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Mini-Workshop: Higher Dimensional Elliptic Fibrations

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Abstract. Elliptic fibrations play a central role in the geometry of complex surfaces, and there is a comprehensive array of theory and examples. They arise also as a tool in many applications, such as the construction of rational points in arithmetic, metrics in differential geometry and certain string dualities in physics. In higher dimensional geometry, the foundational results of the past 30 years have not yet developed into a practical collection of everyday tools, as they have in the surface case. Nevertheless, the applications already work in higher dimensions – a glance at the literature shows the extent to which practical calculations in physics alone now far outpace the existing theory. This workshop brings together geometers, physicists and others to compare applications of elliptic fibrations and the state of the general theory.

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Introduction by the Organisers

This meeting was about elliptic fibrations. The intention was to compare points of view of the uses of elliptic fibrations in different areas of mathematics and physics. This included the methods, results and ambitions of different approaches to elliptic fibrations, and also work with other closely related fibration structures. The contrast between elliptic fibrations in physics and in algebraic, differential and arithmetic geometry was at the forefront.

Several recent advances were prominent. First, the analysis of singular fibres in higher dimensions and their relation to physics, appearing in the work of Grassi and Morrison for example. Second, the non-existence of birational elliptic fibrations on certain Fano 3-folds, proved by Cheltsov and Park, and the biregular descriptions of fibrations when they do exist, arising in the work of Ryder and others. Third, various results in arithmetic geometry by many authors, such as work on Mordell–Weil rank by Kloosterman, and naturally including the recent work of Prendergast-Smith on extremal elliptic fibrations. Finally, several new developments which can be viewed as generalisations of elliptic fibrations, like T-varieties by Altmann and more general torus fibrations by Melnikov.

The meeting had short series of talks by some of the experts, more traditional research talks by all remaining participants, and 'after hours' sessions where detailed calculations and discussion took place as a group at the board.

1. What is an elliptic fibration?

The basic common definition used throughout the workshop was this: an elliptic fibration is a proper morphism between complex algebraic varieties $f\colon X \to Y$ whose general fibre is an elliptic curve. There are many different ways to refine this. Among the optional extras one can choose from are:

- X and Y are both smooth.
- X and Y have $\mathbb{Q}$-factorial terminal (or canonical) singularities.
- f has a section.
- f is flat.
- The discriminant of f is a normal crossing divisor.
- f has relative Mordell–Weil rank 0.

Different combinations of these, among other refinements, appear in the literature and in the talks.

There are several major foundational results in higher dimensions. Miranda describes good nonsingular models of elliptic fibrations (together with a classification of the possible singular fibres), Grassi constructs relative minimal models and Nakayama constructs birational relative canonical models. Nakayama also provides a detailed analysis of the local structure of elliptic fibrations and Dolgachev and Gross analyse multiple fibres. Much of the activity at the meeting involved studying cases of these results in particular examples related to applications.

2. Themes of the meeting

From the wide range of topics, four themes were prominent, and we characterise them as they arose in workshop sessions.

2.1. **Motivation from physics and differential geometry.**

These were pronounced most clearly in two talks by Melnikov, two by Previato, and one each by McIntyre and Singer.

In his talks, Melnikov explained the apparitions of torus fibrations in string theory. He focused on compactications of the heterotic string theory, and presented his recent classification of all such geometric compactifications that preserve $N = 2$ space-time supergravity. He pointed out that it would be a very interesting problem to give a description of the dual backgrounds via the conjectured heterotic/type II duality. One particular motivation for this problem is that in the case of heterotic theory compactified on a $K3 \times T^2$, this leads to elliptically-fibered Calabi-Yau 3-folds which were discussed in the talks by Grassi.

Previato gave two talks. The first introduced the audience to the questions and techniques in the field of integrable systems. Her second talk presented the emergence of elliptic fibrations in this area of research as moduli spaces of solutions to certain integrable differential equations.

Starting from a spectral analysis question on an elliptic curve, McIntyre showed how one can begin from the determinant formula for the Laplacian on an elliptic curve (a formula which depends on the elliptic moduli) and generalize it to Riemann surfaces of higher genus, and then how to interpret the terms in the formula as geometric quantities on an associated infinite-volume hyperbolic 3-manifold.

Singer's talk focused on the differential geometric and Kähler aspects of elliptic fibrations. He presented a program for constructing constant scalar curvature Kähler metrics on elliptically fibered surfaces based on generalizing Fine's technique of constructing such metrics in the case when all the fibers are smooth.

2.2. **Concrete calculations coming from physics.** These were discussed in two talks each by Grassi, Katzarkov and Scheidegger, and after hours sessions by Degeratu and Wendland.

Grassi's first talk, which was also the opening talk of the Miniworkshop, was an introductory talk in the topic of elliptic fibrations. She focused on the case of elliptic fibration with a section, for which she showed how to find the corresponding Weierstrass model. Then she showed that in the case of elliptic surfaces the singular fibers are of ADE type, so each has an associated simply-laced Lie group. This led to the topic of her second talk: the geometry of higher-dimensional elliptic fibrations. In the case of an elliptically fibered Calabi-Yau 3-fold, she showed how all the Lie groups (simply-laced as well as non-simply-laced) can show up in the singular fiber of the fibrations, and used this information associated to the degeneration of the elliptic fibers to define a new invariant of the 3-fold related to the Euler characteristic, as well as to define the "charge matter representation" – a quantity predicted by the heterotic/F-theory string-string duality.

In an after hours session, starting from this heterotic/F-theory duality, Degeratu and Wendland showed how one can associate a concrete elliptically fibered Calabi-Yau 3-fold to a given heterotic theory compactified on a $K3$ surface. Focusing then on a particular example of Grassi's talk, they computed all the quantitites that Grassi introduced (neutral and charged hypermultiplets, vector multiplets) and showed how they match the corresponding quantitites on the heterotic side, providing a concrete check for this conjectured duality.

Katzarkov and Scheidegger talked on related topics around Fano and Calabi–Yau varieties. The driving questions concern the rationality of certain Calabi–Yau n-folds, where new spectral invariants have been introduced and techniques from non-commutative geometry and Hodge theory become crucial. This included calculations of gaps in the spectra of categories and relations to mirror symmetry as well as modular forms, along with the counting results described below.

2.3. **Motivation from surfaces and higher-dimensional arithmetic geometry.** These arose in talks by Kloosterman, Nikulin and Schütt.

Nikulin outlined the lattice-based theory that is known for surfaces, a great deal of which comes from his own work in the subject (summarised more comprehensively in a recent preprint). Some parts of this were applied by Schütt in combination with a quadratic twist manoeuvre to provide a general construction that can be applied in various situations: he explained several cases, including a new construction of Enriques surfaces and the construction of certain Calabi–Yau 3-folds.

Kloosterman described a great deal of current work on 3-folds from an arithmetic point of view. The main motivation was to compute Hodge numbers for elliptic fibrations, and he explained formulas for the Mordell–Weil rank which arises in its contribution to $h^{1,1}$, together with applications to concrete calculations and results on average ranks.

2.4. **Broader ideas from algebraic geometry.** These appeared in talks and after hours sessions by Altmann, Park, Shepherd-Barron and Brown.

There are several generalisations of elliptic fibration that arise immediately. Holding on to the condition of trivial relative canonical class, the first two are either to consider fibrations by higher-dimensional abelian varieties, or by K3 surfaces or other varieties of Kodaira dimension zero. On the first of these, Shepherd-Barron explained his results on canonical models of moduli of Abelian varieties, including the most recent results on the structure of the exceptional loci.

Park explained his results with Cheltsov on elliptic fibrations birational to certain Fano 3-folds. Since the Fano 3-folds V they work with have Picard rank 1, there cannot be elliptic fibrations $X \to Y$ with $X = V$, but elliptic fibrations are very common once one relaxes this to allow X birational to V. Cheltsov and Park make birational constructions when V is a weighted hypersurface, and Brown described how to extend this to cases in higher codimension. Perhaps more strikingly, Cheltsov and Park also prove that certain Fano 3-folds do not admit birational elliptic fibrations.

Brown's calculations took place in toric varieties of rank ≥ 2, and Altmann outlined a general setup of toric fibrations that might be ambient spaces for elliptic (or more general) fibrations. His description was based on the foundational theory of T-varieties, by Altmann, Hausen, Süss and others, but with the theory directed towards fibrations – it remains to construct substantial examples of elliptic fibrations in this framework.

Mini-Workshop: Higher Dimensional Elliptic Fibrations

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Fibrations via divisors with special coefficients

Klaus Altmann

1. Abelian Galois covers via A-divisors

Together with Lars Petersen, we gave in [AlPe] the following construction of abelian Galois coverings of $\mathbb{P}^1_{\mathbb{C}}$: Let A be a finite, abelian group. Then, every degree 0 divisor $E \in \operatorname{Div}^0_A \mathbb{P}^1 := A \otimes_{\mathbb{Z}} \operatorname{Div}^0 \mathbb{P}^1$ can be understood as a linear map $A^* \to \operatorname{Div}^0_{\mathbb{Q}/\mathbb{Z}} \mathbb{P}^1$, $u \mapsto E_u$ with $A^* := \operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z})$ denoting the dual group. Then, we choose arbitrary lifts $\widetilde{E}_u \in \operatorname{Div}^0_{\mathbb{Q}} \mathbb{P}^1$ and, afterwards, rational functions $f_{u,v} = f_{v,u} \in \mathbb{C}(\mathbb{P}^1)$ satisfying

$$\widetilde{E}_u + \widetilde{E}_v + \operatorname{div}(f_{u,v}) = \widetilde{E}_{u+v}.$$

Correcting the functions f_{uv} with suitable constants, we may additionally assume that $f_{u,v+w}\, f_{v,w} = f_{u,v}\, f_{u+v,w}$. Thus, defining a multiplication via

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^1}(\widetilde{E}_u) \otimes \mathcal{O}_{\mathbb{P}^1}(\widetilde{E}_v) & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(\widetilde{E}_{u+v}) \\ f \quad \otimes \quad g & \mapsto & f g\, f_{u,v}^{-1} \end{array}$$

provides an associative and commutative $\mathcal{O}_{\mathbb{P}^1}$-algebra structure of the sheaf $\mathcal{O}(E) := \oplus_{u \in A^*} \mathcal{O}_{\mathbb{P}^1}(\widetilde{E}_u)$. The Galois covering associated to E is now defined as the relative spectrum $C = C(E) := \operatorname{Spec}_{\mathbb{P}^1} \mathcal{O}(E)$. The curve C is indeed smooth, and the structure map $\pi : C \to \mathbb{P}^1$ is ramified at most in the points of $\operatorname{supp} E$. Actually, the ramification index of $p \in C$ equals the order of the E-coefficient of $\pi(p)$ inside A.

Example. If $D \in \operatorname{Div}_{\mathbb{Z}} \mathbb{P}^1$ is an effective divisor with $n | \deg D$, then the associated, well-known cyclic n-fold covering of $\mathbb{P}^1$ is given, via the above recipe, by understanding D as an element of $\operatorname{Div}^0_{\mathbb{Z}/n\mathbb{Z}} \mathbb{P}^1$.

2. Toric fibrations via polyhedral divisors

The following construction stems from [AlHa] (joint with Jürgen Hausen): Let M, N be two mutually dual, free abelian groups of rank k. For convex polyhedra $\Delta \subseteq N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ ($\cong \mathbb{Q}^k$), we denote by $\operatorname{tail}(\Delta) := \{a \in N_{\mathbb{Q}} | a + \Delta \subseteq \Delta\}$ the so-called tail cone of Δ. Note that Δ is the Minkowski sum of $\operatorname{tail}(\Delta)$ and the (compact) polytope Δ^c obtained as the convex hull of the vertices of Δ. For a fixed polyhedral cone $\sigma \subseteq N_{\mathbb{Q}}$ we denote by $\operatorname{Pol}(N_{\mathbb{Q}}, \sigma)$ the Grothendieck group of the semigroup

$$\operatorname{Pol}^+(N_{\mathbb{Q}}, \sigma) := \{\Delta \subseteq N_{\mathbb{Q}} |\, \operatorname{tail}(\Delta) = \sigma\}$$

with respect to Minkowski addition. Then, a "polyhedral divisor" $\mathcal{D} \in \operatorname{Pol}(N_{\mathbb{Q}}, \sigma) \otimes_{\mathbb{Z}} \operatorname{CaDiv}(Y)$ on a semi-projective variety Y can be understood as a piecewise linear map $\sigma^\vee \cap M \to \mathrm{CaDiv}_{\mathbb{Q}}(Y)$, $u \mapsto \mathcal{D}(u)$. A certain positivity assumption for $\mathcal{D}$ translates into the convexity of this function, i.e. into the relation

$$\mathcal{D}(u) + \mathcal{D}(v) \le \mathcal{D}(u+v).$$

Thus, $\mathcal{O}_Y(\mathcal{D}) := \oplus_{u\in\sigma^\vee\cap M}\mathcal{O}_Y(\mathcal{D}(u))$ becomes an $\mathcal{O}_Y$-algebra and gives rise to the relatively affine scheme $\widetilde{X}(\mathcal{D}) := \mathrm{Spec}_Y\, \mathcal{O}_Y(\mathcal{D}) \to Y$. The M-grading of its regular functions translates into an action of the torus $T := N \otimes_{\mathbb{Z}} \mathbb{C}^*$ on $\widetilde{X}$. Moreover, the map $\widetilde{X} \to Y$ is a fibration with general fiber being the affine toric variety $\mathbb{TV}(\sigma)$. If $\mathcal{D}$ is represented as $\sum_i \Delta_i \otimes D_i$ with prime divisors D_i and polyhedra Δ_i, then the first describe where degenerations might occur, and the latter tell how they look like. For example, vertices of Δ_i correspond to irreducible components of the fibers over general points of D_i.

Remark. The affine scheme $X := X(\mathcal{D}) := \mathrm{Spec}\,\Gamma(Y, \mathcal{O}(\mathcal{D}))$ is a contraction of $\widetilde{X}$ and carries a T-action, too. The main point of [AlHa] was to show that all affine, normal T-varieties occur via this construction.

3. Replacing the polyhedra

In both constructions, we have encoded certain fibrations by divisors on the base space which carry rather unusual coefficients. Moreover, the polyhedra Δ_i of the second construction can be understood as semiample divisors on a certain toric variety Z being projective over the affine $\mathbb{TV}(\sigma^\vee)$. Then, the former multidegrees $u \in \sigma^\vee \cap M$ turn into germs of curves $u : (\mathbb{C}, 0) \to Z$, and $\mathcal{D}(u)$ associates, as a coefficient, to each divisor D_i on Y the multiplicity of the pullback of the divisor Δ_i on Z via u. This formulation gives some hope that also other, non-toric fibrations allow a similar description via divisors on both the base Y and some other space Z being linked to the general fiber.

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Elliptic fibrations; Anomalies, Group Representations and the Euler Characteristic of Elliptic Calabi-Yau Threefolds

Antonella Grassi
(joint work with David Morrison)

1. Lecture I: introduction.

Let X and B algebraic varieties. An *elliptic fibration* of X over B is a morphism $\pi : S \to B$ such that the generic fiber is a smooth connected curve of genus 1. The variety X, together with the fibration π is called an *elliptic variety*.

We focus on the case of elliptic fibrations with section; then the generic elliptic fiber is an elliptic curve with a marked point.

Note that X and B do not have to be smooth: we looked at various classical examples, in particular at $X = \mathbb{P}^2$ blown up at the intersection of 2 cubics, in different positions, and the induced elliptic fibration with base $B = \mathbb{P}^1$. We worked out the formula the canonical divisor of X, as the pullback of the canonical divisor of B and a $\mathbb{Q}$-divisor supported on the ramification locus of the fibration, that is the locus where the elliptic fiber becomes singular.

Nakayama [7] shows that elliptic fibrations with section are birationally equivalent to a Weierstrass model $p : W \to B$, with canonical singularities:
Theorem Let $\sigma(B) = C$; $\mathcal{L}^{-1} = R^1\pi_*\mathcal{O}_X = \mathcal{O}_X(C)_{|C}$. Then $W \subset \mathbf{P} := \mathbb{P}_B(\mathcal{O}_B \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$, defined by the equation $y^2z = x^3 + axz^2 + bz^3$, is the Weierstrass model of X, where $a \in H^0(B, \mathcal{L}^4)$ and $b \in H^0(B, \mathcal{L}^6)$. The canonical divisor, up to $\mathbb{Q}$-equivalence is $K_W \equiv_{\mathbb{Q}} p^*(K_B + \Delta)$.

Next we considered the case of $\dim W = 2$; the local form of the Weierstrass model is $W : y^2 = x^3 + a(u)x + b(u)$, $u \in \mathbb{C}$, and $a(u)$ and $b(u)$ determine the type of singularity of W and the type of the resolved singular fiber. It is well known that these singularities are the A-D-E (being canonical) [6]; the exceptional curves on each singular fiber are in fact dual to the Dynkin diagram of the simply laced group $SU(n), SO(n), E_6, E_7, E_8$ [1]. In this context we looked also at the Kodaira classification of the singular fibers. We also defined the Mordell-Weil group of section and its basic properties.

We finished by considering some examples of Calabi-Yau hypersurfaces in toric Fano varieties which are also elliptic varieties.

2. Lecture II: Group and anomalies-Work in collaboration with Dave Morrison

In this lecture we present some results in two projects with D. Morrison [2] and [3]; from now on X is a Calabi-Yau 3-fold which is a resolution of a Weierstrass model; we assume that the resolution is still a flat elliptic fibration.

In the previous lecture we noted that the singularities of a 2-dimensional Weierstrass model are classified by the Dynkin diagrams of the simply laced Lie groups of type A_n, D_n, E_6, E_7, E_8. Furthermore the ranks of the A-D-E groups contribute to the rank of the Picard group of the minimal resolution of the Weierstrass model, as well as its topological Euler characteristic.

When $\dim(X) = 3$ the singularities are only generically rational double points, yet it is possible to associate a group G to the singularities, obtaining all the Dynkin diagrams (including the non-simply laced ones). The 3-fold also determines a specific representation of G (known in the physics literature as the "matter representation") whose irreducible summands can be described in terms of the degenerations of the general singularity; conversely, once one chooses the representations which might occur, the geometry of the Calabi–Yau is completely determined by some relations in representation theory.

Following [2] we defined an invariant related to topological Euler characteristic and show how certain representations of G appear in the invariant when the "general" double point degenerates to a worse singularity. Then, following [3] we showed how the geometry of the Calabi–Yau and its degenerations are, at least in some cases naturally, yet surprisingly, related to the same representations. We presented the original motivation of these projects, coming from string theory. For the theory to be consistent, gauge and gravitational anomalies must vanish. Some of these anomalies can be "cancelled" by an analogue of the Green–Schwarz mechanism, while others (which occur as certain coefficients in a formal expression in the curvature) are required to vanish identically.

It should be noted that the setting presented here applies to an elliptically fibered Calabi–Yau manifold of arbitrary dimension.

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Elliptic fibrations, singularities and linear systems

Remke Kloosterman

Let V and B be smooth projective varieties. Let $\pi : V \to B$ be a non-trivial elliptic fibration with section $\sigma_0 : B \to V$ and suppose $\dim B > 1$. Let $\mathcal{L}$ be an ample line bundle on B. With this fibration one can associate a Weierstrass fibration

$$W = \{y^2z = x^3 + axz^2 + bz^3\} \subset \mathbb{P}(\mathcal{O} \oplus \mathcal{L}^{-2k} \oplus \mathcal{L}^{-3k}).$$

for some $(a, b) \in H^0(\mathcal{L}^{4k}) \times H^0(\mathcal{L}^{6k})$ and $k > 0$. Note that W in general is not smooth.

The problem under consideration is that given (a, b) determine the rank of $MW(\pi)$, the group of rational sections of π, or, equivalently, the $K(B)$-valued point of the generic fiber.

The first result is a Lefschetz-type result

Proposition 1. *There is a natural isomorphism*

$$\operatorname{Pic}(W) \cong \pi^* \operatorname{Pic}(B) \oplus \mathbb{Z}.$$

The second factor is generated by the divisor $z = 0$.

The proof of this fact exploits both that $\dim B$ is at least 2 and that $\mathcal{L}$ is ample.

Consider the map ι which sends fiberwise $(x : y : z) \to (x : -y : z)$. Then ι acts on both $\mathbb{P}(\mathcal{E})$ and W. The above Proposition shows that $\iota^* : \operatorname{Pic}(W) \to \operatorname{Pic}(W)$ is the identity map.

Consider now Weil divisors on W. Let W_η be the generic fiber. Then we have a natural group homomorphism

$$\operatorname{Div}(W) \to \operatorname{Div}(W_\eta) \cong \operatorname{MW}(\pi) \times \mathbb{Z}.$$

The subgroup of $\operatorname{Div}(W_\eta)$ fixed by ι is isomorphic to $\operatorname{MW}(\pi)[2] \times \mathbb{Z}$. From this we deduce:

Proposition 2. *There is an injective group homomorphism*

$$\operatorname{MW}(\pi) \otimes_{\mathbb{Z}} \mathbb{Q} \to \operatorname{Div}(W)/\operatorname{Pic}(W) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

If the base variety is a surface, and if we work over an algebraically closed field of characteristic zero, we have a stronger result, namely

Theorem 3. *Suppose* $\dim B = 2$. *We have an isomorphism*

$$\operatorname{MW}(\pi) \otimes_{\mathbb{Z}} \mathbb{Q} \to (\operatorname{Div}(W)/\operatorname{Pic}(W)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

To prove that this map is surjective we compare $W \to B$ with the so-called Miranda model, and keep track how the divisor group changes under the modification one has to apply to obtain the Miranda model. On the Miranda model one can express the rank of $\operatorname{MW}(\pi)$ in terms of the rank of the Néron-Severi group of that model and a few easy to calculate other invariants. (The so-called Shioda-Tate formula.)

A very practical result is the following

Theorem 4. *Suppose* $\dim B = 2$.

$$\operatorname{rank} \operatorname{MW}(\pi) = \operatorname{rank}(H^{2,2}(H^4(W, \mathbb{C})) \cap H^4(W, \mathbb{Z})) - \rho(B) - 1.$$

In many cases one can calculate the right hand side. Assume the Weierstrass equation is minimal. Then W is normal, hence is smooth in codimension 1. In codimension 2 we have that W is either smooth or has transversal ADE surface singularities. In particular these singularities are weighted homogeneous.

The right hand side in Theorem 4 can be calculated if we put some mild additional assumptions on W.

Suppose first that W has isolated singuarlities and that each singularity is weighted homogeneous. Then Dimca [1] gave a method to compute $H^4(W, \mathbb{C})$ together with its mixed Hodge structure. In particular, this mixed Hodge structure turns out to be pure. Together with Klaus Hulek we [3] extended this method to the case where each singularity of W (isolated or not) is weighted homogeneous, W is smooth in codimension 1 and W has transeversal ADE singularities in codimension 2.

To explain this result, let $\Sigma \subset W_{\text{sing}}$ be the set of points such that (W, p) is not a transversal ADE surface singularity. Then $H^4_{\text{prim}}(W)$ is the cokernel of

$$H^4(\mathbb{P}(\mathcal{E}) \setminus W) \to H^3(W \setminus W_{\text{sing}}) \to H^4_{W_{\text{sing}}}(W) \to \oplus_{p\in\Sigma} H^4_p(W).$$

The first map is the residue map, the second map is the boundary map in the exact sequence of the pair $(W, W \setminus W_{\text{sing}})$.

Since $\mathbb{P}(\mathcal{E}) \setminus W$ is the blow-up of an affine variety Spec A, we can use the above description of H^4_{prim} to define two linear systems L_1, L_2 such that the defect of L_1 gives $h^{3,1}$ of H^4_{prim}, and the defect of L_2 gives the $h^{2,2}$. We can summarize our result as follows:

Theorem 5. *Suppose L_1 has no defect then the rank of* $\text{MW}(\pi)$ *equals the defect of L_2.*

Example 6. [4] Let $f = 0$ be a reduced planar curve of degree $6k$. Suppose that all singularites are of type A_{3m}, A_{3n+1}, D_s. Then $H^4_p(W) = 0$ for all $p \in \Sigma$. Hence the Mordell-Weil rank of $y^2 = x^3 + f$ equals zero.

Suppose now that $f = 0$ has additionally cusps (A_2-singularities) at point $p_1, \ldots p_k$. Then the above method yields that

$$\operatorname{rank} \text{MW}(\pi) = 2 \left(\operatorname{codim} \mathbb{C}[z_0, z_1, z_2]_{5k-3} \stackrel{\oplus \text{ev}_{P_i}}{\longrightarrow} \oplus_{p_i} \mathbb{C} \right).$$

In case $k = 1$ one obtains

$$\operatorname{rank} \text{MW}(\pi) = \begin{cases} 0 & \#\Sigma \leq 5 \\ 2 & \#\Sigma = 7 \\ 4 & \#\Sigma = 8 \\ 6 & \#\Sigma = 9 \end{cases}$$

In the case of 6 cusps the rank depends on the position of the cusps, i.e., we have rank 2 if the cusps lie on a conic, and rank 0 otherwise.

Remark 7. Cogolludo and Libgober [2] remarked that in the above example one has that $\operatorname{rank} \text{MW}(\pi)$ equals the degree of the Alexander polynomial of the curve $f = 0$. (The Alexander polynomial is an invariant associated to the $\pi_1(\mathbb{P}^2 \setminus C)$.)

They proved for a large class of curves $\{f = 0\}$ such that the degree of f is divisible by 6 that the Mordell-Weil rank of $y^2 = x^3 + f$ equals the degree of the Alexander polynomial.

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Determinants of Laplacians on Riemann surfaces

Andrew McIntyre

(joint work with Leon Takhtajan, Lee Peng Teo, Jinsung Park)

It follows from Kronecker's first limit formula that the regularized determinant of the Laplacian on a Riemann surface of genus 1 is given by

$$\det{}'\Delta_0(\tau) = 4\mathrm{Im}\tau\, |\eta(\tau)|^4 ,$$

where $\eta(\tau) = q^{\frac{1}{24}} \prod_{m=1}^{\infty}(1 - q^m)$, $q = e^{2\pi i\tau}$ is the Dedekind eta function. This was generalized in 1986 to genus $g > 1$ by D'Hoker and Phong:

$$\det{}'\Delta_n = \begin{cases} c_{1,g} Z'(1) & : n = 1 \\ c_{n,g} Z(n) & : n > 1 \end{cases}$$

where Δ_n acts on the nth power of the canonical bundle in the hyperbolic metric, $c_{n,g}$ depends only on n and g, and $Z(s) = \prod_{[\gamma]} \prod_{m=0}^{\infty} \left(1 - q_\gamma^{s+m}\right)$ is the Selberg zeta function (the first product is over primitive conjugacy classes in a Fuchsian group $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ uniformizing the Riemann surface; $0 < |q_\gamma| < 1$ is the multiplier of γ, so γ is conjugate to $z \mapsto q_\gamma z$). This generalization loses the holomorphic dependence on moduli apparent in genus 1. It was first noticed in 1986 by Martinec and proved (for $n = 1$) in 1997 by Zograf that the genus 1 formula has a more direct holomorphic analogue (suggested by the Belavin-Knizhnik holomorphic factorization, and ideas of Quillen). In 2004, Leon Takhtajan and I proved an extended version of this result:

$$\frac{\det{}'\Delta_n}{\det N_n} = c_{g,n} \exp\{-\frac{6n^2 - 6n + 1}{12\pi} S\} \, |F(n)|^2 ,$$

where $[N]_{ij} = \langle \phi_i, \phi_j \rangle$ for a normalized basis $\{\phi_i\}$ of $\ker \Delta_n$, S is the "classical Liouville action" (a Kähler potential for the Weil-Petersson metric), and

$$F(n) = R(n, \Gamma) \prod_{[\gamma]} \prod_{m=0}^{\infty} (1 - q_\gamma^{n+m})$$

is an analogue of the Selberg zeta, but this time the product is taken over primitive conjugacy classes in a *Schottky* group $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$ uniformizing the Riemann surface. $F(n)$ is holomorphic as a function over the Teichmüller space (and in fact the Schottky space) of compact marked (resp. Schottky marked) surfaces of genus g. (The normalization of $\{\phi_i\}$ is a generalization of the Riemann normalization for $n = 1$; $R(n, \Gamma)$ is a finite product of Euler factors dependent on the choices in this normalization. For $n = 1$ the product is convergent on a proper subset of Schottky space.)

In 2004, Lee-Peng Teo and I generalized this result to quasi-Fuchsian groups. The quasi-Fuchsian result suggests that this formula is better seen as a relation between the spectral invariant $\det' \Delta_n / \det N_n$ of the Riemann surface, and geometric quantities on the infinite-volume hyperbolic 3-manifold $M^3 = \Gamma \backslash \mathbb{H}^3$. It was first shown in 2000 by K. Krasnov, and established in more generality by Takhtajan and Teo, that the complicated expression used by Takhtajan and Zograf in 1987 to define S actually simplifies to a fairly straightforward regularization of the volume of M^3. The product $F(n)$ is more similar to a traditional Selberg zeta function when considered on M^3: $-\log |q_\gamma|$ is the length of the associated closed geodesic in M^3, and the argument of q_γ can be seen as coming from a representation of the holonomy ($F(n)$ is still not identically a Selberg zeta though, for reasons not yet understood).

It was recently proved by Jinsung Park that the argument of $F(n)$ also has a geometric interpretation: it is essentially the Atiyah-Patodi-Singer eta invariant of M^3 (which turns out to be well-defined without regularization). Based on this, J. Park and I have defined a regularized Chern-Simons invariant of M^3, and we are working on proving analogues of results of T. Yoshida, who defines a regularized Chern-Simons invariant for cusped hyperbolic 3-manifolds of finite volume.

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Heterotic torus fibrations

Ilarion Melnikov
(joint work with Ruben Minasian)

A torus fibrations is an important ingredient in many compactifications of string theory. Perhaps the most familiar way in which such a fibration can arise is in IIB/F-theory compactifications. In this case, the fibration $\pi : X \to B$ encodes the variation of the axio-dilaton field over the base manifold B. Since we start with a ten-dimensional superstring theory, compactification on a d(complex)-dimensional B leads to a $10-2d$-dimensional uncompactified space-time. These constructions have been receiving much attention lately, as reviewed in presentations by Grassi and by Degeratu and Wendland.

There is, however, another natural setting for a torus fibration. This is provided by a compactification of a heterotic string on X. The simplest example of such a compactification is given by taking the trivial case of $X = K3 \times T^2$. The resulting 4-dimensional compactification preserves $N = 2$ space-time supersymmetry and has played a prominent role in the study of heterotic/type II duality. In my talk I reported on a classification of all geometric compactifications of the heterotic string that preserve $N = 2$ space-time supersymmetry. We pursued the problem from the world-sheet point of view, where the required conditions are of (0,2)+(0,4) world-sheet supersymmetry. Analysis of these conditions showed that the most general geometry consistent with $N = 2$ space-time supersymmetry is where X

is a principal T^2 bundle over $B = K3$. Fortunately, this is precisely the class for which it is known that solutions to the full supergravity equations of motion can be found.

It is good to contrast the two situations. In the first, the torus fibration is holomorphic and degenerates over a discriminant locus over the base. The base, moreover, is certainly not a Calabi-Yau manifold. In the $N = 2$ heterotic case the base is $K3$, and the fibration is non-degenerate—we actually have a T^2 bundle over B. The relatively rigid structure of the heterotic theory crucially relies on having a geometric solution with $N = 2$ space-time supersymmetry. Relaxing either of the requirements leads to a much larger class of solutions, as perhaps might have been anticipated by heterotic/type II duality. It would be very interesting to describe the type II duals of the heterotic $N = 2$ backgrounds.

Elliptic fibrations on K3 surfaces

Viacheslav Nikulin

This talk was given in Oberwolfach on workshop "Multi-dimensional Elliptic Fibrations", 3—9 October 2010. The most interesting are elliptic fibrations of Fano and Calabi–Yau varieties. Thus, it is interesting to look on K3 surfaces which are 2-dimensional Calabi–Yau manifolds.

This is mainly a review of my results related to elliptic fibrations on K3 surfaces. See [1]—[6]. For simplicity, we consider K3 surfaces X over $\mathbb{C}$, but, almost all results are valid for K3 surfaces over arbitrary algebraically closed fields.

According to Piatetsky-Shapiro and Shafarevich [7], elliptic fibrations on a K3 surface X are in one to one correspondence with primitive isotropic nef elements $c \in S_X$ where S_X is the Picard lattice of X. It follows that a K3 surface X has elliptic fibrations if and only if S_X has isotropic elements. In particular, it is valid if the Picard number $\rho(X) = \operatorname{rank} S_X$ is at least 5.

In [1], we showed that a K3 surface X with $\rho(X) \geq 6$ has an elliptic fibration with infinite automorphism group (equivalently, with infinite Mordell–Weil group) if and only if its full automorphism group Aut X is infinite, equivalently, the Picard lattice S_X is different from a finite number of hyperbolic lattices S_X of rank ≥ 6 (corresponding to X with finite automorphism group) which all were found in [1]. The same is valid for $\rho(X) = 5$ if one excludes two infinite series of Picard lattices: $S_X = \langle 2^k\rangle \oplus D_4$, $k \geq 7$, and $\langle 2\cdot 3^{2m-1}\rangle \oplus 2A_2$, $m \geq 2$, in notations of [1]. K3 surfaces with Picard lattices from these two series have infinite automorphism groups, but all their elliptic fibrations have finite automorphism groups. For $\rho(X) = 3$ and 4 similar results are unknown. For $\rho(X) = 2$ all elliptic fibrations on X have finite automorphism groups. For $\rho(X) = 1$ there are no elliptic fibrations.

We remark that all K3 surfaces with finite automorphism group were found in [7] for $\rho(X) = 1$ or 2, in [1] for $\rho(X) \geq 5$, in [8] for $\rho(X) = 4$, and in [4] for $\rho(X) = 3$. They are characterized by their Picard lattices; for $\rho(X) \geq 3$ their number is finite, and they were found in [1], [4] and [8]. See [5] for the review of these results, they are related to the general problem of classification of arithmetic hyperbolic reflection groups.

An element $x \in S_X$ is called *exceptional (with respect to the automorphism group* Aut X*)* if its stabilizer subgroup $(\mathrm{Aut}\ X)_x$ has finite index in Aut X; equivalently, the orbit Aut $X(x)$ is finite. All exceptional elements of the Picard lattice S_X define the exceptional sublattice $E(X) \subset S_X$. In [4], we showed that for $\rho(X) \geq 3$ the exceptional sublattice $E(X)$ is non-zero if and only if the Picard lattice S_X belongs to a finite list of hyperbolic lattices. For example, all Picard lattices of X with finite Aut X and $\rho(X) \geq 3$ belong to this list; then $E(X) = S_X$. Applying this general result, we obtain the following statements.

If $\rho(X) \geq 3$ and X has at least one elliptic fibration, then X has infinite number of elliptic fibrations except finite number of Picard lattices S_X when X has either finite Aut X, or X has a unique elliptic fibration with infinite automorphism group and no other elliptic fibrations. If $\rho(X) \geq 3$ and X has at least one elliptic fibration with infinite automorphism group, then X has infinite number of elliptic fibrations with infinite automorphism group except finite number of Picard lattices S_X when X has a unique elliptic fibration with infinite automorphism group.

Cases when $\rho(X) = 2$ or 1 are trivial for similar problem. Then all elliptic fibrations on X have finite automorphism group, and their number cannot be more than 2.

As we know, these results are important, for example, for the dynamics of Aut X, and for the arithmetic of X.

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Elliptic fibrations on weighted Fano threefold hypersurfaces

Jihun Park
(joint work with Ivan Cheltsov)

The 95 families of quasi-smooth anticanonically embedded weighted Fano threefold hypersurfaces with terminal singularities were introduced by A.R. Iano Fletcher ([7]). These families can be described by four positive integers $a_1 \leq a_2 \leq a_3 \leq a_4$ as follows:

$$X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4),$$

where X_d is a hypersurface defined by a quasi-smooth quasi-homogeneous polynomial of degree $d = a_1 + a_2 + a_3 + a_4$.

Geometry on a general threefold of each family has been studied intensively by I.Cheltsov, A. Corti, J. Johnson, J. Kollár, A. Pukhlikov, M. Reid, D. Ryder, and myslef ([1], [2], [3], [4], [5], [6], [8], [9], [10]). To be precise, birational rigidity, group of birational automorphisms, elliptic fibrations, K3 fibrations, and global log canonical threshold were fully investigated.

The goal of my talk is to explain general members of which families allow elliptic fibrations and general members of which families do not. Also, I will introduce methods to get elliptic fibrations in the cases when they allow elliptic fibrations.

I show that general members of 89 families among the 95 families have elliptic fibration structures. Their birational rigidity enables us to easily construct elliptic fibrations on them. In the most cases, the projection to the first three coordinates gives us an elliptic fibration, which is the best way to obtain polynomials of low degrees for general fibers. Moreover, the birational rigidity tells us that a general fiber of the projection cannot be a rational curve.

On the other hand, the general members of the remaining 6 families

- $X_6 \subset \mathbb{P}(1, 1, 1, 1, 3)$;
- $X_{24} \subset \mathbb{P}(1, 4, 5, 6, 9)$;
- $X_{30} \subset \mathbb{P}(1, 4, 5, 6, 15)$;
- $X_{36} \subset \mathbb{P}(1, 7, 8, 9, 12)$;
- $X_{40} \subset \mathbb{P}(1, 5, 7, 8, 20)$;
- $X_{50} \subset \mathbb{P}(1, 7, 8, 10, 25)$

cannot have elliptic fibration structures. To prove this, we suppose that we have a mobile linear system $\mathcal{M} \subset |-nK_X|$ inducing an elliptic fibration. Due to birational rigidity, we are able to assume that the pair $(X, \frac{1}{n}\mathcal{M})$ has at worst canonical singularities. The assumption on the mobile linear system $\mathcal{M}$ implies that the pair cannot be terminal, *i.e.*, it must have a strictly canonical singularity. However, using various geometric properties of general members of these six families, one can show that such pairs have only terminal singularities. Consequently, general members of these 6 families cannot have elliptic fibration structures.

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Elliptic Fibrations in Integrable Systems and PDEs

Emma Previato

Given upon request, this was a minicourse on integrable systems/PDEs. By this still loosely defined term one refers to the observation, in the early 1970, that a surprising number of non-linear problems, such as the Korteweg-de Vries (KdV) equation of "solitons", $u_t = 6uu_x - u_{xxx}$, have families of exact solutions which can be built out of classical special functions. Through some 20 years of mathematical discoveries including the creation of new objects, 'integrability' was explained, by linearizing the flows of (originally non-linear!) motion, by algebraic geometry (both on a Jacobi or Prym-Tyurin variety and on an infinite-dimensional Grassmannian), symplectic geometry (Poisson reduction), and representation theory. It is surprising, however, that the issue of elliptic fibrations hardly received attention. The challenge in these talks was to identify possible connections.

The principles and the main remaining open problems of algebro-geometric solution of integrable hierarchies were presented (Lecture I), along with special cases giving examples of elliptic fibrations that are moduli spaces of solutions to integrable, or conjecturally non-integrable, equations (Lecture II).

Given the size limitations, this abstract omits several topics covered and the background references except for the little-known or current ones, recommending instead the inspirational and comprehensive [BBT].

Lecture I: Geometric Methods of Integrability.

I.1 Linearization via spectral curve. A key property of non-linear wave equations are the "conserved quantities", differential (in x) polynomials in the solution (for KdV, for example, $C_0 = \int u dx, C_2 = \int u^2 dx, ...$ and the whole sequence can be written recursively in terms of Schur polynomials by expanding a heat operator [I]) which P.D. Lax (1968) was able to explain by spectral invariance.

When the "spectral curve" is algebraic, the "Krichever map", an inverse spectral problem, allows us to write solutions in terms of theta functions.
Example. Given a compact Riemann surface X of genus g, $P \in X$ and the Abel map A from X to its Jacobian, $u(\underline{t}) = -2\partial_x^2 \log \vartheta(\sum_{j\geq 1} t_j U_j + A(P) + \delta) + \text{const.}$ solves the KP equation: $u_{yy} = (u_{xxx} - 6uu_x)_x$, where $U_i \in \mathbb{C}^g$ are suitable vectors and δ is Riemann's constant. As customary, we set $t_1 = x, t_2 = y, t_3 = t$.
Remark. A constant rescaling of the solution adjusts the coefficients in KP; however, the constant shift depends on the moduli of the curve, in fact is a projective structure [T]. (2) The U_i correspond to linear flows $\sum_j u_{ij}\partial/\partial z_j$ on Jac(X), so we have linearized the flows of the KP hierarchy. Geometrically, U_1 is the tangent vector to the curve $A(X)$ at $A(\infty)$, and U_j are the j^{th} hyperosculating vectors.

I.2 The σ function. Klein generalized the (genus-1) Weierstrass σ function to a genus-g hyperelliptic curve, $X : y^2 = x^{2g+1} + \lambda_{2g}x^{2g} + \cdots + \lambda_0$ as associated to a ϑ function with characteristics $\delta = \delta'' + \delta'\omega'^{-1}\omega''$,

$$\sigma(u) = \pi^{g/2} D^{-1/4} \det\omega'^{-1/2} \exp\left(-u^T\eta'\omega'^{-1}u/2\right) \vartheta \begin{bmatrix}\delta'' \\ \delta'\end{bmatrix} \left(\omega'^{-1}u/2\,;\,\omega'^{-1}\omega''\right),$$

where D is the discriminant of the curve and the matrices $\omega', \omega'', \eta'', \eta''$ are the periods of differentials of the 1st and 2nd kind, $x^{i-1}dx/(2y)$; $\sum_{k=j}^{2g-j}(k+1-j)\lambda_{k+1+j}x^k dx/(2y)$, $(i, j = 1, \ldots, g)$ over the a, b cycles of a standard homology basis for X. They satisfy the generalized Legendre relation, $\omega'\eta''^T - \omega''\eta'^T = \imath\pi/2$. The advantage of σ over ϑ is that it transforms by a (multiplicative) universal constant under the action of the modular group.

H.F. Baker aimed at characterizing σ by PDEs (and discovered the solutions to KdV without being aware of the equation), in particular he generalized the Weierstrass equation for the elliptic curve by a 3×3 determinant that gives the equation of the Kummer surface in $\mathbb{P}^3$. Pioneering work by Bukhstaber, Ènol'skiĭ and Leĭkin generalized σ to all the curves known (in number theory, generalizing the Weierstrass cubic) as C_{ab} curves: $f(x, y) = y^a + x^b + f_{a-1,b-1}(x, y) = 0$, where a and b are coprime positive integers, and if $f_{a-1,b-1}(x, y)$ contains a monomial $x^i y^j$, then $ia + jb < ab$. The curve can be completed by a smooth point ∞ to a compact Riemann surface X of genus $g = (a-1)(b-1)/2$. The plane model makes it possible to find, case-by-case, an appropriate basis of differentials.

These curves arise in differential algebra as differential resultants [P].

I.3 Open questions. The Grassmannian model, due to M. Sato, for linearizing the flows was given in the talk. In this language, the two remaining challenges of algebraic complete integrability are arguably, in the differential-algebra model, the characterization of commutative rings of partial differential operators [LP], and in the Grassmannian model, the explicit linearization of isospectral deformation of vector bundles over the spectral curve [P]. A model [vGP] of completely integrable Hamiltonian system due to N.J. Hitchin, over the cotangent space to the moduli space of vector bundles over a curve, was also given in the talk.

Lecture II: Elliptic fibrations in integrable systems and PDEs.

II.1 Spectral surfaces. Recently [KP] a differential resultant in several variables was implemented in *Mathematica* and calculated for three commuting operators in two variables (the smallest case), in the separable case that naïvely couples ordinary differential operators whose commutator is the ring of affine functions on two elliptic curves E_i, $\mathbb{C}[\wp_i, \wp_i']$. One expects that their centralizer will be the ring of functions of $E_1 \times E_2$, an elliptic fibration, but the coordinate space $\mathbb{P}^8$ of the Segre embedding still defeats computation, so we tweaked the commuting triple and obtained a surface of degree 9 in $\mathbb{P}^3$; by fixing a point of one elliptic curve, we obtained the equation of the 'fibre' and split it into the equation of the other elliptic curve and a degree-6 curve. In one variable, letting $\partial = d/dx$, $k_1, k_2 \in \mathbb{C}$,

$$L\left(x, \partial; k_1, k_2\right) = \partial^2 - 2\wp_{k_1,k_2}(x), \;\; M\left(x, \partial; k_1, k_2\right) = \partial^3 x - 3\wp_{k_1,k_2}(x)\partial - (3/2)\wp'_{k_1,k_2}(x)$$

where $f(z) = \wp_{k_1,k_2}(z)$ is the $\wp$-function of the elliptic curve $E_{k_1,k_2}(x,y) = y^2 - 4x^3 + k_1 x + k_2 = 0$. Then $[L, M] = 0$ and the operators identically satisfy the relationship $E_{k_1,k_2}(L, 2M) = 0$. In soliton theory, this L satisfies $(L^{3/2})_+ = L^{3/2}$, a stationary (i.e. time-independent) KdV solution, time deformations are defined as $\partial_i L = [(L^{i/2})_+, L]$. We construct the three mutually commuting operators in the variables x_1 and x_2:

$$A_1 = L\left(x_1, \partial_1; k_1, k_2\right) + L\left(x_2, \partial_2; a_1, a_2\right), \;\; A_2 = M\left(x_1, \partial_1; k_1, k_2\right) \;\; A_3 = M\left(x_2, \partial_2; a_1, a_2\right),$$

where we abbreviated $\partial/\partial x_i = \partial_i$. We construct a matrix whose minor determinants are polynomials in the variables μ_1, μ_2 and μ_3 that are equal to zero upon the substitution $\mu_i = A_i$; the coefficients will be differential polynomials in the coefficients of the operators. The differential resultant, à la Macaulay (elimination theory), will be the gcd of all maximal minors. One of these is:
$P(\mu_1, \mu_2, \mu_3) = a_1^3(k_1\mu_1 + k_2 - 4\mu_1^3 + 4\mu_2^2) - a_2(-a_1^2(k_1 - 12\mu_1^2) + 2a_1(k_1^2 - 12k_1\mu_1^2 - 6\mu_1(k_2 + 8\mu_1^3 + 4\mu_2^2 - 8\mu_3^2)) - k_1^3 + 12k_1^2\mu_1^2 - 12k_1\mu_1(k_2 + 4(-\mu_1^3 + \mu_2^2 + 4\mu_3^2)) + 12(k_2^2 + 4k_2(7\mu_1^3 + 2(\mu_2^2 + \mu_3^2)) + 16(\mu_1^6 + (7\mu_2^2 - 2\mu_3^2)\mu_1^3 + (\mu_2^2 + \mu_3^2)^2))) - 2a_1^2(k_1^2\mu_1 + k_1(k_2 + 2\mu_1^3 + 4\mu_2^2 - 2\mu_3^2) + 6\mu_1^2(k_2 + 4(-\mu_1^3 + \mu_2^2 + \mu_3^2))) - 12a_2^2(a_1\mu_1 - 2k_1\mu_1 + k_2 - 4\mu_1^3 + 4\mu_2^2 + 4\mu_3^2) + a_1(k_1^3\mu_1 + k_1^2(k_2 - 4\mu_1^3 + 4\mu_2^2 - 8\mu_3^2) + 24k_1\mu_1^2(k_2 + 2\mu_1^3 + 4\mu_2^2 + 4\mu_3^2) + 24\mu_1(k_2^2 + k_2(-2\mu_1^3 + 8\mu_2^2 + 2\mu_3^2) - 8(\mu_1^6 + (\mu_2^2 - 2\mu_3^2)\mu_1^3 - 2\mu_2^4 + \mu_3^4 - \mu_2^2\mu_3^2))) - 4a_2^3 - 4(k_1^3(\mu_1^3 - \mu_3^2) + 3k_2(k_1^2\mu_1^2 - 4k_1(2\mu_1^3 - 2\mu_2^2 + \mu_3^2)\mu_1 + 16(\mu_1^6 + (7\mu_3^2 - 2\mu_2^2)\mu_1^3 + (\mu_2^2 + \mu_3^2)^2)) - 12k_1^2\mu_1^2(\mu_1^3 - \mu_2^2 - \mu_3^2) + 3k_2^2(k_1\mu_1 + 4(-\mu_1^3 + \mu_2^2 + \mu_3^2)) + 48k_1\mu_1(\mu_1^6 + (\mu_3^2 - 2\mu_2^2)\mu_1^3 + \mu_2^4 - 2\mu_3^4 - \mu_2^2\mu_3^2) + k_2^3 - 64(\mu_1^9 - 3(\mu_2^2 + \mu_3^2)\mu_1^6 + 3(\mu_2^4 - 7\mu_3^2\mu_2^2 + \mu_3^4)\mu_1^3 - (\mu_2^2 + \mu_3^2)^3))$.

The connection between this polynomial and the polynomial E_{k_1,k_2} is as follows: Let (α, β) be the coordinates of a point on the elliptic curve $E_{k_1,k_2}(x,y) = 0$. Then the polynomial $P(x + \alpha, \beta/2, y/2) \in \mathbb{C}[x, y]$ factors, $P(x + \alpha, \beta/2, y/2) = E_{a_1,a_2}(x,y) \times \left(T_2(x,y)E_{a_1,a_2}(x,y) + T_3(x,y)\right)$, where $T_2(x,y) = -4(3a_1\alpha + a_1 x + a_2 - 36\alpha^2 x - 9\alpha k_1 - 36\alpha x^2 - 6k_1 x - 4x^3) - 4y^2$ and $T_3(x,y) = -12a_1^2\alpha^2 + a_1^2 k_1 + 144a_1\alpha^4 + 432a_1\alpha^3 x + 12a_1\alpha^2 k_1 + 288a_1\alpha^2 x^2 - 36a_1\alpha k_1 x - 2a_1 k_1^2 - 24a_1 k_1 x^2 - 1728\alpha^6 - 5184\alpha^5 x + 432\alpha^4 k_1 - 6912\alpha^4 x^2 + 864\alpha^3 k_1 x - 5184\alpha^3 x^3 - 36\alpha^2 k_1^2 + 720\alpha^2 k_1 x^2 - 1728\alpha^2 x^4 - 36\alpha k_1^2 x + 432\alpha k_1 x^3 + k_1^3 - 12k_1^2 x^2 + 144k_1 x^4$.

II.2 Elliptic/co-elliptic solitons. Examples of elliptic fibrations might also be given by special spectral curves of the KP hierarchy. It is a beautiful question originally due to Weierstrass, and a theme of ongoing research, which rational integrals 'reduce' to elliptic, namely a genus-g Jacobian contains an elliptic curve.

In [EEP] we give a non-hyperelliptic example, a "Halphen" curve in terms of differential algebra, meaning that its affine ring is the centralizer (as in the KdV case) of an operator of order 3, reprising the classical study of ODEs with elliptic coefficients: $L = \partial^3 - (n^2-1)\wp\partial - (n^2-1)/2\wp'$.

Requiring also that the curves X and E be tangent in some point of the Jacobian gives rise to the "elliptic solitons" of the KP hierarchy. If so (as in the Halphen case above), a Calogero-Moser-Krichever system will carry as Lagrangian fibration a family of "splittable" Jacobians, over a g-dimensional family of spectral curves; can this be viewed as an elliptic fibration? The natural way to otain the fibration would be to project each Jacobian to the complement of the elliptic curve, however, Poincaré's reducibility only splits the Jacobian up to isogeny, moreover, the projection would have to be done consistently through the family. If the fibration is there, it would seem to have great geometric interest, one feature being the singular fibres which would be expected to compactify generalized Jacobians of singular curves.

A similar construction was suggested in the lecture for the "coelliptic solitons" [DP], where the elliptic curve appears as quotient of a Lagrangian family of Jacobians, in a Hitchin system.

II.3 The Fermat quartic. This K3 surface was viewed as an elliptic fibration in [PSS]. Moreover, by embedding a modular curve by the four Jacobi theta functions, $\Phi(z,\tau) : \mathbb{C}\times\mathbb{H} \mapsto (\vartheta_3(2z,\tau), \vartheta_4(2z,\tau), \vartheta_2(2z,\tau), \vartheta_1(2z,\tau)) \in \mathbb{P}^3$, which satisfy Jacobi's identity, it can be checked [H] that: The image of Φ is the quartic $X_0^4 + X_3^4 - X_1^4 - X_2^4 = 0$ in $\mathbb{P}^3$, a finite quotient of the moduli space of complex tori $\mathbb{H}/\Gamma_4$; it is an elliptic fibration over the conic $X_0^2 = X_0^2 + X_1^2$. The elliptic curves are the inverse images of the map Φ for fixed τ, except over the 6 cusps of $\mathbb{H}/\Gamma_4$; each of the 6 singular fibers has four components ($\cong \mathbb{P}^1$) [PSS]. It becomes of interest then, to ask whether the geodesic flow induced by the Fubini-Study metric is integrable, and along the way to compute various curvatures, which happen to yield new theta indentities [H].

Conclusions. We advocate using σ in the study of elliptic fibrations ([BB] make, out of σ, a two-(elliptic-)variable function). I would ask for a modular study of the coefficients of the expansion of σ, which are related to Schur polynomials [BLE, N], as well as an alternative projective model of the Fermat quartic, devised using σ in the way that re-expresses the Jacobi identity [B], so that perhaps the curvatures and Hamiltonian flows become more tractable.

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Counting holomorphic curves on elliptically fibered Calabi–Yau threefolds

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(joint work with Murad Alim)

Let X be a smooth, projective variety, Σ_g a Riemann surface of genus g, and $f : \Sigma_g \to X$ a stable holomorphic map. For $\beta \in H_2(X,\mathbb{Z})$, we denote by $\mathcal{M}_g(X,\beta) = \{f : \Sigma_g \to X | f_*([\Sigma_g]) = \beta\}$ the moduli stack of stable holomorphic maps. If $c_1(X) = 0$ and $\dim_{\mathbb{C}} X = 3$ then there exists a virtual fundamental class $[\overline{\mathcal{M}}_g(X,\beta)]^{\text{vir}} \in H_0(\overline{\mathcal{M}}_g(X,\beta),\mathbb{Q})$. Hence, in the following we will focus on the case of X being a Calabi–Yau threefold. The Gromov–Witten invariant of stable holomorphic maps with class $\beta \in H_2(X,\mathbb{Z})$ is defined by $N_\beta^{(g)}(X) = \int_{[\overline{\mathcal{M}}_g(X,\beta)]^{\text{vir}}} 1 \in \mathbb{Q}$.

We denote by t the local coordinates on the complexified Kähler moduli space $\mathcal{M}_{\text{Kähler}}(X)$ whose tangent space is $H^2(X,\mathbb{C})$. For a complexified Kähler class $\omega(t) \in H^2(X,\mathbb{C})$ we set $q^\beta = \mathrm{e}^{2\pi i\langle\beta,\omega(t)\rangle}$. Since the Gromov–Witten invariants can be thought of as counting the number of stable holomorphic maps, it is natural to introduce the generating functions $F^{(g)}(X,t) = \sum_{\beta\in H^2(X,\mathbb{Z})} N_\beta^{(g)}(X) q^\beta$ and $F(X,t,\lambda) = \sum_{g\geq 0} F^{(g)}(X,t)\lambda^{2g-2}$. The latter is called free energy or Gromov–Witten potential. Finally, we define the (topological string) partition function $Z(X,t,\lambda) = \exp(F(X,t,\lambda))$. The physical meaning of the partition function is that it is the τ–function of some integrable system underlying topological string theory. There are various conjectures from physics involving this function. It is expected to be some theta function since it can be shown that it satisfies a heat equation [1]. Furthermore, it is expected to be an automorphic function for a particular congruence subgroup $\Gamma \subset \mathrm{Sp}(2n,\mathbb{Z})$, $n = h^{1,1}(X)+1$, acting on $\mathcal{M}_{\text{Kähler}}(X)$. Hence, the goal is to compute $Z(X,t,\lambda)$ as a "function" of t and λ.

There are various approaches to compute this partition function. One approach consists of localization on $\overline{\mathcal{M}}_g(X,\beta)$. If X is compact, this has been developed only for $g=0$ and $g=1$ [2], [3] Another approach pursued in physics is based on mirror symmetry and the holomorphic anomaly equation [4]. This approach requires further input which strongly depends on X. In particular, if X is elliptically fibered, then this further input can be described more easily.

As a first step we notice that the Gromow–Witten invariants are impractical since they are not integral. They can, however, be resummed in the following way [5]:

$$F(X,t,\lambda) = \frac{c(t)}{\lambda^2} + l(t) + \sum_{\beta}\sum_{m>0}\sum_{r\geq 0}\frac{1}{m}n_\beta^{(g)}(X)\left(2\sin\left(\tfrac{m\lambda}{2}\right)\right)^{2g-2}q^{m\beta}$$

where $c(t)$ and $l(t)$ are a cubic and linear polynomials depending on X, respectively. This expansion implicitly defines the Gopakumar–Vafa invariants $n_\beta^{(g)}(X)$. They are conjectured to be integers since they are supposed to be dimensions of cohomology groups associated to certain moduli spaces of curves embedded in X. However, since these cohomology groups are difficult to define, these invariants are currently only defined through this resummation.

Now, let X be an elliptically fibered Calabi–Yau threefold $\pi : X \to B$ where the fiber $\pi^{-1}(p) \cong E$ is an elliptic curve, $p \in B$. We consider the variation of Hodge structure for the family of mirror Calabi–Yau threefolds $f : \mathcal{X}^* \to \mathcal{M}$ where $\mathcal{M} = \mathcal{M}_{\text{cplx}}(\mathcal{X}^*) = \mathcal{M}_{\text{Kähler}}(X)$ is the complex structure moduli space [6]. The Gauss–Manin connection for $\mathrm{R}^3 f_*\mathbb{Z}_{\mathcal{M}}$ has monodromy group $\Gamma \in \mathrm{Aut}(H^3(X_z^*,\mathbb{Z}))$, $z \in \mathcal{M}$. Since X is an elliptic fibration, there is a distinguished subgroup of Γ isomorphic to $\mathrm{SL}_2(\mathbb{Z})$ and the variation of Hodge structure contains a variation of sub–Hodge structures coming from the elliptic fiber. The former is governed by the period integrals $\pi(z) = \int_\gamma \Omega(z)$ and $F^{(0)}(X^*,z) = Q(\pi(z),\pi(z))$, where Q is the symplectic form on $H^3(X_z^*,\mathbb{Z})$ and $\Omega(z) \in H^{3,0}(X_z^*)$. The study of the sub–Hodge structure yields the extra input alluded to above.

Finally, let us briefly discuss the holomorphic anomaly equation in the polynomial formulation due to [7]. Let G be the Weil–Petersson metric on $\mathcal{M}$ with Kähler form α and Kähler potential K. Locally, we have $G_{i\bar{\jmath}} = \partial_i\partial_{\bar{\jmath}}K$ and we use $G_{i\bar{\jmath}}\mathrm{e}^{-K}$ to raise and lower indices. If $\mathcal{L} \to \mathcal{M}$ is a line bundle with $\mathrm{c}_1(\mathcal{L}) = \alpha$, then $F^{(g)} \in \Gamma(\mathcal{M},\mathcal{L}^{2g-2})$, $g>1$. We define the following quantities: $C_{ijk} = \int_{X^*}\Omega\wedge\partial_i\partial_j\partial_k\Omega$, $K_i = \partial_i K$, $\partial_{\bar{\imath}}S^{ij} = \overline{C}_{\bar{\imath}}^{ij}$, $\partial_{\bar{\imath}}S^i = G_{\bar{\imath}j}S^{ij}$, and $\partial_{\bar{\imath}}S = G_{\bar{\imath}j}S^j$. We associate degrees 1, 1, 2, and 3 to K_i, S^{ij}, S^i and S, respectively. Then $F^{(g)}$ is a polynomial of degree $3g-3$ in K_i, S^{ij}, S^i, S and satisfies the following recursion relations [8]

$$\frac{\partial F^{(g)}}{\partial S^{ij}} = \frac{1}{2} D_i D_j F^{(g-1)} + \sum_{h=0}^{g} D_i F^{(h)} D_j F^{(g-h)}$$

$$\frac{\partial F^{(g)}}{\partial K_i} + S^i \frac{\partial F^{(g)}}{\partial S} + S^{ij} \frac{\partial F^{(g)}}{\partial S^i} = 0$$

where D_i is a covariant derivative with respect to $\mathrm{Sym}^n(T^*\mathcal{M}) \otimes \mathcal{L}^r$ for appropriate values of n and r. This allows to determine the $F^{(g)}$ recursively up to a holomorphic function. This holomorphic function is expected to be determined through the extra input coming from the structure of the elliptic fibration. This is work in progress.

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Enriques surfaces and jacobian elliptic K3 surfaces

Matthias Schütt

(joint work with Klaus Hulek)

Overview. Enriques surfaces are a classical object of algebraic geometry. In this talk we will mostly concentrate on varieties over $\mathbb{C}$. Then an Enriques surface Y is defined as quotient of a K3 surface X by a fixed point free involution τ:

$$Y = X/\tau.$$

Conversely the K3 surface X can be recovered as the universal covering of Y:

$$\pi : X \to Y.$$

This covering opens the way for lattice theory as we can pull back the Néron-Severi group on Y to obtain a primitive embedding

$$U(2) + E_8(-2) \hookrightarrow \mathrm{NS}(X). \tag{1}$$

Conversely, such an embedding implies that X has an Enriques involution unless there are (-2)-vectors in the orthogonal complement in $\mathrm{NS}(X)$. Thus we obtain a precise abstract description of Enriques surfaces through K3 surfaces with the lattice polarisation (1), giving a ten-dimensional moduli space with a hypersurface removed. However, this method does not give us a good control of explicit geometric constructions as soon as the Picard number of X is increased. On the other hand, there are many classical geometric constructions, the first dating back to Enriques, where we cannot control the moduli very well. Here we propose a novel construction that aims to balance the geometric and moduli theoretic aspects, see [3].

Set-up. We start with a rational elliptic surface $S \to \mathbb{P}^1$ with section. Consider a quadratic map $f : \mathbb{P}^1 \to \mathbb{P}^1$. For shortness we only study the generic situation in order to avoid special cases. Then the pull-back of S by f leads to a K3 surface X. On X we have the following involutions: the deck transformation $\imath$ and the hyperelliptic involution $-id$ defined fiberwise. Their composition $\jmath = -id \circ \imath$ is a Nikulin involution, and the quotient $X/\jmath$ has a K3 surface X' as minimal desingularisation. X' is actually the quadratic twist of S with respect to the base change f, having I_0^* fibers at the two ramification points. We thus have a diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & \downarrow & \searrow & \\ S & & \mathbb{P}^1 & & X' \\ \downarrow & \overset{f}{\swarrow} & & \overset{f}{\searrow} & \downarrow \\ \mathbb{P}^1 & & = & & \mathbb{P}^1 \end{array}$$

Generically X does not admit an Enriques involution as $\mathrm{NS}(X) = U + E_8(-2)$. To overcome this, we impose a section P' on X'. Pulling back to X, we obtain a $\imath^*$-anti-invariant section P on X. In consequence, translation by P composed with $\imath$ defines an involution τ on X. This involution is fixed point free if P does not specialise to the zero section O on the two fixed fibers of $\imath$. For instance, this holds if P is two-torsion. Lattice theoretically we can encode this information on the quotient X': the section P' has to meet both I_0^* fibers at other components than O'. That is, the decomposition $\mathrm{NS}(X') = U + L$ induced by the elliptic fibration does not admit a direct summand D_4 inside the negative-definite lattice L. In summary, we can encode the Enriques involution τ on X in a lattice polarisation on X' while retaining a good control about the geometry. Note that our geometric construction is neither limited to K3 surfaces nor to base fields of characteristic zero.

Applications. We illustrate the above construction by discussing some applications.

Nodal Enriques surfaces. An Enriques surface Y is called nodal if it contains a (-2)-curve. This curve features as a bisection for an elliptic fibration on Y. On the K3 cover X, this bisection splits into two disjoint sections O, P as above (cf. [5]).

Through the above construction, we can work out the induced elliptic fibration on X explicitly [3, Lemma 3.2]. Immediately, we deduce the unirationality of the moduli space of nodal Enriques surfaces.

Shioda–Inose structures. The construction applies to the framework of Shioda–Inose structures. Here X' is a Kummer surface of product type, and isogenies of the elliptic curves give rise to sections P' as above. In particular, we obtain Enriques involutions on singular K3 surfaces and study the arithmetic of the quotient surfaces such as fields of definition and Galois actions on divisors [4].

Brauer groups. An Enriques surface has $\mathrm{Br}(Y) = \mathbb{Z}/2\mathbb{Z}$. Beauville showed that generically $\pi^* \mathrm{Br}(Y) = \mathbb{Z}/2\mathbb{Z}$ and gave lattice theoretic conditions on $\mathrm{NS}(X)$ for the pull-back of $\mathrm{Br}(Y)$ to vanish. The base change construction enables us to produce explicit examples over $\mathbb{Q}$ or number fields [3] or in interesting families [2].

Enriques Calabi-Yau threefolds. Starting from a K3 surface X with an Enriques involution τ, one can construct Calabi-Yau threefolds by pairing X with an elliptic curve E and dividing the product $X \times E$ by the fixed point free involution $\tau \times (-id)$. With our methods we can derive interesting Calabi-Yau threefolds by endowing X with specific structures such as big Picard number, aiming for instance at arithmetic properties (e.g. modularity) or Picard-Fuchs equations.

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Towards Kaehler metrics of constant scalar curvature on elliptic fibrations

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(joint work with Joel Fine)

Let M be a compact complex surface equipped with a holomorphic map $\pi : M \to B$, where B is a smooth complex curve. Suppose that every fibre of M is a smooth curve. It has been known since the mid-1960's [1, 3] that it is possible for π to be non-trivial (in the sense that the curve $\pi^{-1}(b)$ moves non-trivially in the moduli space of curves) if the genus of the fibre is ≥ 2.

In his PhD thesis, Joel Fine showed how to construct Kähler metrics of constant scalar curvature (cscK metrics) on such M by an adiabatic limit [2]. In brief, one starts with a smooth $(1,1)$ form ω_0, say, on M whose restriction to each fibre is the standard hyperbolic metric. While ω_0 need not be a Kähler form, if we take any Kähler form ω_B on B and consider

$$\omega_\epsilon = \omega_0 + \epsilon^{-2}\pi^*\omega_B, \epsilon > 0$$

on M, this will be a Kähler form if ϵ is sufficiently small. (This is an adiabatic family of metrics, in which lengths in the base are very large compared with lengths in the fibres.) In general the curvature of such a metric is dominated by that of the fibre, because the metric $\epsilon^{-2}\omega_B$ is 'nearly flat' if ϵ is small enough.

In particular, one computes that the scalar curvature of ω_ϵ is equal to $-1 + O(\epsilon^2)$ so that in a certain sense ω_ϵ is already approximately cscK. By suitable choice first of ω_B and then an iterative construction, Fine improved this metric by constructing, for each $N > 0$,

$$\omega_N = \omega_\epsilon + dd^c\phi_{N,\epsilon}$$

such that

$$\mathrm{scal}(\omega_N) = \sum_{j=0}^{N} \lambda_j \epsilon^{2j} + O(\epsilon^{2N+2})$$

where the λ_j are constants. It is not a good idea to try to take the limit as $N \to \infty$ here, but one can show that once N is sufficiently large, a perturbation of ω_N will be cscK (implicit function theorem).

Now let $\pi : M \to B$ be as above, but allow a finite number of singular fibres in π. In particular, π could be an elliptic fibration (assumed to have a holomorphic section), or it could arise as a Lefschetz pencil. A very attractive problem is to try to generalize Fine's technique to construct cscK metrics on such M. (As an aside, I remark that there is also a range of *linear* problems from geometric analysis that deserve attention in this context (e.g. adiabatic limits of η-invariants or analytic torsion when the 'fibration' has singularities.)

If M is an elliptic fibration with section and a finite number of the simplest nodal singularities, and relatively minimal, let B^* be the base of the 'regular part' of the fibration, so $\pi^{-1}(b)$ is smooth iff $b \in B^*$. Let $M^* = \pi^{-1}(B^*)$. We have shown at least that on M^* there exist cscK adiabatic metrics ω_ϵ with $\mathrm{scal}(\omega_\epsilon) = \lambda\epsilon^2$, where λ is a constant determined by the geometry of the fibration π. This is expected to be a basic ingredient in the construction of cscK adiabatic metrics on M, but more work is needed to complete this project.

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