Mathematisches Forschungsinstitut Oberwolfach

Report No. 47/2010

DOI: 10.4171/OWR/2010/47

Arbeitsgemeinschaft: Topological Robotics

Organised by
Michael Farber, Durham
Jesus Gonzalez, Mexico
Dirk Schütz, Durham

October 10th – October 16th, 2010

Abstract. The purpose of the Arbeitsgemeinschaft was to enable PhD students and researchers to study Topological Robotics, a new field investigating topological problems motivated by robotics and engineering as well as problems of practical robotics requiring topological tools. The topics broadly fell into the areas of Topology of configuration spaces, Topological complexity of robot motion planning algorithms and Stochastic topology.

Mathematics Subject Classification (2000): 05C80, 52C35, 55M30, 55R80, 58D29, 58E05, 60D05, 68T40, 70B15.

Introduction by the Organisers

Topological robotics is a new mathematical discipline studying topological problems inspired by robotics and engineering as well as problems of practical robotics requiring topological tools. It is a part of a broader newly created research area called "computational topology". The latter studies topological problems appearing in computer science and algorithmic problems in topology.

Problems of topological robotics can roughly be split into three main categories: (A) studying special topological spaces, configuration spaces of important mechanical systems; (B) studying new topological invariants of general topological spaces, invariants which are motivated and inspired by applications in robotics and engineering; (C) studying algebraic topology of random topological spaces which arise in applications as configuration spaces of large systems of various nature. The meeting focussed on studying major problems and results of these areas.

A. **Topology of configuration spaces.** Topology of classical configuration spaces (i.e. varieties of mutually distinct points of a given manifold) is an important subject of modern algebraic topology interacting with many sub-disciplines (the theory of knots and braids, embeddings and immersions of manifolds, topology of subspace arrangements). The talks included the Totaro spectral sequence which computes the cohomology algebras of configuration spaces, and a beautiful theorem of Światosław R. Gal which gives a general formula for Euler characteristics of configuration spaces $F(X,n)$ of n distinct particles moving in a polyhedron X, for all n.

Other talks studied the topology of configuration spaces of mechanical linkages, a remarkable class of manifolds which appear in several fields of mathematics as well as in molecular biology and in statistical shape theory. Methods of Morse theory, enriched with new techniques based on properties of involutions, allow effective computation of their Betti numbers. The recent solution of the conjecture raised by Kevin Walker in 1985 was also surveyed. This conjecture asserts that the relative sizes of bars of a linkage are determined, up to certain equivalence, by the cohomology algebra of the linkage configuration space.

Another topic was the unknotting theorem for planar robot arms proven recently by R. Connelly, E.Demaine and G. Rote.

B. **Topological complexity of robot motion planning algorithms.** The concept of topological complexity of the robot motion planning problem $\mathsf{TC}(X)$ is an interesting topological invariant which measures navigational complexity of topological spaces and has obvious relevance to various robotics applications. It is a special case of the notion of Schwarz genus, a very general classical concept including also the Lusternik - Schnirelmann category $\mathsf{cat}(X)$.

Computing the topological complexity $\mathsf{TC}(X)$ meets serious difficulties in some cases. Very useful are general upper bounds in terms of the dimension and connectivity, and also the homotopy invariance of $\mathsf{TC}(X)$. The lower bounds for $\mathsf{TC}(X)$ use the structure of the cohomology algebra $H^*(X)$. These estimates can sometimes be significantly improved by applying the theory of weights of cohomology classes. One may also use stable cohomology operations to improve lower bounds on the topological complexity based on products of zero-divisors.

Surprisingly the number $\mathsf{TC}(\mathbf{RP}^n)$ computes the immersion dimension of the real projective space $\mathbf{RP}^n$ (with a few exceptions). Similarly, the symmetric topological complexity of $\mathbf{RP}^n$ computes its embedding dimension (again, with a few exceptions). Finally, algorithms for collision free motion of multiple particles in space and along graphs were discussed and the complexities of these problems were computed.

C. Stochastic algebraic topology. While dealing with large systems in application one cannot assume that all parameters of the system are known or can be measured without errors. A typical situation of this kind appears when one studies the configuration space of a linkage with a large number of sides $n \to \infty$. In such a case the topology of the configuration space depends on a large number of random parameters and it turns out that one may predict many statistical properties of the space with high confidence. Moreover, similar to the situation occurring in statistical physics, the statistical predictions concerning the topology of a random space are extremely precise for large n.

We discussed in detail the study of random linkages (random polygon spaces) where one may predict the asymptotics of their Betti numbers. We also considered other probabilistic models producing random complexes of various dimensions, such as random graphs and random 2-dimensional complexes.

The meeting brought together about 50 researchers from various backgrounds wanting to learn about topological robotics. In particular there were many doctoral students and recent postdocs, who were also giving the majority of the talks. In order to motivate participants into making their own contributions, the activities included a Problem Session where a number of open problems were described. The lively atmosphere and the overall excellent presentations led to better understanding and will allow the participants to delve deeper into this new and exciting part of mathematics.

Arbeitsgemeinschaft: Topological Robotics

Table of Contents

Abstracts

Classical Configuration Spaces

Viktoriya Ozornova

1. Introduction

In my talk, I summarized some results on classical configuration spaces, especially those ones concerning homotopy groups and cohomology of the configuration spaces of $\mathbb{R}^m$ and their minimal CW structure. For this presentation, I followed [1].

By $\mathbb{F}_k(X) = \{(x_1, \ldots, x_k) \in X | x_i \neq x_j \text{ for } i \neq j\}$ we denote the ordered **configuration space** of the topological space X. It carries the subspace topology of X^k. If we start with a manifold X, the configuration space is a manifold as well.

The first important observation to make is the fundamental fiber sequence of the configuration spaces:

One can show that the natural projection $\mathbb{F}_k(M) \to \mathbb{F}_r(M), (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_r)$ for $r \leq k$ is a fibration with a fiber $\mathbb{F}_{k-r}(M \setminus \{q_1, \ldots, q_r\})$ for any connected manifold M, where $Q_r := \{q_1, \ldots, q_r\}$ is a fixed set of r distinct points in M. Now we iterate this for $M = \mathbb{R}^{n+1}$, choosing always the first point (we fix distinct points $q_1, q_2, \ldots$ for the rest suitably). We obtain a sequence of fibrations

$$\begin{array}{ccccccc}
\mathbb{F}_k(\mathbb{R}^{n+1}) & \longleftarrow & \mathbb{F}_{k-1}(\mathbb{R}^{n+1} \setminus \{q_1\}) & \longleftarrow & \mathbb{F}_{k-2}(\mathbb{R}^{n+1} \setminus \{q_1, q_2\}) & \longleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{R}^{n+1} & & \mathbb{R}^{n+1} \setminus \{q_1\} & & \mathbb{R}^{n+1} \setminus \{q_1, q_2\} & & \cdots
\end{array}$$

This is called the fundamental fiber sequence of Fadell and Neuwirth.

Now one can show all the fibrations above admit sections. A first consequence of it is the following theorem which uses only the long exact sequence of a fibration:

Theorem 1.1. *For $n > 1$, there is an isomorphism of groups*

$$\pi_* \mathbb{F}_k(\mathbb{R}^{n+1}) \cong \bigoplus_{r=1}^{k-1} \pi_*(\bigvee_r S^n)$$

In particular, $\mathbb{F}_k(\mathbb{R}^{n+1})$ is $(n-1)$-connected.

Furthermore, we will specify the generators of $\pi_n \mathbb{F}_k(\mathbb{R}^{n+1})$ for $n \geq 2$. For doing so, consider for $1 \leq s \neq r \leq k$ maps

$$\begin{array}{rcl}
\alpha'_{r,s} : S^n & \to & \mathbb{F}_{k-r+1}(\mathbb{R}^{n+1} \setminus Q_{r-1}) \subset \mathbb{F}_k(\mathbb{R}^{n+1}) \\
\xi & \mapsto & (q_s + \xi, q_r, q_{r+1}, \ldots, q_{k-1})
\end{array}$$

(This means, we have r-th point moving around either a puncture or another chosen point q_s). Put $\alpha_{r,s}$ to be the homotopy class of the map $\alpha'_{r,s}$. We now have:

Proposition 1.2. *The elements* $\{\alpha_{r,s} | 1 \leq s < r \leq k\}$ *generate the group* $\pi_n \mathbb{F}_k(\mathbb{R}^{n+1})$. *We also have* $\alpha_{s,r} = (-1)^{n+1}\alpha_{r,s}$.

Now we know what the homotopy group $\pi_n \mathbb{F}_k \mathbb{R}^{n+1}$ looks like additively. There are two remarks on the further structure to make. First, note that the symmetric group Σ_k acts on the configuration space $\mathbb{F}_k\mathbb{R}^{n+1}$ by permuting the points. One can show that the generating set from above is invariant under this action:

Theorem 1.3. *For any* $\alpha_{r,s}$ *with* $1 \leq s \neq r \leq k$*, we have* $\sigma_*(\alpha_{r,s}) = \alpha_{\sigma(r),\sigma(s)}$.

The second point are the so-called **Yang-Baxter-relations** for Whitehead products, described by the following theorem:

Theorem 1.4. *For all* $\sigma \in \Sigma_k$*, we have the following identities in* $\pi_* \mathbb{F}_k \mathbb{R}^{n+1}$*:*

$$[\alpha_{\sigma(2),\sigma(1)}, \alpha_{\sigma(3),\sigma(1)} + \alpha_{\sigma(3),\sigma(2)}] = 0 \textit{ for } k \geq 3,$$
$$[\alpha_{\sigma(2),\sigma(1)}, \alpha_{\sigma(4),\sigma(3)}] = 0 \textit{ for } k \geq 4.$$

Before we leave the homotopy groups of configuration spaces, one further remark should be made. For the case $n = 1$, the configuration space of the plane, the fundamental fibre sequence shows that it's a $K(\pi, 1)$-space, and its fundamental group is the pure braid group. Furthermore, dividing out the action of the symmetric group then gives a classifying space for the whole braid group.

2. Cohomology of configuration spaces

Now we turn to calculating the cohomology rings of the configuration spaces of $\mathbb{R}^n$. First, note that by Hurewicz theorem we already have a set of generators for the n-th homology. The maps $g_{r,s} : \mathbb{F}_k\mathbb{R}^{n+1} \to S^n$ given by

$$(x_1, \dots, x_k) \mapsto \frac{x_r - x_s}{\|x_r - x_s\|}$$

provide us elements $\alpha^*_{r,s} = g^*_{r,s}(\iota^*_n)$ which are, as the notation suggests, Kronecker duals to the $\alpha_{r,s}$.

We now use the fibrations above again to show:

Theorem 2.1. *The algebra* $H^*(\mathbb{F}_{k-r}(\mathbb{R}^{n+1} \setminus Q_r))$ *is the universal commutative graded algebra generated by the set*

$$\mathcal{A}^*_{k-r,r} = \{\alpha^*_{t,s} | 1 \leq s < t \leq k,\ t > r\}$$

subject to cohomological Yang-Baxter-relations:

$$(\alpha^*_{t,s})^2 = 0$$

$$\alpha^*_{t,s}\alpha^*_{t,u} = \begin{cases} \alpha^*_{u,s}(\alpha^*_{t,u} - \alpha^*_{t,s}) & \textit{for } u > r \\ 0 & \textit{for } u \leq r \end{cases}$$

In the proof, one first applies Leray-Hirsch theorem to fundamental fibrations as above to investigate the additive structure. Then, one has to check that the relations hold. This induces natural maps from the universal algebra with these properties into the cohomology ring. To complete the proof, one proceeds by induction using short exact sequences. The necessary cohomology short exact sequences are obtained by closer examination of the Leray-Serre spectral sequence.

Remark 2.2. The cohomology generators are also well-behaved under the action of the symmetric group: For $\sigma \in \mathfrak{S}_k$ and $\alpha^*_{r,s} \in \mathcal{A}_k$, we have $\sigma^*(\alpha^*_{r,s}) = \alpha^*_{\sigma^{-1}(r),\sigma^{-1}(s)}$.

3. Minimal CW structure for configuration spaces

Now recall that, for simply-connected spaces, we can construct a minimal CW structure out of knowing the homology of this space. In this part, we stick to $n > 1$. By using the homological version of Leray-Hirsch Theorem, we have a very similar proof showing that additively, homology has in each degree pn same number of free generators as the cohomology, we denote them by $\alpha_{r_1,s_1} \rtimes \ldots \rtimes \alpha_{r_p,s_p}$, defined inductively. Then one observes that the right and left inductive definitions coincide and this element is Kronecker dual of $\alpha^*_{r_1,s_1} \ldots \alpha^*_{r_p,s_p}$ (we use that both Leray-Hirsch isomorphisms are adjoint w.r.t. Kronecker pairing).

Definition 3.1. Let $\beta : S^n_1 \times \ldots \times S^n_m \to \mathbb{F}_k(\mathbb{R}^{n+1})$ be an imbedding. We write $\beta(\xi) = (q_1(\xi), \ldots, q_k(\xi))$. Then for $1 \leq s < r \leq k$ define **perturbation** of β by $\alpha_{r,s}$:

$$\beta \bowtie \alpha_{r,s} : S^n_1 \times \ldots \times S^n_m \times S^n_{m+1} \to \mathbb{F}_k(\mathbb{R}^{n+1})$$
$$(\xi, \xi_{m+1}) \mapsto (q_1(\xi), q_2(\xi), \ldots, q_{r-1}(\xi), q_s(\xi) + 2^{-\nu}\xi_{m+1}, q_r(\xi), \ldots)$$

where $2^{-\nu} < \min_{\xi, s \neq r} |q_r(\xi) - q_s(\xi)|$.

We can now formulate some properties of the cell decomposition:

Theorem 3.2. *There is a CW-complex X_k and a homotopy equivalence $h_k : X_k \to \mathbb{F}_k\mathbb{R}^{n+1}$ with the following properties:*

1. *For all $m \geq 0$, we have $X^{(mn)} \setminus X^{((m-1)n)} = \coprod_{\omega \in \mathcal{A}_k^{\wedge m}, |\omega| = m} \mathring{D}^n \times \ldots \times \mathring{D}^n$*
2. *For each $\omega \in \mathcal{A}_k^{\wedge m}$ there is a map $\chi_\omega : S^n_1 \times \ldots \times S^n_m \to X_k$ s.t. $h_k \circ \chi_\omega$ is homotopic to a map, built up by perturbations $\varphi_{\omega_m} : S^n_1 \times \ldots \times S^n_m \to \mathbb{F}_k\mathbb{R}^{n+1}$ and mapping $\iota_1 \times \ldots \times \iota_m$ to ω_m.*

Example 3.3. Consider $\mathbb{F}_3\mathbb{R}^{n+1}$, $n > 1$. We have generators $\alpha_{2,1}, \alpha_{3,1}, \alpha_{3,2}$ in degree n and $\alpha_{2,1} \ltimes \alpha_{3,1}, \alpha_{2,1} \ltimes \alpha_{3,2}$. The first three can be used to get a map

$$S^n_{21} \vee S^n_{31} \vee S^n_{32} \to \mathbb{F}_3\mathbb{R}^{n+1}$$

Since this map is $(2n-1)$-connected, we can now use the restriction of maps $\alpha_{2,1} \bowtie \alpha_{3,1}$ and $\alpha_{2,1} \bowtie \alpha_{3,1}$ to $S^n \vee S^n$ to glue the $2n$-cells in.

References

[1] E.R. Fadell, S.Y. Husseini, *Geometry and Topology of Configuration Spaces*, Springer-Verlag, 2001

A survey of the Lusternik-Schnirelmann category. Category weight

Hellen Colman

The *category* of a topological space was introduced by Lusternik and Schnirelmann [11] in the early 30's. The reason for introducing this notion was that the category of a manifold M provides a lower bound on the number of critical points for any smooth function on M. Besides this original motivation, category became over the years the subject of much research in algebraic topology [4, 9]. There had been revivals of interest with the development of new techniques and with the introduction of new variations of the classical concept adapted to different fields [10, 5, 12, 2, 3, 1].

In this talk we describe the original notion of Lusternik-Schnirelmann category as well as some of the variations with special emphasis on the *category weight* introduced by Fadell and Husseini [6].

Given a topological space X, a subset U of X is *categorical* if U is contractible in X. The *category* of X, $\mathrm{cat}X$, is the least number of categorical open sets required to cover X. If no such covering exists, the category is said to be infinite. If A is a subset of X, the *relative category* of A in X, $\mathrm{cat}_X A$, is the least number of categorical open sets required to cover A. In particular $\mathrm{cat}_X X = \mathrm{cat}X$.

One of the fundamental properties of the category is that it is a homotopical invariant. We have that if $f : X \to Y$ is a homotopy equivalence, then $\mathrm{cat}(X) = \mathrm{cat}(Y)$.

The category of a space is in general hard to calculate. We introduce next two classical estimators to approximate the category of a space.

Covering dimension provides an upper bound for category. If X is connected and $\mathrm{cat}X$ is finite then $\mathrm{cat}X \leq 1 + \dim X$, where $\dim X$ denotes its covering dimension. Useful lower bounds for category are obtained by considering a reduced cohomology theory with any coefficient ring. We have that $\mathrm{cat}X \geq \mathrm{nil}\tilde{H}^*(X)$ where the nilpotence of $\tilde{H}^*(X)$ is the least integer r such that $(\tilde{H}^*(X))^r = 0$.

The Lusternik-Schnirelmann's main theorem asserts that if M is a compact manifold and f is a smooth function on M then the number of critical points of f is at least $\mathrm{cat}M$.

Category has the following multiplicative property: if X and Y are connected and paracompact, then $\mathrm{cat}(X \times Y) < \mathrm{cat}X + \mathrm{cat}Y$. In particular if $S_1, \ldots, S_k$ are spheres, we have that $\mathrm{cat}(S_1 \times \ldots \times S_k) = k+1$. Ganea [7] conjectured in 1971 that $\mathrm{cat}(X \times S^k) = \mathrm{cat}X + 1$ for any space X. Iwase [8] provided a counterexample to this conjecture in 1998. If we consider a more general situation than the product, namely a fibration, we have the following result [15]: if $F \to E \to B$ is a fibration with connected base B then $\mathrm{cat}E \leq \mathrm{cat}F\mathrm{cat}B$.

The category of a map $f : X \to Y$, $\mathrm{cat}(f)$, is the least number of open sets $\{U_1, \ldots, U_n\}$ required to cover X such that $f|_{U_i} = 0$. In particular $\mathrm{cat}X = \mathrm{cat}(id_X)$ and $\mathrm{cat}_X A = \mathrm{cat}(i_A)$ where $id_X : X \to X$ is the identity map and $i_A : A \to X$ is the inclusion map.

The cohomological estimates can in some cases be improved by applying the theory of weights of cohomology classes. The notion of *weight* was introduced by Fadell and Husseini [6] and more recently refined by Rudyak [13] and Strom [14] who introduced a homotopy invariant version called the *(strict) category weight*. The original concept of weight of a cohomology class was motivated by the observation that the classical cohomological lower bound for the category can be improved by considering *weights* associated to each cohomology class in the cup length estimate. The (strict) category weight of a class $u \in \tilde{H}^*(X)$ is defined to be

$$\mathrm{wgt}(u) = \sup\{k \mid f^*(u) = 0 \text{ for all maps } f : A \to X \text{ with } \mathrm{cat}(f) \leq k\}.$$

If $u = 0$ we say that $\mathrm{wgt}(u) = \infty$. Category weight is a lower bound for the category of a space: if $u \neq 0$ then $\mathrm{cat}X > \mathrm{wgt}(u)$. The multiplicative property for category weights asserts that $\mathrm{wgt}(u \cdot v) \geq \mathrm{wgt}(u) + \mathrm{wgt}(v)$. One then may improve on the classical cohomological bound by finding indecomposable cohomology classes of category weight more than one. If $0 \neq u = u_1 \cdots u_k$ then

$$\mathrm{cat}(X) > \mathrm{wgt}(u) = \mathrm{wgt}(u_1 \cdots u_k) \geq \sum_i \mathrm{wgt}(u_i).$$

References

[1] H. Colman, The Lusternik-Schnirelmann category of a Lie groupoid, Trans. Amer. Math. Soc. 362 (2010), 5529-5567.

[2] H. Colman and E. Macias-Virgós, The transverse Lusternik-Schnirelmann category of a foliated manifold, *Topology,* Vol. **40** (2) (2001), 419-430.

[3] H. Colman and E. Macias-Virgós, Tangential Lusternik-Schnirelmann Category of Foliations, J. London Math. Soc. (2002) 65 (3), 745-756.

[4] O. Cornea, G. Lupton, J. Oprea, D. Tanré, Lusternik-Schnirelmann category, AMS, Mathematical Surveys and Monographs, vol. 103 (2003).

[5] E. Fadell, The equivariant Ljusternik-Schnirelmann method for invariant functionals and relative cohomological index theories, In *Méthodes Topologiques en Analyse Non-Lineaire,* ed. A. Granas, Montreal, 1985.

[6] E. Fadell, S. Husseini, Category weight and Steenrod operations, Boletín de la Sociedad Matemática Mexicana, 37 (1992), 151-161.

[7] T. Ganea, Some problems on numerical homotopy invariants, Symposium in Algebraic Topology, Seattle 1971, Edited by P. Hilton (Lect. Notes in Math., vol. 249), Berlin Heidelberg New York: Springer, 1971, 23-30.

[8] N. Iwase, Ganea's conjecture on Lusternik-Schnirelmann category, Bulletin of the London Mathematical Society 30 (1998), no. 6, 623-634.

[9] I. M. James, On category, in the sense of Lusternik-Schnirelmann, Topology 17 (1978), 331-348.

[10] I.M. James and J. Morris, Fibrewise category, *Proceedings of the Royal Society of Edinburgh,* 119A (1991), 177-190.

[11] L. Lusternik and L. Schnirelmann, *Méthodes topologiques dans les Problèmes Variationnels,* Hermann, Paris, 1934.

[12] J. Oprea and Y. Rudyak, Detecting elements and Lusternik-Schnirelmann category of 3-manifolds, Contemp. Math. 316, Amer. Math. Soc. (2002) 181-191.
[13] Y. B. Rudyak, On category weight and its applications, Topology 38 (1999), 37-55.
[14] J. Strom, Category Weight and Essential Category Weight, University of Wisconsin-Madison Thesis, (1997).
[15] K. Varadarajan, On fibrations and category, Math. Z. 88 (1965), 267-273.

Introduction to Random Graphs

Anna Gundert

The aim of this talk is to give a brief introduction to the theory of random graphs. We approach the topic by looking at some basic examples and calculations which employ the first and second moment methods. These are quite simple tools that are yet among the most commonly used.

After discussing the two basic models, namely $G(n,p)$ and $G(n,m)$, we consider the containment problem: For a fixed graph H, what is the probability that $G(n,p)$ contains a copy of H as a subgraph? As an example we determine the threshold for the property "$G(n,p)$ contains a K_4", a calculation which already contains the ideas necessary to solve the problem in general.

We then explore the concept of thresholds, briefly mentioning Friedgut's surprising result that the containment of fixed subgraphs is in some sense "the" property with a non-sharp threshold. As an example we look at the famous sharp threshold of $\frac{\log(n)}{n}$ for the connectivity of $G(n,p)$.

We mainly follow chapters 1 and 3 in [3]. Other important basic references are [1] and [2].

References

[1] Noga Alon and Joel H. Spencer, *The Probabilistic Method*, third ed., Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons Inc., 2008.
[2] Béla Bollobás, *Random Graphs*, second ed., Cambridge Studies in Advanced Mathematics, vol. 73, Cambridge University Press, 2001.
[3] Svante Janson, Tomasz Łuczak, and Andrzej Ruciński, *Random Graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, 2000.

The cohomology of the complement of a hyperplane arrangement

Jelena Grbić

Following Orlik-Terao's book [1] on hyperplane arrangements and Yuzvinsky's [2] survey paper on Orlik-Solomon algebras, I reported on some topological invariants of arrangements, concentrating particularly on the cohomology ring of the complement of a hyperplane arrangement.

A hyperplane arrangement $\mathcal{A}$ in an affine space $V \cong K^l$, $l \geq 1$, over an arbitrary field K is a finite set of affine hyperplanes in V. We denote the cardinality of $\mathcal{A}$ by n and without loss of generality to our problems we impose an arbitrary linear order on $\mathcal{A}$ and write $\mathcal{A} = (H_1, \dots, H_n)$.

In studying the topology of arrangements one usually investigates topological invariants of the complement of an arrangement $M(\mathcal{A}) := K^l \setminus \bigcup_{i=1}^n H_i$. In this talk I looked into the integral cohomology ring of the complement $M(\mathcal{A})$ and its relations to the Orlik-Solomon (OS) algebra $A(\mathcal{A})$ associated to the arrangement $\mathcal{A}$.

This talk had as one of its aims to illustrate the rich interplay between the geometry, combinatorics, algebra, and topology associated to hyperplane arrangements.

To follow easier the geometry of hyperplane arrangements, we choose a linear basis $(x_1, \dots, x_l)$ of V^* and to each hyperplane H in K^l we associate a unique (up to a constant) degree 1 polynomial $\alpha_H \in K[x_1, \dots, x_l]$ such that H is the zero locus of α_H. Thus a hyperplane arrangement $\mathcal{A}$ is uniquely determined by the defining polynomial $Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H$.

As an example of a hyperplane arrangement one can mention the braid arrangement defined by $Q(\mathcal{A}) = \prod_{1 \leq i < j \leq l}(x_i - x_j)$ which historically set off the study of hyperplane arrangements. On the other hand, the complex braid arrangement $\mathcal{A} = \{H_{i,j} = \mathrm{Ker}(z_i - z_j)\}_{1 \leq i < j \leq l}$ is directly connected to the main topic of this workshop, that is, to configuration spaces, as it can be shown that $M(\mathcal{A}) = \mathbf{C}^l \setminus \bigcup H_{i,j} = F(\mathbf{C}, l)$.

By the combinatorics of a hyperplane arrangement $\mathcal{A}$ we mean the intersection poset $L(\mathcal{A})$ of $\mathcal{A}$ which is a set of non-empty subspaces of V that are intersections of some hyperplanes of $\mathcal{A}$, including V itself as the intersection of the empty set of hyperplanes. This set is partially ordered by reverse inclusion of subspaces, that is, for $X, Y \in L(\mathcal{A})$ we have $X \leq Y$ if and only of $Y \subseteq X$. In particular, $L(\mathcal{A})$ always has a unique minimal element V. It has a unique maximal element $\bigcap_{i=1}^n H_i$ if and only if the intersection is not empty in which case $\mathcal{A}$ is said to be central. Central arrangements in general tend to be technically easier to handle; for example the intersection poset of a central arrangement is a geometric lattice.

One would like to have an algebraic object which will nicely detect the combinatorics of hyperplane arrangements; that was achieved precisely by Orlik-Solomon algebras. To define the Orlik-Solomon algebra $A(\mathcal{A})$ of a hyperplane arrangement $\mathcal{A}$ we do not need to know all its hyperplanes; this algebra is constructed using only the intersection poset $L(\mathcal{A})$. Given a hyperplane arrangement $\mathcal{A} = (H_1, \dots, H_n)$, the Orlik-Solomon algebra $A(\mathcal{A})$ was originally defined as the quotient of the exterior algebra over an arbitrary commutative ring k generated by elements $e_1, \dots, e_n$ of degree 1 by the Orlik-Solomon ideal $I(\mathcal{A})$. By e_S we denote the monomial $e_{i_1} e_{i_2} \cdots e_{i_p}$ for $S = \{i_1, i_2, \dots, I_p\} \subset [n]$. Now the Orlik-Solomon ideal is generated by all e_S if $\bigcap S \neq \emptyset$ and ∂e_S with dependent subset $S \subset [n]$.

Orlik-Solomon algebras play an important role in the theory of multivariable hypergeometric functions, conformal field theory, the theory of cohomology of Milnor fibres of non-isolated singularities, and the theory of Alexander invariants of projective curves. Results about OS algebras usually involve several areas of mathematics such as pure algebra, combinatorics, topology, differential geometry, and algebraic geometry. Yuzvinsky [2] wrote a very nice survey paper, which should be useful for both experts and novices, in which he studies OS algebras per se and gives some of their applications to topology and combinatorics.

In this talk I calculated the OS algebra for several hyperplane arrangements of different type, illustrating how the OS algebra sees the difference in the combinatorics of these arrangements.

The main theorem of the talk states that the Orlik-Solomon algebra $A(\mathcal{A})$ of a hyperplane arrangement $\mathcal{A}$ is isomorphic, as an algebra, to the cohomology algebra of the complement $M(\mathcal{A})$ of the arrangement $\mathcal{A}$. To prove the theorem an algebraic and topological induction were established and linked together. Namely, we first established a short exact sequence of the OS algebras associated to the deletion-restriction triple $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ where $\mathcal{A}'$ is an arrangement in $V \cong K^l$ with $|\mathcal{A}| = n-1$, and $\mathcal{A}''$ is an arrangement in $V' \cong K^{l-1}$ with $|\mathcal{A}''| \leq l-1$. This way we set up induction on the cardinality of an arrangement as well as on the dimension of the ambient space. To establish the topological induction, we consider the complements of the deletion-restriction triple, that is, $(M(\mathcal{A}'), M(\mathcal{A}), M(\mathcal{A}''))$ and the long exact sequence in cohomology of the pair $(M(\mathcal{A}'), M(\mathcal{A}))$. Using the Thom isomorphism, we identify the cohomology groups $H^*(M(\mathcal{A}''))$ with the cohomology groups $H^{*+2}(M(\mathcal{A}'), M(\mathcal{A}''))$ and therefore obtain a long exact sequence connecting the cohomology groups of the triple $(M(\mathcal{A}'), M(\mathcal{A}), M(\mathcal{A}''))$.

To an arrangement $\mathcal{A}$ we can associate the k-algebra $R(\mathcal{A})$ generated by the differential forms $\omega_H = d\alpha_H/\alpha_H$. Note that this algebra is not a purely combinatorial object, as the defining polynomials α_H enter the definition. However one can prove that there is an isomorphism of algebras $A(\mathcal{A}) \cong R(\mathcal{A})$. This on the one hand gives that $R(\mathcal{A})$ depends only on $L(\mathcal{A})$ and on the other hand gives a description of the cohomology classes of $M(\mathcal{A})$ in terms of differential forms.

References

[1] Orlik, Peter and Terao, Hiroaki, *Arrangements and hypergeometric integrals*, Second, MSJ Memoirs, vol. 9, Mathematical Society of Japan, Tokyo, 2007.

[2] Yuzvinskiĭ, S., *Orlik-Solomon algebras in algebra and topology*, Uspekhi Mat. Nauk 56 (2001), no. 2(338), 87-166, DOI 10.1070/RM2001v056n02ABEH000383.

Motion planning algorithms, topological complexity and Schwarz genus, part 1

Aleksandra Franc

We consider a path-connected topological space X as a configuration space of a mechanical system. The states of the system correspond to the points in X and continuous motions of the system correspond to continuous paths in X. The motion planning algorithm takes as input a pair of points – the initial and the final state of the system – and returns a continuous path between the two. In other words, a motion planning algorithm is a section $s\colon X \times X \to PX$ of the endpoint fibration $\pi\colon PX \to X \times X$, $\gamma \mapsto (\gamma(0), \gamma(1))$. However, a continuous section exists if and only if X is contractible.

Definition 1. *Topological complexity* $\mathrm{TC}(X)$ *is the minimal* k *such that there exists an open cover*

$$U_1 \cup \ldots \cup U_k = X \times X$$

with the property that for all $i = 1, \ldots, k$ π *admits a continuous section* $s_i \colon U_i \to PX$ *over* U_i.

There are several alternative definitions of topological complexity (see [2],[4]). Our definition is a special case of the notion of *Schwarz genus* ([5]). Another special case of Schwarz genus is Lusternik-Schnirelmann category, which provides a nice lower and upper bound for topological complexity.

Proposition 2. *For any topological space* X *we have*

$$\mathrm{cat}(X) \leq \mathrm{TC}(X) \leq \mathrm{cat}(X \times X).$$

For any connected Lie group G we have the equality $\mathrm{TC}(G) = \mathrm{cat}(G)$ (see Lemma 8.2 of [1]). The spheres S^1 and S^2 serve as an example to show that in general, each of the two inequalities of Proposition 2 can be strict. We show that $\mathrm{TC}(S^n) = 2$ for n odd and $\mathrm{TC}(S^n) = 3$ for n even. The upper bound is obtained by explicitly constructing continuous sections corresponding to open covers with two and three elements, respectively. The lower bound for odd n follows from Proposition 2 since $\mathrm{cat}(S^n) = 2$. The lower bound for even n is obtained using Theorem 5.

An upper bound can also be formulated in terms of dimension and connectivity.

Theorem 3. *Let* X *be an* r*-connected polyhedron,* $r \geq 0$. *Then*

$$\mathrm{TC}(X) < \frac{2\dim(X) + 1}{r + 1} + 1.$$

For a subset $A \subset X \times X$ we introduce a relative version of topological complexity. It has all of the basic properties one would expect and can be used to prove that topological complexity is a homotopy invariant (see [2]).

Definition 4. The *relative topological complexity* $\mathrm{TC}_X(A)$ is the Schwarz genus of the fibration $\pi \colon P_A X \to A$. Here $P_A X \subset PX$ is the space of all paths γ in X such that $(\gamma(0), \gamma(1)) \in A$.

Finally, we give another lower bound hidden in the structure of the cohomology ring $H^*(X; R)$. First, we consider the case where $R = k$ is a field. Then $H^*(X; k)$ is a graded k-algebra with the multiplication

$$\smile \colon H^*(X; k) \otimes H^*(X; k) \to H^*(X; k)$$

and $H^*(X; k) \otimes H^*(X; k)$ is a graded k-algebra with the multiplication

$$(u_1 \otimes v_1) \cdot (u_2 \otimes v_2) = (-1)^{|v_1||u_2|} u_1 u_2 \otimes v_1 v_2.$$

The $\smile$-product is an algebra homomorphism. Its kernel $\ker \smile$ is the *ideal of zero-divisors* of $H^*(X; k)$. The *zero-divisors-cup-length* of $H^*(X; k)$, $\mathrm{zcl} H^*(X; k)$, is the length of the longest non-trivial product in $\ker \smile$. As we show later in the general case, we have

Theorem 5. $\mathrm{TC}(X) > \mathrm{zcl}H^*(X;k)$.

We can now show that $\mathrm{TC}(S^n) = 3$ for n even. Very similar calculations also work for connected graphs and orientable surfaces. Details can be found in [2] and [3].

Now, let R be an arbitrary coefficient system on $X \times X$ and let $u \in H^*(X \times X; R)$ be a cohomology class. We say that u *has weight* $k \geq 0$, $\mathrm{wgt}(u) = k$, if k is the largest integer with the property that for any open subset $A \subset X \times X$ with $\mathrm{TC}_X(A) \leq k$ one has $u|_A = 0$. The weight of the zero cohomology class equals ∞. As an immediate consequence of this definition we get:

Proposition 6. *If there exists a nonzero cohomology class* $u \in H^*(X \times X; R)$ *with* $\mathrm{wgt}(u) \geq k$*, then* $\mathrm{TC}(X) > k$.

Hence, our goal is to find nonzero cohomology classes of highest possible weight. It is not difficult to see that $\mathrm{wgt}(u) \geq 1$ if and only if $u|_{\Delta_X} = 0 \in H^*(X; R|_{\Delta_X})$. The cohomology classes with this property are called *zero-divisors*. To obtain cohomology classes of higher weight we use the following lemma:

Lemma 7. *Let* $u \in H^n(X \times X; R)$ *and* $v \in H^m(X \times X; R')$*. Then* $u \smile v \in H^{n+m}(X \times X; R \otimes R')$ *and*

$$\mathrm{wgt}(u \smile v) \geq \mathrm{wgt}(u) + \mathrm{wgt}(v).$$

As an easy corollary of the previous proposition we now obtain a useful lower bound for topological complexity:

Corollary 8. *If the cup-product of* k *zero-divisors* $u_i \in H^*(X \times X; R_i)$*,* $i = 1, \ldots, k$*, is nonzero, then* $\mathrm{TC}(X) > k$.

References

[1] M. Farber, *Instabilities of robot motion*, Topology Appl. 140 (2004), 245–266.

[2] M. Farber, *Invitation to Topological Robotics*, Zürich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008

[3] M. Farber, *Topological complexity of motion planning*, Discrete Comput. Geom. 29 (2003), 221–221.

[4] M. Farber, *Topology of robot motion planning*, in: Morse Theoretic Methods in Nonlinear Analysis and in Symplectic Topology, (P. Biran et all (eds)), 2006, 185–230.

[5] A.S. Švarc, *The genus of a fiber space*, Amer. Math. Soc. Transl. 55 (1966), 49–140.

Linkages and their Configuration Spaces

Moritz Rodenhausen

In the talk I gave a computation of the Betti numbers of the configuration space M_l of planar n-gons with fixed side lengths $l_1, \ldots, l_n > 0$. This space is given by

$$M_l := \left\{ (u_1, \ldots, u_n) \in S^1 \times \ldots \times S^1 \,\middle|\, \sum_{i=1}^{n} l_i u_i = 0 \right\} \Big/ \mathrm{SO}(2),$$

where SO(2) acts diagonally, i.e. we consider n-gons only up to rotation. Figure 1 shows some polygons representing points in M_l.

A subset $J \subset \{1, \dots, n\}$ is called *short*, if $\sum_{i \in J} l_i - \sum_{i \notin J} l_i$ is negative, *median*, if this sum is zero, and *long*, if it is positive. The length vector $l = (l_1, \dots, l_n)$ is called *generic*, if there is no median subset. If l is generic, the configuration space M_l is a closed manifold of dimension $n-3$, otherwise it has singularities.

To state the main theorem, fix i with $l_i = \max_j l_j$. For each k, let a_k (resp. b_k) be the number of short (resp. median) subsets $J \subset \{1, \dots, n\}$ of cardinality $k+1$ such that $i \in J$. We then have:

Theorem. *For $0 \leq k \leq n-3$, the homology group $H_k(M_l; \mathbb{Z})$ is free abelian of rank $a_k + b_k + a_{n-3-k}$.*

For example, the number $b_0(M_l)$ of connected components is $a_0 + b_0 + a_{n-3}$. Assuming $l_1 \geq \dots \geq l_n$, a computation shows that M_l is empty iff $\{1\}$ is long. Otherwise, it has one component, if $\{2, 3\}$ is short or median, and it has two components, if $\{2, 3\}$ is long.

Another application of the theorem is the equilateral case $l_1 = \dots = l_n = 1$: Here the sum of the Betti numbers is

$$\sum_{k=0}^{n-3} b_k(M_l) = B_n := 2^{n-1} - \binom{n-1}{r}, \qquad \text{where } n = 2r+1 \text{ or } n = 2r+2.$$

In fact, B_n is the maximal value of the Betti number sum for any length vector l. If l is generic, then the maximal value is $2B_{n-1}$.

The idea of the proof of the theorem is an investigation of the "robot arm distance map" $f_l : W \to \mathbb{R}$, where $W = \left(S^1\right)^n / \operatorname{SO}(2)$ and

$$f_l(u_1, \dots, u_n) := -\left| \sum_i l_i u_i \right|^2$$

measures the distance of the end points of the robot arm as in Figure 2.

f_l is a Morse function whose critical points are exactly the collinear configurations of the robot arm. There is an involution $\tau : W \to W$ given by reflection of the robot arm. Choosing $a < 0$ such that the interval $(a, 0)$ does not contain critical values, we define $W^a := f^{-1}(-\infty, a]$. For each $J \subset \{1, \dots, n\}$ there is a submanifold given by integrating all l_i, $i \in J$, into one link, i.e.

$$W_J := \{(u_1, \dots, u_n) \,|\, u_i = u_j \text{ for all } i, j \in J\} / \operatorname{SO}(2) \subset W.$$

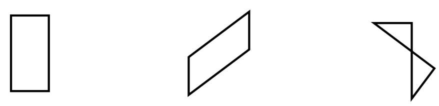


FIGURE 1. Points in M_l: The sides of the polygons have fixed lengths and are allowed to cross, where the angles are ariable.

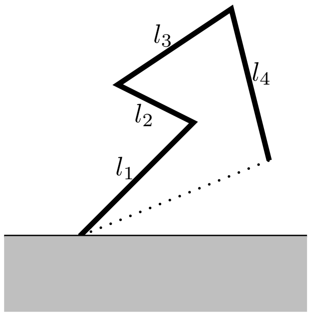


FIGURE 2. The distance map f_l is determined by the length of the dotted line.

Using Morse theory on manifolds with involution, we find that the fundamental classes $[W_J]$ freely generate the homology $H_*(W^a)$ and $H_*(W)$, where J ranges over all long subsets in the case of $H_*(W^a)$ and over all $\{1\} \subset J \subset \{1,\dots,n\}$ for $H_*(W)$. An explicit computation of the map $H_*(W^a) \to H_*(W)$ leads to the relative homology $H_*(W, W^a)$. Since $N := f_l^{-1}[a, 0]$ deformation retracts onto $M_l = f_l^{-1}(0)$, we can use Poincaré duality to obtain

$$H_*(W, W^a) \cong H_*(N, \partial N) \cong H^*(N) \cong H^*(M_l).$$

An easy calculation finishes the proof of the theorem.

References

[1] M. Farber, D. Schütz: *Homology of Planar Polygon Spaces*, Geom. Dedicata 125 (2007), 75-92

Motion planning algorithms, topological complexity and Schwarz genus, part 2

ALBERT RUIZ

The aim of this talk is give more tools to compute the topological complexity of a space:

Definition 1. The *Topological complexity of* X is the minimal k such that there exist open subsets:

$$X \times X = V_1 \cup V_2 \cup \cdots \cup V_k = X \times X$$

and continuous sections $s_i \colon V_i \to PX$ of the map $PX \to X \times X$ defined by $\gamma \mapsto (\gamma(0), \gamma(1))$.

The tool that we introduce depends on the cohomology operations, and more precisely on the *excess* of a cohomology operation. The main general result is:

Theorem 2. *Let θ be stable cohomology operation in degree i and excess $e(\theta) \geq n$. Then for any cohomology class $u \in H^n(X; R)$, $\mathrm{wgt}(\overline{\theta(u)}) \geq 2$.*

Usually the definition of excess if given in the case of admissible Steenrod operations, but it can be generalised as:

Definition 3. Consider θ a stable cohomology operation of degree i acting on $H^*(X;R)$. The *excess* of θ, denoted as $e(\theta)$, is the largest integer n such that $\theta(u) = 0$ for all u such that $|u| < n$.

As example, we will use that the excess of the Bockstein operator is 1.

We can apply this tool to compute the topological complexity of the lens spaces:

Definition 4. Consider the cyclic group $\mathbb{Z}_m = \{1, \omega, \omega^2, \ldots, \omega^{m-1}\}$, where ω is a primitive m-root of unity, acting on $S^{2n+1} \subset \mathbb{C}^{n+1}$ by pointwise multiplication. Define the *lens space*:

$$L_m^{2n+1} = S^{2n+1}/\mathbb{Z}_m$$

These spaces fit in a locally trivial fibration:

$$S^1 \to L_m^{2n-1} \to \mathbb{C}P^n$$

and this allows us to compute a good upper bound for the topological complexity:

Theorem 5.

$$\mathbf{TC}(L_m^{2n+1}) \leq 4n + 2\,.$$

The lower bounds are not so easy, and these can be computed in some especial cases. The next theorem show us that the upper bound we got before is in fact the topological complexity in some cases:

Theorem 6. *Let p be an odd prime and n an integer such that its p-adic expansion*

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_k p^k \text{ (with } 0 \leq n_i < p)$$

satisfy that $n_i \leq (p-1)/2$ for all i. Then:

$$\mathbf{TC}(L_p^{2n+1}) = 4n + 2\,.$$

The proof of this theorem uses that the mod-p cohomology of the lens space $H^*(L_p^{2n+1}; \mathbb{Q}_p)$ has a generator in degree 2 which is the image of a generator in degree 1 by a Bockstein operation. This allows us to apply Theorem 2 to get that:

$$\mathbf{TC}(L_p^{2n+1}) \geq 4n + 2\,.$$

We finish the talk giving bounds for the topological complexity of a product. For this case we deal with the *Euclidean Neighbourhood Retracts*:

Definition 7. A topological space X is a *Euclidean Neighbourhood Retract (ENR)* if it can be embedded into an Euclidean space $X \subset \mathbb{R}^n$ such that $\exists U$ open with $X \subset U \subset \mathbb{R}^n$ and a retraction $r\colon U \to X$ (i.e. $r|_X = \mathrm{Id}_X$).

From the definition of topological complexity, considering explicit open subsets for $X \times X$ and $X' \times X'$ we are able to find an upper bound of the topological complexity of $X \times X'$:

Theorem 8. *Let X and X' be Euclidean Neighbourhood Retracts. Then*

$$\mathbf{TC}(X \times X') \leq \mathbf{TC}(X) + \mathbf{TC}(X') - 1\,.$$

If we consider the reduced typological complexity, defined as $\tilde{\mathbf{TC}}(X) = \mathbf{TC}(X) - 1$ this formula applied to a finite product tells us:

$$\tilde{\mathbf{TC}}(\prod_{i=1}^{n} X_i) \leq \sum_{i=1}^{n} \tilde{\mathbf{TC}}(X_i)\,.$$

To find a lower bound for the topological complexity of a product we use that the zero divisors cup length in rational cohomology has the property:

$$\mathrm{zcl}_{\mathbb{Q}}(X \times Y) \geq \mathrm{zcl}_{\mathbb{Q}}(X) + \mathrm{zcl}_{\mathbb{Q}}(Y).$$

This allows us to prove that the topological complexity of a product on n spaces grows linearly on n:

Theorem 9. *Suppose that one controls simultaneously k systems having path-connected configurations spaces $X_1, \ldots, X_k$. Assume that $\tilde{H}(X_i, \mathbb{Q}) \neq 0$ and there is a constant M such that for all i $\tilde{\mathbf{TC}}(X_i) \leq M$. Then*

$$k \leq \tilde{\mathbf{TC}}(X_1 \times \cdots \times X_k) \leq kM\,.$$

References

[1] M. Faber, *Invitation to topological robtics.* Zurich Lectures in Advanced Mathematics. EMS, Zrich, 2008.
[2] M. Farber and M. Grant, *Robot motion planning, weights of cohomology classes, and cohomology operations.* Proceedings of the AMS 136 (2008), 3339–3349.
[3] M. Farber, S. Tabachnikov and S. Yuzvinsky, *Topological robotics: motion planning in projective spaces,* International Mathematics Research Notices 34 (2003), 1853–1870.

Survey on random graphs, part 2

Benjamin Matschke

This talk is subdivided into two independent parts. The first part deals with the evolution of random graphs, see section 1. In the second part we give examples of how to apply probabilistic ideas to graph theory, see section 2 and 3.

Much more about random graphs and the probabilistic method can be found in the wonderful books [1, 2, 6, 7, 3].

1. The Erdős–Rényi phase transition

Let $G(n,p)$ be the probability space of random graphs on n vertices where we let the edges appear independently from each other with probability $p = p(n)$. We say that $G(n,p)$ has a property A *asymptotically almost surly* (a.a.s.) if the probability that (a random sample of) $G(n,p)$ has property A converges to 1 for $n \to \infty$. Let L_i be the size of the ith largest component of $G(n,p)$. We write $a(n) \sim b(n)$ if $\lim \frac{a(n)}{b(n)} = 1$.

Erdős and Rényi [5] studied the component sizes of $G(n,p)$ and their structure and they observed a phase transition at $p = \Theta(n)$.

If $c < 1$ is constant and $p = \frac{c}{n}$, then a.a.s. all components have at most one cycle and $L_1 \sim \ldots \sim L_k = \Theta(\log n)$ for fixed k. This is the *very subcritical region.*

If $c > 1$ is constant and $p = \frac{c}{n}$, then a.a.s. there is one giant component of size $L_1 \sim yn$ where $0 < y < 1$ solves $e^{-cy} = 1 - y$. Furthermore, $L_2 = \Theta(\log n)$, and all but the giant component have at most one cycle. This is the *very supercritical region.*

In the case $p \sim n$ one uses the fine parameterization $p = \frac{1+\lambda n^{-\frac{1}{3}}}{n}$. If λ is a constant, then a.a.s. the largest k components are of size $\Theta(n^{\frac{2}{3}})$. This is the *critical region.*

If $\lambda \to -\infty$ and $\lambda n^{-\frac{1}{3}} \to 0$, then a.a.s. $L_1 \sim \ldots \sim L_k = \Theta(n^{\frac{2}{3}}\lambda^{-2}\log|\lambda|)$ for fixed k and all components have at most one cycle. This is the *barely subcritical region.*

If $\lambda \to \infty$ and $\lambda n^{-\frac{1}{3}} \to 0$, then a.a.s. there is a giant component of size $L_1 \sim 2\lambda n^{\frac{2}{3}}$ and all other components have at most one cycle and $L_2 = \Theta(n^{\frac{2}{3}}\lambda^{-2}\log\lambda)$. This is the *barely supercritical region.*

In the talk we explain a connection to the Poisson branching process which gives intuition for the results in the very subcritical and the very supercritical region.

2. Ramsey theory

The probabilistic method can sometimes be very powerful in proving the *existence* of interesting examples. Ramsey theory offers beautiful and simple applications of this method.

The *Ramsey number* $R(k)$ is the smallest number n such that the complete graph K_n contains a monochromatic k-clique for any 2-coloring of its edge set. Random graphs can now be used to find lower bounds on $R(k)$. In this talk we presented the following three elementary but useful methods.

2.1. **First moment method.** Fix n and k, and color the edges of K_n randomly, for example i.i.d., with two colors. Calculate the expected value E of the number of monochromatic k-cliques. If $E < 1$ then there exists a coloring without monochromatic k-clique; hence $R(k) > n$.

2.2. **Alteration.** Again color the edges of K_n randomly with two colors and calculate E, which depends on n. Therefore there is a graph G with at most E monochromatic k-cliques. We can remove E vertices from G such that the remaining graph has no monochromatic k-clique. Hence $R(k) > n - E$. Now we maximize $n - E$ analytically to get a good lower bound.

2.3. **Lovász local lemma.** For every k-clique $S \subset [n]$ in a randomly 2-colored K_n, let B_S be the "bad" event that S is monochromatic. The family of events $(B_S)_S$ contains many independencies. The following tool provides a sufficient condition that with non-zero probability none of the bad events occurs, in which case $R(k) > n$.

Theorem 2.4 (Lovász local lemma (symmetric case))**.** *Let $B_1, \ldots, B_n$ be events in a fixed probability space such that each B_i is mutually independent of a set of all the other events but at most d of them. Suppose that* $\Pr[B_i] \le p$ *for all i and $ep(d+1) \le 1$, where e is Euler's number. Then* $\Pr[\bigcup_i \overline{B_i}] > 0$.

3. Graphs with high chromatic number and high girth

Another instance of a probabilistic existence proof is a famous theorem of Erdős (1959) [4]. This was omitted in the talk due to lack of time.

Let G be a graph. Its chromatic number $\chi(G)$ is the minimal number of independent subsets of $V(G)$ that you need to partition this vertex set. The girth of G is the length of the shortest cycle in G (or ∞ if there is no cycle).

Theorem 3.1 (Erdős)**.** *For every number k there exists a graph G with $\chi(G) \geq k$ and* $\operatorname{girth}(G) \geq k$*.*

See [4] or [1] for a proof. It shows that the chromatic number is in some sense far from being a local graph invariant.

References

[1] N. Alon, J. H. Spencer. *The probabilistic method*, Wiley-Interscience, 3rd ed., 2008.
[2] B. Bollobás. *Random Graphs*, 2nd ed., Vol. 73, Cambridge University Press, 2001.
[3] R. Durrett. *Random Graph Dynamics*, Cambridge University Press, 2007.
[4] P. Erdős. *Graph theory and probability*, Can. J. Math. 11, 34-38.
[5] P. Erdős, A. Rényi. *On the evolution of random graphs*, Magyar Tud. Akad. Mat. Kutató Int. Közl. 5, 17-61.
[6] S. Janson, T. Łuczak, A. Ruciński. *Random Graphs*, Wiley-Interscience, 2000.
[7] M. Penrose. *Random Geometric Graphs*, Oxford University Press, 2003.

Walker's Conjecture

Viktor Fromm

A *planar polygonal linkage* is a mechanism consisting of several bars of fixed lengths connected by revolving joints into a single chain in the plain. Identifying two such chains if one can be obtained from the other by an orientation-preserving affine isometry, we obtain what is called the *configuration space* M_ℓ of the linkage. Here ℓ is a vector whose entries are the lengths of the individual bars of the link-

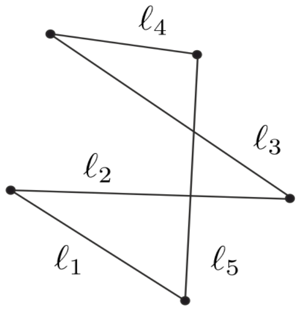


Figure 1. A Planar Polygon

age. The length vector ℓ determines the structure of the space M_ℓ completely and there are numerous important results which relate combinatorics of ℓ to topological properties of M_ℓ.

It is not difficult to show that for most choices of ℓ the space M_ℓ is a smooth manifold. In this generic situation one can classify the configuration spaces by so-called *chambers*. To explain this, let us observe that ℓ can be assume to lie in the unit simplex. The non-generic length vectors lie on certain affine subspaces of the simplex, called *walls*. The connected components of the complement of that union are called chambers. One can prove that whenever two length vectors lie in the same chamber up to a permutation of the entries, the corresponding configuration spaces can be identified.

K. Walker in [4] discovered a formula which computes the Betti numbers of the spaces M_ℓ explicitly in terms of the length vector ℓ. He observed that in low dimensions, the Betti numbers distinguish up to permutation between different chambers and thus determine uniquely the configuration space. However, already starting with dimension three this classification breaks down and there are examples of chambers so that the corresponding configuration spaces are distinct although their Betti numbers are equal. K. Walker conjectured that the graded isomorphism type of the integral *cohomology ring* classifies the spaces M_ℓ in the general case.

In my talk I presented a partial proof of the conjecture, due to M. Farber, J.-Cl. Hausmann and D. Schuetz ([1]). A rough outline of the argument is as follows. Firstly, one notes that M_ℓ can be viewed as a subspace of a torus W. The cohomology ring of W is an exterior algebra and the goal is to demonstrate that under suitable assumptions on the length vector ℓ, a subring of the cohomology ring of M_ℓ can be computed explicitly as the quotient of this exterior algebra by a certain monomial ideal. To show this, one studies the homological long exact sequence of the pair $(W, W - M_\ell)$. Using symmetry with respect to a certain involution $\tau : W \to W$ and a Morse-theoretic argument, one can give an explicit homology basis for the complement $W - M_\ell$. Studying intersection theory on W, one determines the image of the map $H_*(W - M_\ell) \to H_*(W)$ induced by inclusion. This allows to compute the so-called *balanced subalgebra* of $H^*(M_\ell)$, this is the subalgebra of all the classes a with $\tau^* a = (-1)^{|a|} a$. The subalgebra turns out to be a quotient of the desired form. Finally one can show that for a large class of length vectors the balanced subalgebra coincides with the subalgebra generated by $H^1(M_\ell)$ and is thus recovered by the graded isomorphism type of $H^*(M_\ell)$. Applying an algebraic result of J. Gubeladze ([2]) concludes the argument.

The general case of the conjecture was recently established by D. Schuetz ([3]) using a detailed study of the cohomology rings of those spaces M_ℓ which are not covered by the theorem described above.

References

[1] M. Farber, J.-Cl. Hausmann, D. Schuetz, *On the Conjecture of Kevin Walker*, J. of Topology and Anlysis 1 (2009), 65-86

[2] J. Gubeladze, *The Isomorphism Problem for Commutative Monoid Rings*, Journal of Pure and Applied Algebra 129 (1998), 35-65

[3] D. Schuetz, *The Isomorphism Problem for Planar Polygon Spaces*, Journal of Topology (2010) 3(3), 713-742

[4] K. Walker, *Configuration Spaces of Linkages*, Bachelor's Thesis, Princeton (1985)

Higher Topological Complexity and Higher Symmetric Topological Complexity

Ibai Basabe

In [2, 3], Michael Farber introduced the notion of topological complexity (TC). Here, we present some of the work in [1], in particular, the higher versions of topological complexity ($\mathrm{TC}_n(X)$) of a path connected space X as well as two symmetric versions of topological complexity (TC^S and TC^Σ) and their higher analogues (TC_n^S and TC_n^Σ).

In January of 2010, Yuli Rudyak presented the following higher versions of TC [5]: we can consider a motion planning problem whose input is not only a pair of initial and final states, but also an additional set of $n-2$ ordered intermediate states.

Define the *Schwarz genus* of a fibration $p : E \to B$ ($\mathfrak{genus}\,(p)$) to be the minimum number k such that there is an open covering $U_0, U_1, \ldots, U_k$ of B for which the restriction of p over each U_i, $i = 0, 1, \ldots, k$ has a continuous section.

Definition. Let X be a path-connected space. The n^{th} *topological complexity* of X, denoted by $\mathrm{TC}_n(X)$, is the Schwarz genus of the fibration

$$e_n^X = e_n : X^{J_n} \to X^n, \quad e_n(\gamma) = (\gamma(1_1), \ldots, \gamma(1_n))$$

where J_n is the wedge of n closed intervals $[0, 1]$ (each with $0 \in [0, 1]$ as the base point), and 1_i stands for 1 in the i^{th} interval.

We have $\mathrm{TC}(X) = \mathrm{TC}_2(X)+1$ and just as TC, $\mathrm{TC}_n(X)$ is a homotopy invariant of X for all n. We also discuss some other results in [5], such as the proof of the fact that $\mathrm{TC}_n(S^k)$ is $n-1$ for k odd and n for k even, and the definition of two functions $f_a(n) = \min_X\{\mathrm{TC}_n(X)|\,\mathrm{TC}_2(X) = a\}$ and $g_a(n) = \max_X\{\mathrm{TC}_n(X)|\,\mathrm{TC}_2(X) = a\}$ with $a \in \mathbb{N}$.

Note that for a connected finite not contractible CW space,

$$n - 1 \le f_a(n) \le g_a(n) \le na$$

since we have the following proposition from [5]:

Proposition. If X is a connected finite CW-space that is not contractible, then $\mathrm{TC}_n(X) \ge n-1$.

The Schwarz genus of a fibration over X does not exceed $\mathrm{cat}(X)$ [6]. Thus, $\mathrm{TC}_n(X) \le \mathrm{cat}(X^n) \le n\,\mathrm{cat}(X) \le n \dim(X)$.

On the other hand, we can also bound $\mathrm{TC}_n(X)$ from below using the LS-category.

Proposition. For any path-connected space X, $\mathrm{cat}(X^{n-1}) \leq \mathrm{TC}_n(X)$.

For G a path-connected topological group we can give a complete characterization of $\mathrm{TC}_n(G)$ by proving the equality $\mathrm{TC}_n(G) = \mathrm{cat}(G^{n-1})$.

Alternatively, we can look at the growth of TC_n in terms of the difference of any two consecutive values of n. For any path-connected topological group G, $\mathrm{TC}_n(G) - \mathrm{TC}_{n-1}(G) \leq \mathrm{cat}(G)$. In general, for any path-connected space X, $\mathrm{TC}_n(X) - \mathrm{TC}_{n-1}(X) \leq \mathrm{cat}(X^2)$.

In the following definition (modified cup-length of a space X) we consider cohomology with local coefficients.

Definition. Given a space X, a natural number n and the diagonal map $d_n^X = d_n : X \to X^n$, define the d_n^X*-cup-length*, denoted by $\mathrm{cl}(X, n)$, to be the maximum m with the following property: there exist cohomology classes $u_i \in H^*(X^n; A_i)$ such that $d_n^* u_i = 0$ for $i = 1, \ldots, m$ and $u_1 \smile \cdots \smile u_m \neq 0 \in H^*(X^n; A_1 \otimes \cdots \otimes A_m)$.

The following, which follows directly from Theorem 4 in [6], gives a lower bound for TC_n in terms of $\mathrm{cl}(X, n)$. For any path-connected space X we have the inequality $\mathrm{cl}(X, n) \leq \mathrm{TC}_n(X)$.

We have the following inequality for path-connected paracompact Hausdorff spaces X and Y, $\mathrm{TC}_n(X \times Y) \leq \mathrm{TC}_n(X) + \mathrm{TC}_n(Y)$.

Next, we give the higher topological complexity of concrete families of spaces.

Corollary. $\mathrm{TC}_n(T^k) = k(n-1)$.

Proposition. $\mathrm{TC}_n(S^{k_1} \times S^{k_2} \times \cdots \times S^{k_m}) = m(n-1) + l$ where l is the number of even dimensional spheres.

Proposition. For every closed simply connected symplectic manifold M^{2m} we have $\mathrm{TC}_n(M) = nm$.

Farber and Grant suggested in [4] a symmetrized version of topological complexity (TC^S). Symmetric motion planning comes into play in the case in which the motion from configuration a to configuration b must be the time-reverse movement of that from b to a. Here we start by proposing a slight modification TC^Σ.

Consider the involutions $\tau : X \times X \to X \times X$ and $\overline{\tau} : PX \to PX$ defined by $\tau(x, y) = (y, x)$ and $\overline{\tau}(\gamma)(t) = \gamma(1-t)$, for $(x, y) \in X \times X$ and $\gamma \in PX$.

Definitions. A subset A in $X \times X$ is *symmetric* if $\tau A = A$. A function $s : A \to PX$ is *symmetric* if $\overline{\tau}(s(a)) = s(\tau(a))$ for $a \in A$, where A is a symmetric subset of $X \times X$. Now, we define $\mathrm{TC}^\Sigma(X)$ as the minimum number k such that $X \times X = A_0 \cup A_1 \cup \cdots \cup A_k$ where each A_i is open, symmetric, and has a continuous symmetric section $s_i : A_i \to PX$ of the map e_2.

We now proceed by defining TC^S. The symmetric group Σ_n acts on $e_n^{-1}(C_n(X))$ and $C_n(X)$ by permuting paths in the former case, and by permuting coordinates in the latter (where $C_n(X)$ stands for the configuration space of n ordered distinct points in a space X). These actions are free and the restricted fibration $e_n \colon e_n^{-1}(C_n(X)) \to C_n(X)$ is equivariant. There is a resulting fibration $\varepsilon_n^X =$

$\varepsilon_n \colon Y_n(X) \to B_n(X)$ at the level of orbit spaces, where $Y_n(X) = e_n^{-1}(C_n(X))/\Sigma_n$ and $B_n(X) = C_n(X)/\Sigma_n$.

In our terms, Farber and Grant's definition of symmetric topological complexity of a space X amounts to setting $\mathrm{TC}^S(X) = 2 + \mathfrak{genus}\,(\varepsilon_2)$. But, in accordance with the normalization, we have $\mathrm{TC}_2^S(X) = 1 + \mathfrak{genus}\,(\varepsilon_2)$.

For each ENR X, $\mathrm{TC}^\Sigma(X)$ and $\mathrm{TC}_2^S(X)$ are related by $\mathfrak{genus}\,(\varepsilon_2) \leq \mathrm{TC}^\Sigma(X) \leq 1 + \mathfrak{genus}\,(\varepsilon_2) = \mathrm{TC}_2^S(X)$.

The construction of TC^Σ can be generalized as follows.

Definitions. A subset A in X^n is *symmetric* if $\sigma A = A$ for all $\sigma \in \Sigma_n$. For a symmetric $A \subset X^n$, a function $s : A \to X^{J_n}$ is *symmetric* if $\sigma(s(a)) = s(\sigma(a))$ for all $a \in A$ and $\sigma \in \Sigma_n$. We then define $\mathrm{TC}_n^\Sigma(X)$ as the minimum number k such that $X^n = A_0 \cup A_1 \cup \cdots \cup A_k$ where each A_i is open, symmetric and has a continuous symmetric section $s_i : A_i \to X^{J_n}$ for e_n.

It is important to emphasize that $\mathrm{TC}_n^\Sigma(X)$ is a homotopy invariant of X, while $\mathrm{TC}_n^S(X)$ is not a homotopy invariant of X for any n.

The following conjecture suggests the extension of $\mathrm{TC}_2^S(X)$ that follows it next.

Conjecture. We have: $\mathfrak{genus}\,(\varepsilon_n) \leq \mathrm{TC}_n^\Sigma(X) \leq \mathfrak{genus}\,(\varepsilon_n) + \cdots + \mathfrak{genus}\,(\varepsilon_2) + n - 1$.

Definition. For $n \geq 2$ set $\mathrm{TC}_n^S(X) = \mathfrak{genus}\,(\varepsilon_n) + \cdots + \mathfrak{genus}\,(\varepsilon_2) + n - 1$.

For a sphere, we obtain the following upper bound for $\mathrm{TC}_n^S(S^k)$.

Proposition. For integers $k > 0$ and $n > 1$,

$$\mathrm{TC}_n^S(S^k) \leq \Big[(n+2)(k-1) + 4\Big]\,(n-1)/2k.$$

References

[1] Basabe, I.; González, J.; Rudyak, Y.; Tamaki, D,: Higher topological complexity and homotopy dimension of configuration spaces on spheres, Preprint arXiv:1009.1851v1 (2010).

[2] Farber, M.: Topological complexity of motion planning. *Discrete Comput. Geom.* **29** (2003) 211–221.

[3] Farber, M.: *Invitation to topological robotics.* Zurich Lectures in Advanced Mathematics, EMS, Zürich, 2008, x+133 pp.

[4] Farber, M.; Grant, M.: Symmetric motion planning. In: Topology and Robotics, *Contemp. Math.* **438**, Amer. Math. Soc., Providence, RI (2007) 85–104.

[5] Rudyak, Yu.: On higher analogs of topological complexity. *Topology and its Applications* **157** (5) (2010) 916–920.

[6] Švarc (Schwarz), A.: The genus of a fiber space. *Amer. Math. Soc. Transl.* Series 2 **55** (1966) 49–140.

Universility Theorems for Linkages

Marco Hamann

Moduli Space of Planar Linkages. In this contribution, the moduli spaces of planar linkages are considered. From a physical point of view, they have only rotational links which are connected by fixed joints. Mathematically, they can be linked to weighted graphs, see [2].

Definition 1. An *abstract marked linkage* $\mathcal{L}$ is a triple (L, l, W) consisting of

(a) a graph L with vertices $\mathcal{V}(L)$ and edges $\mathcal{E}(L)$
(b) an ordered subset $W \subset \mathcal{V}(L)$, the marking (fixed vertices)
(c) $l : \mathcal{E}(L) \to \mathbb{R}_+$ (weights).

The marking might be also empty; the linkage $\mathcal{L}$ is then called an *abstract linkage*, while for the special marking $W = \{v_1, v_2\}$ it is called an *abstract based linkage*. A planar realisation of $\mathcal{L}$ is a map $\phi : \mathcal{V}(L) \to \mathbb{R}^2$ such that

$$|\phi(v) - \phi(w)|^2 = (l[vw])^2 \tag{0.1}$$

for any $[vw] \in \mathcal{E}(L)$. We use the natural identification of $\mathbb{R}^2 \cong \mathbb{C}$. Clearly, for based linkages one can assume $\phi(v_1) = (0, 0)$ and $\phi(v_2) = (l(e^\star), 0)$ for the edge $e^\star = [v_1, v_2]$. The set of all planar realisations of a based linkage is called the *moduli space* $\mathcal{M}(\mathcal{L})$, for an abstract linkage the configuration space $C(\mathcal{L}, Z)$ where Z is the required image of W under ϕ.

Clearly, the conditions (0.1) determine a real algebraic set in $\mathbb{R}^{2n}$ associated with $\mathcal{M}(\mathcal{L})$, where n denotes the number of vertices of L. Here the ideal I generated by (0.1) is contained in the polynomial ring $\mathbb{R}[x_1, y_1, \ldots, x_n, y_n]$, where $(x_i, y_i) = \phi(v_i)$. It determines further an affine subscheme $\mathfrak{M}(\mathcal{L})$ of $\mathbb{R}^{2n}$ of all polynomials vanishing on the set $\mathcal{M}(\mathcal{L})$.

Functional Linkages. A particular class of abstract linkages are so-called functional linkages. For their definition, we choose m distinguished vertices $P_1, \ldots, P_m \in \mathcal{V}(\mathcal{L})$ as *input vertices* and n distinguished vertices $Q_1, \ldots, Q_n \in \mathcal{V}(\mathcal{L})$ as *output vertices*. Furthermore, we choose two mappings

$$p : \mathcal{M}(\mathcal{L}) \to \mathbb{C}^m, \quad q : \mathcal{M}(\mathcal{L}) \to \mathbb{C}^n \tag{0.2}$$

as *input* and *output mapping* recording the positions of the inputs and outputs under the realisations ϕ.

Definition 2. The linkage $\mathcal{L}$ is called *functional* for a given mapping $f : \mathbb{C}^m \to \mathbb{C}^n$ if $f = q \circ p^{-1}$ exists and p is a regular topological branched cover of a bounded domain $\mathcal{O} \subset \mathbb{C}^m$.

Occasionally, we refer to $\mathbb{R}$-functional linkages if the mapping can be specified to $f : \mathbb{R}^m \to \mathbb{R}^n$ using $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$ and the identification $\mathbb{C} \cong \mathbb{R}^2$.

Functionality Theorems. The operation of *fiber sum* of linkages can be used to construct new linkages. Formally, it is equal to the generalised free products of groups, see [1]. More detailed, let $\mathcal{L}' = (L', l', W')$ and $\mathcal{L}'' = (L'', l'', W'')$ be abstract marked linkages and

$$\beta : S' \subset \mathcal{V}(L') \to S'' \subset \mathcal{V}(L'').$$

with $S', S'' \neq \emptyset$. For given Z', Z'' we require $\phi'(w_j) = \phi''(\beta(w_j))$ for each $w_j \in W'$ and $\phi' \in C(\mathcal{L}', Z')$, $\phi'' \in C(\mathcal{L}'', Z'')$. Then by

(1) the weighted graph (L, l) obtained from $(L', l') \sqcup (L'', l'')$ by identifying v and $\beta(v)$, $v \in S'$
(2) the image of marking $Z = (\ldots, \phi(w_j), \ldots)$ with $w_j \in W$ and ϕ in $C(\mathcal{L}', Z')$ or in $C(\mathcal{L}'', Z'')$, where W is the image of $W' \sqcup W''$ in L

the fiber sum $\mathcal{L} = \mathcal{L}' *_\beta \mathcal{L}''$ is constructed. Evidently, we may consider also the *self-fiber sum* $\mathcal{L}' *_\beta$ for $\mathcal{L}' = \mathcal{L}''$ when replacing $L' \sqcup L''$ in 1 by L'.

Essentially, the functionality theorems state that the fiber sum of two linkages, which are functional for the (vector-valued) functions f and g, is again a linkage, which is functional for the composition of f and g.

Theorem 3. *Let $\mathcal{L}'$ and $\mathcal{L}''$ functional linkages for the functions f and g, furthermore $\mathcal{L} = \mathcal{L}' *_\beta \mathcal{L}''$ the fiber sum of these linkages with respect to β. If $\mathrm{int}(\mathcal{O}') \cap g^{-1}\mathrm{int}(\mathcal{O}'') \neq \emptyset$ then $\mathcal{L}$ is a functional linkage for the composition $h = f \circ g$, see* [1].

Moreover, self-fiber sums provide a technique to construct closed functional linkages. Elementary linkages are derived by modifications of classical linkages such as the rigidified pantograph and the rigidified inversor. They are functional for $z \mapsto \lambda z$ and $z \mapsto t^2/\bar{z}$, for example, and serve as building blocks for functional linkages associated with polynomial mappings.

Universality Theorems. After developing useful techniques for the construction of real functional linkages from complex ones and the expansion of domains by functional linkages, we prove

Theorem 4. *Let $f : k^m \to k^n$ be a polynomial map where k is either $\mathbb{C}$ or $\mathbb{R}$. Let further $\mathbf{o} \in k^m$ be a point and $r > 0$. Then there is a functional linkage $\mathcal{L}$ for f such that the ball $\mathcal{B}_r(\mathbf{o})$ is in the interior of the image of p and p is an analytically trivial covering over $\mathcal{B}_r(\mathbf{o})$, see* [1].

For polynomial maps with not necessary real coefficients, one has to use relative configuration space $C(\mathcal{L}, Z)$ instead of the moduli space $\mathcal{M}(\mathcal{L})$. However, one can easily extend Theorem 4 to compact real-algebraic sets, that is, to the zero sets of polynomial mappings $f : \mathbb{R}^m \to \mathbb{R}^n$, see [1].

References

[1] M. Kapovich and J. J. Millson, *Universality theorems for configuration spaces of planar linkages*, Topology **41** (2002), 1051-1107.

[2] D. Jordan and M. Steiner, *Configuration spaces of mechanical linkages*, Discrete Comput. Geom. **22** (1999) 297-315. MR1698549 (2000f:52030)

TC and the immersion and embedding problems I

Mark Grant

The talk described an intriguing connection between the topological complexity of real projective spaces and their immersion dimension. Recall that the Topological Complexity $\mathbf{TC}(X)$ of a space X is defined to be the Scwharz genus $\mathfrak{genus}(\pi)$ of the free path fibration on X,

$$\pi\colon\ X^I \to X \times X, \qquad \pi(\gamma) = \big(\gamma(0), \gamma(1)\big),$$

which takes a path in X to its pair of initial and final points. For more information on this invariant, the reader may consult for instance [3], [4] or the earlier abstracts in this volume. For any natural number $n \geq 1$, the n-dimensional real projective space P^n is the smooth manifold resulting from identifying antipodal points on the n-sphere. The *immersion dimension* $I(n)$ of P^n is the smallest natural number k such that P^n admits an immersion in $\mathbb{R}^k$ (we use the notation $P^n \looparrowright \mathbb{R}^k$).

Theorem 1 (Farber, Tabachnikov, Yuzvinsky [6])**.** *We have*

$$\mathbf{TC}(P^n) = \begin{cases} I(n)+1 & & \text{for } n \neq 1,3,7 \\ I(n) & = n+1 & \text{for } n = 1,3,7. \end{cases}$$

The immersion problem (compute $I(n)$ as a function of n) is a well-known classical open problem which has been the catalyst for important advances in Algebraic and Differential Topology (for a survey of known results and a bibliography on this and the related embedding problem, see Don Davis' web site [2]). It is therefore surprising that it should be equivalent to a fundamental problem in topological robotics, concerning motions of a line which pivots about the origin in Euclidean space. By way of contrast, we mention that the topological complexity of complex projective spaces satisfy the simple equality $\mathbf{TC}(\mathbb{C}P^n) = 2n+1$, whilst the immersion dimension for these manifolds is still largely unknown. We also mention that Jesus González [7] has discovered a connection between the topological complexity of lens spaces and their immersion dimension, which was not exposited here.

The proof of Theorem 1 proceeds by relating motion planning algorithms on P^n with certain types of axial and non-singular maps.

Definition 1. *A map $f\colon P^n \times P^n \to P^m$ (where $n \leq m$) is called* axial *if each restriction $f|_{P^n\times\{*\}}\colon P^n \times \{*\} = P^n \to P^m$, $f|_{\{*\}\times P^n}\colon \{*\} \times P^n = P^n \to P^m$ is homotopic to the standard inclusion $P^n \subseteq P^m$. (Equivalently, f is axial if $f^*(a) = 1 \times a + a \times 1 \in H^1(P^n \times P^n; \mathbb{Z}_2)$, where $a \in H^1(P^m; \mathbb{Z}_2)$ is the generator.)*

Definition 2. *A map $\varphi\colon \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{m+1}$ is called* non-singular* *if it satisfies the following conditions:*

(1) $\varphi(\lambda u, \mu v) = \lambda\mu\varphi(u, v)$ for all $\lambda, \mu \in \mathbb{R}$ and $u, v \in \mathbb{R}^{n+1}$;
(2) $\varphi(u, v) = 0$ implies that $u = 0$ or $v = 0$;

(3) If $\varphi = (\varphi_1, \ldots, \varphi_m)$ is written in terms of its coordinate functions, then $\varphi_1(u,u) > 0$ for all $u \in \mathbb{R}^{n+1} \setminus \{0\}$.

Axial maps are related to immersions by the following classical result.

Theorem 2 (Adem, Gitler, James [1])**.** *Let $n < k$. There exists an axial map $P^n \times P^n \to P^k$ if and only if $P^n \looparrowright \mathbb{R}^k$.*

The existence of axial maps was related to the existence of non-singular* maps by the authors of [6].

Lemma 1 ([6])**.** *Let $n < k$. There exists an axial map $P^n \times P^n \to P^k$ if and only if there exists a non-singular* map $\varphi\colon \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$.*

Theorem 1 now follows from the following statement, whose proof was outlined in the talk.

Theorem 3 ([6])**.** *The following are equivalent:*

a) $\mathbf{TC}(P^n) \leq k$;
b) The k-fold Whitney sum $k(\gamma \boxtimes \gamma) \to P^n \times P^n$ admits a nowhere vanishing section (here γ is the canonical Hopf line bundle over P^n, and $\boxtimes$ denotes the exterior tensor product of vector bundles);
c) There exists a non-singular map $\varphi\colon \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^k$.*

Remarks.
(i) In the exceptional cases $n = 1, 3, 7$ the projective space P^n has the structure of a (not necessarily associative) topological group, with multiplication induced by the non-singular maps given by multiplication in $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ respectively. Thus $\mathbf{TC}(P^n) = \mathrm{cat}(P^n) = n + 1$ in these dimensions.
(ii) Let $A(n)$ denote the smallest natural number k for which there exists an axial map $P^n \times P^n \to P^k$. Then Jesus González has pointed out to me that Theorem 1 can be expressed in the more compact form $\mathbf{TC}(P^n) = A(n) + 1$.
(iii) As a corollary of the above proof, we found that $\mathbf{TC}(P^n) = \mathfrak{genus}(q)$, where

$$q\colon\ S^n \times_{\mathbb{Z}_2} S^n \to P^n \times P^n, \qquad q([v,w]) = ([v],[w])$$

is the the double cover of $P^n \times P^n$ by the quotient of $S^n \times S^n$ under $\mathbb{Z}_2$ acting diagonally by the antipodal involution in each factor.

Finally we offered some evidence that existing non-immersion results could be framed in terms of **TC**-weights of cohomology classes, and speculated that this may lead to improved lower bounds for the immersion dimension. Recall that for $u \in h^*(X \times X)$, where h^* is any multiplicative cohomology theory, the **TC**-*weight* of u is a number $\mathrm{wgt}(u) \in \mathbb{N}_0 \cup \{\infty\}$ satisfying the following properties:

(1) $\mathbf{TC}(X) > \mathrm{wgt}(u)$ whenever u is nonzero;
(2) $\mathrm{wgt}(u_1 \cdots u_\ell) \geq \sum_{i=1}^{\ell} \mathrm{wgt}(u_i)$;
(3) $\mathrm{wgt}(u) \geq 1$ if and only if $0 = \triangle_X^*(u) \in h^*(X)$ (so u is a *zero-divisor*).

Lemma 2. *Let h^* be a multiplicative complex-oriented cohomology theory, with formal group law $+_h$ expressing the behaviour of Euler classes of complex line bundles under tensor product. Let $e = e(\gamma_{\mathbb{C}}) \in h^2(P^n)$ be the Euler class of the complexification of the Hopf line bundle $\gamma \to P^n$. Let $e_i = p_i^*(e) \in h^2(P^n \times P^n)$, where $p_i\colon P^n \times P^n \to P^n$ is the projection onto the i-th factor for $i = 1, 2$. Then*

$$\mathrm{wgt}(e_1 +_h e_2) \geq 2.$$

Proof. The Euler class of $\gamma_{\mathbb{C}} \boxtimes \gamma_{\mathbb{C}}$ in ordinary cohomology $H^*(-;\mathbb{Z})$ is

$$e_1 +_H e_2 = 1 \times b - b \times 1 = 1 \times \beta(a) - \beta(a) \times 1 \in H^2(P^n \times P^n; \mathbb{Z}),$$

where $a \in H^1(P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$ and $b \in H^2(P^n; \mathbb{Z}) \cong \mathbb{Z}_2$ are the generators and $\beta\colon H^1(-;\mathbb{Z}_2) \to H^2(-;\mathbb{Z})$ is the Bockstein operator. It follows from Theorem 6 of [5] that $\mathrm{wgt}(e_1 +_H e_2) \geq 2$. Now by the naturality of the Euler class we have $\mathrm{wgt}(e_1 +_h e_2) \geq 2$. □

References

[1] J. Adem, S. Gitler, and I. M. James, *On axial maps of a certain type*, Bol. Soc. Mat. Mexicana (2) 17 (1972), 59-62.

[2] D. M. Davis, *Table of immersions and embeddings of projective spaces*, `http://www.lehigh.edu/~dmd1/immtable`

[3] M. Farber, *Topological complexity of motion planning*, Discrete Comput. Geom. 29 (2003), 211–221.

[4] M. Farber, *Instabilities of robot motion*, Topology Appl. 140 (2004), 245–266.

[5] M. Farber and M. Grant, *Robot motion planning, weights of cohomology classes, and cohomology operations*, Proc. Amer. Math. Soc. 136 (2008), 3339-3349.

[6] M. Farber, S. Tabachnikov and S. Yuzvinsky, *Topological robotics: motion planning in projective spaces*, Int. Math. Res. Not. (2003), no. 34, 1853–1870.

[7] J. González, *Topological robotics in lens spaces*, Math. Proc. Cambridge Philos. Soc. 139 (2005), no. 3, 469–485.

Random 2-dimensional complexes, Part I

Thilo Kuessner

Random 2-dimensional complexes are generalizations of random graphs. One generates a random 2-dimensional complex $Y(n,p)$ by considering the full 1-dimensional skeleton of the simplex on vertices $\{1, \ldots, n\}$ and adding 2-dimensional faces independently with probability p. In particular $Y(n,p)$ contains the full 1-skeleton. For a given function $p : \mathbf{N} \to [0,1]$ we say that $Y(n,p)$ has property E asymptotically almost sure (a.a.s.) if

$$\lim_{n\to\infty} P\left(Y\left(n, p\left(n\right)\right) \text{ has property } E\right) = 1.$$

Theorem 1. *: If $\lim_{n\to\infty} \omega(n) = \infty$ and*

$$p(n) \geq \sqrt{\frac{3\log n + \omega(n)}{n}},$$

then $Y(n,p)$ is a.a.s. simply connected.

Proof: If p is as stated, then for every pair of vertices $\{a, b\}$ the following two conditions hold a.a.s.:
- there is some vertex d such that a, b, d are vertices of a 2-simplex in Y,
- the intersection $lk\,(a) \cap lk\,(b)$ is connected.
(The second condition holds because $lk\,(a) \cap lk\,(b)$ is a random graph $G\left(n-2, p^2\right)$ to which the Erdös-Renyi threshold for connectivity of random graphs applies.)

These two conditions imply easily that for each triple of vertices $\{a, b, c\}$ the corresponding 3-cycle bounds a simplicial disk. *QED*

Theorem 2. *: If* $\epsilon > 0$ *and*

$$p\,(n) = O\left(\frac{1}{n^{\frac{1}{2}+\epsilon}}\right),$$

then $\pi_1 Y\,(n, p)$ *is a.a.s. nontrivial and word-hyperbolic.*

Proof: For a 2-complex Z and $i \in \{0, 1, 2\}$ we denote $f_i\,(Z)$ the number of i-simplices of Z. A 2-complex on vertices $\{1, \ldots, n\}$ is $(\epsilon, 3)$-admissible if every subcomplex Z with $\{1, 2, 3\} \subset Z$ satisfies $f_0\,(Z) - 3 \geq \left(\frac{1}{2} + \epsilon\right) f_2\,(Z)$. It is $(\epsilon, m, 3)$-sparse if this condition holds for each subcomplex with at most m vertices. A straightforward computation shows that for each $m \in \mathbf{N}, \epsilon > 0, p = O\left(\frac{1}{n^{\frac{1}{2}+\epsilon}}\right)$, the random 2-complex $Y\,(n, p)$ is $(\epsilon, m, 3)$-sparse a.a.s.

The main topological result from [1] is that each $(\epsilon, 3)$-admissible, finite, 2-dimensional simplicial complex is homotopy equivalent to a disjoint union of wedges of circles, spheres and projective planes. In particular the fundamental group is a free product of copies of $\mathbf{Z}$ and $\mathbf{Z}/2\mathbf{Z}$ and thus is word-hyperbolic.

Gromov has proven that a group Γ is word-hyperbolic if and only if every finite simplicial complex X with $\pi_1 X \cong \Gamma$ satisfies a linear isoperimetric inequality. In particular, if Y is an $(\epsilon, m, 3)$-sparse 2-complex and Z is a subcomplex Z which contains the vertices $\{1, 2, 3\}$ but has at most m vertices, then Z must be $(\epsilon, 3)$-admissible, hence has hyperbolic fundamental group, hence satisfies a linear isoperimetric inequality.

However, there is a local-to-global principle for isoperimetric inequalities (for groups due to Gromov and generalized to 2-complexes in [1]) which allows then to deduce a linear isoperimetric inequality (and hence word-hyperbolic fundamental group) for the whole $(\epsilon, m, 3)$-sparse 2-complex Y.

Finally the linear isoperimetric inequality together with the definition of $(\epsilon, m, 3)$-sparsity allows to deduce that the 3-cycle with vertices $\{1, 2, 3\}$ is a.a.s. not 0-homotopic in $Y\,(n, p)$. In particular $\pi_1 Y\,(n, p) \neq 0$ a.a.s. *QED*

References

[1] E. Babson, C. Hoffman, M. Kahle, *"The fundamental group of random 2-complexes"*, appears in J. AMS

Totaro's spectral sequence

Andrés Angel

The main source for this talk is [2]. The goal was the study of a spectral sequence that abuts to the cohomology of the configuration spaces $F(X,n)$ of a space X, where $F(X,n)$ is the set of n distinct points in X.

The spectral sequence is the Leray spectral sequence for the inclusion $F(X,n) \subset X^n$. For orientable manifolds the E_2 term and the first non-trivial differential can be explicitly identified.

In the case of a projective algebraic variety, the spectral sequence has only one non-trivial differential if the coefficients are taken in a field of characteristic zero and we can determine from the spectral sequence the rational cohomology ring of the configuration space of n-tuples of distinct points in X. The answer depends only on the cohomology ring of X.

In this talk I described the paper of Burt Totaro "Configuration spaces of algebraic varieties", where the Leray spectral sequence for the inclusion $F(X,n) \subset X^n$ is studied. This spectral sequence converges to the cohomology of the ordered configuration spaces with integral coefficients. It was described earlier by Cohen and Taylor in [1] algebraically by a filtration of a dga that calculates the cohomology of the configuration spaces. Taking field coefficients,

Theorem 1. *Let X be an oriented manifold. Then there is a spectral sequence of S_n-algebras converging to $H^*(F(X,n);\mathbb{K})$. The E_2 term is the quotient of the graded-commutative $\mathbb{K}$-algebra*

$$H^*(X^n;\mathbb{K})[A_{a,b}] \qquad \text{for } 1 \le a \ne b \le n$$

where $H^i(X^n,\mathbb{K})$ has degree $(i,0)$ and the $A_{a,b}$ are of degree $(0,m-1)$, subject to the relations,

$$\begin{aligned} A_{a,b} &= (-1)^m A_{b,a} \\ A_{a,b}^2 &= 0 \\ A_{a,b}A_{a,c} + A_{b,c}A_{b,a} + A_{c,a}A_{c,b} &= 0 \text{ for } k < j < i \\ p_a^*(x)A_{a.b} &= p_b^*(x)A_{a,b} \text{ for } a \ne b, x \in H^*(X;\mathbb{K}) \end{aligned}$$

The first non-trivial differential is given by

$$dA_{a,b} = p_{a,b}^*(\Delta)$$

where $\Delta \in H^m(X^2;\mathbb{K})$ is the diagonal class. The action of S_n is induced from the action on $H^*(X^n;\mathbb{K})$ and $\sigma A_{a,b} = A_{\sigma(a),\sigma(b)}$.

The Leray spectral sequence associated to a continuous map $f : X \to Y$, can be seen as a special case of the Grothendieck spectral sequence for the derived functor of the composition of two left exact functors. The global section functors Γ and the direct image f_*, $Sheaves(X) \xrightarrow{f_*} Sheaves(Y) \xrightarrow{\Gamma} Abelian$. In our case for the inclusion $F(X,n) \subset X^n$, and the locally constant sheaf $\underline{\mathbb{Z}}$ on $F(X,n)$, it is a spectral sequence, $H^p(X^n; R^q j_* \underline{\mathbb{Z}}) \Rightarrow H^{p+q}(F(X,n),\mathbb{Z})$ converging to the cohomology of the configuration space. The $E_2^{p,q}$ is the cohomology of the n-fold product with coefficients in the higher direct images of the locally constant sheaf $\underline{\mathbb{Z}}$ under the inclusion $j : F(X,n) \subset X^n$.

The higher direct image sheaf $R^q j_* \underline{\mathbb{Z}}$ is the sheafication of the presheaf $U \to H^q(U \cap F(X,n);\mathbb{Z})$, and then when X is a manifold, we use the local structure of X to calculate the stalk. For $x \in X^n$, suppose (after permutation of indices) that x has the form $x = (x_1,\ldots,x_1,\ldots,x_s,\ldots,x_s)$ with $x_1,\ldots,x_s \in X$ and $\sum i_j = n$, then the stalk at x of the higher direct image sheaf $R^q j_* \underline{\mathbb{Z}}$ is,

$$H^q(F(T_{x_1}X, i_1) \times \ldots \times F(T_{x_s}X, i_s);\mathbb{Z})$$

and therefore we are led to the study of the cohomology of products of ordered configuration spaces of euclidean spaces. To state the main results let us introduce some notation, recall that a partition I of a set $\{1,\ldots,n\}$ is a non-empty collection of subsets of $\{1,\ldots,n\}$ that are disjoint and whose union is $\{1,\ldots,n\}$.

Definition 2. To a partition I into k blocks, ($|I| = k$), we associate the diagonal subspace $X_I^k \subseteq X^n$,

$$X_I^k := \{(x_1,\ldots,x_n) \in X^n \mid x_i = x_j \text{ if } i,j \text{ are in the same block of } I\ \}$$

We say that a partition J refines a partition I ($J \leq I$) if for every $A \in J$ there exists $B \in I$ with $A \subseteq B$.

By the results of Cohen we know that the cohomology algebra of ordered configuration spaces of n points in $\mathbb{R}^m$ is concentrated on multiples of $m-1$ and the top dimensional cohomology group is equal to $\mathbb{Z}^{(n-1)!}$ in dimension$(n-1)(m-1)$, similarly for a product of the form $F(R^m, i_1) \times \cdots \times F(R^m, i_k)$, the cohomology is zero except in dimensions divisible by $m-1$. Each factor $F(\mathbb{R}^m, i_s)$ has top non-zero cohomology in dimension $(i_s-1)(m-1)$ and therefore for the product $F(R^m, i_1) \times \cdots \times F(R^m, i_k)$ the top non-zero cohomology is in dimension $\sum_s (i_s-1)(m-1) = (n-k)(m-1)$. $\mathbb{Z}^{c_I}$ where, $c_I := \prod_{A\in I}(|A|-1)! = \prod_s (i_s-1)!$. Suppose that I is a partition with k blocks of sizes $i_1,\ldots,i_k$, and J a refinement of I with blocks of sizes $j_1,\ldots,j_{n-r}$. We have a natural restriction map, $\prod_s^k F(X,i_s) \to \prod_s^{n-r} F(X,j_s)$. By adding the induced maps on cohomology over all refinements,

Lemma 3. *For $0 \leq r \leq n-k$ we have an isomorphism,*

$$\bigoplus_J H^{r(m-1)}(F(\mathbb{R}^m, j_1)\times\cdots\times F(\mathbb{R}^m, j_{n-r});\mathbb{Z}) \cong H^{r(m-1)}(F(R^m, i_1)\times\cdots\times F(R^m, i_k);\mathbb{Z})$$

where the sum is over all partitions of J of $\{1,...,n\}$ with $n-r$ blocks such that $J \leq I$.

This lemma shows that the $r(m-1)$ dimensional classes are pulled back from classes that are top dimensional for some refinement of I. For example, a basis for $H^{r(m-1)}(F(\mathbb{R}^m, n);\mathbb{Z})$ is given by monomials,

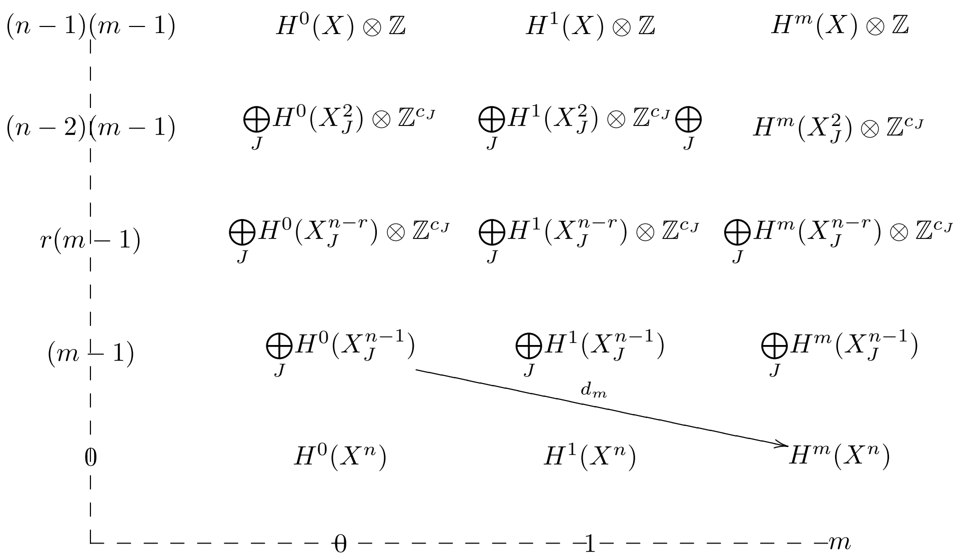
$$A_{a_1,b_1}\cdots A_{a_r,b_r}, a_1 < \cdots < a_r \text{ and } b_s < a_s, 1 \leq s \leq r$$

and we can define a partition J of $\{1,\ldots,n\}$ with $n-r$ blocks by, $a_s \sim b_s$, then it is clear that the element $A_{a_1,b_1}\cdots A_{a_r,b_r}$ lies in the image of the map

$$H^{r(m-1)}(F(\mathbb{R}^m,j_1)\times\cdots\times F(\mathbb{R}^m,j_{n-r});\mathbb{Z})\to H^{r(m-1)}(F(R^m,n);\mathbb{Z})$$

From this lemma follows that the higher direct images sheaves are sums of locally constant sheaves supported on the diagonals and if we assume that X is oriented we have an isomorphism of sheaves, $R^{r(m-1)}j_*\underline{\mathbb{Z}}\cong\sum_{|J|=n-r}\mathbb{Z}^{c_J}_{X_J^{n-r}}$. Since $H^i(X^n;\mathbb{Z}^{c_J}_{X_J^{n-r}})\cong H^i(X_J^{n-r};\mathbb{Z})$ and cohomology commutes with direct sums, we obtain the following description of the E_2 term of the Leray spectral sequence with $\mathbb{Z}$ coefficients.$E_2^{p,r(m-1)}\cong\bigoplus_J H^p(X_J^{n-r};\mathbb{Z})\otimes\mathbb{Z}^{c_J}$.

To describe the first non-trivial differential note that the E_2 term is generated as an algebra by the first row and the the group in the position $(1,m-1)$. The differential d_m is zero on the bottom row, by dimensional reasons. Therefore it is determined by the map $d_m:\bigoplus_{|J|=n-1}H^0(X_J^{n-1};\mathbb{Z})\to H^m(X^n;\mathbb{Z})$, which is the sum of the Gysin maps in cohomology associated to the inclusion $X_J^{n-1}\subseteq X^n$.



For X smooth projective variety of complex dimension l, when we take the coefficients to be a field of characteristic zero, this is the only non-trivial differential of the spectral sequence. Even more the cohomology of the configuration space $F(X,n)$ is isomorphic to the cohomology of the dga $E_2\otimes\mathbb{Q}$ with the differential d_{2l}.

References

[1] Cohen, F. R. and Taylor, L. R., *Configuration spaces: applications to Gelfand-Fuks cohomology*, Bull. Amer. Math. Soc., 84, 1978 , 134–136.
[2] Totaro, Burt., *Configuration spaces of algebraic varieties*, Topology, 35,1996, 1057–1067.

Symmetric Topological Complexity and Embedding Problems for Projective Spaces

Mijail Guillemard

This note describes the main components for characterizing the relation of the symmetric topological complexity and the embedding dimension of r-dimensional projective spaces $\mathbb{RP}^r$. The key property relating these concepts are $\mathbb{Z}_2$-equivariant maps defined on $\mathbb{RP}^r$, together with fundamental ideas of the Haefliger's metastable range theorem. These results have been announced in [2] and are in relation to previous investigations, discussed in [1], relating the nonsymmetric topological complexity and the immersion dimension of projective spaces.

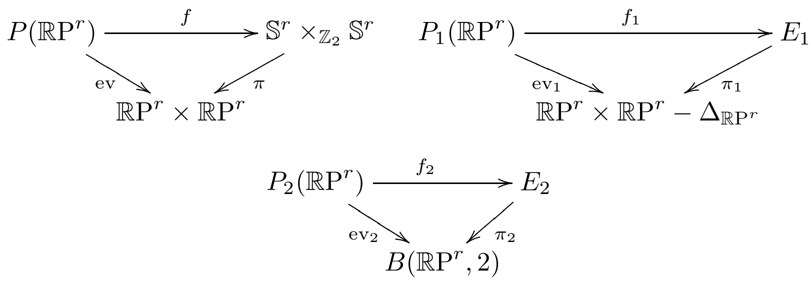
We first recall some basic definitions. Given a fibration $p : E \to B$, the *Schwarz genus* of p, denoted as genus(p), is defined as the smallest number of open sets $\{U_i\}$ covering B such that p admits a continuous section on each U_i. The *topological complexity* of a topological space X is defined as the Schwarz genus of the endpoints evaluation map $\mathrm{ev} : P(X) \to X \times X$, $\mathrm{ev}(\gamma) = (\gamma(0), \gamma(1))$, $\gamma \in P(X)$, where $P(X)$ is the path space $X^{[0,1]}$ with compact open topology. The *symmetric topological complexity* of X is defined as $\mathrm{TC}^{\mathrm{S}}(X) := \mathrm{genus}(\mathrm{ev}_2) + 1$, where $\mathrm{ev}_2 : P_2(X) \to B(X, 2)$, is a fibration with $P_2(X) := P_1(X)/\mathbb{Z}_2$, and $B(X, 2) := (X \times X - \Delta_X)/\mathbb{Z}_2$, $\Delta_X := \{(x, x), x \in X\}$. Here, we use the fibration $\mathrm{ev}_1 : P_1(X) \to X \times X - \Delta_X$, $\mathrm{ev}_1 = \mathrm{ev}|_{P_1(X)}$, with $P_1(X) := \{\text{paths } \gamma \in X^{[0,1]}, \gamma(0) \neq \gamma(1)\}$. The orbit spaces $P_2(X)$ and $B(X, 2)$ are defined with the actions of $\mathbb{Z}_2$ which reverse the direction of the paths $\gamma \in P_1(X)$, and interchange the coordinates of the elements in $X \times X - \Delta_X$. We discuss now the following theorem:

Theorem (Gonzáles and Landweber, 2009, [2]). *The symmetric topological complexity of the r dimensional projective space $\mathbb{R}P^r$, denoted as $TC^{\mathrm{S}}(\mathbb{R}P^r)$, is related to $E(r)$, the Euclidean embedding dimension of $\mathbb{R}P^r$, as $TC^{\mathrm{S}}(\mathbb{R}P^r) = E(r)+1$, $r \in \{1, 2, 4, 8, 9, 13\}$, $r > 15$.*

In the following, we sketch the two main components in the proof strategy. On the one hand, a relation is established between the symmetric topological complexity of $\mathbb{RP}^r$ with the level of an involution defined by considering $\mathbb{Z}_2$-equivariant maps using $\mathbb{RP}^r$. On the other hand, we use the identification, as described in the Haefliger's metastable theorem, between isotopy classes of smooth embeddings of a manifold $M \subset \mathbb{R}^m$ and homotopy classes of $\mathbb{Z}_2$-equivariant maps $M \times M - \Delta_M \to \mathbb{S}^{m-1}$.

The level of an involution given by a $\mathbb{Z}_2$-action on X is denoted as $\mathrm{level}(X, \mathbb{Z}_2)$, and is defined as the minimum $\ell > 0$, such that there exists an $\mathbb{Z}_2$-equivariant map $X \to \mathbb{S}^{\ell-1}$. The theorem that relates the level of an involution to the symmetric topological complexity for projective spaces has been presented in [2], and ensures that for all values of r, $\mathrm{TC}^{\mathrm{S}}(P^r) = \mathrm{level}(P^r \times P^r - \Delta_{P^r}, \mathbb{Z}_2) + 1$. There are three main components for proving this result. First, we need a fundamental property, presented in [3], of the Schwarz genus of a canonical projection which guarantees that for an $\mathbb{Z}_2$-action on X which admits a $\mathbb{Z}_2$-equivariant map $X \to \mathbb{S}^{n-1}$, and for the canonical projection $p : X \to X/\mathbb{Z}_2$, we have $\mathrm{genus}(p) = \mathrm{level}(X, \mathbb{Z}_2)$.

The second property characterizes genus(ev_i) for $\mathrm{ev}_1 : P_1(X) \to X \times X - \Delta_X$. Finally, we characterize also genus(ρ) for $\rho : \mathbb{R}\mathrm{P}^r \times \mathbb{R}\mathrm{P}^r - \Delta_{\mathbb{R}\mathrm{P}^r} \to B(\mathbb{R}\mathrm{P}^r, 2)$ the canonical projection. More precisely, we consider the property that for $i \in \{1, 2\}$ genus(ev_i) = genus(π_i), defined by constructing (commutative) diagrams:



For defining the map f (and proving the commutativity of the diagrams), we consider a path $\gamma \in P(\mathbb{R}\mathrm{P}^r)$, and $\hat{\gamma} : [0, 1] \to \mathbb{S}^r$ any lifting through the canonical projection $\mathbb{S}^r \to \mathbb{R}\mathrm{P}^r$, then $f(\gamma)$ is the class of $(\hat{\gamma}(0), \hat{\gamma}(1))$ in the Borel construction $\mathbb{S}^r \times_{\mathbb{Z}_2} \mathbb{S}^r := (\mathbb{S}^r \times \mathbb{S}^r)/(-x, y) \sim (x, -y)$. Now, the commutativity of these diagrams ensures that part of the equalities genus(ev_i) = genus(π_i) are valid. In order to analyze the missing inequalities, we use similar ideas by constructing additional commutative diagrams, using an $\mathbb{Z}_2$-equivariant map $g_1 : E_1 \to P_1(\mathbb{R}\mathrm{P}^r)$, where g_1 run backwards with respect to f_1. The explicit construction of g_1 uses a model for E_1 as the set $(\mathbb{S}^r \times \mathbb{S}^r - \tilde{\Delta})/(x, y) \sim (-x, -y)$, $\tilde{\Delta} := \{(x, y) \in \mathbb{S}^r \times \mathbb{S}^r | x \neq \pm y\}$, and g_1 maps the class of a pair (x_1, x_2) into the curve $[0, 1] \to \mathbb{S}^r \to \mathbb{R}\mathrm{P}^r$ with the first map given by $t \mapsto v(tx_1 + (1 - t)x_2)$, and v is the normalization map.

Using similar ideas, we can also characterize genus(ρ), for $\rho : \mathbb{R}\mathrm{P}^r \times \mathbb{R}\mathrm{P}^r - \Delta_{\mathbb{R}\mathrm{P}^r} \to B(\mathbb{R}\mathrm{P}^r, 2)$ the canonical projection, with the property genus(ρ) = genus(π_2). With all these steps, we have a rough synthesis of some basic ideas for proving the theorem:

Theorem. *For all values of r,* $TC^{\mathbb{S}}(\mathbb{R}P^r) = \mathit{level}(\mathbb{R}P^r \times \mathbb{R}P^r - \Delta_{\mathbb{R}P^r}, \mathbb{Z}_2) + 1$.

The second part of the proof of the property relating the symmetric topological complexity and the embedding dimension uses the celebrated Haefliger's metastable range theorem:

Theorem (Haefliger's metastable range)**.** *Let M be a smooth n-dimensional manifold and $2m \geq 3(n + 1)$, then there is a surjective map from the set of isotopy classes of smooth embeddings $M \subset \mathbb{R}^m$ onto the set of $\mathbb{Z}_2$-equivariant homotopy classes of maps $M^* \to \mathbb{S}^{m-1}$, $M^* := M \times M - \Delta_M$.*

In our particular case, we only use from the Haefliger's metastable range the fact that the existence of a smooth embedding $M \subset \mathbb{R}^m$ is equivalent to the existence of a $\mathbb{Z}_2$-equivariant map $M^* \to \mathbb{S}^{m-1}$. Notice that the we have an explicit construction for the surjective map used in the Haefliger's metastable range by considering for any embedding $g : \mathbb{R}\mathrm{P}^r \to \mathbb{R}^d$, a $\mathbb{Z}_2$-equivariant map $\tilde{g} : \mathbb{R}\mathrm{P}^r \times \mathbb{R}\mathrm{P}^r - \Delta_{\mathbb{R}\mathrm{P}^r} \to \mathbb{S}^{d-1}$: $\tilde{g}(a, b) := (g(a) - g(b))/(||g(a) - g(b)||)$.

The components required for relating the embedding dimension of a projective space with the level of the involution for $\mathbb{Z}_2$-equivariant map $\tilde{g} : \mathbb{R}\mathrm{P}^r \times \mathbb{R}\mathrm{P}^r - \Delta_{\mathbb{R}\mathrm{P}^r} \to \mathbb{S}^{d-1}$ are the following two properties from Gonzáles and Landweber, complementing results from Haefliger and Hirsch (1961, 1962):

Proposition (Gonzáles and Landweber, 2009)**.** *For $r \in \{8, 9, 13\}$ or $r > 15$, an axial map $\mathbb{R}P^r \times \mathbb{R}P^r \to \mathbb{R}P^s$ can exist only when $2s \geq 3(r+1)$.*

Theorem (Gonzáles and Landweber, 2009)**.** *The existence of a symmetric axial map $\mathbb{R}P^r \times \mathbb{R}P^r \to \mathbb{R}P^s$ implies the existence of a smooth embedding $\mathbb{R}P^r \subset \mathbb{R}^{s+1}$ provided $2s > 3r$.*

Theorem (Haefliger, Hirsch, 1961, 1962)**.** *The existence of a smooth embedding $\mathbb{R}P^r \subset \mathbb{R}^s$ implies the existence of a symmetric axial map $\mathbb{R}P^r \times \mathbb{R}P^r \to \mathbb{R}P^s$.*

In order to analyze the missing cases outside the metastable range, $r \leq 15$, we consider lower and upper bounds of the symmetric topological complexity. This can be achieved by considering the inequalities $\mathrm{TC}(\mathbb{R}\mathrm{P}^r) \leq \mathrm{TC}^{\mathrm{S}}(\mathbb{R}\mathrm{P}^r) \leq E(\mathbb{R}\mathrm{P}^r)+1$, and $\mathrm{TC}(\mathbb{R}\mathrm{P}^r) \leq \mathrm{TC}^{\mathrm{S}}(\mathbb{R}\mathrm{P}^r) \leq E_{\mathrm{TOP}}(\mathbb{R}\mathrm{P}^r)+1$, where E_{TOP} is defined for embeddings which are non necessarily smooth. These inequalities can be proved by considering the property we already discussed $\mathrm{TC}^{\mathrm{S}}(\mathbb{R}\mathrm{P}^r) = \mathrm{level}(\mathbb{R}\mathrm{P}^r \times \mathbb{R}\mathrm{P}^r - \Delta_{\mathbb{R}\mathrm{P}^r}, \mathbb{Z}_2) + 1$.

We finally remark that the corresponding result for complex projective spaces is significantly simpler to proof that the real case. The main property is $\mathrm{TC}^{\mathrm{S}}(\mathbb{C}\mathrm{P}^n) = 2n+1$. As it is know from [1], $\mathrm{TC}(\mathbb{C}\mathrm{P}^n) = 2n+1$, and therefore, we need to verify that $\mathrm{TC}^{\mathrm{S}}(\mathbb{C}\mathrm{P}^n) \leq 2n+1$. This inequality can be verified with the following diagram of pullback squares

$$\begin{array}{ccccc} P(\mathbb{C}\mathrm{P}^r) & \longleftarrow & P_1(\mathbb{C}\mathrm{P}^n) & \longrightarrow & P_2(\mathbb{C}\mathrm{P}^n) \\ {\scriptstyle \mathrm{ev}}\downarrow & & {\scriptstyle \mathrm{ev}_1}\downarrow & & {\scriptstyle \mathrm{ev}_2}\downarrow \\ \mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n & \longleftarrow & \mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n - \Delta_{\mathbb{C}\mathrm{P}^n} & \longrightarrow & B(\mathbb{C}\mathrm{P}^n, 2) \end{array}$$

which guarantees that a common fiber for the fibrations $\mathrm{ev}, \mathrm{ev}_1$ and ev_2 is the path connected loop space $\Omega\mathbb{C}\mathrm{P}^n$. Now, with the Theorem 5 in [3], which estimates the genus of a fibration using the homotopy type of the base and connectivity of the fiber, we obtain the following inequality: $\mathrm{TC}^{\mathrm{S}}(\mathbb{C}\mathrm{P}^n) = \mathrm{genus}(ev_2) + 1 \leq \dim(Y)/2 + 2$, where Y is a CW-complex with the same homotopy type of $B(\mathbb{C}\mathrm{P}^n, 2)$. We can conclude our remark using an observation by Farber and Grant that for M being a smooth closed m-dimensional manifold, $B(M, 2)$ has the homotopy type of a $(2m-1)$-dimensional CW-complex.

References

[1] M. Farber, S. Tabachnikov, and S. Yuzvinsky, *Topological robotics: motion planning in projective spaces*, Int. Math. Res. Not., no. 34, 1853-1870 (2003).

[2] J. González and P. Landweber, *Symmetric topological complexity of projective and lens spaces*, Algebraic & Geometric Topology, no. 9 473-494 (2009).

[3] A. S. Schwarz, *The genus of a fiber space*, Amer. Math. Soc. Transl., Ser. 2, vol. 55 49-140 (1966).

Random Polygon Spaces

Liz Hanbury

In this talk we consider the configuration spaces of closed polygonal linkages in Euclidean space. For fixed $l = (l_1, \ldots, l_n) \in \mathbb{R}^n_{>0}$ we consider the space of all possible configurations, in either $\mathbb{R}^2$ or $\mathbb{R}^3$, of a polygonal linkage having bars of length $l_1, \ldots, l_n$. A configuration of such a linkage can be specified by giving the directions in which each of the n bars points. Thus we have that the configuration spaces of closed linkages with length vector l in $\mathbb{R}^2$ and $\mathbb{R}^3$ are given by:

$$M_l = \{(u_1, \ldots, u_n) \in (S^1)^n : \sum_{i=1}^n l_i u_i = 0\}/SO(2),$$
$$N_l = \{(v_1, \ldots, v_n) \in (S^2)^n : \sum_{i=1}^n l_i v_i = 0\}/SO(3).$$

These configuration spaces appear in applications, for example in mathematical biology, in statisitical shape theory and in topological robotics. In these applications it is natural to assume that the length vector l is unknown or is known with some error and that the number of bars is large. Thus it is natural to think of the length vector not as a fixed vector, but as a random vector, and to think of the number of bars tending to infinity.

In this talk we consider the length vector to be a random vector so that M_l and N_l are random manifolds and their Betti numbers are integer-valued random variables. The main result presented gives an asympototic estimate for the expected values of these Betti numbers.

Since $M_l \equiv M_{tl}$ and $N_l \equiv N_{tl}$ for all $t > 0$, we may assume that the length vector l lies in the open simplex

$$\Delta^{n-1} = \{(l_1, \ldots, l_n) \in \mathbb{R}^n : l_i > 0 \text{ for all } i \text{ and } \sum_{i=1}^n l_i = 1\}.$$

We consider different probability measures μ on the space Δ^{n-1} of length vectors. The expected values of the pth Betti numbers of M_l and N_l (with respect to the measure μ) are given by:

$$E(b_p(M_l)) = \int_{\Delta^{n-1}} b_p(M_l) d\mu,$$
$$E(b_p(N_l)) = \int_{\Delta^{n-1}} b_p(N_l) d\mu.$$

The main theorem presented applies to a large class of probablity measures called *admissable measures*. A sequence of measures μ_n on Δ^{n-1} is called admissable if each μ_n is uniformly continuous with respect to the usual Lebesgue measure on Δ^{n-1} and a certain technical condition is satisfied. For details see [2].

The main result is the following:

Theorem 1. [2, 3] *For any integer $p \geq 0$ and any admissable sequence of probability measures μ_n on Δ^{n-1} there exist constants $C, \tilde{C} > 0$ and $a, \tilde{a} \in (0,1)$ such that*

$$\left| E(b_p(M_l)) - \binom{n-1}{p} \right| < Ca^n \text{ for all } n,$$

$$\left| E(b_{2p}(N_l)) - \sum_{i=0}^{p} \binom{n-1}{i} \right| < \tilde{C}\tilde{a}^n \text{ for all } n.$$

Thus the theorem gives asymptotic values for the expected Betti numbers and states that the Betti numbers approach these asymptotic values exponentially fast. Note that there is a surprising universality phenomenon appearing: the expected Betti numbers do not depend on the choice of admissable measures.

In the case of M_l, the proof of the main theorem proceeds by showing that there is a subspace of Δ^{n-1} on which the Betti numbers are precisely $\binom{n-1}{p}$ and that the volume of the complement of this subspace tends to zero exponentially fast with n. The proof in the case of N_l is similar. Expressions for the Betti numbers of M_l and N_l for fixed l were given in [4] and [5, 6] respectively.

In the last part of the talk we also mention results of [1] where several generalizations are given. These include calculations of the asymptotic expected values of the Poincaré polynomials of M_l and N_l and asymptotic expected values of the Betti numbers $b_p(M_l)$ and $b_p(N_l)$ when p varies with n.

References

[1] Dombry, C. and Mazza, C.. *Some remarks on Betti numbers of random polygon spaces.* Random Structures Algorithms **37** (2010), no. 1, 67–84.

[2] Farber, M. *Topology of random linkages.* Algebr. Geom. Topol. **8** (2008), no. 1, 155–171.

[3] Farber, M. and Kappeler, T. *Betti numbers of random manifolds.* Homology, Homotopy Appl. **10** (2008), no. 1, 205–222.

[4] Farber, M. and Schütz, D. *Homology of planar polygon spaces.* Geom. Dedicata **125** (2007), 75–92.

[5] Hausmann, J.-C. and Knutson, A. *The cohomology ring of polygon spaces.* Ann. Inst. Fourier (Grenoble) **48** (1998), no. 1, 281–321.

[6] Klyachko, A. A. *Spatial polygons and stable configurations of points in the projective line.* Algebraic geometry and its applications (Yaroslavl, 1992), 67–84, Aspects Math., E25, Vieweg, Braunschweig, 1994.

The Euler–Gal Power Series

Kenneth Deeley

Let $B(X,n)$ denote the unordered n-point configuration space of a finite, connected polyhedron X. The talk given at the Topological Robotics Arbeitsgemeinschaft described an elegant theorem due to S. Gal [3] which provides an explicit expression for the Euler characteristics $\{\chi(B(X,n))\}_{n=0}^{\infty}$. The theorem states that the *Euler–Gal power series*

$$\mathfrak{eu}_X(t) := \sum_{n=0}^{\infty} \chi(B(X,n))t^n \in \mathbb{Z}[[t]]$$

represents a rational function $\mathfrak{eu}_X(t) = p(t)/q(t)$, where $p(t), q(t) \in \mathbb{Z}[t]$ satisfy $p(0) = q(0) = 1$. Moreover, the polynomials $p(t), q(t)$ may be computed explicitly in terms of local topological properties of X:

$$p(t) = \prod_{\dim\sigma \text{ even}} (1 + t(1 - \chi(L_\sigma))), \quad q(t) = \prod_{\dim\sigma \text{ odd}} (1 - t(1 - \chi(L_\sigma))).$$

Here, L_σ denotes the link of the closed cell $\sigma \subset X$, and the products are taken over all even-dimensional and odd-dimensional cells, respectively. Gal's theorem therefore provides an algorithm for computing $\chi(B(X,n))$ for each $n \geq 0$. It also implies that the coefficients $\{\chi(B(X,n))\}_{n=0}^{\infty}$ of the Euler–Gal power series satisfy an integer recurrence relation. A detailed exposition of the original paper [3] may be found in Chapter 2 of the book [2].

After stating Gal's theorem and its immediate implications, we described an application to the study of configuration spaces of graphs. If $X = \Gamma$ is a finite, connected, simple graph, then the Euler–Gal power series of Γ is

$$\mathfrak{eu}_\Gamma(t) = (1-t)^{-E} \prod_{v \in V(\Gamma)} (1 + t(1 - \mu(v))),$$

where E is the number of edges, $V(\Gamma)$ is the vertex set and $\mu(v)$ is the number of edges incident to $v \in V(\Gamma)$. Extracting the coefficient of t^2 from this expression and multiplying by 2 gives the formula

$$\chi(F(\Gamma,2)) = \chi(\Gamma)^2 + \chi(\Gamma) - \sum_{v \in V(\Gamma)} (\mu(v) - 1)(\mu(v) - 2),$$

a result which was also obtained by K. Barnett and M. Farber [1] using a different method.

We then outlined the main steps in the proof of Gal's theorem, following [3] and [2]. The most important of these steps is establishing the following recursive formula:

$$\chi(F(X,n)) = \sum_{\sigma} \chi(F(X - \langle\sigma\rangle, n-1))(1 - \chi(L_\sigma)), \quad \forall n \geq 1, \tag{1}$$

where $\langle\sigma\rangle \cong \operatorname{int}\sigma \times (CL_\sigma - L_\sigma)$ is a sufficiently small, open, contractible neighbourhood of an interior point of σ. The sum is taken over all closed cells $\sigma \subset X$. Formula (1) is equivalent to the differential equation

$$\mathfrak{eu}'_X(t) = \sum_{\sigma} \mathfrak{eu}_{X - \langle\sigma\rangle}(t)(1 - \chi(L_\sigma)).$$

As a motivation for (1), suppose X is a connected manifold. Then the projection $\pi : F(X,n) \to X$ onto the first factor is a fibration with fibre $F(X - \star, n-1)$, so we have the multiplicative formula $\chi(F(X,n)) = \chi(F(X - \star, n-1))\chi(X)$ relating the Euler characteristics of the fibre, base and total space. Paper [3] shows this formula may be adapted to the general case when X is not a manifold and the fibre of π changes.

The next step discussed in the talk was the construction of a *cut and paste ring* $\mathfrak{C}\&\mathfrak{P}$. Two polyhedra X, Y are related by cut and paste surgery if one can find a collared subspace $S \subset X$ such that Y is obtained from X by cutting along S and pasting back using a PL homeomorphism $S \to S$. We denote the equivalence class of X by $[X]$, and make the set of equivalence classes into a semiring via the operations

$$[X] + [Y] := [X \sqcup Y], \quad [X] \cdot [Y] := [X \times Y].$$

The standard Grothendieck construction produces the ring $\mathfrak{C}\&\mathfrak{P}$ from this semiring; elements of $\mathfrak{C}\&\mathfrak{P}$ are represented by formal differences $[X] - [Y]$. We have $\mathfrak{eu}_X(t) \equiv \mathfrak{eu}_Y(t)$ if X and Y are related by cut and paste surgery; moreover, the map $\mathfrak{eu} : \mathfrak{C}\&\mathfrak{P} \to \mathbb{Z}[[t]]^\times$ sending $[X] - [Y]$ to $\mathfrak{eu}_X(t)/\mathfrak{eu}_Y(t)$ is a homomorphism from the additive group of $\mathfrak{C}\&\mathfrak{P}$ to the multiplicative group of units $\mathbb{Z}[[t]]^\times$ of $\mathbb{Z}[[t]]$.

An important example of cut and paste surgery is *amputation*: if $X = A \cup B$ for closed subspaces A, B such that $S = A \cap B$ has a collar U such that $(U \cap A, S) \cong (S \times [-1, 0], S \times \{0\})$ and $(U \cap B, S) \cong (S \times [0, 1], S \times \{0\})$, then

$$[X] = [A] + [B] - [S \times [-1, 1]] \text{ in } \mathfrak{C}\&\mathfrak{P}.$$

We then stated the particularly simple relationship between the Euler–Gal power series of a cone and a cylinder over the same base space X, namely

$$\frac{\mathfrak{eu}_{CX}(t)}{\mathfrak{eu}_{X\times[0,1]}(t)} = 1 + t(1 - \chi(X)).$$

The final part of the talk described how to combine the above steps, together with simple topological facts such as $\partial\langle\sigma\rangle \cong S^{\dim\sigma-1} * L_\sigma$, to obtain Gal's remarkably elegant formula

$$\mathfrak{eu}_X(t) = \frac{p(t)}{q(t)} = \prod_\sigma \left[1 + (-1)^{\dim\sigma} t(1 - \chi(L_\sigma))\right]^{(-1)^{\dim\sigma}}.$$

References

[1] K. Barnett and M. Farber (2009), **Topology of Configuration Space of Two Particles on a Graph, I**, Algebraic and Geometric Topology, Volume 9, Issue 1, 593–624.

[2] M. Farber (2008), **Invitation to Topological Robotics**, European Mathematical Society.

[3] S. R. Gal (2001), **Euler characteristic of the configuration space of a complex**, Colloquium Mathematicum Volume 89, Issue 1, 61–67. Available at `http://arxiv.org/abs/math/0202143`arXiv:math/0202143v1.

Collision Free Motion Planning

Miguel A. Xicoténcatl

In this talk we discuss the problem of finding the topological complexity of the configuration space $F_n(X)$ of n distinct points in X, in the case when X is $\mathbb{R}^m$, a compact orientable surface Σ_g or a graph Γ. All these problems can be viewed as instances of the problem of simultaneous control of multiple objects avoiding collisions with each other.
The problem of computing $TC(F_n(\mathbb{R}^m))$ was first considered by M. Farber and S. Yuzvinsky in [6], where they treated the cases $m = 2$ and m odd, and was finished by M. Farber and M. Grant in [5]. Notice the case $m = 2$ corresponds to computing the topological complexity of a hyperplane arrangement. Namely, $F_n(\mathbb{C})$ is the complement of the braid arrangement $\mathcal{A}_n$ in $\mathbb{C}^n$, which happens to be of rank $n-1$. In this direction, the following result was proven in [6]:

Theorem. *If $\mathcal{A}$ is a complex central arrangemente of rank r, then*
- *(a)* $TC(M(\mathcal{A})) \leq 2r$.
- *(b) Assume there are hyperplanes $H_1, \ldots, H_{2r-1} \in \mathcal{A}$ such that: $H_1, \ldots, H_r$ are independent and $H_j, H_{r+1}, \ldots, H_{2r-1}$ are independent $\forall j = 1, \ldots, r$. Then $TC(M(\mathcal{A})) = 2r$.*

It follows from here that $TC(F_n(\mathbb{C})) = 2n-2$. More generally, the topological complexity for the configuration spaces of euclidean spaces is given as follows.

Theorem. *For $n, m \geq 2$*

$$TC\left(F_n(\mathbb{R}^m)\right) = \begin{cases} 2n-2 & \text{for } m \text{ even,} \\ 2n-1 & \text{for } m \text{ odd.} \end{cases}$$

A sketch of the proof follows. Recall from [2] that the cohomology $H^*(F_n(\mathbb{R}^m))$ is given by generators $A_{i,j}$ $(1 \leq j < i \leq n)$ all in degree $m-1$, subject to the relations:

$$A_{i,j}^2 = 0,$$
$$A_{i,k}A_{i,j} = A_{j,k}(A_{i,j} - A_{i,k}) \quad \text{for} \quad k < j < i.$$

Moreover, the space $F_n(\mathbb{R}^m)$ is $(m-2)$-connected and for $m \geq 3$ it is homotopy equivalent to a CW-complex of dimension $(m-1)(n-1)$. Thus, the zero-divisors-cup-length and the dimension and connectivity inequalities give:

$$2n-2 \leq TC(F_n(\mathbb{R}^m)) \leq 2n-1$$

For m odd, the lower bound can be improved as follows. Notice the $A_{i,j}$'s are even and thus setting $e_{i,j} = 1 \otimes A_{i,j} - A_{i,j} \otimes 1$ one gets $e_{i,j}^2 = (1 \otimes A_{i,j} - A_{i,j} \otimes 1)^2 = -2A_{i,j} \otimes A_{i,j}$. Thus

$$e_{2,1}^2 \cdot e_{3,1}^2 \cdots e_{n,1}^2 = (-2)^{n-1} A_{2,1} \ldots A_{n,1} \otimes A_{2,1} \ldots A_{n,1} \neq 0$$

and therefore $TC(F_n(\mathbb{R}^m)) \geq 2n-1$. This gives the result for m odd. For m even, one uses a sharp upper bound result proven in [5] and the case $m = 2$.

The topological complexity for the collision free motion planning on surfaces, was studied by D. Cohen and M. Farber in [1].

Theorem. *If Σ_g denotes a compact, connected, orientable surface of genus g, then*

$$TC\left(F_n(\Sigma_g)\right) = \begin{cases} 3 & \textit{for} \quad g=0, \quad n \leq 2 \\ 2n-2 & \quad g=0, \quad n \geq 3 \\ 2n+1 & \quad g=1, \quad n \geq 1 \\ 2n+3 & \quad g \geq 2, \quad n \geq 1. \end{cases}$$

The main tool in the proof of this result is the Totaro spectral sequence, describing the cohomology of configuration spaces of algebraic varieties [7].

Finally, some important results about collision free motion planning on graphs were obtained by M. Farber in [3]. Recall that an essential vertex in a graph Γ is a vertex which is incident to three or more edges.

Theorem. *Let Γ be a connected graph having an essential vertex. Then*

$$TC(F_n(\Gamma)) \leq 2m(\Gamma) + 1$$

where $m(\Gamma)$ denotes the number of essential vertices in Γ.

This follows from a result of R. Ghrist which states that $F_n(\Gamma)$ has the homotopy type of a cell complex of dimension $\leq m(\Gamma)$. The previous inequality is sharp in the following case:

Theorem. *Let Γ be a tree having an essential vertex. Let n be an integer satisfying $n \geq 2m(\Gamma)$. In the case $n = 2$ assume additionally that Γ is not homeomorphic to the letter Y. Then $TC(F_n(\Gamma)) = 2m(\Gamma) + 1$.*

References

[1] D. Cohen, M. Farber, *Topological complexity of collision-free motion planning on surfaces*, preprint 2009. arXiv: 0901.0877

[2] F. Cohen, *On configuration spaces, their homology, and Lie algebras.* J. of Pure and Applied Algebra 100 (1995), 19–42.

[3] M. Farber, *Collision free motion planning on graphs*, preprint 2004. arXiv:math/0406361v1.

[4] M. Farber, *Invitation to topological robotics.* Zürich Lectures in Advanced Mathematics. European Mathematical Society, 2008.

[5] M. Farber, M.Grant, *Topological complexity of configuration spaces.* Proc. Amer. Math. Soc. 137 (2009), no. 5, 1841–1847.

[6] M. Farber, S. Yuzvinsky, *Topological robotics: Subspace arrangements and collision free motion planning.* Transl. of AMS, 212 (2004), 145–156.

[7] B. Totaro, Configuration spaces of algebraic varieties. Topology 35 (1996), no. 4, 1057–1067.

Configuration spaces of graphs - Part I

Sebastian Grensing

1. Introdution

For a natural number n the space $F_n\Gamma$ of unordered n-point configurations in a graph Γ is the quotient of the space $C_n\Gamma = \big\{(x_1,\dots,x_n) \bigm| x_i \neq x_j \text{ for } i \neq j\big\}$ by the action of S_n, the symmetric group on n symbols. Let $b(\Gamma)$ denote the number of vertices of valence at least three.

Theorem 1. *Let Γ be a finite graph and n a natural number. There exists a cube complex $K_n\Gamma$ of dimension $\min\{b(\Gamma), n\}$ which embeds as a deformation retract into $F_n\Gamma$.*

In what follows, we will illustrate the construction of the cube complex $K_n\Gamma$ and calculate both F_2Q and K_2Q when Q is the Y-shaped graph consisting of three edges incident to a common vertex. This example shall be employed in part II in order to detect free abelian subgroups in the fundamental groups of configuration spaces of graphs.

2. Construction of $K_n\Gamma$

Denote the edge set of Γ by E_Γ and the vertex set by V_Γ. We may assume, without loss of generality, each vertex to be either free, i.e., of valence one, or essential, i.e., of valence at least three; let B_Γ denote the set of vertices of the latter kind. Furthermore, assume an orientation of the edges of Γ be given; denote for each oriented edge e its terminal vertex by v_e and its underlying unoriented edge by $|e|$.

Definition 1. For $k \in \mathbb{N}_0$ let $P_n^{(k)}\Gamma$ be the set of all pairs (f, S) such that:

(1) $f : E_\Gamma \cup B_\Gamma \to \mathbb{N}_0$ is a function;
(2) $S = \{s_1, \dots, s_k\}$ is a set of k distinct oriented edges;
(3) $v_{s_i} \in B_\Gamma$ for each $i \leqslant k$ and $v_{s_i} \neq v_{s_j}$ if $i \neq j$;
(4) $f(b) \in \{0, 1\}$ for each $b \in B_\Gamma$ and $f(v_{s_i}) = 0$ for each $i \leqslant k$;
(5) $\sum_{|a| \in E_\Gamma \cup B_\Gamma} f(|a|) = n - k$.

For two elements (f, S) and $(g, S \cup \{e\})$ in $P_n^{(k)}\Gamma$ and $P_n^{(k+1)}\Gamma$ resp. say that $(f, S) \prec (g, S \cup \{e\})$ if either one of the following conditions holds:

(a) $f(|a|) = \begin{cases} g(v_e) + 1 & \text{if } |a| = v_e\,, \\ g(|a|) & \text{otherwise;} \end{cases}$

(b) $f(|a|) = \begin{cases} g(|e|) + 1 & \text{if } |a| = |e|\,, \\ g(|a|) & \text{otherwise.} \end{cases}$

Let $P_n\Gamma$ be the graded poset given by $\big(P_n^{(0)}, \dots, P_n^{(k)}, \dots\big)$, endowed with the partial order generated by $\prec$.

Observe, that if $G = \big(g, \{e_1, \ldots, e_k\}\big)$ is a k-face, the sub-poset $\{F \prec G\}$ is isomorphic to the face poset of a k-dimensional cube: G has exactly the $2k$ codimension one faces

$$\partial_j^{\pm} G = \big(g_j^{\pm}, \{e_1, \ldots, \hat{e_k}, \ldots, e_k\}\big)$$

where

$$g_j^{+}(v_{e_j}) = 1,\ g_j^{-}(|e_j|) = g(|e_j|) + 1\ .$$

Thus $P_n\Gamma$ is the face poset of a uniquely determined cube complex, which we denote by $K_n\Gamma$.

Lemma 2.1. $\dim K_n\Gamma = \min\{b(\Gamma), n\}$

Proof. Let (f, S) be face of $P_n\Gamma$. As the function f is non-negative it follows from Definition 1 5 that $k \leqslant n$. Since there are at most $b(\Gamma)$ choices of edges with distinct essential terminal vertices, see 3 of Definition 1, we have that $k \leqslant b(\Gamma)$. Hence $\dim K_n\Gamma \leqslant \min\{b(\Gamma), n\}$.

For $k = \min\{b(\Gamma), n\}$ we can choose k edges $s_i \in E_\Gamma$ such that their terminal vertices v_{s_i} are pairwise distinct. Then $f(|s_1|) = n - k$ and $f(|a|) = 0$, for $a \neq s_1$, defines a k-face of $P_n\Gamma$. □

Remark. A 0-face $(f, \emptyset)$ of $P_n\Gamma$ is a function that counts how many particles of a configuration lie on each edge or essential vertex. Two such configurations are joined by a 1-face if a particle moves from the interior of an edge into an essential vertex adjacent to it. If, for example, e is an oriented edge of Γ which terminates in an essential vertex v_e, the two 0-faces $\big(|e| \mapsto 2, \emptyset\big)$ and $\big({}^{|e| \mapsto 1}_{v_e \mapsto 1}, \emptyset\big)$ are joined by the edge $G = (|e| \mapsto 1, \{e\})$.

The higher dimensional k-faces of $P_n\Gamma$ describe such movements of k distinct particles which result in the same quantitative distribution of particles, regardless of the order the individual particles are moved in.

Remark. A different, but closely related approach which traces back to Abrams [2] is discussed in [4] and [3]:

Consider the space of ordered configurations $C_n\Gamma$. In a first step it is shown that $C_n\Gamma$ deformation retracts to the subcomplex of Γ^n one obtains from Γ^n by removing all those product cells whose closure intersect the generalized diagonal. This cube complex is of non-positive curvature, hence a $K(\pi, 1)$, and can also be shown to further retract to a subcomplex of dimension at most $b(\Gamma)$.

While on the one hand the cubical structure in this case is, due to construction, quite obvious, there are, on the other hand, some technical subtleties to overcome, e.g. concerning connectivity.

3. Example: F_2Q and K_2Q

Let Q denote the graph with three edges e_1, e_2, e_3 terminating in a single essential vertex v. The three distinct initial vertices of the edges e_i are free. Regard Q as a metric space by letting each edge be of length one.

The topology of F_2Q is easy to describe: For each pair of distinct edges of Q the space of configurations with exactly one particle on each of them can be parametrized by the distances of the particles to the central vertex v, i.e., a unit square with the origin removed.

Likewise, each subspace of configurations with both particles lying on the same edge is a standard 2-simplex with one closed edge removed.

Having three edges this yields six subspaces for six types of configurations. Identifying the configurations these six subspaces have in common results in the following figure:

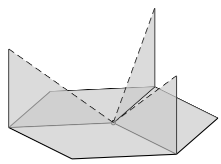


FIGURE 1. A realization of F_2Q.

By Lemma 2.1, the cube complex K_2Q is of dimension one. Each of its 1-cells corresponds to a pair $(g : E_Q \cup B_Q \to \mathbb{N}_0, S)$ where S contains a single edge e_k. Then, by 5 of Definition 1, $g(v) = 0$. Since, according to 5, $\sum_i g(|e_i|) = 1$, there is an l such that $g(|e_i|) = \delta_{il}$ for all $i \leqslant 3$. Hence there are nine 1-cells

$$G_{kl} = \big(|e_l| \mapsto 1, \{e_k\}\big)$$

in K_2Q. They are adjacent to the nine distinct 0-cells

$$\begin{aligned} \partial^+ G_{kl} &= \left(\begin{smallmatrix} v \ \mapsto 1 \\ |e_l| \mapsto 1 \end{smallmatrix}, \emptyset\right), \\ \partial^- G_{kl} &= \left(\begin{smallmatrix} |e_k| \mapsto 1 \\ |e_l| \mapsto 1 \end{smallmatrix}, \emptyset\right) = \partial^- G_{lk}, \quad \text{for } k \neq l, \\ \partial^- G_{kk} &= \left(\begin{smallmatrix} |e_k| \mapsto 2 \end{smallmatrix}, \emptyset\right), \end{aligned}$$

where ∂^+ corresponds to condition (a) of Definition 1 and ∂^- to (b). The cube complex K_2Q is therefore a hexagon with three intervalls attached:

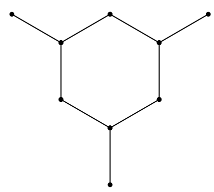


FIGURE 2. The cube complex K_2Q.

References

[1] J. Swiatkowski, *Estimates for homological dimension of configuration spaces of graphs*, Colloq. Math. **89** (2001), 69–79.
[2] A. Abrams, *Configuration spaces and braid groups of graphs*, Ph.D. thesis, UC Berkeley, 2000.
[3] A. Abrams & R. Ghrist, *Finding topology in a factory: configuration spaces*, Amer. Math. Monthly **109** (2002), no. 2, 140–150.
[4] R. Ghrist, *Configuration spaces and braid groups on graphs in robotics*, in: "Knots, braids, and mapping class groups", papers dedicated to Joan S. Birman (New York, 1998), pp. 29–40, AMS/IP Stud. Adv. Math., 24, Amer. Math. Soc., Providence, RI, 2001.

Random 2-dimensional complexes: Part II

Thorranin Thansri

In this talk, we considered the probability p of which asymptotically almost surely simplicial complexes in the model $Y(n,p)$ of random 2-dimensional complexes are $\mathbb{F}_2$-homological 1-connected through an elegant idea originated with Nathan Linial and Roy Meshulam in [LR06].

For a natural number n and a function $p = p(n) \in [0,1]$, the model of random graphs $G(n,p)$, introduced by Erdös and Renyi, is the probability space of all graphs G on the vertex set $[n] = \{1,\dots,n\}$ such that each edge appears independently on G with probability p. A monotone increasing property P of graphs is a class of graphs closed under isomorphism and invariant under the addition of edges. For example, connectivity of graphs is a monotone increasing property. We also say that *asymptotically almost surely (a.a.s) G has a property $\mathcal{P}$* if the probability $\lim_{n\to\infty} \Pr[G \text{ has } \mathcal{P}] = 1$. Erdös and Renyi found that as the probability p increases, a.a.s. G has a monotone increasing property suddenly. For connectivity of graphs, they found that, for any function $\omega(n)$ with $\lim_{n\to\infty} \omega(n) = \infty$, the probability $p = (\log n + \omega(n))/n$ gives the connectivity of a.a.s. G.

As a natural extension of $G(n,p)$, Linial and Meshulam defined the model $Y(n,p)$ as the probability space of all 2-dimensional simplicial complexes Y on vertex set $[n] = \{1,2,\dots,n\}$ such that the 1-skeleton $Y^{(1)}$ of Y is just the complete graph K_n on n vertices and each 2-simplex appears independently on Y with probability p. The notion of *asymptotically almost surely* is defined analogously.

For a 2-dimensional simplicial complex Y with $Y^{(1)} = K_n$, an edge $e \in \binom{[n]}{2}$ is said to be *isolated* in Y if there is no 2-simplices in Y containing e as its face. We can obtain a fundamental result concerning with the existence of isolated edges as follows:

Proposition 1. *Let $\omega(n)$ be any function which satisfies $\lim_{n\to\infty} \omega(n) = \infty$. Then we find that*
(1) if $p = (2\log n - \omega(n))/n$, then a.a.s. Y has isolated edges;
(2) if $p = (2\log n + \omega(n))/n$, then a.a.s. Y has no isolated edges.

As a generalization of the notion of graph connectivity, Linial and Meshulam had considered the $\mathbb{F}_2$-homological 1-connectivity, that is, the vanishing of the first homology with coefficients in $\mathbb{F}_2$. They found the following result, which asserts that the probability for a.a.s. $H^1(Y;\mathbb{F}_2) = 0$ coinsides with the one for a.a.s. Y has no isolated edges.:

Theorem 2 (Linial-Meshulam [LR06])**.** *Let $\omega(n)$ be any function which satisfies $\lim_{n\to\infty} \omega(n) = \infty$. Then the followings hold:*
(1) if $p = (2\log n - \omega(n))/n$, then a.a.s. $H^1(Y, \mathbb{F}_2) \neq 0$;
(2) if $p = (2\log n + \omega(n))/n$, then a.a.s. $H^1(Y, \mathbb{F}_2) = 0$.

We explain here a beautiful idea in the proof by Linial and Meshulam: for $p = (2\log n + \omega(n))/n$, we have to show that $\lim_{n\to\infty} \Pr[H^1(Y, \mathbb{F}_2) \neq 0] = 0$. Denoted by d_1 the coboundary operator from the group of all 1-cochains $C^1(X)$ of a simplicial complex X into $C^2(X)$. If we let $B(f)$ be the number of all $\sigma \in \binom{[n]}{3}$ such that $d_1(f)(\sigma) = 1$, we can find that this probability can be bounded from above by the sum of $(1-p)^{B(f)}$ over all 1-cochain f of Y.

They gave an idea of mapping each 1-cochain f which can be interpreted as a map on the collection $\binom{[n]}{2}$ of all 2-subsets of $[n]$ to a graph

$$G_f = \left([n], \left\{e \in \binom{[n]}{2} \,:\, f(e) \neq 0\right\}\right).$$

This correspondence is one-to-one, and each graph G can be mapped to a map f_G, and we let $B(G) = B(f_G)$. Linial and Meshulam found that the probability of $H^1(Y, \mathbb{F}_2) \neq 0$ can be bounded from above by the sum of $(1-p)^{B(G)}$ over all graphs G whose 1-cochains f_G has the smallest number of elements of $\left\{e \in \binom{[n]}{2} \,:\, f_G(e) \neq 0\right\}$ within all cohomologous cocycles and has exactly one connected component which is not an isolated point. In particular, they found that there exists $c \geq 1/120$ such that for such a graph G, $B(G) \geq cn|E(G)|$. Moreover, they found an upper bound for the number of graphs G with $B(G) = (1-\theta)n|E(G)|$, where θ is bounded away from zero. These two results give us that the summation has the limit 0 as $n \to \infty$.

The above idea was also applied by Meshulam and Wallach in the study of the vanishing of $(k-1)$-dimensional homology group of random k-dimensional complexes for a general fixed k [MW09]. Furthermore, as an extension of these studies, the study of the vanishing of the k-dimensional homology group of random k-dimensional complexes was introduced by Kozlov in [Koz]. After the talk, Prof. Roy Meshulam introduced a result on the model $Y(n,p)$ of random 2-dimensional complexes that, if we let $p = c/n$, as $n \to \infty$, the probability of $H_2(Y;\mathbb{Z}_2) \neq 0$ has the limit 1 if $c > 2.74$; and has the limit $\exp(-c^4/41)$ if $c < 2.45$. He suggested us considering its probability for $2.45 \leq c \leq 2.74$.

References

[Koz] D. N. Kozlov. The threshold function for vanishing of the top homology group of random d-complexes, preprint, arXiv:0904.1652.

[LR06] N. Linial and R.Meshulam, Homological connectivity of random 2-complexes, *Combinatorica* **26** (4) (2006), 475-487.

[MW09] R. Meshulam and N. Wallach, Homological connectivity of random k-dimensional complexes, *Random Structures Algorithms* **34** (3) (2009), 408-417.
[Mun84] J. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, 1984.

Configuration spaces of graphs - part II

Luba Stein

We will study topological properties of configuration spaces of graphs following [2]. Further we apply techniques from metric geometry, see [1]. Let Γ be a graph as in the preceding abstract "Configuration spaces of graphs - part I" whose notation is used throughout this text. We should prove the following.

Theorem 1.
(1) $K_n(\Gamma)$ is a $K(\pi, 1)$.
(2) The fundamental group $\pi_1(K_n(\Gamma))$ contains a subgroup isomorphic to $\mathbb{Z}^k$ where $k = \min(b(\Gamma), [\frac{n}{2}])$.

Definition 2. Let X be a metric space. A triangle T in X satisfies the CAT(0) inequality if the distance between any pair of its points is not bigger than the distance of the corresponding points of the comparison triangle in the euclidean space.
If every triangle satisfies the CAT(0) inequality then X is a CAT(0) space.
A space which is locally CAT(0) is called non-positively curved (see [1]).

Theorem 3 ([1]).
(1) Every CAT(0) *space is contractible.*
(2) Let X be a complete, connected, non-positively curved metric space. Then its universal cover $\tilde{X}$ is CAT(0).

Corollary 4. *Every complete, connected, non-positively curved metric space is a $K(\pi, 1)$.*

Theorem 5 ([1]). *A connected, finite dimensional cube complex is complete and geodesic with respect to the naturally induced metric.*

It follows that the cube complex $K_n(\Gamma)$ is a complete, geodesic metric space. In order to prove the first part of theorem 1 it suffices to show that $K_n(\Gamma)$ is non-positively curved. Therefor we will apply the following theorem by M. Gromov.

Theorem 6. *Every finite dimensional cube complex is non-positively curved if and only if the link of each of its vertices is a flag complex.*

Proof (Theorem 1, (1)). Let $x \in P_n(\Gamma)$ be a vertex, thus x is represented by $(\phi, \emptyset)$. The link L_x consists of all cells (f, S) such that $(\phi, \emptyset) \prec (f, S)$. Those pairs satisfy the following properties.

(L1) v_s is branched for all $s \in S$.
(L2) $v_{s_1} \neq v_{s_2}$ for $s_1 \neq s_2$.
(L3) $\phi(|s|) + \phi(v_s) \geq 1$.

(L4) $\phi(|s|) + \phi(v_s) + \phi(v_{-s}) \geq 2$.

Let P_ϕ be the poset of all (f, S) and K_ϕ the simplicial complex with cells represented by (f, S) satisfying $(L1)$ – $(L4)$. By construction P_ϕ is the face poset of the link L_x, so K_ϕ is isomorphic to L_x.
Note that oriented edges of S satisfy the axoims $(L1)$ – $(L4)$ if and only if each subset of cardinality 1 or 2 does. Thus K_ϕ is a flag complex.

□

For (2) of theorem 1 we will again utilize methods from metric geometry.

Theorem 7. *Let X and Y be complete, connected metric spaces such that X is non-positively curved and Y is geodesic. Suppose there is a local isometry $f : Y \to X$. Then $f_* : \pi_1(Y, y_0) \to \pi_1(X, f(y_0))$ is injective.*

As we work with cube complexes we need a combinatorial property to check if a given map is a local isometry. Following [2] we will make use of the next theorem. This will be a statement on the links of vertices.
Note if for two cube complexes X and Y there is a non-degenerate, combinatorial map $f : Y \to X$ then f induces a map $f_v : L_v \to L_{f(v)}$ on the links for each vertex v of Y.

Theorem 8. *Let X and Y be finite dimensional cube complexes. A non-degenerate, combinatorial map $f : Y \to X$ is a local isometry if and only if for each vertex $v \in Y$ the following conditions hold.*

(I1) $f_v : L_v \to L_{f(v)}$ is an embedding.
(I2) $f_v(L_v)$ is a full subcomplex of $L_{f(v)}$.

Proof (Theorem 1, (2))**.** We will construct a k-torus and a local isometry to $K_n(\Gamma)$. For this let Q be the Y-graph with vertex q. The boundary of a hexagon can be isometrically embedded in K_2Q, thus $(K_2Q)^k$ contains an isometrically embedded k-torus.
Fix $B = \{b_1, \ldots, b_k\} \subseteq B_\Gamma$ and $\psi : B_\Gamma \cup E_\Gamma \to \mathbb{N}_0$ such that $\psi(a) \in \{0, 1\}$ for all $a \in B_\Gamma$, $\psi(b_i) = 0$ for all $1 \leq i \leq k$ and $\sum \psi(a) = n - 2k$ where $a \in B_\Gamma \cup E_\Gamma$.
For each $1 \leq i \leq k$ choose a combinatorial embedding $\delta_i : Q \to \Gamma$ such that $\delta_i(q) = b_i$. For a face representative $((g_1, R_1), \ldots, (g_k, R_k))$ of $(K_2Q)^k$ define

$$k_{B,\psi}((g_1, R_1), \ldots, (g_k, R_k)) = (\psi + \sum (\delta_i)_* g_i, \bigcup \delta_i(R_i))$$

where

$$(\delta_i)_* g_i(a) = \sum_{a' : \delta_i(a') = a} g_i(a').$$

The map $k_{B,\psi}$ is well-defined and a non-degenerate, combinatorial map. Moreover it is proved in [2] that $k_{B,\psi}$ satisfies theorem 8.

□

Corollary 9. *From the second part of theorem 1 we obtain a lower bound for the homological dimension $\dim_h$ of the configuration space of graphs $F_n\Gamma$. More precisely, $\dim_h F_n\Gamma \geq \min(b(\Gamma), \left[\frac{n}{2}\right])$. In the abstract "Configuration spaces of* *graphs - part I" it is shown that* $\dim_h F_n\Gamma \leq \min(b(\Gamma), n)$. *Thus* $\dim_h F_n\Gamma = b(\Gamma)$ *if* $n \geq 2b(\Gamma)$.

References

[1] M. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Springer Verlag (1999)

[2] J. Swiatkowski, *Estimates for homological dimension of configuration spaces of graphs*, Colloq. Math. **89** (2001), 69–79.

Knot theory and robot arms

Bruno Benedetti

This talk served as a brief invitation to Knot Theory and as an exposition of the main result on straigthening robot arms in the plane by Connelly, Demaine and Rote (2003).

1. A few words on Knot Theory

Classical knot theory studies smooth embeddings of the 1-dimensional sphere S^1 in $\mathbb{R}^d$, up to the equivalence relation of *ambient isotopy.* When $d = 2$, the Schönflies theorem says that every simple closed curve divides the plane into two regions, one of which must be homeomorphic to a disc. When $d \geq 4$ all knots in $\mathbb{R}^d$ are known to be isotopic to the "unknot", i.e. the boundary of a disc. So the truly interesting case is given by embeddings of S^1 in $\mathbb{R}^3$.

Knots have been studied since the dawn of human history. They are usually represented via generic planar projections called *diagrams.* A *grid diagram* is an $n \times n$ square grid with n $\mathbb{X}$'s and n $\mathbb{O}$'s placed in distinct squares, so that each row and each column contains exactly one $\mathbb{X}$ and exactly one $\mathbb{O}$ [12]. If we connect each $\mathbb{X}$ with the $\mathbb{O}$ in the same row resp. column by drawing a horizontal resp. vertical string, from the grid diagram we obtain a knot, or possibly a disjoint union of knots. (By convention, whenever two strands overlap, the horizontal one passes under the vertical one.) Any knot can be realized via some grid diagram. Two diagrams D_1 and D_2 represent isotopic knots if and only if D_1 and D_2 are related by a finite sequence of combinatorial moves called *Cromwell moves* [10, 12].

But how can we show that two given knots are non-isotopic? The answer is obtained by considering *knot invariants*, which are maps defined on the set of all knots modulo ambient isotopy. Famous examples include the *knot group*, the *crossing number*, the *Alexander polynomial*, the *Jones polynomial*, *Heegaard–Floer homology* and *Khovanov (co)homology.* Each one of these invariants distinguishes for example the unknot from the trefoil knot, by mapping them into different objects. Heegaard–Floer homology, which can be computed algorithmically from a grid diagram [10], *detects* the unknot, in the sense that it distinguishes it from any other knot [13]. The same holds for Khovanov homology [1, 9], and perhaps (this is an open conjecture) for the Jones polynomial as well.

2. Knots, Robot Arms and Rigid Polygons

A *robot arm* (resp. a *polygon*) is a path without self-intersections (resp. a simple closed curve) in $\mathbb{R}^d$ consisting of a sequence of finitely many segments, called *bars*. Two bars intersect in a *vertex*. Is it always possible to straigthen a robot arm in $\mathbb{R}^d$ via a rigid motion that preserves the bar lengths and avoids self-intersections?

The answer is negative for $d = 3$. To see this, take a trefoil-knotted robot arm in which the first and the last bar are much longer than all the middle bars put together [2]. The idea is that this metric condition keeps the endpoints far away from the actual 'knotted part' of the string, making it impossible to untangle it.

In contrast, all robot arms in $\mathbb{R}^d$ can be straightened if $d \neq 3$ [6, 4]. This was recently proven by solving a related problem: Can any polygon be opened to a convex position via a continuous rigid motion that preserves the bar lengths and avoids self-intersections? (We call such a motion "convexification". Straightening a path contained in the boundary of a convex polygon is easy, so if all polygons can be convexified, then all robot arms can be straightened.) This problem can be viewed as a discrete-geometric analogue of the following topological question: "Is each knot in $\mathbb{R}^d$ isotopic to the unknot?". The answer is in fact analogous: Positive for $d \geq 4$ [4], negative for $d = 3$ [2], and positive again for $d = 2$ [6].

The case $d = 2$, known as "carpenter's rule problem", has been an open question for many years. It was finally solved in 2003 by Connelly, Demaine and Rote [6]; the solution will be discussed in the next paragraphs. The case $d = 3$ is much easier: Any knot in $\mathbb{R}^3$ can be realized as a polygon with sufficiently many edges, but only the unknot can be realized as a *convex* polygon. So, roughly speaking, "knotted polygons" cannot be convexified. Interestingly, some "unknotted polygons" cannot be convexified either: To see this, take the previous example of a robot arm that cannot be straightened, thicken it to a 2-dimensional strip and look at the boundary of the strip [2]. Finally, the case $d \geq 4$ was solved in 2001 [4]. The idea is that a bar in $\mathbb{R}^d$ is free to rotate around one of its endpoints, as long as collisions with other bars are avoided; but if $d \geq 4$ the rotating endpoint has at least 3 degrees of freedom, so any 1-dimensional obstacle can be bypassed [4].

3. The planar case: Infinitesimal Expansions

How to convexify a polygon in the plane? The breakthrough solution [6] is to look for an *expansive* motion, that is, a motion in which the distance between any two points of the polygon is non-decreasing as time goes by. In particular, expansive motions automatically avoid self-intersections. A crucial reduction from infinitely-many to finitely-many conditions is obtained by replacing the word "points" with "vertices" in the definition of expansive motion:

Lemma 3.1. *A motion of a polygon is expansive if and only if the distance between any two vertices is non-decreasing as time goes by.*

The proof is elementary and based on the following observation: If the endpoints of a segment are moving away from us, then also all points of the segment are. Formally, let $p_1, \ldots, p_n$ be the n vertices of the polygon. Let B (resp. S) be the set of all pairs $\{i,j\}$ such that the segment joining p_i and p_j is a bar (resp. is *not* a bar). We will call the elements of S *struts*. A *(strictly) expansive motion* is a differentiable function $\mathbf{p}(t) = (\mathbf{p}_1(t), \dots, \mathbf{p}_n(t))$, such that

$$\begin{cases} \frac{d}{dt}\,||\mathbf{p}_j - \mathbf{p}_i||^2 & = 0 \quad \text{if } \{i,j\} \text{ is in } B, \\ \frac{d}{dt}\,||\mathbf{p}_j - \mathbf{p}_i||^2 & > 0 \quad \text{if } \{i,j\} \text{ is in } S. \end{cases} \tag{3.1}$$

In plain words, expansions increase the strut lengths but maintain the bar lengths. An *infinitesimal expansion* is a $2 \times n$ matrix $\mathbf{v} = (\mathbf{v}_1, \dots,\ \mathbf{v}_n)$ such that

$$\begin{cases} (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) & = 0 \quad \text{if } \{i,j\} \text{ is in } B, \\ (\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{p}_j - \mathbf{p}_i) & \geq 1 \quad \text{if } \{i,j\} \text{ is in } S. \end{cases} \tag{3.2}$$

(The dots above are scalar products of vectors; the system (2) is obtained from (1) simply by differentiating and rescaling.) By the Farkas lemma, the existence of an infinitesimal expansion is equivalent to the non-existence of certain equilibrium stresses. A standard tool to analyze these stresses is the Maxwell–Cremona diagram [7, 11], which 'lifts' a planar bars-and-struts framework to a polyhedral surface in $\mathbb{R}^3$. Combining classical techniques and brilliant intuitions, Connelly, Demaine and Rote were able to achieve the following result [6]:

Theorem 3.2 (Connelly–Demaine–Rote)**.** *Any non-convex polygon in the plane admits an infinitesimal expansive motion.*

The proof [6, pp. 1–19] can be sketched as follows:

(i) Given a non-convex polygon A one can build a *planar* framework of bars and struts $G'_A(\mathbf{p})$. Assuming by contradiction the non-existence of infinitesimal expansions, by the Farkas lemma there is a non-zero equilibrium stress for $G'_A(\mathbf{p})$ that is non-positive on all struts.

(ii) In the Maxwell–Cremona lifting of $G'_A(\mathbf{p})$, one considers the (possibly disconnected) region M in the xy plane where the z value attains it maximum. All edges in ∂M are given a positive stress. In particular, the edges in ∂M cannot be struts, so they are all bars.

(iii) Since the framework comes from a polygon, at most two bars intersect at each vertex. This implies that whenever M contains two incident bars, M contains also one of the two 'pie wedges' they form, namely, the one at the angle bigger than π. (The case of colinear bars can be neglected simply by consolidating them into a longer bar.)

(iv) One concludes that M is a 2-dimensional connected region corresponding to the 'outside' of a *convex* polygon. In particular, ∂M is a convex cycle made of bars. But then ∂M must coincide with the starting polygon A, which was not convex: A contradiction.

4. The planar case: Local to Global

There are several ways to combine the infinitesimal expansions into a global motion [3, 6, 8, 14]. In nature, one observes many expanding motions caused by electrical repulsion phenomena. So, a 'natural' strategy (introduced in [3]) is to define on the polygon a smooth energy functional that (1) decreases under expansions, (2) is infinite when self-intersections occur, (3) is miminum when the polygon reaches a convex position. Following the downhill gradient flow of the energy functional one eventually reaches a minimum energy configuration. In view of Theorem 3.2, this minimum must correspond to a convex position. (For an example of a smooth energy functional, see [3].) A more thorough approach reveals that the smoothness assumption is not needed: One only needs the functional to be C^1 with bounded curvature [3, 8]. For example, one could use the functional

$$E \;=\; \sum_{i=1}^{n} \sum_{j\notin\{i,i-1\},\; j=1}^{n} \frac{1}{\operatorname{dist}(\mathbf{p}_i, e_j)^2}$$

where e_j is the edge between $\mathbf{p}_j$ and $\mathbf{p}_{j+1}$ (and $\mathbf{p}_{n+1} := \mathbf{p}_1$). See [8] for algorithmic consequences.

5. Conclusion

The study of robot arm motions has plenty of applications, ranging from computer graphics to protein folding and engineering. Since many real-life robot arms have “fingers”, one might be tempted to replace the path-like model for robot arms discussed here with a tree-like model. However, in the proof of Theorem 3.2, step (iii), the lack of vertices of degree > 2 is crucial. In fact, some plane metric trees with a single vertex of degree ≥ 2 cannot be brought into a star-shaped configuration via a rigid non-self-intersecting motion [5]. In the plane, the configuration space of a metric tree may thus be disconnected, while that of a path or a cycle is always path-connected.

References

[1] D. Bar-Natan, Fast Khovanov homology computations. J. Knot Theory Ramifications 16 (2007), 243–255.

[2] T. Biedl et al., Locked and unlocked polygonal chains in $3D$. Discrete Comput. Geom. 26 (2001), 269–281.

[3] J. H. Cantarella, E. D. Demaine, H. N. Iben and J. F. O’Brien, An energy-driven approach to linkage unfolding. In Proc. SoCG 2004, New York (2004), 134–143.

[4] R. Cocan and J. O’Rourke, Polygonal chains cannot lock in 4D. Comput. Geom. 20 (2001), 105–129.

[5] R. Connelly, E. D. Demaine and G. Rote, Infinitesimally locked self-touching linkages with applications to locked trees. In Physical Knots: Knotting, Linking, and Folding of Geometric Objects in $\mathbb{R}^3$. J. Calvo, K. Millett and E. Rawdon eds, AMS (2002), 287-311.

[6] R. Connelly, E. D. Demaine and G. Rote, Straightening polygonal arcs and convexifying polygonal cycles. Discrete Comput. Geom. 30 (2003), 205–239.

[7] L. Cremona, Le figure reciproche nella statica grafica, Milano, Bernardoni (1872).

[8] H. N. Iben, J. F. O’Brien and E. D. Demaine, Refolding planar polygons. Discrete Comput. Geom. 41 (2009), 444–460.

[9] P. Kronheimer and T. Mrowka, Khovanov homology is an unknot-detector, `arXiv:1005.4346` (2010).

[10] C. Manolescu, P. S. Ozsváth, Z. Szabó and D. Thurston, On combinatorial link Floer homology. Geom. Topol. 11 (2007), 2339–2412.

[11] J. C. Maxwell, On reciprocal figures and diagrams of forces, Philos. Mag. 4: 27 (1864), 250–261.

[12] L. Ng and D. Thurston, Grid diagrams, braids and contact geometry. Proc. Gokova (2008), 120–136.
[13] P. S. Ozsváth and Z. Szabó, Holomorphic disks and genus bounds. Geom. Topol. 8 (2004), 311–334.
[14] I. Streinu, A combinatorial approach to planar non-colliding robot arm motion planning. In Proc. FOCS (2000), 443–453.

Topological Robotics Arbeitsgemeinschaft, Oberwolfach
Problem Session

The following open problems were discussed in the final meeting of the Topological Robotics Arbeitsgemeinschaft, held at Oberwolfach from 11th–15th October 2010.

(1) (Regular and symmetric TC of high torsion lens spaces). It is known from [2] that the TC of a lens space $L^{2n+1}(m) = S^{2n+1}/\mathbb{Z}_m$ of high torsion (i.e., such that m does not divide the binomial coefficient $\binom{2n}{n}$) is $4n+2$. On the other hand, for even m, it is known from [4] that the TC of a low (i.e., not high) torsion lens space $L^{2n+1}(m)$ is bounded from above by $4n$. In the threshold, i.e. when m divides $\binom{2n}{n}$ but m does not divide $\binom{2n-1}{n}$ (note that $\binom{2n}{n} = 2\binom{2n-1}{n}$), it is known from [4] that $\mathrm{TC}(L^{2n+1}(m)) = 4n$ provided m is even.

On the other hand, for a high torsion lens space $L^{2n+1}(m)$ it is known (see for instance [4]) that $4n+2 \leq \mathrm{TC}^S(L^{2n+1}(m)) \leq 4n+3$.

Questions. (Suggested by J. González, presented by M. Grant).

(a) In the low torsion case, does the upper bound $\mathrm{TC}(L^{2n+1}(m)) \leq 4n$ hold also for odd m?
(b) In the threshold case, does the equality $\mathrm{TC}(L^{2n+1}(m)) = 4n$ hold also for odd m?
(c) What would it be the "next more complicated level of torsion" − and the corresponding TC?
(d) Compute (from the two possibilities above) $\mathrm{TC}^S(L^{2n+1}(m))$ for high torsion lens spaces.

(2) (Higher TC of configuration spaces of linkages in $\mathbb{R}^3$). For a fixed length vector $\ell = (\ell_1, \ldots, \ell_n)$ with $\ell_i > 0$, consider the configuration space

$$N_\ell = \left\{ (v_1, \ldots, v_n) \in S^2 \times \cdots \times S^2 : \sum_{i=1}^{n} \ell_i v_i = 0 \right\} \Big/ \mathrm{SO}(3).$$

Question. (Suggested by J. González, presented by M. Grant). Compute the higher topological complexities $\mathrm{TC}_n(N_\ell)$. It was suggested that this may be possible by considering dimension, connectivity and cohomological bounds. (**Remark**: It is known from [1] that $\mathrm{TC}_n(M) = nm$ for any closed simply connected symplectic manifold M^{2m}.)

(3) (Symmetric TC of P^n and embedding dimension). Let P^n be the real projective space of dimension n. It is known from [4] that

$$\mathrm{TC}^S(P^n) = \mathrm{level}_{\mathbb{Z}_2}(P^n \times P^n - \Delta_{P^n}),$$

where Δ_{P^n} is the diagonal of $P^n \times P^n$, and the configuration space $P^n \times P^n - \Delta_{P^n}$ (also denoted by $B(P^n, 2)$) is equipped with the involution τ permuting the factors. In particular the computation of $\mathrm{TC}^S(P^n)$ (and therefore, in the Haefliger's metastable range, the calculation of the Euclidean embedding dimension of P^n) is equivalent to finding the smallest r such that the following diagram

$$\begin{array}{ccc} & & P^r \\ & \nearrow & \downarrow \iota \\ B(P^n, 2) & \longrightarrow & P^\infty \end{array}$$

can be completed up to homotopy.
Question. (Suggested by J. González, presented by M. Grant). For a complex oriented cohomology theory h^* (the universal case, MU, should suffice), what can it be said about the map $h^*(P^\infty) \to h^*(B(P^n, 2))$? As explained in [3], information of this sort could lead to lower bounds for $\mathrm{TC}^S(P^n)$—and, consequently, to possibly new non-embedding results for P^n.

(4) (TC of a $K(\pi, 1)$). A theorem of Eilenberg–Ganea states that

$$\mathrm{cat}(\pi) = \mathrm{cdim}(\pi) + 1 = \mathrm{geomdim}(\pi) + 1,$$

in all cases, except that there may exist a group π satisfying $\mathrm{cdim}(\pi) = 2$, $\mathrm{cat}(\pi) = \mathrm{geomdim}(\pi) = 3$. The Eilenberg–Ganea conjecture states that no such π exists, that is, every group of cohomological dimension 2 has a 2–dimensional Eilenberg–Maclane space.
  (i) **Question.** (Suggested by M. Farber, presented by M. Grant). Motivated by the conjecture, compute the topological complexity $\mathrm{TC}(\pi) := \mathrm{TC}(K(\pi, 1))$ (since TC is an invariant of homotopy type, $\mathrm{TC}(K(\pi, 1))$ depends only on the group π).
  (ii) **Question.** (Suggested by D. Cohen). For which π do we have $\mathrm{TC}(\pi) = \mathrm{zcl}H^*(\pi) + 1$? (This equality is known to hold for right-angled Artin groups.)
  (iii) **Question.** (Suggested by J. Oprea). Compute the topological complexity of a nilmanifold.
  (iv) **Question.** (Suggested by M. Farber and M. Grant). Let N_g be the non-orientable surface of genus $g \geq 1$. We have the known inequality $4 \leq \mathrm{TC}(N_g) \leq 5$ for all $g \geq 1$. Compute $\mathrm{TC}(N_g)$ for all $g \geq 1$.

(5) (Rational TC). Let X be simply-connected and let $(\Lambda V, d)$ be a minimal model. Then $\mathrm{cat}(X_{\mathbb{Q}})$ is the smallest k such that the projection $p_k : \Lambda V \to \Lambda V/\Lambda^{>k}V$ admits an algebra retraction (up to homotopy). It is easier to

work with $\mathrm{mcat}(X_{\mathbb{Q}})$, which is the smallest k such that p_k admits a ΛV-module retraction (a result of Hess states that mcat = cat). We have that $\Lambda V \otimes \Lambda V$ models $X \times X$ and $\mu : \Lambda V \otimes \Lambda V \to \Lambda V$ models $\Delta : X \to X \times X$. Let $I = \ker \mu$. A result of Jessup, Murillo and Parent states that $\mathrm{TC}(X_{\mathbb{Q}})$ is the smallest k such that the projection $q_k : \Lambda V \otimes \Lambda V \to (\Lambda V \otimes \Lambda V)/I^{k+1}$ has an algebra retraction. Define $\mathrm{MTC}(X_{\mathbb{Q}})$ to be the smallest k such that q_k has a $\Lambda V \otimes \Lambda V$–module retraction.
Question. (Suggested by M. Grant). Motivated by the result of Hess, is MTC = TC?

(6) (TC of the complement of a hyperplane arrangement). Let $\mathcal{A} \subset \mathbb{C}^\ell$ be a hyperplane arrangement. A conjecture of Yuzvinsky states that $\mathrm{TC}(M(\mathcal{A})) = \mathrm{zcl}A^*(\mathcal{A}) + 1$, where $A^*(\mathcal{A})$ is the Orlik-Solomon algebra of $\mathcal{A}$. Evidence for this conjecture includes the known topological complexity of $F(\mathbb{C}, n)$, which is the complement of the braid arrangement, and the known topological complexity of $M(\mathcal{A})$ when $\mathcal{A}$ is a supersolvable arrangement. In the latter case, $M(\mathcal{A})$ is a $K(\pi, 1)$, for $\pi = \pi_1(M(\mathcal{A}))$ (see Problem 4). Further evidence for the conjecture is the knowledge that $\mathrm{TC}(M(\mathcal{A})) = \mathrm{zcl}A^*(\mathcal{A})+1$ for "general position arrangements". Note that here in general we have the strict inequality $\mathrm{TC}(M(\mathcal{A})) \lneq 2 \dim M(\mathcal{A})+1$.
Question. (Suggested by D. Cohen). Determine $\mathrm{TC}(M(\mathcal{A}))$. Navigation functions were suggested as a possible approach for solving this problem.

(7) (Random 2–complexes). The Linial–Meshulam model for random 2–complexes has the following property: if $p \ll n^{-1}$, then random 2–complexes are homotopically 1–dimensional, whereas if $p \gg n^{-1}$, random 2–complexes satisfy $\pi_2 \neq 0$ (asymptotically almost surely).
Question. (Suggested by M. Farber). Find a model producing random *aspherical* 2–dimensional complexes. Determine whether this can also be achieved by modifying the existing Linial–Meshulam model.
Question. (Suggested by M. Farber). Study the behaviour of random 2–complexes for $p = n^\alpha$, where $\alpha \in [-1, -1/2]$. Show that in this range the fundamental group $\pi_1(Y)$ is not free (has homological dimension 2). It is known that for any fixed $\varepsilon > 0$, if $p \ll n^{-1/2-\varepsilon}$, then random 2–complexes have non-trivial hyperbolic fundamental group.

(8) (Random 2–complexes). **Question.** (Suggested by E. Babson). A theorem of Linial–Meshulam–Wallach states that for each prime p, the threshold for vanishing of $H_1(Y(n, n^\alpha), \mathbb{Z}_p)$ occurs at $\alpha = -1$. Is there an α–threshold for the simultaneous vanishing of $H_1(Y(n, n^\alpha), \mathbb{Z}_p)$, p prime? (This is equivalent to the vanishing of $H_1(Y(n, n^\alpha), \mathbb{Z})$).
Question. (Suggested by E. Babson). A result of Meshulam states that for a finite group Γ, the threshold for the non-existence of an epimorphism $\pi_1(Y(n, n^\alpha)) \twoheadrightarrow \Gamma$ occurs at $\alpha = -1$. Is there an α–threshold for the following property: for all finite groups Γ, there is no epimorphism $\pi_1(Y(n, n^\alpha)) \twoheadrightarrow \Gamma$? Conjecture: $\alpha = -1/2$.

(9) (Equivariant TC). Suppose that a group G acts on a space X. There exists an equivariant version cat_G of the Lusternik–Schnirelmann category satisfying the inequality $\mathrm{cat}_G X \geq \mathrm{cat}(X/G)$, where X/G is the orbit space of the action. Equality holds if the action is free.
Question. (Suggested by H. Colman). Provide a suitable definition of equivariant topological complexity, and give an interpretation in the context of topological robotics.

(10) (Cohomology of the 2–point configuration space of a manifold). Given a fibration $p : A \to C$ and a map $f : B \to C$, there is a fibration $p^*(A) \to B$ and a map $f^* : p^*(A) \to A$ such that the following square commutes:

$$\begin{array}{ccc} p^*(A) & \xrightarrow{f^*} & A \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ B & \xrightarrow[f]{} & C \end{array}$$

Let $p : X^I \to X \times X$ be the fibration $\gamma \mapsto (\gamma(0), \gamma(1))$ and let $f : X \to X \times X$ be the diagonal map. The previous square becomes

$$\begin{array}{ccc} LX & \longrightarrow & X^I \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & X \times X \end{array}$$

If X is a closed manifold, the Eilenberg–Moore spectral sequence computes the cohomology of LX.

References

[1] I. Basabe, J. González, Y. B. Rudyak and D. Tamaki, *Higher topological complexity and homotopy dimension of configuration spaces on spheres*, arXiv:1009.1851v1 [math.AT] (2010)

[2] M. Farber and M. Grant, *Robot motion planning, weights of cohomology classes, and cohomology operations*, Proc. Amer. Math. Soc., 136(9) 3339–3349 (2008)

[3] J. González, *Symmetric topological complexity as the first obstruction in Goodwillie's Euclidean embedding tower for real projective spaces*, arXiv:0911.1116v4 [math.AT] (to appear in Transactions of the American Mathematical Society) (2010)

[4] J. González and P. Landweber, *Symmetric topological complexity of projective and lens spaces*, Alg. Geom. Topol., 9(1) 473–494 (2009)

Reporter: Ramón Vera

Participants

Dr. Andres Angel
Hausdorff Center for Mathematics
Universität Bonn
Villa Maria
Endenicher Allee 62
53115 Bonn

Giacomo d'Antonio
Fachbereich 3
Mathematik und Informatik
Universität Bremen
Bibliothekstr. 1
28359 Bremen

Prof. Dr. Eric Babson
Department of Mathematics
University of California, Davis
One Shields Avenue
Davis CA 95616-8633
USA

Dr. Yuliy Baryshnikov
AT & T Bell Laboratories
P.O. Box 636
600 Mountain Avenue
Murray Hill , NJ 07974-0636
USA

Ibai Basabe
Department of Mathematics
University of Florida
358 Little Hall
P.O.Box 118105
Gainesville , FL 32611-8105
USA

Dr. Bruno Benedetti
Institut für Mathematik
MA 6-2
Technische Universität Berlin
Straße des 17. Juni 136
10623 Berlin

Robin Chhabra
University of Toronto
Institute for Aerospace Studies
4925 Dufferin Street
Toronto , Ont. M3H 5T6
CANADA

Prof. Dr. Daniel C. Cohen
Department of Mathematics
Louisiana State University
Baton Rouge LA 70803-4918
USA

Dr. Hellen Colman
Department of Mathematics
Wilbur Wright College
4300 N. Narragansett
Chicago , IL 60634
USA

Dr. Armindo Costa
Dept. of Mathematical Sciences
Durham University
Science Laboratories
South Road
GB-Durham DH1 3LE

Kenneth Deeley
Dept. of Mathematical Sciences
Durham University
Science Laboratories
South Road
GB-Durham DH1 3LE

Barbara Di Fabio
ARCES
Universita degli Studi di Bologna
via Toffano 2/2
I-40135 Bologna

Prof. Dr. Michael Farber
Dept. of Mathematical Sciences
Durham University
Science Laboratories
South Road
GB-Durham DH1 3LE

Prof. Dr. Daniel Farley
Dept. of Mathematics & Statistics
Miami University
Oxford , OH 45056
USA

Prof. Dr. Eva Maria Feichtner
Fachbereich 3
Mathematik und Informatik
Universität Bremen
Bibliothekstr. 1
28359 Bremen

Myriam Finster
Institut für Algebra u. Geometrie
Karlsruhe Institute of Technology
Kaiserstr. 89-93
76128 Karlsruhe

Aleksandra Franc
Institute of Mathematics, Physics
and Mechanics
University of Ljubljana
Jadranska 19
1000 Ljubljana
SLOVENIA

Viktor Fromm
Dept. of Mathematical Sciences
Durham University
Science Laboratories
South Road
GB-Durham DH1 3LE

Dr. Swiatoslaw R. Gal
Institute of Mathematics
Wroclaw University
pl. Grunwaldzki 2/4
50-384 Wroclaw
POLAND

Prof. Dr. Jesus Gonzalez Espino Barros
Departamento de Matematicas
CINVESTAV
Ave. IPN 2508
Mexico City 07360
MEXICO

Dr. Mark Grant
School of Mathematics
University of Edinburgh
James Clerk Maxwell Bldg.
King's Buildings, Mayfield Road
GB-Edinburgh EH9 3JZ

Prof. Dr. Jelena Grbic
Department of Mathematics
The University of Manchester
Oxford Road
GB-Manchester M13 9PL

Dr. Sebastian Grensing
Institut für Algebra u. Geometrie
Fakultät für Mathematik
KIT
Kaiserstr. 89-93
76133 Karlsruhe

Mijail Guillemard
Department Mathematik
Universität Hamburg
Bundesstr. 55
20146 Hamburg

Anna Gundert
Institut für Theoretische Informatik
ETH Zürich
ETH Zentrum
CH-8092 Zürich

Tim Haga
Fachbereich 3
Mathematik und Informatik
Universität Bremen
Bibliothekstr. 1
28359 Bremen

Marco Hamann
Fachrichtung Mathematik
Technische Universität Dresden
Willersbau B 107
Zellescher Weg 12-14
01062 Dresden

Liz Hanbury
Dept. of Mathematical Sciences
Durham University
Science Laboratories
South Road
GB-Durham DH1 3LE

Prof. Dr. Bernhard Hanke
Institut für Mathematik
Universität Augsburg
86135 Augsburg

Danijela Horak
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstr. 22 - 26
04103 Leipzig

Prof. Dr. Thomas Kappeler
Institut für Mathematik
Universität Zürich
Winterthurerstr. 190
CH-8057 Zürich

Mikhail Khristoforov
3-I/10, Nakhimova Str.
199 226 St. Petersburg
RUSSIA

Dr. Stephan Klaus
Mathematisches Forschungsinstitut
Oberwolfach gGmbH
Schwarzwaldstr. 9-11
77709 Oberwolfach

Dr. Thilo Kuessner
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster

Dr. Rongmin Lu
Erwin Schrödinger International
Institute for Mathematical Physics
Boltzmanngasse 9
A-1090 Wien

Prof. Dr. Gregory M. Lupton
Department of Mathematics
Cleveland State University
2121 Euclid Avenue
Cleveland , OH 44115
USA

Dr. Benjamin Matschke
Institut für Mathematik
Technische Universität Berlin
Sekr. MA 6-1
Straße des 17. Juni 136
10623 Berlin

Dr. Roy Meshulam
Department of Mathematics
Technion - Israel Institute of
Technology
Haifa 32000
ISRAEL

Florian Nisbach
Institut für Algebra u. Geometrie
Fakultät für Mathematik
KIT
Kaiserstr. 89-93
76133 Karlsruhe

Dr. Martin Olbermann
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn

Prof. Dr. John F. Oprea
Department of Mathematics
Cleveland State University
2121 Euclid Avenue
Cleveland , OH 44115
USA

Viktoriya Ozornova
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn

Dr. Martin Raußen
Department of Mathematical Sciences
Aalborg University
Fredrik Bajers Vej 7G
DK-9220 Aalborg Ost

Moritz Rodenhausen
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn

Prof. Dr. Albert Ruiz
Departament de Matematiques
Universitat Autonoma de Barcelona
Campus UAB
E-08193 Bellaterra

Dr. Dirk Schütz
Dept. of Mathematical Sciences
Durham University
Science Laboratories
South Road
GB-Durham DH1 3LE

Dipl.Math. Nikita Selinger
School of Engineering and Science
Jacobs University Bremen
Postfach 750561
28725 Bremen

Luba Stein
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn

Thansri Thorranin
Residence Metoba, Room 102
3-1-12 Metoba, Matsumoto-shi
Nagano 390-0806
JAPAN

Prof. Dr. Wilderich Tuschmann
Institut für Algebra u. Geometrie
Fakultät für Mathematik
KIT
Kaiserstr. 89-93
76133 Karlsruhe

Ramon Vera
Dept. of Mathematical Sciences
Durham University
Science Laboratories
South Road
GB-Durham DH1 3LE

Dr. Antonio Viruel
Dept. de Algebra, Geometria y Topologia
Facultad de Ciencias
Universidad de Malaga
E-29071 Malaga

Dr. Uli Wagner
Institut für Theoretische Informatik
ETH Zürich
ETH Zentrum
CH-8092 Zürich

Prof. Dr. Miguel A. Xicotencatl
Departamento de Matematicas
CINVESTAV
Ave. IPN 2508
Mexico City 07360
MEXICO