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Mini-Workshop: Linear Series on Algebraic Varieties

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Abstract. Linear series have long played a central role in algebraic geometry. In recent years, starting with seminal papers by Demailly and Ein-Lazarsfeld, local properties of linear series – in particular local positivity, as measured by Seshadri constants – have come into focus. Interestingly, in their multi-point version they are closely related to the famous Nagata conjecture on plane curves. While a number of important basic results are available by now, there are still a large number of open questions and even completely open lines of research.

Mathematics Subject Classification (2000): 14C20, 14E25.

Introduction by the Organisers

The mini-workshop *Linear Series on Algebraic Varieties* gathered together experts in the subject with a spread of backgrounds and professional experience. Participants came from all over the world: US (Colorado, Illinois, Nebraska), Europe (France, Germany, Italy, Norway, Poland, Spain, Sweden) and Asia (Korea) and ranged from post docs to senior researchers. This variety of experience and background greatly contributed to generating stimulating discussions during the talks and the working group sessions, leading to what we believe will be the basis for several research collaborations.

The theme of the workshop

The theme of the workshop revolved around two related conjectures which in recent years stimulated important developments in the field of linear series, the

Nagata conjecture [19] and the SHGH conjecture [21, 11, 10, 14], sometimes also referred to as the Harbourne-Hirschowitz conjecture. The Nagata conjecture bounds the least degree d of a curve passing through $r \geq 10$ general points in the projective plane with prescribed multiplicities $m_1, \ldots, m_r$. While this conjecture has remained open for half a century, several people observed recently that it could be a piece of a much more general picture, which is by no means specific to $\mathbb{P}^2$ ([3], [2]). The concept which allows passing from a rather special question on the projective plane to a much more general setting is that of Seshadri constants. Recall that given a polarized variety (X, L) and a subscheme Z of X the Seshadri constant of L at Z is the real number $\varepsilon(X, L; Z) := \sup\{\lambda : f^*L - \lambda E \text{ is ample on } Y = Bl_Z X\}$ (see [2] for details). An elegant and uniform way to generalize the Nagata conjecture is to assert that if Z is a union of sufficiently many and sufficiently general reduced points of X, then the Seshadri constant $\varepsilon(X, L; Z)$ is the maximal possible. Recently there were several interesting developments towards approximating from below the numbers $\varepsilon(X, L; Z)$ ([12], [9], [20], [5], [13], [15]). During the mini-workshop various approaches to this problem were presented.

The SHGH conjecture gives a precise prediction for the actual dimension of the linear series $L(d; m_1, \ldots, m_r)$ of curves of degree d passing through r general points of the plane with multiplicities $m_1, \ldots, m_r$, giving a simple geometric condition for a linear series of this type to be special, i.e., of higher dimension than is expected by a naive dimension count. Several approaches have been developed to attack this problem. We first mention the so-called Horace method, a specialization method introduced by Alexander and Hirschowitz [1] and pursued by Mignon [18], Roé [20], and Evain [9]. A second approach is Ciliberto and Miranda's [4, 5, 6] method of degenerations of the underlying variety for linear series on families of planes. A third approach relates the problem to packing type questions in symplectic geometry as observed by McDuff-Polterovich [17] and pursued by Biran [3] and recently by Eckl [8]. Finally there are approaches via symbolic computations, most prominently those of Lorentz-Lorentz [16] and Dumnicki [7].

The structure of the workshop

The aim of the workshop was twofold: to gather together experts working on the three different aspects mentioned above, and to stimulate collaboration by discussing open problems in the field. For this reason every day consisted of two main activities:

- research talks, typically two or three in the morning; and
- working group discussions, in the afternoon.

A list of possible questions to work on during the workshop was discussed via email well ahead of the workshop. During the first day, the final selections were made. Two main areas of interest emerged from the discussion: asymptotic approaches and combinatorial ones. Consequently two working groups were formed. The workshop was just the starting point and an ignition to collaborate on the chosen problem. The working groups continue their efforts. The outcome of these discussions will appear elsewhere.

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Mini-Workshop: Linear Series on Algebraic Varieties

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Open problems on linear systems

Brian Harbourne

One way of stating the SHGH Conjecture [9, 6, 3, 7] is:

Conjecture 1. Let $C \subset X$ be a prime divisor where $X \to \mathbf{P}^2$ is the blow up of generic points $p_1, \dots, p_s$. Then $h^1(X, \mathcal{O}_X(C)) = 0$.

This raises the question of how to extend this conjecture more generally. Joint discussions between the speaker and J. Roé, C. Ciliberto and R. Miranda produced the following possibility:

Conjecture 2. Let X be a smooth projective surface over an algebraically closed field K (where either X is rational or $\mathrm{char}(K) = 0$). Then there exists a constant c_X such that for every prime divisor C [alternatively, for every reduced divisor C] we have $h^1(X, C) \leq c_X h^0(X, C)$.

[Discussions among the mini-workshop participants after the talk show that this is false, but there is still hope that it may hold when X is rational.]

Another question to be discussed in this talk was raised by M. Velasco and D. Eisenbud in an email to the speaker:

Question 3. Consider a divisor $D = dH - \sum_i m_i E_i$ on X where $\pi : X \to \mathbf{P}^N$ is the blow up of a finite set of points $p_1, \dots, p_s$, $E_i = \pi^{-1}(p_i)$ and H is the pullback of a hyperplane. Is there a procedure to determine if D is semi-effective (i.e., is $h^0(X, kD) > 0$ for some $k > 0$, or alternatively, is there a t and a form F of degree td which vanishes at each point p_i to order at least tm_i)?

This question can be partially addressed using an asymptotic invariant defined by Waldschmidt [10]:

Definition 4. Given points $p_i \in \mathbf{P}^N$, let $\alpha(\sum_i m_i p_i)$ be the least t such that there is a form of degree t vanishing at each point p_i to order at least m_i; i.e., the least t such that $h^0(X, tH - \sum_i m_i E_i) > 0$. Then Waldschmidt's constant is defined as:

$$e\Big(\sum_i m_i p_i\Big) = \lim_{k \to \infty} \frac{\alpha(k \sum_i m_i p_i)}{k}.$$

Proposition 5. *Given $D = dH - \sum_i m_i E_i$ on X, then:*

- *$d < e(\sum_i m_i p_i)$ implies $kd < \alpha(k \sum_i m_i p_i)$ for all k, hence $h^0(X, kD) = 0$ for all k; and*
- *$d > e(\sum_i m_i p_i)$ implies $kd > \alpha(k \sum_i m_i p_i)$ for $k \gg 0$, hence $h^0(X, kD) > 0$ for $k \gg 0$.*

By [5], using results of [8], it is possible in principle to compute $e(\sum_i m_i p_i)$ arbitrarily accurately, but it is not known how to compute it exactly. Thus when $d \neq e(\sum_i m_i p_i)$ it is in principle possible to verify the lack of equality computationally, which then allows the above proposition to be applied to answer the Velasco-Eisenbud question. However, when $d = e(\sum_i m_i p_i)$ it is not clear how to verify equality even in principle, nor is it clear how to determine semi-effectivity even if one knew $d = e(\sum_i m_i p_i)$. Moreover, because the computations needed to estimate $e(\sum_i m_i p_i)$ are large and hard to carry to completion, it is of interest to find easier to compute bounds. Lower bounds are of particular interest. This talk will discuss conjectural such bounds due to Chudnovsky [2] and explain applications and discuss additional conjectures related to the problem of determining which symbolic powers of an ideal are contained in a given ordinary power of the ideal [1, 4].

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Emptiness of linear systems and the effect of rescaling

Marcin Dumnicki

Let $p_1, \ldots, p_r \in \mathbb{P}^2 = \mathbb{P}^2(\mathbb{K})$ be distinct points, where $\mathbb{K}$ is a field of characteristic 0. Let $m_1, \ldots, m_r$ be nonnegative integers. By $\mathcal{L}(d; m_1p_1, \ldots, m_rp_r)$ we denote the linear system of plane curves of degree d with multiplicity at least m_j at p_j, $j = 1, \ldots, r$. The (projective) dimension of $\mathcal{L}(d; m_1p_1, \ldots, m_rp_r)$ is upper semicontinuous in the position of imposed base points and reaches minimum for points in general position. This minimum will be denoted by

$$\dim \mathcal{L}(d; m_1, \ldots, m_r).$$

We will also write $\mathcal{L}(d; m_1, \dots, m_r)$ for a system with imposed base points in general position, and $\mathcal{L}(d; m_1^{\times s_1}, \dots, m_r^{\times s_r})$ for repeated multiplicities. Define

$$n\mathcal{L}(d; m_1, \dots, m_r) = \mathcal{L}(nd; nm_1, \dots, nm_r).$$

Our aim is to prove the following:

Theorem 1. *For each $n \geq 0$, the system $n\mathcal{L}(13; 5, 4^{\times 9})$ is empty.*

One of the main ingredients is the *cutting diagram algorithm* from [1]. Briefly, it is proved that in order to show non-speciality of a given system it suffices to find an appropriate finite set of points in $\mathbb{N}^2$ enjoying some combinatorial properties. To be precise, we must first define, for any finite $D \subset \mathbb{N}^2$, the system

$$\mathcal{L}(D; m_1, \dots, m_r)$$

of polynomials with support (supp) in D and with multiplicity at least m_j at p_j, $j = 1, \dots, r$. Formally, we identify $\mathbb{N}^2$ with monomials in $\mathbb{K}[X, Y]$

$$\mathbb{N}^2 \ni (x, y) \mapsto X^x Y^y \in \mathbb{K}[X, Y]$$

and put

$$\mathcal{L}(D; m_1, \dots, m_r) = \{f \in \mathbb{K}[X, Y] : \operatorname{supp}(f) \in D, \operatorname{mult}_{p_j}(f) \geq m_j, j = 1, \dots, k\}.$$

The set $\mathcal{L}(D; m_1, \dots, m_r)$ is a $\mathbb{K}$-linear subspace of $\mathbb{K}[X, Y]$. The system $\mathcal{L}(D; m_1, \dots, m_r)$ is called *empty* if

$$\dim_{\mathbb{K}} \mathcal{L}(D; m_1, \dots, m_r) = 0.$$

Observe that, by dehomogenizing and generality assumption, $\mathcal{L}(d; m_1, \dots, m_r)$ is empty if and only if $\mathcal{L}(D; m_1, \dots, m_r)$ is empty for $D = \{(x, y) : x + y \leq d\}$.

The cutting diagram algorithm is based on the following two theorems:

Theorem 2 (Theorem 14 in [1])**.** *Let $D, D' \subset \mathbb{N}^2$ be finite, let $m_1, \dots, m_r$, $m'_1, \dots, m'_s$ be nonnegative integers. If*

- *$\mathcal{L}(D; m_1, \dots, m_r)$ is empty,*
- *$\mathcal{L}(D'; m'_1, \dots, m'_s)$ is empty,*
- *there exists an affine function $\mathbb{N}^2 : f \ni (a, b) \mapsto q_1 a + q_2 b + q_3 \in \mathbb{Q}$, $q_1, q_2, q_3 \in \mathbb{Q}$ such that f has strictly negative values on D and nonnegative values on D'*

then $\mathcal{L}(D \cup D'; m_1, \dots, m_r, m'_1, \dots, m'_s)$ is empty.

Theorem 3 (Proposition 13 in [1])**.** *Let $D \subset \mathbb{N}^2$, let m be a nonnegative integer. If $\#D = \binom{m+1}{2}$ and D, considered as a set of points in $\mathbb{Q}^2$, does not lie on a curve of degree $m - 1$, then $\mathcal{L}(D; m)$ is empty (for one point p in a general position, which means here that p does not lie on x-axis neither y-axis).*

The proofs are technical but use only simple linear algebra.

To prove emptiness of $\mathcal{L}(13n; 5n, (4n)^{\times 9})$ for a fixed n, we consider the diagram

$$D = \{(x, y) : x + y \leq 13n\}.$$

and cut D with nine lines into ten subsets.

Since vdim $\mathcal{L}(13n; 5n, (4n)^{\times 9}) = -n$, it is enough to show that $\mathcal{L}(D_1; 5n)$ and $\mathcal{L}(D_k; 4n)$ are empty for $k = 2, \ldots, 10$. For each D_k we proceed using Theorem 3.

For the all of subsets $D_1, \ldots, D_{10}$ we can use Bezout theorem to show non-existance of a curve of degree $4n - 1$ ($5n - 1$ for D_1) passing through all points of D_i. I fact, there are always $4n$ lines $L_1, \ldots, L_{4n}$ such that $\#(D_i \cap L_j) \geq j$.

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The effect of points fattening on postulation

Cristiano Bocci
(joint work with Luca Chiantini)

Given a set of points $Z \subset \mathbb{P}^2$, an interesting problem concerns the study of the geometric properties of the fattening $2Z$ and their relations with the properties of Z itself. When Z is general, the postulation of $2Z$ is well understood, but as soon as the points come to some special position, our knowledge on the geometry of $2Z$ becomes less effective ([2],[3]). A specific point of this analysis, consist of considering the first interesting step of the Hilbert function, namely the smallest degree of a curve containing the set of points or its fattening. In more detail, let Z be a configuration of points in $\mathbb{P}^2$ and let $I = I(Z)$ be the homogeneous ideal. Put $I^{(2)} = I(2Z)$. We say that Z has type $(d - t, d)$ if the generators of I and $I^{(2)}$ have minimal degrees $d - t$ and d respectively.

In this talk I present the classification Theorems, given in [1], for set of points $Z \subset \mathbb{P}^2$ of type $(d - t, d)$ with $t = 1$ and $t = 2$.

The classification for $t = 1$ is relatively simple since, in this case, Z is a set of collinear points (and $d = 2$) or Z is a star configuration, i.e. a set of $\binom{d}{2}$ distinct points given by the pair-wise intersection of d lines. More precisely one has

Theorem 1. *Let Z be a set of points in $\mathbb{P}^2$. Then Z has type $(d - 1, d)$ if and only if*

(i) *Z is the set of $\binom{d}{2}$ distinct points given by the pair-wise intersection of d lines or*

(ii) *Z is a set of collinear points and $d = 2$.*

In the classification for $t = 2$ we can find three different cases. In the event that the minimal curve passing through $2Z$ is non-reduced, Z could be a set of points on a conic, or $Z = Z' \cup Y$ where Z' has type $(d - 3, d - 2)$ and Y is a set of collinear points.

The case in which the minimal curve passing through $2Z$ is reduced, presents an interesting large family of sets that are the main topic of the talk.

To describe them, we need to define some specific object. The first object to define is the *tame* curve $C = C_1 + C_2 + \cdots + C_n$, that is a plane curve which satisfies the following conditions:

a) all the components C_i of C are rational;
b) the singular points of each C_i are either nodes or ordinary cusps;
c) no singular point of C_i belongs to some other component C_j;
d) no components C_i, C_j are tangent at two points;
e) for $i \neq j$ the local intersection multiplicity of C_i and C_j at any point is at most two;
f) no components C_i, C_j of C are tangent anywhere if some point of $C_i \cap C_j$ belongs to a third component;
g) for $i < j < k$, the intersection $C_i \cap C_j \cap C_k$ consists of at most one point;
h) if some point of $C_i \cap C_j$ belongs to a third component, then no other components contain points of $C_i \cap C_j$; in particular no point belongs to four components of C.

The singular locus Z_0 of a tame curve, of degree d, is a good candidate to have type $(d-2, d)$. To check if a subset $Z \subset Z_0$ could have still type $(d-2, d)$ we need to define the graph related to the tame curve and its adaption to Z.

Thus, let $C = C_1 + \cdots + C_n$ be a reduced, reducible plane curve of degree d. C defines a (labelled) graph G_C as follows: any component C_i of C corresponds to a vertex $v(i)$ of G_C. For any point P where C_i and C_j meet with multiplicity m, we draw m edges $e_1(i, j, P), \ldots, e_m(i, j, P)$ joining $v(i)$ and $v(j)$.

If Z is a subset of the singular locus of C, then we obtain the subgraph of G_C *adapted to* Z as follow: for each point $P \in Z$ where two or more components meet, we erase *exactly one* edge labelled by P in G_C.

In the last part of the talk, we show that the third case of set of points of type $(d-2, d)$ can be described in terms of this adapted graph. As a matter of fact we say that a subset Z of the singular locus of a tame curve of degree d is *admissible* if Z contains all singular points of any component C_i and any subgraph of G_C adapted to Z is a forest. Thus, the classification Theorem can be finally stated.

Theorem 2. *Let Z be a set of at least four points in $\mathbb{P}^2$, not in the list of Theorem 1. Then Z has type $(d-2, d)$ if and only if either:*

i) *Z is contained in a conic and $d = 4$; or*
ii) *there exists a line L such that the set $Z' := Z - L$ is non empty, of type $(d-3, d-2)$ and there are no curves of degree $d-2$ singular at Z' and passing through $Y := Z \cap L$, neither there are curves of degree $d-3$ passing through Z' and Y; or*
iii) *Z is contained in the singular locus of a tame curve C of degree d, and Z is admissible for C.*

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Linear Systems on Edge-Weighted Graphs

Rick Miranda
(joint work with Rodney James)

Let G be a connected graph with vertex set V and edge set E. We assume that there are no loops or multiple edges. We also assume that there are non-negative weights w_{ij} on the edge joining vertices i and j; these weights are assumed to be real numbers. The *degree* of a vertex is the sum of the weights of the edges on it.

Define the *genus* g of G to be

$$g = 1 - |V| + \sum_{i<j} w_{ij}.$$

Define a *divisor* on G to be an assignment of a real number to every vertex. The space of divisors is an $\mathbb{R}$-vector space. The *degree* of a divisor is the sum of the coordinates. For a real number x, we write $D > x$ if every coordinate of D is bigger than x.

For each vertex i, consider the divisor H_i defined by $H_i(i) = -\deg(i)$, and $H_i(j) = w_{ij}$ for $j \neq i$. Notice that each H_i has degree zero.

We say that a divisor P is *principal* if it is a $\mathbb{Z}$-linear combination of the H_i's.

We define two divisors to be *linearly equivalent* if their difference is a principal divisor; we write $D_1 \equiv D_2$ in that case.

The *canonical divisor* K is defined by $K(i) = \deg(i) - 2$ The canonical divisor has degree $2g - 2$.

We define the *linear system* of D, denoted by $|D|$, to be

$$|D| = \{E \mid E > -1, E \equiv D\}.$$

Define

$$\ell(D) = \min\{e \mid \text{ there exists } E \geq 0, \deg(E) = e, |D - E| = \emptyset\}$$

Theorem: $\ell(D) - \ell(K - D) = \deg(D) + 1 - g$.

This Riemann-Roch theorem generalizes a similar statement for non-edge-weighted graphs (allowing multiple edges) proved by Baker and Norine in [1].

The proof relies on three statements. Let $A(G)$ be the set of divisors with an empty linear system.

(1). There is a discrete set B of divisors in $A(G)$ such that

$$D \in A(G) \text{ if and only if there exists } N \in B \text{ such that } D \leq N.$$

(2) The set B is symmetric with respect to K:

$$N \in B \text{ if and only if } K - N \in B.$$

(3) Every N in B has degree $g - 1$.

For a divisor D, define D^+ to be the least non-negative divisor satisfying $D^+ \geq D$. (This is obtained from D by setting all negative coordinates equal to zero.)

Statement (1) then easily implies that

$$\ell(D) = \min_{N \in B} \{\deg(D - N)^+\}.$$

Statement (2) permits a change of coordinates in the above, and implies that

$$\ell(K - D) = \min_{M \in B} \{\deg(M - D) + \deg(D - M)^+\}.$$

Statement (3) allows us to take the first term out of the above min, and gives the result.

The proofs of the three statements rely on a normal form for divisors, up to linear equivalence. This normal form is obtained by fixing a vertex (call it 0). A divisor D is in normal form if

$$(*) \qquad D(i) > -1 \text{ for every } i > 0,$$

and in addition, for every positive linear combination $P = \sum_{i>0} c_i H_i$ with non-negative integer coefficients c_i, the divisor $D + P$ violates (*).

One proves that every linear equivalence class contains a unique divisor in normal form, and a relatively straightforward analysis of the inequalities that are implied by that, gives the set B as the maximal divisors in normal form, that have an empty linear system.

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Hilbert Functions and Initial Degrees of Fat Points

Susan Marie Cooper

The Hilbert function of the homogeneous ideal of a scheme has played a central role in attacking many intriguing problems. Although Hilbert functions of ideals defining reduced 0-dimensional schemes are well understood, much less is known about symbolic powers of such ideals which define non-reduced schemes called *fat point schemes* (for some partial results see [1, 3, 4, 5, 6]). In [1] we give combinatorially defined upper and lower bounds for the Hilbert function of a fat point scheme $\mathbb{A}$ in projective n-space using nothing more than the multiplicities of the points and information about which subsets of the points are collinear. When $n = 2$ we obtain the bounds by repeatedly trimming down the fat point scheme via taking residuals with respect to lines. In this case we give the bounds explicitly and show that they are easy to calculate. In addition, when $n = 2$, we give an easy to check sufficient condition for the upper and lower bounds to be equal.

Related to the Hilbert function of a fat point scheme $\mathbb{A}$ is the initial degree. The *initial degree* of $\mathbb{A}$, denoted $\alpha(\mathbb{A})$, is defined to be the least degree t for which there is a non-zero homogeneous polynomial in the ideal of $\mathbb{A}$. In projective 2-space, the upper and lower bounds on the Hilbert function of $\mathbb{A}$ from [1] lead to natural upper and lower bounds on the initial degree of $\mathbb{A}$. Recently, unpublished work of Harbourne and Huneke suggests a conjectural relationship between the initial degrees of fat point schemes and the corresponding support set. More precisely:

Conjecture 1. Let $\mathbb{X}$ be a finite set of points in projective n-space, r be any positive integer and $\mathbb{Y}$ be the fat point scheme supported on $\mathbb{X}$ with each point having multiplicity $rn - n + 1$. Then $\alpha(\mathbb{Y}) \geq r\alpha(\mathbb{X}) + (r-1)(n-1)$.

Work of Chudnovsky shows that for a finite set $\mathbb{B}$ in $\mathbb{P}^2$ there is a subset $\mathbb{A} \subseteq \mathbb{B}$ such that $\alpha(\mathbb{A}) = \alpha(\mathbb{B}) = \operatorname{reg}(I(\mathbb{A}))$. As a consequence the Hilbert function of $\mathbb{A}$ must be of the form achieved by special point sets called *line count configurations of type* $(t, t-1, \ldots, 3, 2, 1)$. In general, we say that $\mathbb{X} = \mathbb{X}_1 + \cdots + \mathbb{X}_t \subset \mathbb{P}^2$ is a *line count configuration of type* $c = (c_1, \ldots, c_t)$ if each $\mathbb{X}_i$ consists of c_i points on a line $\mathbb{L}_i$ where the lines $\mathbb{L}_1, \ldots, \mathbb{L}_t$ are distinct and no point of $\mathbb{X}$ occurs where two of the lines $\mathbb{L}_i$ meet. After re-indexing, we assume that $c_1 \geq c_2 \geq \cdots \geq c_t$.

In joint work of S. Cooper and S. G. Hartke [2], we have

Theorem 2. *Conjecture 1 is true for* $r \geq 75$ *when* $\mathbb{X}$ *is a line count configuration of type* $(t, t-1, \ldots, 3, 2, 1)$.

Restricting to projective 2-space, the first part of this talk will focus on the bounds arising from the work of [1]. We then apply the bounds to investigate the initial degree relationship from Conjecture 1.

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Combinatorial bounds for Hilbert functions and graded Betti numbers of fat point schemes in the plane

Zach Teitler

(joint work with Susan Cooper, Brian Harbourne)

Let $A = m_1p_1 + \cdots + m_rp_r$ be a fat point scheme in $\mathbb{P}^2$, so that the ideal $I_A \subset R = k[\mathbb{P}^2]$ is given by $I_A = I(p_1)^{m_1} \cap \cdots \cap I(p_r)^{m_r}$. We give upper and lower bounds for the Hilbert function h_A, along with a condition for them to coincide, generalizing a result of Geramita–Migliore–Sabourin [1] for double point schemes (each $m_i = 2$) with support points in certain configurations. In the case they do coincide, thus computing h_A exactly, we give upper and lower bounds for the graded Betti numbers of the ideal I_A, along with a condition for these bounds to coincide.

In many naturally arising geometric situations, one does not know the support points p_i explicitly. Instead, one has some information about curves on which some of the p_i lie. We focus on the information of which sets of p_i are collinear. That is, we are given some *collinear subsets* $S_1, \ldots, S_k \subseteq \{p_1, \ldots, p_r\}$ such that each S_i is the intersection of A with some line L_i. There may be other collinear subsets; our results use only the given $S_1, \ldots, S_k$.

Given this data, we describe a *reduction procedure* which, at each step, reduces the fat point scheme A by reducing the multiplicities at all the support points in a collinear subset S_i without changing the support points, until all multiplicities have been reduced to zero. This corresponds to residuating A with respect to the line L_i. At each step we record the total of the multiplicities which are reduced in that step; the resulting sequence, which we call a *reduction vector for A*, is the output of the reduction procedure. At each step of the reduction procedure there may be a choice of collinear subsets along which to reduce, and in general different choices may yield a different reduction vector and different bounds for the Hilbert function h_A.

For each vector $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n_{\geq 0}$ we define functions $f_{\mathbf{d}}, F_{\mathbf{d}} : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$. We have the following:

Theorem 1. *Let A be a fat point scheme and let $\mathbf{d}$ be a reduction vector for A.*

(1) *We have $f_{\mathbf{d}}(t) \leq h_A(t) \leq F_{\mathbf{d}}(t)$ for all $t \geq 0$.*
(2) *If $\mathbf{d}$ has non-zero, non-increasing entries then $f_{\mathbf{d}} = F_{\mathbf{d}}$ if and only $\mathbf{d}$ satisfies the following equivalent conditions:*
  - *For $1 \leq i < j \leq n$, $d_i - d_j \geq j - i - 1$.*
  - *Between any two 0 entries in the vector $\Delta\mathbf{d}$ of successive differences, there is an entry which is less than or equal to -2.*
  - *There is no subsequence in $\mathbf{d}$ of the form (a, a, a) or of the form $(a_i, \ldots, a_{i+j+1})$ for $j > 1$ where $a_i = a_{i+1}$, $a_{i+j} = a_{i+j+1}$, and $a_{i+1}, \ldots, a_{i+j}$ are consecutive integers.*

Finite sequences $\mathbf{d}$ of non-zero, non-increasing integers meeting the condition in the second part of the theorem are called *GMS* or *generalized monotone sequences*. Note that the condition $d_i - d_j \geq 0$ is equivalent to the sequence being non-increasing while the condition $d_i - d_j \geq j - i$ is equivalent to the sequence being strictly decreasing, so every strictly decreasing sequence is GMS. The GMS condition was used in [1] to characterize when the Hilbert functions of certain double point schemes are uniquely determined by combinatorial data.

The bounds $f_{\mathbf{d}} \leq h_A \leq F_{\mathbf{d}}$ arise from a quite general principle by considering, for each step in the reduction procedure, the natural short exact sequence of sheaves on $\mathbb{P}^2$ coming from residuation with respect to the line L_i. Rank considerations in the long exact sequence of cohomology bound the Hilbert function at each step in terms of the next step.

A similar strategy gives bounds for Hilbert functions in $\mathbb{P}^N$ in terms of both a reduction vector, and the restrictions of the fat point scheme to various hyperplanes $H \cong \mathbb{P}^{N-1}$. Ultimately bounds are obtained by induction on the dimension. While one can readily work out specific examples, the combinatorial analysis required to give a general statement is quite difficult.

Similarly, using curves in $\mathbb{P}^2$ of higher degree in place of lines, bounds can often be worked out in specific examples, usually giving tighter bounds than would result from using lines alone. However because the restriction of a set of points to a curve may be a special divisor, in order to give good bounds in a general statement one must either keep track of speciality at each step (along with the reduction vector and the sequence of the degree and genus of the curve used at each step), or one must use only rational curves, on which speciality never arises. In either case it is again quite difficult to give a general statement.

Returning to the case of $\mathbb{P}^2$ and reduction along lines, we consider the graded Betti numbers, which are a more subtle invariant than the Hilbert function. For each t, the number of generators of I_A of degree t (in a minimal generating set) is denoted ν_t. (From h_A and ν_t one can recover the number of syzygies in each degree.) We obtain the following.

Theorem 2. *Let A be a fat point scheme in $\mathbb{P}^2$ with GMS reduction vector $\mathbf{d}$.*

(1) *There are explicit lower and upper bounds for the graded Betti numbers ν_t of the ideal I_A, determined by $\mathbf{d}$.*

(2) *These bounds coincide if and only if $\mathbf{d}$ has one of the following forms:*
- *The entries of $\mathbf{d}$ are strictly decreasing.*
- $\mathbf{d} = (m, m, m-1, \dots, 2, 1)$ *for some $m \geq 1$.*
- $\mathbf{d} = (d_1, \dots, d_k, m, m, m-1, \dots, 2, 1)$*, where $d_1 > \cdots > d_k \geq m+2$.*

As examples one can obtain a GMS, or even strictly decreasing, reduction vector in many combinatorially interesting examples, including certain fat point schemes supported at general points on each of a set of lines, and certain fat point schemes supported at the pairwise intersections of a set of lines.

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Asymptotic Syzygies of Higher Dimensional Varieties

Lawrence Ein

(joint work with Robert Lazarsfeld)

In this note, we give a very preliminary report on some of my recent joint work with Rob Lazarsfeld on the asymptotic syzygies of higher dimensional varieties. In the 80's Green and Lazarsfeld began a systematic study of the syzygies of smooth projective curves. One of the main driving problems in this area is the important conjecture of Green which predicts that the behavior of the syzygies of a canonical curve is determined by the Clifford index of the curve. See [5] and [6] for more details. In an important recent breakthrough, Voisin ([9] and [10]) proved that Green's conjecture is true for a generic curve of genus g. Combined with the result of Texidor Bigas [8], they show that Green's conjecture also holds for a general $p-$gonal curve.

It is natural to ask whether we can generalize these results and questions to higher dimensional varieties. In this report, we'll discuss my joint work with R. Lazarsfeld where we begin studying asymptotic behavior of the syzygies of higher dimensional varieties. It is well known that the canonical ring of a minimal projective variety of general type may require very high degree generators. This leads us to study the syzygies of X when it is embedded by a sufficiently very ample line bundle instead.

First we'll recall some basic notations. Let $S = \mathbb{C}[x_0, x_1, \cdots, x_r]$ be the polynomial ring of $r+1$ variables. Suppose that G is a finitely generated graded S-module. We consider a minimal free resolution of G.

$$0 \longrightarrow E_s \longrightarrow \ldots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow G \longrightarrow 0$$

Let $\mathbb{C} = S/\mathfrak{m}$, where $\mathfrak{m}$ is the homogenous maximal ideal of S. By Nakayama's lemma, one observes that

$$\mathrm{Tor}_p(G, \mathbb{C}) = E_p \otimes \mathbb{C}.$$

Observe that $\mathrm{Tor}_p(G, \mathbb{C})$ is a graded vector space. Using the notations of Green, we set the Koszul group $K_{p,q}(G)$ to be the homogenous degree $p+q$ piece of $\mathrm{Tor}_p(G, \mathbb{C})$. Let X be a closed subvariety of dimension n in $\mathbb{P}^r$. In the following we assume that the restriction map gives an isomorphism between $H^0(\mathcal{O}_{\mathbb{P}^r}(1))$ and $H^0(\mathcal{O}_X(1))$. Let $\mathcal{F}$ be a coherent sheaf on X and F be its associated graded $S-$module. We will denote by

$$K_{p,q}(\mathcal{F}) \quad \text{for} \quad K_{p,q}(F).$$

We say that the pair $(X, \mathcal{O}_X(1))$ satisfies the property N_0, if $|\mathcal{O}_X(1)|$ gives a projectively normal embedding of X in $\mathbb{P}^r$. Note that this is equivalent to $K_{0,q}(\mathcal{O}_X) = 0$ for $q > 0$. For $p > 0$, inductively we say that $(X, \mathcal{O}_X(1))$ satisfies the property N_p, if it satisfies N_{p-1} and $K_{p,q}(\mathcal{O}_X) = 0$ for $q \geq 2$.

Let X be a smooth projective variety of dimension n and $\mathcal{O}_X(1)$ be a very ample line bundle on X and B be another line bundle on X. Choose $d >> 0$ with respect to B. We consider the coordinate ring

$$R = \oplus_{m=0}^{\infty} H^0(O_X(md)).$$

We consider the S-module

$$N_{B,d} = \oplus_{m=0}^{\infty} H^0(\mathcal{O}_X(md) \otimes B).$$

We would like to investigate the asymptotic behavior of the syzygy groups

$$K_{p,q}((X, \mathcal{O}_X(d)); B) = \mathrm{Tor}_p(\mathbb{C}, N_{B.d})_{p+q}$$

as $d \to \infty$. Set $L = \mathcal{O}_X(d)$ and $r+1 = h^0(\mathcal{O}_X(d))$. We would like to be able to predict the rough shapes of the minimal resolutions $\mathcal{O}_X$ and $N_{B,d}$ in $\mathbb{P}^r$ in some asymptotic sense. Consider the rank r vector bundle M_L, which is defined as the kernel of the natural surjective map from $H^0(L) \otimes \mathcal{O}_X \to L$. It is well know that the Koszul groups can be computed by the cohomologies of the vector bundles of the form $\bigwedge^p M_L \otimes B \otimes L^{\otimes q}$.

First we describe the situation, when X is a smooth projective curve of genus g and L is line bundle on X of degree d, where d is sufficiently large compared to $2g$. Then the complete linear system $|L|$ embeds X into a non-degenerate in $\mathbb{P}^r$, where $r = d - g$. Then the line bundle L is $2-$regular with respect to $\mathcal{O}_{\mathbb{P}^r(1)}$. Then the Betti table of $\mathcal{O}_C$ has only three rows and the length of the minimal resolution is $r - 1$. It follows from a theorem of Green that $K_{p,2}((X, L); \mathcal{O}_C) \neq 0$ if and only if $d - (2g + 1) < p \leq r - 1 = d - g - 1$ [5]. It is conjectured by Green and Lazarsfeld that $K_{p,1}((X, L); \mathcal{O}_X) \neq 0$ if and only if $1 \leq p \leq r - 1 - \delta$, where δ is the gonality of X. In particular, one can read off the gonality of X from the shape of the minimal resolution of $\mathcal{O}_X$ in $\mathbb{P}^r$. The conjecture is known for generic δ-gonal curve, when gonality is relatively large [1] and [2]

Our knowledge of syzygies of higher dimensional varieties is fairly minimal. Only a few years ago Ottaviani and Paoletti obtained the following result for $\mathbb{P}^2$.

Theorem 1. [7] *$(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ for $d \geq 3$ satisfies N_{3d-3} but not N_{3d-2}.*

See also [4] and [3] for different proofs. The following is a version of the non-vanishing theorem that we obtain.

Theorem 2. *(Ein and Lazarsfeld). Let X be a smooth projective variety of dimension n and $\mathcal{O}_X(1)$ is a very ample line bundle on X. Consider $L = \mathcal{O}_X(d)$, where d is sufficiently large. Then*
(a) $K_{p,q}((X, L); \mathcal{O}_X) = 0$ if $p > n + 1$.
(b) For each q, where $2 \leq q \leq n$, there are two constants $b_q = O(d^{q-1})$ and $c_q = O(d^n)$, such that $K_{p,q}((X, L); \mathcal{O}_X) \neq 0$ for every p between b_q and c_q.

If X is a projective space, or more generally a Fano manifold, we have obtained more precise result. For projective space, we even have a very precise conjecture.

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Seshadri numbers along diagonals and linear syzygies

Jun-Muk Hwang
(joint work with Wing-Keung To)

For a given projective manifold X and an ample line bundle L, it is not easy to check Green's linear syzygy condition (N_p) for the line bundle L. In a recent joint work with Wing-Keung To, we obtained a numerical criterion for (N_p)-property of the adjoint bundle $K_X \otimes L$ in terms of Seshadri numbers along diagonals. Recall that for a given submanifold $Z \subset Y$ of a projective manifold Y and a line bundle M on Y, the Seshadri number of M along Z is the real number

$$\varepsilon(M; Z) := \sup\{\varepsilon \in \mathbf{R}, \pi^*M - \varepsilon E \text{ is nef and big.}\}$$

where $\pi : \mathrm{Bl}_Z(Y) \to Y$ is the blow-up of Y along Z and E is the exceptional divisor. Our result says

Theorem 1. *Let X be a projective manifold of dimension n. Let L be an ample line bundle on X with $K_X \otimes L$ nef. Let $D \subset X \times X$ be the diagonal and $p_i : X \times X \to X$ be the projection for $i = 1, 2$. If $\varepsilon(p_1^*L \otimes p_2^*L; D) \geq n(p+1)$, then $K_X \otimes L$ satisfies $(N_p), p \geq 0$.*

When $p = 0$, this is essentially proved by Bertram-Ein-Lazarsfeld in [1]. Our proof generalizes their approach to higher $p > 0$ by using the result of Inamdar [2] on linear syzygies and the work of Li [4] on blow-ups along union of subvarieties, following the suggestion in [3]. We can apply Theorem 1 to complex hyperbolic manifolds as follows. Let X be a compact quotient of the unit ball $\mathbf{B}^n$. By Kodaira's embedding theorem, it is well-known that X is a projective manifold and K_X is ample. However, very little is known about the defining equations of X under the embedding given by powers of K_X. In our work, we prove a lower bound for $\varepsilon(p_1^*L \otimes p_2^*L; D)$ in terms of the injectivity radius of X with respect to the hyperbolic metric on $\mathbf{B}^n$. By Theorem 1, this implies that for a hyperbolic manifold with a large injective radius, the line bundle $K_X^{\otimes m}, m \geq 2$, satisfies (N_p) for an explicit p depending on m and the injectivity radius.

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Linear syzygies and linear series on a blowup of $\mathbb{P}^2$ at the singular points of a line arrangement

Hal Schenck

For a hyperplane arrangement

$$\mathcal{A} = \bigcup_{i=1}^{d} \subseteq \mathbb{C}^n$$

with complement $M = \mathbb{C}^n \setminus \mathcal{A}$, conjectures of Suciu [7] relate $\pi_1(M)$ to the first resonance variety $R^1(\mathcal{A})$ of $H^*(M, \mathbb{Z})$. Work of Orlik-Solomon [5] shows that $H^*(M, \mathbb{Z})$ is a quotient of an exterior algebra on generators $e_1, \ldots, e_d$. For a one form $\lambda = \sum a_i e_i$ it is possible to define a chain complex $H(A, \lambda)$, with i^{th} term $H^i(M, \mathbb{Z})$, and differential $\cdot\lambda$. The first resonance variety

$$R^1(\mathcal{A}) = \{(a_1 : \cdots : a_d) \in \mathbb{P}(H^1(M, \mathbb{Z})) \text{ for which } H^1(A, \lambda) \neq 0.$$

The geometry of $R^1(\mathcal{A})$ is analyzed in detail in work of Falk, Libgober, and Yuzvinsky [2], [4], [8]. In [6], we describe a connection between the first resonance variety and the Orlik-Terao algebra $C(\mathcal{A})$ of the arrangement; $C(\mathcal{A})$ is a commutative analog of $H^*(M, \mathbb{Z})$. In particular, we show that non-local components of $R^1(\mathcal{A})$ give rise to determinantal syzygies of $C(\mathcal{A})$. As a result, $Proj(C(\mathcal{A}))$ lies on a scroll, placing geometric constraints on $R^1(\mathcal{A})$. The key observation is that $C(\mathcal{A})$ is the homogeneous coordinate ring associated to a nef but not ample divisor $D_{\mathcal{A}}$ on a blowup of $\mathbb{P}^2$ at the singular points of $\mathcal{A}$. Non-local components of $R^1(\mathcal{A})$ actually yield a decomposition $D_{\mathcal{A}} = A + B$; results of Harbourne [3] allow us to show that $h^0(A) = 2$, and then work of Eisenbud [1] gives the the connection to determinantal varieties. The talk closed with several questions:

(1) Find formulas relating $Tor_i^R(C(A), \mathbb{C})_j$ to combinatorics.
(2) For an arrangement in $\mathbb{P}^n$, $n \geq 3$, is $\bigoplus_{t \geq 0} H^0(D_{\mathcal{A}}(t)) \simeq C(\mathcal{A})$?
(3) Understand the intersection theory on a blowup of $\mathbb{P}^n$ at the codimension two singular locus of a hyperplane arrangement. The Hilbert series of $C(\mathcal{A})$ is known, so Riemann-Roch provides a link to the previous question.

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Linear systems of plane curves with A-D-E singularities

Joaquim Roé

A fair amount of work has been devoted in recent years to the study of linear systems of plane curves with imposed multiple points in general position, motivated mainly by the Nagata and the Segre-Harbourne-Gimigliano-Hirschowitz conjectures (see [7, 6, 3] and other talks in this workshop). Linear systems of plane curves with *non-ordinary* singularities in general position, in contrast, have been much less studied, even though there is a strong motivation for them: G. M. Greuel, C. Lossen and E. Shustin in [5] have shown that such systems serve as a means to *construct* irreducible plane curves with given singularity types, and that one often obtains curves with the lowest known degree.

The singularity theory of plane curve germs classifies them by equisingularity, which can be roughly defined as putting two germs in the same class when their embedded resolutions are combinatorially equal (this can be made precise with the notions of *multiplicity* at an infinitely near point, and *proximity* between infinitely near points, see [2]). This is a discrete classification, rather coarse, which happens to coincide with the topological classification of germs of curve in a complex surface. The theory goes further with the *analytic* classification, much finer, which provides a whole moduli space with often complicated structure for each equisingularity type. There are a few equisingularity types for which no analytic moduli exists (i.e., every curve in one of these equisingularity types is analytically isomorphic to a fixed one, which can be used as a "normal form"). These are called *simple* singularities, and are known, after Arnold, by the names A_μ (normal form: $y^2 - x^{\mu+1} = 0, \mu \geq 1$), D_μ (normal form: $xy^2 - x^{\mu-1} = 0, \mu \geq 4$), E_6 (normal form: $y^3 - x^4 = 0$), E_7 (normal form: $y^3 - yx^3 = 0$), and E_8 (normal form: $y^3 - x^5 = 0$). The subindex μ given for these singularities stands for the Milnor number, a topological invariant easily computed for any given equisingularity type.

It is generally not known for a given set of singularity types (either by the equisingularity or by the analytic classification) what is the minimal degree of a plane projective curve with those singularities. Restricting to ordinary nodes (A_1 in Arnold's notation), the classical result by Severi of course tells that the necessary and sufficient condition is that $\binom{d-1}{2}$ is at least equal to the number of nodes, where d is the degree. But already for nodes and cusps such a complete answer is not known (although a number of restrictions and constructions are available). The most general sufficient condition is the following theorem, due to E. Shustin [10].

Theorem 1. *Let $\mathcal{S}_1, \dots, \mathcal{S}_r$ be singularity types (topological or analytic), and denote μ_i the Milnor number of the ith type. If $d^2 - 2d + 3 \geq 9 \sum \mu_i$ then there exist irreducible reduced plane curves C of degree d with* $\operatorname{Sing} C \sim \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_r$.

Our purpose is to get sharper bounds by restricting to the class of singularities of types A, D and E. The improvement consists in getting rid of the factor 9 in the formula. Theorem 1 was obtained by proving that certain linear systems of plane curves have the expected dimension, and that their general member has the prescribed singularity type (via a Bertini-type theorem). For equisingularity types, the relevant linear systems can be described as push downs of some complete linear systems on particular blowups of the plane. Thus, if $\pi : X \to \mathbb{P}^2$ is a composition of point blowups whose centers $p_{11}, \dots, p_{1k_1}, \dots, p_{rk_r}$ satisfy the proximities dictated by the types $\mathcal{S}_1, \dots, \mathcal{S}_r$, then general members of $\pi_*|d\pi^* L - \sum(\operatorname{mult}_{p_{ij}} \mathcal{S}_i) E_{ij}|$ are expected to have the required singularities for large d (where as usual L denotes a line and E_{ij} denotes the exceptional divisor above p_{ij}). Denote

$$C(\mathcal{S}_1, \dots, \mathcal{S}_r) = \sum \binom{\operatorname{mult}_{p_{ij}} \mathcal{S}_i + 1}{2}$$

the "expected" number of conditions imposed by the singularities on the linear system. It is natural in this context to propose the following generalization of the Segre-Harbourne-Gimigliano-Hirschowitz conjecture, due to Greuel, Lossen and Shustin:

Conjecture 2. Let $\mathcal{S}_1, \dots, \mathcal{S}_r$ be equisingular singularity types, and let m_1, m_2, m_3 be the three largest associated multiplicities. For a composition of point blowups $\pi : X \to \mathbb{P}^2$ which is general among those whose centers $p_{11}, \dots, p_{1k_1}$, $\dots, p_{rk_r}$ satisfy the proximities dictated by the given types, and for every $d \geq m_1 + m_2 + m_3$,

$$\dim |d\pi^* L - \sum(\operatorname{mult}_{p_{ij}} \mathcal{S}_i) E_{ij}| = \max \left\{ -1, \binom{d+2}{2} - C(\mathcal{S}_1, \dots, \mathcal{S}_r) - 1 \right\}.$$

It is not hard to show that if $\mathcal{S}_1, \dots, \mathcal{S}_r$ satisfy the conjecture, then a sufficient condition for existence as in Shustin's theorem follows, where the lower bound on d^2 is only of the same order as the sum of the Milnor numbers (i.e., one gets rid of the factor 9). We are able to prove the conjecture for singularities of type A_μ, D_μ or E_μ (with an extra assumption on the degree that is not restrictive for the application to the existence problem):

Theorem 3. *Let $\mathcal{S}_1, \dots, \mathcal{S}_r$ be equisingular singularity types, where each $\mathcal{S}_i$ is of type A_μ, D_μ or E_μ. For a composition of point blowups $\pi : X \to \mathbb{P}^2$ which is general among those whose centers $p_{11}, \dots, p_{1k_1}, \dots, p_{rk_r}$ satisfy the proximities dictated by the given types, and for every $d \geq 21$,*

$$\dim |d\pi^* L - \sum(\operatorname{mult}_{p_{ij}} \mathcal{S}_i) E_{ij}| = \max \left\{ -1, \binom{d+2}{2} - C(\mathcal{S}_1, \dots, \mathcal{S}_r) - 1 \right\}.$$

Ingredients of the proof. We exploit the genericity of the points by a specialization argument similar to that of [8] (where the case of singularities A_μ was treated) which leads to a linear system of curves containing a zero-dimensional scheme supported at a single point. This scheme is a generic embedding of a monomial scheme.

In [9], the "differential Horace method" of J. Alexander-A. Hirschowitz and L. Évain [1, 4] was developed in a way suitable to the study of generic embeddings of a monomial scheme as obtained here. Successive applications of the method reduce the problem to the computation of the dimension of a new linear system $|d\pi^*L - \sum(\mathrm{mult}_{p_j} \mathcal{S})E_j|$ with a single singularity $\mathcal{S}$, for which only one point appears with multiplicity > 1. This can be done by hand. □

We expect to be able to lower the hypothesis $d \geq 21$ to $d \geq 9$ by treating the (finitely many) lower degree cases with ad-hoc methods, and thus effectively prove the conjecture by Greuel-Lossen-Shustin for $A - D - E$ singularities.

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Arithmetic properties of volumes of divisors

Alex Küronya

(joint work with Victor Lozovanu, Catriona Maclean)

The volume of a divisor on an irreducible projective variety describes the asymptotic rate of growth of the number of global sections as we take higher and higher multiples. Our purpose here is to look at volumes of divisors from the point of view of arithmetic. We work over the complex number field.

Along with stable base loci, the volume is one of the first asymptotic invariants of line bundles that have been studied. It has first appeared in some form in [2] (and is elegantly explained in [7]), where Cutkosky used the irrationality of a volume of a divisor to show the non-existence of birational Zariski decompositions with rational coefficients. For a complete account, the reader is invited to look at [7] and the recent paper [8].

Since it is invariant with respect to numerical equivalence of divisors, the volume can be considered as a function on the Néron–Severi space. There it is homogeneous, log-concave, and extends in a continuous fashion to divisors classes with real coefficients. The volume function can be often described in terms of additional structures on the underlying varieties. It can be explcitly determined on toric varieties [4], on surfaces [1], and on abelian varieties and homogeneous spaces for example. In every case, the volume reveals a fair amount of the underlying geometry.

Our main focus here is the multiplicative submonoid of the non-negative real numbers consisting of volumes of integral divisors. As a starting point, we take the fact that the volume of a divisor with finitely generated section ring is rational. Looking at the low-dimensional situation, an immediate consequence of Zariski decomposition on surfaces gives that every divisor there — even the ones with non-finitely generated section ring — has rational volume. Conversely, a simple application of Cutkosky's construction from [2] provides us with examples that every non-negative rational number can be displayed as the volume of an integral divisor.

Moving on to higher dimensions, we have seen above that the volume of an integral divisor need not be rational, although the example Cutkosky obtains is algebraic, leaving a considerable gap, and the question whether an arbitrary non-negative real number can be realized as the volume of a line bundle. This is the issue that we intend to address from two somewhat complimentary directions.

Theorem 1. *Let $\mathcal{V}$ denote the set of non-negative real numbers that occur as the volume of a line bundle. Then*

(1) *$\mathcal{V}$ is countable;*
(2) *$\mathcal{V}$ contains transcendental elements.*

Let us briefly give an idea why the above results appearing originally in [6] hold. The transcendency of volumes of integral divisors is an application of Cutkosky's principle to look for examples among projectivized vector bundles over varieties whose behaviour we understand quite well. In our particular case we consider $\mathcal{O}(1)$ of the projectivization of a rank three vector bundle on the self-product of a general elliptic curve. We exploit the non-linear shape of the nef cone on an abelian surface to arrive at the required transcendency of the volume of $\mathcal{O}(1)$. As a by-product of our reasoning one can also see that divisors with transcendental volume show up quite naturally and often in a non-finitely generated setting.

As far as the cardinality of $\mathcal{V}$ is concerned, it is a direct consequence of a much stronger countability result: building on the existence of multigraded Hilbert schemes as proved in [3], we establish the fact that there exist altogether countably many volume functions and ample/nef/big/pseudo-effective cones for all irreducible varieties in all dimensions.

Getting back to the issue of transcendental volumes, it is an interesting fact that the irregular values obtained so far by Cutkosky's construction have all been produced by evaluating integrals of polynomials over algebraic domains. In fact, all volumes computed to date can be put in such a form quite easily. Such numbers are called periods, and are studied extensively in various branches of mathematics, including number theory, modular forms, and partial differential equations. An enjoyable account of periods can be found in [5].

To some degree the phenomenon that all known volumes are periods is explained and accounted for by the existence of Okounkov bodies. Expanding earlier ideas of Okounkov [9, 10], Lazarsfeld and Mustaţă associate a convex body to any divisor with asymptotically sufficiently many sections. The actual Okounkov body depends on the choice of an appropiate complete flag of subvarieties, however, it is not difficult to see that the volume of a divisor D on an n-dimensional irreducible projective variety X equals up to a constant of $n!$ the n-dimensional Lebesgue measure of the corresponding Okounkov body. Consequently, whenever the Okounkov body of a divisor with respect to a judiciously chosen flag is an algebraic domain, the volume of D will be a period, which indeed happens in all known cases.

This gives rise to the following question: is it true that the volume of a line bundle on a smooth projective variety is always a period?

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Rank two vector bundles on Fano manifolds

Roberto Muñoz
(joint work with Gianluca Occhetta, Luis E. Solá-Conde)

In this talk we present some results on the classification of rank two vector bundles on Fano manifolds. In fact, we will introduce a numerical invariant $t \in \mathbb{R}$, called the *Fano threshold*, which seems to be of interest for our purposes. We relate this invariant, via a family of examples, with the so called Seshadri constants of points in the plane. A bunch of results on the Fano threshold and tools for the classification are presented.

Setup. We will work with a pair (X, E) where X is a complex projective Fano manifold of index i_X and E is a rank two vector bundle on X. On X we also assume:

(i) $\mathrm{Pic}(X) \simeq \mathbb{Z}H$, being H ample;
(ii) $H^4(X, \mathbb{Z}) \simeq \mathbb{Z}H^2$;
(iii) there exists an unsplit covering family $\mathcal{M}$ of rational curves on X;
(iv) $H\ell = 1$ for $\ell \in \mathcal{M}$.

We will normalize E by twisting with proper powers of H and we will use the following notation for its Chern classes and discriminant Δ:

$$c_1(E) = c_1 H \ (c_1 = 0, -1), \ c_2(E) = c_2 H^2 \ (c_2 \in \mathbb{Z}), \ \Delta = c_1^2 - 4c_2.$$

Our general goal on this topic would be to give a complete classification of rank two vector bundles on X. Inspired by the Harshorne conjecture on complete intersections ($X = \mathbb{P}^n$ and $n \geq 5$ implies E splits, i.e., is the direct sum of line bundles) our more concrete goal is to produce splitting criteria, that is, conditions on (X, E) implying that E splits. We look for these conditions with the help of the two following ingredients:

- **Ingredient 1**. Rational curves on X.
- **Ingredient 2**. Nef cone of $\mathbb{P}(E)$.

1. The uniruledness of X assures the existence of rational curves on X. Hence we can restrict E to a rational curve $\ell \subset X$ to get a pair of integers $a_\ell \leq b_\ell$ called the splitting type of E with respect to ℓ. If this splitting type does not depend on $\ell \in \mathcal{M}$ then E is called uniform with respect to $\mathcal{M}$ and some results of classification of uniform vector bundles are known (on linear spaces, quadrics, Grassmannians and other Fano manifolds) providing that, with the exception of some concrete examples, uniformity of E implies that E splits.

2. The Picard number of the projective bundle $\pi : \mathbb{P}(E) \to X$ is two and we would like to describe the nef cone of $\mathbb{P}(E)$, being known one of the two extremal rays, namely π^*H. If $\mathbb{P}(E)$ is Fano (more information on the cone is known) we will say that E is Fano. For linear spaces and quadrics is known that, with the exception of some concrete examples, E being Fano implies E splits.

Trying to classify Fano vector bundles on Grassmannians of lines we realized that the restriction of a Fano vector bundle to a subvariety is possibly non-Fano.

Anyways we can twist the anticanonical bundle by a power of π^*H in order to get ampleness. This suggests us the following definition.

Definition. The *Fano threshold* of (X, E) is the infimum t such that the divisor $-K_{\mathbb{P}(E)} + t\pi^*H$ is ample.

It is inmediate to check that being Fano is equivalent to $t < 0$. It is natural to put some questions:

Q1. When $t \in \mathbb{Q}$?
Q2. If $t \in \mathbb{Q}$, when $-K_{\mathbb{P}(E)} + t\pi^*H$ is semiample?
Q3. When does exist a rational curve ℓ such that $(-K_{\mathbb{P}(E)} + t\pi^*H)\ell = 0$?

Example ([3]). Blow up the projective plane in d points $\sigma : B \to \mathbb{P}^2$ and construct a vector bundle $E' = \sigma^*E$ on B which fits in the following exact sequence:

$$0 \to \mathcal{O}(C) \to E' \to \mathcal{O}(-C) \to 0,$$

where C is the exceptional divisor. The Fano threshold of this vector bundle is $-3 + 2\varepsilon$, being ε the Seshadri constant of the set of blown up points. This relates the problems of the rationality of t and ε. In particular when $d = 9$ and the points are in general position, $t = 3$ and there are no rational curves for which $(-K_{\mathbb{P}(E)} + t\pi^*H)\ell = 0$, showing that the rational curve searched in Q3 does not exist in these example.

This is our toolbox to deal with the problem of the study of the Fano threshold:

- **Tool 1.** Splitting types and positivity.
- **Tool 2.** Vanishing results and Castelnuovo-Mumford regularity.
- **Tool 3.** Minimal sections.

1. We just use that $E(\frac{-c_1+i_X+t}{2})$ is nef to provide two different type of results: some control on the splitting types (giving a lower bound for t, reached only if E is trivial); and, via Schur polynomials or intersections with effective cycles, inequalities or results of the following type:

(1.1) $\tan\left(\frac{\pi}{n+1}\right) < \sqrt{4 - c_1^2}/(i_X + t)$ implies $\Delta \geq 0$,
(1.2) $(i_X + t)^2 \geq |\Delta|$.

2. It is possible to show that to provide effective divisors is of interest to refine 1. We use the vanishing results of Le Potier and Griffiths to get $E(\alpha)$ globally generated for α big enough (under the extra hypothesis on the linear system $|H|$ to be base-point free) via Castelnuovo-Mumford regularity. For $\Delta > 0$ we get results of the type:

(2.1) $t \leq \{\sqrt{\Delta} + 3i_X - 2n - 6, \frac{\sqrt{\Delta}+i_X}{3} - 8\}$ implies E splits.

3. The family of lines through a point $x \in X$ is stratified by their different splitting types. Using the Euler relative sequence it was known ([1] and [2]) that if there is a compact curve parametrizing minimal sections of the same type through a point p on $\pi^{-1}(x)$ then E splits. This says that there is no room for t to bee too negative when E is not split:

(3.1) E Fano implies that either $t = -2$ and there is another structure of $\mathbb{P}^1$-bundle for $\mathbb{P}(E)$ or $t \geq -1$.

Finally we can apply these results to get splitting results for Fano vector bundles on low index Fano manifolds.

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Seshadri constants for toric vector bundles on toric varieties

Kelly Jabbusch

Fix an algebraically closed field k. Let X be a smooth, complete d-dimensional toric variety determined by the fan Σ in $N \simeq \mathbb{Z}^d$. Let M be the dual lattice and $T := Spec(k[M])$ be the torus acting on X. Let $\rho_1, \ldots, \rho_n \in \Sigma(1)$ be the 1-dimensional cones of Σ. Each ρ_i corresponds to a prime T-invariant divisor D_i. For each maximal cone $\sigma = \langle \rho_1, \ldots, \rho_d \rangle$, we get a fixed point $x(\sigma)$ that lies in the intersection of the invariant divisors $D_1, \ldots, D_d$. For any $\tau \in \Sigma(d-1)$, we get a T-invariant curve $C = V(\tau)$, and each of these is isomorphic to $\mathbb{P}^1$.

A *toric vector bundle* $\mathcal{E}$ on X is a locally free sheaf of finite rank on X with a T-action on $\mathbb{V}(\mathcal{E}) = Spec(Sym(\mathcal{E}))$ such that the projection $\varphi : \mathbb{V}(\mathcal{E}) \to X$ is equivariant and T acts linearly on the fibers. Note that $\mathbb{P}(\mathcal{E})$ has a T-action such that the projection $\pi : \mathbb{P}(\mathcal{E}) \to X$ is equivariant, however neither $\mathbb{V}(\mathcal{E})$ nor $\mathbb{P}(\mathcal{E})$ is a toric variety in general. Similarly one can define $\mathbb{Q}$-twisted toric vector bundles.

Hering, Mustaţă and Payne showed that a toric vector bundle is ample, respectively nef, if and only if its restriction to each invariant curve is ample, respectively nef [3]. Similarly, the Seshadri constants of toric vector bundles can computed by restricting to invariant curves. More precisely, let $\mathcal{E}$ be a nef toric vector bundle and let $x \in X$ be a point. Let $p : \widetilde{X} \to X$ be the blow-up at x, with exceptional divisor F. The Seshadri constant of $\mathcal{E}$ at x is

$$\varepsilon(\mathcal{E}, x) := \sup\{\lambda \in \mathbb{Q} | p^*\mathcal{E}\langle -\lambda F \rangle \text{ is nef}\},$$

and the global Seshadri constant is

$$\varepsilon(\mathcal{E}) := \inf_{x \in X} \varepsilon(\mathcal{E}, x).$$

On the other hand, for each invariant curve, we have a decomposition

$$\mathcal{E}|_C \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r).$$

Define $\tau(\mathcal{E}, x) := \min\{a_i\}$, where the minimum ranges over all a_i, and over all invariant curves passing through x. We then define $\tau(\mathcal{E}) := \min_x \tau(\mathcal{E}, x)$, where the minimum is taken over all fixed points of X.

Proposition 1 ([2] for line bundles, [3] for toric vector bundles)**.** *If X is a smooth complete toric variety, then $\varepsilon(\mathcal{E}, x) = \tau(\mathcal{E}, x)$ for each fixed point $x = x(\sigma) \in X$. Furthermore, $\varepsilon(\mathcal{E}) = \tau(\mathcal{E})$.*

We also can relate the Seshadri constant of a toric vector bundle and k-jet ampleness. We first recall the notion of k-jet ample for vector bundles as introduced in [1]. Let X be a smooth projective variety (not necessarily toric) and $\mathcal{E}$ a vector bundle on X. A vector bundle $\mathcal{E}$ is k-jet spanned at $x \in X$ if the evaluation map

$$X \times \Gamma(X, \mathcal{E}) \to \Gamma(X, \mathcal{E} \otimes \mathcal{O}_X/\mathfrak{m}_x^{k+1})$$

is surjective; and $\mathcal{E}$ is is k-jet spanned if it is k-jet spanned at all $x \in X$. Let $x_1, \ldots, x_t$ be distinct points of X with maximal ideal sheaves $\mathfrak{m}_i$, $1 \leq i \leq t$. Consider the 0-cycle $Z = x_1 + \cdots + x_t$. A vector bundle $\mathcal{E}$ is k-jet ample at Z if for every t-ple $(k_1, \ldots k_t)$ of positive integers such that $\sum_{i=1}^{t} k_i = k + 1$, the evaluation map

$$\Gamma(X, \mathcal{E}) \to \Gamma(X, \mathcal{E} \otimes (\mathcal{O}_X/\otimes_{i=1}^{t} \mathfrak{m}_i^{k_i}))$$

is surjective; $\mathcal{E}$ is is k-jet ample if it is k-jet ample at all Z.

Returning now to the toric case, we remark if X is toric, then $\mathcal{E}$ is k-jet spanned if and only if it is k-jet spanned at all fixed points $x(\sigma) \in X$, and similarly for k-jet ampleness. For line bundles on toric varieties, we have the following characterization

Theorem 2. [2] *Let L be a line bundle on a smooth projective toric variety, then the following are equivalent:*

(1) *L is k-jet ample,*
(2) *$L \cdot C \geq k$ for any invariant curve C,*
(3) *$\varepsilon(L) \geq k$.*

To generalize this to a higher rank toric vector bundle $\mathcal{E}$ consider the following conditions:

(1) $\mathcal{E}$ is k-jet ample,
(2) $\tau(\mathcal{E}) \geq k$,
(3) $\varepsilon(\mathcal{E}) \geq k$.

We've seen (2) and (3) are equivalent, and clearly (1) implies (2). If $\mathcal{E}$ is nef, then $\tau(\mathcal{E}) \geq 0$, however $\mathcal{E}$ may not be globally generated, i.e. $\mathcal{E}$ is not 0-jet ample [3, Ex 4.15]. In recent work in progress, [4], we conjecture that for $k \geq 1$, (2) implies $(1')$ where $(1') := \mathcal{E}$ is k-jet spanned.

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Isolated smooth curves in Calabi-Yau threefolds

Andreas Leopold Knutsen

In the paper [5], Kley developed a framework for showing existence of curves of certain genera and degrees in Calabi-Yau complete intersection ($CICY$) threefolds. The paper built on the original idea in the case of genus zero curves of Clemens [1] (then used also in [4], [10] and [3]): one starts with a $K3$ complete intersection surface X containing a smooth rational curve C, embeds the surface in a nodal $CICY$ of suitable intersection type Y and proves that under a general deformation Y_t of $Y_0 = Y$, the rational curve deforms to an isolated curve in the deformation. In the higher genus case, the curve C is replaced by a complete linear system $|\mathcal{L}|$ of curves on the surface of dimension equal to the genus, and the idea is to prove that only finitely many of these deform to the deformation Y_t and possibly also that these are smooth and isolated. The main existence result Theorem 1 in [5] claims that for any $d \geq 3$, the general $CICY$ threefold contains smooth, isolated elliptic curves of degree d, except for degree 3 curves in the $CICY$ of type $(2, 2, 2, 2)$.

A crucial point in the proof is to show that the curves on X do not acquire any additional deformations when considered as curves in Y, precisely that

$$h^0(\mathcal{N}_{C/X}) = h^0(\mathcal{N}_{C/Y}) \text{ for all } C \in |\mathcal{L}|.$$

Unfortunately, the proof of this step contains a serious gap, which also influences the proof of a later corollary.

In the talk I gave an outline of the main results in my paper [7], which can be summarized as follows:

- We give criteria for a continuous family of curves on a regular surface in a nodal threefold Y with trivial canonical bundle to deform to a scheme of finitely many smooth isolated curves in a general deformation Y_t of $Y_0 = Y$, using results from [2] and ideas from the unpublished preprint [6] of Kley, see Theorem 1.
- We apply these results to prove existence of smooth, isolated curves of low genera in the various $CICY$ threefold types, see Theorem 2 (of which [5, Thm. 1] is a special case).

The first main result is the following. It is an improvement under slightly stronger hypotheses of a result in the preprint [6] of Kley.

We first state the assumptions.

Setting and assumptions. Let P be a smooth projective variety of dimension $r \geq 4$ and $\mathcal{E}$ a vector bundle of rank $r-3$ on P that splits as a direct sum of line bundles

$$\mathcal{E} = \oplus_{i=1}^{r-3} \mathcal{M}_i.$$

Let

$$s_0 = s_{0,1} \oplus \cdots \oplus s_{0,r-3} \in H^0(P, \mathcal{E}) = \oplus_{i=1}^{r-3} H^0(P, \mathcal{M}_i)$$

be a regular section, where $s_{0,i} \in H^0(P, \mathcal{M}_i)$ for $i = 1, \ldots, r-3$. Set

$$Y = Z(s_0) \text{ and } Z = Z(s_{0,1} \oplus \cdots \oplus s_{0,r-4})$$

(where $Z = P$ if $r = 4$).

Let $X \subset Y$ be a smooth surface with $H^1(X, \mathcal{O}_X) = 0$ and $\mathcal{L}$ a line bundle on X. We make the following additional assumptions:

(1) Y has trivial canonical bundle;
(2) Z is smooth along X and the only singularities of Y which lie in X are ℓ nodes $\xi_1, \ldots, \xi_\ell$. Furthermore, $\ell \geq \dim |\mathcal{L}| + 2$;
(3) $|\mathcal{L}| \neq \emptyset$ and the general element of $|\mathcal{L}|$ is a smooth, irreducible curve;
(4) for every $\xi_i \in S := \{\xi_1, \ldots, \xi_\ell\}$, if $|\mathcal{L} \otimes \mathcal{I}_{\xi_i}| \neq \emptyset$, then its general member is nonsingular at ξ_i;
(5) $H^0(C, \mathcal{N}_{C/X}) \simeq H^0(C, \mathcal{N}_{C/Y})$ for all $C \in |\mathcal{L}|$;
(6) $H^1(C, \mathcal{N}_{C/P}) = 0$ for all $C \in |\mathcal{L}|$;
(7) the image of the natural restriction map $H^0(P, \mathcal{M}_{r-3}) \to H^0(S, \mathcal{M}_{r-3} \otimes \mathcal{O}_S) \simeq \mathbb{C}^\ell$ has codimension one.

Let $s \in H^0(P, \mathcal{E})$ be a general section. Then our result is the following:

Theorem 1. *Under the above setting and assumptions (1)-(7), the members of $|\mathcal{L}|$ deform to a length $\binom{\ell-2}{\dim |L|}$ scheme of curves that are smooth and isolated in the general deformation $Y_t = Z(s_0 + ts)$ of $Y_0 = Y$. In particular, Y_t contains a smooth, isolated curve that is a deformation of a curve in $|\mathcal{L}|$.*

The main ingredient of the proof involves a careful study of condition (5). The crucial result states that condition (5) is equivalent to the condition

(5)' The set of nodes S imposes independent conditions on $|\mathcal{L}|$, and the natural map $\gamma_C : H^0(C, \mathcal{N}_{X/Y}\mathcal{O}_C) \to H^1(C, \mathcal{N}_{C/X})$ is an isomorphism for all $C \in |\mathcal{L}|$.

The first of the two conditions in (5)' assures that the locus of curves in $|\mathcal{L}|$ passing through at least one node is a simple normal crossing (SNC) divisor (consisting of ℓ hyperplanes). This enables us to identify a certain sheaf $\mathcal{Q}$ of obstructions to deformation as the locally free sheaf of differentials with logarithmic poles along an SNC divisor, which is a crucial point to assure that *smooth and isolated* curves survive in a general deformation Y_t of $Y_0 = Y$.

Our main application is Theorem 2 right below, of which [5, Thm. 1] is the special case with $g = 1$.

Theorem 2. *Let $d \geq 1$ and $g \geq 0$ be integers. Then in any of the following cases the general Calabi-Yau complete intersection threefold Y of the given type contains an isolated, smooth curve of degree d and genus g:*

(a) $Y = (5) \subset \mathbb{P}^4$*:* $g = 0$ *and* $d > 0$*;* $g = 1$ *and* $d \geq 3$*;* $2 \leq g \leq 6$ *and* $d \geq g+3$*;* $7 \leq g \leq 9$ *and* $d \geq g + 2$*;* $g = 10$ *and* $d \geq 11$*;* $11 \leq g \leq 22$ *and* $d \geq \frac{g+13}{2}$.
(b) $Y = (4, 2) \subset \mathbb{P}^5$*:* $g = 0$ *and* $d > 0$*;* $g = 1$ *and* $d \geq 3$*;* $g = 2$ *and* $d \geq 5$*;* $3 \leq g \leq 8$ *and* $d \geq g + 4$*;* $9 \leq g \leq 11$ *and* $d \geq g + 3$*;* $12 \leq g \leq 15$ *and* $d \geq \frac{g+16}{2}$.
(c) $Y = (3, 3) \subset \mathbb{P}^5$*:* $g = 0$ *and* $d > 0$*;* $g = 1$ *and* $d \geq 3$*;* $g = 2$ *and* $d \geq 5$*;* $3 \leq g \leq 7$ *and* $d \geq g + 4$.

(d) $Y = (3,2,2) \subset \mathbb{P}^6$: $g = 0$ *and* $d > 0$; $g = 1$ *and* $d \geq 3$; $g = 2$ *and* $d \geq 5$; $g = 3$ *and* $d \geq 7$; $4 \leq g \leq 10$ *and* $d \geq g + 5$.
(e) $Y = (2,2,2,2) \subset \mathbb{P}^7$: $g = 0$ *and* $d > 0$; $g = 1$ *and* $d \geq 4$; $g = 2$ *and* $d \geq 6$; $g = 3$ *and* $d \geq 7$.

We remark that the genus zero case of the theorem is already known by [4, 10, 3]. In [6] an existence result similar to Theorem 2 was claimed, but only for *geometrically rigid, connected curves* (not necessarily smooth and isolated). But the proof of that result also relied on [5, Thm. 3.5].

The proof follows by applying Theorem 1 to the case of $K3$ surfaces in complete intersection Calabi-Yau threefolds. For each of the complete intersection types in Theorem 2, there is a standard construction allowing to embed a $K3$ surface of one (or more) of the three complete intersection types (4) in $\mathbb{P}^3$, $(2,3)$ in $\mathbb{P}^4$ and $(2,2,2)$ in $\mathbb{P}^5$ into a *nodal* $CICY$ threefold. We are then in the setting of Theorem 1 with X the $K3$ surface, Y the $CICY$, P a projective space and $\mathcal{E}$ the vector bundle corresponding to the complete intersection type of Y.

The existence of smooth curves of certain degrees and genera on the three types of complete intersection $K3$ surfaces is given by the existence results in [9] and [8].

The numerical conditions we end up with are due to conditions (1)-(7) in Theorem 1.

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Collisions of zero dimensional schemes and applications

Laurent Evain

Let X be a smooth algebraic variety and $z_1, \dots, z_k$ be zero-dimensional subschemes of X. We are interested in determining the possible collisions of the z_i's on X.

There are two main applications of these computations, namely the description of the Hilbert functions of generic unions of fat points in the projective plane, and constructions of compactifications. Accordingly, our main interest is when X is a surface S and when each subscheme z_i is a fat point. However, other cases are to be considered in some applications.

We denote by $Coll(n_1, \dots, n_k)(S)$ the closed irreducible subvariety of the Hilbert scheme $Hilb^{\sum \frac{n_i(n_i+1)}{2}}(S)$ whose generic point parametrizes the generic union of k fat points of order $n_1, \dots, n_k$. We denote by p^n the fat point with support p and order n.

The following theorem explains that constructing collisions is a useful tool to study the Hilbert function of generic unions of fat points.

Theorem 1. [1] *The following two conditions are equivalent:*

- *There exists an integer r, two consecutive fat points p^r and p^{r+1}, a collision Z in $Coll(n_1, \dots, n_k)(S)$ such that $p^r \subset Z \subset p^{r+1}$.*
- *The Hilbert function of a generic union of fat points $p_1^{n_1} \cup \dots \cup p_k^{n_k}$ is $H_Z(d) = min(\frac{(d+1)(d+2)}{2}, \sum \frac{n_i(n_i+1)}{2})$.*

This theorem suggests that computing every collision is a far too difficult collision, since for instance it would give a description of the ample and nef cones of the generic blow-up of the projective plane, a problem more difficult than the still unproved Nagata's conjecture. Note that even for simple points, $Coll(1, \dots, 1)(S) = Hilb^k(S)$ is equivalent to the irreduciblity of the Hilbert scheme on a surface, which is not a trivial result.

Since computing all collisions is too difficult, a more sensible goal is the following:

- Construct good collisions sufficient to estimate the dimensions of linear series.
- Classify the collisions for a small number of points.

In connection with his counter-example of the fourteenth Hilbert problem, Nagata has proved the following:

Theorem 2. [5] *There is no projective plane curve of degree $d = km$ through k^2 points in general position with multiplicity m.*

An improvement of this result is:

Theorem 3. [2] [4] *The Hilbert function of the generic union of k^2 fat points of multiplicity m in the plane is $H_Z(d) = min(\frac{(d+1)(d+2)}{2}, k^2\frac{m(m+1)}{2})$.*

A generalization of Nagata's result in higher dimension is the following:

Theorem 4. [3] *There is no projective hypersurface of degree $d = km$ through k^r points of the projective space $\mathbb{P}^r$ in general position with multiplicity m.*

When the number of points is small, namely at most three, the collsions are well understood. It is possible to describe the collisions individually, but also to describe completly the variety $Coll(n_1, n_2, n_3)(S)$ as an abstract variety. It is related to the variety of complete triangles $T(S)$ constructed by Schubert/Le Barz [6] [7] for enumerative purposes. When the n_i's are equal, one has to consider quotients of $T(S)$.

Theorem 5. *If $n_1 > n_2 > n_3 > 1$ and $n_1 < n_2 + n_3$, $Coll(n_1, n_2, n_3(S)) \simeq T(S)$. The symetric group σ_3 acts on $T(S)$ and, when $n \geq 2$, $Coll(n, n, n)(S) \simeq T(S)/\sigma_3$.*

When the number of points is at most four, not all the collisions have been computed, but the collisions are understood when the points approach successivly the origin. These collisions can be described by explicit divisors on the blow-up of the surface. In particular:

Theorem 6. [4] *A collision of 4 fat points in a smooth surface is defined by an integrally closed ideal.*

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New remarks on Seshadri constants

Tomasz Szemberg

(joint work with Thomas Bauer)

This is report on recent works [2] and [5].

Let X be a smooth projective variety and L a nef line bundle on X. Recall that the number

$$\varepsilon(L; x) := \inf \frac{L \cdot C}{mult_x C}$$

is the *Seshadri constant of L at the point $x \in X$* (the infimum being taken over all irreducible curves C passing through x). Its multi-point counterpart $\varepsilon(L; x_1, \ldots, x_r)$ is defined similarly.

We are interested in possible values of Seshadri numbers. Building upon examples due to Miranda and Viehweg [3, Examples 5.2.1 and 5.2.2] we show first that all positive rational numbers appear as Seshadri constant of some ample line bundle L on a variety of given dimension. There is not a single example of an irrational Seshadri constant known and it would be extremely interesting to know if there are any. If Nagata and Harbourne-Hirschowitz conjectures hold true, then in fact most of Seshadri constants should be irrational, at least in the multi-point setting.

Using result of Angehrn and Siu [1, Theorem 0.1] we show that there is a lower bound on Seshadri constants of adjoint line bundles depending only on dimension of X.

Theorem 1. *Let X be a smooth projective variety of dimension n. Let L be a nef line bundle on X and assume that the adjoint line bundle $K_X + L$ is ample. Then*

$$\varepsilon(K_X + L) \geq \frac{2}{n^2 + n + 4} \ .$$

In particular not all rational numbers may appear as Seshadri constants of adjoint line bundles. In case of algebraic surfaces one can be surprisingly precise about possible values.

Theorem 2. *Let X be a smooth projective surface and L a nef line bundle such that $K_X + L$ is ample. If for some point $x \in X$ the Seshadri constant $\varepsilon(K_X + L, x)$ lies in the interval $(0, 1)$, then*

$$\varepsilon(K_X + L, x) = \frac{m-1}{m}$$

for some integer $m \geq 2$.

Even more can be said in the hyper-adjoint case.

Theorem 3. *Let X be a smooth projective surface and L a very ample line bundle on X such that $K_X + L$ is ample. Then*

a) *$\varepsilon(K_X + L) \geq 1$.*
b) *If $\varepsilon(K_X + L, x) = 1$ for all $x \in X$, then either $(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4))$ or X is a ruled surface. In the latter case, one has $L = -3C_0 + s \cdot f$, where C_0 is a section, f a fiber of the ruling, and s a positive integer.*

The proof of the last part uses nice adjoint-theoretic arguments and the structure theorem established in [4, Theorem 0.1].

Passing to the multi-point Seshadri constants we show the parallel picture. Whereas again all positive rational numbers may appear as Seshadri constant of an ample line bundle, their values for adjoint line bundles are subject to strong restrictions. For surfaces we get the following very precise statement.

Theorem 4. *We fix an integer $r \geq 2$. Let X be a smooth projective surface and let L be a nef line bundle on X such that $K_X + L$ is ample. If for some distinct points $x_1, \ldots, x_r \in X$ the Seshadri constant $\varepsilon(K_X + L; x_1, \ldots, x_r)$ lies in the interval $(0, \frac{1}{r})$, then*

$$\varepsilon(K_X + L; x_1, \ldots, x_r) = \frac{1}{r+1} \quad or \quad \frac{1}{r+2} \ ,$$

unless $r = 2$ and $\varepsilon(K_X + L; x_1, x_2) = \frac{2}{5}$.

We provide examples showing that all presented results are sharp.

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