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Komplexe Analysis

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Abstract. The aim of this workshop was to discuss recent developments in several complex variables and complex geometry. Special emphasis was put on the interaction between model theory and the classification theory of complex manifolds. Other topics included Kähler geometry, foliations, complex symplectic manifolds and moduli theory.

Mathematics Subject Classification (2000): 32xx, 14xx.

Introduction by the Organisers

The meeting Komplexe Analysis attracted 52 mathematicians from 11 countries. It was the aim of the conference to cover a wide spectrum, thus enabling in particular the younger mathematicians to get an overview of the most recent important developments in the subject. Apart from research-oriented talks the organizers chose to put special emphasis on the subject of "Model Theory and Complex Analysis". For this purpose three mathematicians mostly working in logic, namely K. Tent, A. Pillay, and B. Zilber, were invited to give two talks each. They gave an introduction to model theory and its applications to complex analysis, in particular highlighting the perspectives opened by this theory in the study of complex manifolds of the Fujiki class $\mathcal{C}$, as well as the relationships with certain questions in differential algebra or arithmetic geometry (fields of definition, cycles, Schanuel's conjecture, etc.). These lectures stimulated a lively discussion.

Several lectures were concerned with recent developments in the study of Kähler geometry and Monge–Ampère equations, in relation with Donaldson's program or the existence of Kähler–Einstein metrics on Fano manifolds. Eyssidieux explained his recent results on the continuity of solutions of degenerate complex Monge–Ampère equations, based on viscosity techniques. Keller gave an example showing that balanced metrics may not converge towards constant scalar curvature Kähler metrics, thus considerably refining the depiction of the precise Tian–Donaldson stability conditions which would be needed to warrant the existence of CSCK metrics. Sano presented new results on the calculation of multiplier ideal sheaves which appear as obstructions to the existence of Kähler–Einstein metrics on toric Fano varieties. Schumacher computed the curvature of the higher direct images of relative canonical bundles for deformations of canonically polarized projective varieties, and derived some consequences towards the Shafarevich hyperbolicity conjecture for the corresponding moduli spaces.

There were also other talks, analytic in nature, e.g. in the direction of singularities. Sibony explained his work in complex dynamics, especially his recent results with Dinh and Nguyen on laminations by Riemann surfaces: using heat equation and harmonic current techniques, one can obtain general geometric ergodicity theorems for compact laminations with isolated singularities. Barlet lectured on "themes" of vanishing periods for an isolated singularity, providing new tools to compute monodromy invariants via his theory of (a, b)-modules. Mustaţă presented new deep results in collaboration with M. Jonsson on valuations of function fields and asymptotic invariants of singularities, in relation with the openness conjecture of Demailly-Kollár for singularities of plurisubharmonic functions. Finally Greb considered actions of complex reductive Lie groups on Stein manifolds and reported on joint work with Miebach, establishing a meromorphic quotient in this setup with a very detailed description of the quotient map.

Irreducible symplectic manifolds appeared in two talks: Verbitsky outlined his recent important work on the global Torelli theorem for these manifolds, while Markman presented a proof (joint with Charles) of the standard conjectures for irreducible symplectic manifolds which are deformation equivalent to Hilbert schemes on $K3$ surfaces. Rigidity theorems for Fano manifolds were the topic of Hwang's talk. Several talks discussed manifolds with trivial canonical bundle. Prendergast-Smith discussed recent results on the movable cone conjecture for Calabi–Yau manifolds, Halle presented reported on joint work with Nicaise on degenerations of Calabi–Yau manifolds and the motivic monodromy conjecture, and Mukai discussed his recent work on (semi-)symplectic automorphisms of Enriques surfaces and the connection to Mathieu groups. Moduli spaces were the topic of three talks. Van der Geer explained his calculations with Kouvidakis of the class of a certain geometrically interesting divisor on the moduli space of stable curves of even genus, Bauer discussed moduli of Burniat surfaces (joint work with Catanese) and Grushevsky presented results (with Hulek) on the cycle of intermediate Jacobians of cubic threefolds in the moduli space of abelian varieties.

On Tuesday evening young participants (shortly before or after their Ph.D.) were invited to give short presentations of their work. Such presentations were given by M. Gulbrandsen, S. Krug, P. Larsen, and F. Schrack. This session was well attended by the senior participants and led to new contact between younger participants and senior researchers.

Workshop: Komplexe Analysis

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Abstracts

Analysis on laminations by Riemann Surfaces

Nessim Sibony
(joint work with T.-C. Dinh, V.-A. Nguyen)

Consider the polynomial differential equation in $\mathbb{C}^2$

$$\frac{dz}{dt} = P(z,w), \qquad \frac{dw}{dt} = Q(z,w).$$

The polynomials P and Q are holomorphic, the time is complex. We want to study the global behavior of the solutions. It is convenient to consider the extension as a foliation in the projective plane $\mathbb{P}^2$. There are however singular points. When the line at infinity is invariant, Il'yashenko has shown that generically leaves are dense and that the foliation is ergodic. This follows from the study of the holonomy on the invariant line. But generically on the vector field, there is no invariant line and even no invariant algebraic surface as shown by Jouanolou. This example is a special case of a lamination (with singularities) by Riemann surfaces. In particular, one can consider similar questions in any number of dimensions.

In order to understand their dynamics we will need some analysis on such objects.

We will discuss the following topics.

1. Singularities of holomorphic foliations, linearization problems. Algebraic solutions. Examples.

2. Holonomy, closed directed currents, harmonic currents ($\partial\overline{\partial}$ -closed currents) directed by the lamination.

3. Heat equation on a lamination. The directed $\partial\overline{\partial}$ -closed current replaces the manifold and we solve the heat equation with respect to the current. This is useful because in presence of singularities, the leaves are not of bounded geometry. We can develop a Hilbert space theory with respect to that equation using Lax-Milgram and Hille-Yosida theorems. More precisely, given u_0 in the domain $\mathrm{Dom}(\Delta)$ of Δ, we solve

$$\frac{du}{dt} = \Delta u \quad \text{and} \quad u(0,\cdot) = u_0$$

with $u(t,\cdot) \in \mathrm{Dom}(\Delta)$. The theory is sufficient to get an ergodic theorem for that diffusion. So, we rather get the heat equation in the space $(M, \mathcal{F}, m)$. The Laplacians are not necessarily symmetric operators in $L^2(m)$ and the natural ones depend on m.

4. Geometric ergodic theorems. In the second part, for compact Riemann surface laminations with singularities we get an ergodic theorem with more geometric

flavor than the ones associated to a diffusion. Let $(X, \mathcal{L}, E)$ be a lamination by Riemann surfaces. Assume for simplicity that the singularity set E of $\mathcal{L}$ is a finite set of points Then every hyperbolic leaf L is covered by the unit disc Δ. Let $\phi_a : \Delta \to L_a$ denote a universal covering map of the leaf L_a passing through a with $\phi_a(0) = a$. We consider the associated measure

$$m_{a,R} := \frac{1}{M_R}(\phi_a)_*\big(\log^+ \frac{r}{|\zeta|}\omega_P\big) \quad \text{with} \quad R := \log\frac{1+r}{1-r}$$

which is obtained by averaging until "hyperbolic time" R along the leaves. Here, M_R is a constant to normalize the mass. Recall that ω_P denotes the Poincaré metric on Δ and also on the leaves of X.

Theorem 1 (Dinh-Nguyen-Sibony [1]). *Let $(X, \mathcal{L}, E)$ be a compact lamination with isolated singularities in a complex manifold M and ω_P the Poincaré metric on the leaves. Let T be an extremal positive harmonic current of Poincaré mass 1 on $(X, \mathcal{L}, E)$ without mass on the union of parabolic leaves. Then for almost every point $a \in X$ with respect to the measure $m_P := T \wedge \omega_P$, the measure $m_{a,R}$ defined above converges to m_P when $R \to \infty$.*

For holomorphic foliations in in the projective plane $\mathbb{P}^2$, with only hyperbolic singularities and without algebraic leaves (this is the generic case) it was shown by J.-E. Fornaess and the author that one has unic ergodicity ,i.e., there is a unique $\partial\overline{\partial}$-closed current, of mass one, directed by the foliation. So the weighted averages considered above have a unique limit.

The new results are based on recent joint work with T.-C. Dinh and V.-A. Nguyen. We give below some references.

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Geometric Model Theory, I and II

Katrin Tent

These talks were meant as an introduction to model theory in order to explain some applications of model theory to compact complex spaces. Starting from the basics, a structure on a universe M is given by a family $\mathrm{Def}_0(M)$ of subsets of $M^n, n \in \mathbb{N}$, containing diagonals $\Delta_k = \{(a,a), a \in M^k\}$ for all k, and closed under finite unions, intersections, complements (these are the *Boolean operations*), projections onto coordinates and Cartesian products. Sets in $\mathrm{Def}_0(M)$ are called 0-definable. We obtain $\mathrm{Def}_M(M)$ by also allowing fibres of 0-definable sets $X_c = \{b\colon (bc) \in X\}$ for $b \in M^k$, where X_b is said to be definable over the *parameter* b.

Examples: (i) If M is a reduced irreducible compact complex analytic space, we may start from the analytic subsets of $M^n, n \in \mathbb{N}$ and close under complements, intersections, unions and products. By an observation of Zilber, this is already closed under projections. Since elements of M^n are analytic subsets themselves, we obtain nothing new by allowing fibres. So we have in this case $\mathrm{Def}_0(M) = \mathrm{Def}_M(M)$.

(ii) For $M = \mathbb{C}$, we can take as $\mathrm{Def}_0(\mathbb{C})$ the constructible sets over the rationals. Again by results of Tarski and Chevalley, this is closed under projections. In this case $\mathrm{Def}_{\mathbb{C}}(\mathbb{C})$ consists of the subsets of $\mathbb{C}^n$ constructible over $\mathbb{C}$.

The second example shows a certain functoriality: for any field F containing $\mathbb{C}$, the sets $\mathrm{Def}_0(F), \mathrm{Def}_{\mathbb{C}}(F)$ still make sense by evaluating the polynomials used in the construction of a constructible set in F.

This functoriality is more easily captured by talking about a *language* (or signature) for the structure M. In many cases, there is a natural choice of language L associated to the structure M:

Examples: (i) $L_{gp} = \{\circ, e\}$ language for groups;
(ii) $L_{fields} = \{+, \cdot, 0, 1\}$ language for fields;
(iii) for a given compact complex manifold M we may take the language $L_{ccm}(M)$ to consist of predicates P_X for each analytic subset X of M^n.

L-formulas are constructed in this language with respect to the obvious syntax rules using the connectives $\neg, \wedge, \vee, =$ (corresponding to the Boolean operations) and quantifiers $\exists x, \forall x$ ranging over elements of M.

We write $\phi(x_1, \ldots x_n)$ for an L-formula in which at most $x_1, \ldots x_n$ appear unquantified. An L-sentence is an L-formula without unquantified variables. To say that M is an L-structure means that the symbols of L have *some* meaning or interpretation on the set M (respecting the arity etc.). If M is an L-structure and ϕ is an L-sentences, it makes sense to say that ϕ holds in M, $M \models \phi$. If $\phi(x_1, \ldots x_n)$ is an L-formula, this makes sense after naming elements in place of variables, so for $b = (b_1, \ldots b_n) \in M^n$ it makes sense to write $M \models \phi(b)$.

We now take as $\mathrm{Def}_0(M)$ all sets of the form

$$\phi(M) = \{b \in M^n \colon M \models \phi(b)\}.$$

So $\mathrm{Def}_M(M)$ consists of all sets of the form

$$\phi(M, c) = \{b \in M^n \colon M \models \phi(b, c)\}$$

for $c \in M^k$.

In this way we identify a formula with its set of realisations in a given structure. Note that under this interpretation the existential quantifier translates into a projection. We say that a structure M has *quantifier elimination* (QE) if every definable set can be defined by a formula without quantifiers.

An L-theory is just a set of L-sentences. In particular, for any L-structure M, we define $Th(M)$ to be the set of L-sentences ϕ such that $M \models \phi$. For any compact complex manifold M we let $\mathfrak{A}(M)$ denote the theory of M in the language $L_{ccm}(M)$ defined above.

A very important fact, which lies at the basis of all of model theory, is the following:

Compactness Theorem. *A set T of L-sentences has a model M (i.e. $M \models \phi$ for all $\phi \in T$) if every finite subset $T_0 \subset T$ has a model.*

Definition 1. *Let κ be a cardinal.*
(i) We say that M is κ-compact if for any family $(X_i : i \in I), |I| < \kappa$ having the finite intersection property with $X_i \in \mathrm{Def}_M(M)$ we have $\bigcap_{i\in I} X_i \neq \emptyset$.
(ii) M is κ-saturated if for all $A \subseteq M, |A| < \kappa, (X_i : i \in I), X_i \in \mathrm{Def}_A(M)$ with the finite intersection property we have $\bigcap_{i\in I} X_i \neq \emptyset$.

By work of Zilber and Lojasewicz, a compact complex manifold M is $\aleph_1$-compact. Note however, that it is not even 0-saturated: namely, let $X_a = M \setminus \{a\}, a \in M$. Then X_a is definable without parameters (since a is itself an analytic subset) and the family $(X_a; a \in M)$ certainly has the finite intersection property. However, clearly $\bigcap_{a\in M} X_a = \emptyset$.

Clearly, every κ-saturated structure is κ-compact. Using elementary considerations it is not hard to see that for $\kappa > |L|$ the converse holds as well.

If $\bar{M}$ is a 'very saturated' model of the theory T, $A \subseteq M \subset \bar{M}, a \in \bar{M}$, then the *type* of a over A is defined as

$$tp(a/A) = \{X \in \mathrm{Def}_A(\bar{M})\colon a \in X\}.$$

Note that if M is κ-saturated, then for every type $tp(a/A), |A| < \kappa$, there is some element $b \in M$ in the intersection of the definable sets in $tp(a/A)$.

In what follows assume that $\bar{M}$ is very saturated. The *Morley rank* is a model theoretic notion of dimension which turns out to be very useful in the context of compact complex manifolds. The main part of the definition is that for a definable set $X \subset \bar{M}$ we have $RM(X) \geq 0$ if $X \neq \emptyset$ and $RM(X) \geq \alpha + 1$ if there are disjoint definable subsets $X_0, X_1, \ldots, i \in \mathbb{N}$ with $RM(X_i) \geq \alpha$ (where α is a not necessarily finite ordinal). If the Morley rank of all definable sets is an ordinal number (rather than ∞), the theory is called *totally transcendental.* In compact complex manifolds, one has $RM(X) \leq dim(X)$. It can be shown that in algebraically closed fields, the Morley rank of a definable set X equals the dimension of the Zariski closure of X.

Definition 2. *Assume that X is a definable set over parameters A in a totally transcendental theory. Then $a \in X$ is called* generic *over A if $a \notin Y$ for any A-definable set Y with $RM(Y) < RM(X)$.*

Note that if we choose a countable language for a compact complex manifold as we may do for Kähler manifolds, then by $\aleph_1$-compactness and saturation, every definable set contains generic points over any countable set, so a point which is not contained in any analytic subset of smaller dimension. The properties of such a generic point will then automatically hold of a dense subset of the manifold.

The main point of this definition is that we obtain an inductive setting, hoping to understand the sets of Morley rank 1 and to be able to analyse sets of higher rank in terms of rank 1 sets: a definable set in a totally transcendental theory is called *strongly minimal* if every definable subset of X is either finite or its complement is finite. Any strongly minimal set carries a combinatorial (pre-)geometry:

Suppose that X is strongly minimal, $A \subseteq X$. Let

$$\mathrm{acl}(A) = \{b\colon \text{ there is an } A\text{-definable finite set } Z \subseteq X \text{ containing } b\}.$$

Then acl satisfies the *exchange property*, i.e. if $a \in \mathrm{acl}(Ab) \setminus \mathrm{acl}(A)$ then $b \in \mathrm{acl}(Aa)$. This allows us to define a dimension of the associated matroid.

We call a strongly minimal set X *modular* if for any $A, B \subset X$ we have

$$dim(A) = dim(B) = dim(A \cup B) - dim(A \cap B)$$

and locally modular if the same is true after naming a point. A strongly minimal set is said to be *trivial* if for $a \in \mathrm{acl}(b_1, \dots b_n)$ there is some $i \leq n$ with $a \in \mathrm{acl}(b_i)$. Certainly, in a trivial strongly minimal set there could not be a definable group since the group law directly contradicts triviality.

Zilber conjectured that any non-locally modular strongly minimal set X is *almost internal* to an algebraically closed fields, meaning that there should be a definable algebraically closed field K and a definable relation $R \subseteq K^n \times X$ which is many-to-finite and onto.

This conjecture was disproved by Hrushovski in the general context. However, in the context of compact complex manifolds it was proved to hold by Hrushovski and Zilber:

Theorem 3. *If M is a compact complex manifold, then any non locally modular strongly minimal set is a projective curve.*

On the other hand, any locally modular strongly minimal set which is not trivial 'contains' an infinite definable group.

Using a structure theorem by Pillay and Scanlon on meromorphic groups, these results then yield the following trichotomy:

Theorem 4 (Hrushovski–Zilber, Pillay–Scanlon)**.** *Any simple compact complex manifold is either*
(i) a smooth projective algebraic curve (so non-modular);
(ii) essentially (i.e. up to generically finite-to-finite correspondence) a generic complex torus;
(iii) or 'geometrically trivial'.

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Global Torelli theorem for hyperkähler manifolds

Misha Verbitsky

A mapping class group of an oriented manifold is a quotient of its diffeomorphism group by the isotopies. We compute a mapping class group of a hypekähler manifold M, showing that it is commensurable to an arithmetic lattice in $SO(3, b_2 - 3)$. A Teichmüller space of M is a space of complex structures on M up to isotopies. We define **a birational Teichmüller space** by identifying certain points corresponding to bimeromorphically equivalent manifolds. We show that the period map gives the isomorphism between connected components of the birational Teichmüller space and the corresponding period space $SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$. We use this result to obtain a Torelli theorem identifying each connected component of the birational moduli space with a quotient of a period space by an arithmetic group. When M is a Hilbert scheme of n points on a K3 surface, with $n - 1$ a prime power, our Torelli theorem implies the usual Hodge-theoretic birational Torelli theorem (for other examples of hyperkähler manifolds, the Hodge-theoretic Torelli theorem is known to be false).

A hyperkähler manifold is a compact, holomorphically symplectic manifold of Kähler type, simply connected and with $H^{2,0}(M) = \mathbb{C}$. We shall say that a complex structure I on M is **of hyperkähler type** if (M, I) is a hyperkähler manifold.

Let (M, I) be a compact hyperkähler manifold, $\mathfrak{I}$ the set of oriented complex structures of hyperkähler type on M, and $\operatorname{Diff}_0(M)$ the group of isotopies. The quotient space $\operatorname{Teich} := \mathfrak{I}/\operatorname{Diff}_0(M)$ is called **the Teichmüller space** of (M, I), and the quotient of Teich over a whole oriented diffeomorphism group **the coarse moduli space of** (M, I).

Let (M, I) be a simple hyperkähler manifold, and Teich its Teichmüller space. For any $J \in \operatorname{Teich}$, (M, J) is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional. Consider a map $\operatorname{Per} : \operatorname{Teich} \longrightarrow \mathbb{P}H^2(M, \mathbb{C})$, sending J to a line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. Clearly, Per maps Teich into an open subset of a quadric, defined by

$$\mathbb{P}\mathrm{er} := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \ \mid \ q(l, l) = 0, q(l, \bar{l}) > 0\}.$$

The map $\operatorname{Per} : \operatorname{Teich} \longrightarrow \mathbb{P}\mathrm{er}$ is called **the period map**, and the set $\mathbb{P}\mathrm{er}$ **the period space**.

Theorem 1 (Bogomolov)**.** *Let M be a simple hyperkähler manifold, and* Teich *its Teichmüller space. Then the period map* $\operatorname{Per} : \operatorname{Teich} \longrightarrow \mathbb{P}\mathrm{er}$ *is locally a unramified covering (that is, an etale map).*

Bogomolov's theorem implies that Teich is smooth. However, it is not necessarily Hausdorff (and it is non-Hausdorff even in the simplest examples).

In many cases, the moduli of complex structures on M can be described in terms of Hodge structures on cohomology of M. Such results are called *Torelli theorems.* In this note, we state a Torelli theorem for hyperkähler manifolds, using the language of mapping class group and Teichmüller spaces. We describe the mapping class group and, separately, the Teichmüller space.

Let Ω be a holomorphic symplectic form on M. Bogomolov and Beauville defined the following bilinear symmetric 2-form on $H^2(M)$:

$$\tilde{q}(\eta,\eta') := \int_M \eta\wedge\eta'\wedge\Omega^{n-1}\wedge\bar{\Omega}^{n-1} - \\ - \frac{n-2}{n}\frac{\left(\int_M \eta\wedge\Omega^{n-1}\wedge\bar{\Omega}^{n}\right)\left(\int_M \eta'\wedge\Omega^{n}\wedge\bar{\Omega}^{n-1}\right)}{\int_M \Omega^n\wedge\bar{\Omega}^n}$$

where $n = \dim_{\mathbb{H}} M$.

Theorem 2 ([V])**.** *Let M be a compact, simple hyperkähler manifold, and* $\Gamma = \operatorname{Diff}/\operatorname{Diff}_0$ *its mapping class group. Then Γ acts on $H^2(M,\mathbb{R})$ preserving the Bogomolov–Beauville–Fujiki form. Moreover, the corresponding homomorphism $\Gamma \longrightarrow O(H^2(M,\mathbb{Z}),q)$ has finite kernel, and its image has finite index in the group $O(H^2(M,\mathbb{Z}),q)$.*

Theorem 3 ([V])**.** *Let M be a compact, simple hyperkähler manifold, and* Teich *its Teichmüller space. Consider the* **birational Teichmüller space** Teich_b *obtained from* Teich *by gluing non-Hausdorff points. Then the period map* Per : $\text{Teich}_b \longrightarrow \mathbb{P}\text{er}$ *induces a homeomorphism for each connected component of* Teich_b.

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Degenerations of CY-varieties, and the motivic monodromy conjecture

Lars Halvard Halle

(joint work with Johannes Nicaise)

Let K be a complete discretely valued field with ring of integers R and algebraically closed residue field k of characteristic p. We say that X is a *CY*-variety if X is a smooth proper geometrically connected K-variety with trivial canonical sheaf.

In a joint project with Johannes Nicaise (K.U. Leuven) we introduce and study the so called *motivic zeta function* $Z_X(T)$ of a CY-variety X. This is a formal power series in T with coefficients in $\mathcal{M}_k = K_0(Var_k)[\mathbb{L}^{-1}]$ which, loosely stated, measures how degenerations of X vary under ramified extensions of K. Here $K_0(Var_k)$ denotes the Grothendieck ring of k-varieties and $\mathbb{L}$ is the class of the affine line. It is defined via the motivic integration theorey of Loeser and Sebag [6], which associates to any smooth K-variety X and any gauge form Ω a motivic integral $\int_X |\Omega| \in \mathcal{M}_k$. These motivic zeta functions can be seen as global analogues of the motivic zeta functions associated to complex hypersurface singularities by Denef and Loeser (cf. [3]).

Motivic generating series Let $\omega \in \Omega^g_{X/K}(X)$ be a *gauge form*, i.e., a nowhere vanishing differential form of maximal degree on X. In [7] Nicaise and Sebag associated a motivic generating series $S(X,\omega;T)$ to the pair (X,ω). In order to state their construction, we first need some notation. For any $d \in \mathbb{N}' := \{n \in \mathbb{N} | (n,p) = 1\}$ we write $X(d) = X \times_K K(d)$ and denote by $\omega(d)$ the pullback of ω to $X(d)$.

Definition: *The motivic generating series of* (X,ω) *is the series*

$$S(X,\omega;T) = \sum_{d \in \mathbb{N}'} (\int_{X(d)} |\omega(d)|) T^d \in \mathcal{M}_k[[T]].$$

We would like to say a few words about how to interpret the integral in the above definition. One can always find a *weak Néron model* for $X(d)$, by which we mean a smooth $R(d)$-scheme $\mathcal{Y}(d)$ with $\mathcal{Y}(d) \times_{R(d)} K(d) \cong X(d)$ such that

$$\mathcal{Y}(d)(R(d)) \cong X(d)(K(d)).$$

Loeser and Sebag showed in [6] that

$$\int_{X(d)} |\omega(d)| = \mathbb{L}^{-g} \cdot \sum_{C \in \pi_0(\mathcal{Y}(d)_s)} [C] \cdot \mathbb{L}^{-ord_C \omega(d)} \in \mathcal{M}_k.$$

It should be noted that this integral does not depend on the choice of weak Néron model for $X(d)$. One should also remark that Loeser and Sebag formulated their results in terms of formal models of rigid $K(d)$-varieties, but this level of generality is not necessary for this report.

In general little is known about this series, but when $char(k) = 0$ Nicaise and Sebag could show in [7] that $S(X,\omega;T)$ is rational for any gauge form ω on X. In fact, they showed that by choosing a sncd-model $\mathcal{X}/R$ for X with special fiber $\mathcal{X}_s = \sum_{i \in I} N_i E_i$ and writing $\mu_i := ord_{E_i}(\omega)$ then

$$S(X,\omega;T) = P(T)/Q(T)$$

where P and Q are polynomials in T depending on the integers N_i, μ_i and some geometrical data associated to $\mathcal{X}_s$. It follows from their presentation that the poles of $S(X,\omega;\mathbb{L}^{-s})$ (with s a formal parameter) all are on the form $s = -\mu_j/N_j$ with $j \in J \subset I$. It is in general quite subtle to determine $J \subset I$, since this description is independent of the choice of sncd-model.

Motivic zeta functions From now on we assume that $X(K) \neq \emptyset$. In that case there exists a *distinguished* gauge form ω, by which we mean a gauge form s.t.

$$min\{ord_C \omega | C \in \pi_0(\mathcal{Y}_s)\} = 0$$

for any weak Néron model $\mathcal{Y}$ of X. In [4] we made the following definition:

Definition: *Let ω be a distinguished gauge form of the CY-variety X/K. The motivic generating series*

$$Z_X(T) = \mathbb{L}^g \cdot S(X, \omega; T)$$

is called the motivic zeta function of X.

It can be seen that the definition of $Z_X(T)$ is independent of the choice of distinguished gauge form, and hence is an invariant for X. We have studied these zeta functions in the papers [4] and [5], with particular emphasis on abelian varieties. Other types of CY-varieties as well as more general aspects will be discussed in upcoming work.

In analogy with Denef and Loeser's motivic monodromy conjecture for local zeta functions, we formulated in [4] the following conjecture:

Conjecture (The global monodromy conjecture): *Let X/K be a CY-variety, and fix a topological generator σ of the tame monodromy group $Gal(K^t/K)$. Then there exists a finite subset $\mathcal{S}$ of $\mathbb{Z} \times \mathbb{Z}_{>0}$ such that*

$$Z_X(T) \in \mathcal{M}_k \left[T, \frac{1}{1 - \mathbb{L}^a T^b} \right]_{(a,b) \in \mathcal{S}}$$

and such that for each $(a, b) \in \mathcal{S}$ the cyclotomic polynomial $\Phi_{\tau(a/b)}(t)$ divides the characteristic polynomial of σ on $H^i(X \times_K K^{sep}, \mathbb{Q}_\ell)$ for some $i \in \mathbb{Z}_{\geq 0}$.

Here $\tau(a/b)$ denotes the order of a/b in the group $\mathbb{Q}/\mathbb{Z}$ and K^t is the tame closure of K in K^{sep}. Our main piece of evidence comes from our study of abelian varieties, in which case we have established the following theorem:

Theorem 1. *Let A/K be a tamely ramified abelian variety. Then the global monodromy conjecture holds for A. Moreover, $Z_A(\mathbb{L}^{-s})$ has a unique pole of order $t_{pot}(A) + 1$ at $s = c(A)$.*

Here $t_{pot}(A)$ denotes the potential toric rank of A, and $c(A)$ denotes Chai's *base change conductor* of A (cf. [1]). We refer to [4] and [5] for a complete discussion of the case of abelian varieties.

In light of Theorem 1, it is natural to hope that the Global Monodromy Conjecture has an affirmative answer also for more general varieties X/K with $K_X = 0$. The proof of Theorem 1 relies heavily on the machinery of Néron models, which works also in positive characteristic. In the general case, we must probably restrict ourselves to the case $k = \mathbb{C}$.

In fortunate situations, there exist "distinguished" sncd-models. This is for instance the case if X is a surface with trivial canonical sheaf allowing a triple-point-free degeneration. In [2], Crauder and Morrison produced normal forms for such degenerations, and classified the possible degenerate fibers. In an ongoing project, we use the classification in [2] with the aim to prove the monodromy conjecture for surfaces allowing triple-point-free degenerations.

Final remarks The motivic zeta functions mentioned in this note are invariants for algebraic varieties with trivial canonical sheaf, and are related to degenerations of such objects. In particular, we have a huge load of "global" results and machinery at at our disposal. Therefore, we believe that the global motivic monodromy conjecture should be more accessible than its local counterpart, and we hope that insight into the global case will have applications also for local zeta functions.

Motivic integration seems to be a very useful tool for studying degenerations, and in our work so far we have found that it provides a bridge between arithmetic and geometric properties of abelian varieties. It will be interesting to see whether this picture holds in general.

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Viscosity solutions of complex Monge-Ampère equations

Philippe Eyssidieux

(joint work with V. Guedj, A. Zeriahi)

The talk is a report on a joint work with V. Guedj and A. Zeriahi [4].

In the last 4 years, there has been some interest on weak solutions of degenerate complex Monge-Ampère equations on manifolds initiated by the authors and independantly G. Tian and Z. Zhang.

Let X be a n-dimensionnal compact connected Kähler manifold and ω be a semi-positive closed $(1,1)$-form with $\int_X \omega^n = 1$. Denote by $PSH(X,\omega)$ the set of locally L^1 usc functions with values in $[-\infty,+\infty[$ such that $\omega + i\partial\bar{\partial}\phi \geq 0$ in the sense of currents.

Let $v \geq 0$ be a probability measure on X having a $L^p, p > 1$ density. Let $\epsilon \geq 0$ be a real parameter. In [3], it is proved that the equation:

$$(\omega + i\partial\bar{\partial}\phi)^n = e^{\epsilon\phi} v$$

has a unique weak solution $\phi \in PSH(X,\omega) \cap L^\infty$ with $\int_X \phi v = 0$. This weak solution is understood in the sense of pluripotential theory the Monge-Ampère measure being defined as in [1].

In the work under report, we developped an alternative approach to the basics of the theory based on the clasical viscosity solutions to degenerate elliptic PDE.

We define viscosity sub/supersolutions of the above Monge-Ampère equations, when v has a continuous density. A subsolution is an usc function such that a C^2-smooth supertest function q at some point should satisfy $(\omega + i\partial\bar{\partial}q)^n \geq e^{\epsilon q}v$ at that point. A supersolution is lsc and satisfies an analogous condition.

We relate the subsolutions to ordinary pluripotential theory, supersolutions being more mysterious. We formulate the viscosity comparison principle that a subsolution is always less or equal than a supersolution and prove it when $\epsilon > 0$ and $v > 0$. This boils down to a sophisticated Alexandroff-type maximum principle for semi-continuous functions.

It implies that a continuous viscosity solution exists, constructed as an enveloppe by Perron's method. It can be identified with the above pluripotential solution which is thus continuous. Using pluripotential stability estimates from [3], we extend this to the general case.

Continuity of the pluripotential solution of a degenerate complex Monge-Ampère equation is a subtle technical point that was only settled in the projective case (under some additionnal technical condition) by [2].

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Model theory and the class $\mathcal{C}$ of compact complex Kähler type spaces, I and II

Anand Pillay

I gave two lectures around the model-theoretic perspective on bimeromorphic geometry.

Around 1990 Zilber oberserved that a compact complex manifold X, considered as a first order structure by adjoining predicates for (closed) analytic subspaces of Cartesian powers X, $X \times X$, etc. is well-behaved model-theoretically, namely its first order theory has "finite Morley rank".

About 10 years later, in 1999-2000, myself and Tom Scanlon (later joined by Rahim Moosa) began to explore in some more detail the ramifications of Zilber's observation, and its possible relevance for the classification of compact complex manifolds. There was a rather intense period of activity from 2000 to 2004, including Moosa's 2001 Ph. D. thesis, and work continues up to the present moment, the most recent being the preprint [1].

I will mention a few of the themes of the talk:
(a) The model theoretic perspective brings interesting analogies, for example between compact complex manifolds and algebraic differential equations.
(b) Among the main model-theoretic results is a trichotomy statement for arbitrary "simple" compact complex spaces.
(c) We conjecture that the (first order theory of) the class $\mathcal{C}$ (viewed as a many sorted structure) is "almost $\aleph_1$-categorical". This is the next best thing after $\aleph_1$-categoricity.

The theorem in (b), due to myself and Scanlon, says that if X is a "simple" compact complex manifold, then essentially (up to correspondence), exactly one if the following holds:
(i) X is an algebraic curve,
(ii) X is a nonalgebraic complex torus, without proper subtori.
(iii) There is no positive dimensional "meromorphic family" of self correspondences on X.

Concerning (c), "almost $\aleph_1$-categoricity" of a countable theory T means (in model theoretic language) that T is stable and nonmultidimensional, and that every U-rank 1 type is nonorthogonal to a Morley rank 1 type. When translated into the complex analytic framework it says that first: whenever $f : X \to Y$ is a fibration in $\mathcal{C}$ with generic fibre simple, then any two generic fibres are in correspondence, and secondly, if X is a simple compact complex manifold then there exists a maximal positive dimensional closed analytic subspace of X.

Work of Campana in 2005 gave support to this conjecture. Another basic conjecture relating (b) and (c) is that a simple compact complex manifold $X \in \mathcal{C}$ should be close to a an irreducible hyperkähler manifold.

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Hurwitz Spaces and the Class of a Divisor on the Moduli of Curves of Even Genus

Gerard van der Geer
(joint work with Alexis Kouvidakis)

In their celebrated paper [4] Harris and Mumford considered the divisor in the moduli space M_g of curves of genus $g = 2k-1$ that parametrizes curves admitting a pencil of degree k to $\mathbb{P}^1$ and they determined the class of the corresponding divisor in $\overline{M}_g$. They used this to determine the Kodaira dimension of M_g for $g \geq 25$. In his continuation of this work [3] Harris generalized this by looking at the locus of smooth curves of genus g that are expressable as k-sheeted covers of

$\mathbb{P}^1$ with an ℓ-fold ramification point with $\ell = 2k - g + 1$. Such divisors come from Hurwitz spaces; e.g. the first comes from

$$\mathcal{H}_{g,k} = \{f : C \stackrel{k:1}{\longrightarrow} \mathbb{P}^1, p_1, \ldots, p_b\},$$

the space of smooth curves of genus g that are covers of degree k of $\mathbb{P}^1$ with $b = 2g - 2 + 2k$ marked ordinary branch points. This allows a compactification $\overline{\mathcal{H}}_{g,k}$ by so-called admissible covers of degree k. We get in such a way a natural map from a compactified Hurwitz space to a Deligne-Mumford moduli space of stable curves.

Faber and Pandharipande showed in [1] that the cycle classes of such loci belong to the tautological rings.

In our work, a byproduct of some other investigation, we looked at M_g with $g = 2k$ even. In this case the generic curve is in finitely many ways a degree $k+1$ cover of $\mathbb{P}^1$ with simple ramification points. So there is a generically finite morphism $\overline{\mathcal{H}}_{g=2k,d=k+1} \to \overline{M}_g$. In this situation there are two important divisors: D_2 and D_3. The divisor D_2 is the closure of the locus of curves having a pencil of degree $k+1$ with two ramifications points in one fibre; the divisor D_3 is the closure of the locus of curves having a pencil of degree $k+1$ where two ramification points collide, i.e., that have one triple ramification point. These divisor appear already in Harris' paper [3] and Harris determined the class of D_3. For this he used test curves. The class of D_2 remained undetermined however. We determine this class using global methods. The result is:

Theorem 1. *Let $g = 2k$ be an even natural number. The class of D_2 on $\overline{M}_g$ can be written as $c_\lambda \lambda + \sum_{j=0}^{k} c_j \delta_j$ with the coefficients c_λ and c_j given by*

$$c_\lambda = 6\,N \frac{6k-1}{2k-1}(k-2)(k+3),$$

and

$$c_0 = -\frac{2N}{2k-1}(k-2)(3k^2+4k-1),$$

and for $1 \le j \le k$

$$c_j = -3N\,\frac{j(2k-j)}{2k-1}(6k^2-4k-7) + \frac{9}{2}\,j(2k-j)\,\alpha(k,j).$$

Here $N = \binom{2k}{k+1}/k$ and $\alpha(k,j)$ is the combinatorial expression

$$\alpha(k,j) = \frac{j(2k-j)+k}{k(k+1)}\binom{j}{[j/2]}\binom{2k-j}{k-[j/2]} \qquad \textit{for } j \textit{ even}$$

and

$$\alpha(k,j) = \frac{(j+1)(2k-j)}{k(k+1)}\binom{j+1}{1+[j/2]}\binom{2k-j-1}{k-1-[j/2]} \qquad \textit{for } j \textit{ odd}.$$

We prove the theorem by studying the images of boundary divisors on the Hurwith space $\overline{\mathcal{H}}_{2k,k+1}$ in the moduli space $\overline{M}_{2k}$ under the natural map $\pi : \overline{\mathcal{H}}_{2k,k+1} \to$

$\overline{M}_{2k}$. In particular, we look at divisors in the boundary $\overline{\mathcal{H}}_{2k,k+1} - \mathcal{H}_{2k,k+1}$ that map dominantly to a divisor in $\overline{M}_{2k}$.

We study the number of irreducible components of such divisors and the degree of the morphism π restricted to these. Another important ingredient is a recent result of Kokotov, Korotkin and Zograf [5] who gave a formula for the first Chern class λ of the Hodge bundle in terms of boundary divisors on the Hurwitz space.

As a check one observes that our formula gives the well-known relation $10\lambda = \delta_0 + \delta_1$ for $k = 1$, gives $0 = 0$ for $k = 2$ and satisfies $c_\lambda + 12c_0 - c_1 = 0$ where c_λ, c_0 and c_1 are the coefficients of λ, δ_0 and δ_1, a relation that one knows should be satisfied.

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The Standard Conjectures for holomorphic symplectic varieties deformation equivalent to Hilbert schemes of $K3$ surfaces

Eyal Markman

(joint work with François Charles)

An *irreducible holomorphic symplectic manifold* is a simply connected compact Kähler manifold X, such that $H^0(X, \Omega^2_X)$ is generated by an everywhere non-degenerate holomorphic two-form (see [Be1, Hu1]).

Let S be a smooth compact Kähler $K3$ surface and $S^{[n]}$ the Hilbert scheme (or Douady space) of length n zero dimensional subschemes of S. Beauville proved in [Be1] that $S^{[n]}$ is an irreducible holomorphic symplectic manifold of dimension $2n$. If X is a smooth compact Kähler manifold deformation equivalent to $S^{[n]}$, for some $K3$ surface S, then we say that X is of $K3^{[n]}$*-type*. The variety X is then an irreducible holomorphic symplectic manifold. The odd Betti numbers of X vanish [Gö].

The moduli space of Kähler manifolds of $K3^{[n]}$-type is smooth and 21-dimensional, if $n \geq 2$, while that of $K3$ surfaces is 20-dimensional [Be1]. It follows that if S is a $K3$ surface, a general Kähler deformation of $S^{[n]}$ is not of the form $S'^{[n]}$ for a K3 surface S'. The same goes for projective deformations. Indeed, a general projective deformation of $S^{[n]}$ has Picard number 1, whereas for a projective S, the Picard number of $S^{[n]}$ is at least 2.

In this note, we prove the standard conjectures for projective varieties of $K3^{[n]}$-type. Let us recall general facts about the standard conjectures.

In the paper [Gr] of 1968, Grothendieck states those conjectures concerning the existence of some algebraic cycles on smooth projective algebraic varieties over an algebraically closed ground field. Here we work over $\mathbb{C}$. The Lefschetz standard conjecture predicts the existence of algebraic self-correspondences on a given smooth projective variety X of dimension d that give an inverse to the operations

$$H^i(X) \to H^{2d-i}(X)$$

given by the cup-product $d-i$ times with a hyperplane section, for all $i \leq d$. Above and throughout the rest of the paper, the notation $H^i(X)$ stands for singular cohomology with rational coefficients.

Over the complex numbers, the Lefschetz standard conjecture implies all the standard conjectures. If it holds for a variety X, it implies that numerical and homological equivalence coincide for algebraic cycles on X, and that the Künneth components of the diagonal of $X \times X$ are algebraic. We refer to [K1] for a detailed discussion.

Though the motivic picture has tremendously developed since Grothendieck's statement of the standard conjectures, very little progress has been made in their direction. The Lefschetz standard conjecture is known for abelian varieties, and in degree 1, where it reduces to the Hodge conjecture for divisors. The Lefschetz standard conjecture is also known for varieties X, for which $H^*(X, \mathbb{Z})$ is isomorphic to the Chow ring $A^*(X)$, see [K2]. Varieties with the latter property include flag varieties, and smooth projective moduli spaces of sheaves on rational Poisson surfaces [ES, Ma1].

In the paper [Ar], Arapura proves that the Lefschetz standard conjecture holds for uniruled threefolds, unirational fourfolds, the moduli space of stable vector bundles over a smooth projective curve, and for the Hilbert scheme $S^{[n]}$ of every smooth projective surface ([Ar], Corollaries 4.3, 7.2 and 7.5). He also proves that if S is a $K3$ or abelian surface, H an ample line-bundle on S, and $\mathcal{M}$ a smooth and compact moduli space of Gieseker-Maruyama-Simpson H-stable sheaves on S, then the Lefschetz standard conjecture holds for $\mathcal{M}$ ([Ar], Corollary 7.9). Those results are obtained by showing that the motive of those varieties is very close, in a certain sense, to that of a curve or a surface. Aside from examples obtained by taking products and hyperplane sections, those seem to be the only cases where a proof is known.

The main result of this note is the following statement.

Theorem 1. *The Lefschetz standard conjecture holds for every smooth projective variety of* $K3^{[n]}$*-type.*

Since the Lefschetz standard conjecture is the strongest standard conjecture in characteristic zero, we get the following corollary.

Corollary 2. *The standard conjectures hold for any smooth projective variety of* $K3^{[n]}$*-type.*

Note that by the remarks above, Arapura's results do not imply Theorem 1, as a general variety of $K3^{[n]}$-type is not a moduli space of sheaves on any K3 surface.

Theorem 1 is proven in [CM]. The degree 2 case of the Lefschetz standard conjecture, for projective varieties of $K3^{[n]}$-type, has already been proven in [Ma2] as a consequence of results of [C].

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Invariant meromorphic functions on Stein manifolds

Daniel Greb
(joint work with Christian Miebach)

One of the fundamental results relating invariant theory to the geometry of algebraic group actions is Rosenlicht's Theorem [Ros56, Thm. 2]: for any action of a linear algebraic group on an algebraic variety there exists a finite set of invariant rational functions that separates orbits in general position. Moreover, there exists a rational quotient, i.e., a Zariski-open invariant subset on which the action admits a geometric quotient. In my talk I discussed meromorphic functions invariant under holomorphic group actions and the construction of quotients of Rosenlicht-type in the analytic category.

While such quotients for Hamiltonian group actions on compact Kähler spaces were constructed by Fujiki [Fuj78] and Lieberman [Lie78], not much is known about the corresponding problem on non-compact Kähler manifolds. The initial impetus for the project that led to the results presented in the talk is a recent result by Gilligan, Miebach and Oeljeklaus [GMO10] stating the analyticity of orbit closures of Hamiltonian actions on possibly non-compact Kähler manifolds.

As a natural starting point in the non-compact case we consider group actions on spaces with rich function theory such as Stein spaces. Actions of reductive groups and their subgroups on these spaces are known to possess many features of algebraic group actions. However, while the holomorphic invariant theory in this setup is well understood mainly due to work of Heinzner [Hei91] and his coworkers, invariant meromorphic functions have been studied less until now.

Our following main result provides a natural generalisation of Rosenlicht's Theorem to Stein spaces with actions of complex-reductive groups.

Theorem 1. *Let G be a complex-reductive Lie group, let X be a Stein G-space, and let H be an algebraic subgroup of G. Assume that X is H-irreducible. Then, there exists an H-irreducible Zariski-open dense subset Ω in X and a holomorphic map $p\colon \Omega \to Q$ to an irreducible complex space Q such that*

(1) *the map p is a geometric quotient for the H-action on Ω,*
(2) *the map p is universal with respect to H-stable analytic subsets of Ω,*
(3) *the map p extends to a weakly meromorphic map (in the sense of Stoll) from X to Q,*
(4) *for every H-invariant meromorphic function $f \in \mathcal{M}_X(X)^H$, there exists a unique meromorphic function $\bar{f} \in \mathcal{M}_Q(Q)$ such that $f|_U = \bar{f} \circ p$,*
(5) *the H-invariant meromorphic functions separate the H-orbits in general position in X,*
(6) *the map p realises Ω as a locally trivial topological fibre bundle over Q.*

The idea of proof is first to establish a weak equivariant analogue of Remmert's and Narasimhan's [Nar60] embedding theorem for Stein spaces. More precisely, given a G-irreducible Stein G-space we prove the existence of a G-equivariant holomorphic map into a finite-dimensional G-representation space V that is a proper embedding when restricted to a big Zariski-open G-invariant subset. Since the G-action on V is algebraic, we may then apply Rosenlicht's Theorem to this linear action. Subsequently, a careful comparison of algebraic and holomorphic geometric quotients allows us to carry over the existence of a Rosenlicht quotient from V to X.

Applications of our main result include an estimate for the number of analytically independent meromorphic functions invariant under a compact group action and a refinement of Richardson's result on principal orbit types in the case of generic reductive stabiliser groups. Furthermore, using the newly constructed Rosenlicht-type quotient we clarify the relation between meromorphic and holomorphic invariants of holomorphic group actions on Stein spaces.

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On multiplier ideal sheaves and test configurations on toric Fano manifolds (Obstructive divisors to Kähler-Einstein metrics)

Yuji Sano

Let X be an n-dimensional Fano manifold. Let $\omega_0 \in c_1(M)$ be a Kähler form and $h_\omega \in C^\infty_{\mathbb{R}}(X)$ be the Ricci discrepancy function of ω_0 defined by

$$\mathrm{Ric}(\omega_0) - \omega_0 = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}h_{\omega_0}, \quad \int_X e^{h_{\omega_0}}\omega_0^n = \int_X \omega_0^n.$$

Here $\mathrm{Ric}(\omega_0)$ denotes the Ricci form of ω_0. Let G be a compact subgroup of the group $\mathrm{Aut}(X)$ of the holomorphic automorphisms. Assume that ω_0 is G-invariant. Let φ_t be the potential of the normalized Kähler-Ricci flow starting at ω_0

$$\frac{d}{dt}\omega_t = -\mathrm{Ric}(\omega_t) + \omega_t, \quad t \in [0,\infty)$$

where $\omega_t = \omega_0 + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_t$. If X does not admit Kähler-Einstein metrics, then φ_t will not converge in C^∞. However, by taking a subsequence, we have a weak limit $\varphi_\infty = \lim_{t_i\to\infty}(\varphi_{t_i} - \sup\varphi_{t_i})$ in current sense. From [4], [1], [7], the following is known.

Fact 1. *For every $\gamma \in (n/(n+1), 1)$, the multiplier ideal sheaf $\mathcal{I}(\gamma\varphi_\infty)$ is $G^{\mathbb{C}}$-invariant and $\mathcal{I}(\gamma\varphi_\infty) \neq \mathcal{O}_X$.*

Here $G^{\mathbb{C}}$ denotes the complexification of G, and $\mathcal{I}(\gamma\varphi_\infty)$ denotes a coherent ideal sheaf induced by the presheaf consisting of

$$\Gamma(U, \mathcal{I}(\gamma\varphi_\infty)) = \{f \in \mathcal{O}_U \mid |f|^2 e^{-\gamma\varphi_\infty} \in L^1_{\mathrm{loc}}\}$$

with respect to each open subset $U \subset X$. Comparing to Fact 1, the following is also known (cf. [3]).

Fact 2. *Let X, G, ω_0 be as above. For every $\varepsilon > 0$, there exist a sufficiently large $m \in \mathbb{N}$ and a G-invariant linear system $|\Sigma| \subset |-mK_X|$ such that $|\Sigma|$ has a member D satisfying that the pair $\left(X, \frac{n+\varepsilon}{(n+1)m}D\right)$ is not KLT.*

In Fact 2, an obstruction to the existence of KE metrics is described by the singularities of the divisors, while it is of the ideal sheaves (or the associated subschemes) in Fact 1. Questions asked in the talk are

Question 3. *Let $\mathcal{V}(\gamma\varphi_\infty)$ be the subscheme cut out by $\mathcal{I}(\gamma\varphi_\infty)$.*

(1) *Is $\mathcal{V}(\gamma\varphi_\infty)$ also a divisor?*
(2) *Is there any difference between $\mathcal{V}(\gamma\varphi_\infty)$ and D in Fact 2 (if Question (1) is affirmative)?*
(3) *Which violates stability of X (in the sense of Ross-Thomas [6])?*

I discussed the above questions in toric case. Let X be an n-dimensional toric Fano manifold. Let G be the compact subgroup generated by $(S^1)^n$ and the Weyl group $\mathcal{W}(X)$ of $\mathrm{Aut}(X)$ with respect to $(\mathbb{C}^\times)^n$. It is known that $\mathcal{W}(X)$ is a finite subgroup of $\mathrm{GL}(n, \mathbb{Z})$ which preserves the momentum polytope of X. Then, the main result of the talk gives an affirmative partial answer to Question (1).

Theorem 4 ([9]). *Suppose that X does not admit any G-invariant Kähler-Einstein metric. Then, the support of $\mathcal{V}(\gamma\varphi_\infty)$ contains a ($(S^1)^n$-invariant) subvariety of codimension one for any $\gamma \in (1/2, 1)$.*

By computing the complex singularity exponent of φ_∞, we can define the divisor

$$D_{\varphi_\infty} = \sum d_i D_i,$$

where D_i is an $(S^1)^n$-invariant divisor contained in $\mathrm{Supp}(\mathcal{V}(\gamma\varphi_\infty))$ for all $\gamma \in (1/2, 1)$ and d_i is the inverse of the complex singularity exponent of φ_∞ along D_i. Note that the computation of the complex singularity exponent of φ_∞ implies that $d_i \geq 2$ and $d_i \in \mathbb{Z}$.

Instead of φ_∞, I compared D_{φ_∞} to $\frac{1}{m}D$ in Fact 2. The latter is numerically equivalent to the anticanonical divisor $-K_X$, while the former is not necessarily so. For example, when $X = \mathbb{CP}^2 \sharp \overline{\mathbb{CP}^2}$ the blown-up surface of $\mathbb{CP}^2$ at one point, we can check that $D_{\varphi_\infty} = 2E$ where E is the exceptional divisor (cf. [8]). It is easy to see that it is not numerically equivalent to $-K_X$. On the other hand, when $X = \mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}^2}$, we can check that $D_{\varphi_\infty} = 3E_0 + 2(E_1 + E_2)$, where E_1, E_2 are the exceptional divisors of the blow up of $\mathbb{CP}^2$ at two points p_1, p_2, and E_0 is the proper transformation of the line $\overline{p_1 p_2}$ (cf. [8]). In this case, D_{φ_∞} is numerically equivalent to $-K_X$.

Question (3) is motivated by the conjecture which says

Conjecture 5 (Yau, Tian, Donaldson)**.** *The existence of constant scalar curvature Kähler metrics on a polarized manifold* (X, L) *is equivalent to* K*-polystability of* (X, L)*.*

See [2] for K-stability. From the above conjecture, we could expect that obstructions to KE metrics also obstruct stability, but we know that the answer to Question 3 is negative in general. As for $\frac{1}{m}D$ in Fact 2, it is not difficult to see that any ($\mathbb{Q}$-)divisor numerically equivalent to $-K_X$ does not destabilize a Fano manifold $(X, -K_X)$ in the sense of [6]. As for D_{φ_∞}, we have a counter-example. In fact, any subscheme does not destabilize $\mathbb{CP}^2\sharp 2\overline{\mathbb{CP}^2}$ polarized by $-K_X$ (cf. [5]). As a possibility to overcome 1this phenomenon, I observed the following.

Observation 6 ([9])**.** *Let* $X = \mathbb{CP}^2\sharp 2\overline{\mathbb{CP}^2}$. *We iterate the toric blow up of* $X \times \mathbb{C}$ *as follows. Let*

$$Z_0 = \mathcal{V}\big(\frac{1}{3}D_{\varphi_\infty}\big) = E_0 \subset Z_1 = \mathcal{V}(D_{\varphi_\infty}) = D_{\varphi_\infty}.$$

The first is to blow up $X \times \mathbb{C}$ *along* $Z_1 \times \{0\}$*. The second is to blow up* $\mathrm{Blw}_{Z_1\times\{0\}}(X \times \mathbb{C})$ *along the proper transformation of* $Z_0 \times \{0\}$*. Then, we can get a destabilizing test configuration.*

In the above, $\mathcal{V}(D)$ is the subscheme cut out by the algebraic multiplier ideal $\mathcal{I}(D)$ of an effective $\mathbb{Q}$-divisor D.

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Deformations of the space of lines on the 5-dimensional hyperquadric

Jun-Muk Hwang

Let $\mathbf{Q}^n \subset \mathbf{P}^{n+1}$ be the n-dimensional hyperquadric over $\mathbf{C}$. Denote by $\mathbf{F}^n$ the space of lines lying on $\mathbf{Q}^n$. When $n \geq 5$, $\mathbf{F}^n$ is a Fano manifold of Picard number 1, homogeneous space under the group $\mathbf{PSO}(n+2)$. In [Hw97] and also in [HM02], among others, the following was proved for $n \geq 6$.

Theorem 1. *Let $\pi : \mathcal{X} \to \Delta$ be a smooth projective morphism from a complex manifold $\mathcal{X}$ to the unit disc $\Delta \subset \mathbf{C}$. If the fiber $\pi^{-1}(t)$ is biregular to $\mathbf{F}^n$ for each $t \in \Delta \setminus \{0\}$, then the central fiber $\pi^{-1}(0)$ is biregular to $\mathbf{F}^n$, too.*

In the statement of Theorem 1 in [Hw97] and [HM02], the case of $n = 5$ was sloppily included, although neither the proof in [Hw97] nor the slightly different proof in [HM02] work when $n = 5$. In fact, Pasquier and Perrin recently found a counter-example to the statement of Theorem 1 when $n = 5$ in Proposition 2.3 of [PP]. This simple but remarkable example exists because of the fact that $\mathbf{Q}^5$ is homogeneous under a smaller subgroup of $\mathbf{PSO}(7)$, a group of type G_2. Then $\mathbf{F}^5$ is a quasi-homogeneous variety under G_2 and there exists a deformation of $\mathbf{F}^5$ to another G_2-quasi-homogeneous smooth projective variety, called X^5 in [PP].

This counter-example naturally raises the question: which variety can occur as a smooth projective degeneration of $\mathbf{F}^5$. In this talk, we report our solution to this question:

Theorem 2. *Let $\pi : \mathcal{X} \to \Delta$ be a smooth projective morphism from an 8-dimensional complex manifold $\mathcal{X}$ to the unit disc $\Delta \subset \mathbf{C}$. If the fiber $\pi^{-1}(t)$ is biregular to $\mathbf{F}^5$ for each $t \in \Delta \setminus \{0\}$, then the central fiber $\pi^{-1}(0)$ is biregular to either $\mathbf{F}^5$ or the variety X^5 in* [PP].

One interesting feature of our approach to Theorem 2 is that *we do not need to know even the definition of the variety X^5* to prove it. In fact, what we really prove can be stated as follows, which implies Theorem 2 by the concrete example of [PP].

Theorem 3. *Let $\pi : \mathcal{X} \to \Delta$ be a smooth projective morphism from an 8-dimensional complex manifold $\mathcal{X}$ to the unit disc $\Delta \subset \mathbf{C}$. If the fiber $\pi^{-1}(t)$ is biregular to $\mathbf{F}^5$ for each $t \in \Delta \setminus \{0\}$, then up to biholomorphism only two projective manifolds may appear as the central fiber $\pi^{-1}(0)$.*

To be precise, all we need to know about X^5 is just what its variety of minimal rational tangents looks like. To explain this, let us discuss the nature of the errors in [Hw97] and [HM02].

Heuristically, the essence of the errors lie in the description of the embedded deformation of the Segre embedding $\mathbf{P}^1 \times \mathbf{Q}^{n-4} \subset \mathbf{P}^{2n-5}$. To be precise, consider a family of smooth projective varieties

$$\{\mathcal{C}_t \subset \mathbf{P}^{2n-5}, t \in \Delta\}$$

such that for $t \neq 0$, the submanifold $\mathcal{C}_t \subset \mathbf{P}^{2n-5}$ is the Segre embedding. We ask the question whether the embedding $\mathcal{C}_0 \subset \mathbf{P}^{2n-5}$ is isomorphic to the Segre embedding. It is not hard to see that this is so if $n \geq 6$. But when $n = 5$, one additional possibility of $\mathcal{C}_0 \subset \mathbf{P}^{2n-5}$ exists. Let $\mathcal{H}$ be the Hirzebruch surface $\mathbf{P}(\mathcal{O}(-1) \oplus \mathcal{O}(-3))$ where $\mathcal{O}(k)$ denotes the line bundle on $\mathbf{P}^1$. By the dual tautological line bundle on this projectivized bundle over $\mathbf{P}^1$, this surface $\mathcal{H}$ can be embedded in the projective space

$$\mathbf{P}H^0(\mathbf{P}^1, \mathcal{O}(3) \oplus \mathcal{O}(1))^* \cong \mathbf{P}^5.$$

One can easily construct a family of surfaces $\mathcal{C}_t \subset \mathbf{P}^5$ which is the Segre embedding of $\mathbf{P}^1 \times \mathbf{Q}^1$ for $t \neq 0$, but $\mathcal{C}_0 \subset \mathbf{P}^5$ is isomorphic to $\mathcal{H} \subset \mathbf{P}^5$. This additional possibility is overlooked in [Hw97] and [HM02].

In the setting of Theorem 1, the above problem of embedded deformations arises in the following way. Take a section $\sigma : \Delta \to \mathcal{X}$ of π such that $\sigma(0)$ is a general point of the central fiber. Then consider the family $\nu : \mathcal{K} \to \Delta$ such that the fiber $\mathcal{K}_t$ corresponds to the normalized Chow space of minimal rational curves on $\pi^{-1}(t)$ passing through $\sigma(t)$. It is known that ν is a smooth projective morphism and the general fiber $\nu^{-1}(t)$ is biregular to $\mathbf{P}^1 \times \mathbf{Q}^{n-4}$. The proofs in [Hw97] and [HM02] started from establishing that $\nu^{-1}(0)$ is also biregular to $\mathbf{P}^1 \times \mathbf{Q}^{n-4}$. By looking at the varieties of minimal rational tangents, the nature of the problem then is essentially equal to the above embedded deformation problem.

Generalizing the argument of [HM02], one can determine the variety of minimal rational tangents of the central fiber. When the central fiber is different from $\mathbf{F}^5$, the variety of minimal rational tangents at a general point of $\pi^{-1}(0)$ is exactly the Hirzebruch surface

$$\mathcal{H} = \mathbf{P}(\mathcal{O}(-1) \oplus \mathcal{O}(-3)) \subset \mathbf{P}^5 \subset \mathbf{P}^6.$$

Then Theorem 3 follows from

Theorem 4. *There exists (up to biholomorphism) at most one 7-dimensional Fano manifold of Picard number 1 whose variety of minimal rational tangents at a general point is isomorphic to* $\mathcal{H} \subset \mathbf{P}^6$.

The proof of Theorem 4 is modeled on the proof of the following result of Mok's [Mk].

Theorem 5. *Let X be a Fano manifold of Picard number 1 whose variety of minimal rational tangents at a general point is isomorphic to that of* $\mathbf{F}^n, n \geq 5$. *Then X is biholomorphic to* $\mathbf{F}^n$.

The proof of Theorem 5 consists of two parts. The first part is to extend the geometric structure defined by the variety of minimal rational tangents isomorphic to that of $\mathbf{F}^n$ to a neighborhood of a general minimal rational curve on X. This part is the main result of [Mk]. The second part is to show that this extended geometric structure is flat. This second part was essentially done by Hong in [Ho], using the work of Tanaka [Ta] and Yamaguchi [Ya].

Our proof of Theorem 4 follows the same approach as Mok's. The extension of geometric structures can be carried out by modifying the argument of [Mk]. The major difference is in the step corresponding to [Ho]. The problem is that the geometric structure defined by the Hirzebruch surface does not belong to the semi-simple theory treated in [Ta] and [Ya], and, to our knowledge, has never been studied in differential geometry. Thus a large part of our work is devoted to the study of the Cartanian geometry of this geometric structure using the more general framework of Morimoto [Mr].

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Valuations and invariants of sequences of ideals

Mircea Mustaţă

(joint work with Mattias Jonsson)

We study asymptotic versions of invariants of singularities such as the log canonical threshold, or more generally, the jumping numbers for the multiplier ideals. Let X be a fixed smooth variety over an algebraically closed field of characteristic zero. Recall that if $\mathfrak{a}$ is a nonzero ideal on X, then one defines the *log canonical threshold* $\mathrm{lct}(\mathfrak{a})$ of $\mathfrak{a}$ in terms of a log resolution of singularities of the pair $(X, \mathfrak{a})$. In terms of the multiplier ideals $\mathcal{J}(\mathfrak{a}^{\lambda})$ of $\mathfrak{a}$, this can be described as the smallest λ such that $\mathcal{J}(\mathfrak{a}^{\lambda}) \neq \mathcal{O}_X$ (with the convention that $\mathrm{lct}(\mathfrak{a}) = \infty$ if $\mathfrak{a} = \mathcal{O}_X$). More generally, if $\mathfrak{q}$ is an auxiliary ideal, then $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a})$ is the smallest λ such that $\mathfrak{q} \not\subseteq \mathcal{J}(\mathfrak{a}^{\lambda})$. If we let $\mathfrak{q}$ vary, then the numbers obtained in this way are precisely the jumping numbers for the multiplier ideals of $\mathfrak{a}$.
Given a prime divisor E over X, we have an associated valuation ord_E of the function field of X. The log discrepancy $A(\mathrm{val}_E)$ of this valuation is the coefficient of E in $K_{Y/X}$, plus one (here Y is a model such that E is a divisor on Y). It follows from definition that

$$\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}) = \min_E \frac{A(\mathrm{ord}_E) + \mathrm{ord}_E(\mathfrak{q})}{\mathrm{ord}_E(\mathfrak{a})}.$$

In fact, the minimum is achieved by some divisor E on a log resolution of $\mathfrak{a} \cdot \mathfrak{q}$.

Suppose now that $\mathfrak{a}_\bullet = (\mathfrak{a}_p)_p$ is a graded sequence of ideals on X, that is, $\mathfrak{a}_p \cdot \mathfrak{a}_q \subseteq \mathfrak{a}_{p+q}$ for all $p, q > 0$. The main geometric example is the following: X is projective, and $\mathfrak{a}_m$ is the ideal defining the base locus of L^m, where $L \in \mathrm{Pic}(X)$ is a line bundle with $h^0(L) \geq 1$. It is easy to see that if ν is a valuation of the function field of X, having center on X, then

$$\nu(\mathfrak{a}_\bullet) := \inf_m \frac{\nu(\mathfrak{a}_m)}{m} = \lim_{m\to\infty} \frac{\nu(\mathfrak{a}_m)}{m}.$$

Similarly, if $\mathfrak{q}$ is a fixed ideal, then

$$\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) := \sup_m m \cdot \mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_m) = \lim_{m\to\infty} m \cdot \mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_m).$$

One can show using the asymptotic multiplier ideals of $\mathfrak{a}_\bullet$ that we have

$$\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \inf_E \frac{A(\mathrm{ord}_E) + \mathrm{ord}_E(\mathfrak{q})}{\mathrm{ord}_E(\mathfrak{a}_\bullet)}.$$

However, it is easy to see that in this setting the above infimum might not be achieved. One can check that we may alternatively take the above infimum over all valuations in Val_X, the space of real valuations of $K(X)$ with center on X (with a suitable definition of $A(\nu)$ for $\nu \in \mathrm{Val}_X$). The following is our main result.

Theorem 1. *Given $\mathfrak{a}_\bullet$ and $\mathfrak{q}$ as above, there is $v \in \mathrm{Val}_X$ that computes $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a})$, that is, such that*

$$\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet) = \frac{A(\nu) + \nu(\mathfrak{q})}{\nu(\mathfrak{a}_\bullet)}.$$

It would be of great interest to show that one can find ν as in the above theorem that is quasi-monomial (sometimes also called an *Abhyankar valuation*). A valuation $\nu \in \mathrm{Val}_X$ is quasi-monomial if there is a birational morphism $Y \to X$, with Y smooth, and $y_1, \ldots, y_n$ local coordinates at a point $p \in Y$, and $\alpha_1, \ldots, \alpha_n \in \mathbf{R}_+$ such that for $f \in \mathcal{O}_{Y,p}$ with $f = \sum_{u \in \mathbf{Z}^n_{\geq 0}} c_u y^u$ in $\widehat{\mathcal{O}_{Y,p}}$ we have

$$\nu(f) = \min\{\sum_i u_i \alpha_i \mid c_u \neq 0\}.$$

The following is a basic open problem in this direction.

Conjecture 2. *With the notation in Theorem 1, there is always a quasi-monomial valuation in Val_X that computes $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$.*

We expect that under mild conditions, every valuation as in Theorem 1 is quasimonomial. More precisely, we make the following conjecture.

Conjecture 3. *Suppose that $X = \mathbf{A}^n_k$, with k an algebraically closed field of characteristic zero. Suppose that $\nu \in \mathrm{Val}_X$ has center at one point, and transcendence degree zero. If ν computes $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ for some graded sequence $\mathfrak{a}_\bullet$, then ν is quasi-monomial.*

We show that given $\nu \in \mathrm{Val}_X$ and $\mathfrak{q}$, there is a graded sequence $\mathfrak{a}_\bullet$ such that ν computes $\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet)$ if and only if for every $\mu \in \mathrm{Val}_X$ such that $\mu(\mathfrak{b}) \geq \nu(\mathfrak{b})$ for every ideal $\mathfrak{b}$ on X, we have $A(\mu) + \mu(\mathfrak{q}) \geq A(\nu) + \nu(\mathfrak{q})$.

Theorem 4. *If Conjecture 3 holds, then Conjecture 2 holds as well.*

Theorem 5. *Conjecture 3 holds if* $n = 2$.

The proof of Theorem 5 relies on ideas close to the ones used in [3] to prove the Openness Conjecture of Demailly and Kollár. However, unlike the proof in [3], our argument avoids the use of the valuation tree.

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Model theory and geometry of complex transcendental functions

Boris Zilber

By a structure (in a Bourbaki–Tarski sense) we mean a set with distinguished relations and functions on it. An example is the structure

$$\mathbf{C}_F = (\mathbf{C}, +, \times, f_1, \dots, f_n), \quad F = \{f_1, \dots f_n\}$$

where $f_1, \dots, f_n$ are some classical meromorphic functions on $\mathbf{C}$. A particularly interesting case is the structure $\mathbf{C}_{\exp}$ corresponding to $n = 1, f_1 = \exp$. Other interesting functions are the Weierstrass function with a fixed τ.

One of the problems we discuss is whether such a structure satisfies the following **quasi-minimality** property:
given an F-analytic subset $S \subset \mathbf{C}^m$ (that is given by equations made with the use of $+, \times, f_1, \dots, f_n$ only), the image of S under the projection map $(z_1, \dots, z_m) \mapsto z_m$ is either a countable subset of $\mathbf{C}$ or its complement is countable.

Another problem of considerable interest is the intersection theory of F-analytic sets. It can be essentially reduced to the following problem, presented here in case F consists of just one one-variable function f.

Let W be an algebraic subvariety of $\mathbf{C}^{2m}$ of dimension m written in variables $x_1, \dots, x_m, y_1, \dots, y_m$. When is

$$W \cap \{\bigwedge_i y_i = f(x_i)\} \neq \emptyset?$$

Model theory sheds some light on these and other related problems. We noticed that Hrushovski's construction of "new stable structures", a source of many counter-examples in model theory, can be interpreted as a construction of structures that look strikingly similar to the analytic structures $\mathbf{C}_F$ as above. In particular, a crucial ingredient of the construction, the Hrushovski predimension inequality, can be interpreted as a very general form of Schanuel's conjecture, which claims that for any $\mathbf{Q}$-linearly independent complex numbers $x_1, \dots, x_m$

$$\text{transc.degree}(x_1, \dots, x_m, e^{x_1}, \dots, e^{x_m}) \geq m.$$

Using Hrushovski's construction we build an analogue $K_{\rm exp}$ of the structure $\mathbf{C}_{\rm exp}$ and study its properties. It is an algebraically closed field with a function exp satisfying $\exp(x_1 + x_2) = \exp(x_1) \cdot \exp(x_2)$. By construction Schanuel's conjecture holds in $K_{\rm exp}$ (property Sch). We also find out that the quasi-minimality property holds in $K_{\rm exp}$. Moreover, the intersection $W \cap \{\bigwedge_i y_i = \exp(x_i)\}$ is nonempty as long as this does not immediately contradict Schanuel's property for the given W. This property of $K_{\rm exp}$ formulated in a precise form is a direct analogue of algebraic closedness of the field of complex numbers, we call it **exponential-algebraic closedness**, EC for short.

One more property, called the **countable closure** property, $K_{\rm exp}$ shares with $\mathbf{C}_{\rm exp}$: the system $W \cap \{\bigwedge_i^m y_i = \exp(x_i)\}$ has at most countably many solutions for generic enough W of dimension m.

Main Theorem. *There is a unique, up to isomorphism, field $K_{\rm exp}$ of cardinality continuum (and of any given uncountable cardinality) with abstract exponentiation satisfying properties (Sch), (EC) and (CC).*

A natural corollary of the theorem is the conjecture that *the classical* $\mathbf{C}_{\rm exp}$ *is isomorphic to* $K_{\rm exp}$ *of cardinality continuum.* This, of course, contains the Schanuel conjecture and the new conjecture of exponential-algebraic closedness of $\mathbf{C}_{\rm exp}$. We discuss some evidence supporting the latter.

Analogues of the main theorem are known by now for some other classical transcendental functions.

The cone conjecture for Calabi–Yau pairs

Artie Prendergast-Smith

A basic problem in algebraic geometry is to understand the nef cone $\overline{A(X)}$ and movable cone $\overline{M(X)}$ of a projective variety X. These cones are important invariants because they encode all morphisms from X (the nef cone) and all birational maps from X which extract no divisors (the movable cone).

For Fano varieties, these cones are well-behaved: the nef cone is rational polyhedral by the Cone Theorem, and the movable cone is rational polyhedral by recent celebrated results of Birkar–Cascini–Hacon–McKernan. From this point of view, Fano varieties can be considered the simplest class of projective varieties.

Calabi–Yau varieties can be considered the next simplest class of varieties, but already here the nef and movable cones can fail to be well-behaved. For example, the nef cone of a $K3$ surface may have infinitely many isolated extremal rays, or it may be a 'round' cone.

In general, therefore, we cannot hope for such a simple description of the nef and movable cones. However, one could still a hope for nice answer if we take into account the action of automorphisms on the vector space $N^1(X)$ of numerical classes of divisors: automorphisms of X define automorphisms of $N^1(X)$ preserving the nef cone, and we could ask for a well-behaved fundamental domain for that action. This hope is made precise by the following conjecture, due to Morrison [Mor92].

Conjecture 1 (Morrison)**.** *Let X be a Calabi–Yau manifold. Then there is a rational polyhedral fundamental domain* Π *for the action of* $\mathrm{Aut}(X)$ *on* $\overline{A(X)}^e$.

Here $\overline{A(X)}^e$ means the intersection of the nef cone $\overline{A(X)}$ with the cone Eff(X) spanned by the classes of effective Cartier divisors. (It is easy to see that Eff(X) is also preserved by automorphisms.) We consider the cone $\overline{A(X)}^e$ in the conjecture rather than the full nef cone, because in light of the example mentioned above of a 'round' nef cone, it is clear that in general no countable union of translates of a polyhedral cone can cover the whole nef cone.

There is an analogous conjecture on the movable cone. To state it, define a *pseudo-automorphism* of X to be a birational map $X \dashrightarrow X$ which is an isomorphism in codimension 1. Note that a pseudo-automorphism defines an automorphism of $N^1(X)$ via proper transforms, and such an automorphism preserves the movable cone and the effective cone. So the following conjecture makes sense:

Conjecture 2 (Morrison)**.** *Let X be a Calabi–Yau manifold. Then there is a rational polyhedral fundamental domain* Π *for the action of the group* $\mathrm{PsAut}(X)$ *of pseudo-automorphisms on* $\overline{M(X)}^e$.

These conjectures were generalised by Kawamata [Kaw97] and Totaro [Tot10] in the spirit of the minimal model program. To state the most general form of the conjecture, suppose $X \to S$ is a projective surjective morphism. For an **R**-divisor Δ on X, we say $(X/S, \Delta)$ is a *klt Calabi–Yau pair over S* if (X, Δ) is a klt pair and $K_X + \Delta$ is numerically trivial over S. The generalised conjecture says that for $(X/S, \Delta)$ a klt Calabi–Yau pair, there are rational polyhedral fundamental domains for the action of $\mathrm{Aut}(X/S, \Delta)$ on $\overline{A(X/S)}^e$ and for the action of $\mathrm{PsAut}(X/S, \Delta)$ on $\overline{M(X/S)}^e$. (Here we are considering automorphisms or pseudo-automorphisms over the base which preserve the divisor Δ, and the cones in question are appropriate cones of relative divisor classes.)

The conjectures seem out of reach at present, essentially because it is unclear where the necessary automorphisms or pseudo-automorphisms should come from. Nevertheless, some cases are known. The nef cone conjecture was proved in dimension 2 by Sterk [Ste85] ($K3$ surfaces), Namikawa [Nam85] (Enriques surfaces), Kawamata [Kaw97] (all other Calabi–Yau surfaces) and Totaro [Tot10] (all klt Calabi–Yau pairs of dimension 2). Also, Kawamata [Kaw97] proved the conjectures for all 3-dimensional Calabi–Yau fibre spaces over a positive-dimensional base. For 'honest' Calabi–Yau varieties of dimension 3, not much is known, though there are verifications in special cases, due to Grassi–Morrison, Borcea, and Fryers.

In [PS09] the conjectures were proved in the following case. This seems to be the first verification for a klt Calbi–Yau pair of dimension 3 (over $S = \mathrm{Spec}\, k$) with $\Delta \neq 0$.

Theorem 3. *Let X be the blowup of $\mathbf{P}^3$ in the base locus of a net of quadrics with no reducible member. Then there is a divisor Δ for which (X, Δ) is a klt Calabi–Yau pair, and the nef and movable cone conjectures hold for X.*

The idea of the proof is that there is an elliptic fibration $f : X \to \mathbf{P}^2$, and translations by points of the generic fibre provide enough pseudo-automorphisms to verify the movable cone conjecture. (The nef cone turns out to be rational polyhedral.) See [PS09] for details.

In [PS10] these ideas were developed further, to prove a weak form of the movable cone conjecture for a more general class of 3-folds. The result is the following:

Theorem 4. *Let X be a terminal Gorenstein 3-fold with $-K_X$ semiample of positive Iitaka dimension. Then $\overline{M(X)}^e$ decomposes as the union of the nef effective cones of the small modifications of X, and this decomposition is finite up to the action of* $\mathrm{PsAut}(X)$.

Again the basic point of the proof is that we have a fibration $X \to S$, given by sections of some power of the line bundle $-K_X$. Putting together Kawamata's results on the relative cone conjecture for $X \to S$ and Mori's contraction theorem, one gets the theorem. If we restrict to the case when S has dimension 2, we get more precise results about the relative movable cone; see [PS10] for details.

To conclude, we mention the following result of [PS10b] on the cone conjecture for abelian varieties. This is one of the few known results on the conjecture in arbitrary dimension. Unsurprisingly, the proof here is very different from the cases above; in particular, it uses the fact that the ample cone of an abelian variety is a homogeneous self-dual convex cone.

Theorem 5. *Let X be an abelian variety over an algebraically closed field. Then the cone conjecture holds for X.*

For an abelian variety the nef and movable cones coincide, so there is only one statement to verify in this case.

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Moduli spaces of Burniat surfaces

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(joint work with Fabrizio Catanese)

1. What are Burniat surfaces?

Burniat surfaces are surfaces of general type with geometric genus $p_g(S) = 0$, and were constructed by Pol Burniat in 1966 in [4], where the method of singular bidouble covers was introduced in order to solve the geography problem for surfaces of general type.
The birational structure of Burniat surfaces is rather simple to explain:
let $P_1, P_2, P_3 \in \mathbb{P}^2$ be three non collinear points (which we assume to be the points $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$), and let $D_i = \{\Delta_i = 0\}$, for $i \in \mathbb{Z}/3\mathbb{Z}$, be the union of three distinct lines through P_i, including the line $D_{i,1}$ which is the side of the triangle joining the point P_i with P_{i+1}.
We furthermore assume that $D = D_1 \cup D_2 \cup D_3$ consists of nine different lines.

Definition 1. *A Burniat surface S is the minimal model for the function field*

$$\mathbb{C}\left(\sqrt{\frac{\Delta_1}{\Delta_2}}, \sqrt{\frac{\Delta_1}{\Delta_3}}\right).$$

Proposition 2. *Let S be a Burniat surface, and denote by m the number of points, different from P_1, P_2, P_3, where the curve D has multiplicity at least three. Then $0 \leq m \leq 4$, and the invariants of the smooth projective surface S are:*

$$p_g(S) = q(S) = 0, \; K_S^2 = 6 - m.$$

The heart of the calculation, based on the theory of bidouble covers, as illustrated in [5], is that the singularities where the three curves have multiplicities $(3, 1, 0)$ lower K^2 and $p_g - q$ both by 1, while the singularities where the three curves have multiplicities $(1, 1, 1)$ lower K^2 by 1 and leave $p_g - q$ unchanged.

One understands the biregular structure of S through the blow up W of the plane at the points $P_1, P_2, P_3, \dots P_m$ of D of multiplicity at least three.

W is a weak Del Pezzo surface of degree $6 - m$ (i.e., a surface with nef and big anticanonical divisor).

Proposition 3. *The Burniat surface S is a finite bidouble cover (a finite Galois cover with group $(\mathbb{Z}/2\mathbb{Z})^2$) of the weak Del Pezzo surface W. Moreover the bicanonical divisor $2K_S$ is the pull back of the anticanonical divisor $-K_W$. The bicanonical map of S is the composition of the bidouble cover $S \to W$ with the anticanonical quasi-embedding of W, as a surface of degree $K_S^2 = K_W^2$ in a projective space of dimension $K_S^2 = K_W^2$.*

2. The main classification theorem

Fixing the number $K_S^2 = 6 - m$, one sees immediately that the Burniat surfaces are parametrized by a rational family of dimension $K_S^2 - 2$, and that this family is irreducible except in the case $K_S^2 = 4$.

Definition 4. *The family of Burniat surfaces with $K_S^2 = 4$ of nodal type is the family where the points P_4, P_5 are collinear with one of the other three points P_1, P_2, P_3, say P_1.*
The family of Burniat surfaces with $K_S^2 = 4$ of non-nodal type is the family where the points P_4, P_5 are never collinear with one of the other three points.

Our main classification result of Burniat surfaces is summarized in the following table giving information concerning the families of Burniat surfaces: more information will be given in the subsequent theorems.

K^2	dimension	is conn. comp.?	is rational?	π_1
6	4	yes	yes	$1 \to \mathbb{Z}^6 \to \pi_1 \to (\mathbb{Z}/2\mathbb{Z})^3$
5	3	yes	yes	$\mathbb{H} \oplus (\mathbb{Z}/2\mathbb{Z})^3$
4, non-nodal	2	yes	yes	$\mathbb{H} \oplus (\mathbb{Z}/2\mathbb{Z})^2$
4, nodal	2	no: $\subset$ 3-dim. irr. conn. component	yes	$\mathbb{H} \oplus (\mathbb{Z}/2\mathbb{Z})^2$
3	1	no: $\subset$ 4-dim. irr. component	yes	$\mathbb{H} \oplus \mathbb{Z}/2\mathbb{Z}$
2	0	no: $\in$ conn. component of standard Campedelli	yes	$(\mathbb{Z}/2\mathbb{Z})^3$

Theorem. *The three respective subsets of the moduli spaces of minimal surfaces of general type $\mathfrak{M}^{can}_{1,K^2}$ corresponding to Burniat surfaces with $K^2 = 6$, resp. with $K^2 = 5$, resp. Burniat surfaces with $K^2 = 4$ of non-nodal type, are irreducible connected components, normal, rational of respective dimensions 4,3,2.*
Moreover, the base of the Kuranishi family of such surfaces S is smooth.

Observe that the above result for $K^2 = 6$ was first proven by Mendes Lopes and Pardini in [6]. We showed in [1] the stronger theorem

Primary Burniat's theorem. *Any surface homotopy equivalent to a Burniat surface with $K^2 = 6$ is a Burniat surface with $K^2 = 6$.*

For $K^2 = 2$ another realization of the Burniat surface is as a special element of the family of Campedelli surfaces, Galois covers of the plane with group $(\mathbb{Z}/2\mathbb{Z})^3$ branched on seven lines (one for each non trivial element of the group). For the Burniat surface we have the special configuration of a complete quadrilateral together with its three diagonals.
For $K^2 = 3$ work in progress of the authors shows that the general deformation of a Burniat surface is a Galois covering with group $(\mathbb{Z}/2\mathbb{Z})^2$ of a cubic surface with at least three singular points, and with branch locus equal to three plane sections. It is still an open question whether the closure of this set is again a connected component of the moduli space.

3. Nodal Burniat surfaces

Let S be the minimal model of a nodal Burniat surface with $K_S^2 = 4$ or $K_S^2 = 3$, and let X be its canonical model. Observe that for $K^2 = 4$, X has one ordinary node, while for $K^2 = 3$, X has three ordinary nodes.
We denote by $Def(S)$ (resp. $Def(X)$) the base of the Kuranishi family of S (resp. of X). Moreover, for $G = (\mathbb{Z}/2\mathbb{Z})^2$, we denote by $Def(S, G)$ (resp. $Def(X, G)$) the infinitesimal variations of complex structures which are G-invariant.
Observe that there is a natural Galois covering $\rho\colon Def(S) \to Def(X)$, in particular ρ is surjective. Then we have the following result
Theorem. *The deformations of nodal Burniat surfaces with $K^2 = 4$ (resp. $K^2 = 3$) to the 3-dimesional (resp. 4-dimensional) family of* extended *nodal Burniat surfaces exists and yield examples, where $Def(S, G) \neq Def(S)$, and the map $Def(S, G) \to Def(X, G) = Def(X)$ is not surjective.*

The reason for this phenomenon can already be seen locally. In fact, $G = \{1, \sigma_1, \sigma_2, \sigma_3 = \sigma_1 + \sigma_2\}$ acts on the family $\mathcal{X}_t = \{w^2 = uv + t\}$, by $\sigma_1(u, v, w) = (u, v, -w)$, $\sigma_2(u, v, w) = (-u, -v, w)$.

$\mathcal{X}_t$ admits a simultaneous resolution only after the base change $\tau^2 = t$:and then we have two small resolutions

$$\mathcal{S} := \{((u, v, w, \tau), \xi) \in \mathcal{X} \times \mathbb{P}^1 | \frac{w - \tau}{u} = \frac{v}{w + \tau} = \xi\},$$

$$\mathcal{S}' := \{((u, v, w, \tau), \eta) \in \mathcal{X} \times \mathbb{P}^1 | \frac{w + \tau}{u} = \frac{v}{w - \tau} = \eta\}.$$

Then it is easy to see that G has several liftings to $\mathcal{S}$, but

- either G acts only as a group of *birational* automorphisms on $\mathcal{S}$ and leaves τ fixed,
- or, G acts as a group of *biregular* automorphisms on $\mathcal{S}$ and does not leave τ fixed.

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Positivity of relative canonical bundles and applications

Georg Schumacher

Given an effectively parameterized family of canonically polarized manifolds the Kähler–Einstein metrics on the fibers induce a hermitian metric on the relative canonical bundle.

We use a global elliptic equation to show that this metric is strictly positive. For degenerating families we show that the curvature form on the total space can be extended as a (semi-)positive closed current. By fiber integration it is shown that the generalized Weil-Petersson form on the base possesses an extension as a positive current. In in this situation, the determinant line bundle associated to the relative canonical bundle on the total space can be extended. As an application the quasi-projectivity of the moduli space $\mathcal{M}_{\mathrm{can}}$ of canonically polarized varieties follows.

The direct images $R^{n-p}f_*\Omega^p_{\mathcal{X}/S}(\mathcal{K}^{\otimes m}_{\mathcal{X}/S})$, $m > 0$, carry induced natural hermitian metrics. We prove an explicit formula for the curvature tensor of these direct images. We apply it to the morphisms $S^p\mathcal{T}_S \to R^pf_*\Lambda^p\mathcal{T}_{\mathcal{X}/S}$ induced by the Kodaira-Spencer map and obtain a differential geometric proof for hyperbolicity properties of $\mathcal{M}_{\mathrm{can}}$.

Similar results hold for families of polarized Ricci-flat manifolds (dissertation in progress). References are listed in [SCH1, SCH2].

Theorem 1. *Let $\mathcal{X} \to S$ be a holomorphic family of canonically polarized, compact, complex manifolds, which is nowhere inifinitesimally trivial. Then the hermitian metric on $\mathcal{K}_{\mathcal{X}/S}$ over the total space $\mathcal{X}$ that is induced by the Kähler-Einstein metrics on the fibers is strictly positive.*

The construction is functorial, i.e. compatible with base change and descends to the moduli space $\mathcal{M}_{can}$ in the orbifold/stack theoretic sense. Let $\omega_{\mathcal{X}}$ be the curvature form on the total space. Then the fiber integral $\int_{\mathcal{X}/S}\omega_{\mathcal{X}}^{n+1}$ is equal to the generalized Weil-Petersson form on S up to a numerical constant, if n denotes the dimension of the fibers. It was shown in [F-S] that this form is equal to the Chern form of the determinant line bundle $\lambda = \det f_!((\mathcal{K}^{\otimes m}_{\mathcal{X}/S} - (\mathcal{K}^{-1}_{\mathcal{X}/S})^{\otimes m})^{n+1})$, equipped with a Quillen metric.

However, general theory only yields that the determinant line bundle can only be extended *locally* to a compactification of $\mathcal{M}_{can}$.

Using Yau's C^0-estimates for the solution of Monge-Ampre equations we show:

Theorem 2. $(\mathcal{K}_{\mathcal{X}/\mathcal{H}}, h)$ *be the relative canonical bundle on the total space over the Hilbert scheme, equipped with the hermitian metric induced by the Kähler-Einstein metrics on the fibers. Then the curvature form extends to the total space over the compact Hilbert scheme $\overline{\mathcal{H}}$ as a positive, closed current $\omega^{KE}_{\mathcal{X}}$.*

The quasi-projectivity of the moduli space $\mathcal{M}_{can}$, which was first established by Viehweg, now follows from:

Theorem 3. *The generalized Weil-Petersson form on the moduli stack of canonically polarized varieties is strictly positive. It is the Chern form of a determinant* *line bundle, equipped with a Quillen metric. A tensor power of the line bundle metric extends to a compactification, and the Quillen metric extends as a (semi-)positive singular hermitian metric (in the orbifold sense).*

The proof follows from the statements below. Based upon Siu's structure theorem for closed, positive currents and the theorem of Bombieri-Skoda we first prove an extension theorem.

Theorem 4. *Let Y be a normal space and $Y' \subset Y$ the complement of a closed analytic, nowhere dense subset. Let L' be a holomorphic line bundle on Y' together with a (semi-)positive hermitian metric h', which also may be singular. Assume that the curvature current can be extended to Y as a positive, closed current and that the line bundle L' possesses local holomorphic extensions to Y. Then there exists a holomorphic line bundle (L, h) with a singular, positive hermitian metric, whose restriction to Y' is isomorphic to (L', h').*

The reduction to normal space is done by means of:

Proposition. *Let Y be a reduced complex space, and $A \subset Y$ a closed analytic subset. Let $\mathcal{L}$ be an invertible sheaf on $Y \backslash A$, which possesses a holomorphic extension to the normalization of Y as an invertible sheaf. Then there exists a reduced complex space Z together with a finite map $Z \to Y$, which is an isomorphism over $Y \backslash A$ such that $\mathcal{L}$ possesses an extension as an invertible sheaf to Z.*

The other application concerns the direct image sheaves $R^{n-p} f_* \Omega^p_{\mathcal{X}/S}(\mathcal{K}^{\otimes m}_{\mathcal{X}/S})$. These are equipped with a natural hermitian metric that is induced by the L^2-inner product of harmonic tensors on the fibers of f. We will give an explicit formula for the curvature tensor assuming that these are locally free. Then they can be equipped with a local basis $\{\psi^i\}$ of $\overline{\partial}$-closed forms, whose restrictions to the fibers are harmonic.

Let $A^\alpha_{i\overline{\beta}}(z,s)\partial_\alpha dz^{\overline{\beta}}$ be a harmonic Kodaira-Spencer form. Then for $s \in S$ the cup product together with the contraction defines

$$\begin{aligned} A^\alpha_{i\overline{\beta}}\partial_\alpha dz^{\overline{\beta}} \cup \lrcorner : \mathcal{A}^{0,n-p}(\mathcal{X}_s, \Omega^p_{\mathcal{X}_s}(\mathcal{K}^{\otimes m}_{\mathcal{X}_s})) &\to \mathcal{A}^{0,n-p+1}(\mathcal{X}_s, \Omega^{p-1}_{\mathcal{X}_s}(\mathcal{K}^{\otimes m}_{\mathcal{X}_s})) \\ A^{\overline{\beta}}_{\overline{\jmath}\alpha}\partial_{\overline{\beta}} dz^{\alpha} \cup \lrcorner : \mathcal{A}^{0,n-p}(\mathcal{X}_s, \Omega^p_{\mathcal{X}_s}(\mathcal{K}^{\otimes m}_{\mathcal{X}_s})) &\to \mathcal{A}^{0,n-p-1}(\mathcal{X}_s, \Omega^{p+1}_{\mathcal{X}_s}(\mathcal{K}^{\otimes m}_{\mathcal{X}_s})). \end{aligned}$$

We will apply the above product to harmonic $(0, n-p)$-forms. In general the result is not harmonic. We denote the pointwise L^2 inner product by a dot.

Theorem 5. *The curvature tensor for $R^{n-p} f_* \Omega^p_{\mathcal{X}/S}(\mathcal{K}^{\otimes m}_{\mathcal{X}/S})$ is given by*

$$\begin{aligned} R_{i\overline{\jmath}}{}^{\overline{\ell}k}(s) &= m \int_{\mathcal{X}_s} (\Box + 1)^{-1} (A_i \cdot A_{\overline{\jmath}}) \cdot (\psi^k \cdot \psi^{\overline{\ell}}) g dV \\ &\quad + m \int_{\mathcal{X}_s} (\Box + m)^{-1} (A_i \cup \psi^k) \cdot (A_{\overline{\jmath}} \cup \psi^{\overline{\ell}}) g dV \\ &\quad + m \int_{\mathcal{X}_s} (\Box - m)^{-1} (A_i \cup \psi^{\overline{\ell}}) \cdot (A_{\overline{\jmath}} \cup \psi^k) g dV. \end{aligned}$$

The potentially negative third term is not present for $p = n$, i.e. for $f_*\mathcal{K}_{\mathcal{X}/S}^{\otimes(m+1)}$. We give estimates that only depend upon the diameter of the fibers.

By Serre duality we get the following version, which contains for $p = 1$ the curvature formula for the generalized Weil-Petersson metric. Again a tangent vector of the base is identified with a harmonic Kodaira-Spencer form A_i, and ν_k stands for a section of the relevant sheaf:

Theorem 6. *The curvature for $R^p f_* \Lambda^p \mathcal{T}_{\mathcal{X}/S}$ equals*

$$\begin{aligned} R_{i\overline{\jmath}k\overline{\ell}}(s) &= -\int_{\mathcal{X}_s} (\Box + 1)^{-1} (A_i \cdot A_{\overline{\jmath}}) \cdot (\nu_k \cdot \nu_{\overline{\ell}}) g dV \\ &\quad - \int_{\mathcal{X}_s} (\Box + 1)^{-1} (A_i \wedge \nu_{\overline{\ell}}) \cdot (A_{\overline{\jmath}} \wedge \nu_k) g dV \\ &\quad - \int_{\mathcal{X}_s} (\Box - 1)^{-1} (A_i \wedge \nu_k) \cdot (A_{\overline{\jmath}} \wedge \nu_{\overline{\ell}}) g dV. \end{aligned}$$

We introduce Kodaira-Spencer maps of higher order

$$\rho^p : S^p \mathcal{T}_S \to R^p f_* \Lambda^p \mathcal{T}_{\mathcal{X}/S}$$

sending a symmetric power

$$\frac{\partial}{\partial s^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial s^{i_p}}$$

to the class of

$$A_{i_1} \wedge \ldots \wedge A_{i_p} := A^{\alpha_1}_{i_1 \overline{\beta}_1} \partial_{\alpha_1} dz^{\overline{\beta}_1} \wedge \ldots \wedge A^{\alpha_p}_{i_p \overline{\beta}_p} \partial_{\alpha_p} dz^{\overline{\beta}_p}.$$

We apply Demailly's measure theoretic version of Ahlfors' lemma and show:

Theorem 7. *On any relatively compact subset of the moduli space of canonically polarized manifolds there exists a Finsler orbifold metric, i.e. a Finsler metric on the moduli stack, whose holomorphic curvature is bounded from above by a negative constant.*

As an immediate application of the curvature formula (or the above theorem) we get a known case of the Shafarevich hyperbolicity conjecture.

If a compact curve C parameterizes a non-isotrivial family of canonically polarized manifolds, its genus must be greater than one.

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Balanced metrics may not converge towards constant scalar curvature Kähler metrics

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(joint work with Julius Ross)

Let us consider a smooth projective complex manifold M, and $L \to M$ an ample holomorphic line bundle on M. A famous conjecture of Yau [Ya] asserts that there exists a constant scalar curvature Kähler metric (CSCK metric in short) if and only if (M, L) is "stable" in the G.I.T (Geometric Invariant Theory) sense. Various notions of stability have been tested to sastisfy this conjecture. We refer to [Bi, PS, Th] as excellent surveys on that topic.
Let us recall briefly the notion of asymptotic Chow stability. For k sufficiently large, one can consider the Kodaira embedding

$$\iota_k : M \hookrightarrow \mathbb{P}(H^0(M, L^k)^*)$$

where $\mathbb{P}(H^0(M, L^k)^*)$ consists in the set of all hyperplanes in $H^0(M, L^k)$ through the origin. If we denote d_k the degree of $\iota_k(M)$ in the projective space and $n = \dim_{\mathbb{C}} M$, then there is an action of $SL_{\mathbb{C}}(H^0(M, L^k))$ on the space

$$\mathcal{C}how_k = Sym^{d_k}(H^0(M, L^k))^{\otimes n+1}$$

where Sym^d denotes the d-th symmetric tensor product. Let us consider an element $\dot{M}_k \in \mathcal{C}how_k^*$ such that the associated element $[\dot{M}_k] \in \mathbb{P}\mathcal{C}how_k^*$ is the Chow point of the algebraic cycle $\iota_k(M)$. Then, one says that (M, L^k) is Chow stable if the $SL_{\mathbb{C}}(H^0(M, L^k))$-orbit of $\dot{M}_k$ is closed in $\mathcal{C}how_k$ and its stabilizer is finite. Polystability is obtained when one drops the assumption on the stabilizer group. Furthermore, (M, L) is said to be asymptotically Chow stable if (M, L^k) is Chow stable for k large enough.
From the results of [Zh, Lu], we obtain way to express Chow stability using differential geometry and a metric approach. This is built on the classical relationship between G.I.T quotients and symplectic quotients. Let us recall that for a hermitian metric $h \in Met(L^k)$, the Bergman function (or distorsion function) ρ_h is the restriction to the diagonal of the kernel of the L^2 projection from $C^\infty(M, L^k)$ to $H^0(M, L^k)$. If we set $N = h^0(M, L^k)$, one can write explicitly this function as

$$\rho_h = \sum_{i=1}^{N} |s_i|_h^2 \in C^\infty(M, \mathbb{R}_+)$$

with $(s_i)_{i=1,..,N}$ an orthonormal basis of $H^0(M, L^k)$ with respect to the L^2-inner product $\int_M h(.,.)\frac{c_1(h^{1/k})^n}{n!}$.

Theorem 1 ([Zh, Lu]). *(M, L) is asymptotically Chow polystable if and only if for all $k >> 0$ there exists a balanced metric $h_k \in Met(L^k)$, i.e a metric such that the Bergman function ρ_{h_k} is constant over M.*

Now, S.K. Donaldson proved in [Do] the following important result.

Theorem 2. *Assume that $Aut(M,L)/\mathbb{C}^*$ is discrete. If there is a CSCK metric ω_∞ in $c_1(L)$ then (M,L) is asymptotically stable and for the sequence of balanced metrics $h_k \in Met(L^k)$, one has $\lim_{k\to\infty} c_1(h_k)/k = \omega_\infty$ in smooth topology.*

Thus it is natural to wonder if a sequence of balanced metrics *necessarily* converge. Note that in that case, the limit would be a CSCK metric (this is a consequence of [Ti1]). We give a *negative* answer to this question. In order to do so, we construct an example by considering the projectivisation of a Gieseker stable bundle which is not Mumford stable over a CSCK manifold.

Step 1. Construction of balanced metrics. We aim to prove the following theorem which extends [Se, Mo] (work in progress).

Theorem 3. *Let us consider $\pi : E \to M$ a Gieseker stable bundle such that its Harder-Narasimhan filtration is given by subbundles. Let us assume that (M,L) has a CSCK metric in $c_1(L)$ and $Aut(M,L)/\mathbb{C}^*$ is discrete. Let us define the polarisation $\mathcal{L}_m = \mathcal{O}_{P(E^*)}(1) \otimes \bar{\pi}^* L^m$ on the projective bundle $\bar{\pi} : \mathbb{P}(E^*) \to M$ and fix m_0 sufficiently large. Then for all $m \geq m_0$ and $k > 0$, there exists a balanced metric on $\mathcal{L}_m^k$ and in particular $\mathcal{L}_m$ is asymptotically Chow stable.*

A key ingredient of the proof is to use the sequence of metrics constructed by Leung in [Le], that have bounded curvature. Thus, it is possible to get an asymptotic of the Bergman kernel for holomorphic sections $E \otimes L^m$ and also $Sym^k(E\otimes L^m)$. We relate this Bergman kernel to the Bergman function by the natural isomorphism $Sym^k V \simeq H^0(\mathbb{P}V^*, \mathcal{O}_{PV^*}(k))$ and extending techniques developped by R. Seyyedali in [Se]. Finally, we use crucially that the Bergman function is a moment map, following the lines of [Do].

Step 2. Computation of the Donaldson-Futaki invariant for the projectivisation. Let us consider now a vector bundle E and a subbundle F of E. Extending the work of [RT2], we compute the Futaki invariant of the test configuration $\mathcal{X}$ induced by the deformation to the normal cone of $\mathbb{P}(F^*)$, i.e the test configuration obtanined by blowing up $\mathbb{P}(E^*) \times \mathbb{C}$ along $\mathbb{P}(F^*) \times \{0\}$. We get a formula for the Futaki invariant at the second order.

Theorem 4. *Let us assume that M is a surface and $rk(E) = 2$. Then,*

$$Fut(\mathcal{X}) = \gamma_1 m^3 + \gamma_2 m^2 + O(m)$$

where, if one denotes $\mu(F) = deg_L(F)/rk(F)$ the slope of F with respect to L,

$$\begin{aligned}
\gamma_1 &= \frac{c_1(L)^2}{6}(\mu(E) - \mu(F)) \\
\gamma_2 &= \frac{c_1(L)^2}{24}\left(\frac{c_1(E)}{2} - c_1(F)\right) c_1(M) + \frac{c_1(L)^2}{6}\left(ch_2(F) - \frac{ch_2(E)}{2}\right) \\
&\quad + \frac{1}{12}(\mu(F) - \mu(E))(c_1(M)c_1(L) + 2c_1(E)c_1(L)).
\end{aligned}$$

Here $ch_2(F)$ stands for second Chern character of F.

Step 3. An example. If we choose a Gieseker stable bundle E of rank 2 such that there exists F subbundle of E of rank 1 with $\mu(F) = \mu(E)$, then the Futaki invariant of the test configuration given by Step 2 is given by

$$Fut(\mathcal{X}) = m^2 \frac{c_1(L)^2}{6} \left(\frac{\chi(F \otimes L^k)}{rk(F)} - \frac{\chi(E \otimes L^k)}{rk(E)} \right) + O(m)$$

Obviously, $Fut(\mathcal{X})$ is strictly negative for $m >> 0$ because of the Gieseker stability assumption. This shows that $\mathcal{L}_m$ cannot carry a CSCK metric [Fu]. An example of such a vector bundle can be found on $\mathbb{CP}^2$ blown up at q points (with $q \geq 4$) in generic position. Note that from a result of G. Tian [Ti2], this Fano manifold is endowed with a Kähler-Einstein metric. Also note that $Aut(\mathbb{P}(E^*), \mathcal{L}_m)/\mathbb{C}^*$ is discrete. Eventually, the bundle E is constructed as a non trivial extension of a line bundle F such that $c_1(E)c_1(M) = c_1(F)c_1(M) = 0$ and $c_1(F)^2 << 0$.

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The class of the locus of intermediate Jacobians of cubic threefolds

Samuel Grushevsky
(joint work with Klaus Hulek)

In this talk we presented our joint work with Klaus Hulek, on computing the class of the locus of intermediate Jacobians of cubic threefolds in the cohomology of the moduli space $\mathcal{A}_5$ of principally polarized abelian varieties (ppav) of dimension 5, and of its closure — in the perfect cone toroidal compactification $\overline{\mathcal{A}_5}$. The detailed results will appear as [GH10].

Recall that in the foundational work [CG72] Clemens and Griffiths studied the intermediate Jacobian of a cubic threefold (a smooth degree 3 hypersurface in $\mathbb{P}^4$). This intermediate Jacobian is a five-dimensional ppav such that its theta divisor has a unique triple point, the tangent cone at which gives the cubic threefold — thus proving the Torelli theorem for cubic threefolds. In [CMF05, CM08] Casalaina-Martin and Friedman showed that within the locus of indecomposable (those that are not products of lower-dimensional ppav) abelian 5-folds the existence of a triple point on the theta divisor characterizes intermediate Jacobians of cubic threefolds.

Since the theta divisor Θ is symmetric under the involution -1 on the abelian variety A and has only one triple point, the triple point must be invariant under -1, and so is a two-torsion point $m \in A[2]$ (which is to say that $2m = 0$). Since the theta divisor is given as the zero locus of the theta function $\theta(\tau, z)$, where τ is the period matrix of A, and $z \in A$, we are thus looking for an odd point $m \in A[2]$ where the multiplicity of the theta function is equal to three. The corresponding theta function with characteristic, $\theta_m(z)$, is then an odd function of z, and thus the condition for it to vanish to order three is that its gradient, $\operatorname{grad}_z \theta_m(z)|_{z=0}$, vanishes.

For any genus g we thus define the locus $G^{(g)} \subset \mathcal{A}_g$ as the locus of ppav for which the theta divisor is singular (i.e. has multiplicity at least three) at an odd two-torsion point. By the above description we have then

$$G^{(g)} = \{\tau \in \mathcal{A}_g \mid \exists m \in A[2]_{\text{odd}};\ \operatorname{grad}_z \theta_m(z)|_{z=0} = 0\}.$$

By definition the locus G, being the zero locus of g components of the gradient, has codimension at most g. The following statement was conjectured in [GSM09]:

Conjecture 1. *The locus $G^{(g)} \subset \mathcal{A}_g$ is purely of codimension g.*

This conjecture is known to hold for genus up to 5, in which cases combining the known results about the singularities of theta divisors we have

$$G^{(1)} = G^{(2)} = \emptyset;\quad G^{(3)} = Sym^3(\mathcal{A}_1);\quad G^{(4)} = \mathcal{A}_1 \times \theta_{\text{null}}^{(3)};\quad G^{(5)} = I \cup (\mathcal{A}_1 \times \theta_{\text{null}}^{(4)})$$

where I is the locus of intermediate Jacobians of cubic threefolds, and $\theta_{\text{null}}^{(g)} \subset \mathcal{A}_g$ is the theta-null divisor — the locus of ppav for which there exists an even two-torsion point lying on the theta divisor (equivalently for which $\theta_m(\tau, 0) = 0$ for some even $m \in A[2]$).

It is known (and can be easily verified from the theta transformation formula) that the gradient of the theta function with characteristic at zero is a section of the bundle $\mathbb{E} \otimes \det \mathbb{E}^{\otimes 1/2}$ on the level cover $\mathcal{A}_g(4,8)$, where $\mathbb{E}$ denotes the rank g Hodge bundle — with the fiber over A being the space of abelian differentials $H^{1,0}(A, \mathbb{C})$. From the above definition of the locus G as the union of the vanishing loci of the gradients for all the $2^{g-1}(2^g - 1)$ odd 2-torsion points we thus get by computing the top Chern class $c_g(\mathbb{E} \otimes \det \mathbb{E}^{\otimes 1/2})$ the following

Theorem 2. *If the conjecture above holds, then the class of the locus $G^{(g)}$ is equal to*

$$\left[G^{(g)}\right] = 2^{g-1}(2^g - 1) \sum_{i=0}^{g} \lambda_{g-i} \left(\frac{\lambda_1}{2}\right)^i \in CH^*(\mathcal{A}_g)$$

where $\lambda_i := c_i(\mathbb{E})$ are the Chern classes of the Hodge bundle.

In particular using the description of the tautological ring obtained by van der Geer in [vdG99] we get $[G^{(5)}] = 16 \cdot 63 \cdot (3\lambda_1^5/32 + 3\lambda_1^2\lambda_3/4)$ and a new formula in genus 4. In genus 3 our formula agrees with van der Geer's [vdG98] computation of the class of $Sym^3(\mathcal{A}_1)$, and we get 0 in genus 2 as expected.

We then consider the problem of computing the class of the closure $\overline{G^{(g)}}$ in a suitable toroidal compactification $\overline{\mathcal{A}_g}$. This problem we study only for the perfect cone toroidal compactification — for which the complement of Mumford's partial compactification (obtained by adding semiabelian varieties of torus rank 1) is codimension two. To this end we recall that the Hodge bundle extends as a vector bundle to any toroidal compactification, and then show that the gradients of theta functions with characteristics also extend to the level cover perfect cone compactification $\overline{\mathcal{A}_g(4,8)}$.

However, their extensions will vanish generically on some irreducible components of the boundary. For technical reasons it is easier to work on $\overline{\mathcal{A}_g(8)}$, where we recall that the boundary divisor (of the level perfect cone toroidal compactification) has irreducible components D_n labeled by primitive vectors $n \in (\mathbb{Z}/8\mathbb{Z})^{2g}$. By studying the behavior of the gradients near the boundary explicitly we show that the extension of $\operatorname{grad}_z \theta_m(z)|_{z=0}$ vanishes identically, with vanishing order one, on a boundary component $D_n \subset \partial\overline{\mathcal{A}_g(8)}$ if and only if $n_2 + m$ is odd, where n_2 denotes the image of n in $(\mathbb{Z}/2\mathbb{Z})^{2g}$. For any odd $m \in (\mathbb{Z}/2\mathbb{Z})^{2g}$ we denote by Z_m the set of primitive vectors $n \in (\mathbb{Z}/8\mathbb{Z})^{2g}$ such that $m + n_2$ is odd, where n_2 is the image of n in $(\mathbb{Z}/2\mathbb{Z})^{2g}$. We can then consider the extensions of the gradients on $\overline{\mathcal{A}_g(8)}$ as sections of the vector bundle

$$\mathbb{E} \otimes \det \mathbb{E}^{1/2} \otimes \mathcal{O}\left(-\sum_{n \in Z_m} D_n\right)$$

Our second main result is then the following:

Theorem 3. *For $g \leq 5$ the class of the closure $\overline{G^{(g)}}$ of the locus $G^{(g)}$ is*

$$[\overline{G^{(g)}}] = \frac{1}{N} \sum_{m \text{ odd}} \sum_{i=0}^{g} \overline{p}_* \left(\lambda_{g-i} \left(\frac{\lambda_1}{2} - \sum_{n \in Z_m} \delta_n \right)^i \right),$$

where $\overline{p} : \overline{\mathcal{A}_g(2)} \to \overline{\mathcal{A}_g}$ is the level cover, δ_n is the class of D_n, and N is the order of the group $Sp(2g, \mathbb{Z}/8\mathbb{Z})$.

The formulas obtained from the theorem in genera 4 and 5 are new, while in genus 2 we still get zero as expected, and in genus 3 we get the same answer as in [vdG98]. Our answer also leads to further questions about computing the classes of intersections of boundary divisors of $\overline{\mathcal{A}_g(8)}$ that are of interest for trying to define and understand a suitable tautological ring for $\overline{\mathcal{A}_g(8)}$.

The difficulty in proving this theorem is in showing that there is no extra vanishing of the extensions of the gradients to the boundary — for example that there are no components of the zero loci of their extensions that are contained in the boundary. To prove this, we in fact describe explicitly the semiabelic theta divisors for *all* semiabelic varieties of torus rank up to 3, compute explicitly the limits of two-torsion points on these semiabelic varieties, the limits of the gradients of the theta function at these two-torsion points, and check that there is indeed no extra vanishing. These results take a lot of work, produce very nice explicit description of principally polarized, sometimes reducible, semiabelic varieties, and are of an independent interest.

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The theme of a vanishing period

Daniel Barlet

First recall what is a vanishing period. Let X be a complex manifold of dimension $n+1$, and $f : X \to D$ be a proper holomorphic function on a disc D in $\mathbb{C}$ such that $\{df = 0\} \subset \{f = 0\}$. We consider f as a family of compact complex manifolds of dimension n with a degeneration at $s = 0$.
For a smooth $(p+1)-$differential form ω on X such that $d\omega = 0 = df \wedge \omega$ and for $\gamma \in H_p(X_{s_0}, \mathbb{C})$ we define the holomorphic multivalued function

$$F_\gamma(s) := \int_{\gamma_s} \omega/df$$

where $(\gamma_s)_{s\in D^*}$ is the (multivalued) horizontal family of $p-$cycles in the fibers of f with value γ at $s = s_0 \in D^*$.
It is known that such a function has a convergent expansion at $s = 0$ of the form

$$F_\gamma(s) = \sum_{j=0}^{n}\sum_{i=1}^{r} c_{j,i}.s^{\alpha_i}.(Log\, s)^j$$

where $\alpha_1, \dots, \alpha_r$ are in $\mathbb{Q}\cap]-1,0]$, and $c_{j,i} \in \mathbb{C}\{s\}$ are holomorphic germs. This is, by definition, an **vanishing period**. Note that the restriction to $f^{-1}(s), s \neq 0$ of ω/df is a de Rham cohomology class of degree p for each $s \in D^*$, so for a given vanishing cycle γ, this function describes the way the corresponding period "vanishes" when $s \to 0$.
Now let $\tilde{\mathcal{A}} := \{\sum_{\nu}^{\infty} P_\nu(a).b^\nu\}$ be the $\mathbb{C}-$algebra defined by the relation $a.b - b.a = b^2$ and the fact that left and right multiplication by a is continuous for the $b-$adic topology. This algebra acts on the set Ξ of formal asymptotics expansions

$$\Xi := \oplus_{\alpha\in\mathbb{Q}\cap]0,1]}\ \Xi_\alpha$$
$$\Xi_\alpha := \oplus_{j=0}^{\infty}\mathbb{C}[[a]].s^{\alpha-1}.(Log\, s)^j = \oplus_{j=0}^{\infty}\mathbb{C}[[b]].s^{\alpha-1}.(Log\, s)^j$$

where $a := \times s$ and $b := \int_0^s$.
A **theme** is then, by definition, a left $\tilde{\mathcal{A}}-$module of the form $\tilde{\mathcal{A}}.\varphi \subset \Xi$ where $\varphi \in \Xi$ is any given element. This means that $\tilde{\mathcal{A}}.\varphi$ is "the minimal filtered differential equation" satisfied by φ. Then the theme associated to a vanishing period is the image in Ξ of the function F_γ via its asymptotic expansion at $s = 0$. A theme will be called $[\lambda]-$**primitive** when it is contained in Ξ_λ for some $\lambda \in]0,1] \cap \mathbb{Q}$. This will be the case if we choose γ in the generalized eigenspace of the monodromy corresponding to the eigenvalue $exp(2i\pi.\lambda)$. Our aim is to classify isomorphism classes (in the sens of left $\tilde{\mathcal{A}}-$modules) of $[\lambda]-$primitive themes.
In this talk we gave some stability properties and characterizations of themes as left $\tilde{\mathcal{A}}-$modules. For instance $[\lambda]-$primitive themes are characterized by the uniqueness of there Jordan-Hölder sequence. Then we show how to construct a standard family, which has a finiete dimensional parameter space, of $[\lambda]-$primitive themes with given fundamental invariants. This means that, in these holomorphic deformations, a precise version of the Bernstein polynomial is fixed. The standard

families are always versal, but not, in general, universal. Then we discuss the universality of a standard family and show that it is universal when all $[\lambda]-$primitive themes with these fundamental invariants are "monodromy invariant" (note that the monodromy acts on Ξ_λ.). This condition is satisfied in many cases.

Enriques surfaces with many (semi-)symplectic automorphisms

Shigeru Mukai

An automorphism of a K3 surface X is *sympletic* if it acts on $H^0(\mathcal{O}_X(K_X))$ trivially. All finite groups which have symplectic actions on K3 surfaces are classified in terms of the Mathieu group M_{24} by Mukai [4] and Kondo [2]. An automorphism of an Enriques surface S is *semi-symplectic* if it acts on $H^0(\mathcal{O}_S(2K_S))$ trivially. A smart classification similar to K3 surfaces is desirable for semi-symplectic actions of Enriques surfaces but still far from complete investigation. Here I propose a restricted class of semi-symplectic actions.

Definition An effective semi-symplectic action of a finite group G on an Enriques surface is *M-semi-sympletic* if the Lefschetz number of g equals 4 for every automorphism $g \in G$ of order 2 and 4.

Here the Lefschetz number of an automorphism σ is the Euler number of the fixed point locus Fix σ, and equals to the trace of the cohomology action of σ on $H^*(S, \mathbb{Q})$.

M-semi-symplectic actions are closely related to the symmetric group S_6 of degree 6 via the Mathieu group M_{12} though S_6 itself has no semi-symplectic actions. It is known that S_6 has six maximal subgroups upto conjugacy, and four modulo automorphisms. The four subgroups are

(1) the alternating group A_6,
(2) the symmetric group S_5 of degree 5,
(3) $(C_3)^2.D_8$, the normalizer of a 3-Sylow subgroup, and
(4) the direct product $S_4 \times C_2$,

where C_n and D_n denote a cyclic and a dihedral group of order n, respectively.

Theorem 1. *The three maximal subgroups $A_6, S_5, (C_3)^2.D_8$ and the abelian group $(C_2)^3$ have M-semi-symplectic actions on Enriques surfaces.*[1]

Remark By Kondo [1], there are two Enriques surfaces whose automorphism groups are isomorphic to S_5. One is called type VII and the other is the quotient of the Hessian of a special cubic surface (type VI). The action of S_5 is M-semi-symplectic for the former and not for the latter.

The action of the three maximal subgroups are constructed refining the method of [5]. We use

(1) embeddings of S_6 into the Mathieu group M_{12},

[1]In my talk at MFO I stated $A_4 \times C_2$ in place of $(C_2)^3$ by mistake. I do not know whether an M-semi-symplectic action of $A_4 \times C_2$ exists or not.

(2) the action of $M_{12} \times C_2$ on the Leech lattice, and
(3) Torelli type theorem for Enriques surfaces.

An Enriques surface $S = Km(E_1 \times E_2)/\varepsilon$ of Lieberman type has a semi-symplectic action of $(C_2)^4$ by translation by 2-torsion points, where both E_1 and E_2 are ellliptic curves and ε is a free involution of the Kummer surface $Km(E_1 \times E_2)$ (see [3]). One involution in $(C_2)^4$ is *numerically trivial*, that is, its Lefschetz number attains the maximum ($= 12$) but the others are M-semi-symplectic. Hence S has an M-semi-symplectic action of $(C_2)^3$.

Question Is a finite group embeddable into the symmetric group S_6, if it has an (effective) M-semi-symplectic action on an Enriques surface?

The definition of M-semi-symplectic action is modeled on the permutation group M_{12} of degree 12. The permutation type of $g \in M_{12}$ depends only on its order n when it has a fixed point. The type and the number of fixed points $\mu_+(n)$ are as follows.

n	1	2	3	4	5	6	8	11
permutation type	(1)	$(2)^4$	$(3)^3$	$(4)^2$	$(5)^2$	(6)(3)(2)	(8)(2)	(11)
$\mu_+(n)$	12	4	3	4	2	1	2	1

It is well known that a symplectic involution of a K3 surface have exactly 8 fixed points. But for an involution σ of an Enriques surface, the fixed point set $\mathrm{Fix}\,\sigma$ is not necessarily finite and the Lefschetz number varies from -4 to 12. (Note that every involution of an Enriques surface is semi-symplectic.) The required value 4 in our definition is one half of 8, the mean of -4 and 12, and equals to both $\mu_+(2)$ and $\mu_+(4)$. A semi-symplectic action of a group G on an Enriques surface is M-semi-symplectic if and only if the Lefschetz number and μ_+ are the same on G since the order of semi-symplectic automorphism is either ≤ 6 or ∞ by H. Ohashi.

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