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Nonlinear Waves and Dispersive Equations

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Abstract. The aim of the workshop was to discuss current developments in nonlinear waves and dispersive equations from a PDE based view. The talks centered around rough initial data, long time and global existence, perturbations of special solutions, and applications.

Mathematics Subject Classification (2000): 35xx.

Introduction by the Organisers

Nonlinear dispersive equations are models for nonlinear waves in a wide range of physical contexts. They display the competing or cooperating effects of linear dispersion and nonlinear interactions, which may be focussing or defocussing. They are linked to diverse areas of mathematics and physics, ranging from nonlinear optics over Fourier analysis to integrable systems.

Despite a huge range of different dispersive equations there are a number of recurrent themes. Current research aims at a conceptual understanding of phenomena across different classes of equations, and a detailed understanding of features of classes of solutions. The following themes were covered.

(1) *Rough initial data* Scaling often determines a critical Hilbert space of initial data. Wellposedness may or may not hold up to the critical space. During the last years an almost complete understanding of wave maps in 2 space dimensions and of critical nonlinear Schrödinger equations with power nonlinearities has been gained in $\mathbb{R}^n$. With these insights the Schrödinger maps are intensively studied. The situation is different for the Korteweg-de-Vries hierarchy and the Nonlinear Schrödinger equation, where specific

results throw some light upon the gap between scaling and wellposedness. A conceptual understanding of this gap seems to be out of reach. The insights gained in recent years allowed to attack global wellposedness in a number of surprising directions, including global wellposedness for a quintic nonlinear Schrödinger equation on a compact three dimensional manifold, global existence for supercritical 'stochastic' initial data and global existence for several problems from physics.

(2) *Dynamics near solitons* What happens to solitons in perturbated media? What can one say about solutions near solitons? The set of solitons appears to be surprising stable. Can one describe the interaction between modal parameters and a 'wave' part of the solution? A particular important case is blow-up along the set of solitons. The study of this mechanism gave insights into blow-up dynamics.

(3) *Applications and relations to different fields* There are prominent examples of dispersive integrable PDEs, including Korteweg-de-Vries, Nonlinear Schrödinger and Camassa-Holmes. They are asymptotic equations for example for waterwaves models. The Einstein equations in general relativity naturally lead to linear and nonlinear hyperbolic equations. There is considerable progress on the understanding of decay for linear wave equations in the Kerr geometry- potentially an important step towards stability of the Kerr family.

There is a large number of promising young mathematicians working in this area. As in previous meetings the organizers gave a strong preference to talks by young researchers.

Workshop: Nonlinear Waves and Dispersive Equations

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Abstracts

Existence and uniqueness of minimal blow-up solutions to an inhomogeneous mass critical NLS

Jérémie Szeftel
(joint work with Pierre Raphaël)

In this note, we present the results obtained in [10] for a two dimensional focusing mass critical nonlinear Schrödinger equation with an inhomogeneous nonlinearity:

$$(NLS) \quad \begin{cases} i\partial_t u = -\Delta u - k(x)|u|^2 u, & (t,x) \in \mathbb{R} \times \mathbb{R}^2, \\ u(t_0,x) = u_0(x), & u_0 : \mathbb{R}^2 \to \mathbb{C}, \end{cases} \tag{1}$$

for some smooth bounded inhomogenity $k : \mathbb{R}^2 \to \mathbb{R}_+^*$ and some real number $t_0 < 0$. This is a canonical model to break the large group of symmetries of the $k \equiv 1$ homogeneous case.

1. The homogeneous case

Let us start with recalling some well-known facts in the homogeneous case $k \equiv 1$:

$$\begin{cases} i\partial_t u = -\Delta u - |u|^2 u, & (t,x) \in \mathbb{R} \times \mathbb{R}^2, \\ u(t_0,x) = u_0(x), & u_0 : \mathbb{R}^2 \to \mathbb{C}, \end{cases} \tag{2}$$

where $t_0 < 0$. A large group of H^1 symmetries leaves the flow invariant: if $u(t,x)$ solves (2), then $\forall(\lambda_0, \tau_0, x_0, \beta_0, \gamma_0) \in \mathbb{R}_*^+ \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$, so does

$$v(t,x) = \lambda_0^{\frac{N}{2}} u(t+\tau_0, \lambda_0 x + x_0 - \beta_0 t) e^{i\frac{\beta_0}{2}\cdot(x - \frac{\beta_0}{2}t)} e^{i\gamma_0}. \tag{3}$$

A last symmetry is not in the energy space H^1 but in the virial space $\Sigma = \{xu \in L^2\} \cap H^1$, the pseudo conformal transformation: if $u(t,x)$ solves (2), then so does

$$v(t,x) = \frac{1}{|t|^{\frac{N}{2}}} \overline{u}\left(\frac{1}{t}, \frac{x}{t}\right) e^{i\frac{|x|^2}{4t}}. \tag{4}$$

Following [4], [6], let Q be the unique H^1 nonzero positive radial solution to

$$\Delta Q - Q + Q^3 = 0, \tag{5}$$

then the variational characterization of Q ensures that initial data $u_0 \in H^1$ with $\|u_0\|_{L^2} < \|Q\|_{L^2}$ yield global and bounded solutions $T = +\infty$, [11]. On the other hand, finite time blow-up may occur for data $\|u_0\|_{L^2} \geq \|Q\|_{L^2}$. At the critical mass threshold, the *pseudo conformal symmetry* (4) applied to the periodic solitary wave solution $u(t,x) = Q(x)e^{it}$ yields a *minimal mass blow-up solution*:

$$S(t,x) = \frac{1}{t} Q\left(\frac{x}{t}\right) e^{i\frac{|x|^2}{4t} - \frac{i}{t}}, \quad \|S(t)\|_{L^2} = \|Q\|_{L^2} \tag{6}$$

which blows up at $t = 0$. In [7], Merle proves the *uniqueness* of the critical mass blow-up solution: a solution $u \in H^1$ with $\|u_0\|_{L^2} = \|Q\|_{L^2}$ and blowing up at $t = 0$ is equal to $S(t)$ up to the symmetries of the flow. Through the conformal invariance, this results yields a complete dynamical classification of the solitary wave as the only nondispersive solution in Σ with critical mass.

2. The inhomogeneous case

We now come back to (1). The canonical effect of the inhomogenity is to completely destroy the group of symmetry (3)[1], and in this sense (1) is a toy model to analyze the properties of NLS systems in the absence of symmetries.
From standard variational techniques and a virial type argument, Merle has derived in [9] the criterion of global existence for (1). Since k is bounded, we may assume without loss of generality that:

$$\sup_{x\in\mathbb{R}^2} k(x) = 1.$$

Then initial data with $\|u_0\|_{L^2} < \|Q\|_{L^2}$ yield global and H^1 bounded solutions while finite time blow-up may occur for $\|u_0\|_{L^2} > \|Q\|_{L^2}$. Moreover, Merle gives necessary conditions for the existence of a minimal blow-up element u: if such a solution u exists, there is $x_0 \in \mathbb{R}^2$ such that u concentrates all of its mass at x_0 at the blow-up time, and the concentration must occur at a point where k reaches its maximum, and hence in particular:

$$\nabla k(x_0) = 0.$$

In the presence of a very smooth and flat k at x_0, the existence of critical elements may be derived using a brutal fixed point argument, [9], [8], [3]. This argument has been recently sharpened by Banica, Carles, Duyckaerts [1] for (1) by adapting the method designed by Bourgain and Wang [2] and further revisited by Krieger and Schlag [5]: after linearizing the problem close to the explicit $S(t)$ approximate solution, one uses modulation theory and energy estimates to treat perturbatively the unstable modes and integrate the system backwards from the singularity. This allows one to lower the flatness of k and indeed the existence of critical mass blow-up solution is obtained under the flatness assumption:

$$\nabla k(x_0) = \nabla^2 k(x_0) = 0.$$

Let us say that in this approach, the problem is treated perturbatively from the homogeneous case and x_0 can be taken to be any point where k is flat enough. The case of smooth k with nondegenerate Hessian at x_0 i.e.

$$\nabla^2 k(x_0) < 0$$

is out of reach with these techniques because it generates large deformations *which live at the same scaling like $S(t)$*.

Let us also say that even when existence is known, uniqueness in the energy class as proved in [7] for the homogeneous case does not follow. The strength and the weakness of Merle's pioneering proof for $k \equiv 1$ is that it fundamentally relies on the *pseudoconformal symmetry* (4) which is lost here. The challenge is hence

[1] up to phase invariance

to provide a more *dynamical proof* of the classification theorem in the absence of symmetry.

3. Statement of the result

Let us assume that k is smooth enough: $k \in \mathcal{C}^5 \cap W^{1,\infty}$. In [10], we obtain the following necessary conditions for the existence of a critical mass blow-up solution which is based on a slight refinement of the variational techniques introduced in [9]:

Proposition 1 (Necessary condition for the existence of a critical blow-up element)**.** *Let u with $\|u\|_{L^2} = \|Q\|_{L^2}$ be a solution to* (1) *which blows up at time $T = 0$, then there exists $x_0 \in \mathbb{R}^2$ such that $k(x_0) = 1$ and u blows up at x_0 in the sense:*

$$|u(t)|^2 \rightharpoonup \|Q\|_{L^2}^2 \delta_{x=x_0} \quad \text{as} \quad t \to 0. \tag{7}$$

Moreover, the energy E_0 of u satisfies:

$$E_0 + \frac{1}{8}\int \nabla^2 k(x_0)(y,y)Q^4 > 0. \tag{8}$$

In order to classify critical mass blow-up solutions, we hence pick $x_0 \in \mathbb{R}^2$ such that $k(x_0) = 1$ and focus onto the case of a nondegenerate Hessian $\nabla^2 k(x_0) < 0$ which we expect to be the most delicate one. We claim that the lower bound (8) is sharp, and the following theorem is the main result of [10]:

Theorem 2 (Existence and uniqueness of a critical element at a nondegenerate critical point)**.** *Let $x_0 \in \mathbb{R}^2$ with*

$$k(x_0) = 1 \quad \text{and} \quad \nabla^2 k(x_0) < 0.$$

Then for all E_0 satisfying (8)*, there exists a time $t_0 < 0$ and a unique up to phase shift $u \in \mathcal{C}([t_0, 0), H^1(\mathbb{R}^2))$ solution to* (1) *with critical mass $\|u\|_{L^2} = \|Q\|_{L^2}$ which blows up at time $T = 0$ and at the point x_0 in the sense of* (7)*, with energy E_0. Moreover,*

$$\lim_{t\to 0} Im\left(\int \nabla u \overline{u}\right) = 0. \tag{9}$$

In other words, for k smooth, there exists a critical mass finite time blow-up solution if and only if the supremum of k is attained, and the corresponding minimal blow-up elements at a non degenerate blow-up point are completely classified.

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On energy critical nonlinear Schrödinger equations with (partially) periodic initial data in H^1

Sebastian Herr
(joint work with Daniel Tataru and Nikolay Tzvetkov)

Consider the well-posedness problem for energy critical nonlinear Schrödinger IVPs

$$\begin{aligned}(i\partial_t - \Delta)u &= \pm|u|^{\frac{4}{d-2}}u \text{ in } (-T,T)\times M\\ u(0,x) &= \phi(x), \quad x \in M\end{aligned}$$

posed on specific d-dimensional Riemannian manifolds M, namely $M = \mathbb{T}^3$ (where $d = 3$) and $M = \mathbb{T}\times\mathbb{R}^3, \mathbb{T}^2\times\mathbb{R}^2$ (where $d = 4$). Here, $\mathbb{T}$ denotes the flat torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$.

In [3, 4] we prove the local well-posedness and the small data global well-posedness in the above mentioned set-up with initial data in the scaling critical energy space $H^1(M)$. These results include the existence of strong solutions, uniqueness in a specified subspace, local Lipschitz-continuity of the map $\phi \mapsto u$, and the persistence of higher initial Sobolev regularity. As our global results are restricted to small data they apply both in the focusing and defocusing case.

The proofs are based on refinements of sharp multilinear Strichartz inequalities and critical function space theory. More precisely, for solutions to the linear equations which are Fourier localized to rectangular sets (with one short side) we prove scale invariant Strichartz estimates which take into account the smallness of one side, which extend previous estimates of Bourgain [1]. In combination with almost orthogonality arguments, this allows us to obtain sharp bi- and tri-linear Strichartz type estimates in critical function spaces, where it is possible to perform the Picard iteration argument. A crucial ingredient are function spaces which are sensitive to a finer than dyadic Fourier decomposition. For this purpose we use the U^p and V^p type spaces of Tataru and Koch-Tataru (cf. [2] for details) on the unit scale.

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Effective evolution equations from many body quantum dynamics

Benjamin Schlein

Systems of interest in physics are typically composed by a huge number of elementary particles N. Dilute samples of Bose-Einstein condensates contain $N \simeq 10^3 - 10^6$ atoms (and, strictly speaking, each atom contains many elementary components). The number of molecules contained in chemical samples is typically of the order of Avogadro's number, $N \simeq N_A \simeq 6 \cdot 10^{23}$. Systems of relevance in astronomy and cosmology (like stars and galaxies) are composed by up to $N \simeq 10^{60}$ elementary component.

In principle, the dynamics of these systems can be determined by solving fundamental evolution equations like the Newton equation or the many-body Schrödinger equation. These are partial differential equations in N coupled variables. In practice, fundamental equations are impossible to solve (neither analytically nor numerically) when so many particles are involved (unless the interaction among the particles is neglected). Moreover, observers are typically not interested in following the precise evolution of every particle in the system. Instead, they need a prediction for the macroscopically measurable properties of the dynamics (which result from averaging over the many particles in the system). For this reason, it is very important to find effective evolution equations which, on the one hand, can be easily solved (numerically), and, on the other hand, accurately predict the macroscopic behavior of the system under consideration. One of the main goal of statistical mechanics consists therefore in the development of effective theories approximating the solutions of fundamental evolution equations in the relevant regimes.

In my talk, I am going to discuss two examples of systems of interest in physics, for which the derivation of effective evolution equations can be made rigorous in a mathematical sense. In the first part, which is based on the results of [4], a joint work with A. Michelangeli, I will illustrate the derivation of the semi-relativistic Hartree equation for the evolution of boson stars and for the description of the phenomenon of stellar collapse. In the second part of my talk, based on [1, 2, 3], a series of works in collaboration with L. Erdős and H.-T. Yau, I will sketch the derivation of the Gross-Pitaevskii equation for the dynamics of initially trapped Bose-Einstein condensates. In both cases, the starting point of our analysis is the fundamental many-body Schrödinger equation.

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On the Camassa-Holm equation: physical relevance and mathematical properties

Adrian Constantin

The Camassa-Holm equation

$$u_t - u_{txx} + \kappa u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx},$$

with κ a parameter, was first derived as a bi-Hamiltonian system in [4], and a physical derivation of it as a model for shallow water waves was proposed in [1]. In [1] Camassa-Holm also found a Lax pair and discovered that for $\kappa = 0$ the solitary wave solutions have a peak at their crest and interact like solitons. Subsequently the equation attracted a lot of attention due its remarkable structure: for a large class of initial data it is a completely integrable infinite-dimensional Hamiltonian system (and an inverse spectral as well as an inverse scattering approach are by now available), its solitary waves are solitons (smooth for $\kappa \neq 0$), it is a re-expression of geodesic flow for the H^1 right-invariant metric on the diffeomorphism group (if $\kappa = 0$) and on the Bott-Virasoro group (if $\kappa \neq 0$). Certain smooth data develop into solutions that exist for all times while others develop singularities in finite time: the slope becomes unbounded while the solution itself remains bounded ("wave breaking"). The success of the qualitative investigations of the equation led to an increased interest in the physical relevance of the equation. It turns out (see the discussion in [5]) that the physical derivation as a model for the unidirectional propagation of shallow water waves in water with a flat bed presented in [1] is not consistent with the governing equations for water waves. An alternative derivation was proposed in [5] and was put on a firm mathematical basis in [2]. It turns out that rather than being an equation for the average horizontal fluid velocity as well as for the surface water wave (as is the case for the classical Korteweg-de Vries equation), the Camassa-Holm equation is an equation for the horizontal fluid velocity at a certain depth, and the free surface is given by a nonlinear transformation of u. This occurs in the regime of shallow water waves of moderate amplitude while the Korteweg-de Vries equation arises as a model for shallow water waves of small amplitude. The fact that waves of moderate amplitude are admissibile in the Camassa-Holm regime is of major interest: the equation can be used to study breaking waves, whereas for the Korteweg-de Vries equation all hydrodynamically relevant solutions do not develop singularities in finite time. While an evolution equation for the horizontal fluid velocity can be derived at any fixed depth ratio, the specific depth at which the Camassa-Holm equation arises is characteriszed by the fact that this is one of only two depths at which one obtains an integrable model. The other integrable model arising this way is the Degasperis-Procesi equation [3]

$$u_t - u_{txx} + \kappa u_x + 4uu_x = 3u_x u_{xx} + uu_{xxx},$$

an equation that presents structural properties quite analogous to those discovered in the context of the Camassa-Holm equation.

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Dynamics of soliton-like solutions for slowly varying, generalized KdV equations

Claudio Muñoz

In this report we describe our recent work concerning the dynamics of a soliton for some generalized Korteweg-de Vries equations (gKdV). Indeed, our objective was the study of the global behavior of a *generalized soliton solution* for the following subcritical, variable coefficients gKdV equation:

$$u_t + (u_{xx} - \lambda u + a(\varepsilon x)u^m)_x = 0, \text{ in } \mathbb{R}_t \times \mathbb{R}_x, \quad m = 2, 3 \text{ and } 4. \tag{1}$$

Here $u = u(t, x)$ is a real-valued function, $\varepsilon > 0$ is a small number, $\lambda \geq 0$ a fixed parameter, and the *slowly varying potential* $a(\cdot)$ is a smooth, positive function satisfying $a \in C^3(\mathbb{R})$ and $\lim_{r\to-\infty} a(r) = 1$ and $\lim_{r\to+\infty} a(r) = 2$, among other mild assumptions.

The above equation represents in some sense a simplified model of *long dispersive waves in a channel with variable depth*, which takes in account *large* variations in the shape of the solitary wave. The primary physical model, and the dynamics of a generalized soliton-solution, was formally described in [6, 7, 8], with further results in [4, 10]. See [16] and references therein for a detailed physical introduction to this model.

The main novelty in the works above cited was the discovery of a *dispersive tail* behind the soliton, with small height but large width, as a consequence of the lack of conserved quantities such as mass or energy. However, no mathematically rigorous proof of this phenomenon was given.

On the other hand, from a mathematical point of view, equation (1) is a variable coefficients version of the gKdV equation

$$u_t + (u_{xx} - \lambda u + u^m)_x = 0, \quad \text{in} \quad \mathbb{R}_t \times \mathbb{R}_x; \quad m \geq 2 \text{ integer}, \tag{2}$$

(note that (1) formally behaves as a gKdV equation (2), with constant coefficients 1 and 2, as $x \to \pm\infty$.) This last equation is important because of the existence of localized, exponentially decaying and smooth solutions called *solitons*. Given real numbers x_0 and $c > 0$, solitons are solutions of (2) of the form

$$u(t,x) := Q_c(x - x_0 - (c-\lambda)t), \quad \text{with} \quad Q_c(s) := c^{1/(m-1)}Q(c^{1/2}s), \tag{3}$$

and where Q is the unique –up to translations– function satisfying the second order nonlinear ordinary differential equation

$$Q'' - Q + Q^m = 0, \quad Q > 0, \quad Q \in H^1(\mathbb{R}).$$

In this case, the solution belongs to the Schwartz class and it is explicit. In particular, if $c > \lambda$, (3) represents a *solitary wave*, of scaling c and velocity $(c-\lambda)$, defined for all time moving to the right without *any change* in shape, velocity, etc. In other words, a soliton represents a *pure*, traveling wave solution with *invariant profile*. In addition, equation (2) allows soliton solutions with negative velocities, moving to the left direction, provided $c < \lambda$. Finally, for the case $c = \lambda$, one has a stationary soliton solution, $Q_\lambda(x)$. These two last soliton solutions do not exist in the standard model of inviscid gKdV (namely $\lambda = 0$.)

Coming back to (1), the corresponding Cauchy problem has been considered in [14]; in particular, we showed global well-posedness for $H^1(\mathbb{R})$ initial data, even in the absence of some standard conserved quantities. Indeed, equation (1) is not invariant anymore under scaling and spatial translations, and therefore, the standard **mass** is not conserved, but it varies slowly. There exists a generalized **energy**, conserved for H^1-solutions of (1). The proof of the global well-posedness result is an adaptation of the fundamental work of Kenig, Ponce and Vega [9].

One fundamental question related to (2) is how to **generalize** a soliton-like solution to more complicated models. Very little is known in the case of an inhomogeneous nonlinearity, as in (1). In a general situation, no elliptic, time-independent ODE can be associated to the soliton solution, unlike the standard autonomous case studied in [1]. Other methods are needed.

Concerning some time dependent, generalized KdV and mKdV equations ($m = 2$ and $m = 3$), Dejak-Jonsson, and Dejak-Sigal [2, 3] studied the dynamics of a soliton for not too large times, of $O(\varepsilon^{-1})$. Recently, Holmer [5] has improved some of the Dejak-Sigal results in the KdV case, up to the Ehrenfest time $O(|\log\varepsilon|\varepsilon^{-1})$. In their model, the perturbation is of linear type, which do not allow large variations of the soliton shape, different to the scaling itself.

Finally, in [14, 15] we have described the soliton dynamics for *all time* in the case of the time independent, perturbed gKdV equation (1).

Description of the dynamics. Let us be more precise. In [14, 15] we have proved, among other things, the following results.

Theorem 3 (Existence of a soliton-like solution, see also [11, 14])**.**
Suppose $m = 2, 3$ *and* 4*, and let* $0 \leq \lambda < 1$ *be a fixed number. There exists a small constant* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$ *the following holds. There exists a solution* $u \in C(\mathbb{R}, H^1(\mathbb{R}))$ *of (1), global in time, such that*

$$\lim_{t \to -\infty} \|u(t) - Q(\cdot - (1-\lambda)t)\|_{H^1(\mathbb{R})} = 0.$$

This solution is unique *in the following cases:* (i) $m = 3$*; and* (ii) $m = 2, 4$*, provided* $\lambda > 0$*.*

The next step is the description of the interaction soliton-potential. In this case, we have at least two different behaviors, depending on a fixed number $\tilde{\lambda} \in (0, 1)$.

Theorem 4 (Interaction soliton-potential and refraction, case $0 < \lambda < \tilde{\lambda}$, see [14, 15])**.**
Suppose $0 < \lambda < \tilde{\lambda}$*. There exist constants* $K, \tilde{T}, c^+, c_\infty(\lambda) > 0$*, with* $\lambda < c_\infty(\lambda) < +\infty$*; and a smooth function* $\rho(t) \in \mathbb{R}$ *such that the function*

$$w^+ := u(t) - 2^{-\frac{1}{m-1}} Q_{c^+}(\cdot - \rho(t))$$

satisfies, for all $t \geq \tilde{T}$*,*

$$\|w^+(t)\|_{H^1(\mathbb{R})} + |\rho'(t) - c_\infty(\lambda) + \lambda| + |c^+ - c_\infty| \leq K\varepsilon^{1/2}. \tag{4}$$

Let us remark that in this case the soliton is perturbed by the potential, in a non trivial form, but it still exits the interaction region by the right hand side.

Now we consider the case $\tilde{\lambda} < \lambda < 1$. Here a new behavior is described. The soliton solution is, in this case, a *reflected* solitary wave.

Theorem 5 (Interaction soliton-potential and reflection, case $\tilde{\lambda} < \lambda < 1$, [15])**.**
Suppose now $\tilde{\lambda} < \lambda < 1$*. Then there exist constants* $K, \tilde{T}, c^+, c_\infty(\lambda) > 0$*, with* $0 < c_\infty(\lambda) < \lambda$*; and a smooth function* $\rho(t) \in \mathbb{R}$ *such that*

$$w^+ := u(t) - Q_{c^+}(\cdot - \rho(t))$$

satisfies, for all $t \geq \tilde{T}$*, (4).*

Remark 1 (Case $\lambda = \tilde{\lambda}$). The behavior of the solution in the case $\lambda = \tilde{\lambda}$ remains an interesting open problem.

Strategy of the proof. The proof of these results (Theorems 4 and 5) requires the introduction of an approximate solution, up to first order in ε, since the interaction can be seen as a *soliton-potential collision.* In particular, we generalize the recent framework developed by Martel and Merle in [12, 13] to this new situation.

Roughly speaking, the solution $u(t)$ behaves like a well modulated soliton-solution, plus a small order term, namely

$$u(t,x) \sim \mu(t)Q_{c(t)}(x-\rho(t)) + \varepsilon\nu(t)A_{c(t)}(x-\rho(t)), \tag{5}$$

where $c(t), \rho(t)$ are the scaling and position parameters, and $\mu(t), \nu(t)$ and A_c are unknown functions, to be found. Moreover, we have proved that this is a good approximation of the dynamics, provided (c,ρ) follow a well defined dynamical system, of the form:

$$\begin{cases} c'(t) \sim \varepsilon f_1(t), \quad c(-T_\varepsilon) \sim 1, \\ \rho'(t) \sim c(t) - \lambda, \quad \rho(-T_\varepsilon) \sim -(1-\lambda)T_\varepsilon, \end{cases}$$

for a given function $f_1(t) = f_1(\mu(t), c(t), \rho(t))$ and some explicit time $T_\varepsilon \gg \frac{1}{\varepsilon}$. Therefore, the infinite dimensional dynamics reduces to a simple finite dimensional problem, which describes the main properties of the soliton solution. Once this system is understood, the main problem reduces to an advanced form of stability argument, in the spirit of Weinstein [17, 12, 14]. The bound $O(\varepsilon^{1/2})$ above is just a consequence of the emergency of a *dispersive tail behind the soliton*, which is mathematically described by the function A_c in (5) above.

In addition, by using a contradiction argument and the L^1-conservation law, it was proved the non existence of pure soliton-like solutions for this regime:

Theorem 6 (Non-existence of pure soliton-like solution for (1), [14, 15])**.**

Suppose $m = 2,3,4$, *with* $0 < \lambda < 1$, $\lambda \neq \tilde{\lambda}$. *There exists* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$,

$$\limsup_{t\to+\infty} \|w^+(t)\|_{H^1(\mathbb{R})} > 0.$$

Remark 2. From the above results we do not discard the existence of small solitary waves traveling *to the left* (since a small soliton moves to the left), at least for the case $m = 2$. In the cubic and quartic cases, we believe there are no such soliton solutions.

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Superradiance, trapping, and decay of waves on Kerr spacetimes in the general subextremal range $|a| < M$

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(joint work with Igor Rodnianski)

The stability problem for the Kerr family $(\mathcal{M}, g_{a,M})$ of black hole solutions to the Einstein vacuum equations

$$R_{\alpha\beta}(g) = 0, \tag{1}$$

the system of nonlinear hyperbolic equations governing general relativity, is one of the most important unresolved issues in the theory. For background on the geometry of Kerr black holes and more generally on the Cauchy problem for (1), we refer the reader to our [8].

A model problem for the above is to consider linear *scalar* perturbations, i.e. solutions ψ of

$$\Box_g \psi = 0, \tag{2}$$

on a fixed Kerr exterior spacetime $(\mathcal{M}, g = g_{a,M})$. Equation (2) can be thought of as a poor man's substitute for the more complicated problem of *gravitational* perturbations, obtained by linearising (1) around the Kerr family.

In the very slowly rotating case $|a| \ll M$, first boundedness [7] and then decay [8, 19, 2] has been shown for solutions of (2) in rapid developments over the last two years, following earlier progress in the Schwarzschild case $a = 0$. We refer the reader to [8] for a complete discussion of the status of the problem starting from the pioneering work of Regge–Wheeler from the 1950's up to and including the Kerr results for $|a| \ll M$, with many references. In this talk, we shall provide the essential elements of our recent proof of boundedness and decay for the general subextremal case $|a| < M$.

Theorem 7. *Let ψ be a solution of the wave equation $\Box_g \psi = 0$ on the Kerr background $g = g_{M,a}$ for arbitrary $|a| < M$, with sufficiently regular initial data on a Cauchy hypersurface Σ and let $\{\Sigma_\tau\}_{\tau \geq 0}$ be a suitably defined hyperboloidal foliation of the exterior. Then, with $|\partial\psi|^2$ denoting an appropriate square sum of non-degenerate derivatives*[1] *we have*

- *Boundedness of energy*

$$\int_{\Sigma_\tau} |\partial\psi|^2 \leq C \int_{\Sigma_0} |\partial\psi|^2 \tag{3}$$

- *Integrated local energy decay: for arbitrary $r_+ \doteq M + \sqrt{M^2 - a^2} < R_1 < \infty$*

$$\int_0^\infty \int_{\Sigma_\tau \cap \{r_+ \leq r \leq R_1\}} \left(\chi|\partial\psi|^2 + |\psi|^2\right) \leq C_{R_1} \int_{\Sigma_0} |\partial\psi|^2 \tag{4}$$

and

$$\int_\tau^{2\tau} \int_{\Sigma_\tau \cap \{r_+ \leq r \leq R_1\}} |\partial\psi|^2 \leq C_{R_1} D \tau^{-2}. \tag{5}$$

- *Polynomial-time decay of the energy-flux:*

$$\int_{\Sigma_\tau} |\partial\psi|^2 \leq C D \tau^{-2}. \tag{6}$$

Here, χ is a smooth cutoff function which vanishes in a neighborhood of the physical space projection of the trapped set[2]*, and D denotes the square of an appropriate (weighted Sobolev) norm of the data, involving higher derivatives of ψ.*

Combining energy boundedness and decay with commutation arguments gives pointwise boundedness and decay statements. Besides the usual stationary Killing field as a commutator, one must commute with the axisymmetric Killing field and a transversal Killing field to the horizon[3], to ensure that one has a timlike direction in the span of the commutators. We give here an example of the decay statements that follow:

[1] Specifically, this expression is the flux density through Σ_τ of the energy associated to a vector field which is strictly timelike and which coincides with ∂_t for large r. In particular, at any given point this quantity is comparable to the sum of all derivatives in local regular coordinates and is thus non-degenerate at the horizon.

[2] The actual statement proven is stronger but its formulation would require to introduce a microlocal χ.

[3] The possibility of such a use of a transversal vector field is a manifestation of the red-shift effect and was introduced in [7]. Cf. the use of a vector field multiplier to capture the red-shift effect, introduced in [6].

Corollary 1. *We have the following statement of pointwise decay for* ψ

$$|r^{\frac{1}{2}}\psi| \leq C\sqrt{D}\tau^{-1}, \qquad |r\psi| \leq C\sqrt{D}\tau^{-\frac{1}{2}}. \tag{7}$$

Moreover, for any $\eta > 0$

$$|\psi| \leq C_\eta\sqrt{D}\tau^{-\frac{3}{2}+\eta}. \tag{8}$$

The above decay statements are in principle sufficient for non-linear applications and are robust in that exact stationarity is not used to derive them, given the most primitive statements (3) and (4). See [9] which in fact gives a general method for obtaining (6) and thus the above Corollary starting from (3), (4), and suitable behaviour of the metric far out. The method of [9] not only is independent of exact stationarity, but avoids all weights (on either commutators or multipliers) growing in t for fixed r. This is important from the perspective of the non-linear stability problem for (1). For an alternative method of obtaining the refinement (8) using commutation with the scaling vector field, see [15, 16]. Even more refined pointwise decay statements for a class of very regular data vanishing faster at spatial infinity can be obtained–again, given our (3) and (4)–from the recent [18] using estimates for the resolvent, but these statements would depend essentially on the exact stationarity of the metric.

We turn now to a discussion of the proof of statements (3) and (4) of Theorem 7.

We have discussed at length elsewhere the two main difficulties of the Schwarzschild case, namely, the importance of understanding the red-shift effect and the role of trapped null geodesics. See [6, 5, 8]. These difficulties can be captured by suitably constructed energy currents. A fundamental role is also played by the energy current corresponding to the static Killing field. In view of the fact that this Killing field is timelike everywhere outside the horizon, its associated energy flux is nonnegative.

Considering now the Kerr case, there are two main additional difficulties: superradiance (due to the failure of the stationary Killing field ∂_t to be everywhere timelike in the exterior), and the fact that the structure of trapped null geodesics is more complicated, at least when viewed purely in physical space. These two problems are in some sense coupled, and this coupling can be viewed as an additional, third difficulty. It turns out that to understand these difficulties, it is indeed useful to reconsider Carter's formal separation (see [4]) of solutions ψ of (2) into modes, but viewed from a slightly more sophisticated perspective: as a mircolocalisation intimately tied to Kerr geometry.

We briefly recall Carter's separation: Remarkably, although the Kerr metric for $a \neq 0$ only has a 2-dimensional algebra of Killing fields, the wave equation (2) on Kerr still admits a non-trivial complete separation which allows one to define mode-type solutions of the form

$$\psi^{\omega}_{m\ell}(r)S_{m\ell}(a\omega, \phi)e^{im\phi}e^{i\omega t} \tag{9}$$

so that $\psi^{\omega}_{m\ell}(r)$ again satisfies an ODE that can be rewritten as

$$u'' + \omega^2 u - V(a\omega)u = 0, \tag{10}$$

where the potential V depends, in addition to the labels (m, ℓ), on the frequency ω. Here, (t, r, θ, ϕ) denote Boyer-Lindquist coordinates. The $S_{m\ell}$ are not explict but can be characterized as the eigenvectors of an associated operator depending on $a\omega$, which reduces to the spherical Laplacian if $a\omega = 0$. The above separation is related to the complete integrability of geodesic flow, discovered previously by Carter, and in fact has its origin in the existence of an additional "hidden" symmetry of the Kerr metric.

We term 'superradiant' those modes (9) for which

$$0 \leq m\omega/a < \frac{m^2}{2Mr_+}. \tag{11}$$

It is precisely in the frequency range (11) that the sign of the energy flux through the event horizon of a solution of the form (9) is negative.

Let us note that the difficulty of whether a priori it is possible to decompose general solutions ψ as superpositions of modes can be sidestepped with the help of a cutoff function, by restricting consideration to a finite time slab. In such a slab, one can indeed represent a given finite-energy solution ψ of (2) as a superposition of modes (9) localised at fixed frequency-triple $\omega \in \mathbb{R}$, m, ℓ. If one is able to recover a suitable *quantitative* estimate on the solution in this timeslab, then by a bootstrap argument the estimate can easily be extended to be valid for all time.

Recall from our brief mention of the Schwarzschild case that the key to showing quantitative bounds is constructing suitable energy currents which capture the usual energy conservation, the red-shift, and the obstruction of trapped null orbits. Key to the original boundedness result [7] on very slowly rotating Kerr were two observations:

- In the case $|a| \ll M$, superradiant frequencies are not trapped.
- When superradiance is controlled by a small parameter, then it can be absorbed with the help of the redshift.

The two observations taken together allowed an understanding of boundedness without understanding trapping. In fact, the boundedness results were obtained not just for exactly Kerr metrics but for general axisymmetric stationary spacetimes whose metric is C^1-close to Schwarzschild.[4] The decay result [8] (which is now restricted to exactly Kerr spacetimes) in the $|a| \ll M$ case (see also [19, 2]) replaces the first observation above with a complete treatment of trapping. Here the full potential of the microlocalisation provided by (9) is used: In the range of frequencies relevant for trapping, virial currents are chosen separately for each frequency-triple (ω, m, ℓ), so as to degenerate at a unique value of r depending on this triple. This r-value can be related to a null orbit with conserved quantities which are in turn associated to the frequency triple. Behind this construction is the

[4] These assumptions are sufficient to separate the superradiant part of the solution from the non-superradiant part. Note that the weak regularity assumptions allow the behaviour of geodesic flow to be very different from in Schwarzschild. In particular, one does not expect to be able to prove quantitative decay under these assumptions.

close connection between the separation of the wave equation and the separation of the Hamilton-Jacobi equations.

The main new idea necessary for the general $|a| < M$ case is to revisit also the first observation above, made in the original boundedness paper [7]. In the small $|a| \ll M$ case, the fact that superradiant frequencies are not trapped follows essentially from the fact that all future-trapped null geodesics eventually leave the ergoregion. For general $|a| < M$ the latter is no longer the case. Nonetheless, remarkably,

- Superradiant frequencies are not trapped for the entire $|a| < M$ range!

We note that in the special case of frequency $\omega = 0$, the above is related to the fact that there are no *trapped* null geodesics orthogonal to ∂_t. This latter fact plays a role in recent work on black hole uniqueness [1].

A more precise embodiment of the above property can be given as follows. We may partition the potential V of (10) as $V = V_0 + V_1$ where

$$V_0 = \frac{4Mram\omega - a^2m^2 + \Delta(\lambda_{m\ell} + \omega^2 a^2)}{(r^2+a^2)^2},$$
$$V_1 = \frac{\Delta(3r^2 - 4Mr + a^2)}{(r^2+a^2)^3} - \frac{3\Delta^2 r^2}{(r^2+a^2)^4}$$

and $\Delta = r^2 - 2Mr + a^2$. In the high frequency regime, V_0 dominates V_1. We have

Lemma 1. *For the values $0 \le a < M$ and*

$$0 \le m\omega \le \frac{am^2}{2Mr_+},$$

we have that V_0

$$\omega^2 < V_0(r^0_{\max}),$$

where $r^0_{\max}$ denotes the r-value where V_0 achieves its maximum.

The above lemma allows us to adapt the second observation discussed above in the context of the original boundedness result [7] to the case where superradiance is not a small parameter and still couple the conserved energy estimate, the redshift and understanding of trapping (the latter again via the microlocalisation achieved by the classical separation) so as to obtain the desired estimates.

We have singled out for disussion here the regime of superradiant high frequencies (see $\mathcal{G}^\sharp$ below) because this is the regime which required a new insight into the geometry of Kerr. For the proof, one in fact needs separate constructions for each of the following regimes:

- $\mathcal{G}^\sharp = \{(\omega, m, \ell) : \omega^2 + \Lambda \ge \lambda_1,\ m\omega \in [0, \frac{am^2}{2Mr_+}]\}$,
- $\mathcal{G}_\sharp = \{(\omega, m, \ell) : |\omega| \ge \omega_1,\ \Lambda < \lambda_2\omega^2, m\omega \notin [0, \frac{am^2}{2Mr_+}]\}$,
- $\mathcal{G}_\mathbb{q} = \{(\omega, m, \ell) : |\omega| \ge \omega_1,\ \lambda_2\Lambda > \omega^2, m\omega \notin [0, \frac{am^2}{2Mr_+}]\}$,
- $\mathcal{G}_\natural = \{(\omega, m, \ell) : |\omega| \ge \omega_1,\ \lambda_2\Lambda \le \omega^2 \le \lambda_2^{-1}\Lambda, m\omega \notin [0, \frac{am^2}{2Mr_+}]\}$,
- $\mathcal{G}_\flat = \{(\omega, m, \ell) : |\omega| \le \omega_1,\ \Lambda \le \lambda_1\}$.

Here $\Lambda = \lambda_{m\ell}(a\omega) + a^2\omega^2$, and the parameters λ_1, ω_1, λ_2 must be chosen accordingly. We have given a detailed account of the microlocal constructions in each of the above regimes in our recent [11]. The complete proof will appear in our forthcoming [12].

The resolution of the linear stability problem in the $|a| < M$ case thus brings us full circle. Though based on energy methods, the proof reconnects with the classical mode analysis, now considered from a more sophisticated point of view as a tool for the 'microlocalisation' of energy currents. Moreover, the method highlights certain properties of individual modes (9), indeed of the potential functions $V(\omega)$, which do not appear to have been noticed previously, but turn out to play a fundamental part in the stability mechanism. Thus, we hope that this argument offers something new even for the reader impatient with some of the more technical difficulties of the proof.

With Theorem 7 and its corollaries, the problem of linear stability, at least for scalar perturbations, has been essentially completely understood, leaving only the extremal $|a| = M$ case. In view of recent work of Aretakis [3] on extremal Reissner-Nordström, for extremal Kerr one in fact expects *instabilities* on the horizon. Higher dimensional analogues of the decay problem for Schwarzschild have been considered in [17, 14]. The problem of gravitational perturbations is the subject of forthcoming work [13].

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Dynamics of near-harmonic Schrödinger and Landau-Lifshitz maps

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(joint work with Kenji Nakanishi, Tai-Peng Tsai)

We address questions of singularity formation, global existence, and long-time behaviour for the *Landau-Lifshitz* (or *Landau-Lifshitz-Gilbert*) family of equations, which describe dynamics in a 2D isotropic ferromagnet:

$$\vec{u}_t = a_1(\Delta\vec{u} + |\nabla\vec{u}|^2\vec{u}) + a_2\vec{u}\times\Delta\vec{u}, \qquad a_1 \geq 0, \ \ a_2 \in \mathbb{R} \tag{1}$$

where the *magnetization vector* $\vec{u} = \vec{u}(t,x) = (u_1, u_2, u_2)$ is a 3-vector with normalized length, so can be considered a map into the 2-sphere $\mathbb{S}^2$:

$$\vec{u} : [0,T)\times\mathbb{R}^2 \to \mathbb{S}^2 := \{\vec{u}\in\mathbb{R}^3 \mid |\vec{u}| = 1\}. \tag{2}$$

The family of PDE (1) includes the well-known geometric evolution equations

- $a_2 = 0$: *harmonic map heat-flow* into $\mathbb{S}^2$
- $a_1 = 0$: *Schrödinger map* (or *Schrödinger flow*) into $\mathbb{S}^2$.

Introducing the tangent plane

$$T_{\vec{u}}\mathbb{S}^2 := \vec{u}^{\perp} = \{\vec{\xi}\in\mathbb{R}^3 \mid \vec{u}\cdot\vec{\xi} = 0\} \tag{3}$$

to the sphere $\mathbb{S}^2$ at $\vec{u}\in\mathbb{S}^2$, and the operations

- $P^{\vec{u}} := Id - \vec{u}\vec{u}\cdot \ = -\vec{u}\times\vec{u}\times$, the orthogonal projection : $\mathbb{R}^3 \to T_{\vec{u}}\mathbb{S}^2$,
- $J^{\vec{u}} := \vec{u}\times$, a $\pi/2$ rotation (complex structure) : $T_{\vec{u}}\mathbb{S}^2 \to T_{\vec{u}}\mathbb{S}^2$,

as well as the *energy*

$$\mathcal{E}(\vec{u}) := \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\vec{u}|^2 dx, \tag{4}$$

equation (1) may be written as

$$\vec{u}_t = -(a_1 + a_2 J^{\vec{u}})\mathcal{E}'(\vec{u}) \ , \tag{5}$$

a form which clearly displays (1) as a combination of a gradient flow (harmonic map heat-flow) and a Hamiltonian flow (Schrödinger map). In two space dimensions, equation (1) is *energy critical*: the energy $\mathcal{E}$ is left invariant by the coordinate scaling transformation which maps solutions to solutions.

We consider a symmetric sub-class of maps which is preserved by the dynamics (1), namely the *m-equivariant* maps ($m \in \mathbb{Z}^+$) from $\mathbb{R}^2$ to $\mathbb{S}^2$:

$$\Sigma_m := \{\vec{u} = e^{m\theta R}\vec{v}(r) \mid \mathcal{E}(\vec{u}) < \infty,\ \vec{v}(0) = -\vec{k},\ \vec{v}(\infty) = \vec{k}\}, \tag{6}$$

with notations:

- $\vec{k} := (0, 0, 1)$ (north pole of $\mathbb{S}^2$)
- $R := \vec{k}\times$ (generator of rotations about $\vec{k}$)
- $x_1 + ix_2 = re^{i\theta}$ (polar coordinates on $\mathbb{R}^2$)
- $\vec{v} : [0,\infty) \to \mathbb{S}^2$ (radial *profile map*)

Maps in this class have topological (Brouwer) *degree m*:

$$\vec{u} \in \Sigma_M \implies \text{degree}(\vec{u}) = m \in \mathbb{Z}^+. \tag{7}$$

A standard integration by parts shows that for a map $u \in \Sigma_m$,

$$\mathcal{E}(\vec{u}) \geq 4\pi|\text{degree}(\vec{u})| = 4\pi m, \tag{8}$$

with equality holding if and only if $\vec{u}$ is the explicit harmonic map

$$\vec{u} = e^{m\theta R}\vec{h}(r), \quad \vec{h} = (h_1, 0, h_3), \quad h_1 = \frac{2}{r^m + r^{-m}}, \quad h_3 = \frac{r^m - r^{-m}}{r^m + r^{-m}}, \tag{9}$$

or a rotation or spatial scaling thereof; that is, for some $s > 0$ (length scale) and $\alpha \in \mathbb{R}$ (rotation angle),

$$\vec{u} = e^{m\theta R}\vec{h}[\mu], \quad \vec{h}[\mu] := e^{\alpha R}\vec{h}^s, \quad \vec{h}^s := \vec{h}(r/s), \quad \mu := m \log s + i\alpha. \tag{10}$$

Of course, any member of this two-parameter family of harmonic maps satisfies $\mathcal{E}'(\vec{u}) = 0$, and in particular is a static solution of (1).

It is not known if solutions of the Landau-Lifshitz equations can form finite-time singularities, except for in the heat-flow case ($a_2 = 0$), where the following dichotomoy is well-established: if $\mathcal{E}(\vec{u}_0) \leq 4\pi$ (the lowest energy of a non-trivial harmonic map) solutions are global ([6]), while there are initial data $\vec{u}_0$, for any $\mathcal{E}(\vec{u}_0) > 4\pi$, leading to finite-time blow-up (as follows from the subsolution construction of [1] – see [2]). It is worth noting that for the *wave map* equation – the wave analogue (1) (which is also energy ciritical) the same dichotomoy was recently established ([5, 4]). Furthermore, in both the heat and wave cases, the examples of finite-time blow-up lie in the m-equivariant class of maps ($m = 1$ for heat-flow; any $m \geq 1$ for wave maps).

One might expect the same dichotomoy to hold for the full Landau-Lifshitz family (1) (including the Schrödinger map), but this remains open. However, for energies slightly above the threshhold $4\pi m$, some closely related results were obtained in [3]:

Theorem 8. [3] *Let $m \geq 3$. There exists $\delta > 0$ such that for any $\vec{u}(0, x) \in \Sigma_m$ with $\mathcal{E}(\vec{u}(0)) \leq 4m\pi + \delta^2$, we have a unique global solution $\vec{u} \in C([0,\infty); \Sigma_m)$*

of (1)*, satisfying* $\nabla\vec{u} \in L^2_{t,loc}([0,\infty); L^\infty_x)$*. Moreover, for some* $\mu \in \mathbb{C}$ *we have*

$$\|\vec{u}(t) - e^{m\theta R}\vec{h}[\mu]\|_{L^\infty_x} + a_1\mathcal{E}(\vec{u}(t) - e^{m\theta R}\vec{h}[\mu]) \to 0 \quad \text{as } t \to \infty. \tag{11}$$

This result, which includes the Schrödinger map case $a_1 = 0$, says that every solution with energy close to the minimum converges to one of the harmonic maps uniformly in x as $t \to \infty$.

The paper [3] had also a result for the case $m = 2$, but only in the special case of the harmonic map heat-flow ($a_2 = 0$), with the further restriction that the image of the radial profile map lies on a great circle:

Theorem 9. [3] *Let* $m = 2$*, and* $a = a_1 > 0$*. There exists* $\delta > 0$ *such that for any* $\vec{u}(0,x) = e^{2\theta R}\vec{v}(0,r) \in \Sigma_2$ *with* $\mathcal{E}(\vec{u}(0)) \leq 8\pi + \delta^2$*, and* $v_2(0,r) \equiv 0$*, we have a unique global solution* $\vec{u} \in C([0,\infty);\Sigma_2)$ *satisfying* $\nabla\vec{u} \in L^2_{t,loc}([0,\infty); L^\infty_x)$*. Moreover, for some continuously differentiable* $s : [0,\infty) \to (0,\infty)$ *we have*

$$\|\vec{u}(t) - e^{m\theta R}\vec{h}(r/s(t))\|_{L^\infty_x} + \mathcal{E}(\vec{u}(t) - e^{m\theta R}\vec{h}(r/s(t))) \to 0 \quad \text{as } t \to \infty. \tag{12}$$

In addition, we have the following asymptotic formula for $s(t)$*:*

$$(1 + o(1))\log(s(t)) = \frac{2}{\pi}\int_1^{\sqrt{at}} \frac{v_1(0,r)}{r}dr + O_c(1), \tag{13}$$

where as $t \to \infty$*,* $o(1) \to 0$ *and* $O_c(1)$ *converges to some finite value.*

This result shows that when $m = 2$ (and in this very special case) solutions do converge asymptotically to the *family* of harmonic maps. But in contrast to the higher degree ($m \geq 3$) cases, the strong asymptotic stability result of Theorem 8 no longer holds; indeed, it follows from formula (13) that more complex behaviour such as infinite-time blow-up ($s(t) \to 0$) or even "eternal oscillations" ($\liminf s(t) < \limsup s(t)$) is possible, if the initial data has a "long tail".

In more recent work, we have established the global existence in degree $m = 2$ for the full Landau-Lifshitz family with dissipation ($a_1 > 0$) – by exploiting the dissipation in an essential way to obtain sufficiently strong decay estimates for the linearized evolution. The result is:

Theorem 10. *Let* $m = 2$*, and* $a_1 > 0$*. There exists* $\delta > 0$ *such that for any* $\vec{u}(0,x) \in \Sigma_2$ *with* $\mathcal{E}(\vec{u}(0)) \leq 8\pi + \delta^2$*, we have a unique global solution* $\vec{u} \in C([0,\infty);\Sigma_2)$ *satisfying* $\nabla\vec{u} \in L^2_{t,loc}([0,\infty); L^\infty_x)$*. Moreover, for some continuously differentiable* $\mu : [0,\infty) \to \mathbb{C}$ *we have*

$$\mathcal{E}(\vec{u}(t) - \vec{h}[\mu(t)]) \to 0 \quad \text{as } t \to \infty. \tag{14}$$

Of course these results still leave open the possibility of finite-time singularity formation in degree $m = 1$ (as is known for the heat-flow), or even in degree $m = 2$ for the Schrödinger map.

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Near soliton evolution for equivariant Schrödinger Maps in two spatial dimensions

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(joint work with Daniel Tataru)

We consider the Schrödinger map equation in $\mathbb{R}^{2+1}$ with values into $\mathbb{S}^2$,

$$u_t = u \times \Delta u, \qquad u(0) = u_0 \tag{1}$$

This equation admits a conserved energy,

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx$$

and is invariant with respect to the dimensionless scaling

$$u(t,x) \to u(\lambda^2 t, \lambda x).$$

The energy is invariant with respect to the above scaling, therefore the Schrödinger map equation in $\mathbb{R}^{2+1}$ is *energy critical*.

The global well-posedness and scattering were established in [1] for initial data which is small in the energy space $\dot{H}^1$. However, such a result cannot hold for large data. In particular there exists a collection of families $\mathcal{Q}^m$ of finite energy stationary solutions, indexed by integers $m \geq 1$. To describe these families we begin with the maps Q^m defined in polar coordinates by

$$Q^m(r,\theta) = e^{m\theta R}\bar{Q}^m(r), \qquad \bar{Q}^m(r) = \begin{pmatrix} h_1^m(r) \\ 0 \\ h_3^m(r) \end{pmatrix}, \qquad m \in \mathbb{Z}\setminus\{0\}$$

with

$$h_1^m(r) = \frac{2r^m}{r^{2m}+1}, \qquad h_3^m(r) = \frac{r^{2m}-1}{r^{2m}+1}.$$

Here R is the generator of horizontal rotations, $Ru = \overrightarrow{k} \times u$. The families $\mathcal{Q}^m$ are constructed from Q^m via the symmetries of the problem, namely scaling and isometries of the base space $\mathbb{R}^2$ and of the target space $\mathbb{S}^2$. The elements of $\mathcal{Q}^m$

are harmonic maps from $\mathbb{R}^2$ into $\mathbb{S}^2$, and admit a variational characterization as the unique energy minimizers, up to symmetries, among all maps $u : \mathbb{R}^2 \to \mathbb{S}^2$ within their homotopy class.

In the above context, a natural question is to study Schrödinger maps for which the initial data is close in $\dot{H}^1$ to one of the $\mathcal{Q}^m$ families. We confine ourselves the class of equivariant Schrödinger maps. These are indexed by an integer m called the equivariance class, and consist of maps of the form

$$u(r,\theta) = e^{m\theta R}\bar{u}(r)$$

The maps Q^m above are m-equivariant. Intersecting the full set $\mathcal{Q}^m$ with the m-equivariant class and with the homotopy class of Q^m we obtain the two parameter family $\mathcal{Q}_e^m$ generated from Q^m by rotations and scaling,

$$\mathcal{Q}_e^m = \{Q^m_{\alpha,\lambda}; \alpha \in \mathbb{R}/2\pi\mathbb{Z}, \lambda \in \mathbb{R}^+\}, \qquad Q^m_{\alpha,\lambda}(r,\theta) = e^{\alpha R} Q^m(\lambda r, \theta)$$

Their energy depends on m as follows:

$$E(Q^m_{\alpha,\lambda}) = 4\pi m := E(\mathcal{Q}^m)$$

The energy conservation suffices to confine solutions to a small neighborhood of $\mathcal{Q}_e^m$ due to the inequality

$$\mathrm{dist}_{\dot{H}^1}(u, \mathcal{Q}_e^m)^2 = \inf_{\alpha,\lambda} \|Q^m_{\alpha,\lambda} - u\|^2_{\dot{H}^1} \lesssim E(u) - E(\mathcal{Q}^m),$$

which holds for all m-equivariant maps $u : \mathbb{R}^2 \to \mathbb{S}^2$ in the homotopy class of $\mathcal{Q}_e^m$ with $0 \le E(u) - E(\mathcal{Q}^m) \ll 1$. One can interpret this as an orbital stability result for $\mathcal{Q}_e^m$. However, this does not say much about the global behavior of solutions since these soliton families are noncompact; thus one might have even finite time blow-up while staying close to a soliton family.

The case of large m was considered in prior work by Gustafson, Kang, Tsai in [3] and Gustafson, Nakanishi, Tsai in [4]. Their main result asserts that

Theorem 11 ([3] for $m \ge 4$, [4] for $m = 3$)**.** *The solitons $Q^m_{\alpha,\lambda}$ are stable in the $\dot{H}^1$ topology within the m-equivariant class.*

In our work we begin the study of the more difficult case $m = 1$, and establish a very different type of behavior. To track the evolution of an 1-equivariant Schrödinger map $u(t)$ along $\mathcal{Q}_e^1$ we use functions $(\alpha(t), \lambda(t))$ describing trajectories in $\mathcal{Q}_e^1$. We will be content with any choice $(\alpha(t), \lambda(t))$ satisfying

$$\|u - Q^1_{\alpha(t),\lambda(t)}\|^2_{\dot{H}^1} \lesssim E(u) - E(\mathcal{Q}^1)$$

For our main result we introduce a slightly stronger topology X with the property that

$$H^1_e \subset X \subset \dot{H}^1_e.$$

where $\dot{H}^1_e, H^1_e$ are the one dimensional equivariant versions of $\dot{H}^1, H^1$,

$$\|f\|^2_{\dot{H}^1_e} = \|\partial_r f\|^2_{L^2(rdr)} + \|r^{-1} f\|^2_{L^2(rdr)}, \quad \|f\|^2_{H^1_e} = \|f\|^2_{\dot{H}^1_e} + \|f\|^2_{L^2(rdr)}$$

X is defined in terms of the spectral resolution of the linearized evolution around the soliton. The spectral analysis for the linearized operator was developed by

Krieger, Schlag and Tataru in [5] and plays a key role in our analysis. In a nutshell, the X norm penalizes the behavior near frequency zero. Our first result below asserts that the soliton $Q^1_{0,1}$ is stable in the X topology (which applies to $\bar{Q}^1_{0,1}$).

Theorem 12. *Let $m = 1$ and $\gamma \ll 1$. Then for each 1-equivariant initial data u_0 satisfying*

$$\|\bar{u}_0 - \bar{Q}^1_{0,1}\|_X \leq \gamma$$

there exists a unique global solution u so that $\bar{u} - \bar{Q}^1_{0,1} \in C(\mathbb{R}; X)$ and

$$\|\bar{u} - \bar{Q}^1_{0,1}\|_{C(\mathbb{R};X)} \lesssim \gamma$$

Furthermore, this solution has a Lipschitz dependence on the initial data in X, uniformly on compact time intervals.

The above result holds true if $\bar{Q}^1_{0,1}$ is replaced by $\bar{Q}^1_{\alpha,\lambda}$, which implies that the solitons $Q^1_{\alpha,\lambda}$ are stable in the X topology. However, our second result asserts that the solitons $Q^1_{\alpha,\lambda}$ are unstable in the $\dot{H}^1$ topology:

Theorem 13. *For each $\epsilon, \gamma \ll 1$ and (α, λ) so that*

$$|\alpha| + |\lambda - 1| \approx \gamma$$

there exists a solution u as in Theorem 12 with the additional property that

$$\|u(0) - Q^1_{\alpha,\lambda}\|_{\dot{H}^1} \lesssim \epsilon\gamma$$

while

$$\lim \sup_{t \to \pm\infty} \|u - Q^1_{0,1}\|_{\dot{H}^1} \lesssim |\log \epsilon|^{-1}\gamma$$

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Review on blow-up for a semilinear wave equation in one space dimension

Frank Merle
(joint work with Hatem Zaag)

We consider the one dimensional semilinear wave equation

$$\begin{cases} \partial_t^2 u = \partial_x^2 u + |u|^{p-1}u, \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases} \tag{1}$$

where $u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}$, $p > 1$, $u_0 \in \mathrm{H}^1_{\mathrm{loc,u}}$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc,u}}$ with $\|v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}} = \sup_{a\in\mathbb{R}} \int_{|x-a|<1} |v(x)|^2 dx$ and $\|v\|^2_{\mathrm{H}^1_{\mathrm{loc,u}}} = \|v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}} + \|\nabla v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}}$.
The Cauchy problem for equation (1) in the space $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}$ follows from the finite speed of propagation and the wellposedness in $\mathrm{H}^1 \times \mathrm{L}^2$ (see Ginibre, Soffer and Velo; the precise references of the articles we cite here can be found in [4]).
If the solution is not global in time, then we call it a blow-up solution (the term "blow-up" will be fully justified in Proposition 17 below). The existence of blow-up solutions is guaranteed by ODE techniques together with the finite speed of propagation, or also by Levine's energy-based blow-up criterion. More blow-up results can be found in papers by Caffarelli and Friedman, Alinhac, Kichenassamy and Littman.

If u is a blow-up solution of (1), we define a 1-Lipschitz graph $\Gamma = \{(x, T(x))\}$ such that the maximal influence domain of u (or the domain of definition of u) is written as

$$D = \{(x,t) \mid t < T(x)\}. \tag{2}$$

$\bar{T} = \inf_{x\in\mathbb{R}} T(x)$ and Γ are called the blow-up time and the blow-up graph of u.
A point a is a non characteristic point (or a *regular* point) if there are $\delta_0 \in (0,1)$ and $t_0 < T(a)$ such that u is defined on $\mathcal{C}_{a,T(a),\delta_0} \cap \{t \ge t_0\}$ where

$$\mathcal{C}_{\bar{x},\bar{t},\bar{\delta}} = \{(x,t) \mid t < \bar{t} - \bar{\delta}|x - \bar{x}|\}.$$

If not, then we call a a characteristic point (or a *singular* point). Naturally, we denote by $\mathcal{R}$ (resp. $\mathcal{S}$) the set of non characteristic (resp. characteristic) points.
Note then that

$$\mathcal{R} \cup \mathcal{S} = \mathbb{R}.$$

In our papers [1], [2], [3] and [4], we made several contributions to the study of blow-up solutions of (1), namely the description of its blow-up graph and blow-up behavior in selfsimilar variables.

1. The blow-up graph of equation (1)

It is clear from rather simple arguments that $\mathcal{R} \neq \emptyset$ for *any* blow-up solution $u(x,t)$ (if $T(x)$ achieves its minimum at a, then $a \in \mathcal{R}$; if the infimum of $T(x)$ is at infinity, then there exists a large a such that $a \in \mathcal{R}$ as we state in the remark following Theorem 1 in [2]). On the contrary, the situation was unclear for

$\mathcal{S}$, and it was commonly conjectured before our contributions that $\mathcal{S}$ was empty. In particular, that was the case in the examples constructed by Caffarelli and Friedman. In [3], we prove that the conjecture was false. More precisely, we proved the following (see Proposition 1 in [3]):

Proposition 14 (Existence of initial data with $\mathcal{S} \neq \emptyset$)**.** *If the initial data* (u_0, u_1) *is odd and* $u(x,t)$ *blows up in finite time, then* $0 \in \mathcal{S}$.

For general blow-up solutions, we proved the following facts about $\mathcal{R}$ and $\mathcal{S}$ in [2] and [3] (see Theorem 1 (and the following remark) in [2], see Theorems 1 and 2 in [4]):

Theorem 15 (Geometry of the blow-up graph)**.**
(i) $\mathcal{R}$ *is a non empty open set, and* $x \mapsto T(x)$ *is of class* C^1 *on* $\mathcal{R}$*;*
(ii) $\mathcal{S}$ *is made of isolated points, and given* $a \in \mathcal{S}$*, if* $0 < |x-a| \le \delta_0$*, then*

$$(3)\quad \frac{|x-a|}{C_0|\log(x-a)|^{\frac{(k(a)-1)(p-1)}{2}}} \le T(x) - T(a) + |x-a| \le \frac{C_0|x-a|}{|\log(x-a)|^{\frac{(k(a)-1)(p-1)}{2}}}.$$

for some $\delta_0 > 0$ *and* $C_0 > 0$*, where* $k(a) \ge 2$ *is an integer. In particular,* $T(x)$ *is right and left differentiable at* a*, with* $T_l'(a) = 1$ *and* $T_r'(a) = -1$.

From (3), we see that the blow-up set is corner-shaped near a. In particular, there exists no solution of the semilinear wave equation (1) with a characteristic point a such that $T(x)$ is differentiable at $x = a$.
Note from (3) that the blow-up set never touches the backward light cone with vertex $(a, T(a))$ (except of course at a), and that the distance between them is bounded from above and from below by the same rate, which is quantified in terms of the integer $k(a) \ge 2$. In particular, from the shape of the solution near $(a, T(a))$, we can recover the integer $k(a) \ge 2$, and $k(a) - 1$ is the number of sign changes of the solution near $(a, T(a))$ as we will see in (ii) of Theorem 16 below. In one word, the shape of the solution near $(a, T(a))$ gives the topology of the solution and conversely.

Remark. The fact that the elements of $\mathcal{S}$ are isolated points is not elementary. Direct arguments give no more than the fact that $\mathcal{S} \neq \mathbb{R}$ (a point a such that $T(a)$ is the blow-up time is non characteristic). The first step of the proof is done in [3] where we proved that $\mathcal{S}$ has an empty interior and that in similarity variables, the solution splits in a non trivial decoupled sum of (at least 2) solitons with alternate signs (see (ii) of Theorem 16 below for a statement). The second step is done in [4]. It consists in using this decomposition and a good understanding of the dynamics of the equation in similarity variables (see equation (5) below) near a decoupled sum of “generalized” solitons. In fact, this is the first time where flows near an unstable sum of solitons are used and where such a result is obtained.

Remark. The fact that $\mathcal{S}$ is made of isolated points certainly does not hold in general for quasilinear wave equations. Indeed, Alinhac gives an explicit solution $u(x,t)$ for the following nonlinear wave equation

$$\partial_t^2 u = \partial_x^2 u + \partial_x u \partial_t u,$$

where $\mathcal{S} = \mathbb{R}^*$.

2. Asymptotic behavior near the blow-up graph

As one may guess from the above description, the asymptotic behavior will not be the same on $\mathcal{R}$ and on $\mathcal{S}$. In both cases, we need to use the similarity variables which we recall in the following. Let us stress the fact that the keystone of our work is the existence of a Lyapunov functional in similarity variables.

Given some $a \in \mathbb{R}$, we introduce the following self-similar change of variables:

$$w_a(y,s) = (T(a)-t)^{\frac{2}{p-1}} u(x,t), \quad y = \frac{x-a}{T(a)-t}, \quad s = -\log(T(a)-t). \tag{4}$$

The function $w = w_a$ satisfies the following equation for all $y \in B = B(0,1)$ and $s \ge -\log T(a)$:

$$\partial^2_{ss} w = \mathcal{L}w - \frac{2(p+1)}{(p-1)^2} w + |w|^{p-1} w - \frac{p+3}{p-1}\partial_s w - 2y\partial^2_{y,s} w \tag{5}$$

$$\text{where } \mathcal{L}w = \frac{1}{\rho}\partial_y\left(\rho(1-y^2)\partial_y w\right) \text{ and } \rho(y) = (1-y^2)^{\frac{2}{p-1}}. \tag{6}$$

From Antonini and Merle, we know the existence of the following Lyapunov functional for equation (5):

$$E(w) = \iint \left(\frac{1}{2}(\partial_s w)^2 + \frac{1}{2}\left(\partial_y w\right)^2(1-y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1}|w|^{p+1}\right)\rho dy, \tag{7}$$

defined for $(\partial_s w, w) \in \mathcal{H}$ where

$$\mathcal{H} = \left\{ q \;\mid\; \|q\|^2_{\mathcal{H}} \equiv \int_{-1}^{1} \left(q_1^2 + \left(q_1'\right)^2(1-y^2) + q_2^2\right)\rho dy < +\infty \right\}. \tag{8}$$

Using this energy structure, interpolation and the Gagliardo-Nirenberg estimate, we proved in our earlier papers and in [1] that $(w_a(s), \partial_s w_a(s))$ is bounded in the energy space $\mathcal{H}$. Moreover, if $a \in \mathcal{R}$, then the bound holds in $H^1 \times L^2(-1,1)$ as well by a covering technique.

From Proposition 1 in [1], we know that the only stationary solutions of (5) in the space $\mathcal{H}$ are $q \equiv 0$ or $w(y) \equiv \pm\kappa(d,y)$, where $d \in (-1,1)$ and

$$\kappa(d,y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \text{ where } \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}} \text{ and } |y| < 1. \tag{9}$$

As a matter of fact, there is convergence for w_a when $a \in \mathcal{R}$ as we see from the following result (see Corollary 4 in [1] and Theorem 6 in [3]):

Theorem 16 (Asymptotic behavior near the blow-up graph). *(i)* **Case where $a \in \mathcal{R}$: Existence of an asymptotic profile**. *There exist $\delta_0(a) > 0$, $|e(a)| = 1$, $s_0(a) \ge -\log T(a)$ such that for all $s \ge s_0$:*

$$\left\| \begin{pmatrix} w_a(s) \\ \partial_s w_a(s) \end{pmatrix} - e(a) \begin{pmatrix} \kappa(T'(a),.) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \le C_0 e^{-\mu_0(s-s_0)} \tag{10}$$

for some positive μ_0 and C_0 independent from a. Moreover, $E(w_a(s)) \to E(\kappa_0)$ as $s \to \infty$.

(ii) **Case where $a \in \mathcal{S}$: Decomposition into a sum of decoupled solitons**. *It holds that*

(11)
$$\left\| \begin{pmatrix} w_a(s) \\ \partial_s w_a(s) \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{k(a)} e_i^*(a)\kappa(d_i(s),\cdot) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ and } E(w_a(s)) \to k(a)E(\kappa_0)$$

as $s \to \infty$, for some

$$k(a) \geq 2, \quad e_i^*(a) = e_1^*(a)(-1)^{i+1} \tag{12}$$

and continuous $d_i(s) = -\tanh \zeta_i(s) \in (-1,1)$ for $i = 1, ..., k(a)$. Moreover, for some $C_0 > 0$, for all $i = 1, ..., k(a)$ and s large enough,

$$\left| \zeta_i(s) - \left(i - \frac{(k(a)+1)}{2} \right) \frac{(p-1)}{2} \log s \right| \leq C_0. \tag{13}$$

As a consequence of our analysis, particularly the lower bound on $T(x)$ in (3), we have the following result on the blow-up speed in L^∞ of the section of the backward light cone with vertex $(a, T(a))$ where $a \in \mathbb{R}$ (Corollary 3 and the following remark in [4]):

Proposition 17 (Blow-up speed in the section of the backward light cone)**.**
(i) **Case of a non characteristic point**. *If $a \in \mathcal{R}$ and $t \in [0, T(a))$, then*

$$\frac{(T(a)-t)^{-\frac{2}{p-1}}}{C} \leq \sup_{|x-a|<T(a)-t} |u(x,t)| \leq C(T(a)-t)^{-\frac{2}{p-1}}.$$

(ii) **Case of a characteristic point**. *If $a \in \mathcal{S}$ and $t \in [0, T(a))$, then*

$$\frac{|\log(T(a)-t)|^{\frac{k(a)-1}{2}}}{C(T(a)-t)^{\frac{2}{p-1}}} \leq \sup_{|x-a|<T(a)-t} |u(x,t)| \leq \frac{C|\log(T(a)-t)|^{\frac{k(a)-1}{2}}}{(T(a)-t)^{\frac{2}{p-1}}}.$$

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Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear Schrödinger equation when $d \geq 3$

Benjamin Dodson

In this talk we study the defocusing, L^2 critical initial value problem

$$iu_t + \Delta u = |u|^{4/d}u, \\ u(0,x) = u_0 \in L^2(\mathbf{R}^d). \tag{1}$$

We prove (1) is globally well-posed and scattering for all $u_0 \in L^2(\mathbf{R}^d)$. The principal new ingredient is a frequency localized interaction Morawetz estimate obtained by an induction on frequency argument. We use the induction on frequency argument to estimate the Strichartz norm at high frequencies, which in turn are used to estimate the errors that arise in the Morawetz estimates when truncating in frequency.

We prove global well-posedness for the mass critical nonlinear Schrodinger initial value problem, (1), by means of the concentration compactness argument. It is well known that (1) is globally well-posed and scattering when $\|u_0\|_{L^2}$ is sufficiently small. Furthermore, the set of m, such that there exists u_0 with $\|u_0\|_{L^2} = m$ and global well-posedness fails is a closed set. In particular, this set must have a least element m_0. Further analysis then shows that $u(t,x)$ must lie in a precompact set modulo symmetries.

Because $u(t,x)$ must lie in a precompact set, a sequence of $u(t_n,x)$ must have a convergent subsequence. Taking this limit, we obtain a minimal mass solution which blows up on $[0,\infty)$ and has scaling factor $N(t)$, $N(0) = 1$, and $N(t) \leq 1$ for $t \geq 0$. From there we can analyze two situations separately. If $\int N(t)^3 dt = \infty$ we use a frequency localized interaction Morawetz estimate to exclude this case. When $\int N(t)^3 dt < \infty$ we prove the solution possesses additional regularity, and then exclude this scenario by a conservation of energy argument. In either case, the key is an estimate on an interval $[0,T]$, $\int_0^T N(t)^3 dt = K$, for $N \leq K$,

$$\|P_{|\xi-\xi(t)|>N} u\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,T]\times\mathbf{R}^d)} \lesssim_{m_0,d} \left(\frac{K}{N}\right)^{1/2} \rho(N), \tag{2}$$

with $\rho(N) \leq 1$, $\lim_{N\to\infty} \rho(N) = 0$. We use this to estimate the errors of the frequency localized interaction Morawetz estimate and to prove additional regularity.

The energy super-critical wave in three dimensions

Rowan Killip
(joint work with Monica Vişan)

We consider the Cauchy problem for the nonlinear wave equation

$$u_{tt} - \Delta u + u^7 = 0 \tag{1}$$

where $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$. The seventh power in the nonlinearity is merely archetypal of the problems that we consider. That the nonlinearity is defocusing (i.e. appears with a plus sign) is essential for our arguments.

Equation (1) possesses a scaling symmetry; specifically, if $u(t, x)$ is a solution then so is $u^\lambda(t, x) := \lambda^{1/3} u(\lambda t, \lambda x)$. This scaling symmetry dictates a critical Sobolev index and, correspondingly, a natural notion of size:

$$\mathcal{E}(u(t)) := \|u(t)\|^2_{\dot{H}^{7/6}_x(\mathbb{R}^3)} + \|u_t(t)\|^2_{\dot{H}^{1/6}_x(\mathbb{R}^3)}.$$

We emphasize that this is *not* conserved by the evolution.

By comparison, the energy

$$E(u) = \int_{\mathbb{R}^3} \tfrac{1}{2}|u_t(t, x)|^2 + \tfrac{1}{2}|\nabla u(t, x)|^2 + \tfrac{1}{8}|u(t, x)|^8 \, dx \tag{2}$$

is conserved in time, but not by the scaling. Indeed, $E(u^\lambda) = \lambda^{-1/3} E(u)$. As λ appears here to a negative power we say that our equation is *energy super-critical.* Physically, it means that energy conservation provides very little control over the small length-scale behaviour of solutions. As no other conserved quantity (or related technique) provides *a priori* control over the short length scale behaviour, we may drop the qualification and simply refer to the equation as *super-critical.*

We have focussed here on the short-scale behaviour since that is precisely where the nemeses of blow-up and turbulence appear.

Theorem 18 (See [2])**.** *Let u be a maximal life-span solution to* (1) *with initial data chosen so that $\mathcal{E}(u(0)) < \infty$, then either*
(a) *$\mathcal{E}(u(t))$ is unbounded; or*
(b) *$u(t)$ is global and there exist solutions $u_\pm(t)$ to the linear wave equation so that*

$$\|u(t) - u_\pm(t)\|^2_{\dot{H}^{7/6}_x(\mathbb{R}^3)} + \|\partial_t[u(t) - u_\pm(t)]\|^2_{\dot{H}^{1/6}_x(\mathbb{R}^3)} \longrightarrow 0$$

as $t \to \pm\infty$.

It is not difficult to show that item (b) is equivalent to the finiteness of global space-time norms. Indeed we prove Theorem 1 by showing the stronger statement

$$\int_I \int_{\mathbb{R}^3} |u(t, x)|^{12} \, dx \, dt \le C\Big(\sup_{t \in I} \mathcal{E}(t)\Big) \tag{3}$$

for all local solutions $u : I \times \mathbb{R}^3 \to \mathbb{R}$ and some fixed (but not effective) function $C : [0, \infty) \to [0, \infty)$.

The fundamental difficulty with super-critical problems is controlling the possible growth of critical norms. The formulation of our results in (3) makes it clear that we do not address this issue. Rather, we discuss consequences of boundedness of the critical norm. This makes the problem *critical* and so amenable to the technology developed over the past decade or so for treating such problems.

The impetus to consider this specific problem comes from recent work [1] of Kenig and Merle who proved (3) (and so also Theorem 1) for spherically symmetric solutions. In their work, as ours, the key point however is the proof of good decay properties for certain solutions. To better explain this point, we need to delve a little into the proof, which is what we do next.

The first step in the proof is to show that should (3) fail, there would be a 'minimal criminal', that is, a solution $u : I \times \mathbb{R}^3 \to \mathbb{R}$ with

$$\int_I \int_{\mathbb{R}^3} |u(t,x)|^{12}\, dx\, dt = \infty$$

that is minimal in the following sense: if $v : J \times \mathbb{R}^3 \to \mathbb{R}$ is another solution with $\sup_{t\in J} \mathcal{E}(v(t)) < \sup_{t\in I} \mathcal{E}(u(t))$ then the solution v has finite $L^{12}_{t,x}$ norm.

The proof that there is such a minimal criminal u proceeds via concentration compactness methods and yields a stronger result, namely, that u is almost periodic modulo symmetries. This means that there are two functions $x : I \to \mathbb{R}^3$ and $N : I \to (0,\infty)$ so that

$$\Big\{ y \mapsto N(t)^{-\frac{1}{3}} u\big(t, x(t) + yN(t)^{-1}\big) : t \in I \Big\}$$

is a precompact set of $\dot{H}^{7/6}_x(\mathbb{R}^3)$; u_t also enjoys a similar compactness property. This is a mathematically precise formulation of the notion that our solution u consists of a single wave packet that does not disperse. The function $x(t)$ gives its location, while $1/N(t)$ gives its characteristic length scale.

If we knew that such a minimal counter-example had finite energy, it would not be difficult to use the almost periodicity of the solution to prove that it does not exist (and so obtain Theorem 1). To be precise, one employs conservation of energy to show that $N(t)$ must remain approximately constant and then the Morawetz identity to show that this is also impossible. Note that even if one only wishes to understand (1) for Schwartz initial data, when passing to the minimal counterexample one can only retain control over critical norms and so finiteness of the energy may be lost. Note also that since they lie in $\dot{H}^{7/6}_x$, our solutions have finite energy on any compact set; the problem is to show adequate decay at infinity.

Euler–Lagrange equations play a central role in understanding minimizers for the classical problems of the Calculus of Variations. A recurring theme in our recent work is that the following reduced Duhamel formulae play the same role in the context of dispersive PDE:

$$u(t) = \int_t^{\sup I} \frac{\sin\big((t-s)|\nabla|\big)}{|\nabla|} u(s)^7\, ds = -\int_{\inf I}^t \frac{\sin\big((t-s)|\nabla|\big)}{|\nabla|} u(s)^7\, ds \tag{4}$$

whenever $u : I \times \mathbb{R}^3 \to \mathbb{R}$ is a maximal-lifespan almost periodic solution to (1). Both integrals should be interpreted as converging in the sense of distributions, that is, when tested against Schwartz functions. As a first step toward using (4) to prove anything, we first need to show that these integrals converge in some absolute sense. This is effected via the energy flux identity.

The majority of the talk was devoted to showing how to use (4) to prove the required decay of u. This is a rather long and involved process and we can only give a caricature of it here. From (4) we obtain two representations of each of $\nabla u(0,x)$ and $u_t(0,x)$, one involving integration over the future and another involving integration over the past. Taking inner products of these representations yields a formula for the energy, well, at least the first two terms in (2). Since we do not yet know that the energy is finite, we need to localize this representation in space (with the ultimate intention of summing the localized pieces).

The key to estimating the space-localized energy, via the 'double Duhamel' representation just described, is to control and then exploit the space-time geometry of the curve $x(t)$, which represents the motion of our wavepacket in time.

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Dynamics of solutions to dispersive equations above the ground state energy

Wilhelm Schlag

(joint work with Kenji Nakanishi)

This talk is intended as an introduction to, and exposition of, some of the phenomena that solutions of dispersive equations exhibit at energy levels strictly above that of the ground state solution. These phenomena were recently established in several papers: in [7], [9] the authors studied the radial as well as nonradial nonlinear cubic Klein-Gordon equation in $\mathbb{R}^3$, and in [8] they treated the radial nonlinear cubic Schrödinger equation in $\mathbb{R}^3$. The energy critical wave equation in $\mathbb{R}^3$ and $\mathbb{R}^5$ was studied in [6] jointly with J. Krieger.

To be more specific, consider the subcritical Klein-Gordon equation

$$\ddot{u} - \Delta u + u = u^3 \tag{1}$$

in $\mathbb{R} \times \mathbb{R}^3$. It admits stationary solutions Q which are characterized as minimizers of

$$J(\varphi) := \int \Big(\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}|\varphi|^2 - \frac{1}{4}|\varphi|^4\Big)\, dx$$

subject to the constraint, with $\varphi \neq 0$,

$$K_0(\varphi) := \int \Big(|\nabla\varphi|^2 + |\varphi|^2 - |\varphi|^4\Big)\, dx = 0$$

It follows that the regions

$$\begin{aligned} \mathcal{PS}_+ &= \{(u,\dot u) \mid E(u,\dot u) < J(Q),\ K_0(u) \geq 0\} \\ \mathcal{PS}_- &= \{(u,\dot u) \mid E(u,\dot u) < J(Q),\ K_0(u) < 0\} \end{aligned} \tag{2}$$

are invariant under the nonlinear flow in the phase space $H^1 \times L^2$ where

$$E(u,\dot u) = \int \Big(\frac{1}{2}|\dot u|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 - \frac{1}{4}|u|^4\Big)\, dx$$

is the conserved energy for (1). It is a classical result of Payne, Sattinger [10] that solutions in $\mathcal{PS}_+$ are global, whereras those in $\mathcal{PS}_-$ blow up in finite time. In particular, the stationary solution Q is unstable, see also Shatah [11] and Berestycki, Cazenave [1]. Scattering in $\mathcal{PS}_+$ was only recently shown by Ibrahim, Masmoudi, and Nakanishi [4].

We are concerned with solutions whose energies satisfy

$$J(Q) \leq E(u,\dot u) < J(Q) + \varepsilon \tag{3}$$

for some small $\varepsilon > 0$. Since comparatively little is known about solutions in the regime $E(u,\dot u) > J(Q)$, it seems natural to turn to numerical investigations in order to obtain some idea of the nature of the blowup/global existence dichotomy. Roland Donninger and the second author have conducted such computer experiments at the University of Chicago, see [3]. This work consists of numerical computations of radial solutions to (1) whose data belong to a two-dimensional surface (such as a planar rectangle) in the infinite dimensional phase space $\mathcal{H} := H^1 \times L^2$ (of course the data are chosen to belong to a fine rectangular grid on that surface). Each solution is then evaluated with regard to blowup/global existence and a red dot is placed on the data rectangle if global existence is observed, whereas the dot is left blank otherwise. In the talk we shall show several pictures which are obtained in this fashion.

In the case where the data are perturbation of $(Q,0)$ (or $(-Q,0)$), the two Payne-Sattinger regions clearly meet in conic type singularity. In fact, these two regions are reminiscent of the set $\xi^2 - \eta^2 \leq 0$. This is due to the fact that the energy near Q takes the form of a saddle surface, which is well-known. In fact, there is a codimension one plane around $(Q,0)$ in $\mathcal{H}$ such that locally around that point the energy is positive definite on this plane, whereas it is indefinite on the whole space. An important feature of the red region is the appearance of the boundary: it seems to be a smooth curve. In fact, we show in the aforementioned references that *near* $(Q,0)$ in $\mathcal{H}$ the boundary is indeed a smooth co-dimension 1 manifold $\mathcal{M}$ with the property that solutions with data on that manifold are global and scatter to Q as $t \to \infty$. In dynamical terms this manifold is precisely the *center-stable* one, which contains the 1-dimensional stable manifold. Furthermore, $\mathcal{M}$ is transverse to the 1-dimensional unstable manifold. The latter manifold is characterized by the property that all solutions starting on it converge to $(Q,0)$ as $t \to -\infty$; in fact, this convergence is exponential. Moreover, in positive times solutions on the unique unstable manifold grow exponentially up until the time at which they leave a small neighborhood of the equilibrium $(Q,0)$. This *hyperbolic* nature of the dynamics near $(Q, 0)$, combined with an analysis of the variational structure of J and K_0 near that point allow us to state, moreover, that $\mathcal{M}$ divides a ball around $(Q, 0)$ into two halves, one in which solutions blow up in finite positive time, and another which has the property that solutions starting from it scatter to zero as $t \to \infty$. In other words, one faces a trichotomy in a small ball around $(Q, 0)$ in $\mathcal{H}$: as $t \to \infty$, one either has finite time blowup, or scattering to Q, or scattering to 0. This is the main result of [7] (which covers the radial case, see [9] for the nonradial one). Moreover, loc. cit. contain a complete description of the dynamics of all solutions satisfying (3): the data in $\mathcal{H}$ which obey this condition split into the disjoint union of nine nonempty infinite sets which correspond to all possible combinations of this trichotomy as $t \to \infty$ and $t \to -\infty$, respectively. Here is the precise statement.

Theorem 19. *Consider all solutions of NLKG* (1) *with radial initial data* $(u(0), \dot{u}(0)) \in H^1 \times L^2(\mathbb{R}^3)$ *and such that* $E(u, \dot{u}) < E(Q) + \varepsilon^2$ *for some small* $\varepsilon > 0$. *Then the set of all solutions is decomposed into nine non-empty sets characterized as*

(1) *Scattering to* 0 *for both* $t \to \pm\infty$,
(2) *Finite time blow-up on both sides* $\pm t > 0$,
(3) *Scattering to* 0 *as* $t \to \infty$ *and finite time blow-up in* $t < 0$,
(4) *Finite time blow-up in* $t > 0$ *and scattering to* 0 *as* $t \to -\infty$,
(5) *Trapped by* $\pm Q$ *for* $t \to \infty$ *and scattering to* 0 *as* $t \to -\infty$,
(6) *Scattering to* 0 *as* $t \to \infty$ *and trapped by* $\pm Q$ *as* $t \to -\infty$,
(7) *Trapped by* $\pm Q$ *for* $t \to \infty$ *and finite time blow-up in* $t < 0$,
(8) *Finite time blow-up in* $t > 0$ *and trapped by* $\pm Q$ *as* $t \to -\infty$,
(9) *Trapped by* $\pm Q$ *as* $t \to \pm\infty$,

where "trapped by $\pm Q$*" means that the solution stays in a* $O(\varepsilon)$ *neighborhood of* $\pm Q$ *forever after some time (or before some time). The initial data sets for (1)-(4), respectively, are open.*

This result does not make any mention of the center-stable manifold. In fact, its proof does not use any dispersive estimates for the evolution linearized about $(Q, 0)$. This distinction is relevant since such estimates on the linearized evolution depend on finer spectral information on the linearized operator

$$L_+ = -\Delta + 1 - 3Q^2$$

which is less robust with respect to changing the power in the nonlinearity (more specifically, as one lowers the cubic power slightly, say below 2.8, eigenvalues appear below the threshold of L_+, i.e., 1, which are not present for the cubic case, see [2]). However, for the cubic case (1) we show below that one can refine "trapped by" Q above to "scattering to" Q. This means that the solution takes the form

$$u(t) = \pm Q + v(t) + o_{H^1}(1) \quad t \to \infty$$

where $(v, \dot{v})$ is an energy solution to the free Klein-Gordon equation.

The challenge in proving these theorems lies with the fact that the linearization around $(Q, 0)$ ceases to be meaningful outside of a small neighborhood of that point. Therefore, one requires some form of a global argument to address the long-term behavior of the solutions once they leave such a neighborhood. More precisely, a central role in all aforementioned references is taken by the so-called one-pass theorem. This result states that there can be no almost homoclinic orbit, i.e., a solution which starts near $(\pm Q, 0)$, then leaves a small neighborhood of that point, but eventually returns to such a neighborhood. An example of a homoclinic is given by the ODE

$$(\dot{x}, \dot{y}) = (x - y^2, -y + x^2)$$

The one-pass theorem, which appears in all of the aforementioned references, then allows one to fix the sign of the functional K_0 (as well as other related functionals $K_{\alpha,\beta}$, obtained by applying a two-parameter group of scalings to both the dependent and the independent variables), after the solution exits a small neighborhood of $(\pm Q, 0)$. This is essential, as one can then rely on the classical Payne, Sattinger criterium to conclude the blowup/scattering dichotomy for solutions that are not trapped by the ground states; the scattering then requires more work but can be obtained by means of the Kenig, Merle scheme, see [5] and [4].

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Small data scattering and soliton stability in $\dot{H}^{-\frac{1}{6}}$ for the quartic KdV equation

Jeremy Louis Marzuola
(joint work with Herbert Koch)

We present here results from [5]. The generalized KdV equation

$$\begin{cases} \partial_t \psi + \partial_x(\partial_x^2 \psi + \psi^p) = 0, \ t, x \in \mathbb{R} \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{1}$$

has an explicit soliton solution

$$\psi_c(x,t) = Q_{p,c,c^2t+x_0}(x) := c^{\frac{2}{p-1}} Q_p(c(x-(x_0+c^2t)))$$

with $c > 0$, $x_0 \in \mathbb{R}$ and

$$Q_p = \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}} \left(\frac{p-1}{2} x\right). \tag{2}$$

Well-posedness of the generalized KdV equation was established by Kenig-Ponce-Vega [4] in H^s for some s depending on p. The case $p = 4$ (quartic KdV) is particularly interesting as it is the only subcritical power nonlinearity that does not lead to a completely integrable system. The critical space for the quartic KdV equation is $H^{-\frac{1}{6}}$. Grünrock [2] obtained local wellposedness in H^s, $s > -1/6$ and the endpoint $\dot{H}^{-\frac{1}{6}}$ was reached by Tao [12]. Though wellposedness is not the main focus of this note, we use spaces of bounded p variation and their predual (see [3]) to simplify and strengthen Tao's wellposedness result in the critical space.

We will focus on the case $p = 4$ and omit p in the notation. It seems that any further progress is tied to an understanding of the linearization, or more precisely of the linear equation

$$u_t + \partial_x \mathcal{L} u = 0 \tag{3}$$

and its adjoint

$$v_t + \mathcal{L} \partial_x v = 0, \tag{4}$$

which have the explicit solutions (with $\tilde{Q} = c\partial_c Q_c|_{c=1}$)

$$u = a(\tilde{Q} + 2tQ') + bQ', \ v = cQ,$$

where

$$\tilde{Q} := c\frac{d}{dc} c^{\frac{2}{p-1}} Q_p(cx)\Big|_{c=1} = \frac{2}{p-1} Q_p + xQ', \tag{5}$$

usually evaluated at $c = 1$.

Thus both equations (3) and (4) have linearly growing solutions. It is one of the first contributions of this paper that both equations are uniformly L^2 bounded once we take into account these modes, and, moreover, there are local energy estimates global in time once we remove these modes. In particular the assumption of Pego and Weinstein on the absence of embedded eigenvalues holds.

The goal of this work is to build on the arguments of Weinstein [13] and Martel-Merle [8, 9] to establish some type of asymptotic soliton stability for generalized KdV equations by a direct analysis of the equation itself. We apply a variant of Weinstein's and Martel and Merle's arguments to the linear equations (4) and (4) and their relatives with variable scale and velocity, and control nonlinear terms through estimates for linear equations.

Specifically, we define projection operators related to the spectrum of $\mathcal{L}$:

$$P_{Q'}^{\perp}\psi = \psi - \frac{\langle \psi, Q'\rangle}{\langle Q', Q'\rangle} Q', \quad \tilde{P}\psi = \psi - \frac{\langle \psi, Q\rangle}{\langle Q, \tilde{Q}\rangle}\tilde{Q}. \tag{6}$$

We obtain the main linear estimates which in their simplest form can be written as

Theorem 20. *Let S be the solution operator for* (3) *and S^* the solution operator for* (4)*. Then, we have*

$$\sup_t \|S(t)\tilde{P}^* u_0\|_{L^2} + \|\operatorname{sech}(x)\partial_x P_{Q'}^{\perp} S(t)\tilde{P}^* u_0\|_{L^2(\mathbb{R}^2)} \lesssim \|u_0\|_{L^2}, \tag{7}$$

$$\sup_t \|S^*(t)P_{Q'}^{\perp} v(t)\|_{L^2} + \|\operatorname{sech}(x)\partial_x \tilde{P} S^*(t) P^{\perp}\|_{L^2(\mathbb{R}^2)} \lesssim \|v_0\|_{L^2}. \tag{8}$$

We provide variants of Theorem 20 for linearization at solitons with variable scale and velocity as well as estimates in scales of Banach spaces similar to estimates for the Airy equation.

Even near the trivial solution dominating the nonlinear part globally by the linear parts requires to work in a scale invariant space similar to $\dot{H}^{-\frac{1}{6}}$. On the positive side it will lead to scattering for perturbations of a soliton in $\dot{H}^{-\frac{1}{6}}$, without the smallness condition of Tao in the energy space. The study of the linear equation will lead to a fairly precise understanding of its properties which seems to be new and we hope that it will provide a model for many other questions on the stability of solitons.

As is standard in the study of stability, we take

$$\psi(x,t) = Q_{c(t)}(x - y(t)) + w(x,t).$$

Then, we have

$$\begin{aligned} \partial_t w + \partial_x(\partial_x^2 w + 4Q_c^3 w) = &- \dot{c}(\partial_c Q_c)(x-y) + \dot{y}(Q_c')(x-y) \\ &- \partial_x(\partial_x^2 Q_c - c^2 Q_c + Q_c^4) - c^2(Q_c'(x-y)) \\ &- \partial_x(6Q_c^2(x-y)w^2 + 4Q_c(x-y)w^3 + w^4). \end{aligned} \tag{9}$$

The standard choice of $\dot{c}$ and $\dot{y}$ ensures orthogonality conditions for w. Due to low time regularity we are forced to relax the orthogonality conditions to

$$\frac{\dot{c}}{c}\langle Q_c, \tilde{Q}_c\rangle = \langle w, Q_c\rangle, \tag{10}$$

$$(\dot{y} - c^2)\langle Q_c', Q_c'\rangle = -\kappa\langle w, Q_c'\rangle, \tag{11}$$

where $\kappa \gg 1$.

From an implicit function theorem argument similar to that in the proof of Proposition 1 of [7] there exist unique $c(0)$ and $y(0)$ so that $w(.,0)$ is orthogonal to $Q_{c(0)}(.-y(0))$ and $Q'_{c(0)}(.-y(0))$ provided the distance of ψ to the set of solitons is small in a suitable norm.

We consider the equations above as ordinary differential equations for c and y, coupled with the partial differential equation.

Using the decomposition and linear estimates, we can prove (refering to the result for the definition of the function spaces, with $\dot{B}_\infty^{-1/6,2}$ slightly larger than $\dot{H}^{-1/6}$) the following global result

Theorem 21. *There exists $\epsilon > 0$ and $c > 0$ such that given* (1) *with initial data of the form*

$$\min_{c_0,y_0} \|\psi_0 - Q_{c_0}(x-y_0)\|_{\dot{B}_\infty^{-\frac{1}{6},2}} \leq \epsilon,$$

there exist unique functions c and y with

$$\langle w(0), Q_{c(0)}\rangle = \langle w(0), Q'_{c(0)}\rangle = 0,$$
$$\dot{c} \in L^1 \cap C^0, \quad \dot{y} - c^2 \in L^2 \cap C^0,$$

and a function $w(x,t) \in \dot{X}_\infty^{-\frac{1}{6}}$ such that

$$\psi(x,t) = Q_{c(t),y(t)}(x) + w(x,t)$$

satisfies the quartic KdV equation, and w, c and y satisfy (10),(11) *and* (9). *Moreover,*

$$\|\dot{c}\|_{L^1\cap C^0} + \|\dot{y} - c^2\|_{L^2\cap C^0} + \|w\|_{\dot{X}_\infty^{-\frac{1}{6}}} \leq c\|w_0\|_{\dot{B}_\infty^{-\frac{1}{6},2}}.$$

In addition, there exists a function $z_0 \in \dot{B}_\infty^{-\frac{1}{6},2}$ such that

$$\|w(t) - e^{-t\partial_x^3} z_0\|_{\dot{B}_\infty^{-\frac{1}{6},2}} \to 0$$

and

$$\|w(.) - e^{-\cdot\partial_x^3} z_0\|_{X_\infty^{-1/6}((t,\infty))} \to 0 \qquad \text{as } t \to \infty$$

if $w(0)$ is in the closure of C_0^∞.

In fact, we prove a far stronger result than this, though Theorem 21 captures the main ideas. Finally, we show for a function v, there exists a quantity $J(v)$ such that we have the following

Theorem 22. *Let v_0 be in the closure of C_0^∞ in $\dot{B}_\infty^{-\frac{1}{6},2}$, $c_\infty > 0$, $y_0 \in \mathbb{R}$. Let v be the solution to the linear homogeneous KdV equation. Assume that*

$$J(v) \leq \delta$$

for some $\delta = \delta(\|v_0\|_{\dot{B}_\infty^{-\frac{1}{6},2}})$. Then there exists a solution Ψ to the quartic KdV equation, a function $y \in C^1([0,\infty))$, $c \in C^1([0,\infty),(0,\infty))$ such that $w = \Psi - Q_{c,y}$, *c and y satisfy equations* (10),(11), (9), *and*

$$\langle w(0), Q_{c(0)}(. - y(0))\rangle = \langle w(0), Q'_{c(0)}(. - y(0))\rangle = 0,$$

$$c(t) \to c_\infty, \quad y(0) = y_0, \quad w(t) - v(t) \to 0 \text{ in } \dot{B}_\infty^{-\frac{1}{6},2} \text{ as } t \to \infty.$$

Moreover, if in addition $v_0 \in L^2$*, then* $\Psi \in C(\mathbb{R}, L^2(\mathbb{R}))$ *and*

$$\|v_0\|_{L^2}^2 + \|Q_{c_\infty,0}\|_{L^2}^2 = \|\Psi(t)\|_{L^2}.$$

There exists $\varepsilon > 0$ *such that the assumptions are satisfied if* $\|v_0\|_{\dot{B}_\infty^{-\frac{1}{6},2}} \le \varepsilon$.

Remark 1. The conclusions in Theorems 21 and 22 hold as well in the spaces $\dot{B}_\infty^{-\frac{1}{6},2} \cap \dot{H}^s \cap H^\sigma$ for any $-1 < s \le 0$ and $\sigma \ge 0$, allowing one to prove uniform bounds in higher Sobolev norms. In particular, given initial data in $\dot{B}_\infty^{-\frac{1}{6},2} \cap \dot{H}^s \cap H^\sigma$, J small will imply stability and scattering in $\dot{B}_\infty^{-\frac{1}{6},2} \cap \dot{H}^s \cap H^\sigma$. Specifically, we note one can prove boundedness and scattering in the energy space H^1, intersected with $\dot{B}_\infty^{-\frac{1}{6},2}$.

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Wellposedness of the two and three dimensional full water wave problem

Sijue Wu

We consider the motion of the interface separating an inviscid, incompressible, irrotational fluid, under the influence of gravity, from a region of zero density (i.e. air) in n-dimensional space. It is assumed that the fluid region is below the air region. Assume that the density of the fluid is 1, the gravitational field is $-\mathbf{k}$, where $\mathbf{k}$ is the unit vector pointing in the upward vertical direction, and at time $t \geq 0$, the free interface is $\Sigma(t)$, and the fluid occupies region $\Omega(t)$. When surface tension is zero, the motion of the fluid is described by

$$(1) \qquad \begin{cases} \mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = -\mathbf{k} - \nabla P \qquad \text{on } \Omega(t),\ t \geq 0, \\ \operatorname{div}\mathbf{v} = 0, \qquad \operatorname{curl}\mathbf{v} = 0, \qquad \text{on } \Omega(t),\ t \geq 0, \\ P = 0, \qquad \text{on } \Sigma(t) \\ (1, \mathbf{v}) \text{ is tangent to the free surface } (t, \Sigma(t)), \end{cases}$$

where $\mathbf{v}$ is the fluid velocity, P is the fluid pressure. It is well-known that when surface tension is neglected, the water wave motion can be subject to the Taylor instability [3, 16, 2]. Assume that the free interface $\Sigma(t)$ is described by $\xi = \xi(\alpha, t)$, where $\alpha \in \mathbb{R}^{n-1}$ is the Lagrangian coordinate, i.e. $\xi_t(\alpha, t) = \mathbf{v}(\xi(\alpha, t), t)$ is the fluid velocity on the interface, $\xi_{tt}(\alpha, t) = (\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v})(\xi(\alpha, t), t)$ is the acceleration. Let $\mathbf{n}$ be the unit normal pointing out of $\Omega(t)$. The Taylor sign condition relating to Taylor instability is

$$(2) \qquad -\frac{\partial P}{\partial \mathbf{n}} = (\xi_{tt} + \mathbf{k}) \cdot \mathbf{n} \geq c_0 > 0,$$

point-wisely on the interface for some positive constant c_0. In previous works [17, 18], we showed that the Taylor sign condition (2) always holds for the n-dimensional infinite depth water wave problem (1), $n \geq 2$, as long as the interface is non-self-intersecting; and the initial value problem of the water wave system (1) is uniquely solvable **locally** in time in Sobolev spaces for arbitrary given data. Earlier work includes Nalimov [13], Yosihara [21] and Craig [6] on local existence and uniqueness for small data in 2D. Notice that if surface tension is not zero, or if there is a bottom, or nonzero-vorticity, the Taylor sign condition need not hold. Local wellposedness for water wave motion with the effect of surface tension, bottom and a non-zero vorticity, under the assumption (2) can be found in [1, 4, 5, 9, 11, 12, 14, 15, 22].

In order to understand the long time behavior of the water wave motion, we need to understand the nature of the nonlinearity of the water wave equation. In [19, 20], we showed that the nature of the nonlinearity of the water wave equation (1) is of cubic and higher orders. For water waves in two space dimensions, if initially the amplitude of the interface and the kinetic energy (and finitely many of their derivatives) are of size $O(\epsilon)$ and small, then there exists a unique classical solution of the water wave equation (1) for a time period $[0, e^{c/\epsilon}]$; during this time period, the interface remains small and as regular as the initial interface. Here c

is a constant independent of ϵ (c.f. Theorem 23, [19]). For water waves in three space dimensions, if initially the steepness of the interface and the fluid velocity on the interface (and finitely many of their derivatives) are small, then there exists a unique classical solution of the water wave equation (1) for all time, and the interface remains to have small steepness and is as regular as the initial interface for all time (c.f. Theorem 24, [20]).

Let's state what we have obtained so far in precise terms. Notice that equation (1) is a nonlinear equation defined on moving domains. It is difficult to obtain results directly from it. One key step in our approach is to rewrite (1) into forms from which results and information can be obtained.

For clarity, we mainly write in terms of the 2D water waves. We regard the 2D space as a complex space and use the same notation for complex form $\xi = x + iy$ and $\xi = (x, y)$. So $\bar{\xi} = x - iy$.

Let $\xi = \xi(\alpha, t)$ be the free interface $\Sigma(t)$ at time t in Lagrangian parameter α, $N = i\xi_\alpha$ be the normal vector pointing out of the fluid domain, $\mathbf{n} = \frac{N}{|N|}$ be the unit normal, $\mathfrak{a} = -\frac{1}{|N|}\frac{\partial P}{\partial \mathbf{n}}$. We know from [17, 18] that equation (1) is equivalent to the following system defined on the interface $\Sigma(t)$:

$$\xi_{tt} + i = i\mathfrak{a}\xi_\alpha \tag{3}$$

$$\bar{\xi}_t = \mathfrak{H}\bar{\xi}_t \tag{4}$$

where

$$\mathfrak{H}f(\alpha, t) = \frac{1}{\pi i} p.v. \int \frac{f(\beta, t)\xi_\beta(\beta, t)}{\xi(\alpha, t) - \xi(\beta, t)}\, d\beta \tag{5}$$

is the Hilbert transform on $\Sigma(t) : \xi = \xi(\alpha, t),\ \alpha \in \mathbb{R}$. Notice that (3)-(4) is fully nonlinear. To solve (3)-(4) on a (small) time interval $[0, T]$, we further derived the following equation by taking derivative to t to (3):

$$\begin{cases} \bar{\xi}_{ttt} + i\mathfrak{a}\bar{\xi}_{t\alpha} = -i\mathfrak{a}_t\bar{\xi}_\alpha \\ \bar{\xi}_t = \mathfrak{H}\bar{\xi}_t \end{cases} \tag{6}$$

Using the fact $\bar{\xi}_t = \mathfrak{H}\bar{\xi}_t$, and $\mathfrak{a}$, $\mathfrak{a}_t$ are real valued, we deduced that

$$\begin{aligned} &(I + \mathfrak{K}^*)(\mathfrak{a}_t|\bar{\xi}_\alpha|) = \\ &- \Re\Big(\frac{i\xi_\alpha}{|\xi_\alpha|}\{2[\xi_{tt}, \mathfrak{H}]\frac{\bar{\xi}_{t\alpha}}{\xi_\alpha} + 2[\xi_t, \mathfrak{H}]\frac{\bar{\xi}_{tt\alpha}}{\xi_\alpha} - \frac{1}{\pi i}\int \Big(\frac{\xi_t(\alpha, t) - \xi_t(\beta, t)}{\xi(\alpha, t) - \xi(\beta, t)}\Big)^2 \bar{\xi}_{t\beta}\, d\beta\}\Big) \end{aligned} \tag{7}$$

here $\Re\xi$ indicates the real part of ξ,

$$\mathfrak{K}^* f(\alpha, t) = \int \Re\{\frac{-1}{\pi i}\frac{\xi_\alpha}{|\xi_\alpha|}\frac{|\xi_\beta(\beta, t)|}{(\xi(\alpha, t) - \xi(\beta, t))}\} f(\beta, t)\, d\beta$$

is the adjoint of the double layered potential operator $\mathfrak{K}$ in $L^2(\Sigma(t), dS)$. Notice that $I + \mathfrak{K}^*$ is invertible in $L^2(\Sigma(t), dS)$. Rewriting

$$-i\mathfrak{a}_t\bar{\xi}_\alpha = -i\frac{\bar{\xi}_\alpha}{|\bar{\xi}_\alpha|}\mathfrak{a}_t|\bar{\xi}_\alpha| = \frac{\bar{\xi}_{tt} - i}{|\xi_{tt} + i|}\mathfrak{a}_t|\bar{\xi}_\alpha|,$$

using (7) for $\mathfrak{a}_t|\bar{\xi}_\alpha|$. (6) is now a quasi-linear system with the right hand side of the first equation in (6) consisting of terms of lower order derivatives of $\bar{\xi}_t$.

Let $\mathfrak{u} = \bar{\xi}_t$. Notice that $i\partial_\alpha \mathfrak{u} = \nabla_{\mathbf{n}}\mathfrak{u}$, and the Dirichlet-Neumann operator $\nabla_{\mathbf{n}}$ is a positive operator. By further proving $\mathfrak{a} = -\frac{1}{|N|}\frac{\partial P}{\partial \mathbf{n}} > 0$ for nonself-intersecting interfaces, we showed that (6)-(7) is a quasi-linear equation of weakly hyperbolic type. The local in time wellposedness of (6)-(7) in Sobolev spaces (with $(\mathfrak{u}, \mathfrak{u}_t) \in C([0,T], H^{s+1/2} \times H^s)$, $s \geq 4$) was then proved by energy estimates and a fixed point iteration argument. Through establishing the equivalence of (1) with (6)-(7), we obtained the local in time well-posedness in Sobolev spaces of the full water wave equation (1) (c.f. [17, 18]).

For 3D water waves we introduced the framework of Clifford algebra, or in other words, the algebra of quaternions $\mathcal{C}(V_2)$ [18]. Let $\{1, e_1, e_2, e_3\}$ be the basis of $\mathcal{C}(V_2)$, satisfying $e_i^2 = -1$ for $i = 1, 2, 3$, $e_ie_j = -e_je_i, i \neq j, e_3 = e_1e_2$. Let $\mathcal{D} = \partial_x e_1 + \partial_y e_2 + \partial_z e_3$. By definition, a Clifford-valued function $F : \Omega \subset \mathbb{R}^3 \to \mathcal{C}(V_2)$ is Clifford analytic in domain Ω if $\mathcal{D}F = 0$ in Ω. Therefore $F = \sum_{i=1}^3 f_ie_i$ is Clifford analytic in Ω iff $\operatorname{div}F = 0$ and $\operatorname{curl}F = 0$ in Ω. Furthermore we know F is Clifford analytic in Ω iff $F = \mathfrak{H}_\Sigma F$, where

$$\mathfrak{H}_\Sigma g(\alpha,\beta) = p.v. \iint K(\eta(\alpha',\beta') - \eta(\alpha,\beta))\,(\eta'_{\alpha'} \times \eta'_{\beta'})\,g(\alpha',\beta')\,d\alpha' d\beta'$$

is the 3D version of the Hilbert transform on $\Sigma = \partial\Omega : \eta = \eta(\alpha,\beta),\ (\alpha,\beta) \in \mathbb{R}^2$, with normal $\eta_\alpha \times \eta_\beta$ pointing out of Ω, and $K(\eta) = -2\mathcal{D}\Gamma(\eta) = -\frac{2}{\omega_3}\frac{\eta}{|\eta|^3}$.

All these indicate Clifford analysis can be an effective tool for 3D water waves. Indeed, in the framework $\mathcal{C}(V_2)$, we derived the quasi-linear equation (cf. (5.21)-(5.22) of [18]) for the 3D water waves, and the local in time well-posedness of the 3D full water wave equation was therefore obtained from energy estimates and a fixed point iteration argument to the quasi-linear equation.

We now turn to the question of the long time behavior of the solutions for the water wave equation (1) for small initial data.

Let's state what we found for 2-D water wave ($n = 2$) [19].

Let $U_g f = f \circ g = f(g(\cdot,t),t)$, and for $\kappa : \mathbb{R} \to \mathbb{R}$ a diffeomorphism, let

$$\zeta := \xi \circ \kappa^{-1} = \mathfrak{x} + i\mathfrak{y}, \qquad U_\kappa^{-1} D_t := \partial_t U_\kappa^{-1}, \qquad U_\kappa^{-1}\mathcal{P} := (\partial_t^2 - i\mathfrak{a}\partial_\alpha)U_\kappa^{-1}$$
$$b := \kappa_t \circ \kappa^{-1}, \qquad U_\kappa^{-1}\mathcal{A}\partial_\alpha := \mathfrak{a}\partial_\alpha U_\kappa^{-1}, \qquad U_\kappa^{-1}\mathcal{H} = \mathfrak{H}U_\kappa^{-1},$$

so

$$D_t = (\partial_t + b\partial_\alpha), \qquad \mathcal{P} = D_t^2 - i\mathcal{A}\partial_\alpha \tag{8}$$

In [19], we showed that for any solution $\xi(\alpha,t) = x(\alpha,t) + iy(\alpha,t)$ of (3)-(4), the quantity $\Pi := (I - \mathfrak{H})y$ satisfies the equation

$$\begin{aligned}\mathcal{P}(\Pi \circ \kappa^{-1}) = \frac{2}{\pi i}\int \frac{(D_t\zeta(\alpha,t) - D_t\zeta(\beta,t))(\mathfrak{y}(\alpha,t) - \mathfrak{y}(\beta,t))}{|\zeta(\alpha,t) - \zeta(\beta,t)|^2}\partial_\beta D_t\zeta(\beta,t)d\beta \\ + \frac{1}{\pi i}\int \left(\frac{D_t\zeta(\alpha,t) - D_t\zeta(\beta,t)}{\zeta(\alpha,t) - \zeta(\beta,t)}\right)^2 \partial_\beta \mathfrak{y}(\beta,t)d\beta\end{aligned} \tag{9}$$

Notice that the right hand side of (9) is cubicly small if the velocity $D_t\zeta$ and steepness $\partial_\alpha\mathfrak{y}$ (and their derivatives) are small. Furthermore we found a coordinate change $\kappa:\mathbb{R}\to\mathbb{R}$,

$$\kappa(\alpha,t)=\bar{\xi}(\alpha,t)+\frac{1}{2}(I+\mathfrak{H})(I+\mathfrak{K})^{-1}(\xi-\bar{\xi}) \tag{10}$$

so that equation (9) contains no quadratic nonlinear terms if κ is given by (10).[1] Here $\mathfrak{K}=\Re\mathfrak{H}$ is the double layered potential operator. In other words, the projection of the height function y of the interface into the space of holomorphic functions in the air region, under the change of coordinates κ as given in (10): $\pi:=\Pi\circ\kappa^{-1}=U_\kappa^{-1}(I-\mathfrak{H})y=(I-\mathcal{H})\mathfrak{y}$ satisfies such an equation

$$(\partial_t^2-i\partial_\alpha)\pi=G$$

where G contains no quadratic nonlinear terms.

For 3D water waves, we use the setting of the quaternions $\mathcal{C}(V_2)$. In this setting, we also found that the projection of the height function of the interface into the space of holomorphic functions in the air region, in an appropriate coordinate system, satisfies an equation containing no quadratic nonlinear terms. (c.f. (1.25) or (1.35) and (1.28) of [20] for the 3D counterparts of the equation (9) and the change of coordinates (10).)

The almost global well-posedness of the 2D water waves and the global well-posedness of the 3D water waves are then obtained by applying the method of vector fields to (9) for 2D and to the equation (1.35) in [20] for 3D. We mention that the method of vector fields was first developed by Klainerman [10] for the nonlinear wave equation. The basic steps involved include developing a generalized Sobolev inequality that gives a L^∞ decay with rate $1/t^{\frac{n-1}{2}}$ for n-D water waves bounded by the generalized L^2 Sobolev norms defined by the vector fields for the water wave operator $\partial_t^2-i\partial_\alpha$,[2] an energy estimate and a continuity argument. We state the results:

Theorem 23 (2D water waves, [19]). *Let $\xi_0=(\alpha,y_0(\alpha))$, $\alpha\in\mathbb{R}$ be the initial interface, $\mathbf{v}_0=\mathbf{v}_0(x,y)$, $(x,y)\in\Omega(0)$ be the initial velocity. Assume $y_0(\alpha)=\epsilon f(\alpha)$, $\mathbf{v}_0(x,y)=\epsilon g(x,y)$, where $f\in L^2(\mathbb{R})$ and $g\in L^2(\Omega(0))$ and up to 12 derivatives of f and g are in L^2. Then there is $\epsilon_0>0$, such that for $\epsilon\le\epsilon_0$, there exists a unique classical solution of the 2D water wave equation* (1) *for a time period $[0,e^{c/\epsilon}]$. Here c depends on f, g only. During this time period, the solution stays small and has the same regularity as the initial data; and the L^∞ norm of the steepness of the interface $\partial_\alpha y$ and the velocity on the interface ξ_t decay at rate $1/t^{1/2}$.*

Theorem 24 (3D water waves. [20]). *Let $\xi_0=(\alpha,\beta,z_0(\alpha,\beta))$, $(\alpha,\beta)\in\mathbb{R}^2$ be the initial interface, $\xi_{t,0}=\xi_1(\alpha,\beta)$, $(\alpha,\beta)\in\mathbb{R}^2$ be the initial velocity on the interface. Assume that $|D|^{1/2}z_0=\epsilon f$, $\xi_1=\epsilon g$, $f,g\in L^2(\mathbb{R}^2)$ and 20 derivatives of f and g are in L^2. Then there is $\epsilon_0>0$, such that for $\epsilon\le\epsilon_0$, there exists a unique classical*

[1] It was shown in [19] that $\kappa:\mathbb{R}\to\mathbb{R}$ is a diffeomorphism if $\xi(\alpha,t)-\alpha$ is small.

[2] The water wave operator for 3D is $\partial_t^2-e_2\partial_\alpha+e_1\partial_\beta$.

solution of the 3D water wave equation (1) *for all time* $t \in [0, \infty)$*. During this time, the solution stays small and is as regular as the initial data; and the* L^∞ *norm of the steepness of the interface, the acceleration on the interface and the derivative of the velocity on the interface decay at rate* $1/t$*.*

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Global well-posedness for the cubic wave equation at supercritical regularity

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(joint work with Nicolas Burq)

In our previous work [3] , we developed a general method for obtaining local well-posedness of semi-linear wave equations with data of super-critical regularity. In addition, in [4] we gave a very particular example (based on invariant measures considerations) of global existence with data of supercritical regularity.

In this talk we complete [3, 4] by presenting a general global well-posedness result for a semi-linear wave equation with data of super-critical regularity.

Let (M, g) be $3d$ boundaryless Riemannian manifold with associated Laplace-Beltrami operator Δ_g. Consider the cubic defocusing wave equation

$$(1) \qquad (\partial_t^2 - \Delta_g)u + u^3 = 0, \quad u : \mathbb{R} \times M \to \mathbb{R}$$

with initial data

$$(2) \qquad u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1, \qquad (u_0, u_1) \in H^s(M) \times H^{s-1}(M)\,,$$

where $H^s(M)$ denotes the classical Sobolev spaces on M. By using simple scaling considerations one obtains that $s = 1/2$ is the critical Sobolev regularity associated to (1). It turns out that this regularity is the border line of the deterministic theory, in the sense of local well-posedness in the Hadamard sense (existence, uniqueness and continuous dependence on the data). More precisely, the Cauchy problem (1)-(2) is locally well-posed for data in $H^s \times H^{s-1}$, $s \geq 1/2$ (and even globally for $s \geq 1$). In the opposite direction, for $s \in (0, 1/2)$, the Cauchy problem (1)-(2) is not locally well-posed in $H^s \times H^{s-1}$. For instance one can contradict the continuous dependence by showing that there exists a sequence (u_n) of smooth solutions of (1) such that

$$\lim_{n\to\infty} \|(u_n(0), \partial_t u_n(0))\|_{H^s \times H^{s-1}} = 0$$

and

$$\lim_{n\to\infty} \|(u_n(t), \partial_t u_n(t))\|_{L^\infty([0,T];H^s \times H^{s-1})} = \infty, \quad \forall\, T > 0.$$

The well-posedness can be proved by invoking the Strichartz estimates for the wave equation on a riemannian manifold due to Kapitanskii [6]. For $s > 1/2$ the well-posedness holds in a stronger sense since the time existence can be chosen the same for all data in a fixed bounded set of $H^s \times H^{s-1}$. In the case $s = 1/2$ the situation is more delicate since the existence time depends in a more subtle way on the data. The ill-posedness claim is proved in [3], by using the approaches of Christ-Colliander-Tao and Lebeau [7].

One may however ask whether some sort of well-posedness for (1)-(2) survives for $s < 1/2$. In [3] we have shown that the answer is positive, at least locally in time, if one accepts to randomize the initial data. Moreover, the method of [3] works for a quite general class of randomizations. As already mentioned the approach of [4] to get global in time results is restricted only to very particular randomizations. More precisely, it is based on a global control on the flow given by an invariant measure (see also [1, 2]). In [3], Remark 1.5, we asked whether the globalization argument can be performed by using other global controls on the flow such as conservations laws. In the present work we give a positive answer to this question.

Let us now describe the initial data randomization we use. We suppose that $M = \mathbb{T}^3$ with the flat metric. Starting from $(u_0, u_1) \in H^s \times H^{s-1}$ given by their Fourier series

$$u_j(x) = \sum_{n\in\mathbb{Z}^3} c_{n,j}\, e^{in\cdot x}, \qquad c_{n,j} = \overline{c_{-n,j}}, \quad j = 0, 1,$$

we define u_j^ω by

$$u_j^\omega(x) = \sum_{n\in\mathbb{Z}^3} g_{n,j}(\omega) c_{n,j}\, e^{in\cdot x}, \qquad j = 0, 1,$$

where $(g_{n,j}(\omega))$, $n \in \mathbb{Z}^3$, $j = 0, 1$ is a sequence of complex random variables such that $g_{n,j} = h_{n,j} + i l_{n,j}$ with real valued $h_{n,j}$ and $l_{n,j}$ satisfying $g_{n,j} = \overline{g_{-n,j}}$ (so that u_j^ω remains real valued). In addition, we suppose that the system of random variables $(h_{n,j}, l_{n,j})_{n\in\mathbb{Z}^3, j=0,1}$ contains independent (after ignoring the repetitions) identically distributed real random variables with a joint distribution μ satisfying

$$\exists\, c > 0, \quad \forall\, \gamma \in \mathbb{R}, \quad \Big| \int_{-\infty}^{\infty} e^{\gamma x} d\mu(x) \Big| \leq e^{c\gamma^2}. \tag{3}$$

Typical examples of random variables satisfying (3) are the standard gaussians, i.e. $d\mu(x) = (2\pi)^{-1/2} \exp(-x^2/2)dx$ (with an identity in (3)) or the Bernoulli variables $d\mu(x) = \frac{1}{2}(\delta_{-1} + \delta_1)$ thanks to the inequality $\mathrm{ch}(\gamma) \leq \exp(\gamma^2/2)$. An advantage of the Bernoulli randomization is that it keeps the H^s norm of the original function. The gaussian randomization has the advantage to "generate" a dense set in $H^s \times H^{s-1}$ via the map $(u_0, u_1) \mapsto (u_0^\omega, u_1^\omega)$ for every $(u_0, u_1) \in H^s \times H^{s-1}$. As shown in [3] the considered randomization does not regularize in the scale of the Sobolev spaces (this fact is obvious for the Bernoulli randomization). We can now state our result.

Theorem 25. *Let $M = \mathbb{T}^3$ with the flat metric and $(u_0, u_1) \in H^s \times H^{s-1}$, $s \in (0, 1/2)$. Then the cubic wave equation* (1) *with data (u_0^ω, u_1^ω) is globally well-posed a.s. in ω.*

A similar to Theorem 25 statement holds if we replace the data (u_0^ω, u_1^ω) by $(v_0 + u_0^\omega, v_1 + u_1^\omega)$ with a deterministic (v_0, v_1) covered by the deterministic theory.

In a forthcoming work, we show that similar results could be obtained for general manifolds by modifying accordingly the randomization. We also extend our result to more general nonlinearities.

In the above result by well-posedness we mean existence, uniqueness in a suitable class and a suitable continuity property of the established flow.

In the above result we have a bound on the possible growth in time of the corresponding Sobolev norms.

Let us give an idea of the proof of the existence of the flow. Define the free evolution $S(t)$ by

$$S(t)(u_0,u_1) \equiv \cos(t\sqrt{-\Delta})(u_0) + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}(u_1)$$

with the natural convention concerning the zero Fourier mode. Set

$$w_{lin}^{\omega} \equiv S(t)\big(u_0^{\omega}, u_1^{\omega}\big)\,.$$

We look for the solution under the form

$$u = w_{lin}^{\omega} + v\,.$$

Then v solves the problem

$$(\partial_t^2 - \Delta_{\mathbb{T}^3})v + (v + w_{lin}^{\omega})^3 = 0, \quad (v(0), \partial_t v(0)) = (0,0).$$

Define the natural energy associated to our original problem. Namely

$$E(u) \equiv \frac{1}{2}\int_{\mathbb{T}^3}(|\nabla u|^2 + (\partial_t u)^2) + \frac{1}{4}\int u^4\,.$$

Then

(4)
$$\frac{d}{dt}E(v(t)) = \int_{\mathbb{T}^3}\partial_t v(v^3 - (v+w_{lin}^{\omega})^3) = -\int_{\mathbb{T}^3}\partial_t v(3v^2 w_{lin}^{\omega} + 3v(w_{lin}^{\omega})^2 + (w_{lin}^{\omega})^3)\,.$$

The key point is that thanks to large deviations estimates w_{lin} is in L^∞ almost surely and therefore we can conclude by a simple application of the Gronwall lemma. For instance we can write

$$\Big|\int_{\mathbb{T}^3}\partial_t v\, v^2 w_{lin}^{\omega}\Big| \leq \|\partial_t v\|_{L^2}\|v\|_{L^4}^2 \|w_{lin}^{\omega}\|_{L^\infty} \leq C\,E(v)\,\|w_{lin}^{\omega}\|_{L^\infty}$$

which provides the needed bound for the main term in the right hand-side of (4). All other terms can be treated similarly.

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Global existence for coupled Klein-Gordon equations with different speeds

Pierre Germain

We prove global existence and scattering for the system of Klein-Gordon equations

$$(1) \qquad \begin{cases} \square u^1 + u^1 = Q^1(u^1, u^c) \\ \square_c u^c + u^c = Q^c(u^1, u^c) \\ (u^1, \partial_t u^1)(t=0) = (u^1_0, u^1_1) \\ (u^c, \partial_t u^c)(t=0) = (u^c_0, u^c_1) \end{cases}$$

where u^1, u^c, are real functions of $(t,x) \in \mathbb{R} \times \mathbb{R}^3$, we denoted

$$\square \overset{def}{=} \partial_t^2 - \Delta \quad \text{and} \quad \square_c \overset{def}{=} \partial_t^2 - c^2\Delta,$$

and we make the assumption that Q^1 and Q^c vanish quadratically

$$Q^1(u,v), Q^c(u,v) = O(|u|^2 + |v|^2).$$

The data $u^1_0, u^1_1, u^c_0, u^c_1$ will be chosen small, smooth and localized in space.

This equation models the nonlinear interaction of two types of Klein-Gordon waves, one propagating fast, the other slowly. If c is very large with respect to 1, it is for instance a toy model for the Euler-Maxwell equation describing plasmas; in that case, the fast waves would be electromagnetic, and the slow waves accoustic.

Another source of interest of this equation is mathematical: the space-time resonant structure of (1) has new features. Space-time resonances will be defined in the next section.

1. Space-time resonances

We discuss briefly, on a model problem, the concept of space-time resonance, introduced by Germain, Masmoudi and Shatah.

Transformation of the equation In order to present the idea of space-time resonance, consider a fairly general scalar quadratic nonlinear dispersive equation

$$\begin{cases} i\partial_t u + P(D)u = Q(u, \bar{u}) \\ u(t=0) = u_0, \end{cases}$$

where $P(D)$ is a real Fourier multiplier, and $Q(u,\bar{u})$ is either u^2, or $\bar{u}^2$ or $u\bar{u}$. Switching to the unknown function $f = e^{-itP(D)}u$ (the "profile"), and to the Fourier side, Duhamel's formula can be written

$$(2) \qquad \widehat{f}(t,\xi) = \widehat{u_0}(\xi) + \frac{1}{(2\pi)^{d/2}} \int_0^t \int e^{is\phi(\xi,\eta)} \widehat{f}(s,\eta)\widehat{f}(s,\xi-\eta)\, d\eta\, ds,$$

(where we denoted for simplicity indifferently $\widehat{f}$ for $\widehat{f}$ or $\widehat{\bar{f}}$) with

$$\phi(\xi,\eta) = P(\xi) \pm P(\eta) \pm P(\xi - \eta),$$

where the signs $--$, $++$ and $+-$ correspond respectively to Q being u^2, $\bar{u}^2$, and $u\bar{u}$.

The resonant sets Viewing the integral in (2) from the point of view of the stationary phase lemma, the critical sets are those where $s\phi$ is not oscillating in s, η, or even worse, both:

$$\mathcal{T} \overset{def}{=} \{(\xi,\eta) \text{ such that } \phi(\xi,\eta)=0\} \quad (\text{“time resonances”})$$
$$\mathcal{S} \overset{def}{=} \{(\xi,\eta) \text{ such that } \partial_\eta\phi(\xi,\eta)=0\} \quad (\text{“space resonances”})$$
$$\mathcal{R} \overset{def}{=} \mathcal{S}\cap\mathcal{T} \quad (\text{“space-time resonances”}).$$

The central idea is that the sets $\mathcal{T}$, $\mathcal{S}$, and to a greater extent $\mathcal{R}$, are the obstructions to a linear behaviour of u, for large time, and small data.

The method The method which we apply is straightforward: perform a (time-dependent) cut-off in the (ξ,η) space in order to distinguish three regions. Away from $\mathcal{T}$, integrate by parts in s (which amounts to a normal form transform). Away from $\mathcal{S}$, integrate by parts in η (this is similar to the vector fields method). There remains a neighbourhood of $\mathcal{R}$; it should shrink with t, and one has to take advantage of the smallness of this set.

1.0.1. *Application to our problem.* Computation of the resonances For the problem which is the subject of this talk (equation (1)), one needs to define several phase functions corresponding to all the possible interactions. They read

$$\phi^{k,\ell,m}_{\epsilon_0,\epsilon_1,\epsilon_2}(\xi,\eta) \overset{def}{=} \epsilon_0\langle\xi\rangle_k - \epsilon_1\langle\eta\rangle_\ell - \epsilon_2\langle\xi-\eta\rangle_m,$$

where k,l,m equal 1 or c and $\epsilon_0,\epsilon_1,\epsilon_2$ equal $+$ or $-$. The associated time, space, and space-time resonant sets are

$$\begin{aligned}
\mathcal{T}^{k,\ell,m}_{\epsilon_0,\epsilon_1,\epsilon_2} &\overset{def}{=} \{(\xi,\eta) \text{ such that } \phi^{k,\ell,m}_{\epsilon_0,\epsilon_1,\epsilon_2}=0\}\\
\mathcal{S}^{k,\ell,m}_{\epsilon_0,\epsilon_1,\epsilon_2} &\overset{def}{=} \{(\xi,\eta) \text{ such that } \partial_\eta\phi^{k,\ell,m}_{\epsilon_0,\epsilon_1,\epsilon_2}=0\}\\
\mathcal{R}^{k,\ell,m}_{\epsilon_0,\epsilon_1,\epsilon_2} &\overset{def}{=} \mathcal{T}^{k,\ell,m}_{\epsilon_0,\epsilon_1,\epsilon_2}\cap\mathcal{S}^{k,\ell,m}_{\epsilon_0,\epsilon_1,\epsilon_2}.
\end{aligned} \tag{3}$$

It turns out that space time resonances occur for some interactions. Then $\mathcal{R}^{k,\ell,m}_{\epsilon_0,\epsilon_1,\epsilon_2}$ has dimension 2 and is of the form $\{|\eta|=R\ ,\ \xi=\lambda\eta\}$ for real numbers R and λ. Furthermore (in general), $\partial_\xi\phi$ does not vanish on $\mathcal{R}$.

New difficulties and methods

We summarize below a few of the problems one faces when trying to apply the space-time resonance method to the problem studied here.

- One is led to excise the space-time resonant set $\mathcal{R}$ in order to be able to perform the above manipulations away from it. This is done with the help of bilinear cut-off functions which belong to the class of pseudo-products; but these bilinear cut-off functions must be chosen very steep. One would like to control the norms of the resulting operators, and since the singular set is not flat it turns out to be a difficult task.

- When performing the manipulations explained above, pseudo-products appear which are singular not only close to $\mathcal{R}$, but also at infinity, in the sense that they are not asymptotically homogeneous of degree 0 there. The idea is to treat the high frequencies by an argument independent of resonances (essentially, Strichartz estimates), which gives that u can be controlled in H^N for N very large.
- Separation of resonances: one is led to the following requirement: the frequences which are at the source of a space-time resonance should not be produced by it. This condition is generic in c.

Dispersive estimates for the wave equation in strictly convex domains with boundary

Oana Ivanovici
(joint work with Fabrice Planchon)

Consider the wave equation inside a domain Ω of dimension $d \geq 2$:

$$\begin{cases} (\partial_t^2 - \Delta_g)u(t,x) = 0, \quad x \in \Omega \\ u(0,x) = \delta_a, \ \partial_t u(0,x) = 0, \end{cases} \tag{1}$$

where $a \in \Omega$, δ_a is the Dirac function and Δ_g denotes the Laplace-Beltrami operator on Ω. In the case of a non empty boundary we consider the Diriclet condition $u|_{\partial\Omega} = 0$ on the boundary.

If Ω is the free space $\mathbb{R}^d$ with the Euclidian metric $g_{i,j} = \delta_{i,j}$ and if $u_{\mathbb{R}^d}(t,x)$ is the Green function (i.e. the solution to (1) in $\mathbb{R}^d$) then it is given by

$$u_{\mathbb{R}^d}(t,x) = \frac{1}{(2\pi)^d} \int \cos(t|\xi|) e^{i(x-a)\xi} d\xi$$

and it satisfies the classical dispersive estimates:

$$\|\psi(hD_t)u_{\mathbb{R}^d}(t,.)\|_{L^\infty(\mathbb{R}^d)} \leq C(d)h^{-d} \min\{1, (h/t)^{\frac{d-1}{2}}\}. \tag{2}$$

Here $\psi \in C_0^\infty$ is a smooth function supported outside a neighborhood of 0.

In this note we are interested in domains with boundary: the difficulties arise from the behavior of the singularities of the solutions to (1) near the points of $\partial\Omega$. In the case of a concave boundary, sharp dispersive estimates should follow using the Melrose and Taylor parametrix and the approach in [7]. In the opposite situation of a strictly convex domain, the presence of the gliding rays prevent the construction of such a parametrix.

Gilles Lebeau was the first who described in [6] the dispersive estimates on small time intervals for the solutions of (1) inside a strictly convex domain (Ω, g) of dimension $d \geq 2$. The result he had announced reads as follows:

Theorem 26. *If $a > 0$ is sufficiently small, then there exists $T > 0$, $C > 0$ so that for every $h \in (0,1]$ and $t \in (0,T]$ the solution u to (1) satisfies*

$$|\psi(hD_t)u(t,x)| \leq C(d)h^{-d} \min\{1, (h/t)^{\frac{d-2}{2}+\frac{1}{4}}\}. \tag{3}$$

Remark 2. The estimate (3) means that, compared to the dispersive estimate in the free space (2), there is a loss of a power of $\frac{1}{4}$ of $\frac{h}{t}$ inside a strictly convex domain, and this is due to micro-local phenomena such as caustics generated in arbitrarily small time near the boundary. This loss is optimal.

Remark 3. In [6] Gilles Lebeau sketched the main steps of the proof and gave a full description of the geometry behind. However, many details are missing and therefore, our forthcoming work [5] in collaboration with Fabrice Planchon is intended to complete the analytical part of Gilles Lebeau's result.

Remark 4. The loss of $\frac{1}{4}$ comes only from the dispersion in the normal variable, therefore it will be enough to prove the result in dimension $d = 2$ only.

Remark 5. Theorem 26 allows to prove sharp results in dimension $d \geq 2$ for the spectral projector estimates generalizing the work [8] in dimensions $d \geq 3$ in the case of convex domains. It also gives the sharp range of indices for which optimal Strichartz estimates hold (this is a work in progress, in collaboration with Fabrice Planchon); moreover, using (3) we can prove that the counterexamples constructed in [3, 4] are optimal.

Proof. Before starting the proof, the first thing to understand is the type of concentration phenomena such as *caustics* that may occur near the boundary.

What are caustics? Caustics are envelopes of light rays that appear in a given problem. At the caustic point the intensity of light is singularly large, causing different physical phenomena. Mathematically, caustics could be characterized as points were usual bounds on oscillatory integrals are no longer valid. It is well known that the asymptotic behavior of an oscillatory integral is governed by the number and the order of their critical points which are real. Let

$$(4) \qquad u_h(z) = \frac{1}{(2\pi h)^{1/2}} \int_\zeta e^{\frac{i}{h}\Phi(z,\zeta)} g(z,\zeta,h)d\zeta, \quad z \in \mathbb{R}^d, \quad \zeta \in \mathbb{R}, \quad h \in (0,1].$$

If there are degenerate critical points, known as caustics, then $\|u_h(z)\|_{L^\infty}$ is no longer uniformly bounded. The order of a caustic κ is defined as the infimum of κ' so that $\|u_h(z)\|_{L^\infty} = O(h^{-\kappa'})$. For example, recall that in [3] we considered phase functions of the form $\Phi_F(z,\zeta) = \frac{\zeta^3}{3} + z_1\zeta + z_2$ and obtained a loss in the Strichartz estimates of $\frac{1}{6}$ derivatives. This phase corresponds to a fold and has order precisely $\kappa = \frac{1}{6}$. In the proof of Theorem 26 a crucial role will be played by the Pearcey type integrals, with phase function of the form $\Phi_C(z,\zeta) = \frac{\zeta^4}{4} + z_1\frac{\zeta^2}{2} + z_2\zeta + z_3$ and order $\kappa = \frac{1}{4}$. They correspond to a cusp type singularity; the swallowtail canonical form involves the phase $\Phi_S(z,\zeta) = \frac{\zeta^5}{5} + z_1\frac{\zeta^3}{3} + z_2\frac{\zeta^2}{2} + z_3\zeta + z_4$, with order $k = \frac{3}{10}$.

Let $\Omega = \{(x,y) \in \mathbb{R}^2, x > 0\}$ and $\Delta_g = \partial_x^2 + (1+x)\partial_y^2$ define a strictly convex domain in $\mathbb{R}^2$. A first step in the proof of Theorem 26 consists in a detailed description of the set of points of Ω which can be reached following all the optical rays starting from a of length t. We split the data in packets in such a way that each packet corresponds to a number of reflections on the boundary for a fixed time T. At high frequency $\frac{1}{h}$, the "worst" packets will be those for which $a \simeq h^{1/2}$ and which propagate along directions parallel to $\partial\Omega$. These localized data will involve "swallowtail" type singularities in the wave front set of the solution. Hence it will be sufficient to prove the estimates (3) for the following initial data:

$$u_0(x,y) = \frac{1}{(2\pi h)^2}\int e^{\frac{i}{h}((x-a)\xi+y\eta)}\psi(\eta)\rho(\frac{\xi}{h^{1/4}\eta})d\xi d\eta,$$

where ψ, ρ are smooth functions compactly supported in a neighborhood of 1 and 0, respectively, $\psi \in C_0^\infty(\frac{1}{2}, 2)$, $\rho \in C_0^\infty(-\frac{1}{2}, \frac{1}{2})$. If the initial distance a to the boundary is small, namely $a \leq h^{\frac{1}{2}}$, we use the fact that the essential support of the Fourier transform of u remains small, together with the elementary estimate [6][(2.24)]. For $a > h^{1/2}$ we construct a parametrix u for small time t between 0 and the moment the wave reaches the boundary the first time; we then solve the Airy equation with this data on the boundary. We repeat this construction a number of times $N \simeq \frac{1}{\sqrt{a}}$. We obtain a parametrix of the form

$$U_h(t,x,y) = \sum_{n=0}^{N} u_n(t,x,y),$$

$$u_n(t,x,y) = \int e^{\frac{i}{h}\eta\phi_n(t,x,y,\xi)} g_h^n(x,y,t,\xi)\psi(\eta)\rho(h^{-1/4}(\frac{a}{\xi} - \frac{\xi}{4}))d\xi d\eta.$$

The symbols g_h^n are chosen so that u_n to have almost orthogonal supports in time and so that the Dirichlet condition to be satisfied. We study the asymptotic behavior of the parametrices u_n. We otain the the equivalent of [6][Lemma 3.7]:

Theorem 27. *For every $n \in \{1,..,N\}$, the phase ϕ_n has saddle points of order at most 3; for each $n \in \{1,..,N\}$ there exists a unique time $t = t_{S,n}$ for which $\phi_n(t)$ has a critical point ξ_S of order 3.*

From the above Lemma it follows, using Arnold's classification, that ϕ_n is a Pearcey type integral with order $\frac{1}{4}$. Writing the asymptotic expansion of $u_n(t)$ near $t_{S,n}$, we deduce that a loss of $\frac{1}{4}$ powers of $\frac{|t|}{h}$ is unavoidable for $\|u_n\|_{L^\infty}$.

Theorem 28. *The loss of $\frac{1}{4}$ powers of $\frac{|t|}{h}$ in the dispersive estimates (3) is optimal in any dimension $d \geq 2$.*

The optimality follows from the fact that there is a swallowtail type singularity in the wavefront set $WF_h(u_n)$ for each $n \in \{1,..,N\}$. Then use Remark 4. □

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Cauchy Problem for higher order KdV and mKdV equations

Axel Grünrock

In the talk some recently published results on local and global well-posedness of the Cauchy Problem for higher order KdV and mKdV equation were presented, cf. [2]. In the first part the equations were described briefly:

The KdV hierarchy introduced by Lax in 1968 [7] can most easily be written down as:

$$\partial_t u + \partial_x G_j(u) = 0 \tag{1}$$

with G_j being the gradients of the Hamiltonians

$$H_j(u) = \int P_j(u, \partial_x u. \dots, \partial_x^j u) dx \tag{2}$$

of KdV, which is the first equation in this hierarchy. The polynomial conserved densities P_j are usually ordered in such a way that $(\partial_x^j u)^2$ is the highest derivative term and that the rank (= degree $+\frac{1}{2}$ derivative index) equals $j+2$ in each monomial contained in P_j. The first condition leads to an increasingly dispersive linear part of the equation with $2j+1$ derivatives in the x-variable, while the rank condition implies a scale invariance leading to a joint critical regularity ($s = -\frac{3}{2}$ on the H^s-scale) for all equations in the hierarchy. Moreover, all theses equations obey the same conservation laws as KdV itself, which give a priori control over each integer Sobolev norm ($H^s, s \in \mathbb{N}_0$) of real valued solutions of any of these equations. A similar hierarchy of higher order mKdV equations is connected to the KdV hierarchy by the Minra transform. More details are given in the introduction of [2], which is based on the expositions of [8] and, for the mKdV part, [1], [9]. The Cauchy Problem for the equations in both hierarchies has attracted much interest during the last decades, we refer to [12], [11], [4], [5], [10].

The Cauchy Problem for (1) and its modified counterpart are considered here for data in the generalized Sobolev spaces $\widehat{H^r_s}$, which are defined by their norm

$$\|f\|_{\widehat{H^r_s}} := \left\| \langle \xi \rangle^s \hat{f} \right\|_{L^{r'}_\xi}, \quad \frac{1}{r} + \frac{1}{r'} = 1.$$

Local well-posedness results for s as small as possible depending on $r \in (1,2]$ are obtained by using the contraction mapping principle in generalized Bourgain spaces $X^{r,p}_{s,b}$ with norm

$$\|u\|_{X^{r,p}_{s,b}} = \left\| \langle \tau - \varphi(\xi) \rangle^b \langle \xi \rangle^s \hat{u} \right\|_{L^{r'}_\xi L^{p'}_\tau},$$

where the phase function φ in our case is given by $\varphi(\xi) = \xi^{2j+1}$. Multilinear smoothing estimates for free solutions of the linearized equations are the main tool in the derivation of the crucial estimates of the nonlinearities, the resonance relation plays a major role only for the quadratic terms appearing in the KdV hierarchy.

An extensive result can be achieved for the mKdV hierarchy.

Theorem 29. *The Cauchy problem for the higher order mKdV equation of order j is locally well-posed in $\widehat{H^r_s}(\mathbb{R})$ for $1 < r \leq 2$ and $s \geq \frac{2j-1}{2r'}$.*

Discussion:

- *Sharpness:* The lower bound on s in this theorem is sharp. In fact for each j there is a complex version of the equation, which is ill-posed in the C^0-uniform sense for lower values of s.
- *H^s-theory:* The theorem contains the earlier $H^{\frac{1}{4}}$-result of Kenig-Ponce-Vega on mKdV itself (cf. [3]) as well as the more recent $H^{\frac{3}{4}}$-result of Kwon for $j = 2$ [6] and gives the generalization of these to all higher equations in the hierarchy. Combining this with the conservation laws, we obtain global well-posedness of the Cauchy Problem for the j-th higher order mKdV equation in $H^s(\mathbb{R})$ for $s \geq [\frac{j+1}{2}]$, which is a new result for $j \geq 3$.
- *Improvement of the local theory for $r < 2$:* While on the H^s-scale there is an increasing gap of $\frac{2j+1}{4}$ derivatives between the best possible local well-posedness result and the scaling prediction, this gap can be closed almost completely by considering data in $\widehat{H^r_s}(\mathbb{R})$ with r closed to one. The lower bounds on s all meet at $\widehat{H^1_0}(\mathbb{R})$, which becomes a joint critical space for all equations in the mKdV hierarchy. Unfortunately local well-posedness in this space is out of reach of our arguments.

Secondly, for the higher order KdV equations, the following theorem is shown.

Theorem 30. *The Cauchy Problem for the higher order KdV equation of order $j \geq 2$ is locally well-posed in $\widehat{H^r_s}(\mathbb{R})$, provided $1 < r \leq \frac{2j}{2j-1}$ and $s > j - \frac{3}{2} - \frac{1}{2j} + \frac{2j-1}{2r'}$.*

The lower bounds on s in this theorem are far away from any critical space. The main point here is, that we can at all obtain a result for these equations by the contraction mapping principle, since by an argument of Pilod [Pi08] there is C^2-illposedness in $H^s(\mathbb{R})$ for all $s \in \mathbb{R}$, if $j \geq 2$. A slight modification shows C^2-illposedness as well in $\widehat{H^r_s}(\mathbb{R})$ for any s, if $r > \frac{2j}{2j-1}$.

Finally, when applied to the KdV equation itself, our methods achieve the following slight improvement of the existing local theory.

Theorem 31. *The Cauchy Problem for the KdV equation is locally well-posed in* $\widehat{H^r_s}(\mathbb{R})$ *if* $1 < r < 2$ *and* $s > \max(-\frac{1}{2} - \frac{1}{2r'}, -\frac{1}{4} - \frac{11}{8r'})$.

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Global existence for the Maxwell-Dirac system in two space dimensions

Sigmund Selberg

(joint work with Piero D'Ancona)

The Maxwell-Dirac system (MD) describes the motion of an electron interacting with an electromagnetic field. Expressing Maxwell's equations in terms of a real potential A, and imposing the Lorenz gauge condition

$$\partial^\mu A_\mu = 0 \qquad (\iff \partial_t A_0 = \nabla \cdot \mathbf{A}),$$

the MD system reads

$$\text{(1)} \qquad \begin{cases} (-i\boldsymbol{\alpha}^\mu \partial_\mu + M\boldsymbol{\beta})\,\psi = A_\mu \boldsymbol{\alpha}^\mu \psi, \\ \qquad\qquad \square A_\mu = -\langle\, \boldsymbol{\alpha}_\mu \psi, \psi \,\rangle, \end{cases}$$

where $\psi \colon \mathbb{R}^{1+d} \to \mathbb{C}^{N=N(d)}$ is the Dirac spinor, $M \in \mathbb{R}$ is a constant, $\square = \partial_\mu \partial^\mu = -\partial_t^2 + \Delta_x$ is the D'Alembertian, the $\boldsymbol{\alpha}^\mu$'s and $\boldsymbol{\beta}$ are Dirac matrices, and $\langle\, \cdot, \cdot \,\rangle$ is the standard $\mathbb{C}^N$ inner product. In the 2d case (space dimension $d = 2$), which we are interested in here, the dimension of the spin space is $N = 2$, and for the Dirac matrices one can take the representation $\boldsymbol{\alpha}^0 = \mathbf{I}_{2\times 2}$, $\boldsymbol{\alpha}^1 = \boldsymbol{\sigma}^1$, $\boldsymbol{\alpha}^2 = \boldsymbol{\sigma}^2$ and $\boldsymbol{\beta} = \boldsymbol{\sigma}^3$, where the $\boldsymbol{\sigma}^j$ are the Pauli matrices.

Recently, there has been a lot of progress in the regularity theory for MD and the simpler Dirac-Klein-Gordon system (DKG),

$$\text{(2)} \qquad \begin{cases} (-i\boldsymbol{\alpha}^\mu \partial_\mu + M\boldsymbol{\beta})\,\psi = \phi \boldsymbol{\beta} \psi, \\ \qquad (-\square + m^2)\phi = \langle\, \boldsymbol{\beta}\psi, \psi \,\rangle, \end{cases}$$

where ϕ is real-valued and $m \in \mathbb{R}$ is a constant.

A key question for both systems is whether *global regularity* holds, i.e. starting from smooth initial data, does the solution exist for all time and stay smooth? For small data this has been answered affirmatively by Georgiev [7] in 3d, but for large data there was until quite recently only the 1d result of Chadam [5].

To make progress on the large data question in 2d and 3d, a natural strategy is to study local (in time) well-posedness for rough data and exploit conservation laws to extend the solutions globally. For both DKG and MD there is a conserved energy, but as this lacks a definite sign, the only conserved quantity that appears to be immediately useful is the charge:

$$\int |\psi(t,x)|^2 \, dx = \text{const.}$$

It should be noted that the initial value problems for MD and DKG are charge critical in 3d, and charge subcritical in 2d and 1d (i.e. the scale invariant data regularity is below the charge).

Decisive improvements in the local theory were made possible through the discovery of the complete null structure of DKG and MD in [1] and [3]. In particular, for 2d DKG, local well-posedness below the charge norm was established in [2], which allows to use the charge conservation. This is not enough to get global well-posedness, however, since there is no conservation law for the scalar field ϕ. This difficulty was overcome by Grünrock and Pecher, who proved global well-posedness of 2d DKG in the charge class [8]. The lack of a conservation law for ϕ was compensated for by proving a very precise local existence theorem and applying an iteration idea due to Colliander, Holmer and Tzirakis [6].

In this talk I present recent joint work with Piero D'Ancona [4], where we extend the result of Grünrock and Pecher to the full MD system in 2d. This introduces additional difficulties since MD has a much more complicated structure than DKG. The scheme of Colliander, Holmer and Tzirakis is not directly applicable due to a logarithmic loss in the local theory, and due to the fact that our data norm depends implicitly on the local existence time itself.

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Long time existence for nonlinear wave equations in exterior domains

Jason Metcalfe

These results focus on long-time existence for nonlinear wave equations with small initial data in exterior domains. In particular, the focus is on nonlinear equations where dependence on the solution u, rather than just its first and second derivatives, is permitted at the lowest order.

Two primary problems are studied. In both cases, we shall fix a bounded obstacle $\mathcal{K} \subset \mathbb{R}^n$ with smooth boundary and examine the wave equation exterior to it.

The first problem is an analog of the Strauss conjecture and regards equations of the form

$$\begin{cases} \Box u := (\partial_t^2 - \Delta)u = |u|^p, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \backslash \mathcal{K}, \\ (Bu)|_{\partial\mathcal{K}} = 0, \\ u(0,\,\cdot\,) = f, \quad \partial_t u(0,\,\cdot\,) = g. \end{cases} \tag{1}$$

The obstacle is assumed to be nontrapping, and B is either the identity operator or the normal derivative. The Cauchy data are assumed to satisfy the relevant compatibility condition. The question concerns the values of p for which there is global existence if the data are taken sufficiently small.

In the absence of a boundary, the question of existence was resolved in [4], [13], where it is shown that $p > p_c$, where p_c is the larger root of $(n-1)p_c^2-(n+1)p_c-2 =$

0, guarantees the existence of global solutions. A more thorough history of the problem can also be found therein. The first progress on this problem in exterior domains has come in [2] ($n = 4$), which is a joint work with Du, Sogge, and Zhou, and [5] ($n = 3, 4$), which is a collaboration with Hidano, Smith, Sogge, and Zhou. The key tool is a class of weighted Strichartz estimates

$$\left\| |x|^{\frac{n}{2}-\frac{n+1}{p}-\gamma} u \right\|_{L^p_{t,r} L^2_\omega} \lesssim \|u'(0,\,\cdot\,)\|_{\dot{H}^{\gamma-1}} + \left\| |x|^{-\frac{n}{2}+1-\gamma} \Box u \right\|_{L^1_{t,r} L^2_\omega}, \tag{2}$$

$$2 \le p \le \infty, \quad \frac{1}{2} - \frac{1}{p} < \gamma < \frac{1}{2}, \tag{3}$$

which are obtained by interpolating between a trace lemma and a localized energy estimate which is proved using Plancherel's theorem. The index $\gamma = \frac{n}{2} - \frac{2}{p-1}$ corresponds precisely to $p_c < p < \frac{n+3}{n-1}$. Moreover, for this γ, we have

$$\left\| |x|^{-\frac{n}{2}+1-\gamma} |u|^p \right\|_{L^1} = \left\| |x|^{\frac{n}{2}-\frac{n+1}{p}-\gamma} u \right\|^p_{L^p}.$$

Thus, provided the nonlinearity is sufficiently regular to allow for the Sobolev embeddings which are needed in the angular variables, which corresponds to $n \le 4$, an iteration can be closed to show small data global existence. In order to obtain analogs of (2) when there is a boundary, arguments akin to those of [12], [1], and [9] may be adapted to permit the weighted spaces.

The second class of nonlinear problems which are examined concern quasilinear wave equations with nonlinearities vanishing to second order

$$\begin{cases} \Box u = Q(u, u', u''), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \backslash \mathcal{K}, \\ u|_{\partial\mathcal{K}} = 0, \\ u(0,\,\cdot\,) = f, \quad \partial_t u(0,\,\cdot\,) = g. \end{cases} \tag{4}$$

The novelty here is nonlinearity's dependence on the solution u at the quadratic level, rather than just on u' and u''. For data of size ε, the boundaryless studies [8], [6] establish the goal of showing a c/ε^2 ($n = 3$) and $\exp(c/\varepsilon)$ ($n = 4$) lower bound on the lifespan respectively.

For star-shaped obstacles $\mathcal{K}$ and data satisfying the compatibility conditions, the three dimensional lifespan was proved in [3] and in four dimensions, in the author's joint work [2] with Du, Sogge, and Zhou. The key estimate of [2] is a localized energy estimate that corresponds to the $p = 2$, $\gamma = 0$ endpoint of (2), for which there is a logarithmic blow-up in the length of the time-interval T.

This estimate is combined with a localized energy estimate for perturbations of the d'Alembertian from [10], which holds exterior to star-shaped obstacles. One then iterates in the fashion developed in [7] which uses localized energy estimates to permit one to show long-time existence based on decay in $|x|$ rather than decay in t.

If the additional hypothesis that $(\partial_u^2 Q)(0, 0, 0) = 0$ is imposed on the nonlinearity, which disallows a u^2 term, then a longer lifespan is expected. On $\mathbb{R}^4$, [6] showed that such equations have global solutions for sufficiently small data, and exterior to a star-shaped obstacle, an analogous result was proved by the author and Sogge [11]. Here, one utilizes the $p = 2$ version of (2) with $\gamma = 2\delta$. This is combined with an estimate for equations with forcing terms in divergence form, which says that if $\Box u = \sum_{j=0}^{4} a_j \partial_j G$ with vanishing initial data, then

$$\|\langle x\rangle^{-1/2-\delta} u\|_{L^2_{t,x}} \lesssim \|G(0,\,\cdot\,)\|_{\dot H^{\gamma-1}} + \int_0^T \|G(t,\,\cdot\,)\|_2\, dt \tag{5}$$

provided $0 < \delta < 1/2$. The proof uses techniques which were also employed in [6], [8], but rather than applying them to the energy inequality, they are instead applied to a localized energy estimate

$$\|\langle x\rangle^{-1/2-\delta} u'\|_{L^2_{t,x}} \lesssim \|u'(0,\,\cdot\,)\|_2 + \int_0^T \|\Box u(t,\,\cdot\,)\|_2\, dt. \tag{6}$$

The method of iteration can be illustrated by studying the boundaryless semilinear equation

$$\Box u = u\partial_t u + (\partial_t u)^2 = \frac{1}{2}\partial_t(u^2) + (\partial_t u)^2$$

for smooth initial data of size ε. We set

$$M(T) = \sum_{|\alpha|\le 10} \Big(\|\langle x\rangle^{-1/2-\delta}(Z^\alpha u)'\|_{L^2_{t,x}} + \|\langle x\rangle^{-1/2-2\delta} Z^\alpha u\|_{L^2_{t,x}} \Big),$$

where Z denotes the set of vector fields $\{\partial_k, x_i\partial_j - x_j\partial_i\}$ where $0 \le k \le 4$ and $1 \le i < j \le 4$. Utilizing that $[Z, \Box] = 0$, we may apply (2), (5), and (6) to see that

$$M(T) \lesssim \varepsilon + \sum_{|\alpha|+|\beta|\le 10} \||x|^{-1-2\delta}(Z^\alpha u)'(Z^\beta u)'\|_{L^1_{t,r}L^2_\omega} + \sum_{\substack{|\alpha|+|\beta|\le 10\\ |\mu|,|\nu|\le 1}} \int_0^T \|\partial^\mu Z^\alpha u \partial^\nu Z^\beta u\|_2\, dt.$$

The key is to notice that only quadratic terms involving u' rather than just u appear in the second term in the right. An application of Sobolev embedding on the sphere and the Schwarz inequality to the second term in the right and an application of a standard weighted Sobolev inequality (see [7] and the references therein) which provides $O(\langle x\rangle^{-(n-1)/2})$ decay to the third term in the right shows that $M(T) \lesssim \varepsilon + (M(T))^2$, from which it is easy to construct an iteration to show global existence.

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The Hamiltonian structure of the nonlinear Schrödinger equation and the asymptotic stability of its ground states

Scipio Cuccagna

In this talk we consider nonlinear Schrödinger equations $\mathrm{i}u_t = -\Delta u + Vu + \beta(|u|^2)u$ with $V(x)$ and β smooth, with $x \in \mathbb{R}^3$ and with appropriate growth conditions at infinity for β. We assume the existence of a smooth family of ground states $\phi_\omega(x)$ for ω in an open subset of $\mathbb{R}^+$. We discuss how the necessary conditions for orbital stability introduced by M.Weinstein in [1] imply asymptotic stability for generic equations. It is known that in $H^1(\mathbb{R}^3)$, near the manifold of ground states, solutions of the NLS can be expressed canonically as $u(t,x) = e^{\mathrm{i}\vartheta(t)}(\phi_{\omega(t)}(x) + r(t,x))$, with $\vartheta \in \mathbb{R}$. This yields

$$\begin{aligned}\mathrm{i}r_t = &-\Delta r + Vr + \omega(t)r + \beta(\phi^2_{\omega(t)})r + \beta'(\phi^2_{\omega(t)})\phi^2_{\omega(t)}r + \beta'(\phi^2_{\omega(t)})\phi^2_{\omega(t)}\overline{r} \\ &+ (\dot\vartheta(t) - \omega(t))(\phi_{\omega(t)} + r) - \mathrm{i}\dot\omega(t)\partial_\omega\phi_{\omega(t)} + O(r^2)\end{aligned}$$

and modulation equations $\dot\omega = O(r^2)$, $\dot\vartheta - \omega = O(r^2)$. It is common practice to write, for R the transpose of $(r, \overline{r})$ and Φ_ω the transpose of $(\phi_\omega, \phi_\omega)$, to rewrite the equation for r as

$$\mathrm{i}R_t = \mathcal{H}_{\omega(t)}R + \sigma_3(\dot\vartheta - \omega)(\Phi_{\omega(t)} + R) - \mathrm{i}\dot\omega\partial_\omega\Phi_{\omega(t)} + O(R^2)$$

where

$$\mathcal{H}_\omega = \begin{pmatrix} -\Delta + V + \omega + \beta(\phi^2_\omega) + \beta'(\phi^2_\omega)\phi^2_\omega & \beta'(\phi^2_\omega)\phi^2_\omega \\ -\beta'(\phi^2_\omega)\phi^2_\omega & \Delta - V - \omega - \beta(\phi^2_\omega) - \beta'(\phi^2_\omega)\phi^2_\omega \end{pmatrix}.$$

Weinstein's conditions for orbital stability (i.e. $\partial_\omega \|\phi_\omega\|_{L^2} > 0$, $\ker(L_+) = 0$ and number of negative eigenvalues of $L_+ = 1$, for $L_+ = -\Delta + V + \omega + \beta(\phi_\omega^2) + 2\beta'(\phi_\omega^2)\phi_\omega^2$), imply $\sigma(\mathcal{H}_\omega) \subset \mathbb{R}$ and the existence of a spectral decomposition

$$L^2 = N_g(\mathcal{H}_\omega) \oplus \sum_{\lambda \in \sigma_p \setminus \{0\}} \ker(\mathcal{H}_\omega - \lambda) \oplus L_c^2(\mathcal{H}_\omega)$$

$$N_g(\mathcal{H}_\omega) = \{\sigma_3 \Phi_\omega, \partial_\omega \Phi_\omega\} \text{ and } L_c^2(\mathcal{H}_\omega) := \{N_g(\mathcal{H}_\omega^*) \oplus \sum_{\lambda \in \sigma_p \setminus \{0\}} \ker(\mathcal{H}_\omega^* - \lambda)\}^\perp.$$

Under generic conditions, the restriction of $e^{\mathrm{i}t\mathcal{H}_\omega}$ on the continuous component $L_c^2(\mathcal{H}_\omega)$ satisfies the same dispersive and Strichartz estimates of $e^{\mathrm{i}t\Delta}$. Given $\mathrm{i}\dot{R} = \mathcal{H}_{\omega(t)} R + ...$ and if $0 < \lambda_1(\omega) \le \lambda_2(\omega) \le .... \le \lambda_n(\omega) < \omega$ are the positive eigenvalues of $\mathcal{H}_\omega$, we consider

$$L^2 = N_g(\mathcal{H}_{\omega(t)}) \oplus \sum_{j=1}^n \big(\ker(\mathcal{H}_{\omega(t)} - \lambda_j(\omega(t))) \oplus \ker(\mathcal{H}_{\omega(t)} + \lambda_j(\omega(t))\big) \oplus L_c^2(\mathcal{H}_{\omega(t)})$$

and correspondingly we can split (by modulation $R(t)$ has 0 component in $N_g(\mathcal{H}_{\omega(t)})$)

$$R(t,x) = \sum_{j=1}^n \left(z_j(t)\xi_j(x,\omega(t)) + \overline{z}_j(t)\sigma_1\xi_j(x,\omega(t))\right) + f(t,x),$$

for an appropriately normalized basis of eigenvectors $\{\xi_j(x,\omega)\}$ and for $f(t,x)$ the continuous component. One is naturally led to a system of the following form:

$$\begin{aligned} \mathrm{i}\dot{z}_j(t) - \lambda_j(\omega(t)) z_j(t) &= \sum_{\mu\nu} a_{\mu\nu}^{(j)}(\omega(t)) z^\mu(t) \overline{z}^\nu(t) \\ &\qquad + \sum_{\mu\nu} \nu_j \frac{z^\mu \overline{z}^\nu}{\overline{z}_j} \langle G_{\mu\nu}^{(j)}(x,\omega(t)), f(t,x)\rangle_{L_x^2} + \cdots \\ \mathrm{i}\dot{f} - \mathcal{H}_{\omega(t)} f &= \sum_{\mu\nu} z^\mu(t) \overline{z}^\nu(t) M_{\mu\nu}(x,\omega(t)) + \cdots . \end{aligned}$$

This system has been considered by various authors, [2]–[8] among others. To prove asymptotic stability one needs to prove that $\lim_{t\to\infty} z_j(t) = 0$ and that f scatters asymptotically. It turns out that the $z_j(t)$ lose energy because of nonlinear coupling with the f (a well known fact in the literature) and that the key feature needed to prove this fact is the hamiltonian structure of the NLS (this is the key new insight in [7, 8]). We fix $\omega_0 \in \mathcal{O}$ s.t. $\|\phi_{\omega_0}\|_2^2 = \|u_0\|_2^2$ with u_0 the initial datum. Then $(\vartheta, \omega, z_1, ..., z_n, f)$ with $f \in L_c^2(\mathcal{H}_{\omega_0})$ is a natural system of coordinates s.t.

$$R = \sum_{j=1}^n \left(z_j \xi_j(\omega) + \overline{z}_j \sigma_1 \xi_j(\omega)\right) + P_c(\mathcal{H}_\omega) f\,.$$

Then, for E the energy and Q the charge, and if we set $K = E + \omega Q - \omega \|u_0\|_{L_x^2}^2$, in order to follow the evolution of the key coordinates, (ω, z, f), we can consider the system

$$\dot\omega = \{\omega, K\}\,, \quad \dot f = \{f, K\}\,, \quad \dot z_j = \{z_j, K\}\,, \quad \dot\vartheta = \{\vartheta, K\}.$$

Applying carefully the Darboux Theorem we find a new system of coordinates where the system is of the following form and is still semilinear:

$$q'\dot\omega = \frac{\partial H}{\partial\vartheta} \equiv 0\,, \quad q'\dot\vartheta = -\frac{\partial H}{\partial\omega}\,, \quad \mathrm{i}\dot z_j = \frac{\partial H}{\partial \overline{z}_j}\,, \quad \mathrm{i}\dot f = \sigma_3\sigma_1\nabla_f H.$$

In particular we have reduced the variables since it is enough to focus on the last two equations. We simplify the system by means of canonical changes of variables s.t. the equation for f looks like $\mathrm{i}\dot f - \mathcal{H}f = \sum_{|\lambda(\omega_0)\cdot(\mu-\nu)|>\omega_0} z^\mu \overline{z}^\nu G_{\mu\nu} + \ldots$. We expand $f = -\sum_{|\lambda(\omega_0)\cdot(\mu-\nu)|>\omega_0} z^\mu\overline{z}^\nu R^+_{\mathcal{H}}(\lambda(\omega_0)\cdot(\mu-\nu))G_{\mu\nu} + \cdots$. Substitute in the equation of z_j (here $H_2 = \sum_{j=1}^n \lambda_j|z_j|^2 + \frac{1}{2}\langle\sigma_3\mathcal{H}_{\omega_0}f, \sigma_1 f\rangle$ and Z_0 is such that $\{Z_0, H_2\} = 0$), which is of the form

$$\mathrm{i}\dot z_j = \partial_{\overline{z}_j}(H_2 + Z_0) + \sum_{|\lambda(\omega_0)\cdot(\mu-\nu)|>\omega_0} \nu_j \frac{z^\mu\overline{z}^\nu}{\overline{z}_j}\langle f, \sigma_1\sigma_3 G_{\mu\nu}\rangle + \cdots$$

to get

$$\begin{aligned}\mathrm{i}\dot z_j &= \partial_{\overline{z}_j} H_2(z,f) + \partial_{\overline{z}_j} Z_0(z,f)\\ &\quad - \sum_{\lambda(\omega_0)\cdot\alpha=\lambda(\omega_0)\cdot\nu>\omega_0} \nu_j \frac{z^\alpha\overline{z}^\nu}{\overline{z}_j}\langle R^+_{\mathcal{H}}(\alpha\cdot\lambda(\omega_0))G_{\alpha 0}, \sigma_1\sigma_3 G_{0\nu}\rangle.\end{aligned}$$

Multiplying by $\lambda_j\overline{z}_j$, summing on j and taking imaginary part, we get

$$\begin{aligned}\partial_t \sum_{j=1}^n \lambda_j|z_j|^2 &= -2\sum_{\varrho>\omega_0} \varrho\,\mathrm{Im}\left\langle R^+_{\mathcal{H}}(\varrho)\sum_{\lambda\cdot\alpha=\varrho} z^\alpha G_{\alpha 0}, \sigma_3\overline{\sum_{\lambda\cdot\alpha=\varrho} z^\alpha G_{\alpha 0}}\right\rangle\\ &= -2\pi\left\langle \delta(-\Delta+\omega-\varrho)\sum_{\lambda\cdot\alpha=\varrho} z^\alpha F_\alpha, \overline{\sum_{\lambda\cdot\alpha=\varrho} z^\alpha F_\alpha}\right\rangle \le 0\end{aligned}$$

for some F_α's. Generically, for some $\Gamma > 0$, this yields

$$\partial_t \sum_{j=1}^n \lambda_j|z_j|^2 + \Gamma\sum_{\lambda\cdot\mu>\omega_0} |z^\mu|^2 \le 0.$$

Integrating we get

$$\begin{aligned}&\sum_{j=1}^n \lambda_j|z_j(t)|^2 + \Gamma\sum_{\lambda\cdot\mu>\omega_0}\int_0^t |z^\mu(s)|^2 ds \le \sum_{j=1}^n \lambda_j|z_j(0)|^2 \Rightarrow\\ &\|z^\mu\|_{L^2_t} \le \epsilon \text{ for } \lambda\cdot\mu>\omega_0 \Rightarrow \lim_{t\to+\infty} z(t) = 0.\end{aligned}$$

All of this is done in detail in [8] where we give a template on how to treat the interaction between discrete modes and radiation in hamiltonian systems.

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On the uniqueness of stationary black holes in vacuum

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I discussed some joint work in collaboration with S. Klainerman [3] and [4], and S. Alexakis and S. Klainerman [1] and [2], on the long-standing problem of uniqueness properties of the Kerr family. The Kerr spaces are an explicit family of vacuum space-times in General Relativity which contain a black hole. They are, in fact, the only known explicit solutions that model rotating black holes in vacuum. They depend on two parameters: m (the mass of the black hole) and J (the angular momentum of the black hole). We assume $m > 0$ and $a = J/m \in [0, m)$. The Kerr space of mass m and angular momentum J is asymptotically flat and *stationary*, i.e. it admits a Killing vector field $\mathbf{T}$ which is timelike in the asymptotic region.

A fundamental conjecture in General Relativity (the "no-hair" conjecture) asserts that the domain of outer communication of *any* regular, stationary, four dimensional, vacuum black hole is isometrically diffeomorphic to the domain of outer communication of a Kerr black hole. One expects, due to gravitational radiation, that general, asymptotically flat solutions of the Einstein-vacuum equations settle down, asymptotically, into a stationary regime. Thus the conjecture, if true, would characterize all possible asymptotic states of the general evolution.

So far the conjecture has been resolved, by combining results of S. W. Hawking, B. Carter, and D. C. Robinson, under the additional hypothesis of non-degenerate event horizons and *real analyticity* of the space-time. Our main goal is to remove the real analyticity assumption, which is a key weakness of the current no-hair theorems.

In [4], based on a characterization of the Kerr solution by the vanishing of a covariant complex valued tensor $\mathcal{S}$ (called the Mars–Simon tensor), we were able to remove the analyticity assumption by replacing it with a complex scalar condition to be satisfied on the bifurcation sphere of the horizon. The main idea of the proof was to derive a covariant wave equation for the Mars–Simon tensor $\mathcal{S}$, show that $\mathcal{S}$ vanishes on the bifurcate event horizon of the stationary solution (at this stage we use the identity we assume on the bifurcation sphere), and then use Carleman estimates to deduce that $\mathcal{S}$ must vanish in the entire domain of outer communications. This paper appears to be the first time Carleman estimates and unique continuation results are used in General Relativity.

This approach was further explored in [1] and [2]. The main result in [1] is the first proof, in the class of smooth manifolds, that a stationary black hole solution must posess an additional, rotational Killing field in an open neighborhood of the event horizon. This result was already known in the case of real analytic spacetimes (Hawking's Rigidity Theorem), and plays a key role in proving "no-hair" theorems. More precisely, the main result in [1] is the following:

Theorem 32. *Assume that* $(S, \mathcal{N}, \underline{\mathcal{N}})$ *is a local, regular, bifurcate, non-expanding horizon in a vacuum Einstein space-time* $(\mathbf{O}, \mathbf{g})$ *which possesses a Killing vector-field* $\mathbf{T}$ *tangent to* $\mathcal{N} \cup \underline{\mathcal{N}}$ *and not identically vanishing on* S. *Then there exists an open neighborhood* $\mathbf{O}' \subseteq \mathbf{O}$ *of* S *and a non-trivial rotational Killing vector-field* $\mathbf{Z}$ *in* $\mathbf{O}'$ *which commutes with* $\mathbf{T}$.

We hope that this general local result will play an important role in a future, general, classification of stationary, smooth black holes in vacuum. For the moment, however, the only global result we can prove is the following perturbative one.

Theorem 33. *Any regular stationary black-hole solution of the vacuum Einstein equations, which is a perturbation of a Kerr solution* $\mathcal{K}(a, m)$ *with* $0 \leq a < m$ *is in fact the Kerr solution.*

We prove a precise version of this theorem in [2]. The perturbation condition is expressed geometrically by assuming that the Mars-Simon tensor $\mathcal{S}$ of the stationary space-time is sufficiently small. The proof of Theorem 33 uses Theorem 32 as a first step. We start by defining the Killing vector-field $\mathbf{Z}$ in a neighborhood of the bifurcation sphere S and then extend it to the entire space-time by using the level sets of a canonically defined function y. We show that these level sets are conditionally pseudo-convex as long as the the Mars-Simon tensor $\mathcal{S}$ is sufficiently small, and use a unique continuation argument to extend the Killing vector-field $\mathbf{Z}$ to the entire space-time. Once this extension is achieved, the proof then follows from well known results of Carter and Robinson.

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