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Actions and Invariants of Residually Finite Groups: Asymptotic Methods

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September 5th – September 11th, 2010

Abstract. The workshop brought together experts in finite group theory, L^2-cohomology, measured group theory, the theory of lattices in Lie groups, probability and topology. The common object of interest was residually finite groups, that each field investigates from a different angle.

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Introduction by the Organisers

The workshop 'Actions and Invariants of Residually Finite Groups: Asymptotic Methods' organized by Miklos Abert (Budapest), Damien Gaboriau (Lyon) and Fritz Grunewald (Dusseldorf) was held September 5 - September 11, 2010. Fritz Grunewald tragically and unexpectedly passed away in March 2010. Fritz was a great person and an excellent mathematician and his presence will be missed by all of us. Abert took over as the contact organizer and the MFO has been very efficient with helping him in this new situation.

The wokshop aimed to bring together experts in different fields, like finite group theory, L^2-cohomology, measured group theory, the theory of lattices in Lie groups, probability and topology. The common object of interest was residually finite groups, that each field investigates from a different angle. There are various group invariants whose asymptotic behaviour on the subgroup lattice of a residually finite group is connected to an interesting analytic invariant of the group. These include the rank and various homological and spectral invariants. The usual setting is to consider a normalized limit of the group invariant on a descending chain in the subgroup lattice that approximates the group. The group acts by automorphisms on the corresponding coset tree (a locally finite rooted tree) and this action extends to a measure preserving action on the boundary of the tree. One can connect the dynamics of this boundary action to asymptotic properties of the group.

The meeting was attended by over 50 participants from all over the world and of many fields, including measured group theory, asymptotic group theory, graph theory and topology. There were 24 talks over the five days of the workshop. All talks were 50 minute long except Evija Ribnere's, which was 30 minutes. This way, less people had the opportunity to talk but they could give a more substantial presentation. This setup seemed to work really well for this particular workshop.

As a general rule, the organizers asked people to talk about specific subjects, not just any nice piece of research. In some cases, this meant sacrificing hearing about some new results from excellent mathematicians that were further away from the workshop's main directions. What the workshop gained was a strong focus.

The participant body came from a wide range of areas and people typically did not speak each other's mathematical dialect fluently. To address this, the organizers asked the speakers to put a special emphasis on (and give extra length to) the first, introductory part of their talks. Most of the speakers did this wonderfully. Furthermore, some of the key talks had a major survey aspect, in particular John Wilson gave an introduction to profinite groups, Balazs Szegedy gave an introductory talk on the emerging theory of limits of finite structures, Gabor Elek talked about graph limits in particular and their use in group theory, Narutaka Ozawa gave a survey talk on group measure space von Neumann algebras and measured group theory and Damien Gaboriau gave an introductory talk on orbit equivalence and cost.

On Thursday, some of the talks had a memorial aspect, in particular the talks given by Dan Segal, Martin Bridson and Evija Ribnere put a special emphasis on the mathematics and personality of Fritz Grunewald. Dan Segal kindly agreed to organize a ping-pong tournament in memoriam Fritz, since he liked to play that game in Oberwolfach and was quite good at it. (For the record, Romain Tessera won the tournament with ease.)

There were two talks, where the full proof of the main result could be presented. First, Jan Christoph Schlage-Puchta very recently constructed residually finite, non-amenable torsion p-groups, using rank gradient (note that the first, more complicated construction of this kind is due to Denis Osin). The existence of such groups (not necessarily non-amenable, only infinite) was one of the big open questions that brought Golod-Savarevich groups to life. Puchta's solution is astonishingly simple, in fact it could be presented in a first year undergraduate class without difficulties. Another talk of this kind was from Lukasz Grabowski, a student of Andreas Thom; he has recently found a really transparent way to compute the spectral measure of various group ring elements over the lamplighter group. This topic (versions of the conjectures named after Atiyah) has been considered highly technical and difficult in the past.

Nikolay Nikolov talked about his joint with Miklos Abert on rank gradient of groups, its unexpected connection to cost and how that is related to the rank vs Heegaard genus conjecture.

There were two talks that concentrated on the asymptotic behaviour of invariants for lattices in Lie groups. Tsachik Gelander talked about volume vs. rank of lattices. His main result says that for the family of lattices in a fixed connected semisimple Lie group without compact factors, the rank grows at most linearly in the covolume. Nicolas Bergeron talked about his joint work with Akshay Venkatesh, where they study the growth of torsion in the homology of 3-manifold groups.

Andreas Thom talked about integrality properties of spectral measures. Spectral measures of group ring operators over residually finite groups are weak limits of root distributions of monic polynomials of integer coefficients, and this already means a strong sturctural restriction on them. It is clear that there must be further restrictions but it is not known exactly what kind.

Emmanuel Breuillard talked about his joint work with Ben Green and Terence Tao about approximate groups and Laszlo Pyber talked about his joint work with Endre Szabo on growth in finite simple groups of Lie type. These two talks were very much related, as the results are highly correlated, but independent – it was very interesting to hear the story from two quite different perspectives.

Mikhail Ershov talked about his joint with Andrei Jaikin-Zapirain on groups of positive weighted deficiency. This talk was related to Jan Christoph Schlage-Puchta's talk as it generalized some of the notions there. They proved a very exciting result, namely, the existence of 'residually finite Tarski monsters', that is, infinite finitely generated residually p-groups where every finitely generated subgroup either has finite index or is finite.

Andrei Jaikin-Zapirain talked about his joint work with Fritz Grunewald, Aline Pinto and Pavel Zalesski on profinite completions and 3-manifold groups.

Roman Sauer talked about his joint work with Peter Linnell and Wolfgang Lück, where they generalize the Lück approximation result for arbitrary fields when the fundamental group is amenable. It is an open problem whether such result holds for any group and it is even unclear what exactly to identify the limit with.

Mark V. Sapir talked about his joint work with J. Behrstock and C. Drutu on homomorphisms into the mapping class groups. This work also uses a limit concept, namely asymptotic cones (that uses ultrafilters).

Balint Virag talked about his joint work with Miklos Abert and Yair Glasner. They generalized Kesten's theorem on vertex transitive graphs to random unimodular networks. This resulted in a new rigidity result on the eigenvalue distributions of finite d-regular graphs.

Mikael Pichot talked about his new work on random groups of intermediate rank. The idea is to generalize the Gromov random group model in a novel way, using a high index finite sheeted cover and then omitting some of the new relations randomly.

Finally, Rostislav Grigorchuk talked about a question of Wiegold and torsion images of Coxeter groups and Todor Tsankov talked about unitary representations of oligomorphic groups. These talks were a bit further from the main directions of the workshop but were certainly interesting to hear.

There were some informal and formal evening sessions with a high attendance. These sessions typically lasted for 2-3 hours and people were free to come in and out anytime. In particular Balazs Szegedy spoke about his new results in higher order Fourier theory, Gabor Elek talked more about graph limits and Miklos Abert talked about profinite actions and weak containment. There was a session run by the two organizers on Friday, that went from 9pm to 2am – it started as a problem session and then evolved into a joyful exchange of ideas and questions. The organizers also attempted to run a young researchers session on one of the evenings.

Workshop: Actions and Invariants of Residually Finite Groups: Asymptotic Methods

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Abstracts

Orbit equivalence and cost

Damien Gaboriau

Two probability measure preserving (p.m.p.) actions of countable (infinite) groups on standard Borel spaces $\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$ and $\Lambda \overset{\beta}{\curvearrowright} (Y,\nu)$ are **orbit equivalent** (OE) if there is an isomorphism $f : (X,\mu) \sim (Y,\nu)$ such that $f_*\mu = \nu$ and $f(\Gamma x) = \Lambda f(x)$, μ-a.e. $x \in X$.

The cost is an invariant of OE for such actions: A **measured equivalence relation**, such as the "orbit equivalence relation" of an action $\mathcal{R}_{\Gamma\curvearrowright X} = \{(x,y) :$ x and y are in the same Γ-orbit $\}$ is **generated** by a countable family $\Phi := (\varphi_j : A_j \to B_j)_{j\in J}$ of measure preserving **partial isomorphisms** if $\mathcal{R}$ coincides with $\mathcal{R}_\Phi$: the smallest equivalence relation such that $\forall i \in I$, $\forall x \in A_j$: $x \sim \varphi_j(x)$ (in other words, $x\mathcal{R}_\Phi y$ iff some word m in $\varphi_j^{\pm 1}$ is defined at x and $m(x) = y$).

Definition 1 (Levitt [Lev95]). *The* ***cost of*** Φ= $(\varphi_j : A_j \to B_j)_{j\in J}$ *is the number of generators weighted by the measure of their support:* $\mathrm{cost}(\Phi) = \sum_{j\in J} \mu(A_j)$.

Definition 2 (Levitt [Lev95]). *The* ***cost of*** $\mathcal{R}$ *is the infimum over the costs of its generating graphings:* $\mathrm{cost}(\mathcal{R}) = \inf\{\mathrm{cost}(\Phi) : \Phi$ *generates* $\mathcal{R}\}$.

The cost has been computed in some situations [Gab00], leading to distinguish modulo OE the actions of some groups such as the free group $\mathbf{F}_n$ on n generators ($\mathrm{cost}(\mathcal{R}_{\mathbf{F}_n\curvearrowright X}) = n$) or $\mathrm{SL}(2,\mathbb{Z})$ ($\mathrm{cost}(\mathcal{R}_{\mathrm{SL}(2,\mathbb{Z})\curvearrowright X}) = 1 + \frac{1}{12}$). More generally, for free products:

Theorem 3 (G. [Gab00]). *For free actions of finitely generated groups:*
$\mathrm{cost}(\mathcal{R}_{\Gamma_1*\Gamma_2\curvearrowright(X,\mu)}) = \mathrm{cost}(\mathcal{R}_{\Gamma_1\curvearrowright(X,\mu)}) + \mathrm{cost}(\mathcal{R}_{\Gamma_2\curvearrowright(X,\mu)})$.

However, there are quite few groups for which we know the cost of all free actions, and no counterexample to the following question:

Question 4 (Fixed Price Question). *Does* $\mathrm{cost}(\mathcal{R}_{\Gamma\curvearrowright(X,\mu)})$ *depend on the particular free p.m.p. action?*

An interesting family of examples is that of **profinite actions**: A decreasing sequence $\Gamma = \Gamma_0 \geq \Gamma_1 \geq \Gamma_2 \geq \cdots \geq \Gamma_i \geq \cdots$ of finite index subgroups (a **chain**) delivers the measure preserving **profinite** Γ-action on the space of ends ∂T of the coset tree T (with the cosets as vertices and edges given by inclusions of Γ/Γ_{i+1}-cosets in Γ/Γ_i-cosets). Abert-Nikolov related the cost of this limiting action to the asymptotic behavior of the number of generators (the **rank**) of the Γ_i's, supporting the feeling that the cost counts a number of generators:

Theorem 5 (Abert-Nikolov [AN07]). *If the* Γ*-action on* $\partial T(\Gamma,(\Gamma_i))$ *is free, then*

$$\mathrm{cost}(\Gamma\curvearrowright\partial T) - 1 = \lim_{i\to\infty} \frac{\mathrm{rank}(\Gamma_i) - 1}{[\Gamma : \Gamma_i]} \qquad \Big(=: \underbrace{\mathrm{RG}(\Gamma,(\Gamma_i))}_{\text{Lackenby rank gradient}}\Big)$$

A negative answer to the fixed price question for Γ would imply the independance of the rank gradient on the chain!

Let's now introduce some geometry and cohomology in the picture. Consider a **cocompact** free action $\Gamma \curvearrowright L$ on a simplicial complex L and the associated tower of coverings $\Gamma_0\backslash L \leftarrow \Gamma_1\backslash L \leftarrow \Gamma_2\backslash L \leftarrow \cdots \Gamma_i\backslash L \leftarrow \cdots\cdots L$

The asymptotic behavior of the Betti numbers $b_n(\Gamma_i\backslash L)$ along the tower is related to the ℓ^2-Betti numbers $b_n^{(2)}(L;\Gamma)$ of the action $\Gamma \curvearrowright L$.

Theorem 6 (Lück [Lüc94], Farber [Far98]). *If the* Γ*-action on* $\partial T(\Gamma, (\Gamma_i))$ *is free, then*

$$\lim_{i\to\infty} \frac{b_n(\Gamma_i\backslash L)}{[\Gamma : \Gamma_i]} = b_n^{(2)}(L;\Gamma)$$

The result was obtained by Lück in case the chain is normal and has trivial intersection and extended by Farber to free profinite actions. The general case of any chain of finite index subgroups is investigated in [BG04], the above sequence is shown to still converge (non necessarily toward $b_n^{(2)}(L;\Gamma)$) and the limit is interpreted as a "lamination Betti number" for the profinite action, more precisely, the ℓ^2-Betti numbers for the lamination $\Gamma\backslash(\partial T \times L)$.

In case Γ is finitely presented, the cocompact L can be choosen to be also simply connected. And in this case, $b_1(\Gamma_i\backslash L) = b_1(\Gamma_i)$ and $b_1^{(2)}(L;\Gamma) = b_1^{(2)}(\Gamma)$ are invariants of the group.

Now the cost and first ℓ^2-Betti number are related:

Theorem 7 ([Gab02]). $\beta_1(\Gamma) \leq \mathrm{cost}(\mathcal{R}_{\Gamma \curvearrowright (X,\mu)}) - 1$*, for free actions of* Γ.

And there no counterexample to the following stronger form of the fixed price question:

Question 8. *Is there some infinite group with a free action such that* $\beta_1(\Gamma) < \mathrm{cost}(\mathcal{R}_{\Gamma \curvearrowright (X,\mu)}) - 1$*?*

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Graph limits

Gabor Elek

The goal of the talk was to give a survey of some recent development on the the Benjamini-Schramm type graph limits and its variants. Let Gr_d be the set of simple graphs with vertex degree bound d. A rooted (r,d)-ball is a finite, simple, connected graph H in the class Gr_d such that

- H has a distinguished vertex x (the root).
- $d_G(x,y) \leq r$ for any $y \in V(H)$.

For $r \geq 1$, we denote by $U^{r,d}$ the finite set of rooted isomorphism classes of rooted (r,d)-balls. Let $G(V,E)$ be a finite graph with vertex degree bound d. For $\alpha \in U^{r,d}$, $T(G,\alpha)$ denotes the set of vertices $x \in V(G)$ such that there exists a rooted isomorphism between α and the rooted r-ball $B_r(x)$ around x. Set $p_G(\alpha) := \frac{|T(G,\alpha)|}{|V(G)|}$. Thus one can associate to a graph $G \in Gr_d$ a probability distribution on $U^{r,d}$ for any $r \geq 1$. Let $\{G_n\}_{n=1}^{\infty} \subset Graph_d$ be a sequence of finite simple graphs such that $\lim_{n\to\infty} |V(G_n)| = \infty$. Then $\{G_n\}_{n=1}^{\infty}$ is called *convergent* if for any $r \geq 1$ and $\alpha \in U^{r,d}$, $\lim_{n\to\infty} p_{G_n}(\alpha)$ exists. This notion is also called *localconvergence*

Let (X,μ) be a probability measure space and $\mathbf{\Psi} := \{\Psi_i\}_{i=1}^{n}$ be a family of measure preserving Borel transformations. If $x \neq y$, call x and y adjacent, if $\Psi_i(x) = y$ or $\Psi_i(y) = x$ for some $1 \leq i \leq n$. This graph is a *graphing* $\mathcal{G}$. Now let $\alpha \in U^{r,d}$ be rooted graph, then $p_{\mathcal{G}}(\alpha)$ is defined as the probability measure of points x in X for which the r-neighborhood of x is isomorphic to α. This is called the *local statistic* of the graphing. We say that a sequence of graphs $\{G_n\}$ *converges* to $\mathcal{G}$ if for each $r \geq 1$ and $\alpha \in U^{r,d}$

$$\lim_{n\to\infty} p_{G_n}(\alpha) = p_{\mathcal{G}}(\alpha)\,.$$

We show that for each convergent graph sequence $\{G_n\}_{n=1}^{\infty}$ there exists a limit graphing. The converse is a conjecture of Aldous and Lyons [1]. Following Bollobás and Riordan [2] (and also Lovász and Szegedy) we introduced the notion of global-local convergence. Let $\{1,2,\ldots,q\}$ be a finite set and let $\tau : V(G) \to q$ be a q-labeling of the vertices of G. Denote by $U^{r,d,q}$ the set of q-labeled rooted r-graphs. Observe that any labelling τ induces a distribution on $U^{r,d,q}$. Now let $\Sigma_{d,q}$ denote the set of all countable rooted q-labeled graphs with vertex degree bound d. The set of such graphs form a compact metric space such a way that for any $\alpha \in U^{r,d,q}$, W^α is an open-closed set, where W^α is the set of all rooted labeled countable graphs for which the r-neighborhood of the root is isomorphic to α. If G is a finite graph then the set of q-labellings define a finite set $M(G,q)$ in the space of all probability measures on $\Sigma_{d,q}$. We say that a sequence of finite graphs $\{G_n\}_{n=1}^{\infty}$ is *globally-locally* convergent if for any $q \geq 1$ the associated finite sets $M(G_n,q)$ converge in the Hausdorff topology of the compact space of probability measures on $\Sigma_{d,q}$.

Again, let $\mathcal{G}(X,\mu)$ be a graphing. Observe that any measurable function $\tau : X \to \{1,2,\dots,q\}$ defines an element of the probability measure spaces of $\Sigma_{d,q}$. The closure of these sets in the Hausdorff topology is denoted by $M(\mathcal{G},q)$. This compact set is called the *global-local statistic* of the graphing. We say that $\mathcal{G}$ is the global-local limit of $\{G_n\}_{n=1}^{\infty}$ if for any q the sets $M(G_n,q)$ converge to $M(\mathcal{G},q)$. Following Lovász and Szegedy we showed that for any globally-locally convergent graph sequence there exists a global-local limit. We also show that if $\{G_n\}_{n=1}^{\infty}$ is a covering tower of finite graphs, they converge globally-locally to the inverse limit graphing of the tower.
A graphing parameter is *local* resp. *global-local* if it depends on the local resp. global-local statistic of the graphing. We showed that the L^2-Betti numbers, introduced by Gaboriau [3] are local invariants, on the other hand the cost is a global-local invariant.

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Rank gradient of groups

Nikolay Nikolov
(joint work with Miklos Abert)

Let Γ be a finitely generated group. A *chain* in Γ is a sequence $\Gamma = \Gamma_0 \geq \Gamma_1 \geq \dots$ of subgroups of finite index in Γ. Let $T = T(\Gamma,(\Gamma_n))$ denote the *coset tree*, a rooted tree on the set of right cosets of the Γ_n defined by inclusion. The group Γ acts by automorphisms on T; this action extends to the boundary ∂T of T, the set of infinite rays starting from the root. The boundary is naturally endowed with the product topology and product measure coming from the tree and Γ acts by measure-preserving homeomorphisms on ∂T. We say that a chain (Γ_n) satisfies the *Farber condition*, if the action of Γ on $\partial T(\Gamma,(\Gamma_n))$ is essentially free, that is, almost every element of $\partial T(\Gamma,(\Gamma_n))$ has trivial stabilizer in Γ. This is the case e.g. when the chain consists of normal subgroups of Γ and their intersection is trivial. Note that in this case ∂T is simply the profinite completion of Γ with respect to (Γ_n) endowed with the normalized Haar measure.

For a group G let $d(G)$ denote the minimal number of generators (or rank) of G. Let the *rank gradient* of Γ with respect to (Γ_n) be defined as

$$\mathrm{RG}(\Gamma,(\Gamma_n)) = \lim_{n\to\infty} \frac{d(\Gamma_n)-1}{|\Gamma : \Gamma_n|}$$

This notion has been introduced by Lackenby.

It is a natural question whether this limit depends on the choice of the chain. We make the following

Conjecture 1. *For a finitely generated residually finite group* Γ *and a normal chain* (Γ_n) *with trivial intersection, the value of* $\mathrm{RG}(\Gamma, (\Gamma_n))$ *does not depend on the choice of the chain.*

This Conjecture is motivated by a newly found connection between the rank gradient and the cost of measurable equivalence relations.

Theorem 2. *Let* (Γ_n) *be a chain in* Γ *that satisfies the Farber condition. Then*

$$\mathrm{RG}(\Gamma, (\Gamma_n)) = \mathrm{cost}(E) - 1$$

where $E = E(\partial T(\Gamma, (\Gamma_n)))$ *denotes the orbit equivalence relation on* $\partial T(\Gamma, (\Gamma_n))$ *defined by the action of* Γ.

A group Γ has *fixed price*, if all essentially free Borel-actions of Γ have the same cost. *The Fixed Price Conjecture* asks whether every group has fixed price and if true this will trivially solve Conjecture 1.

Conjecture 1 has a direct application to the 'rank vs. Heegaard genus' problem for hyperbolic 3-manifolds. For a closed 3-manifold M let $g(M)$ denote the Heegard genus of M and let $r(M) = d(\pi_1(M))$ be the rank of the fundamental group of M. Waldhausen asked if it is always true that $d(\pi_1 M) = g(M)$. This was shown not to be true in general but the question remains open for hyperbolic 3-manifolds.

Theorem 3. *If Conjecture 1 is true then the ratio* $g(M)/r(M)$ *gets arbitrarily large for hyperbolic* 3*-manifolds* M.

Profinite groups - a short introduction

John S. Wilson

This lecture was commissioned by the organizers.It began with equivalent definitions and a discussion of the ubiquity of profinite groups in pure mathematics. Some key basic properties, tools used in their study and recent results were described, and their relevance to the study of residually finite groups was illustrated.

Finally, the following result, which holds in both the category of abstract groups and the category of pro-p groups, was discussed: if G has a presentation with n generators and r relations where $n > r$, then any generating set for G has $n - r$ elements that freely generate a free group. My new proof is much simpler and more direct than my earlier one from 2002.

Rank rigidity for simple Lie groups

Tsachik Gelander

Based on the [Ge1].

For a group Γ we denote by $d(\Gamma)$ the minimal cardinality of a generating set. When considering the class of finite index subgroups Γ in some given finitely generated group Δ, it is easy to show that $d(\Gamma)$ is at most $(d(\Delta)-1)[\Delta:\Gamma]+1$ and in particular bounded linearly by the index. We prove the analog statement when Δ is replaced by a connected semisimple Lie group G.

Theorem 1. *Let G be a connected semisimple Lie group without compact factors. Then there is a computable constant $C=C(G)$ such that*

$$d(\Gamma)\leq C vol(G/\Gamma),$$

for every irreducible lattice $\Gamma\leq G$.

Theorem 1 implies in particular the well known but nontrivial fact that every lattice is finitely generated. Originally the finiteness of $d(\Gamma)$ was proved case by case by many different authors, notably is the work of Garland and Raghunathan [GR] for $\mathrm{rank}(G)=1$ and of Kazhdan [K] for the case where all the factors of G have rank ≥ 2. (Note in particular that our proof does not rely on Kazhdan's property (T).)

Note also that Theorem 1 implies the classical Kazhdan–Margulis theorem:

Corollary 2 (Kazhdan–Margulis Theorem [KM])**.** *For any semisimple Lie group without compact factors G, there is a positive lower bound on the covolume of lattices.*

Proof. Indeed,

$$\mathrm{vol}(G/\Gamma)\geq\frac{d(\Gamma)}{C}\geq\frac{2}{C},$$

for every lattice $\Gamma\leq G$. □

In general, the linear estimate in Theorem 1 cannot be improved. For instance, for every $n\geq 2$ the group $G=\mathrm{SO}(n,1)$ admits a lattice Γ which projects onto the free group F_2, hence taking Γ_k in Γ to be the pre-image of an index k subgroup of F_2, we get $d(\Gamma_k)\geq k+1$ while $\mathrm{vol}(G/\Gamma_k)=k\cdot\mathrm{vol}(G/\Gamma)$. On the other hand, in general one cannot give a lower bound on $d(\Gamma)$ which tends to ∞ with $\mathrm{vol}(G/\Gamma)$. Moreover, by [SV], when $\mathrm{rank}_{\mathbb{R}}(G)\geq 2$, every nonuniform lattice in G admits a 3-generated finite index subgroup.

Another immediate consequence of Theorem 1 is that the first Betti number grows at most linearly with the covolume:

Corollary 3. *The first Betti number $b_1(\Gamma,A)$ of a lattice $\Gamma\leq G$, with respect to an arbitrary ring A, is at most $C vol(G/\Gamma)$.*

Proof.

$$b_1(\Gamma,A)\leq d(\Gamma)\leq C\mathrm{vol}(G/\Gamma).$$

□

Corollary 3 extends Theorem 2 from [BGS] to locally symmetric orbifolds rather than manifolds. However, while in [BGS] it is shown that *all* the Betti numbers are bounded by the volume of the manifold, here we obtain a bound only on the first Betti number. It is likely that the higher Betti numbers of orbifolds are bounded by the volume as well. When G has property (T), the first Betti number with coefficients in $\mathbb{Z}$, $b_1(\Gamma, \mathbb{Z})$ vanishes for every lattice $\Gamma \leq G$, however for rings with torsion $b_1(\Gamma, A)$ is typically nonzero.

As another consequence we obtain a concrete upper bound on the number $AL_G(v)$ of conjugacy classes of arithmetic subgroups of G of covolume at most v:

Corollary 4. *Let G be a connected semisimple lie group without compact factors. Then for every $\epsilon > 0$*

$$AL_G(v) \leq v^{(1+\epsilon)Cv},$$

for all $v \gg 0$.

Corollary 4 was proved in [BGLS] for rank one groups (see [BGLS, Theorem 1.1]). It was also shown in [BGLS] that for $\mathrm{SO}(n, 1)$ this estimate is sharp in the sense that $AL_{\mathrm{SO}(n,1)}(v) \geq v^{av}$ for some constant $a > 0$ and v sufficiently large, and for $G = \mathrm{SL}(2, \mathbb{R})$ the following precise asymptotic formula was given $\log AL_{\mathrm{SL}(2,\mathbb{R})}(v) \sim \frac{v}{2\pi} \log v$. When $\mathrm{rank}(G) > 1$ the estimate given in Corollary 4 is somewhat weak. In fact Belolipetsky and Lubotzky [BL] have shown recently that if all lattices in G posses the Congruence Subgroup Property then

$$v^{a \log v} \leq AL_G(v) \leq v^{b \log v},$$

for some $a, b > 0$ and all $v \gg 0$.

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Torsion in the homology of 3-manifold groups

Nicolas Bergeron
(joint work with Akshay Venkatesh)

1. Introduction

Let M be a compact 3-manifold and Γ be its fundamental group. The group Γ is residually finite. Consider a *residual chain*, that is a decreasing sequence of finite index normal subgroups $\Gamma_N \lhd \Gamma_{N-1} \lhd \cdots \lhd \Gamma$ with the property that $\cap_N \Gamma_N = \{1\}$. Let M_N be the finite cover of M corresponding to Γ_N.

As $\mathbb{Z}$-modules, the homology groups $H_1(M_N, \mathbb{Z})$ have a free part $H_1(M_N)_{\mathrm{free}}$ and a torsion part $H_1(M_N)_{\mathrm{tors}}$. We study the asymptotic behavior $\# H_1(M_N)_{\mathrm{tors}}$, as N tends to infinity. This can be surprisingly large.

1.1. (Follows from Silver and Williams [6].) Let $k \subset \mathbb{S}^3$ be a knot. Denote by M_N the N-th cyclic cover of $\mathbb{S}^3 - k$. Then:

$$\lim_{N \to +\infty} \frac{\log \# H_1(M_N)_{\mathrm{tors}}}{N} = m(\Delta_k).$$

Here Δ_k is the Alexander polynomial of k and m is the (logarithmic) Mahler measure $m(P) = \int_{\mathbb{S}^1} \log |P|$ (the integral is taken with respect to the Haar probability measure).

1.2. Numerical experiments ([3], and unpublished data computed by Calegari-Dunfield in connection with [1]) suggest that arithmetic hyperbolic 3-manifolds should have a lot of torsion in their homology. Recently, M. Sengun has carried out [5] extensive computations of the integral cohomology of Bianchi groups. His results appear to lend support to our conjecture stated below. He considers congruence subgroups of the type

$$\Gamma_0(\mathfrak{p}) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathrm{SL}_2(\mathcal{O}_d) \ : \ c \equiv 0 \bmod \mathfrak{p} \right\}.$$

Here $\mathcal{O}_d$ is the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ and $\mathfrak{p} \subset \mathcal{O}_d$ is a prime ideal. He has collected date on the ratio of $\log \# H_1(\Gamma_0(\mathfrak{p}))_{\mathrm{tors}}$ to the covolume of $\Gamma_0(\mathfrak{p})$ in the case of the five Euclidean $\mathcal{O}_d$ and with respect to the norm of $\mathfrak{p}$. Ignoring the first 500 entries in each case, the average ratios read 0.054291, 0.053140, 0.055386, 0.053206, 0.053131 respectively for $d = 1, 2, 3, 7, 11$. The range of his computations were up to norm 45000, 30000, 45000, 21000, 21000 respectively. It is very significant that the ratio is very close to

$$\frac{1}{6\pi} \approx 0.053051.$$

2. Statements of the main results

2.1. Along the same lines as the Silver-Williams' theorem, we can prove the following theorem.

Theorem 1 (B.-Venkatesh, Raimbault, 2010)**.** *Assume that* $\pi_1(M) \twoheadrightarrow \mathbb{Z}$. *Let* M_N *be the corresponding* N*-th cyclic covers. Then:*

$$\lim_{N \to +\infty} \frac{\log \# H_1(V_N)_{\mathrm{tors}}}{N} = \log M(\Delta_i).$$

Here Δ_i *is the first non-vanishing Alexander polynomial of the corresponding* $\mathbb{Z}$*-cover.*

2.2. What about residual chains ? We state a conjecture in the case of arithmetic 3-manifolds.

Let B be a quaternion division algebra over an imaginary quadratic field, and $\mathfrak{o}_B$ a maximal order. Then $\mathfrak{o}_B^{\times}$ embeds into $\mathrm{PGL}_2(\mathbb{C})$, and acts on $\mathbb{H}^3$; let M be the quotient.

Conjecture 2 (B.-Venkatesh, 2010)**.** *Let* M_N *be any residual chain of congruence cover of* M*. Then:*

$$\frac{\log \# H_1(M_N)_{\mathrm{tors}}}{vol(M_N)} \longrightarrow \frac{1}{6\pi}$$

as N *tends to infinity.*

2.3. Conjecture 2 is wide open but we can prove an unconditional result for some (most) twisted local systems. There is a particularly interesting (acyclic) bundle that exists for certain arithmetic hyperbolic 3-manifolds:

Let L be the set of trace-zero elements in $\mathfrak{o}_B$, considered as a $\pi_1(M)$-module via conjugation. Then $H^1(M, L \otimes \mathbb{C}) = 0$. This is a consequence of Weil local rigidity. We can prove the following:

Theorem 3. *Let* M_N *be any sequence of finite coverings of* M *such that the injectivity radius of* M_N *tends to infinity with* N*. Then:*

$$\frac{\log \# H_1(M_N, L)}{vol(M_N)} \longrightarrow \frac{1}{6\pi}.$$

This has the following consequence: Although the defining representations $\pi_1(M_N) \to \mathrm{SL}_2$ do not deform over the complex numbers, they do deform *modulo* p for many p: indeed, if p divides the order of $H_1(\pi_1(M), L)$, it means precisely that there is a nontrivial map

$$\pi_1(M) \to \mathrm{SL}_2(\mathbb{F}_p[t]/t^2).$$

It would be interesting if the existence of "many" such quotients shed any light on the conjectural failure of property τ; cf. [4].

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A p-group with positive rank gradient

Jan-Christoph Schlage-Puchta

Let Γ be a finitely generated group with a presentation $\langle X|R\rangle$. If R is not too large compared with X, one would expect Γ to be big in some sense. The simplest example of such a reasoning is the fact that a group of deficiency ≥ 2 has a subgroup of finite index which projects onto a non-abelian free subgroup. A more difficult statement is the fact that every group having a Golod-Shafarevich presentation is infinite. Here we are interested in the constuction of groups with positive rank gradient. To do so we introduce th p-deficiency of a group as follows. Let F be the free group on X. For a word $w \in F$ define the p-valuation $\nu_p(w)$ as $\max\{k : \exists v \in F : v^{p^k} = w\}$. Define the p-deficiency of a presentation as $|X| - 1 - \sum_{r\in R} p^{-\nu_p(r)}$, and the p -deficiency of Γ as the supremum taken over all presentations of Γ.

We first show that the p-deficiency of Γ is super-multiplicative on normal subgroups of index p, i.e. if $\Delta \lhd \Gamma$, and $|\Gamma/\Delta| = p$, then $\mathrm{def}_p(\Delta) \geq p\mathrm{def}_p(\Gamma)$. Using this together with the fact that the p-deficiency is a lower bound for the rank of a group we find that a group of positive p-deficiency has positive rank gradient.

If $\{w_1, \dots, \}$ is an enumeration of all elements of F, then

$$\langle X|w_1^p, w_2^{p^2}, w_3^{p^3}, \dots\rangle$$

is a p-group with p-deficiency $\geq |X| - \frac{1}{p-1}$, hence, we have constructed a finitely generated p-group with positive rank gradient. In fact, by increasing the occurring powers of p we find d generated p-groups with rank gradient arbitrarily close to the rank gradient of the d-generated free group.

Integrality properties of spectral measures

Andreas Thom

This talk started out with a gentle introduction to the realm of approximation problems. I explained in detail why Lück's results [3] imply that the normalized dimension of the kernel of a matrix if a *local* invariant of the matrix. More precisely,

Theorem 1. *Let $A \in M_n\mathbb{Z}$ be a symmetric positive semi-definite matrix. Let $\kappa > 1$ be an upper bound for the operator norm of A. Then, there exists a constant $C = O(\log(\kappa))$ such that*

$$\left| \frac{\dim \ker A}{n} - \operatorname{tr}\left(\left(1 - \frac{A}{\kappa} \right)^n \right) \right| \leq \frac{C}{\log n}.$$

Here, $\operatorname{tr}\colon M_n\mathbb{R} \to \mathbb{R}$ denotes the normalized trace, i.e. we have $\operatorname{tr}(1_n) = 1$. Clearly, the right side of the difference in the left side of the inequality can be estimated by randomly picking a certain number (not depending on n or A) of indices and exploring the neighborhood of the matrix around the indices; hence it consists of local information. The proof of this theorem (whose idea goes back to [3]) ultimately involves some algebraic number theory. It is not known whether it has any analogue in finite characteristic.

Let us now consider similar questions for operators on an infinite dimensional Hilbert space. Let Γ be a discrete group and $\lambda\colon \Gamma \to U(\ell^2\Gamma)$ be its left-regular representation on the Hilbert space $\ell^2\Gamma$ with orthonormal basis $\{\delta_g \mid g \in \Gamma\}$. We identify the complex group ring $\mathbb{C}\Gamma$ with its image in $B(\ell^2\Gamma)$.

For every $a \in \mathbb{C}\Gamma$, the assignment

$$C_b(\mathbb{R}) \ni f \mapsto \langle f(a)\delta_e, \delta_e \rangle \in \mathbb{R}$$

defines a measure on $\mathbb{R}$ which is called the *spectral measure* of the element $a \in \mathbb{C}\Gamma$. Careful analysis of this spectral measure has been carried out in many special cases, i.e. for random walk operators [2].

In this talk we further study special properties of such a measure under the assumption that $a \in \mathbb{Z}\Gamma$, i.e. a has integral coefficients. We define the notion of *integer measure* which captures the properties of spectral measures of elements in the integral group ring. A measure on $\mathbb{C}$ is said to be an *integer measure* if it the weak limit of zero-distributions of monic integer polynomials with uniformly bounded support. More precisely, there exists a sequence of monic integer polynomials p_n such that the measures

$$\mu_n := \frac{1}{\deg(p_n)} \sum_{z : p_n(z) = 0} \delta_z$$

converge weakly to μ and there exists a constant C, such that for all $n \in \mathbb{N}$, $|z| \geq C$ implies that $p_n(z) \neq 0$.

Our first result (see [4]) shows that for a large class of groups, spectral measures of self-adjoint elements are indeed integer. More precisely:

Theorem 2. *Let Γ be a sofic group and let $a \in \mathbb{Z}\Gamma$ be self-adjoint. Then, μ_a is an integer measure.*

There are two main results concerning integer measures. The first one is:

Theorem 3. *Let μ be an integer measure.*

(1) *The atoms of μ are located at algebraic integers and the set of atoms forms an orbit under the action of the absolute Galois group* $\mathrm{Gal}(\overline{\mathbb{Q}} : \mathbb{Q})$.
(2) *Galois-conjugate atoms appear with the same size.*

This provides a solution of the so-called Algebraic Eigenvalue Conjecture of Dodziuk, Linnell, Matthai, Schick and Yates [1].

We show that, if μ is an integer measure, then its atomic part and its non-atomic part are integer measures as well. It is widely believed that the non-atomic part of the spectral measure of an element in the integral group ring must satisfy special regularity properties. In that respect, we can show the following result (see [5]):

Theorem 4. *Let μ be an integer measure. There exists a constant C, depending only on the spectral radius of μ, such that for all $\beta \in \mathbb{C}$ and $0 < \epsilon < 1$*

$$\mu_a(\{\beta' \in \mathbb{C} \mid |\beta - \beta'| < \epsilon\}) \leq C \cdot |\log 2\epsilon|^{-\frac{1}{2}}.$$

In each situation the proof needs a careful study of the the asymptotic properties of zero-distributions of roots of monic integer polynomials. The results also imply that spectral measures have non-negative logarithmic capacity. Again, the techniques are inspired by the Lück's approach to the approximation problems for ℓ^2-Betti numbers [3].

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Approximate groups

Emmanuel Breuillard
(joint work with Ben Green and Terence Tao)

In the talk, I discussed the notion of *approximate group*. The concept was introduced a few years ago by T. Tao in [19] as a tool to tackle the so-called non-commutative Freiman problem. This problem can be described as asking for a rough classification of finite subsets A of an ambient group G with the property that the size of the product set $AA = \{a_1a_2|a_1, a_2 \in A\}$ is not much bigger that the size of A itself in that $|AA| \leq K|A|$, where $K \geq 1$ is a fixed parameter. Such sets are said to be of *doubling at most* K. Rough means that we will consider two such sets A and B to be essentially equivalent for the purpose of this classification if each set can cover the other set by few (left and right) translates of it, where few means a number less than a constant $f(K)$ depending on K only. The celebrated Freiman Theorem [1, 10] answered this problem in the case when $G = \mathbb{Z}$. For some historical background and a presentation of the non-commutative Freiman problem, we refer the reader to T.Tao's blog.

Definition 1. *An* approximate subgroup *of an ambient group G is a finite subset A of G such that A is symmetric (i.e. stable under inverse $A = A^{-1}$), contains the identity and verifies $AA \subset XA$ for some symmetric subset $X \subset G$ with $|X| \leq K$.*

Using various tools from additive number theory and the combinatorics of sumsets, which we now refer collectively as the (non-commutative) *Ruzsa calculus*, T. Tao [19] reduced the non-commutative Freiman problem to the classification of approximate subgroups of a given ambient group.

Basic examples of approximate groups include finite groups, arithmetic progressions, or more generally nilpotent progressions (a concept introduced and made precise in [2], they are basically homomorphic images of large word balls in the free nilpotent group).

Many authors have by now approached the non-commutative Freiman problem from various angles and for various groups. To name only a few: Breuillard-Green [2, 3, 4] (Lie groups, nilpotent, solvable or compact semisimple), Chang [7] ($\mathrm{SL}_2(\mathbb{Z}), \mathrm{SL}_3(\mathbb{Z})$), Helfgott [11] ($SL_2(\mathbb{Z}/p\mathbb{Z})$ and $SL_3(\mathbb{Z}/p\mathbb{Z})$), Katz-Fisher-Peng (torsion free nilpotent groups), Razborov [17] and Safin [18] (free groups), Tao [20] (general solvable groups).

Precise conjectural statements regarding the problem have been given by different people (including Helfgott and Lindenstrauss). They all say that, in some precise sense, every approximate subgroup of an ambient group is roughly equivalent to an extension of a finite group by a nilpotent progression.

About a year ago, E. Hrushovksi released the preprint [13] in which he addressed the general problem in an arbitrary group by tools that had not been considered before in this context as they pertain to model theory and the theory of stable groups in mathematical logic. Making key use of the Gleason-Montgomerry-Zippin structure theory of locally compact groups, he was able to give the first general structure theorem valid for all approximate groups showing that they always exhibit some nilpotent behavior close to what B. Green and T. Sanders call a *Bourgain system.* He also gave complete answers to the non-commutative Freiman problem in several non-trivial cases: for groups with bounded exponent (there every approximate group is roughy equivalent to a finite subgroup), for subgroups of GL_n over a field (there every approximate group is roughly equivalent to a solvable approximate group -by- a finite semisimple group), for finitely generated groups that are exhausted by an increasing union of K-approximate subgroups (they must be nilpotent-by-finite and this improves Gromov's well-known polynomial growth theorem).

More recently, together with B. Green and T. Tao, we found a different argument than Hrushovski's which treats the case of simple (and, in fact, semisimple) algebraic groups over any field. This argument gives good polynomial bounds in K. Hrushovski's methods on the other hand rely heavily on ultrafilters and model theory and are not likely to give such good bounds. It turns out that such polynomial bounds are crucial for the applications to expanders (see [8], [9]). We prove:

Theorem 2 ([6])**.** *Let $\mathbf{G}$ be an almost simple algebraic group over an algebraically closed field k and $d = dim\mathbf{G}$. Then there are no non trivial approximate subgroups of $\mathbf{G}(k)$. More precisely, there is a constant $C = C(d) > 0$ if $A \subset \mathbf{G}(k)$ is a K-approximate subgroup, then*

1. *either $A \subset \mathbf{H}(k)$, where $\mathbf{H}$ is a proper algebraic subgroup of $\mathbf{G}$ of complexity at most $C(d)$.*
2. *or $|\langle A\rangle| \leq K^C|A|$.*
3. *or $|A| \leq K^C$.*

This statement generalizes the pioneering work of Helfgott [11], which dealt with $\mathrm{SL}(2, \mathbb{Z}/p\mathbb{Z})$. The above theorem was obtained independently by Pyber and Szabo in [16], who considered more specifically the case of $k = \overline{\mathbb{F}_p}$ (which is the hardest case anyway) and gave additional similar statements for the twisted finite simple groups of Lie type.

Our approach was originally inspired by the reading of Hrushovki's paper and one of our guiding principle was to mimic the method of Larsen and Pink [15] for the classification of finite subgroups of algebraic groups and adapt it to approximate subgroups. Ultimately the passage from the approximate subgroup to the close-by genuine subgroup relied on a study of the action of $\mathbf{G}$ on its variety of maximal tori. Although it seems that Pyber and Szabo followed a different route towards the same result, at a certain level our arguments and theirs are similar. We refer the reader to Pyber's report in this volume for a discussion of their work and exciting subsequent developments.

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Growth in finite simple groups of Lie type

László Pyber
(joint work with Endre Szabó)

The diameter, $\mathrm{diam}(X)$, of an undirected graph $X = (V, E)$ is the largest distance between two of its vertices. Given a subset A of the vertex set V the expansion of A, $c(A)$, is defined to be the ratio $|\sigma(X)|/|X|$ where $\sigma(X)$ is the set of vertices at distance 1 from A. A graph is a C-expander for some $C > 0$ if for all sets A with $|A| < |V|/2$ we have $c(A) \geq C$. A family of graphs is an expander family if all of its members are C-expanders for some fixed positive constant C.

Let G be a finite group and S a symmetric (i.e. inverse-closed) set of generators of G. The Cayley graph $\Gamma(G, S)$ is a graph whose vertices are the elements of G and which has an edge from x to y if and only if $x = sy$ for some $s \in S$. Then the diameter of Γ is the smallest number d such that $S^d = G$.

The following classical conjecture is due to Babai [1]

Conjecture 1 (Babai)**.** *For every non-abelian finite simple group L and every symmetric generating set S of L we have* $\operatorname{diam}\big(\Gamma(L,S)\big) \le C\big(\log|L|\big)^c$ *where c and C are absolute constants.*

In a spectacular breakthrough Helfgott [7] proved that the conjecture holds for the family of groups $L = PSL(2,p)$, p a prime. In recent major work [8] he proved the conjecture for the groups $L = PSL(3,p)$, p a prime. We prove the following.

Theorem 2. *Let L be a finite simple group of Lie type of rank r. For every symmetric set S set of generators of L we have*

$$\operatorname{diam}\big(\Gamma(L,S)\big) < \big(\log|L|\big)^{c(r)}$$

where the constant $c(r)$ depends only on r.

A key result of Helfgott [7] shows that generating sets of $SL(2,p)$ grow rapidly under multiplication. His bound on diameters is an immediate consequence.

Theorem 3 (Helfgott)**.** *Let $L = SL(2,p)$ and A a generating set of L. Let δ be a constant, $0 < \delta < 1$.*

(1) *Assume that $|A| < |L|^{1-\delta}$. Then*

$$|A^3| \gg |A|^{1+\varepsilon}$$

where ε and the implied constant depend only on δ

(2) *Assume that $|A| > |L|^{1-\delta}$. Then $A^k = L$ where k depends only on δ.*

It was observed in [11] that a result of Gowers [6] implies that 2) holds for an arbitrary simple group of Lie type L with $k = 3$ for some $\delta(r)$ which depends only on the Lie rank r of L. Hence to complete the proof of our theorem on diameters it remains to prove an analogue of the (rather more difficult) part 1) as was done by Helfgott for the groups $SL(3,p)$ in [8].

We prove the following.

Theorem 4. *Let L be a finite simple group of Lie type of rank r and A a generating set of L. Then either $A^3 = L$ or*

$$|A^3| \gg |A|^{1+\varepsilon}$$

where ε and the implied constant depend only on r.

We also give some examples which show that in the above result the dependence of ε on r is necessary. In particular we construct generating sets of $SL(n,3)$ of size $2^{n-1}+4$ with $|A^3| < 100|A|$ for $n \ge 3$.

Theorem 4 was first announced in [12]. The same day similar results were announced by Breuillard, Green and Tao [4] for finite Chevalley groups. It is noted in [4] that their methods are likely to extend to all simple groups of Lie type, but this has not yet been checked. On the other hand in [4] various interesting results for complex matrix groups are also announced.

Helfgott's work [7] has been the starting point and inspiration of much recent work by Bourgain, Gamburd, Sarnak and others. Let $S = \{g_1, g_2, \dots, g_k\}$ be a symmetric subset of $SL(n,\mathbb{Z})$ and $\Lambda = \langle S\rangle$ the subgroup generated by

S. Assume that Λ is Zariski dense in $SL(n)$. According to the theorem of Matthews-Vaserstein-Weisfeiler [10] there is some integer m_0 such that $\pi_m(\Lambda) = SL(n, \mathbb{Z}/m\mathbb{Z})$ assuming $(m, m_0) = 1$. Here π_m denotes reduction $\mod m$.

It was conjectured in [9], [3] that the Cayley graphs $\Gamma\big(SL(n, \mathbb{Z}/m\mathbb{Z}), \pi_m(S)\big)$ form an expander family, with expansion constant bounded below by a constant $c = c(S)$. This was verified in many cases when $n = 2$. In [2] Bourgain and Gamburd also prove the following

Theorem 5 (Bourgain, Gamburd)**.** *Assume that the analogue of Helfgott's theorem on growth holds for $SL(n, p)$, p a prime. Let S be a symmetric finite subset of $SL(n, \mathbb{Z})$ generating a subgroup Λ which is Zariski dense in $SL(n)$. Then the family of Cayley graphs $\Gamma(SL(n, p), \pi_p(S))$ forms an expander family as $p \to \infty$. The expansion coefficients are bounded below by a positive number $c(S) > 0$.*

By Theorem 4 the condition of this theorem is satisfied hence the above conjecture is proved for prime moduli.

Simple groups of Lie type can be treated as subgroups of simple algebraic groups. In fact, instead of concentrating on simple groups, we work in the framework of arbitrary linear algebraic groups over algebraically closed fields. The following extension of Theorem 4, valid for finite groups obtained from connected linear groups over $\overline{\mathbb{F}_p}$, produces growth within certain normal subgroups.

Theorem 6. *Let G be a connected linear algebraic group over $\overline{\mathbb{F}_p}$ and $\sigma : G \to G$ a Frobenius map. Let G^σ denote the subgroup of the fixpoints of σ and $1 \in S \subseteq G^\sigma$ a symmetric generating set. Then for all $1 > \varepsilon > 0$ there is an integer $M = M_{\text{main}}\big(\dim(G), \varepsilon\big)$ and a real K depending on ε and the numerical invariants of G (notably $\dim(G)$, $\deg(G)$ and the degrees of the multiplication and inverse-element morphisms) with the following property. If $Z(G)$ is finite and*

$$K \leq |S| \leq |G^\sigma|^{1-\varepsilon}$$

then there is a connected closed normal subgroup $H \lhd G$ such that $\deg H \leq K$, $\dim(H) > 0$ and

$$|S^M \cap H| \geq |S|^{(1+\delta)\dim(H)/\dim(G)}$$

where $\delta = \frac{\varepsilon}{128\dim(G)^3}$.

Consider the groups G^σ for simply connected simple algebraic groups G. Central extensions of all but finitely many simple groups of Lie type are obtained in this way, and the centres $\mathcal{Z}(G^\sigma)$ have bounded order. Hence Theorem 6 implies Theorem 4 for both twisted and untwisted simple groups of Lie type in a unified way.

We use Theorem 6 to prove the following the following partial extension of Theorem 4:

Theorem 7. *Let S be a symmetric subset of $GL(n, p)$ satisfying $|S^3| \leq K|S|$ for some $K \geq 1$. Then $GL(n, p)$ has two subgroups $H \geq P$, both normalised by S, such that P is perfect, H/P is soluble, P is contained in S^6 and S is covered by $K^{c(n)}$ cosets of H where $c(n)$ depends on n.*

Understanding the structure of symmetric subsets S of $GL(n,p)$ (or more generally of $GL(n,q)$, q a prime-power) satisfying $|S|^3 \leq K|S|$ is mentioned by Breuillard, Green and Tao as a difficult open problem in [4].

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On a question of Wiegold and torsion images of Coxeter groups

Rostislav I. Grigorchuk

In 2006, J. Wiegold raised the following question in Kourovka Notebook [5, 16.101]: "Do there exist uncountably many infinite 2-groups that are quotients of the group $\Delta = \langle x, y \mid x^2, y^4, (xy)^8 \rangle$?″. The problem is motivated by the following comment by J. Wiegold: "There certainly exists one, namely the subgroup of finite index in Grigorchuk's first group generated by b and ad; see (R. I. Grigorchuk, *Functional Anal. Appl.*, **14** (1980), 41–43))." A positive answer to this question gives

Theorem 1. *The group*

$$\langle x, y \mid x^2, y^4, (xy)^8, (xy^2)^{16}, (xyxy^{-1})^8 \rangle$$

has uncountably many up to quasi-isometry quotients which are branch just-infinite residually finite 2-groups of intermediate growth.

The question of Wiegold inspired us to look at the problem wider. Fixing the number of generators $m \geq 2$ one may be interested in the minimal values of orders of generators, of products of their powers, of products of length 3 etc, that m-generated infinite residually finite torsion group may have. The case of p-groups is of special interest because of many reasons. For $m = 2$ the order 2 for the generators x, y is impossible because the group would be a dihedral group in this case. As follows from the above theorem the orders 2 and 4 and 8 for the product xy are possible values, while the triple $2, 4, 4$ is not possible (because the corresponding group is crystallographic). Starting with $m = 3$ the orders of generators may take the minimal possible value 2, and we come to the question on torsion quotients of Coxeter groups. As Coxeter groups are generated by involutions it is natural to investigate their 2-torsion quotients.

Recall that a Coxeter group can be defined as a group with a presentation

$$\mathcal{C} = \langle x_1, x_2, \ldots, x_n \mid x_i^2, (x_i x_j)^{m_{ij}}, 1 \leq i < j \leq n \rangle,$$

where $m_{i,j} \in \mathbb{N} \cup \{\infty\}$ (the case $m_{i,j} = \infty$ means that there is no defining relator involving x_i and x_j). If $m_{ij} = 2$ this means that x_i and x_j commute. A Coxeter group can be described by a Coxeter graph $\mathcal{Z}$. The vertices of the graph are labeled by the generators of the group $\mathcal{C}$, the vertices x_i and x_j are connected by an edge if and only if $m_{i,j} \geq 3$, and an edge is labeled by the corresponding value m_{ij} whenever this value is 4 or greater. If a Coxeter graph is not connected, then the group $\mathcal{C}$ is a direct product of Coxeter subgroups corresponding to the connected components. Therefore we may focus on the case of connected Coxeter graphs. If we are interested in 2-torsion quotients of $\mathcal{C}$, then one has to assume that m_{ij} are powers of 2 or infinity. In order for $\mathcal{C}$ to have infinite torsion quotients it has to be infinite and not virtually abelian.

Theorem 2. *Let $\mathcal{C}$ be a non virtually abelian Coxeter group defined by a connected Coxeter graph $\mathcal{Z}$ with all edge labels m_{ij} being powers of 2 or infinity. If $\mathcal{Z}$ is not a tree or is a tree with ≥ 4 vertices, or is a tree with two edges with one label ≥ 4 and the other ≥ 8, then the group $\mathcal{C}$ has uncountably many up to quasi-isometry 2-torsion quotients. Moreover these quotients can be chosen to be residually finite, just-infinite, branch 2-groups of intermediate growth and the main property that distinguishes them is the growth type of the group.*

Observe that all cases of connected Coxeter graphs that are excluded by the statement of Theorem 2 are related to finite or virtually abelian crystallographic groups. Indeed, in the case when $\mathcal{Z}$ consists of one edge the corresponding group is a dihedral group, and when $\mathcal{Z}$ has two edges labeled by 4 the corresponding Coxeter group is the crystallographic group $\langle x, y, z \mid x^2, y^2, z^2, (yz)^2, (xy)^4, (xz)^4 \rangle$ generated by reflections in sides of an isosceles right triangle. On the other hand, there are five "critical" Coxeter groups Ξ, Φ, Υ, Π and Γ: that satisfy the requirements of Theorem 2 and play a crucial role in the proof. Their Coxeter graphs are depicted in Figure 1.

The proof of the theorem is based on the properties of the group $\mathcal{G}$ constructed by the author in [4] and of the groups of intermediate growth from the uncountable

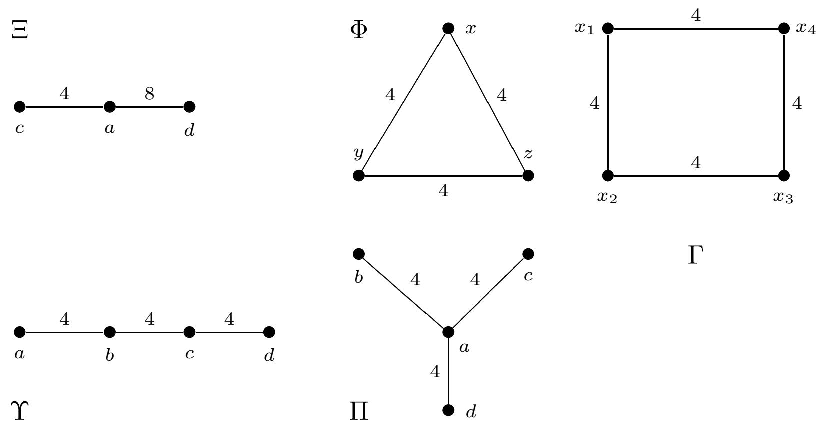

FIGURE 1. Coxeter graphs corresponding to Ξ, Φ, Υ, Π, and Γ

family $\{G_\omega \mid \omega \in \Omega\}$ constructed in [1], which includes (and generalizes) the example $\mathcal{G}$.

The group $\mathcal{G}$ originally was defined in [4] as a group generated by four interval exchange transformations a, b, c, d of order 2 acting on the interval $[0, 1]$ from which the diadic rational points are removed. Later the language of actions on rooted tree started to be used more often and a natural realization of the group is by action on a binary rooted tree [3]. From the definition it immediately follows that the generators satisfy the relations

$$a^2 = b^2 = c^2 = d^2 = [b, c] = [b, d] = [c, d] = bcd = (ad)^4 = (ac)^8 = (ab)^{16} = 1$$

(this list of relations is not complete because the group is not finitely presented). The relation $bcd = 1$ implies that the group $\mathcal{G}$ is 3-generated, but it is usually convenient to work with the generating set $\mathcal{A} = \{a, b, c, d\}$, because together with the identity element it constitutes the so called nucleus of the group, the important tool in the study of self-similar groups [6]. Excluding the generator b we see that $\mathcal{G}$ has a presentation of the form

$$\langle a, c, d \mid a^2, c^2, d^2, (cd)^2, (ad)^4, (ac)^8, \ldots \rangle$$

and therefore is a homomorphic image of the group Ξ.

For the proof of Theorem 2 we also use the construction of an uncountable family of groups G_ω, where $\omega \in \Omega = \{0, 1, 2\}^{\mathbf{N}}$ described in [1] for which the group $\mathcal{G}$ is a particular case corresponding to the sequence $\varsigma = (012)^\infty$. The group G_ω is generated by the set of elements $\mathcal{A}_\omega = \{a, b_\omega, c_\omega, d_\omega\}$ of order 2, with $b_\omega, c_\omega, d_\omega$ commuting and generating the Klein 4-group (i.e. $b_\omega c_\omega d_\omega = 1$) (so indeed the groups G_ω are 3-generated). For the definition of these groups we address the reader to [1, 3].

Let $\Omega_0 \subset \Omega$ be the subset consisting of sequences ω which contain each symbol $0, 1, 2$ infinitely many times, $\Omega_1 \subset \Omega$ be the set of sequences which contain at least two symbols from $\{0,1,2\}$ infinitely many times, and $\Omega_2 = \Omega \setminus \Omega_1$ be the set of sequences $\omega = \omega_1\omega_2\ldots\omega_n\ldots$ such that $\omega_n = \omega_{n+1} = \omega_{n+2} = \ldots$ starting with some coordinate n. Observe that all sets $\Omega_0, \Omega_1, \Omega_2$ are invariant with respect to the shift τ

$$\tau(\omega_1\omega_2\omega_3\ldots) = \omega_2\omega_3\ldots$$

in the space of sequences. The groups G_ω are virtually abelian for $\omega \in \Omega_2$, while the groups $\{G_\omega : \omega \in \Omega_1\}$ are just-infinite, branch groups of intermediate growth. Additionally, the groups G_ω, for $\omega \in \Omega_0$ are 2-groups. Proofs of these facts are provided by Theorems 2.1, 2.2, 8.1, and Corollary 3.2 in [1]. One of important facts that is used in the proof of the Theorem 2 is that the set of growth degrees of groups $\{G_\omega : \omega \in \Omega_0\}$ has uncountable cardinality. The word problem for the family $G_\omega, \omega \in \Omega_1$ can be solved by algorithm with oracle ω (i.e. the algorithm which uses the symbols of the sequence ω in its work), which we call branch algorithm because of its branching nature [3, 1]. Using this algorithm, or directly from the definition of groups G_ω, it is easy to check that if ω begins with prefix 0121 then $(ad_\omega)^4 = 1$ and $(ac_\omega)^8 = 1$, therefore G_ω is a homomorphic image of Ξ. Moreover if two sequences ω and η have the same prefixes of length n then the sets of words of length $\leq 2^n$ over generating set representing the identity element in groups G_ω and G_η coinside. Let $\Omega_3 \subset \Omega_0$ be the set of sequences which begin with prefix 0120 (all groups $G_\omega, \omega \in \Omega_3$ have the same set of relators of length ≤ 16. The proofs of results about growth in [1] allow to conclude that the set of growth degrees of groups from $\{G_\omega, \omega \in \Omega_3\}$ has uncountable cardinality. Moreover the same holds for any set of the form $w\Omega_0$, where w is arbitrary finite binary sequence.

The results from [1] also show that the group $G_\omega, \omega \in \Omega_1$ is abstractly commensurable with $G_{\tau(\omega)} \times G_{\tau(\omega)}$ and therefore the growth of G_ω is equal to the square of the growth of $G_{\tau(\omega)}$.

The strategy of the proof of theorem 2 is the following. First we show that a Coxeter group $\mathcal{C}$, satisfying the condition of the theorem 2 can be mapped onto one of Coxeter groups Ξ, Φ, Υ, Π, or Γ. (Despite the fact that Γ surjects onto Υ in our proof of theorem 2 we need a separate consideration of the case of Γ). This reduces the proof to the case of these groups. Then using the family of groups G_ω we first resolve the case of a group Ξ. Let Q be a subgroup of Ξ generated by the elements $x = a, y = d, z = cac$. It is easy to check that Q has index 2 in Ξ and has a presentation

$$\langle x, y, z \mid x^2, y^2, z^2, (xy)^4, (xz)^4, (yz)^4 \rangle.$$

Therefore Q is isomorphic to Φ. Taking the corresponding subgroups of index 2 in groups G_ω we get uncountably many 2-quotients of Φ. This resolves the second case. The other cases we resolve taking certain subgroup L_ω in the wreath products of $G_\omega \wr \mathbb{Z}_2$, then taking some subgroup M_ω in the permutational wreath product $L_\omega \wr_J K$ where K is a Klein group acting on a set $J = \{1,2,3,4\}$ and finally taking a certain subgroup P_ω in $M_\omega \wr_J K$.

For answering Weigold question we proceed as follows. Let L_ω be defined as a group generated by elements $x = b_\omega, y = ad_\omega$. Then L_ω satisfies the relations

$$1 = x^2 = y^4 = (xy)^8 = (xy^2)^{16} = (xyxy^{-1})^8$$

(the provided list of defining relations in not complete) and we are done with the proof of the first theorem modulo checking the properties of the quotient groups listed in the first theorem. It is unclear if the power 16 in the fourth relation can be replaced by 8, and if the power 8 in fifth relation can be replaced by 4 in such a way that there is an infinite 2-generated 2-group satisfying the corresponding relations.

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Limits of discrete structures

Balázs Szegedy

A classical example for a limit language is due to Fürstenberg. A sequence of growing finite $0-1$ sequences is caled convergent if the frequency of every fixed pattern (for example 0110) is convergent. The limit of such a sequence can be represented by a shift invariant Borel probability measure on the compact space $\{0,1\}^{\mathbb{Z}}$. This creates a correspondence between finite combinatorics and the theory of measure preserving systems.

It turns out that this example is a special case of a more general phenomenon. Similar limit notions exist for structures in which local structures can be explored through random sampling. Another example deals with bounded degree graphs and was developed by Benjamini and Schramm [3].

Let $\mathcal{G}_d$ denote the set of finite graphs with maximum degree at most d. Let $\mathcal{G}^*_{d,r}$ denote the set of finite connected graphs with a distinguished vertex called root such that every vertex is of distance at most r from the root and every vertex has degree at most d. If $G \in \mathcal{G}_d$ then a random r-sample from G is the radius r neighborhood of a random vertex v. We can regard this sample as an element in

$\mathcal{G}^*_{d,r}$. This means that for a fixed grapf $G \in \mathcal{G}_d$ the r-sampling gives a probability distribution on $\mathcal{G}^*_{d,r}$.

A sequence of graphs in $\mathcal{G}_d$ is called convergent if for every fixed r the probability distributions of r-samples converge on $\mathcal{G}^*_{d,r}$. One can represent the limit object as a probability distribution on (possibly) infinite connected rooted graphs with maximum degree d. Benjamini and Schramm proved that the limit of a convergent sequence of planar graphs inside $\mathcal{G}_d$ is recurrent with probability one. We generalized this result with O. Angel for excluded minor families of graphs.

The above sampling process is not useful for graphs with too many edges. There is a complementray theory which deals with dense graphs [1],[2]. We use the following samplig process. Let G be a graph and let $v_1, v_2, \ldots, v_r$ be r randomly chosen vertices from G. Then the induced graph on $v_1, v_2, \ldots, v_r$ is the random r-sample from G. This sampling process gives a probability distribution on graphs in which the vertices are labeled by the first r natural numbers. We say that a sequence of graphs is convergent if for every r the distributions of r-samples converge. The completion of the set of finite graphs with respect to this convergence notion is a compact Hausdorf topological space. Many problems in graph theory can be formulated as analytic statements on this space.

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Groups of positive weighted deficiency

Mikhail Ershov

(joint work with Andrei Jaikin-Zapirain)

In 1964 Golod and Shafarevich found a sufficient condition for a finitely generated group given by generators and relators to be infinite. This criterion, which is applicable to both abstract groups and pro-p groups, was used to solve two outstanding problems: construction of number fields with infinite Hilbert class field towers [3] and construction of infinite finitely generated torsion groups [2]. The groups satisfying the above condition (or rather a slight modification of it) are called Golod-Shafarevich groups. In [1], we consider certain generalizations of Golod-Shafarevich groups, which we call *groups of positive weighted deficiency.* The goal of this generalization is not so much to deal with a larger class of groups, but rather to provide a new point of view on Golod-Shafarevich groups.

Weighted deficiency is defined similarly to the usual deficiency except that generators and relators are counted with suitable weights. Let X be a finite set and $F = F(X)$ the free pro-p group on X. Given any function W_0 from X to the open interval $(0,1)$, we extend W_0 in certain canonical way to a function $W : F \to [0,1)$. Any function $W : F \to [0,1)$ obtained in such a way is called a *weight function on F with respect to X*.

If W is a weight function on $F(X)$ with respect to X and S is any subset of $F(X)$, we define $W(S) = \sum_{s\in S} W(s) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$. Given a pro-$p$ group G, we define the weighted deficiency of G to be the supremum of the set $W(X)-W(R)-1$ where (X,R) runs over all pro-p presentations of G and W runs over weight functions on $F(X)$ with respect to X. If G is a finitely generated abstract group, we define its weighted deficiency (with respect to a fixed prime p) as the weighted deficiency of the pro-p completion of G. Golod-Shafarevich groups are those groups of positive weighted deficiency, for which the quantity $W(X) - W(R) - 1$ can be made positive for a *uniform* weight function W, that is, for a weight function W which is constant on X.

There is an important refinement of the notion of weighted deficiency – the *deficiency with respect to a valuation.* Valuations are certain functions on pro-p groups with values in the interval $[0,1]$ and should be thought of as "multiplicative valuations in characteristic p". Weight functions are special types of valuations on free pro-p groups. If G is a pro-p group and $\pi : F \to G$ is a free presentation of G, then any weight function on F induces a valuation of G, and conversely, any valuation W on G satisfying certain mild assumptions can be lifted to a weight function on F. Thus, we can talk about the deficiency of a pro-p group G with respect to a valuation W, denoted by $def_W(G)$. It is defined in the same way as the weighted deficiency, except that instead of all weight functions we consider only those which induce W. Similarly one can define valuations and deficiency with respect to a valuation for abstract groups.

Given a pro-p group G and a valuation W on G, to each closed subgroup H of G one can associate its W-rank $rk_W(G)$ and W-index $[G : H]_W$. These quantities inherit many properties of the usual rank and index; in particular, there is a direct analogue of the Schreier index formula. As a consequence, we deduce, for instance, that the class of groups of positive weighted deficiency is closed under taking finite index subgroups. This, in turn, implies that groups of positive weighted deficiency are always infinite and gives a new proof of the original criterion of Golod and Shafarevich.

While it is more natural to study valuations and the corresponding deficiency in the category of pro-p groups (not abstract groups), the main applications of this theory presented in [1] are concerned with abstract groups of positive weighted deficiency.

Key Theorem ([1]). *Let G be a finitely generated abstract group, W a valuation on G, and assume that $def_W(G) > 0$. Let H be a finitely generated subgroup of G. Then there exists an epimorphism $\pi : G \to \overline{G}$ such that either*

(i) *$\pi(H)$ is of finite index in $\overline{G}$ and $def_W(\overline{G}) > 0$ or*
(ii) *$\pi(H)$ is finite, and there is a finite index subgroup $\overline{K}$ of $\overline{G}$ such that $def_V(\overline{K}) > 0$ for some valuation V on $\overline{K}$.*

Moreover, if $[G : H]_W < \infty$, then (i) holds and if $[G : H]_W = \infty$, then (ii) holds.

Using Key Theorem (in fact, a slightly more general result), we construct finitely generated groups which "almost deserve" to be called *residually finite Tarski monsters.* By Tarski monsters one usually means infinite finitely generated groups in which every proper subgroup is cyclic of order p (where p is a fixed prime). Such groups were constructed by Ol'shanskii in 1980 for every sufficiently large prime p. They are clearly simple (in particular, not residually finite), and there has been a lot of interest in constructing residually finite groups which come 'as close as possible' to satisfying the above conditions. To make matters more precise, if G is finitely generated, infinite, torsion and residually finite, it must have elements of arbitrarily large order (by Zelmanov's solution to the restricted Burnside problem) as well as subgroups of arbitrarily large finite index. Thus, probably the most natural question is the following:

Question 1. *Does there exist an infinite finitely generated residually finite torsion group in which every subgroup is finite or of finite index?*

We prove that there exist groups satisfying the above condition for all *finitely generated subgroups* (but not necessarily for all subgroups). In fact, any abstract group of positive weighted deficiency has a quotient with this property:

Main Theorem ([1]). *Let p be any prime, and let G be an abstract group of positive weighted deficiency with respect to p. Then G has an infinite residually finite p-torsion quotient $\overline{G}$ in which every finitely generated subgroup is either finite or of finite index.*

Going back to the general problem of studying pro-p groups using valuations, one class of pro-p groups where this approach appears promising is that of Galois groups of unramified pro-p extensions of number fields. Given a number field K, a prime p and a finite set S of primes of K, none of which lies above p, let $K_{p,S}$ be the maximal pro-p extension of K which is unramified outside of S, and let $G_{p,K,S} = Gal(K_{p,S}/K)$ be the corresponding Galois group. There are some natural conditions on the triple (K, p, S) which guarantee that the group $G_{p,K,S}$ is Golod-Shafarevich (this is how the Hilbert class field tower problem was solved in [3]). For each such group $G = G_{p,K,S}$ we have $def_W(G) > 0$ where valuation W comes from a uniform weight function (and therefore is easy to describe). If H is an open subgroup of such G, then $H = G_{p,L,S_L}$ for some other number field L and a set of L-primes S_L, and by the above discussion $def_W(H) > 0$ as well. However, it is not clear how to describe W (restricted to H) in terms of L and S_L, without a reference to the overgroup G or the field K.

Problem 2. *Let K, p, S be as above. Define a class of valuations on $G = G_{K,p,S}$, which can be naturally described in terms of the arithmetic of K. If G has positive weighted deficiency, can one ensure that there is a valuation W in this class for which $def_W(G) > 0$?*

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On the Atiyah problem for the lamplighter groups

Łukasz Grabowski

Let G be a countable discrete group. We say that a non-negative real number r is an *l^2-Betti number arising from G* iff there exists $\theta \in M_m(\mathbb{Q}G)$, a matrix over the rational group ring of G, such that the von Neumann dimension of kernel of θ is r. Motivation for the name is as follows: when G is finitely presented and r is an l^2-Betti number arising from G, then there exists a closed manifold M whose fundamental group is G, and such that one of the l^2-Betti numbers of the universal cover of M is equal to r. See [Eck00] or [Lüc02] for more details.

The following problem is a fine-grained version of a question asked by Atiyah in [Ati76].

Question 1 (The Atiyah problem for a group G)**.** *What is the set of l^2-Betti numbers arising from G?*

Let us call this set the *l^2-complexity of G*, and denote it by $\mathcal{C}(G)$. For a class of groups $\mathbf{G}$ define $\mathcal{C}(\mathbf{G}) = \cup_{G\in\mathbf{G}}\mathcal{C}(G)$.

So far $\mathcal{C}(G)$ has been computed only in cases where $\mathcal{C}(G)$ is a subset of $\mathbb{Q}$. In fact, what is known as the *Atiyah conjecture for torsion-free groups* says that $\mathcal{C}(G) = \mathbb{N}$ for any torsion-free group, and till the article [DS02] of Dicks and Schick it was widely conjectured that $\mathcal{C}(G) \subset \mathbb{Q}$ for every group G. However, Dicks and Schick gave an example of an operator $\theta \in \mathbb{Q}((\mathbb{Z}_{/2} \wr \mathbb{Z})^2)$ together with an heuristic argument showing why $\dim_{vN} \ker \theta$ is probably irrational. Their work was motivated by the article [GŻ01] of Grigorchuk and Zuk.

Only recently Austin has been able to obtain a definite result by proving in [Aus09] that $\mathcal{C}$(Finitely generated groups) is uncountable. The key idea of Austin was to use Pontryagin duality in order to compute $\dim_{vN}$ by analyzing certain dynamical systems.

Subsequently it has been shown independently by the author in [Gra10b] and by Pichot, Schick and Zuk in [PSZ10] that in fact $\mathcal{C}$(Finitely generated groups) $= \mathbb{R}_{\geq 0}$ and that $\mathcal{C}$(Finitely presented groups) $\not\subseteq \mathbb{Q}$. Moreover, in [Gra10b] it is shown that $\mathcal{C}((\mathbb{Z}_{/2} \wr \mathbb{Z})^3) \not\subseteq \mathbb{Q}$.

More recently, Lehner and Wagner showed in [LW10] that $\mathcal{C}(\mathbb{Z}_{/p} \wr F_d)$ contains irrational algebraic numbers, where F_d is the free group on d generators, and $d \geq 2, p \geq 2d - 1$.

In all the articles cited above the following is trivial to check: if it is proven that for a given group G it holds that $\mathcal{C}(G) \not\subseteq \mathbb{Q}$, then there exists p such that $\mathbb{Z}_{/p}\wr\mathbb{Z} \subset G$. In other words, according to the current state of knowledge, $\mathbb{Z}_{/p}\wr\mathbb{Z} \subset G$ could be the necessary condition for $\mathcal{C}(G) \not\subseteq \mathbb{Q}$. We prove that it is a sufficient condition. Indeed, it is very easy to see that if $A \subset B$ are groups then $\mathcal{C}(A) \subset \mathcal{C}(B)$

(see for example Corollary 4.2.2 in [Gra10b]) and here we prove the following theorem.

Theorem 2. *Let $p \geq 2$. Then $\mathcal{C}(\mathbb{Z}_{/p} \wr \mathbb{Z})$ contains transcendental numbers.*

In order to prove Theorem 2 we need to find an operator in $M_m(\mathbb{Q}(\mathbb{Z}_{/p} \wr \mathbb{Z}))$ whose kernel has transcendental von Neumann dimension. However, we show that $|H| \cdot \mathcal{C}(G \times H) = C(G)$, for any group G and any finite group H, so we can as well find such an operator in $\mathbb{Q}(\mathbb{Z}_{/p} \wr \mathbb{Z} \times H)$, where H is some finite group.

Thanks to Pontryagin duality we can exchange the above problem with a question about existence of an operator in the von Neumann algebra $L^\infty(X) \rtimes \Gamma$ whose kernel has transcendental von Neumann dimension, where $X := {\mathbb{Z}_{/p}}^{\mathbb{Z}} \times {\mathbb{Z}_{/2}}^3$, and $\Gamma := \mathbb{Z} \times GL_3(\mathbb{Z}_{/2})$.

Here is a brief description of the computational tool we use. We are given a probability measure space (X, μ), an action $\rho \colon \Gamma \curvearrowright X$ by measure preserving maps, and an operator $T \in L^\infty(X) \rtimes_{\text{alg}} \Gamma$ given as $\sum_{i=1}^n c_i \gamma_i \chi_i$, where $c_i \in \mathbb{C}$, $\gamma_i \in \Gamma$, and χ_i are characteristic functions of some measurable sets X_i. Such T gives us a graphing of X given by the family $(\rho(\gamma_i), X_i)$ of partial isomorphisms. Let g denote a connected component of this graphing. It makes sense to "restrict" T to an operator T^g defined on the Hilbert space $l^2 g$ spanned by vertices of g (i.e. points of X.) Computing $\dim \ker T^g$ turns out to be relatively easy, and to obtain $\dim_{vN} \ker T$ one needs to "integrate" the function $\dim \ker T^g$ over all the connected componenets g of the graphing.

In case of the particular T we consider, connected components of the graphing are finite graphs. They come in three families: $g(k)$, $k = 1, 2 \ldots$, $h(l)$, $l = 1, 2, \ldots$, and $j(k, l)$, $k, l = 1, 2, \ldots$. The point is that $\dim \ker T^{j(k,l)}$ is 2-dimensional iff $l = 2^{k-1} - 1$ and it is 1-dimensional otherwise. After integrating this leads to the result that $\dim_{vN} \ker T$ is equal to

$$p + q \sum_{k=1}^{\infty} r^{k + 2^{k-1}},$$

where p, q and r are non-zero rational numbers. Transcendence of the number above follows from the work [aT02] of Tanaka.

Details of the proof can be found in [Gra10a].

We finish by stating some related questions. The first one summarizes the current state of knowledge on irrational l^2-Betti numbers.

Question 3. *Is it the case that $\mathcal{C}(G) \not\subseteq \mathbb{Q}$ is equivalent to $\mathbb{Z}_{/p} \wr \mathbb{Z} \subset G$ for some p?*

The author thinks it is not true. More sensible question seems to be the following.

Question 4. *Is it the case that $\mathcal{C}(G) \not\subseteq \mathbb{Q}$ is equivalent to $\mathbb{Z}_{/p} \wr K \subset G$ for some p and some infinite finitely generated K?*

However, when K is an infinite finitely generated torsion group then even showing $\mathcal{C}(\mathbb{Z}_{/p} \wr K) \nsubseteq \mathbb{Q}$ seems to be difficult. To apply the method from [Gra10a] one needs to have a good understanding of the Cayley graph of K.

The "easiest" group known so far for which $\mathcal{C}(G) \nsubseteq \mathbb{Q}$ is $\mathbb{Z}_{/2} \wr \mathbb{Z}$, and hence the following question.

Question 5. *What is $\mathcal{C}(\mathbb{Z}_{/2} \wr \mathbb{Z})$?*

This question is related to finite automata and regular languages. To be slightly more precise, the families of graphs $g(k)$, $h(l)$ and $j(k,l)$ in our concrete example can be encoded by a regular language, and the author is convinced that this is the case for any element of the group ring of $\mathbb{Z}_{/p} \wr \mathbb{Z}$ for which the associated graphing consists only of finite graphs.

A concrete related problem is as follows. Let n and C be integers and let A be an automaton whose letters are $n \times n$ matrices with integer coefficients from the set $\{-C, -C+1, \ldots, C\}$. We will say that a matrix M of dimension at least n is recognized by A iff (i) entries of M are 0 except in the n-neighbourhood of the diagonal, and (ii) A recognizes the word consisting of subsequent $n \times n$ minors of M along the diagonal.

Let $M_1, M_2, \ldots$, be the sequence of all matrices which a recognized by A. Let $d(M)$ be the dimension of a matrix M.

Question 6. *What are the possible values of the series*

$$F_A(x) := \sum_{i=1}^{\infty} \dim \ker M_i \cdot x^{d(M_i)}$$

evaluated on rational numbers? In particular can it happen for some automaton A and a rational q that $F_A(q)$ is an irrational algebraic number?

Similar questions can be asked for other wreath products $\mathbb{Z}_{/p} \wr \Gamma$. In this case automata should be replaced with automata operating on the Cayley graph of Γ.

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Fritz Grunewald

DAN SEGAL

Fritz Grunewald was an inspirational mathematician. The breadth of his knowledge was remarkable, and is reflected (though by no means fully represented) in the diverse range of areas where he made important contributions. Mathematics for him was a communal enterprise: all but two of his numerous publications were joint work. He brought a unique vision to these collaborations, and following up his ideas will keep many mathematicians busy for a long time to come. Of his 48 co-authors, those I have spoken to testify to the profound influence working with him had on their own mathematics – this is certainly true in my case.

Although his first research was in group theory, he was from the start a mathematical universalist. His most distinctive contributions to group theory came from approaching it via number theory and geometry; his work in number theory was centred round arithmetic groups. First and foremost he was an indefatigable calculator, both by hand and by computer. Theoretical insights were inspired by hands-on familiarity with the data. Although I don't think he had strong views about the philosophy of mathematics, he was in practice very much a constructivist: to understand a theory, for him, meant to be able to calculate explicit examples. This led on the one hand to insightful conjectures in number theory, and on the other hand to some fundamental decidability theorems in algebra.

Fritz spent the academic year 1973-74 as a post-doc at Queen Mary College, London. When he arrived, the head of department Karl Gruenberg took one look at his long hair and hippyish attire and decided he should share a room with me. We immediately became firm friends and began a 37-year long (intermittent) collaboration. He did many things with many different collaborators: here I will briefly sketch some highlights, from areas where we worked together.

The first topic of our joint work was *polycyclic groups*. Earlier work of Baer and more recent work of Remeslennikov had indicated that these needed to be studied through algebraic number theory; still more recent results of L. Auslander, Borel and Fred Pickel showed that linear algebraic groups were also a key part of the picture. Fritz (unlike me at the time) had a good grounding in all these subjects. Together we proved that polycyclic groups are 'subgroup conjugacy separable' [1], and in a series of papers (some with Pickel) established the 'finite genus' property for these groups [5], [11], [6]. This says that a family of polycyclic groups all with the same profinite completion contains only finitely many isomorphism types; this had recently been established for nilpotent groups by Pickel, and was at the time one of the two main open problems in the area. One of Fritz's essential insights (for this project) was a result in algebraic number theory: we knew that Chevalley's theorem on the congruence subgroup property of groups of algebraic integer units had to be a key step, but it was not sharp enough; Fritz saw how it could be generalized in the relevant manner [3].

Our most important result on this topic was the solution of the *isomorphism problem for nilpotent groups* [8], the other main open problem in the area (later extended to all virtually polycyclic groups [16] – in fact, the general expectation had been for a *negative* solution). That is, the construction of an algorithm to decide whether two nilpotent groups, given by finite presentations, are isomorphic (Adian and Rabin had famously proved that no such algorithm exists for finitely presented groups in general). It had become clear that this problem should be construed as a question about orbits of an arithmetic group: Pickel had settled the genus problem for nilpotent groups by reducing it to a 'local-global' finiteness theorem for such orbits, and the result of [11] was in the same spirit; what was needed now was a *decidability* theorem.

The second of Fritz's two single-authored papers [4] presents an explicit algorithm for solving the *conjugacy problem* in $\mathrm{GL}_n(\mathbb{Z})$, the first general result of this type for arithmetic groups. There was no doubt in his mind that something similar should be possible in much greater generality, and – as often – his intuition was correct. Using more sophisticated tools (Borel and Harish-Chandra's reduction theory) we established the decidability of the 'orbit problem' for any rational action of any arithmetic group on a $\mathbb{Z}$-lattice, and along the way gave an effective procedure for finding finite generating sets for arithmetic groups [2], [7]. Subsequently this was all generalized to the case of S-arithmetic groups [10], [12] – the extra ingredient needed here was the theory of Bruhat-Tits buildings (which, like many other things, Fritz had to explain to me). These results had many implications: roughly speaking, any Diophantine problem involving a linear group action should be decidable. As well as enabling the solution of the isomorphism problem for nilpotent groups, mentioned above, it solves the conjugacy problem in all S-arithmetic groups, and shows the decidability of the equivalence of forms over a ring of algebraic integers (or S-integers). A different Diophantine application concerns the solvability in integers of quadratic equations (in abitrarily many unknowns). Siegel, in a rather difficult paper, had shown that this is a decidable question. The note [9] gives a simple algorithm based on the effective constructibilty of finite generating sets for integral orthogonal groups, a special case of results from [7]. 24 years later, the logician Harvey Friedman asked Fritz if one could decide the solvability in *positive* integers: this (more delicate) question is settled positively in [13]; a key insight here was Fritz's intuition that a Fuchsian group has 'dense orbits at infinity', which lies behind a sharpened Strong Approximation Theorem for orthogonal groups established and used in that paper.

Fritz's two most-cited papers are [14] (with D.S. and Geoff Smith) and [15] (with Marcus du Sautoy). A number-theorist at heart, Fritz was always interested in arithmetical sequences that can be associated to infinite groups. By analogy with the Dedekind zeta function of a number field, one defines the *zeta function* of a group G to be

$$\zeta_G(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

where a_n is the number of subgroups of index n in G. When G is finitely generated and nilpotent, ζ_G splits as an Euler product, the local factor at a prime p being

$$\zeta_{G,p}(s) = \sum_{n=0}^{\infty} a_{p^n} p^{-ns}.$$

The main result of [14] is that for each prime p, $\zeta_{G,p}(s)$ represents a *rational function* $Q_p(p^{-s})$ of p^{-s}. The nature of these rational functions was, and to some extent still is, deeply mysterious; several explicit examples are computed in [14], and several conjectures are formulated. These have generated much work by several authors (notably du Sautoy and Voll), concerning the variation of Q_p as p varies over all primes, the existence of 'local functional equations', and the analytic properties of the global function ζ_G.

Classically, analytic number theory uses the analytic properties of zeta functions to deduce growth properties of numerical sequences. The ground-breaking paper [15] takes this approach to the 'subgroup growth' sequence (s_n) of a finitely generated nilpotent group G (here $s_n = \sum_{j=1}^{n} a_j$). The key idea of this paper is the concept of a *cone integral*, a kind of p-adic integral defined by polynomial data. It is shown that each local factor $\zeta_{G,p}$ is a cone integral. By analysing these and applying Lang-Weil type estimates, Fritz and du Sautoy show first that the abscissa of convergence α_G of the Dirichlet series ζ_G is a rational number; using more algebraic geometry they go on to show that ζ_G can be approximated by a suitable Artin L-function, and deduce that the complex function $\zeta_G(s)$ has a meromorphic analytic continuation a little to the left of the line $\mathrm{Re}(s) = \alpha_G$. Standard methods of analytic number theory (Tauberian theorems) allow them to infer the asymptotic estimate

$$s_n \sim cn^{\alpha_G} (\log n)^{\beta},$$

where $c \in \mathbb{R}$ and β is a non-negative integer, an extraordinary result of genuinely non-commutative arithmetic.

This has been only a sample of Fritz's wide-ranging and influential contributions to group theory.

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Profinite completions and 3-manifold groups

Andrei Jaikin-Zapirain

(joint work with Fritz Grunewald, Aline Pinto and Pavel Zalesski)

Let Γ be a group, $\widehat{\Gamma}$ its profinite completion. The group Γ is called ***p*-good** if the homomorphism of cohomology groups

$$i^n(M): H^n(\widehat{\Gamma}, M) \longrightarrow H^n(\Gamma, M)$$

induced by the natural homomorphism $i: \Gamma \longrightarrow \widehat{\Gamma}$ is an isomorphism for every finite p-primary $\mathbb{Z}[\Gamma]$-module M and for all $n \geq 0$. The group Γ is called **good** if it is p-good for all primes p. This notion was introduced by Serre (see [10, I.2.6]) and has been studied recently in several papers (see, for example, [3]).

Let $\Gamma_{\hat{p}}$ be the pro-p completion of Γ, and let $i_p: \mathbf{G} \to \mathbf{G}_{\hat{p}}$ denote the canonical map. This map induces natural homomorphisms

$$i_p^n(M): H^n(\Gamma_{\hat{p}}, M) \longrightarrow H^n(\Gamma, M)$$

for all $n \geq 0$ and for all finite $\mathbb{Z}[\Gamma]$-modules M of p-power order for which all composition factors are trivial Γ-modules. The group Γ is called **pro-*p* good** if

$i_p(\mathbb{F}_p)$ is an isomorphism for all n. Note that this also implies that $i_p^n(M)$ is an isomorphism for all $n \geq 0$ and for all finite $\mathbb{Z}[\Gamma]$-modules M of p-power order for which all composition factors are trivial Γ-modules.

The following relation between p-goodness and pro-p godness was discovered by Weigel [11]:

Proposition 1. *Let Γ be a group. Assume that all the subgroups of Γ of finite index are pro-p good. Then Γ is p-good.*

In the following theorem we give a necessary condition for goodness.

Theorem 2. *Let $\mathbf{G}$ be finitely presented group. Suppose that $\mathbf{G}$ is virtually residually-p group and the first l^2-Betti number $\beta_1^{(2)}(\Gamma)$ of $\mathbf{G}$ is trivial. Assume also that either*

a) $\mathrm{cd}(\mathbf{G}) = 2$ and it is of deficiency 1 or

b) $\mathbf{G}$ is an orientable Poincare duality group of dimension 3.

Then Γ is p-good. Moreover, if Γ is residually-p, then $\mathbf{G}$ is also pro-p good.

Recall that a lattice in $\mathrm{SL}_2(\mathbb{C})$ is a discrete group (Kleinian group) of finite covolume. One particular family of lattices is a family of arithmetic groups. We recall the definition of an arithmetic group in this case; see [2], [7] for more details. Let k be a number field with exactly one pair of complex places and let A be a quaternion algebra over k which is ramified at all real places. Let ρ be a k-embedding of A into the algebra $\mathrm{M}_2(\mathbb{C})$ of two by two matrices over $\mathbb{C}$ (using one of the complex places). Let $\mathcal{O}$ be the ring of integers of k and let $\mathcal{R}$ be a $\mathcal{O}$-order of A Let $A^1(\mathcal{R})$ the corresponding group of elements of norm one. It is well known that $\rho(A^1(\mathcal{R}))$ is a lattice in $\mathrm{SL}_2(\mathbb{C})$. Then a subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$ is an arithmetic Kleinian group if it is commensurable with some such a $\rho(A^1(\mathcal{R}))$ (groups are commensurable if they have $\mathrm{SL}_2(\mathbb{C})$-conjugate subgroups of finite index). The quotient $\mathrm{SL}_2(\mathbb{C})/\rho(A^1(\mathcal{R}))$ is not compact if k is an imaginary quadratic number field and if $A = \mathrm{M}_2$.

The main application of Theorem 2 is the following result.

Corollary 3. *Let Γ be an arithmetic lattice in $\mathrm{SL}_2(\mathbb{C})$. Then Γ is good.*

Remark 4. Using a recent deep result of D. Wise [13] it is also possible to prove goodness for all fundamental groups of 3-manifolds. Wei give a sketch of the argument.

Let $\mathbf{G} = \pi_1(M)$ be the fundamental group of a 3-manifold M (possibly with a boundary). Without loss of generality we may assume that the boundary of M is incompressible and M is irreducible, because a free product of good groups is also good. From the Geometrization Conjecture proved by Perelman, it follows that we can cut M along a finite collection of incompressible tori so that the resulting pieces $\{M_i\}$ are geometric. Wilton and Zalesski [12] proved that $\pi_1(M)$ is good if $\pi_1(M_i)$ are good. Also they proved that $\pi_1(M_i)$ is good if M_i is a Seifert manifold. Thus, we have to show that $\pi_1(M)$ is good when M is hyperbolic. Hence assume that M is hyperbolic. In the case when M is not virtually Haken the goodness of $\pi_1(M)$ was first observed by Reznikov [9] (see also [4, 11]). If M is virtually

Haken, then by [13] and [1], M is virtually fibered over circle and so $\pi_1(M)$ is also good.

Now, let $\mathbf{G}$ be an arithmetic lattice in $\mathrm{SL}_2(\mathbb{C})$. To define the congruence kernels we assume without loss of generality that $\Gamma = \rho(A^1(\mathcal{R}))$. The congruence kernel $\mathbf{C}(A, \mathcal{R})$ is the kernel of the canonical map from the profinite completion $\widehat{\Gamma}$ of Γ to $\rho(A^1(\widehat{\mathcal{R}}))$. Here $\widehat{\mathcal{R}}$ stands for the profinite completion of the ring $\mathcal{R}$. The congruence subgroup problem (in general, for arithmetic groups) asks whether the congruence kernel is trivial. If the congruence kernel is finite, i.e. the congruence subgroup problem has almost positive solution, one says that Γ has a congruence subgroup property. It is proved by Lubotzky [6] that the congruence kernels $\mathbf{C}(A, \mathcal{R})$ of the arithmetic lattices in $\mathrm{SL}_2(\mathbb{C})$ are infinite. Led by a result of Melnikov [8] in the case $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ we ask:

Question 5. *Is the congruence kernel of an arithmetic lattice in* $\mathrm{SL}_2(\mathbb{C})$ *isomorphic to* $\widehat{F}_\omega$*? Or more generally, what can be said about the congruence kernel in this case?*

Note that from Corollary 3, it follows that the cohomological dimension of $\mathbf{C}(A, \mathcal{R})$ is 1 or 2. Of course the answer to question 5 is negative if the cohomological dimension of the congruence kernel is not one. We shall describe in the following an interesting connection between question 5 and certain cohomological problems.

In [5] it is proved that if Γ is a lattice in $\mathrm{SL}_2(\mathbb{C})$, then for any chain of normal subgroups Γ_i of finite index of Γ with trivial intersection the numbers

$$\frac{\dim H^1(\Gamma_i, \mathbb{Q})}{[\Gamma : \Gamma_i]}$$

tend to zero when i tends to infinity (this means that $\beta_1^{(2)}(\Gamma) = 0$). Let us formulate the following analoguous problem for the dimensions of the first cohomology groups over $\mathbb{F}_p$.

Question 6. *Let* Γ *be an arithmetic lattice in* $\mathrm{SL}_2(\mathbb{C})$ *and* p *a prime number. Do the numbers*

$$\frac{\dim H^1(\Gamma_i, \mathbb{F}_p)}{[\Gamma : \Gamma_i]}$$

tend to zero when i *tends to infinity for any chain of normal* p*-power index subgroups* Γ_i *of* Γ *with trivial intersection.*

We shall now describe a connection beween the problems posed in Questions 5 and 6.

Theorem 7. *Let* Γ *be an arithmetic lattice in* $\mathrm{SL}_2(\mathbb{C})$ *and* p *be a prime number. If the answer to the Question 6 is positive for all the subgroups of* Γ *of finite index, then the* p*-cohomological dimension of the congruence kernel of* Γ *is 1.*

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Group measure space von Neumann algebras and measure group theory

Narutaka Ozawa

I gave a survey talk on group measure space von Neumann algebras and measure group theory, and elaborated a case of profinite actions. Consult the references for more detailed surveys.

Group measure space von Neumann algebras. Murray and von Neumann (1936, 1943) introduced a method of constructing von Neumann algebras, called *group measure space von Neumann algebras*, from group actions on measure spaces by measurable transformations. The most fascinating case is when the group Γ is countable and the action $\Gamma \curvearrowright (X,\mu)$ is probability-measure-preserving, essentially-free and ergodic; and in this case the resulting von Neumann algebra $L^\infty(X,\mu)\rtimes\Gamma$ is so called a type II_1 factor. Murray and von Neumann proposed the classification problem of such von Neumann algebras. Now, it is known that there are uncountably many isomorphism classes and the classification is impossible in the sense that it is impossible to classify all countable discrete groups. Thus, the classification problem becomes: To what extent does the von Neumann algebra $L^\infty(X,\mu)\rtimes\Gamma$ remember the group action $\Gamma \curvearrowright (X,\mu)$?

Orbit Equivalence Relations. In late 1950s, it is realized that the isomorphism class of the von Neumann algebra $L^\infty(X,\mu) \rtimes \Gamma$ depends only on the *orbit equivalence relation* induced by the group action $\Gamma \curvearrowright (X,\mu)$. The orbit equivalence relation is defined as

$$\mathcal{R}_{\Gamma \curvearrowright (X,\mu)} = \{(x,y) \in X \times X : x \text{ and } y \text{ are on the same } \Gamma\text{-orbit}\}.$$

It is uninteresting as a mere equivalence relation, but the point is that $\mathcal{R}_{\Gamma \curvearrowright (X,\mu)}$ is a measurable subset of $X \times X$. (There's an axiomatic treatment of measurable equivalence relations, which we do not elaborate here.) Two actions $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ are said to be *orbit equivalent* (abbreviated as OE) if their orbit equivalence relations are isomorphic, i.e., if there is a measure-preserving isomorphism $F\colon X \to Y$ such that $F(\Gamma x) = \Lambda F(x)$ for μ-a.e. $x \in X$. (The both sides of the equations are countable sets. One may also consider the Borel orbit equivalence by imposing that F is Borel and the equality holds everywhere; this is a subject of descriptive set theory.) Prototypical example of OE actions are lattices of the same locally compact group: Let Γ and Λ are lattices of G with the same covolume. Then, $\Gamma \curvearrowright G/\Lambda$ is OE to $\Lambda \curvearrowright G/\Gamma$. (Hint of the proof: $x\Lambda \sim y\Lambda$ in G/Λ iff $\Gamma x \Lambda = \Gamma y \Lambda$.) Here are some of highlights in the last century. Ornstein and Weiss (1980) proved that all ergodic p.m.p. actions of countable amenable groups are OE to each other, i.e., the orbit equivalence relations completely forget what are groups and what are actions (except for being amenable and ergodic). To the contrary, in late 1980s, Zimmer proved a rigidity theorem for OE between lattices of higher rank Lie groups: OE induces conjugate actions of the ambient Lie groups, and so the lattices of different ranks do not admit OE actions. This result was strengthened by Furman (1999) to the extent that if a group Λ has an OE action to a lattice of higher rank Lie group, then Λ itself has to be (virtually) a lattice of the same Lie group.

Measure(d) Group Theory. In this decade, study of orbit equivalence relations have seen a remarkable progress, driven by Furman, Gaboriau, Monod, Shalom, et al. from the (geometric) group theory side and by Popa and his colleagues from the von Neumann algebra theory side. The subject of orbit equivalence relations even acquired a new name, but consensus is lacking: it is called measure group theory, measured group theory, or measurable group theory. Hoping that it is friendlier to group theorists, I employ Monod's picture here. Let

$$[\Gamma,\Lambda] = \{f\colon \Gamma \to \Lambda,\ f(1)=1\} \cong \Lambda^{\Gamma\setminus\{1\}}$$

be the mapping space. Observe that f is an homomorphism iff $f(t) = f(tg)f(g)^{-1}$ for all $t,g \in \Gamma$. We define a Γ-action on $[\Gamma,\Lambda]$ by $(g\cdot f)(t) = f(tg)f(g)^{-1}$. Then, a Γ-invariant Borel measure on $[\Gamma,\Lambda]$ is called a *randomorphism*. By the above observation, a homomorphism is same as a deterministic (i.e., Dirac) randomorphism. We say a randomorphism is injective, surjective, or bijective if all the fibers are so (up to null sets). Let $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$, and suppose there is $F\colon X \to Y$ such that $F(\Gamma x) \subset \Lambda F(x)$ for all x (after discarding a null set). Then, for each pair $(g,x) \in \Gamma \times X$, there is $\alpha(g,x) \in \Lambda$ such that $F(gx) = \alpha(g,x)F(x)$.

Since Λ action is assumed to be free (a.d.a.n.s.), the element $\alpha(g,x)$ is uniquely determined. Thus $\alpha\colon \Gamma\times X\to\Lambda$ is a measurable map satisfying

$$\alpha(gh,x)=\alpha(g,hx)\alpha(h,x);$$

any such map is called a *cocycle*. Given a cocycle $\alpha\colon \Gamma\times X\to\Lambda$, one obtains a randomorphism α_*, defined to be the push-out of μ through $\alpha\colon X\to[\Gamma,\Lambda]$. I translate some of main theorems in measure group theory.

Ornstein–Weiss (1980)	:	Γ infinite and amenable $\Leftrightarrow \Gamma\cong_{\mathrm{ran}}\mathbb{Z}$.
Adams (1996)	:	Γ hyperbolic $\Rightarrow$ (non-amenable) $\times\,\mathbb{Z}\not\hookrightarrow_{\mathrm{ran}}\Gamma$.
Gaboriau–Lyons (07)	:	Γ non-amenable $\Leftrightarrow \mathbb{F}_2\hookrightarrow_{\mathrm{ran}}\Gamma$.
Jones–Schmidt (1987)	:	Γ not property (T) $\Leftrightarrow \Gamma\twoheadrightarrow_{\mathrm{ran}}\mathbb{Z}$.
(Zimmer &) Furman (1999)	:	$\Gamma\le \mathrm{SL}(3,\mathbb{R})$ and $\Lambda\cong_{\mathrm{ran}}\Gamma\Rightarrow\Lambda\hookrightarrow_{\mathrm{vir}}\mathrm{SL}(3,\mathbb{R})$.
Kida (06)	:	Γ MCG of higher complexity and $\Lambda\cong_{\mathrm{ran}}\Gamma\Rightarrow\Lambda\cong_{\mathrm{vir}}\Gamma$.

Here $\cong_{\mathrm{ran}}$ means that there is a bijective randomorphism, and "vir" means that they are isomorphic up to finite index subgroups and/or factoring by finite normal subgroups. Gaboriau–Lyons result can be viewed as a group measure theoretic solution to the famous von Neumann's problem, and is used by Epstein to prove that every non-amenable group admits uncountably many mutually non-OE actions. Because time is limited, I don't talk at all about group measure theoretic invariants such as ℓ_2-Betti numbers.

Profinite Actions and the classification problem. Let Γ be a residually finite group and $\Gamma=\Gamma_0\supset\Gamma_1\supset\cdots$ be a chain of finite index subgroups. Then, Γ acts on the probability measure space $X=\varprojlim\Gamma/\Gamma_i$ from the left. We assume that the action is essentially-free (e.g. the chain is normal and has trivial intersection). The profinite action $\Gamma\curvearrowright(X,\mu)$ has the following property: $\xi_n=[\Gamma:\Gamma_n]^{1/2}\sum_{A\in\Gamma/\Gamma_n}\chi_{A\times A}$ in $L^2(X\times X)$ is invariant under Γ (diagonal action), and hence is approximately invariant under $[\mathcal{R}_{\Gamma\curvearrowright(X,\mu)}]$, the group of p.m.p. isomorphisms on X which preserve the orbit equivalence relation $\mathcal{R}_{\Gamma\curvearrowright(X,\mu)}$. In general, we say $\Gamma\curvearrowright(X,\mu)$ is weakly compact if there are Γ-approximately invariant vectors in $L^2(X\times X)$ whose supports converge to the diagonal. Weak compactness is an OE invariant. If Γ acts on an infinite abelian group A by automorphisms and $X=\widehat{A}$, then $\Gamma\curvearrowright(X,\mu)$ is weakly compact iff Γ is co-amenable in the semidirect product $A\rtimes\Gamma$. (A subgroup $\Gamma\le\Lambda$ is said to be co-amenable if there is a Λ-invariant mean on Λ/Γ.) As in the group case, many properties of Γ lift to the groupoid $\mathcal{R}_{\Gamma\curvearrowright(X,\mu)}$ if the action is weakly compact.

Suppose that $\Gamma\curvearrowright(X,\mu)$ and $\Lambda\curvearrowright(Y,\nu)$ are OE and $\alpha\colon\Gamma\times X\to\Lambda$ is the associated cocycle. Also assume that Γ has no non-trivial finite normal subgroup. Under these assumptions, Zimmer proved that $\Gamma\cong\Lambda$ and the actions are conjugate iff the cocycle α is equivalent to a homomorphism. Here a cocycle α is equivalent to β if there is $\phi\colon X\to\Lambda$ such that $\beta(g,x)=\phi(gx)^{-1}\alpha(g,x)\phi(x)$; and a cocycle β is a homomorphism if $\beta(g,\,\cdot\,)$ is essentially constant (and hence $g\mapsto\beta(g)$ is a homomorphism). Ioana (07) proved that if Γ has property (T) and $\Gamma\curvearrowright(X,\mu)$ is a profinite action, then any cocycle $\alpha\colon\Gamma\times X\to\Lambda$ into any discrete group is virtually equivalent to a homomorphism, and hence the orbit equivalence relation

$\mathcal{R}_{\Gamma \curvearrowright (X,\mu)}$ virtually remembers what was the group Γ and what was the action $\Gamma \curvearrowright (X, \mu)$. (This kind of "cocycle superrigidity" with arbitrary discrete group target was first obtained by Popa for Bernoulli shift actions.) Ozawa and Popa (07–08) proved that if Γ is a lattice of (a product of) $\mathrm{SO}(n, 1)$ or $\mathrm{SU}(n, 1)$ and $\Gamma \curvearrowright (X, \mu)$ is a profinite action, then the von Neumann algebra $L^\infty(X, \mu) \rtimes \Gamma$ completely remembers the orbit equivalence relation $\mathcal{R}_{\Gamma \curvearrowright (X,\mu)}$. Unfortunately, there is no example for which both Ioana's theorem and Ozawa–Popa's theorem apply. However, subsequently many examples were found by Peterson, Popa–Vaes and Ioana, for which the correspondences

Group Action $\Gamma \curvearrowright (X, \mu)$	$\rightsquigarrow$	OE Relation $\mathcal{R}_{\Gamma \curvearrowright (X,\mu)}$	$\rightsquigarrow$	vN Algebra $L^\infty(X, \mu) \rtimes \Gamma$

are both one-to-one, i.e., there is an example for which the isomorphism class of the von Neumann algebra $L^\infty(X, \mu) \rtimes \Gamma$ completely remembers the group Γ and the action $\Gamma \curvearrowright (X, \mu)$.

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The limit of characteristic p Betti numbers of a tower of finite covers with amenable fundamental groups

Roman Sauer

(joint work with Peter Linnell and Wolfgang Lück)

The limit of characteristic p Betti numbers of a tower of finite covers with amenable fundamental groups
The talk is based on the results in [3]. To formulate the result, we need to review the notion of *Ore localization.*

Consider a torsionfree group Γ and a field k. Let S be the set of non-zero-divisors of $k\Gamma$. This is a multiplicatively closed subset of $k\Gamma$ and contains the unit element of $k\Gamma$. Suppose that $k\Gamma$ satisfies the *zero-divisor conjecture*, i.e., $S = k\Gamma - \{0\}$. Further assume that S satisfies the *left Ore condition*, i.e., for $r \in k\Gamma$ and $s \in S$ there exists $r' \in k\Gamma$ and $s' \in S$ with $s'r = r's$. Then we can consider the *Ore localization* $S^{-1}k\Gamma$. Recall that every element in $S^{-1}k\Gamma$ is a formal fraction of the form $s^{-1} \cdot r$ for $r \in k\Gamma$ and $s \in S$, and one has the usual calculus of fractions. The Ore localization $S^{-1}k\Gamma$ is a skew field and the canonical map $k\Gamma \to S^{-1}k\Gamma$ sending r to $e^{-1} \cdot r$ is injective. The functor $S^{-1}k\Gamma \otimes_{k\Gamma} -$ is exact.

There is no counterexample to the zero-divisor conjecture for torsionfree groups so far. Linnell proved the zero-divisor conjecture for torsionfree elementary amenable groups. If an amenable group satisfies the zero-divisor conjecture, its group ring $k\Gamma$ satisfies the Ore condition with respect to $S = k\Gamma - \{0\}$, where k is an arbitrary field. On the other hand, the Ore condition is violated if the group in question has a free subgroup of rank 2.

Definition. *Let Γ be a torsionfree amenable group such that Γ contains no zero-divisors. The* Ore dimension *of a $k\Gamma$-module M is defined by*

$$\dim_{k\Gamma}^{\mathrm{Ore}}(M) = \dim_{S^{-1}k\Gamma}\big(S^{-1}k\Gamma \otimes_{k\Gamma} M\big).$$

A residual chain $(\Gamma_n)_{i\geq 0}$ in a group Γ is a descending series of normal subgroups of finite index whose intersection is trivial. We formulate the main result only for torsionfree groups. Similar results hold true if the group Γ is virtually torsionfree or has a bound on the orders of finite subgroups.

Theorem. *Let k be a field. Let Γ be a torsionfree amenable group for which $k\Gamma$ has no zero-divisors. Let $(\Gamma_i)_{i\geq 0}$ be a residual chain of Γ. Let X be a finite free Γ-CW-complex. Then we get for all $i \geq 0$*

$$\dim_{k\Gamma}^{\mathrm{Ore}}\big(H_i(X)\big) = \lim_{n\to\infty} \frac{\dim_k\big(H_i(\Gamma_n\backslash X;k)\big)}{[\Gamma:\Gamma_n]}.$$

We would like to mention that the above result can be reduced to Lück's approximation theorem if the ground field is of characteristic zero. The limit above would be equal to the i-th L^2-Betti number of X. So the really interesting case is the one where k has positive characteristic $p > 0$.

G. Elek initiated the study of L^2-theory in the context of arbitrary ground fields [2]. Moreover, he proved that the limit above exists and is independent of the residual chain if Γ is free abelian. More recently it was observed that the existence and independence of the limit for amenable Γ follows from results of Weiss and Lindenstrauss-Weiss (see the appendix in [1]). Our contribution here is that the limit can be expressed in terms of Ore localizations.

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Homomorphisms into the mapping class groups

Mark V. Sapir
(joint work with J. Behrstock and C. Drutu)

Let Γ and G be finitely generated groups. Suppose there are infinitely many pairwise non-conjugate in G homomorphisms $\Gamma \to G$. Then Γ acts non-trivially on an asymptotic cone of G . Hence studying asymptotic cones of groups gives information about the number of homomorphisms $\Gamma \to G$ up to conjugacy. Let G be the Mapping Class Group of a punctured (oriented) surface. Let $\mathcal{C}$ be an asymptotic cone of G. We prove that $\mathcal{C}$ is bi-Lipschitz equivalent to a median space and is bi-Lipschitz equivariantly embedded into a product of finitely many $\mathbb{R}$-tree. This implies that if Γ has infinitely many pairwise non-conjugate homomorphisms into G, then Γ has a finite index subgroup that acts non-trivially on an $\mathbb{R}$-tree. If in addition Γ is finitely presented, then it virtually splits. In particular, Γ cannot have Kazhdan property (T).

The measurable Kesten theorem

Bálint Virág
(joint work with Miklós Abért and Yair Glasner)

Let G be a d-regular, countable, connected undirected graph. Let $\ell^2(G)$ be the Hilbert space of all square summable functions on the vertex set of G and let $M : \ell^2 \to \ell^2$ be the Markov (averaging) operator on $\ell^2(G)$. When G is infinite, we define the *spectral radius* of G, denoted $\rho(G)$, to be the norm of M. When G is finite, we want to exclude the trivial eigenvalues and thus define $\rho(G)$ to be the second largest element in the set of absolute values of eigenvalues of M.

We call G a *Ramanujan* graph, if $\rho(G) \leq \rho(T_d)$ where T_d denotes the d-regular tree. Since T_d covers G, when G is infinite, $\rho(T_d) = 2\sqrt{d-1}/d$ is the minimal possible value for the spectral radius. For finite graphs, this is not the case, but it stays true asymptotically. Namely, the Alon-Boppana theorem [6] says that if (G_n) is a sequence of finite connected d-regular graphs with $|G_n| \to \infty$, then $\liminf \rho(G_n) \geq \rho(T_d)$. The interest in finite Ramanujan graphs comes from that they are best expander graphs, at least in the spectral sense. However, it is not an easy task to find infinite families of finite Ramanujan graphs of a given degree; the most famous construction is by Lubotzky, Philips and Sarnak [3]. Friedman [1] showed that random d-regular graphs are close to being Ramanujan for every d.

The Ramanujan graphs given in the constructions mentioned above all have *large girth*, that is, the minimal size of a cycle tends to infinity with the size of the graph. However, the reason for that is group theoretic and not spectral, and indeed, by [2] it is possible to find a Ramanujan sequence (G_n) of fixed degree such that all G_n contain loops. Our first theorem shows that nevertheless, there is an asymptotic restriction, as Ramanujan graphs necessarily have few cycles of short length.

A sequence (G_n) of graphs has *essentially large girth*, if for all L, we have

$$\lim_{n\to\infty} \frac{c_L(G_n)}{|G_n|} = 0$$

where $c_L(G_n)$ denotes the number of cycles of length L in G.

Theorem 1. *Let (G_n) be a Ramanujan sequence of finite d-regular graphs. Then (G_n) has essentially large girth.*

Theorem 1 follows from a generalization of a theorem of Kesten to random unimodular networks. Let $\mathrm{Cay}(\Gamma, S)$ denote the Cayley graph of the group Γ with respect to the generating set S. In his seminal papers [4] and [5], Kesten proved the following result.

Theorem 2 (Kesten)**.** *Let Γ be a group generated by a finite symmetric set S and let N be a normal subgroup of infinite index. Then the following are equivalent:*
1) $\rho(\mathrm{Cay}(\Gamma/N, S)) = \rho(\mathrm{Cay}(\Gamma, S))$;
2) N is amenable.

In the particular case when Γ is a non-Abelian free group (or a suitable free product), Γ does not have any nontrivial normal amenable subgroups. This implies that infinite Ramanujan Cayley graphs are trees. It is easy to find counterexamples for this result if we omit the vertex transitivity condition. However, the result does hold for d-regular random rooted graphs whose distribution are invariant under moving the root. More precisely, a d-tregular unimodular network is a random rooted graph (G, o) so that if v is a random neighbor of o chosen independently and uniformly, then the triple (G, o, v) has the same distribution as (G, v, o).

Theorem 3. *Let G be a d-regular random unimodular network that is infinite and Ramanujan a.s. Then $G = T_d$ a.s.*

Using the language of Benjamini-Schramm graph convergence, Theorem 3 implies Theorem 1.

Theorem 3 takes care of Ramanujan graphs (or, from another point of view, free groups), but what happens to Kesten's original theorem for other groups? Let Γ be a group generated by a finite symmetric set S and let H be a subgroup of Γ. In this situation, the role of the tree T_d will be played by the Cayley graph $\mathrm{Cay}(\Gamma, S)$ and the role of an arbitrary graph will be played by a Schreier graph $\mathrm{Sch}(\Gamma/H, S)$: here the vertex set is the right coset space of H in Γ and just as for the Cayley graph, the edges encode the action of S on the space.

Theorem 2 fails badly for subgroups in general; for instance, when Γ is a free group on, say 1000 generators and H is the subgroup generated by the first 2 generators of Γ, then $\mathrm{Sch}(\Gamma/H, S)$ is Ramanujan, although H is obviously not amenable. Still, Theorem 2 does hold for a natural random generalization of normal subgroups. By an *invariant random subgroup* of Γ, we mean a probability distribution on subgroups of Γ that is invariant under conjugation by Γ.

Theorem 4. *Let Γ be a group generated by a finite symmetric set S and let H be an invariant random subgroup of Γ such that H has infinite index in Γ a.s. Then*

the following are equivalent:
1) $\rho(\mathrm{Sch}(\Gamma/H, S)) = \rho(\mathrm{Cay}(\Gamma, S))$ *a.s.*
2) H *is amenable a.s.*

Note. This abstract was adapted from the introduction of an upcoming research paper with the same title.

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Unitary representations of oligomorphic groups

Todor Tsankov

Traditionally, representation theory is restricted to studying representations of locally compact groups, and for a good reason: the Haar measure provides an invaluable tool for constructing and analyzing representations. It gives rise to the left-regular representation (so that every locally compact group has at least one faithful representation) but also allows to define convolution of functions and various useful topologies on function spaces on the group. And indeed, many standard theorems of representation theory break down for non-locally compact groups: for example, the group of homeomorphisms of the reals has no non-trivial unitary representations (Megrelishvili [6]), while the group of all measurable maps from $[0, 1]$ to the circle has a faithful unitary representation (by multiplication on $L^2([0, 1])$) but has no irreducible representations (this example is due to Pestov; see [2, Example C.5.10]). Nevertheless, some non-locally compact groups do have a nice representation theory: for example, the infinite symmetric group S_∞ and the unitary group of a separable, infinite-dimensional Hilbert space both have only countably many irreducible representations that separate points and every representation splits as a sum of irreducibles (Lieberman [5] and Kirilov [4]). In both situations, the representation theory is quite similar to the one for compact groups. We present a similar classification result for the representations of oligomorphic permutation groups.

Let S_∞ be the group of *all* permutations (not necessarily of finite support) of a countable infinite set $\mathbf{X}$. It becomes naturally a topological group if equipped with the pointwise convergence topology (where $\mathbf{X}$ is taken to be discrete). For us, a *permutation group* will be a *closed* subgroup of S_∞ equipped with its natural action on the set $\mathbf{X}$. It is well known that the topological groups that can be realized as permutation groups are exactly the *Polish* (separable, completely metrizable) groups that admit a basis at the identity consisting of open subgroups. The basis is given by the stabilizers of finite sets.

A natural way in which permutation groups arise is as *automorphism groups of countable structures* in model theory, i.e. one fixes some relations and functions on the set $\mathbf{X}$ and considers the group of all permutations that preserve them. We will restrict our attention to *oligomorphic* permutation groups which are defined as follows.

Definition 1. *A permutation group $G \curvearrowright \mathbf{X}$ is called* oligomorphic *if the induced action $G \curvearrowright \mathbf{X}^n$ has only finitely many orbits for each n.*

The following property of topological groups will also be relevant for us.

Definition 2. *A topological group G is called* Roelcke precompact *if for every open neighborhood of the identity U, there exists a finite set F such that $G = UFU$.*

In the above definition, one can obviously restrict U to belong to a basis at the identity, so for closed subgroups of S_∞, the definition has the following equivalent form: $G \leq S_\infty$ is Roelcke precompact iff for every open subgroup $V \leq G$, the set of double cosets $\{VxV : x \in G\}$ is finite. One has the following basic characterization.

Proposition 3. *For a closed, non-compact group $G \leq S_\infty$, the following are equivalent:*

(i) *G is Roelcke precompact;*
(ii) *for every continuous transitive action $G \curvearrowright \mathbf{X}$ on a countable set $\mathbf{X}$, the induced action $G \curvearrowright \mathbf{X}^n$ has finitely many orbits for each n.*

Moreover, if $G \curvearrowright \mathbf{X}$ is a closed oligomorphic permutation group, then G is Roelcke precompact.

Note that an oligomorphic group can never be locally compact: if $V \leq G$ is a compact open subgroup, then the union of finitely many V double cosets will be compact.

A standard way to produce structures with oligomorphic automorphism groups is the so-called Fraïssé construction: given a class of finite structures satisfying a certain amalgamation property, there is a way to build an infinite structure that contains all structures in the class as substructures and is moreover homogeneous. We refer the reader to [3] for the general theory and just present a few examples.

- The Fraïssé limit of all finite sets without structure is a countably infinite set $\mathbf{X}$. The corresponding group is S_∞, the group of all permutations of $\mathbf{X}$.
- The Fraïssé limit of all finite linear orders is the countable dense linear order without endpoints $(\mathbf{Q}, <)$. We denote the corresponding automorphism group by $\operatorname{Aut} \mathbf{Q}$.
- The Fraïssé limit of all finite boolean algebras is the countable atomless boolean algebra which is isomorphic to the algebra of all clopen subsets of the Cantor space $2^{\mathbb{N}}$. The corresponding automorphism group is $\operatorname{Homeo}(2^{\mathbb{N}})$, the group of all homeomorphisms of $2^{\mathbb{N}}$.

- The Fraïssé limit of all finite vector spaces over a fixed finite field $\mathbf{F}_q$ is the countable-dimensional vector space over $\mathbf{F}_q$.
- The Fraïssé limit of all finite graphs is the *random graph*, the unique countable graph R such that for every two finite disjoint sets of vertices U, V, there exists a vertex x which is connected by an edge to all vertices in U and to no vertices in V. We denote its automorphism group by $\mathrm{Aut}(R)$.

The main theorem describing the representations of oligomorphic groups is the following.

Theorem 4. *Let G be an oligomorphic group. Then every irreducible unitary representation of G is of the form $\mathrm{Ind}_{N(V)}^{G}(\sigma)$, where $V \leq G$ is an open subgroup, $N(V)$ is the normalizer of V, and σ is an irreducible representation of the* finite *group $N(V)/V$. Moreover, every unitary representation of G splits as a sum of irreducibles.*

As every oligomorphic group has only countably many distinct open subgroups, this means that every oligomorphic group has only countably many irreducible representations.

If one is given a realization of an oligomorphic group as the automorphism group of a countable structure, it is usually possible to give a more concrete description of the representations in terms of the structure. For example, for the random graph, one can take a finite (induced) subgraph $A \subseteq R$ and set V to be the pointwise stabilizer of A. Then $N(V)$ is the setwise stabilizer of A and $N(V)/V \cong \mathrm{Aut}(A)$. As a result, one obtains that irreducible representations of the automorphism group of the random graph are obtained by induction from irreducible representations of automorphism groups of finite graphs (and in fact, this correspondence is one-to-one if one takes care of the obvious identifications).

Once one knows all the representations of a group, it is usually not difficult to verify various representation-theoretic properties like property (T). Using a technique of Bekka [1], one can prove the following.

Theorem 5. *All of the above examples have property (T).*

It is also possible to find explicit (finite) Kazhdan sets and constants.

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Random groups of intermediate rank

Mikaël Pichot

I provide additional information on a recent paper of Sylvain Barré and myself [1]. The goal is to construct a new family of (finitely presented countable) groups which are "almost" lattices in certain higher rank Lie groups over nonarchimedean local fields (for example $K = \mathbb{F}_p((x))$ and $\mathbf{G} = \mathrm{SL}_3$).

We introduce new models of random groups which are parallel to the classical models of Gromov [3, 4]. Let us first describe the latter briefly.

Start with the free group F_2 on 2 generators a, b (or more generally any group that has "many quotients") and let

$$W_n = \text{the set of cyclically reduced words of length } n \text{ in } a^{\pm 1}, b^{\pm 1}.$$

M. Gromov defines several models of random groups, for example:

The few-relators model. Let $\ell \geq 1$ be a fixed integer. Choose uniformly and independently at random ℓ words $w_1, \ldots, w_\ell$ in W_n (we omit the dependance of w_i in the particular event and n) and consider the group presentation

$$\langle F_2 \mid w_1, \ldots, w_\ell \rangle.$$

Say that a property is satisfied with *overwhelming probability* if the probability to satisfy it goes to 1 as n goes to infinity. Gromov [3] proves that for every $\ell \geq 1$, a random group in this model is hyperbolic with overwhelming probability (and more precisely, that it satisfies the $C'(\lambda)$ small cancellation condition for every $\lambda > 0$, as all relations have same length).

The density model. In this model, the number ℓ of randomly chosen relations (again, uniformly and independently) is allowed to vary with n. Typically, M. Gromov chooses $\ell_n = |W_n|^d$ words in W_n, where d is a (density) parameter, and constructs a random group G with presentation

$$\langle F_2 \mid w_1, \ldots, w_{\ell_n} \rangle.$$

A well-known result of [4] asserts that if $d > 1/2$, then G is trivial with overwhelming probability (trivial means $G = \{e\}$ or $G = \mathbb{Z}/2\mathbb{Z}$ here), while if $d < 1/2$ then G is infinite hyperbolic with overwhelming probability.

In the new models, we will rather start with a sequence of finitely presented group G_n, called the *initial data* for the model, and *remove* relations at random from the given presentation of G_n. The number of relations in the given presentation of G_n tends to infinity. Depending on how we remove these relations, we obtain analogs of the Gromov models above (see [1]).

At this level of generality, this is of little use but the point is to choose interesting sequences of finitely presented groups G_n as initial data, which typically arise as a decreasing chain of finite index subgroups of a well–chosen G_1.

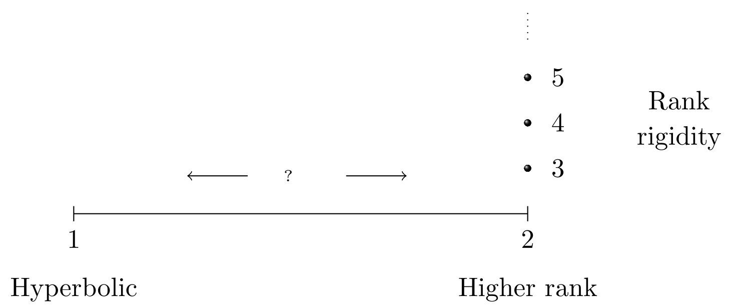
Thus, there are two important points with the new models:

(1) They are "mirror symmetric" to the Gromov models, in the sense that, rather than adding relations at random to groups which have "many quotients", we remove relations at random to groups which have (very few quotients but) "many extensions".
(2) They are *localized* in the space of finitely presented groups: rather than being generic among all finitely presented groups, we will restrict these constructions to a tiny portion of the space of group presentation and prove results there. The model can be seen as a *randomization* of the initial sequence of group presentations.

An example where (2) is not satisfied would be to let the sequence G_n densify in the space of finitely generated groups (endowed with the usual Chabauty topology, i.e. pointed Gromov–Hausdorff).

We are interested in drawing conclusions in "geometry of intermediate rank" (see [2]), whence our initial data are of appropriate geometric (and arithmetic) origin.

Let K be a nonarchimedean local field (e.g. $K = \mathbb{Q}_p$ or $K = \mathbb{F}_p((x))$). Let us choose as initial data for the model certain arithmetic chains $G_1 \geq G_2 \geq \ldots$ of finite index normal subgroups in a uniform lattice of $\mathrm{SL}_3(K)$. For instance, take the Lubotzky–Samuels–Vishne congruence subgroups, which one can have a hold on using strong approximation (see [5]). The Bruhat–Tits buildings X_K of $\mathrm{SL}_3(K)$ is of rank 2, and the randomization of G_n is obtained by removing at random (uniformly and independently) G_n-orbits of chambers in X_K. Equivalently, if the action of G_1 on X_K is free, we remove at random chambers of X_K/G_n. Taking the universal cover (say X) of this (random) space, we obtain a random group G which acts freely uniformly on the CAT(0) space X and which surjects to a uniform lattice in $\mathrm{SL}_3(K)$—namely, G_n (see [1] for details). It is convenient to introduce a notion of "building with chambers missing" to explain the properties of (X, G). As described above, we can then consider analogs of the few-relators model and the density model, called respectively the model with a few chambers missing and the lattice density model.



The idea of *rank interpolation* is quite self-explanatory. Our setting is that of CAT(0) space X (usually with a simplicial structure) endowed with a proper action of a countable group G with compact quotient. In this context, the word "rank" refers to everything related to *flatness* in X. We want to interpolate between "soft" hyperbolic spaces (which have rank 1) and rigid "higher rank" spaces, like Bruhat–Tits buildings or symmetric spaces of rank at least 2. Many concrete examples are given in [2], for example we find there spaces of so-called "rank $\frac{7}{4}$".

Here is a specific open problem: if X contains an isometric copy of $\mathbb{R}^2$, does G contains a copy of $\mathbb{Z}^2$? This question is a version of the flat closing conjecture, which has received a lot of attention over the years. The above-mentioned intermediate rank constructions provide many test spaces (X, G) for it. The randomized lattices too are of intermediate rank. In fact their rank is "very close to 2", in various senses that can be made precise. For example, as far as the above question is concerned, we prove that, in the lattice density model with initial data from [5], the group G contain a copy of $\mathbb{Z}^2$ whenever $d < 1/4$. Like in Gromov's model, there is a phase transition at $d = 1/2$, so that (with overwhelming probability) the group G splits off a free group $G = G_0 * F_k$ if $d > 1/2$, while it has Kazhdan's property T if $d < 1/2$ (if the order of the residue field is large enough). It would be interesting to determine whether the existence of $\mathbb{Z}^2$ still holds for $d > 1/4$, especially for $1/4 < d < 1/2$ (there are some choices of initial data from [5] for which it does). The existence of $\mathbb{R}^2$ in X holds with overwhelming probability whenever $d < 1/2$ in all cases. Several problems are left open. For example: what rigidity properties (Mostow, Margulis, OE,...) do these random groups have? Are they non linear? residually finite?...

Like in many models of random groups, the groups we obtain here have geometric dimension 2. The same construction can be performed in higher dimension but will not lead to new groups. Taking for instance the Ramanujan hypergraphs associated (by [5]) to chains of subgroups of $\mathrm{SL}_n(K)$ for $n \geq 3$, the above randomization does produce random hypergraphs with interesting expansion properties. Again, these random hypergraphs are localized in the space of hypergraphs, as opposed to the usual constructions of random graph, for example, used to show that "random graphs are expanders".

These constructions belong to the usual "structure and randomness" circle of idea, although it goes the opposite way: rather than uncovering the structure of some random/huge object (as in the Szemerédi lemma for example), we start with a structured set of data and add a random component to it.

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