volume of $\mathbb{H}^n/\Gamma_n$. More generally, maximal arithmetic subgroups of $\mathcal{G}$ are principal and one can use the methods of [2] to bound their covolumes from below. This is the main idea to prove the minimality of $\mathrm{vol}(\mathbb{H}^n/\Gamma_n)$. We note that our work can be extended to obtain the value of the second smallest volume of arithmetic orbifolds.

Our work [1] does not give information on the geometry of the smallest arithmetic orbifold $\mathbb{H}^n/\Gamma_n$. In each dimension $n = 2, 3, 4$ the smallest arithmetic orbifold is known to be related to some Coxeter group with a particularly simple presentation. One expects this to be true in dimension 5 as well. A good candidate in this dimension is given by the following cocompact arithmetic hyperbolic Coxeter group:

$\Delta \quad =$ 

Let Δ_0 be the intersection $\mathrm{Isom}^+(\mathbb{H}^5) \cap \Delta$ (which is of index two in Δ). Using Schläfli's differential formula, Ruth Kellerhals could obtain an expression for the volume of the fundamental domain of Δ as an integral of a sum of Lobatchevsky's functions (see [3, §14.4]). This can be compared with the value of $\mathrm{vol}(\mathbb{H}^5/\Gamma_5)$ computed in [1], and we obtain the following numerical approximation:

$$\mathrm{vol}(\mathbb{H}^5/\Delta_0) \quad \approx \quad \mathrm{vol}(\mathbb{H}^5/\Gamma_5). \tag{1}$$

But this approximation is good enough to deduce that the orbifold $\mathbb{H}^5/\Delta_0$ must be $\mathbb{H}^5/\Gamma_5$ (because it is a compact orientable arithmetic orbifold whose volume is clearly smaller than the second smallest volume).

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Structure of quantum 3-manifold invariants

Ruth Lawrence

This was an (incomplete) survey of some constructions and results on the structure of quantum 3-manifold invariants, including integrality of the WRT invariants Z_K at K-th roots of unity, existence of the Ohtsuki series Z_∞ for rational homology spheres, the Habiro cyclotomic expansion of Z_∞, Z_K for integer homology spheres and the 'slope conjecture' on the growth of coefficients in the Ohtsuki series [4]. See also [3].

For the Poincaré homology sphere (PHS), we explained existence of holomorphic extensions of Z_K, Z_∞ [1]. In particular they share the same asymptotic expansion Z_∞ for $q = exp 2\pi i/K$ around $q = 1$, one of which has 'almost' modular properties as a $-1/2$-Eichler integral of a modular form [7], and another which can be considered as the trivial connection contribution to the stationary phase expansion of the Witten-Chern-Simons path integral expression of Z_K, which for Seifert-fibred manifolds is exact [5].

A calculation of the $PSU(3)$-Ohtsuki invariant of PHS was also presented [2], as well as a simple state-sum formula for the Jones function of a knot, from a knot diagram [6]. Parts of the work presented are joint with Jacoby, Ron, Rozansky, Zagier, while other parts are due to many authors, many of whom were also present at the meeting!

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On the AJ conjecutre

Thang Lê
(joint work with Anh Trân)

For a knot K in S^3 let $J_K(n) \in \mathbb{Z}[q^{\pm 1/2}]$ be the colored Jones polynomial [Jo] of K colored by the (unique) n-dimensional simple sl_2-module, normalized so that for the unknot U,

$$J_U(n) = [n] := \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}.$$

The color n can be assumed to take negative integer values by setting $J_K(-n) = -J_K(n)$. In particular, $K_K(0) = 0$. It is known that $J_K(1) = 1$, and $J_K(2)$ is the ordinary Jones polynomial.

Define two operators L, M acting on the set of functions $\{f : \mathbb{Z} \to \mathbb{Z}[q^{\pm 1/2}]\}$ by

$$(Lf)(n) = f(n+1)$$
$$(Mf)(n) = q^{n/2} f(n).$$

It is easy to see that $LM = q^{1/2}ML$. Besides, the inverse operators L^{-1}, M^{-1} are well-defined. One can consider L, M as elements of the quantum torus

$$\mathcal{T} = \mathbb{C}[q^{\pm 1/2}]\langle L^{\pm 1}, M^{\pm 1}\rangle/(LM - q^{1/2}ML),$$

which is not commutative, but almost commutative.

Let $\mathcal{A}_K = \{p \in \mathcal{T}, pJ_K = 0\}$, which is a left ideal of $\mathcal{T}$, called the recurrence ideal of K. It was proved in [GL] that for every knot K, the recurrence ideal $\mathcal{A}_K$ is non-zero. An element in $\mathcal{A}_K$ is called a recurrence relation for the colored Jones polynomial of K.

The ring $\mathcal{T}$ is not a principal ideal domain, i.e. not every left ideal of $\mathcal{T}$ is generated by one element. In [Ga], by adding the inverses of polynomial in $q^{1/2}, M$ to $\mathcal{T}$ one gets a principal ideal domain, and a generator α_K of the extension of $\mathcal{A}_K$. The element α_K can be presented in the form

$$\alpha_K = \sum_{j=0}^{d} a_k(M)\, L^j,$$

where $a_j(M) \in \mathbb{Z}[q^{\pm 1/2, M}]$, and α_K is defined up to a polynomial in $\mathbb{Z}[q^{\pm 1/2, M}]$. Besides, $\alpha_K \in \mathcal{A}_K$, i.e. it is a recurrence relation for the colored Jones polynomial. We call α_K the recurrence polynomial.

S. Garoufalidis [Ga] formulated the following AJ conjecture (see also [FGL]).

0.43. **Conjecture. (AJ conjecture, strong form)** *For every knot K, $\alpha_K|_{q^{1/2}=1}$ is equal to the A-polynomial, up to a factor depending on M only.*

Here in the definition of the A-polynomial [CCGLS], we also allow the abelian component. By definition, the A-polynomial is the generator of a reduced principal ideal in $\mathbb{Z}[M, L]$, and hence does not have repeated factor. Recall that an ideal I in a commutative ring is reduced if I is equal to its radical, $I = \sqrt{I}$.

In connection with the M-factors, we suggest the following conjecture.

0.44. **Conjecture.** *For every knot K the A-polynomial does not have any M-factor.*

Conjecture 0.43 was checked for all torus knots by Hikami [Hi] using explicit formulas of the colored Jones polynomial; it was established for a big class of two-bridge knots, including all twist knots by the first author [Le], using skein theory. Here we have the following stronger result.

0.45. **Theorem.** *Suppose K is a knot satisfying all the following condition*

(i) K is hyperbolic,

(ii) The SL_2-character variety of $\pi_1(S^3 \backslash K)$ is one-dimensional and consists of 2 irreducible components (one abelian and one non-abelian),

(iii) The universal SL_2-character ring of $\pi_1(S^3 \backslash K)$ is reduced.

Then both Conjectures 0.43 and 0.44 hold for K.

For the definition of the universal SL_2-character ring see [LM].

0.46. **Theorem.** *The following knots satisfy all the conditions (i)–(iii) of Theorem 0.45 and hence Conjectures 0.43 and 0.44 hold for them.*

(a) All pretzel knots of type $(-2, 3, 6n \pm 1)$, $n \in \mathbb{Z}$.

(b) All non-torus two bridge knots for which the character variety has exactly 2 irreducible components; these includes all twist knots, double twist knots of the form $J(k, l)$ with $k \neq l$ in the notation of [MPL]*, all two-bridge knots of the form $b(p, m)$ with $m = p - 2$ or $m = 3$, all two-bridge knots $b(p, m)$ with p prime. Here we use the notation $b(p, m)$ from* [BZ].

Actually, Conjecture 0.44 holds for all 2-bridge knots.

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Asymptotics of Quantum Invariants

Kazuhiro Hikami
(joint work with K. Bringmann, J. Lovejoy)

The unified Witten–Reshetikhin–Turaev (WRT) invariant for integral homology sphere M was introduced by Habiro [3]; the unified WRT invariant $I_q(M)$ takes values in $\widehat{\mathbb{Z}[q]} = \varprojlim_n \mathbb{Z}[q]/(q)_n$, and the WRT invariant is given by $\tau_N(M) = I_{q=e^{2\pi i/N}}(M)$. Here $(x)_n = (x;q)_n = \prod_{j=1}^n \left(1 - x\, q^{j-1}\right)$. When M is -1 surgery on knot K whose colored Jones polynomial is given by

$$J_K(N) = \sum_{n=0}^{\infty} C_K(n) \left(q^{1+N}\right)_n \left(q^{1-N}\right)_n$$

the unified WRT invariant $I_q(M)$ is given by [3, 1]

$$(1-q)\, I_q(M) = \sum_{n=1}^{\infty} C_K(n) \left(q^{n+1}\right)_{n+1}$$

For example, we have $C_{\text{trefoil}}(n) = q^n$, and we obtain

$$1 + q\,(1-q)\, I_q(\Sigma(2,3,5)) = \sum_{n=1}^{\infty} q^n \left(q^n\right)_n = M(q)$$

This q-series can be written as

$$M(q) = \sum_{n=0}^{\infty} \chi(n)\, q^{\frac{n^2-1}{120}} \qquad \chi(n) = \begin{cases} 1 & \text{when } n = 1, 11, 19, 29 \mod 60 \\ -1 & \text{when } n = 31, 41, 49, 59 \mod 60 \\ 0 & \text{otherwise} \end{cases}$$

which was studied by Lawrence–Zagier [7] as the Eichler integral of vector-valued modular form. It has a nearly modular property when $\tau = 1/N$, and the asymptotic expansion of the WRT invariant $\tau_N(\Sigma(2,3,5))$ in $N \to \infty$ was studied in detail. Topological invariants such as the Chern–Simons invariant and the Ohtsuki invariant are interpreted from view point of modular form.

In case of other Seifert manifolds, there still exists a relationship between the WRT invariant and the modular forms (see, *e.g.*, [5]). Furthermore it was shown that many mock theta functions coincide asymptotically with the WRT invariants of certain Seifert manifolds [4].

We may expect that a similar property holds for unified WRT invariants. It was shown [6] that the unified WRT invariant for $\Sigma(2,3,6p-1)$, which was given based on the colored Jones polynomial for twist knot [8], has a Hecke-type formula.

We find that the unified WRT invariant for M, which is $+2$ surgery on trefoil, has a direct connection with the Ramanujan mock theta function [2]. We have

$$(1-q)\, I_q(M) = \sqrt{2}\, q^{\frac{1}{4}}\, \phi(-q^{1/2})$$

where

$$2\, q^2\, \phi(q^2) = \psi(q) + \psi(-q)$$

Here $\psi(q)$ is the 3rd order mock theta function whose shadow is $\sum_{n\in\mathbb{Z}}(6n+1)\, q^{n(3n+1)/2}$;

$$\psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}$$

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Algebraic constructions associated with ideal triangulations

Rinat Kashaev

In the papers [2, 3], based on the notion of a Delta-groupoid, ring valued knot invariants were constructed. These constructions originate from the combinatorial descriptions of Teichmüller spaces of punctured surfaces based on Penner coordinates [4], and, in particular, on the *ratio coordinates* of [1]. These rings can also be associated to (topological) ideal triangulations of knot complements. In the case of the B'-ring, the definition is as follows.

Given an ideal triangulation τ of the complement of a knot K. Let $\tilde{\tau}$ be the corresponding cell complex of the exterior of K: $X_K = S^3\backslash N(K)$, where $N(K)$ is an open tubular neighborhood of K. There are two types of 2-cells in $\tilde{\tau}$: triangular and hexagonal faces, the latter being remnants of the faces of τ. There are also two types of edges in $\tilde{\tau}$: *short* edges which bound the triangular faces and *long* edges which are remnants of the edges of τ. The ring $B'(\tau)$ is presented by a set of generators given by associating a pair of generators (u_e, v_e) with each oriented short edge e, and a set of relations:

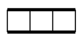
- if $\overline{e}$ is the short edge e taken with opposite orientation, then $u_{\overline{e}} = u_e^{-1}$, $v_{\overline{e}} = -u_e^{-1} v_e$;
- if $\check{e}$ is the unique oriented short edge such that it belongs to the same hexagonal face as e, and the terminal points of e and $\check{e}$ form the boundary of a long edge, then $u_{\check{e}} = v_e$ and $v_{\check{e}} = u_e$;
- if e_1, e_2, e_3 are cyclically oriented short edges constituting the boundary of a triangular face, then $u_{e_1} u_{e_2} u_{e_3} = 1$ and $u_{e_1} u_{e_2} v_{e_3} + u_{e_1} v_{e_2} + v_{e_1} = 0$.

The significance of the B'-ring consists in the fact that there exists a canonical representation of the knot group into the group $GL(2, B'(\tau))$ given by associating the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to each (oriented) long edge, and the matrix $\begin{pmatrix} u_e & v_e \\ 0 & 1 \end{pmatrix}$ to each oriented short edge e. To see how these rings look like, we consider two examples corresponding to the trefoil and the figure-eight knots. In what follows, the following graphical notation is used for an oriented tetrahedron:

where the vertical segments, ordered from left to right, correspond to the faces of the tetrahedron with the order induced from that of the vertices through the bijective correspondence between the faces and vertices obtained by associating to each face the vertex opposite to it.

The trefoil knot. There is an ideal triangulation of the complement of the trefoil knot consisting of two tetrahedra described by the following diagram

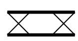
(1)



0.47. **Theorem.** *The B'-ring associated to ideal triangulation* (1) *is isomorphic to the ring $\mathbb{Z}[t, 3^{-1}]/(\Delta_{3_1}(t))$, where $\Delta_{3_1}(t) = t^2 - t + 1$ is the Alexander polynomial of the trefoil knot.*

The figure-eight knot. The standard ideal triangulation of the complement of the figure-eight knot [5] consists of two tetrahedra and it is described by the following diagram

(2)



0.48. **Theorem.** *The B'-ring associated to diagram* (2) *admits the following presentation*

$$\mathbb{Z}\left\langle a, b, c^{\pm 1} \,\middle|\, c = a(a+1),\ c^{-1} = b(b+1),\ c = \left(aba^{-1}b^{-1}\right)^2 \right\rangle.$$

Moreover, any element x can uniquely be written as a linear combination

$$x = m\epsilon + \sum_{\mu \in \{1,w,d\}} \sum_{\nu \in \{1,a,b,ab\}} n_{\mu,\nu}\, \mu\nu,$$

where $m \in \{0, \pm 1, \pm 2\}$ and $n_{\mu,\nu} \in \mathbb{Z}$, and the ring structure is characterized by the conditions that the elements ϵ, w, d are central, and the multiplication rules:

$$\epsilon^2 = 0,\ 5\epsilon = 0,\ \epsilon\mu\nu = \varepsilon_\mu \varepsilon_\nu\, \epsilon, \quad \varepsilon_x = \begin{cases} 1, & \text{if } x = 1; \\ 2, & \text{if } x \in \{w, d, a, b\}; \\ 4, & \text{if } x = ab; \end{cases}$$

$$a^2 = \epsilon - 1 + w - a,\ b^2 = -\epsilon - 1 + w - b,\ ba = d - a - b - ab;$$

$$w^2\nu = 2w\nu,\ d^2\nu = -d\nu + 3w\nu, \quad \nu \in \{1, a, b, ab\};$$

$$wd\nu = \begin{cases} \epsilon + wa + wb + 2wab, & \text{if } \nu = 1; \\ 2\epsilon + w - wa + 2wb - wab, & \text{if } \nu = a; \\ 2\epsilon + w + 2wa - wb - wab, & \text{if } \nu = b; \\ -\epsilon + 2w - wa - wb, & \text{if } \nu = ab; \end{cases}$$

$$a^2 b = 2\epsilon - b + wb - ab,\ ab^2 = -2\epsilon - a + wa - ab;$$

$$aba = 2\epsilon + 1 - w + a + da + b - wb;$$

$$bab = -2\epsilon + 1 - w + a - wa + b + db;$$

$$(ab)^2 = -1 + ab + dab.$$

We remark that in both cases the rings are finite dimensional over $\mathbb{Z}$, and ask the following question: what is the class of ideal triangulations which have finite dimensional B'-rings (over $\mathbb{Z}$)?

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On the complex volume of hyperbolic knots

Yoshiyuki Yokota

Let M be a closed, oriented, hyperbolic 3-manifold. The *Chern-Simons invariant* of M is defined by

$$\mathrm{cs}(M) = \frac{1}{8\pi^2}\int_{s(M)} \mathrm{tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \in \mathbb{R}/\mathbb{Z},$$

where A denotes the connection in the orthonormal frame bundle determined by the metric and $s(M)$ is an orthonormal frame field. In this article, we define the *complex volume* of M by

$$\mathrm{cv}(M) = -2\pi^2\mathrm{cs}(M) + \sqrt{-1}\mathrm{vol}(M) \mod 2\pi^2,$$

which is extended to *cusped* hyperbolic 3-manifolds modulo π^2, see [2].

Conjecture([1],[3]). Let K be a hyperbolic knot in S^3. If N is large,

$$J_K(N; e^{2\pi\sqrt{-1}/N}) \sim e^{\frac{N}{2\pi\sqrt{-1}}\left\{-2\pi^2\mathrm{cs}(S^3\backslash K)+\sqrt{-1}\mathrm{vol}(S^3\backslash K)\right\}},$$

where $J_K(N;q)$ is the N-colored Jones polynomial of K.

This conjecture is still open. However, we can show that

$$J_K(N; e^{2\pi\sqrt{-1}/N}) = \int e^{\frac{N}{2\pi\sqrt{-1}}\{V(x_1,\dots,x_n)+O(1/N)\}}dx_1\cdots dx_n,$$

where $V(x_1,\dots,x_n)$ is a sum of dilogarithms, and the *hyperbolicity equations* for $M = S^3\backslash K$ are given by

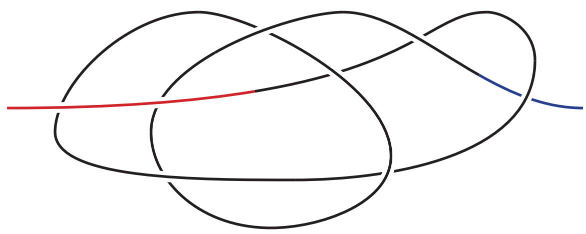
$$x_\nu\frac{\partial V}{\partial x_\nu} = 2\pi\sqrt{-1}\cdot r_\nu, \quad r_\nu \in \mathbb{Z}$$

Then, by using the results of W. Neumann[4] and C. Zickert[6], we can prove

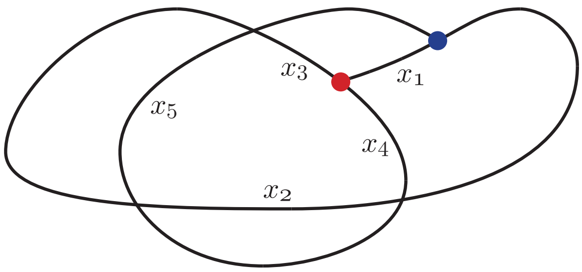
Theorem[5]**.** If $(x_1,\dots,x_n) = (z_1,\dots,z_n)$ is the *geometric* solution to the above,

$$\mathrm{cv}(M) = V(z_1,\dots,z_n) - 2\pi\sqrt{-1}\sum_{\nu=1}^{n} r_\nu \log z_\nu \mod \pi^2.$$

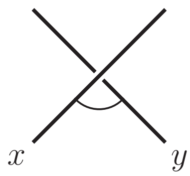
Example. Choose a diagram D of a hyperbolic knot K in S^3, and remove an overpass and an underpass of D which are adjacent.



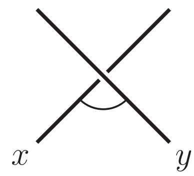
Then, we obtain a subgraph G of D with the *edge variables* x_ν's.



Put *dilogarithm functions* on the interior corners of G.



$\mathrm{Li}_2(y/x) - \pi^2/6$



$\pi^2/6 - \mathrm{Li}_2(x/y)$

Then, the *potential function* $V(x_1, x_2, x_3, x_4, x_5)$ is nothing but the sum of these dilogarithm functions, that is,

$$\begin{aligned}&\mathrm{Li}_2(x_1/x_4) - \mathrm{Li}_2(x_1/x_3) + \mathrm{Li}_2(x_1) - \mathrm{Li}_2(1/x_4)\\ &+ \mathrm{Li}_2(x_2/x_4) - \mathrm{Li}_2(x_2) - \mathrm{Li}_2(1/x_2) + \mathrm{Li}_2(x_5/x_2)\\ &- \mathrm{Li}_2(x_5) - \mathrm{Li}_2(1/x_5) + \mathrm{Li}_2(x_3/x_5) - \mathrm{Li}_2(x_3) + \pi^2/3.\end{aligned}$$

On the other hand, there is an ideal triangulation $\mathcal{S}$ of $M = S^3\backslash K$, such that the hyperbolicity equations for $\mathcal{S}$ are given by

$$\begin{aligned}0 &\equiv x_1\frac{\partial V}{\partial x_1} \equiv \ln\frac{1 - x_1/x_3}{(1 - x_1/x_4)(1 - x_1)},\\ 0 &\equiv x_2\frac{\partial V}{\partial x_2} \equiv \ln\frac{(1 - x_2)(1 - x_5/x_2)}{(1 - x_2/x_4)(1 - 1/x_2)},\\ 0 &\equiv x_3\frac{\partial V}{\partial x_3} \equiv \ln\frac{(1 - x_1/x_3)(1 - x_3)}{1 - x_3/x_5},\\ 0 &\equiv x_4\frac{\partial V}{\partial x_4} \equiv \ln\frac{(1 - x_1/x_4)(1 - x_2/x_4)}{1 - 1/x_4},\\ 0 &\equiv x_5\frac{\partial V}{\partial x_5} \equiv \ln\frac{(1 - x_5)(1 - x_3/x_5)}{(1 - x_5/x_2)(1 - 1/x_5)}\end{aligned}$$

modulo $2\pi\sqrt{-1}\mathbb{Z}$. The solutions to the equations are given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} +1.066 \pm 2.484\sqrt{-1} \\ -1.099 \pm 1.129\sqrt{-1} \\ -0.812 \mp 0.173\sqrt{-1} \\ -0.099 \pm 1.129\sqrt{-1} \\ -1.177 \pm 0.250\sqrt{-1} \end{pmatrix}, \begin{pmatrix} +1.281 \pm 0.392\sqrt{-1} \\ -0.317 \pm 0.618\sqrt{-1} \\ +1.949 \mp 0.441\sqrt{-1} \\ +0.682 \pm 0.618\sqrt{-1} \\ +0.487 \mp 0.110\sqrt{-1} \end{pmatrix}, \begin{pmatrix} 0.304 \\ 0.833 \\ 0.725 \\ 1.833 \\ 1.379 \end{pmatrix}$$

with $r_\nu \equiv 0$. Note that these solutions satisfy

$$\frac{x_1}{x_4}, \frac{x_1}{x_3}, x_1, \frac{1}{x_4}, \frac{x_2}{x_4}, x_2, \frac{1}{x_2}, \frac{x_5}{x_2}, x_5, \frac{1}{x_5}, \frac{x_3}{x_5}, x_3 \notin \{0, 1, \infty\}.$$

The critical values of $V(x_1, x_2, x_3, x_4, x_5)$ are given by

$$11.9099 \pm 4.1249\sqrt{-1},\ 1.85138 \pm 1.10891\sqrt{-1},\ -1.20365,$$

and so $\mathrm{cv}(M) = 11.9099 + 4.1249\sqrt{-1}$.

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Commensurabilty of knots and the Berge conjecture

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(joint work with Michel Boileau, Radu Cebanu, Genevieve Walsh)

We work in the oriented category. In particular we endow the complement of any knot $K \subset S^3$ with the orientation inherited from the standard orientation on S^3. We consider two knots to be equivalent if there is an orientation-preserving homeomorphism of S^3 taking one to the other. Covering maps will be assumed to preserve orientation unless stated otherwise.

Two oriented orbifolds are *commensurable* if they have homeomorphic finite sheeted covers. The *commensurability class* of an orbifold is its equivalence class under this relation. Two knots are commensurable if their complements are commensurable.

In what follows K will denote a hyperbolic knot. Any knot K' commensurable with K is also hyperbolic and by Schwartz [13], two hyperbolic knots are commensurable if and only if their fundamental groups are quasi-isometric. Set

$$\mathcal{C}(K) = \{\text{knots } K' \subset S^3 : K' \text{ is commensurable with } K\}.$$

Equivalently,

$$\mathcal{C}(K) = \{\text{knots } K' \subset S^3 : \pi_1(S^3 \backslash K') \text{ is quasi-isometric to } \pi_1(S^3 \backslash K)\}.$$

A difficult and widely open problem is to describe $\mathcal{C}(K)$. In particular, to determine upper bounds on its cardinality and to characterize those knots for which $|\mathcal{C}(K)| > 1$. Here we describe various results on these problems obtained in [3].

The arithmeticity of a hyperbolic 3-manifold is invariant under taking covers, so as the figure-8 is the only knot with arithmetic complement [11], it is the unique knot in its commensurability class. Hence we assume below that knots are non-arithmetic. A fundamental result of Margulis [8] then implies that there is an orbifold $\mathcal{O}_{min}(K)$ covered by each element of the commensurabilty class of $S^3 \backslash K$.

A *hidden symmetry* of K is an isometry between finite covers of $S^3 \backslash K$ which is not the lift of an isometry of $S^3 \backslash K$. Neumann and Reid [9] have shown that the following statements are equivalent:

- K has no hidden symmetries.
- the cover $S^3 \backslash K \to \mathcal{O}_{min}(K)$ is regular.
- $\mathcal{O}_{min}(K)$ has cusp cross-section a torus or the 2-orbifold $S^2(2,2,2,2)$.

Hyperbolic knots with hidden symmetries appear to be rare. For instance by [9], if K has hidden symmetries then the cusp shape of $\mathcal{O}_{min}(K)$ is contained in $\mathbb{Q}[i]$ or $\mathbb{Q}[\sqrt{-3}]$. Currently, the only knots known to admit hidden symmetries are the figure-8 and the two dodecahedral knots of Aitchison and Rubinstein [1, 9] and each of these has cusp field $\mathbb{Q}[\sqrt{-3}]$. There is one known example of a knot with cusp field $\mathbb{Q}[i]$ and it does not admit hidden symmetries. See Boyd's notes [4, page 17] and Goodman, Heard and Hodgson [7]. Walter Neumann has conjectured that the figure-8 and the two dodecahedral knots are the only knots which admit hidden symmetries.

Reid and Walsh have raised the following conjecture:

0.49. **Conjecture.** (Reid-Walsh [12]) *For a hyperbolic knot $K \subset S^3$, $|\mathcal{C}(K)| \leqslant 3$.*

We prove

0.50. **Theorem.** ([3]) $|\mathcal{C}(K)| \leqslant 3$ *if K is a hyperbolic knot without hidden symmetries.*

The proof of this theorem implies:

0.51. **Theorem.** ([3]) *Knots without hidden symmetries which are commensurable are cyclically commensurable.*

We also determine strong constraints on knots with $|\mathcal{C}(K)| \geqslant 2$.

0.52. **Theorem.** ([3]) *Let K be a hyperbolic knot without hidden symmetries. If $|\mathcal{C}(K)| \geqslant 2$ then:*

(1) *K is a fibred knot.*
(2) *The genus of K is the same as that of any $K' \in \mathcal{C}(K)$.*
(3) *K is not amphichiral.*

The two dodecahedral knots [1] do not satisfy any of the three conditions above: one is fibred, the other isn't; the knots have different genus; the knots are both amphichiral. This underlines the marked difference between the hidden symmetries and no hidden symmetries situations.

An *orbi-lens space* is an orbifold obtained as the quotient of S^3 by a finite cyclic subgroup of $SO(4)$. The fibredness of knots K without hidden symmetries for which $|\mathcal{C}(K)| \geqslant 2$ is a consequence of the following analogue of Ni's fundamental result [10] in the orbifold setting:

0.53. **Theorem.** ([3]) *Let K be a knot in an orbi-lens space $\mathcal{L}$ which is primitive in $|\mathcal{L}|$. If K admits a non-trivial orbi-lens space surgery, then the exterior of K admits a fibring by 2-orbifolds with base the circle.*

Next we turn to the problem of characterizing hyperbolic knots without hidden symmetries for which $|\mathcal{C}(K)| > 1$.

A *Berge-Gabai knot in a solid torus V* is a 1-bridge braid in V which admits a non-trivial cosmetic surgery slope. Such knots and their cosmetic surgeries have been classified [2], [6]. A *Berge-Gabai knot in an orbi-lens space $\mathcal{L}$* consists of a knot $\bar{K}$ and a genus one Heegaard splitting $|\mathcal{L}| = V_1 \cup V_2$ such that $\bar{K}$ is a Berge-Gabai knot in V_1 and the singular locus of $\mathcal{L}$ is a closed submanifold of the core of V_2. An *unwrapped Berge-Gabai knot in S^3* is the inverse image of a Berge-Gabai knot in an orbi-lens space $\mathcal{L}$ under the universal cover $S^3 \to \mathcal{L}$.

A knot K is *periodic* if it admits a non-free symmetry with an axis disjoint from K.

As a consequence of the works of Berge [2] and Gabai [5] we obtain the following characterization of commensurability classes of periodic knots without hidden symmetries:

0.54. **Theorem.** ([3]) *Let K be a periodic hyperbolic knot without hidden symmetries. If $|\mathcal{C}(K)| \geqslant 2$ then:*
(1) *K has a unique axis of symmetry disjoint from K.*
(2) *K is obtained by unwrapping a Berge-Gabai knot $\bar{K}$ in an orbi-lens space. In particular K is strongly invertible.*
(3) *each $K' \in \mathcal{C}(K)$ is determined by unwrapping the Berge-Gabai knot represented by the core of the surgery solid torus in an orbi-lens space obtained by Dehn surgery along $\bar{K}$.*

We also reduce the characterization of non-periodic hyperbolic knots K without hidden symmetries with $|\mathcal{C}(K)| > 1$ to the characterization of primitive hyperbolic knots in a lens space which admit a non-trivial lens space surgery and thus, to a generalization of the Berge problem.

The results described above immediately yield the following result on the quasi-isometry classes of hyperbolic knot complements:

0.55. **Theorem.** ([3]) *Let K be a hyperbolic knot without hidden symmetries. Then, up to mirror images, there are at most three knots K' with group $\pi_1(S^3\backslash K')$ quasi-isometric to $\pi_1(S^3\backslash K)$. Moreover $\pi_1(S^3\backslash K)$ is the unique knot group in its quasi-isometry class in the following cases:*
(i) K is not fibred.
(ii) K is amphichiral.
(iii) K is periodic and is not an unwrapped Berge-Gabai knot; for instance, K is periodic but not strongly invertible.
(iv) K is periodic with two distinct axes of symmetry. □

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The colored Jones polynomial of a knot and representations of its fundamental group

Hitoshi Murakami

Let $J_N(K;q)$ be the colored Jones polynomial of a knot K associated with the N-dimensional irreducible representation of the Lie algebra $sl(2;\mathbb{C})$. We normalize it so that $J_N(U;q)=1$ for the unknot U.

It was conjectured that the asymptotic behavior of $J_N(K;\exp(2\pi\sqrt{-1}/N))$ would determine the volume of the knot complement.

0.56. **Conjecture** (Volume Conjecture (R. Kashaev, J. Murakami, and the author))**.**

$$2\pi \lim_{N\to\infty} \frac{\log |J_N\bigl(K;\exp(2\pi\sqrt{-1}/N)\bigr)|}{N} = \mathrm{Vol}(S^3\backslash K),$$

where Vol *is the simplicial volume, which coincides with the hyperbolic volume if the knot complement possesses a* (*unique*) *complete hyperbolic structure with finite volume.*

In terms of asymptotics this means:

$$(1)\quad \begin{aligned} &J_N\bigl(K;\exp(2\pi\sqrt{-1}/N)\bigr) \\ &\underset{N\to\infty}{\sim} \exp\left(\frac{N}{2\pi}\times \mathrm{Vol}(S^3\backslash K)\right)\times (\text{function of } N \text{ with polynomial growth}). \end{aligned}$$

We would like to study the asymptotic behavior of $J_N\bigl(K;\exp(\theta/N)\bigr)$ for $N\to\infty$, where θ is a complex parameter.

In [2], S. Gukov and the author proposed the following generalization of (1) in the case where K is hyperbolic and θ is close to $2\pi\sqrt{-1}$ as follows.

$$(2)\quad \begin{aligned} &J_N\bigl(K;\exp(\theta/N)\bigr) \\ &\underset{N\to\infty}{\sim} \exp\left(\frac{N}{\theta}\times \mathrm{CS}(\theta)\right)\times\left(\frac{N}{\theta}\right)^{1/2}\times T(\theta)^{1/2}\times(1+\text{lower terms}), \end{aligned}$$

where

- θ defines an irreducible representation $\rho(\theta)\colon \pi_1(S^3\backslash K)\to SL(2;\mathbb{C})$ such that the image of the meridian has eigenvalues $\exp(\pm\theta/2)$,
- $\mathrm{CS}(\theta)$ is the $SL(2;\mathbb{C})$ Chern-Simons invariant associated with $\rho(\theta)$, which coincides with $\sqrt{-1}\,\mathrm{Vol}$ when $\theta = 2\pi\sqrt{-1}$,
- $T(\theta)$ is the Reidemeister torsion of the cochain complex of the universal cover of $S^3\backslash K$ with $sl(2;\mathbb{C})$ coefficients twisted by the adjoint representation of $\rho(\theta)$.

Note that since θ is close to $2\pi\sqrt{-1}$, it defines a small deformation of the unique complete hyperbolic structure of the knot complement.

0.57. **Remark.** *When $\theta = 2\pi\sqrt{-1}$, we need to modify the formula slightly. See* [1, Theorem 1]*, where J. Andersen and S. Hansen proved*

$$J_N\bigl(K;\exp(2\pi\sqrt{-1}/N)\bigr)\underset{N\to\infty}{\sim}\exp\left(\frac{N}{2\pi}\times\mathrm{Vol}(S^3\backslash K)\right)N^{3/2}3^{-1/4}.$$

Let $T(a,b)$ be the torus knot of type (a,b) for coprime integers $a>b>1$. Put

$$\begin{aligned} \tau(z) &:= \frac{2\sinh(z)}{\Delta(T(a,b);\exp(2z))}, \\ S_k(\theta) &:= \frac{-(2k\pi\sqrt{-1}-ab\theta)^2}{4ab}, \\ T_k &:= \frac{16\sin^2(k\pi/a)\sin^2(k\pi/b)}{ab}, \end{aligned}$$

and

$$A_k(\theta;N) := \sqrt{-\pi}\exp\left(\frac{N}{\theta}\times S_k(\theta)\right)\left(\frac{N}{\theta}\right)^{1/2}(T_k)^{1/2},$$

where $\Delta(T(a,b);t)$ is the normalized Alexander polynomial. Note that $\tau(z)$ can be regarded as the abelian Reidemeister torsion, that $S_k(\theta)$ is the Chern–Simons invariant in the sense of [4] associated with the irreducible representation $\rho_k(\theta)$ of $\pi_1(S^3\backslash T(a,b))$ to $SL(2;\mathbb{C})$ parametrized by k (and the image of the meridian has eigenvalues $\exp(\pm\theta/2)$), and that T_k is the twisted Reidemeister torsion associated with $\rho_k(\theta)$. Note also that k parametrizes the irreducible components of the character variety of $S^3\backslash T(a,b)$ and that $\tau(z)$ corresponds to its abelian component.

In a joint work with K. Hikami [3], we obtain the following asymptotic expansions.

0.58. **Theorem.** *Suppose that* $\theta/2 \notin \left\{\frac{k\pi\sqrt{-1}}{ab} \mid k\in\mathbb{Z}, a\nmid k, b\nmid k\right\}$. *Then we have*

$$\begin{aligned} &J_N(T(a,b);\exp(\theta/N)) \\ &\underset{N\to\infty}{\sim} \frac{\exp\left(ab-a/b-b/a\right)\frac{\theta}{4N}}{2\sinh(\theta/2)}\left(\tau(\theta/2)+\sum_{j=1}^{\infty}\frac{\tau^{(2j)}(\theta/2)}{j!}\left(\frac{j}{4abN}\right)^j\right)\end{aligned}$$

if $\Re(\theta)>0$, *and*

$$\begin{aligned} &J_N(T(a,b);\exp(\theta/N)) \\ &\underset{N\to\infty}{\sim} \frac{\exp\left(ab-a/b-b/a\right)\frac{\theta}{4N}}{2\sinh(\theta/2)} \\ &\quad\times\left(\tau(\theta/2)+\sum_{k=1}^{\lfloor ab|\theta|/(2\pi)\rfloor}(-1)^{k+1}A_k(\theta;N)+\sum_{j=1}^{\infty}\frac{\tau^{(2j)}(\theta/2)}{j!}\left(\frac{j}{4abN}\right)^j\right)\end{aligned}$$

if $\Re(\theta)\leqslant 0$.

This theorem says that the colored Jones polynomial of a torus knot can be decomposed into the summation each of whose summand is described in terms of the $SL(2;\mathbb{C})$ Chern–Simons invariant and the Reidemeister torsion associated with a representation.

We expect a similar asymptotic expansion for a generic knot. Then (2) (and the Volume Conjecture) would follow when one looks only at the summand that has the "largest" Chern–Simons invariant.

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The secret algebraic lives of (hyperbolic) 3-manifolds

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(joint work with Shelly Harvey, Constance Leidy)

We consider two different but related questions. Firstly, what properties of 3-manifold groups are invariant modulo perfect subgroups and what invariants detect these properties? Secondly, what is the classification of closed oriented 3-manifolds modulo homology cobordism? The classification of classical knot and link concordance is a special case of the latter.

Specifically, suppose $\mathcal{P}$ is a property enjoyed by certain (homeomorphism classes of) closed oriented 3-manifolds. We say that a given closed oriented 3-manifold M is **secretly** $\mathcal{P}$ if it does not have property $\mathcal{P}$, but there exists some closed oriented 3-manifold N with property $\mathcal{P}$ and a degree one map $f : M \to N$ for which the kernel of

$$f_* : \pi_1(M) \to \pi_1(N)$$

is a perfect group. Note that this forces M and N not only to have the same integral homology, but to have isomorphic homology with $\pi_1(N)$ coefficients. It also forces M and N to not be homeomorphic. We consider such questions as:

1. What hyperbolic 3-manifolds are secretly Seifert-fibered? (or fibered? (or have some particular JSJ decomposition?)
2. What 3-manifolds secretly have torsion in their fundamental groups?

It follows from work of A. Kawauchi that, for any $\mathcal{P}$, there is some hyperbolic 3-manifold that (either has $\mathcal{P}$ or) is secretly $\mathcal{P}$. Our work, which is in a preliminary stage, applies to give invariants showing, for example, that certain Haken hyperbolic 3-manifolds are not secretly Seifert fibered.

Associated to any 3-manifold is a family of *higher-order Alexander modules*, namely the successive quotients, $G^{(n)}/G^{(n+1)}$, of the terms of the (rational) derived series of its fundamental group G. These are modules over the group rings of certain torsion-free solvable groups, $G/G^{(n)}$ (G acts by conjugation) [1, 4]. The isomorphism type of these modules is unchanged if G is taken modulo a normal perfect subgroup, hence these give many potential invariants that can answer questions about secret properties. These are modules over rings that are almost always noncommutative, non-Noetherian and non-UFD, but are Ore domains. This makes it challenging to extract information from them. Despite this we describe a way to talk about certain $p(t_1, ..., t_m)$-torsion in these modules (where p is an ordinary polynomial). We define localization at such a polynomial. In doing so, certain aspects of transcendental number theory and algebraic geometry arise naturally. These notions are used, in conjunction with certain $\ell^{(2)}$-signatures, to distinguish among 3-manifolds (even up to homology cobordism) having "different" types of torsion in their higher-order modules. A huge range of subtle phenomena is realized by "satellite operations" (elementary techniques of torus decomposition). Furthermore, since by Kawauchi's result, for any closed 3-manifold group there are closed *hyperbolic* 3-manifolds which secretly have that fundamental group, all of this algebraic complexity is present even when restricting to the class of hyperbolic 3-manifolds. So far extensive applications have been made to knots (essentially just 3-manifolds with $\beta_1 = 1$) and knot concordance (homology cobordism of 3-manifolds with $\beta_1 = 1$) [2, 3].

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