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Large Scale Stochastic Dynamics

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Abstract. In focus are interacting stochastic systems with many components, ranging from stochastic partial differential equations to discrete systems as interacting particles on a lattice moving through random jumps. More specifically one wants to understand the large scale behavior,large in spatial extent but also over long time spans, as entailed by the characterization of stationary measures, effective macroscopic evolution laws, transport of conserved fields, homogenization, self-similar structure and scaling, critical dynamics, aging, dynamical phase transitions, large deviations, to mention only a few key items.

Mathematics Subject Classification (2000): 60G, 82, 35.

Introduction by the Organisers

"Large Scale Stochastic Dynamics" is at the crossroad of probabilitity theory and statistical physics, the central theme being the stochastic evolution of a system with many interacting components. A prototypical example is the stochastic Ising model: at the sites of a regular lattice one has spins which take values ± 1. A specified spin flips at random times with a rate depending on the current neighboring spin configuration. Such a seemingly simple model has a very rich phenomenology. For example, let us impose that at the initial time the spin values are random according to a Bernoulli measure, whereas the dynamics runs at low temperatures forcing spins to align. Which laws govern the resulting spatial coarsening process? One may modify the dynamics by requiring the number of up spins (= particles) to be conserved, which is implemented by exchanging the spin values for a neighboring pair of spins. This model leads to a system of interacting symmetric random

walks. One can drive the system by a uniform force field making the random walks asymmetric. The variations are without bound. Mathematically one has to focus on a few central issues. Our workshop is like a snap-shot of the current activities, in fact quite distinct from the two previous snap-shots. A partial list of topics reads

- low temperature Ising dynamics
- stochastic growth and the Kardar-Parisi-Zhang equation
- random walks in random environments
- energy transport
- condensation and coarsening phenomena
- hydrodynamic limits

We had 53 participants from 15 countries, mostly probabilists, but also experts from partial differential equations, numerical analysis, and statistical physics. They all enjoyed tremendously the unique and stimulating atmosphere at the Mathematische Forschungsinstitut Oberwolfach and hope to return some day.

Claudio Landim,
Stefano Olla,
Herbert Spohn.

Workshop: Large Scale Stochastic Dynamics

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Abstracts

Current fluctuations in non equilibrium systems

Bernard Derrida

The talk was a review of of recent results on the fluctuations of the current in non-equilibrium diffusive systems.

When a system is maintained in a non-equilibrium steady state by contact with two reservoirs at unequal densities or two heat baths at unequal temperatures, the fluctuations of the current have in general a non-Gaussian distribution which can be computed exactly for diffusive systems [1, 2, 3, 4, 5, 8, 9, 10] by applying to current fluctuations the macroscopic fluctuation theory developed by Bertini, De Sole, Gabrielli, Jona–Lasinio, Landim [6, 7].

For systems at equilibrium on a ring geometry, the cumulants of these fluctuations take a universal scaling form which can be understood by several theoretical approaches such as the Bethe anstazt or fluctuationg hydrodynamics [11]. For driven systems on a ring geometry, one can observe a phase transition in the large deviation function of the current from a phase where the optimal density profile, to generate a certain current, is flat to a phase where it becomes space (and possibly time) dependent [13, 5, 14]. This phase transition is very reminiscent of the phase transition in the ABC model [15, 16].

Some recent results concerning nonsteady state initial conditions were also discussed [17, 18].

One knows [19, 20, 21] that mechanical systems which conserve momentum exhibit an anomalous Fourier's law in one dimension. What the large deviation functions of the current or of the density become for such systems looks to me an interesting and challenging question.

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Diffusion in Hamiltonian systems

Wojciech De Roeck, Antti Kupiainen

Our work concerns the problem of proving diffusion in a system described by Hamiltonian dynamics. The particular system that we study is the following: Consider a particle with a large mass M and an internal degree of freedom. One can, for example, imagine that the particle is a molecule consisting of two atoms and the internal degree of freedom is the relative position of the two atoms. The particle is coupled weakly (coupling strength λ) to a scalar wave field at some positive temperature. This wave field models, for example, the vibration modes of a crystal or the electromagnetic field. In the absence of coupling, the particle moves ballistically, with the evolution of the internal degree of freedom decoupled from its translational motion, and the wave field evolves freely (different wave packets do not interact). When the coupling is switched on, one expects that the particle will emit and absorb wavepackets and, since it can change its momentum in the process of doing so, diffuse in the the long-time limit. Next, we make a modification that should be of no importance to the result, not even to the description given above, but that does make the problem technically a bit easier, at least from the point of view of our techniques: We consider the whole setup to be quantum-mechanical. There are two (closely related) reasons why this is useful: Firstly, it becomes natural to model the internal degree of freedom by a finite dimensional variable (in quantum physics, this is called a 'few level'-system). Secondly, it is natural to consider the particle in discrete space, instead of the continuum. Its free ballistic

motion is now generated by the lattice Laplacian which does not admit arbitrarily high velocities, hence the technical simplification. We stress however that in the transition from classical to quantum, the core of the problem was not lost. The dynamics is still deterministic, the microscopic entropy is constant and the system displays Poincare recurrencies in finite volume.

Let us now focus on a way to make the problem simpler, although this is not what we do. As the coupling constant λ becomes small, the behaviour of the particle starts looking Markovian. One could take the point of view that the particle emits and absorbs wave-packets at a frequency of λ^2, and, because of this small frequency, different emission and absorption events become virtually independent. This picture becomes sharp in the limit where one observes the particle at macroscopic time $\tau \sim \lambda^2 t$ (where t is microscopic time) and one sends $\lambda \to 0$ and $t \to \infty$ with τ fixed. Indeed, the approach to a Markovian process in the time coordinate τ (in this case: the linear Boltzmann equation) has been proven in [1]. However, our aim is to prove diffusion in the long time limit, $t \to \infty$, for a small, but fixed coupling strength λ. The most logical way to attack this problem is to attempt an expansion around the Markovian regime. This is exactly what we do, except that the Markovian regime that we study is a different one: it is obtained by recaling not only time and the coupling, but also the mass M which is chosen to scale as $M \sim \lambda^{-2}$. Hence,

$$t \to \infty,\ \lambda \to 0,\ M \sim \lambda^{-2} \to \infty,\ \lambda^2 t = \tau \text{ fixed}$$

In that limit, the internal degree of freedom plays an important role: the emerging picture is that of the particle emitting and absorbing waves and keeping the energy balance not by changes in its kinetic energy, but by changes in the internal degree of freedom. This is ultimately the reason why we need this degree of freedom. The expansion around this Markovian regime can be viewed as a random walk in a random environment, where the disorder is weak and time-dependent. The time-dependent random environment is of course due to the wave field, more precisely to that part of the wave field that is non-Markovian, and the random walk originates from the free particle together with a Markovian component of the wave field. The random walk in random environment problem is then tackled by a renormalization group treatment that imposes some constraints on the free wave field. The electromagnetic field can be studied in dimension $d \geq 4$, the massive phonon field in dimension $d \geq 3$. Our treatment uses elements of earlier work of the authors, most importantly [2] where the problem is considered for a rather restricted class of wave-fields, and [3], where a renormalization group approach of random walk in time-dependent random environment is performed.

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Entropy of stationary nonequilibrium measures of boundary driven symmetric simple exclusion processes

Cédric Bernardin
(joint work with Claudio Landim)

In the last decade important progress has been accomplished in the understanding of nonequilibrium stationary states through the study of stochastic lattice gases ([5, 9] and references therein).

The simplest nontrivial example of such dynamics is the one-dimensional simple symmetric exclusion process on the finite lattice $\{1, 2, \dots, N-1\}$ with particle reservoirs coupled to the sites 1 and $N-1$. In this model the microscopic states are described by the vector $\eta = (\eta(1), \eta(2), \dots, \eta(N-1))$, where $\eta(i) = 1$ if the site i is occupied and $\eta(i) = 0$ if the site is empty. Each particle, independently from the others, perform a nearest-neighbor symmetric random walk with the convention that each time a particle attempts to jump to a site already occupied the jump is suppressed. At the boundaries, particles are created and destroyed in order for the density to be α at the left boundary and β at the right boundary, $0 \le \alpha,\, \beta \le 1$.

We denote by $\mu^N_{\alpha,\beta}$ the stationary state of this system which is a probability measure in the space of configurations and which can be expressed in terms of a product of matrices [10]. Since the particle number is the only conserved quantity in the bulk, in the scaling limit $N \to \infty$, $i/N \to x \in [0, 1]$, the system is described by a single density field $\rho(x)$, $x \in (0, 1)$. The typical density profile $\bar\rho(x)$ is the stationary solution of a partial differential equation with boundary conditions. In the context of symmetric exclusion processes,

$$\bar\rho(x) \;=\; \alpha(1-x) \;+\; \beta x \;.$$

The nonequilibrium stationary states exhibit long range correlations [14] which are responsible in the large deviations regime for the non locality of the free energy functional [10, 3]. More precisely, if γ stands for a density profile different from the typical one $\bar\rho$, the asymptotic probability of γ is exponentially small and given by

$$\mu^N_{\alpha,\beta}[\gamma(\cdot)] \sim e^{-N V_{\alpha,\beta}(\gamma)} \;,$$

where the so called nonequilibrium free energy $V_{\alpha,\beta}$ is a non local functional.

Since in equilibrium the probability of such large deviations is determined by the induced change in the entropy, it is natural to investigate the entropy of nonequilibrium stationary states.

Denote by $\mathfrak{S}_N(\nu^N)$ the Gibbs–Shannon entropy of a state ν^N:

$$\mathfrak{S}_N(\nu^N) \;=\; -\sum_\eta \nu^N(\eta)\, \log \nu^N(\eta) \;,$$

where the sum is carried over all lattice configurations η. Recently, Bahadoran [1] proved that for a large class of stochastic lattice gases the Gibbs–Shannon entropy of nonequilibrium stationary states has the same asymptotic behavior as the Gibbs–Shannon entropy of local equilibrium states. In our context of boundary driven symmetric simple exclusion processes this result can be stated as follows. Denote by $\nu^N_{\alpha,\beta}$ the product measure

$$\nu^N_{\alpha,\beta}(\eta) \;=\; \prod_{i=1}^{N-1} \bar\rho(i/N)^{\eta(i)}[1-\bar\rho(i/N)]^{1-\eta(i)} \;.$$

Thus, at site i, independently from the other sites, we place a particle with probability $\bar\rho(i/N)$ and leave the site empty with probability $1-\bar\rho(i/N)$. Bahadoran proved that

$$\lim_{N\to\infty} \frac{1}{N}\mathfrak{S}_N(\mu^N_{\alpha,\beta}) \;=\; \lim_{N\to\infty} \frac{1}{N}\mathfrak{S}_N(\nu^N_{\alpha,\beta}) \;.$$

The long range correlations of the nonequilibrium stationary state is therefore not captured by the Gibbs–Shannon entropy.

Derrida, Lebowitz and Speer [11] showed that for the symmetric simple exclusion process the difference

$$\mathfrak{S}_N(\mu^N_{\alpha,\beta}) \;-\; \mathfrak{S}_N(\nu^N_{\alpha,\beta})$$

converges as $N\to\infty$, and that the limit depends on the two points correlation functions. Hence, the long range correlations appear in the first order correction to the Gibbs–Shannon entropy.

In this talk we examine the entropy of the stationary nonequilibrium states $\mu^N_{\alpha,\beta}$. In the classical Boltzmann–Gibbs theory of equilibrium statistical mechanics [5], the steady state $\mu^N_\beta(\eta)$ of a microstate η is given by

$$\mu^N_\beta(\eta) = \frac{1}{Z_N(\beta)}\exp(-\beta H(\eta)) \tag{1}$$

where β is the inverse of the temperature, $H(\eta)$ the energy of η and $Z_N(\beta)$ the partition function. The Boltzmann entropy is then defined as the limit, when the degrees of freedom N of the system converges to infinity, of $1/N$ times the logarithm of the number of microstates with a prescribed energy:

$$S(E) \;=\; \lim_{\delta\to 0}\lim_{N\to\infty} N^{-1}\log\Big(\sum_\eta \mathbf{1}\{|H(\eta)-NE|\le\delta N\}\Big) \;,$$

where the summation is performed over all configurations η and where $\mathbf{1}\{A\}$ is the indicator of the set A. The pressure $P(\beta)$ is defined by

$$P(\beta) = \lim_{N\to\infty}\frac{1}{N\beta}\log Z_N(\beta)$$

and the Boltzmann entropy is related to the pressure function by

$$S(E) = \inf_{\beta>0}\{\beta P(\beta)+\beta E\} \;.$$

In view of (1) and by analogy, we define the energy of a microstate η as $-\log \mu^N_{\alpha,\beta}(\eta)$ and the entropy of the stationary nonequilibrium measure $\mu^N_{\alpha,\beta}$ by

$$S_{\alpha,\beta}(E) \;=\; \lim_{\delta\to 0}\lim_{N\to\infty} N^{-1}\log\Big(\sum_{\eta\in\Omega_N} \mathbf{1}\{|N^{-1}\log\mu^N_{\alpha,\beta}(\eta)+E|\le\delta\}\Big)\,.$$

There is also a probabilistic interpretation of this function. Consider the sequence of random variables $Y_N(\eta) = -N^{-1}\log\mu^N_{\alpha,\beta}(\eta)$ defined on the probability space Ω_N equipped with the probability measure $\mu^N_{\alpha,\beta}$. Then if the sequence $(Y_N : N \ge 1)$ satisfies a large deviations principle, the corresponding rate function $J_{\alpha,\beta}$ is given by $J_{\alpha,\beta}(E) = E - S_{\alpha,\beta}(E)$.

Our first result is the following formula for $S_{\alpha,\beta}$

$$S_{\alpha,\beta}(E) = \sup_{m\in\mathcal{M}} \big\{\mathbb{S}(m) : V_{\alpha,\beta}(m) + \mathbb{S}(m) = E\big\}\,,$$

where $\mathcal{M}$ is the set of macroscopic profiles $m : [0,1] \to [0,1]$ and

$$\mathbb{S}(m) \;=\; -\int_0^1 s(m(x))\,dx\,, \quad s(q) = q\log q + (1-q)\log(1-q)$$

is the Gibbs-Shannon entropy corresponding to the macroscopic profile m. The derivation of this formula is based on a strong form of local equilibrium and heuristics arguments are proposed in [2] to show the validity of this assumption. We guess that the variational formula for $S_{\alpha,\beta}$ in terms of the nonequilibrium free energy $V_{\alpha,\beta}$ and Gibbs-Shannon entropy $\mathbb{S}$ is valid for generic diffusive systems.

Since in the context of the boundary driven symmetric simple exclusion process the nonequilibrium free energy can be computed exactly ([4]) one can derive a more explicit formula for the entropy. Let $P_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}_+$ be the real function defined by

$$P_{\alpha,\beta}(\theta) = \theta\,\log\Big(\frac{1}{\beta-\alpha}\int_\alpha^\beta \frac{dx}{[x^\theta + (1-x)^\theta]^{1/\theta}}\Big)$$

then we have

$$S_{\alpha,\beta}(E) \;=\; \inf_{\theta\in\mathbb{R}} \big\{\theta E - P_{\alpha,\beta}(\theta)\big\}\,.$$

We also show that the entropy of the nonequilibrium stationary states $\mu^N_{\alpha,\beta}$ is different from the entropy of the local equilibrium states $\nu^N_{\alpha,\beta}$. From he exact computation of $S_{\alpha,\beta}$ and its probabilistic interpretation (see above) we then recover some of the results of [1] and some of [11].

These results can be found in [2].

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Energy transport in a fast-slow hamiltonian system

Carlangelo Liverani
(joint work with Dmitry Dolgopyat)

One of the central problems in the study of non-equilibrium statistical physics is the derivation of transport equations for conserved quantities, in particular energy transport, from first principles, (see [1], and references therein, or [9], for a more general discussion on the derivation of macroscopic equations from microscopic dynamics).

We consider a *microscopic* dynamics determined by a (classical) Hamiltonian describing a finite number of weakly interacting strongly chaotic systems and we explore the following strategy to derive a macroscopic evolution: first one looks at times for which there is an effective energy exchange between interacting systems, then takes the limit for the strength of the interaction going to zero and hopes to obtain a self-contained equation describing the evolution of the energies only. We call such an equation *mesoscopic* since most of the degrees of freedom have been averaged out. Second, one performs on such a mesoscopic equation a thermodynamic limit to obtain a *macroscopic* evolution. In particular, one can consider a scaling limit of the diffusive type in order to obtain a non linear heat equation as in the case of the so called *hydrodynamics limit* for particle systems, see [7, 10] for more details. A similar strategy has been carried out, at a heuristic level, in [5, 6].

In this talk I will describe the *first step* of such a program.

For $\nu \in \mathbb{N}$, we consider a lattice $\mathbb{Z}^\nu$ and a finite connected region $\Lambda \subset \mathbb{Z}^\nu$. Associated to each site in Λ we have the cotangent bundle T^*M of a $\mathcal{C}^\infty$ compact Riemannian d-dimensional manifold M of strictly negative curvature and the associated geodesic flow g^t. We have then the phase space $\mathcal{M} = (T^*M)^\Lambda$ and we designate a point as (q_x, p_x), $x \in \Lambda$. It is well known that the geodesic flows is an Hamiltonian flow. If we define $\boldsymbol{i} : T^*M \to TM$ to be the natural isomorphism defined by $w(v) = \langle \boldsymbol{i}(w), v\rangle_G$, G being the Riemannian metric, then the Hamiltonian reads[1] $H_0 = \sum_{x\in\Lambda} \frac{1}{2} p_x^2$.

Note that, by the Hamiltonian structure, $e_x := \frac{1}{2}p_x^2$ is constant in time for each $x \in \Lambda$. It is then natural to use the variables (q_x, v_x, e_x), where $v_x := (p_x^2)^{-1}\boldsymbol{i}(p_x)$ belongs to the unit tangent bundle T_1M of M.

Next we want to introduce a small energy exchange between particles. To describe such an exchange we introduce a symmetric, non constant, function (potential) $V \in \mathcal{C}^\infty(M^2, \mathbb{R})$ and, for each $\varepsilon > 0$, consider the flow g_ε^t determined by the Hamiltonian $H_\varepsilon = \sum_{x\in\Lambda} \frac{1}{2}p_x^2 + \frac{\varepsilon}{2} \sum_{|x-y|=1} V(q_x, q_y)$. The equation of motions reads

$$
\begin{aligned}
&\dot q_x = \sqrt{2e_x}\, v_x \\
&\dot v_x = \sqrt{2e_x} F(q_x, v_x) + \frac{\varepsilon}{\sqrt{2e_x}} \sum_{|y-x|=1} \{v_x L_x V - \nabla_{q_x} V(q_x, q_y)\} \\
&\dot e_x = -\varepsilon \sqrt{2e_x} \sum_{|x-y|=1} L_x V,
\end{aligned}
\tag{1}
$$

where $\langle \nabla V, w\rangle_G = dV(w)$, F is an homogeneous function of degree two in the second variable and $L_x = v_x \partial_{q_x} + F(q_x, v_x)\partial_{v_x}$ denotes the generator associated to the geodesic flow of the x particle on T_1M.

We will consider *random* initial conditions of the following type

$$
\begin{aligned}
&\mathbb{E}(f(q(0), v(0)) = \int_{(T_1M)^\Lambda} f(q, v)\rho(q, v) dm, \quad \forall f \in \mathcal{C}^0((T_1M)^\Lambda, \mathbb{R}) \\
&e_x(0) = E_x > 0,
\end{aligned}
\tag{2}
$$

where m is the Riemannian measure on $(T_1M)^\Lambda$ and $\rho \in \mathcal{C}^1$.

Since the currents L_xV have zero average with respect to the microcanonical measure, one expects that it will take a time of order ε^{-2} in order to see a change of energy of order one. It is then natural to introduce the process $e_x(\varepsilon^{-2}t)$ and to study the convergence of such a process in the limit $\varepsilon \to 0$.

Our main result is the following.

Theorem [3]. *The process* $\{e_x(\varepsilon^{-2}t)\}$ *defined by* (1) *with initial conditions* (2) *converges in law to a random process* $\{\mathcal{E}_x(t)\}$ *with values in* $\mathbb{R}_+^\Lambda$ *which satisfies the*

[1] By p_x^2 we mean $\langle \boldsymbol{i}(p_x), \boldsymbol{i}(p_x)\rangle_{G(q_x)} = \langle p_x, p_x\rangle_{\tilde G}$ where $\tilde G = \boldsymbol{i}_*(G)$.

stochastic differential equation

$$
\begin{aligned}
d\mathcal{E}_x &= \sum_{|x-y|=1} \boldsymbol{a}(\mathcal{E}_x, \mathcal{E}_y)dt + \sum_{|x-y|=1} \boldsymbol{b}(\mathcal{E}_x, \mathcal{E}_y)dB_{xy} \\
\mathcal{E}_x(0) &= E_x > 0,
\end{aligned} \tag{3}
$$

where B_{xy} are standard Brownian motions which are independent except that

$$B_{xy} = -B_{yx}.$$

The coefficients have the following properties: $\boldsymbol{b}$ is symmetric and $\boldsymbol{a}$ is antisymmetric; $\boldsymbol{b} \in \mathcal{C}^0([0,\infty)^2, \mathbb{R}_+)$ and $\boldsymbol{b}(a,b)^2 = abG(a,b)$ where $G \in \mathcal{C}^\infty((0,\infty)^2, \mathbb{R}_+) \cap \mathcal{C}^1((0,\infty)\times[0,\infty), \mathbb{R}_+)$ and $G(a,0) = A(2a)^{-\frac{3}{2}}$ for some $A > 0$. Moreover,

$$\boldsymbol{a} = \frac{1}{2}(\partial_{\mathcal{E}_x} - \partial_{\mathcal{E}_y})\boldsymbol{b}^2 + \frac{d-1}{2}(\mathcal{E}_x^{-1} - \mathcal{E}_y^{-1})\boldsymbol{b}^2. \tag{4}$$

In addition, (3) has a unique solution and the probability for one energy to reach zero in finite time is zero.

A direct computation shows that the measures $\prod_{x\in\Lambda} \mathcal{E}_x^{\frac{d}{2}-1} e^{-\beta\mathcal{E}_x}$ are invariant for the above process for each $\beta \in \mathbb{R}_+$.

It is interesting to note that the mesoscopic equation that we obtain seems to have some very natural and universal structure since it holds also when starting from different models. Indeed, essentially the same equation is obtained in [8] for a system of coupled nonlinear oscillators in the presence of an energy preserving randomness. In addition, such an equation is almost identical to the one studied in [10] apart from the necessary difference that the diffusion is a degenerate one. Indeed, since it describes the evolution of energies, and energies are positive, the diffusion coefficients must necessarily be zero when one energy is zero.

Since, due to the weak interaction, the energies vary very slowly, once the time is rescaled so that the energies evolve on times of order one all the other variables will evolve extremely fast. Thus our result is an example of *averaging theory* for slow-fast systems. Yet, in our case the currents have zero average which means that standard averaging theory (such as, e.g. [4]) cannot suffice. It is necessary to look at longer times when the fluctuation play a fundamental role. The study of such longer times can in principle be accomplished thanks to the theory developed in [2].

Unfortunately, the results in [2] do not apply directly and we are forced to a roundabout in order to obtain the wanted result. Not surprisingly, the trouble takes place at low energies. Indeed, when the energy of a particle is of order ε, then the interaction is no longer a small perturbation and we may loose the control of the microscopic dynamics. We have thus to investigate with particular care the behavior of the system at low energies. In particular, we prove that for each strictly positive initial condition and each $T, \epsilon > 0$ there exists δ such that the probability that the energy of any particle becomes less than δ in the time interval $[0, \varepsilon^{-2}T]$ is smaller that ϵ for each ε small enough. Hence, the unreachability of zero for the limiting process.

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Quenched invariance principle for random walks in balanced random environment

Ofer Zeitouni

(joint work with Xiaoqin Guo)

This work explores the extent to which the assumption of uniform ellipticity can be dropped in the non-reversible, but balanced, setupi of random walks in random environments. Not surprisingly, it turns out that some moment assumptions on the ellipticity constant suffice to yield the invariance principle in the ergodic environment setup, after some analytical effort has been expanded in obtaining a-priori estimates. What is maybe more surprising is that for i.i.d. environments, no assumptions of uniform ellipticity are needed at all.

Let $\mathcal{M}$ be the space of all probability measures on $V = \{v \in \mathbb{Z}^d : |v| \leq 1\}$, where $|\cdot|$ denotes the l^2-norm. We equip $\mathcal{M}$ with the weak topology on probability measures, which makes it into a Polish space, and equip $\Omega = \mathcal{M}^{\mathbb{Z}^d}$ with the induced Polish structure. Let $\mathcal{F}$ be the Borel σ-field of Ω and P a probability measure on $\mathcal{F}$.

A random *environment* is an element $\omega = \{\omega(x,v)\}_{x\in\mathbb{Z}^d, v\in V}$ of Ω with distribution P. The random environment is called i.i.d. if $\{\omega(x,\cdot)\}_{x\in\mathbb{Z}^d}$ are i.i.d. across the sites x under P. The random environment is called *balanced* if

$$P\{\omega(x,e_i) = \omega(x,-e_i) \text{ for all } i \text{ and all } x \in \mathbb{Z}^d\} = 1,$$

and *elliptic* if $P\{\omega(x,e) > 0$ for all $|e| = 1$ and all $x \in \mathbb{Z}^d\} = 1$.

The random walk in the random environment $\omega \in \Omega$ (RWRE) started at x is the canonical Markov chain $\{X_n\}$ on $(\mathbb{Z}^d)^{\mathbb{N}}$, with state space $\mathbb{Z}^d$ and law P_ω^x specified by

$$P_\omega^x\{X_0 = x\} = 1,$$
$$P_\omega^x\{X_{n+1} = y + v | X_n = y\} = \omega(y, v), \quad v \in V.$$

The probability distribution P_ω^x on $((\mathbb{Z}^d)^{\mathbb{N}}, \mathcal{G})$ is called the *quenched law*, where $\mathcal{G}$ is the σ-field generated by cylinder functions.

Define the canonical shifts $\{\theta^y\}_{y\in\mathbb{Z}^d}$ on $(\Omega, \mathcal{F})$ by $(\theta^y\omega)(x, v) = \omega(x + y, v)$. Assume that the system $(\Omega, \mathcal{F}, P)$ is ergodic with respect to the group of shifts $\{\theta^y\}$ and that the environment is balanced and elliptic.

Let $o = (0, \cdots, 0)$ denote the origin and

$$X_t^n := \frac{1}{\sqrt{n}} X_{\lfloor tn \rfloor} + \frac{tn - \lfloor tn \rfloor}{\sqrt{n}} (X_{\lfloor tn \rfloor + 1} - X_{\lfloor tn \rfloor}), \quad t \geq 0.$$

We say that the *quenched invariance principle* holds with nondegenerate covariances if for P-almost every $\omega \in \Omega$, the P_ω^o law of the path $\{X_t^n\}_{t\geq 0}$ converges weakly to a Brownian motion on $\mathbb{R}^d$ with covariance matrix $(a_i\delta_{ij})_{1\leq i,j\leq d}$, $a_i > 0$, as $n \to \infty$.

Lawler [3] proved a quenched invariance principle for random walks in a balanced random environment under the assumption that the random environment is *uniformly elliptic*, i.e.

$$P\{\omega(x, e) \geq \varepsilon_0 \text{ for all } |e| = 1\} = 1 \qquad \text{for some } \varepsilon_0 > 0.$$

The paper studies the extent to which the uniform ellipticity assumption can be dropped. Let

$$\varepsilon(x) = \varepsilon_\omega(x) := [\prod_{i=1}^{d} \omega(x, e_i)]^{\frac{1}{d}}. \tag{1}$$

Theorem 1. *Assume that the random environment is ergodic, elliptic and balanced.*

(i) *If $E\varepsilon(o)^{-p} < \infty$ for some $p > d \geq 2$, then the quenched invariance principle holds with a nondegenerate limiting covariance.*

(ii) *If $E[(1 - \omega(o, o))/\varepsilon(o)]^q < \infty$ for some $q > 2$ and $d \geq 3$, then the RWRE is transient P-almost surely.*

Recurrence for $d = 2$ under the condition $E\varepsilon(0)^{-p} < \infty$ follows from the quenched invariance principle and ergodicity by an unpublished argument of Kesten detailed in [5, Page 281]. That some integrability condition on the tail of $\varepsilon(o)$ is needed for part (i) to hold is made clear by the (non-Gaussian) scaling limits of random walks in Bouchaud's trap model, see [1]. In fact, it follows from that example that Theorem 1(i), or even an annealed version of the CLT, cannot hold in general with $p < 1$.

The proof of Theorem 1 is based on a sharpening of the arguments in [3, 4, 5]; in particular, refined versions of the maximum principle for walks in balanced environments and of a mean value inequality play a crucial role.

Next, the iid setup is considered.

Theorem 2. *Assume that the random environment is i.i.d., elliptic and balanced.*

(i) *If* $P\{\max_{|e|=1} \omega(o,e) \geq \xi_0\}=1$ *for some positive constant* ξ_0*, then the quenched invariance principle holds with a non-degenerate limiting covariance.*

(ii) *When* $d \geq 3$*, the RWRE is transient* P*-almost surely.*

The proofs combine percolation arguments with Theorem 1. Because the transience or recurrence of the random walks does not change if one considers the walk restricted to its jump times, one concludes, using Kesten's argument and the invariance principle, compare with Theorem 1, that for $d = 2$, a random walk in a balanced elliptic i.i.d. random environment is recurrent P-a.s.

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Bose-Einstein condensation and probabilistic methods for nonlinear Schrödinger equations

Kay Kirkpatrick

(joint work with Sourav Chatterjee)

Near absolute zero, a gas of quantum particles can condense into an unusual state of matter, called Bose-Einstein condensation, that behaves like a giant quantum particle. Recently we've been able to make the rigorous probabilistic connection between the physics of the microscopic dynamics and the mathematics of the macroscopic model, the cubic nonlinear Schrodinger equation (NLS).

I'll discuss new work with Sourav Chatterjee about a phase transition for invariant measures of the focusing NLS. Using techniques from probability theory, we show that the thermodynamics of the NLS are exactly solvable in dimensions three and higher. A number of explicit formulas are derived, with implications for some open questions about blow-up and statistical mechanics of the NLS.

1. Motivation

Studying the statistical mechanics of the nonlinear Schrödinger equation (NLS) is crucial to understanding certain physical phenomena that the NLS models: Bose-Einstein condensation, Langmuir waves in plasmas, nonlinear optics, and envelopes of water waves [8, 9, 11, 22]. Water wave systems in particular have interesting features called rogue waves: non-tsunami waves around sixty meters tall that occur on the open ocean and are thought to appear in finance and be replicable in BECs [1]. Rogue waves cannot be explained by a traditional Rayleigh distribution of wave heights and instead should be linked to statistical mechanical properties of the NLS.

One celebrated approach to understanding these phenomena is via the NLS's invariant measures, which ignore transient states and focus instead on the long-time probabilistic behavior of the system. Studying invariant measures was initiated in the seminal work of Lebowitz, Rose, and Speer [12], for the one-dimensional focusing NLS on $[0, L]$ with periodic boundary conditions:

$$i\partial_t \phi = -\partial_{xx}\phi - |\phi|^{p-1}\phi. \tag{1}$$

They conjectured that, at least with a mass cutoff, the Hamiltonian energy functional,

$$H(\phi) := \frac{1}{2}\int_0^L |\phi'(x)|^2 dx - \frac{1}{p+1}\int_0^L |\phi(x)|^{p+1} dx, \tag{2}$$

gives natural formal invariant (or Gibbs) measures with inverse temperature parameter β:

$$e^{-\beta H(\phi)} \prod_x d\phi(x).$$

Here the product has to be interpreted properly, and numerical simulations led them to make a conjecture about a possible phase transition in the invariant measures.

Bourgain continued the investigation [2] by proving that indeed, with an L^2 truncation, the following weighted Wiener measure is invariant for the periodic 1D focusing cubic ($p = 3$) NLS:

$$d\nu_{\beta,B} = Z(\beta, B)^{-1} \exp\left(\frac{\beta}{4}\|\phi\|_4^4\right) \mathbf{1}_{\{\|\phi\|_2^2 \le B\}} d\mu_\beta(\phi). \tag{3}$$

(Here μ_β is the Wiener measure with scale parameter β.) Simulations in [7] suggested that there is no phase transition, and then Rider, following on the works of McKean and Vaninsky [14, 15, 16], confirmed this for the 1D infinite-volume focusing NLS by proving that the thermodynamic limit is trivial [17, 18]. Bourgain also studied invariant measures of the 1D infinite-volume defocusing NLS [3] and of the 2D defocusing NLS [4] (see also the review article [5]), as did Tzvetkov [20].

Limitations of this approach became apparent at criticality, as when Brydges and Slade studied the 2D focusing NLS [6] and saw that the natural construction cannot produce an invariant measure for large coupling coefficients of the (critical) nonlinearity. We present an alternative approach to the supercritical NLS by

replacing the continuum with a discrete grid, where invariant measures are clear and amenable to techniques from discrete probability, and by letting the grid size go to zero eventually.

This approach has been investigated previously in the physics literature (see e.g., Rumpf [19] and references therein), as well as in mathematics (e.g., Weinstein [21]). However, it seems unlikely that this approach will yield useful information about the continuous system, because the mass and energy levels attained by the discrete system Gibbs measures are typically too low to be relevant for the continuous NLS. Still, the discrete system is interesting in its own right and the tractability of the Gibbs measures in dimensions ≥ 3, which is our main result, is remarkable.

2. Our results

We prove that in the discrete setting in dimensions three and higher, the natural invariant measure of the cubic focusing NLS is "exactly solvable" in the limit as the grid size goes to zero. The exact solvability is manifest in a number of ways:

- We are able to compute the limit of the partition function explicitly.
- Analysis of the partition function yields a first-order phase transition: if $\tilde{\mu}_{\beta,B}$ is now a discrete multidimensional version of (3), then for βB^2 less than an explicit critical threshold (approximately 2.455407, independent of dimension), a random function with $\tilde{\mu}_{\beta,B}$ as its probability law must be uniformly small with high probability. And if βB^2 is bigger than this threshold, then such a random function must have exactly one highly localized mode, sometimes called 'discrete breather'. This is the so-called anti-integrable regime (see e.g., MacKay and Aubry [13] and Weinstein [21]).
- Besides the partition function and the critical point, we are also able to compute the exact size of the breather and the energy density, and the limiting probability distributions of individual coordinates.
- Additionally, we show that the localized mode persists at one site for an exponentially long time.

The study of discrete breathers has a long history in the physics literature, with applications to crystals, biological macro-molecules, and optical waveguides (see [21] and references therein). It is of interest to understand when these structures occur, and moreover, how prevalent and concentrated they are. Flach, Kladko, and MacKay made conjectures about positive minimum energies necessary for discrete breathers for (roughly) supercritical lattice models, e.g., discrete cubic NLS in 3D [10]. Weinstein proved one of the conjectures, that in the supercritical case, ground state standing waves exist if and only if the total power is larger than some strictly positive threshold [21].

Putting this in the context of statistical mechanics, it was previously known that there are localized modes at zero temperature ($\beta = \infty$) for mass going to infinity; we show that such modes exist for finite mass and energy and non-zero temperature – and not only exist, but are typical.

Our results may be connected to the phenomenon of blow-up in supercritical NLS, suggesting that blow-up is typical in 3D, at least for the discrete NLS – and we address how energy and mass concentrate.

The two-dimensional periodic analog is open and should be more interesting from the probabilistic point of view.

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Approximations to the Stochastic Burgers Equation

Jochen Voss
(joint work with Martin Hairer)

This talk studies several finite difference schemes for stochastic partial differential equations (SPDEs) of the form

$$\partial_t u(x,t) = \nu\,\partial_x^2 u + g(u)\,\partial_x u + \sigma\,\xi(x,t), \qquad x \in [0,2\pi], \quad t \geq 0 \tag{1}$$

where $\nu, \sigma > 0$ are constants. In this equation, ξ denotes space-time white noise, that is the centred, distribution-valued Gaussian random variable such that $\mathbb{E}\big(\xi(x,t)\xi(y,s)\big) = \delta(t-s)\delta(x-y)$. This equation is equipped with periodic boundary conditions and we consider solutions u taking values either in $\mathbb{R}$ or in $\mathbb{R}^d$ (in which case g is matrix-valued). For $d = 1$ and $g(u) = -u$, the SPDE (1) is known as the *viscous stochastic Burgers equation*.

We study the effect of finite difference discretisations on this equation and in particular show that many discretisation schemes which look 'reasonable' at a first glance are in fact not even consistent.

In contrast to the situation presented here, the solution of the stochastic Burgers equation (or rather the integrated process which solves the corresponding KPZ equation) arising as the fluctuations process in the weakly asymmetric exclusion process [3, 4] is driven by the *derivative* of space-time white noise. As a consequence, it does *not* solve an SPDE that is well-posed in the classical sense and can currently only be defined via the Hopf-Cole transform. Such a process is even rougher (by "one derivative") than the process considered here and one would expect the 'wrong' numerical approximation schemes to fail there in an even more spectacular way.

An extended version of the results presented here is submitted for publication as [1]. Some of the conjectures presented in the talk have in the meantime been proved in [2].

To discretise the SPDE (1) we consider the approximating equation

$$\partial_t u(x,t) = \nu\,\partial_x^2 u + g(u)\,D_\delta u + \sigma\,\xi(x,t), \tag{2}$$

where we define the approximate derivative D_δ by

$$D_\delta u(x) = \frac{u(x + a\delta) - u(x - b\delta)}{(a+b)\delta}$$

for some $a, b \geq 0$. In the absence of the noise term ξ, it would be a standard result in numerical analysis that the solution of (2) converges to the solution of (1) as $\delta \downarrow 0$. Here we will argue that, if ξ is taken to be space-time white noise, the limit of (2) as $\delta \to 0$ depends on the values a and b and is equal to (1) only if either g is constant or $a = b$. More precisely, we conjecture that, as $\delta \to 0$, the solution of (2) converges to the solution of

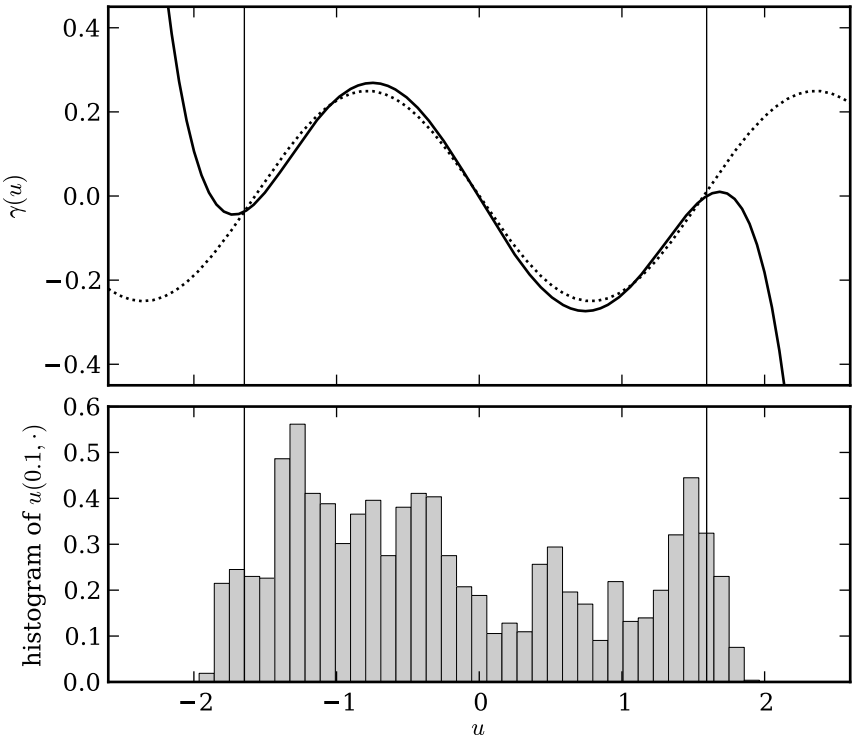
$$\partial_t u(x,t) = \nu\,\partial_x^2 u + g(u)\,\partial_x u - \frac{\sigma^2}{4\nu}\frac{a-b}{a+b}\,g'(u) + \sigma\,\xi(x,t). \tag{3}$$

Currently, this result is mostly supported by numerical evidence. To validate the conjecture, we perform numerical experiments like the following: We numerically solve the "approximating" equation (2) until a fixed time T and store the solution u. Then, for the same instance of the noise and a function $\gamma\colon \mathbb{R} \to \mathbb{R}$, we use a known-good discretisation scheme to solve the "corrected" SPDE

$$\partial_t u_\gamma(x,t) = \nu\,\partial_x^2 u_\gamma + g(u_\gamma)\,\partial_x u_\gamma - \frac{\sigma^2}{4\nu}\frac{a-b}{a+b}\,\gamma(u_\gamma) + \sigma\,\xi(x,t). \tag{4}$$

Finally, we numerically optimise the correction term γ in (4) (using some parametric form) to minimise the distance $\|u(T,\,\cdot\,) - u_\gamma(T,\,\cdot\,)\|_2$. If the conjecture is correct, one would expect the minimum to be attained for a function γ which is close to the predicted correction term g' from (3).

The result of a simulation for the one-dimensional example $g(u) = \sin(u)^2$, using the right-sided discretisation $a = 1, b = 0$, is shown in the following figure (taken from [1]):



Here we use a fifth-order polynomial for the correction term γ. The top panel shows the resulting fitted correction term $-\sigma^2\gamma/4\nu$ (full line) together with the conjectured correction term $-\sigma^2 g'(u)/4\nu$ (dotted line). To give an idea which range of arguments is actually used in the computation, the lower panel shows the histogram of the values of u (the vertical bars indicate the 5% and 95% quantiles).

In the interval between these quantiles, the graph shows a good fit between the numerically determined and conjectured correction terms, thus giving support to the conjecture presented here.

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Crossover to the KPZ equation

Patrícia Gonçalves, Milton Jara

1. Introduction

We consider the one-dimensional weakly asymmetric simple exclusion process (wasep), i.e. our microscopic dynamics is given by a stochastic lattice gas with hard core exclusion. This process arises as a simple model for the growing of random interfaces. The presence of a weak asymmetry in the microscopic dynamics, breaks down the detailed balance condition, which implies the system to exhibit a non trivial behavior even in the stationary situation. The dynamical scaling exponent has been established by the physicists as being $z = 3/2$ and one of the challenging problems is to establish the limit distribution for the density and the current of particles, see [9]. We take the process with asymmetry given by $an^{2-\gamma}$ and we want to analyze the effect of strengthening the asymmetry in the limit distribution of the density field.

The wasep was studied in [2] and in [3], for $\gamma = 1$; and in [1] for $\gamma = 1/2$. The equilibrium density fluctuations (for $\gamma = 1$) are given by an Ornstein-Uhlenbeck process. For $\gamma = 1/2$ (which corresponds to strength asymmetry n^z), [3] used the Cole-Hopf transformation to derive the non-equilibrium fluctuations of the current of particles. By removing the drift to the system, there is no effect of the strength of the asymmetry on the limit distribution of the density field. By strengthening the asymmetry the limit distribution "feels" the effect of this strengthening, by developing a non linear term in the limit distribution. In this case the limit density field is a solution of the Kardar-Parisi-Zhang (KPZ) equation. The KPZ equation was proposed in [8] to model the growth of random interfaces. Denoting by h_t the height of the interface, this equation reads as $\partial_t h = D\Delta h + a(\nabla h)^2 + \sigma \mathcal{W}_t$, where D, a, σ are constants related to the thermodynamical properties of the interface and $\mathcal{W}_t$ is a space-time white noise. According to z, a non-trivial behavior occurs under re-scaling $h_n(t,x) = n^{-1/2} h(tn^{3/2}, x/n)$. This means, roughly speaking, that in our case, for $\gamma = 1/2$ a non trivial behavior is expected even in the stationary situation and in this case, the model belongs to the universality class of

the KPZ equation. Here we provide the characterization of the transition from the Edwards-Wilkinson class to the KPZ class, for the wasep. We prove that the transition depends on the strength of the asymmetry without having any other intermediate state and by establishing precisely the strength in order to have the crossover. From this result we obtain the crossover regime for the current of particles across a characteristic. Our method relies on a stronger Boltzmann-Gibbs Principle introduced in [4] and is robust enough in the sense that it can be applied for general interacting particle systems.

2. Equilibrium fluctuations

Let η_t be the wasep evolving on $\mathbb{Z}$ and with space state $\Omega = \{0,1\}^{\mathbb{Z}}$. In this process, each particle waits a mean one exponential time after which jumps to an empty neighboring site according to a transition rate that has a weak asymmetry to the right. The process is taken on the diffusive time scale n^2 so that $\eta_t^n = \eta_{tn^2}$ and the transition rate to the right is $1/2 + 1/n^\gamma$ and to the left is $1/2 - 1/n^\gamma$. We notice that if we decrease the value of γ, this corresponds to speeding up the asymmetric part of the dynamics on longer time scales as $n^{2-\gamma}$. A stationary measure for this process is the Bernoulli product measure on Ω of parameter ρ, that we denote by $\{\nu_\rho : \rho \in [0,1]\}$.

We denote by π_t^n the empirical measure as the positive measure in $\mathbb{R}$ defined by $\pi_t^n(dx) = \frac{1}{n}\sum_{x\in\mathbb{Z}} \eta_t^n(x)\delta_{x/n}(dx)$, where for $u \in \mathbb{R}$, δ_u is the Dirac measure at u. We are interested in establishing the fluctuations of the empirical measure from the stationary state ν_ρ. From now on, fix a density ρ and take η_t^n moving in a reference frame with constant velocity given by $(1-2\rho)n^{2-\gamma}$. Let $\mathcal{Y}_t^n$ be the *density fluctuation field* on $H \in \mathcal{S}(\mathbb{R})$ as:

$$\mathcal{Y}_t^{n,\gamma}(H) = \frac{1}{\sqrt{n}} \sum_{x\in\mathbb{Z}} T_t^\gamma H_x \big(\eta_t^n(x) - \rho\big), \tag{1}$$

where $T_t^\gamma H(\cdot) = H(\cdot - (1-2\rho)tn^{1-\gamma})$. From now on we use the denotation $H_x = H(x/n)$.

For $\gamma = 1$, it is not hard to show that $\{\mathcal{Y}_t^{n,\gamma}; n \in \mathbb{N}\}$ converges to $\mathcal{Y}_t^\gamma$ solution of the Ornstein-Uhlenbeck equation:

$$d\mathcal{Y}_t^\gamma = \frac{1}{2}\Delta \mathcal{Y}_t^\gamma dt + \sqrt{\chi(\rho)}\nabla d\mathcal{W}_t, \tag{2}$$

where $\mathcal{W}_t$ is a space-time white noise. So, for $\gamma = 1$ the system belongs to the Edwards-Wilkinson universality class.

In order to see the effect of the asymmetry in the limit density field we increment the strength of the asymmetry by decreasing the value of γ. According to [3], the effect of the asymmetry is presented in the limit field when $\gamma = 1/2$ and in that case $\mathcal{Y}_t^\gamma$ has a very different qualitatively behavior from the one obtained for $\gamma = 1$, namely the solution of (2).

In this work, we characterize the limit field $\mathcal{Y}_t^\gamma$ for the intermediate state, i.e. for $\gamma \in (1/2, 1)$, by showing that for this range of the parameter it also solves

(2). As a consequence, for $\gamma \in (1/2, 1)$ the system still belongs to the Edwards-Wilkinson universality class. The idea of the proof of last result is to use Dynkin's formula so that

$$\mathcal{Y}_t^{n,\gamma}(H) = M_t^{n,\gamma}(H) + \mathcal{Y}_0^{n,\gamma}(H) + \mathcal{I}_t^{n,\gamma}(H) + \mathcal{A}_t^{n,\gamma}(H)$$

where $M_t^{n,\gamma}(H)$ is a martingale with respect to the natural filtration and the integral terms $\mathcal{I}_t^{n,\gamma}(H)$ and $\mathcal{A}_t^{n,\gamma}(H)$ are given (respectively) by

$$\int_0^t \frac{1}{2\sqrt{n}} \sum_{x\in\mathbb{Z}} \Delta^n T_s^\gamma H_x(\eta_s^n(x) - \rho) ds,$$

$$\int_0^t \frac{n^{1-\gamma}}{\sqrt{n}} \sum_{x\in\mathbb{Z}} \nabla^n T_s^\gamma H_x \Big\{ \eta_s^n(x)(1 - \eta_s^n(x+1)) - \chi(\rho) - (1-2\rho)(\eta_s^n(x) - \rho) \Big\} ds,$$

and Δ^n, ∇^n are the discrete laplacian and the discrete derivative, respectively.

Now we analyze the asymptotic behavior of the martingale and the integral terms above. The hard programme of this approach is to analyze the limit of the integral term $\mathcal{A}_t^{n,\gamma}(H)$. For that purpose, we derive a *stronger Boltzmann-Gibbs principle* as in Corollary 7.4 of [4], which implies that $\mathcal{A}_t^{n,\gamma}(H)$ vanishes as $n \to \infty$. In [4] the result was obtained for the symmetric simple exclusion but is also true for the process we consider here. In fact that result can be stated as: if $\psi : \Omega \to \mathbb{R}$ is a local function, $\gamma \in (1/2, 1)$ and if $H \in \mathcal{S}(\mathbb{R})$ then

$$\int_0^t \frac{n^{1-\gamma}}{\sqrt{n}} \sum_{x\in\mathbb{Z}} H_x \Big\{ \tau_x \psi(\eta_s^n) - E_{\nu_\rho}[\psi(\eta)] - \partial_\rho E_{\nu_\rho}[\psi(\eta)](\eta_s^n(x) - \rho) \Big\} ds, \tag{3}$$

vanishes as $n \to \infty$ in $L^2(\mathbb{P}_{\nu_\rho})$. Last result together with some computations on the quadratic variation of the martingale, gives us that $\mathcal{Y}_t^\gamma$ is solution of (2).

From the previous arguments we have seen that if we want to see the effect of the asymmetry in the limit field, we go towards decreasing the value of γ, which, as mentioned above, corresponds to speeding up the asymmetric part of the dynamics. This is in agreement with the result of [7] which says that for $\gamma = 1/2$ that is indeed the case.

Recently in [5], it was shown that for $\gamma = 1/2$, $\{\mathcal{Y}_t^{n,\gamma}; n \in \mathbb{N}\}$ is tight and any limit point is an energy solution of the KPZ equation:

$$d\mathcal{Y}_t^\gamma = \frac{1}{2}\Delta \mathcal{Y}_t^\gamma dt + \nabla(\mathcal{Y}_t^\gamma)^2 dt + \sqrt{\chi(\rho)}\nabla d\mathcal{W}_t. \tag{4}$$

Since we are in the presence of a stronger asymmetry, the result in (3) is no longer true. In order to establish last result, a *second order Boltzmann-Gibbs principle* was derived in [5]. The ingredients invoked in order to derive this stronger replacement is a multi-scale argument introduced in [4] combined with some fundamental features of the model: as a sharp spectral gap bound for the dynamics restricted to finite boxes, plus a second order expansion on the equivalence of ensembles.

The results beyond the hydrodynamic time scale, the crossover at $\gamma = 1/2$ and the KPZ class, are in fact true for a general class of weakly asymmetric exclusion processes see [6].

As a consequence of last result and by relating the current of particles with the density field, we can obtain the crossover on the fluctuations of the current and we obtain that for $\gamma \in (1/2, 1)$ the limit is Gaussian, while for $\gamma = 1/2$ the limit is written in terms of the KPZ equation. For details we refer the interested reader to [6].

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Universality of KPZ equation and renormalization techniques in interacting particle systems

Milton Jara, Patrícia Gonçalves

1. Introduction

In the middle eighties, Kardar, Parisi and Zhang in [7] proposed a phenomenological model for the stochastic evolution of the profile of a growing interface $h_t(x)$. The Kardar, Parisi and Zhang (KPZ) equation has the following form in one dimension: $\partial_t h = D\Delta h + a\big(\nabla h\big)^2 + \sigma \mathcal{W}_t$, where $\mathcal{W}_t$ is a space-time white noise and the constants D, a, σ are related to some thermodynamic properties of the interface. The quantity $h_t(x)$ represents the *height* of the interface at the point $x \in \mathbb{R}$. From a mathematical point of view, this equation is ill-posed, since the solutions are expected to look locally like a Brownian motion, and in this case the nonlinear term does not make sense, at least not in a classical sense.

In dimension $d = 1$, a conservative version of the KPZ equation can be obtained by defining $\mathcal{Y}_t = \nabla h_t$: $\partial_t \mathcal{Y}_t = D\Delta \mathcal{Y}_t + a\nabla \mathcal{Y}_t^2 + \sigma \nabla \mathcal{W}_t$. This equation has spatial white noise as an invariant solution. In this case is even clearer that some procedure is needed in order to define $\mathcal{Y}_t^2$ in a proper way. It is widely believed in the physics community that the KPZ equation governs the large-scale properties of

one-dimensional, weakly asymmetric, conservative systems in great generality. The microscopic details of each model should only appear through the values of the constants D, a and σ. In this work we provide a new approach which is robust enough to apply for a wide family of one-dimensional weakly asymmetric systems. As a stochastic partial differential equation, the main problem with the KPZ equation is the definition of the square $\mathcal{Y}_t^2$.

Our first contribution is the notion of *energy solutions* of the KPZ equation, which we introduce in order to state in a rigorous way our second contribution. Take a one-dimensional, weakly asymmetric conservative particle system and consider the rescaled space-time fluctuations of the density field $\mathcal{Y}_t^n$. When the strength of the asymmetry is of order $1/\sqrt{n}$, we prove that any limit point of $\mathcal{Y}_t^n$ is an energy solution of the KPZ equation. The only ingredients needed in order to prove this result are a sharp estimate on the spectral gap of the dynamics of the particle system restricted to finite boxes and a strong form of the equivalence of ensembles for the stationary distribution. Therefore, our approach works, modulo technical modifications, for any one-dimensional, weakly asymmetric conservative particle system satisfying these two properties. As a consequence, we say that energy solutions of the KPZ equation are *universal*, in the sense that they arise as the scaling limit of the density in one-dimensional, weakly asymmetric conservative systems satisfying fairly general, minimal assumptions.

In order to prove this result, we introduce a new mathematical tool, which we call *second-order Boltzmann-Gibbs principle*. The usual Boltzmann-Gibbs principle, introduced in [1] and proved in [3] in our context, basically states that the space-time fluctuations of any field associated to a conservative model can be written as a linear functional of the density field $\mathcal{Y}_t^n$. A stronger Boltzmann-Gibbs Principle was derived in [4], which implies that for strength asymmetry less than $1/\sqrt{n}$, the limit field falls into the Edwars-Wilkinson universality class [5]. Our second-order Boltzmann-Gibbs principle states that the first-order correction of this limit is given by a singular, quadratic functional of the density field. It has been proved that in dimension $d \geq 3$, this first order correction is given by a white noise [2]. As a consequence for strength asymmetry $1/\sqrt{n}$ the system fall into the KPZ universality class [6].

2. The results

2.1. **The process.** Let $\Omega = \{0,1\}^{\mathbb{Z}}$ be the state space of a continuous-time Markov chain η_t which we will define as follows. We say that a function $f : \Omega \to \mathbb{R}$ is *local* if there exists $R = R(f) > 0$ such that $f(\eta) = f(\xi)$ for any $\eta, \xi \in \Omega$ such that $\eta(x) = \xi(x)$ whenever $|x| \geq R$. Let $c : \Omega \to \mathbb{R}$ be a non-negative function. We assume the following conditions on c:

i) *Ellipticity*: There exists $\epsilon_0 > 0$ such that $\epsilon_0 \leq c(\eta) \leq \epsilon_0^{-1}$ for any $\eta \in \Omega$.
ii) *Finite range*: The function $c(\cdot)$ is local.
iii) *Reversibility*: For any $\eta, \xi \in \Omega$ such that $\eta(x) = \xi(x)$ whenever $x \neq 0, 1$, $c(\eta) = c(\xi)$.

For any $x \in \mathbb{Z}$ let $\tau_x f(\eta) = f(\tau_x \eta)$ for any $\eta \in \Omega$, where $\tau_x \eta$ denotes the space translation by x. We will also assume a fourth condition, which is the most restrictive one:

iv) *Gradient condition*: There exists a local function $h : \Omega \to \Omega$ such that $c(\eta)(\eta(1) - \eta(0)) = \tau_1 h(\eta) - h(\eta)$ for any $\eta \in \Omega$.

In this work, we consider the Markov process $\{\eta_t^n; t \geq 0\}$ generated by the operator L_n acting over local functions $f : \Omega \to \mathbb{R}$ as

$$L_n f(\eta) = n^2 \sum_{x \in \mathbb{Z}} \tau_x c(\eta) \big\{ p_n \eta(x)(1 - \eta(x+1)) + q_n \eta(x+1)(1 - \eta(x)) \big\} \nabla_{x,x+1} f(\eta),$$

where $n \in \mathbb{N}$, $\nabla_{x,x+1} f(\eta) = f(\eta^{x,x+1}) - f(\eta)$, p_n and q_n are non-negative constants such that $p_n + q_n = 1$ (and $p_n - q_n = a/\sqrt{n}$ with $a \neq 0$) and $\eta^{x,x+1}$ is given by $\eta^{x,x+1}(x) = \eta(x+1)$, $\eta^{x,x+1}(x+1) = \eta(x)$, otherwise $\eta^{x,x+1}(z) = \eta(z)$.

For $\rho \in [0,1]$ let ν_ρ be the Bernoulli product measure in Ω of parameter ρ. Under condition iii), the measures $\{\nu_\rho; \rho \in [0,1]\}$ are invariant and reversible with respect to the evolution of η_t^n. Under condition i), these measures are also ergodic with respect to the evolution of η_t^n.

2.2. **Equilibrium Fluctuations.** In this work we are interested in a central limit theorem for the density of particles starting from the equilibrium state ν_ρ. Let us fix a density $\rho \in (0,1)$ and let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of test functions and let $\mathcal{S}'(\mathbb{R})$ be the space of tempered distributions in $\mathbb{R}$, which corresponds to the topological dual of $\mathcal{S}(\mathbb{R})$. The fluctuation field $\{\mathcal{Y}_t^n; t \geq 0\}$ is defined as the $\mathcal{S}'(\mathbb{R})$-valued process given by

$$\mathcal{Y}_t^n(G) = \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} \big(\eta_t^n(x) - \rho\big) G(x/n - v(\rho) t n^{1/2})$$

where $v(\rho) = a\beta'(\rho)$. Our main result in this work says that the sequence of processes $\{\{\mathcal{Y}_t^n; t \in [0,T]\}; n \in \mathbb{N}\}$ is tight in $\mathcal{D}([0,T], \mathcal{S}'(\mathbb{R}))$ and any limit point is a stationary energy solution of the KPZ equation:

$$d\mathcal{Y}_t = \frac{\varphi'(\rho)}{2} \Delta \mathcal{Y}_t dt - \frac{a\beta''(\rho)}{2} \nabla \mathcal{Y}_t^2 dt + \sqrt{\chi(\rho)\varphi'(\rho)} \nabla d\mathcal{W}_t. \tag{1}$$

2.3. **Energy solutions of the KPZ equation.** The space $\mathcal{C}([0,T], \mathcal{S}'(\mathbb{R}))$ is the space on which the solutions of the KPZ equation (1) will live. For $\epsilon > 0$ we define $i_\epsilon(x) : \mathbb{R} \to \mathbb{R}$ by $i_\epsilon(x)(y) = \epsilon^{-1} \mathbf{1}(x < y \leq x + \epsilon)$. We say that a process $\{\mathcal{Y}_t; t \in [0,T]\}$ with trajectories in $\mathcal{C}([0,T], \mathcal{S}'(\mathbb{R}))$ and adapted to some natural filtration $\{\mathcal{F}_t; t \in [0,T]\}$ is a *weak solution* of (1) if:

i) There exists a process $\{\mathcal{A}_t; t \in [0,T]\}$ with trajectories in $\mathcal{C}([0,T], \mathcal{S}'(\mathbb{R}))$ and adapted to $\{\mathcal{F}_t; t \in [0,T]\}$ such that for any $G \in \mathcal{S}(\mathbb{R})$,

$$\lim_{\epsilon \to 0} \int_0^t \int_{\mathbb{R}} \mathcal{Y}_s(i_\epsilon(x))^2 \frac{G(x+\epsilon) - G(x)}{\epsilon} dx ds = \mathcal{A}_t(G). \tag{2}$$

ii) For any function $G \in \mathcal{S}(\mathbb{R})$ the process

$$M_t(G) = \mathcal{Y}_t(G) - \mathcal{Y}_0(G) - \frac{\varphi'(\rho)}{2}\int_0^t \mathcal{Y}_s(G'')ds - \frac{a\beta''(\rho)}{2}\mathcal{A}_t(G) \tag{3}$$

is a martingale of quadratic variation $\chi(\rho)\varphi'(\rho)t\int G'(x)^2dx$.

Now we introduce a stronger notion of solution, which captures well some of the particularities of the solutions of (1). Let $\{\mathcal{Y}_t; t \in [0,T]\}$ be a weak solution of (1). For $0 \le s < t \le T$, let us define the fields

$$\mathcal{I}_{s,t}(G) = \int_s^t \mathcal{Y}_u(G'')du,$$

$$\mathcal{A}_{s,t}(G) = \mathcal{A}_t(G) - \mathcal{A}_s(G),$$

$$\mathcal{A}^\epsilon_{s,t}(G) = \int_s^t \int_{\mathbb{R}} \mathcal{Y}_u(i_\epsilon(x))^2 \frac{G(x+\epsilon) - G(x)}{\epsilon}dxdu.$$

We say that $\{\mathcal{Y}_t; \in [0,T]\}$ is an *energy solution* of (1) if there exists a constant $\kappa > 0$ such that

$$E[\mathcal{I}_{s,t}(G)^2] \le \kappa(t-s)\int G'(x)^2dx$$

and

$$E[(\mathcal{A}_{s,t}(G) - \mathcal{A}^\epsilon_{s,t}(G))^2] \le \kappa\epsilon(t-s)\int G'(x)^2dx$$

for any $0 \le s < t \le T$, any $\epsilon \in (0,1)$ and any $G \in \mathcal{S}(\mathbb{R})$. We say that a weak solution $\{\mathcal{Y}_t; t \in [0,T]\}$ is a stationary solution if for any $t \in [0,T]$ the $\mathcal{S}'(\mathbb{R})$-valued random variable $\mathcal{Y}_t$ is a white noise of variance $\chi(\rho)$.

An immediate consequence of last result is the existence of weak solutions of the KPZ equation. Let $\mathcal{Y}_t$ be a limit point of $\mathcal{Y}^n_t$. Since the measure ν_ρ is invariant under the evolution of η^n_t, for any fixed time $t \in [0,T]$ the $\mathcal{S}'(\mathbb{R})$-valued random variable $\mathcal{Y}_t$ is a white noise of variance $\chi(\rho)$. As a consequence of the previous result we obtain that for any limit point $\{\mathcal{Y}_t; t \in [0,T]\}$ of $\{\{\mathcal{Y}^n_t; t \in [0,T]\}; n \in \mathbb{N}\}$, there is a finite constant $c > 0$ such that the process $\{\mathcal{A}_t; t \in [0,T]\}$ defined as above satisfies the moment bound $E[\mathcal{A}_{s,t}(G)^2] \le c|t-s|^{3/2}\int G'(x)^2dx$. Moreover, for any $\gamma \in (0,1/4)$ and any $G \in \mathcal{S}(\mathbb{R})$ the real-valued process $\{\mathcal{Y}_t(G); t \in [0,T]\}$ is Hölder-continuous of order γ.

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Scaling exponents for certain 1+1 dimensional directed polymers

Timo Seppäläinen

(joint work with Márton Balázs, Jeremy Quastel, and Benedek Valkó)

This talk describes three 1+1 dimensional models of a directed polymer in a random environment for which the precise values of the two fluctuation exponents ζ and χ have been computed. The directed polymer is a statistical mechanical model of a random walk path in a random potential. The two fluctuation exponents are defined somewhat informally as follows:

- Fluctuations of the path $\{x(t) : 0 \le t \le n\}$ are of order n^{ζ}.
- Fluctuations of the partition function $\log Z_n$ are of order n^{χ}.

The conjectured values for 1+1 dimensions are $\zeta = 2/3$ and $\chi = 1/3$. Next we present a brief description of some results for the three models.

1. Log-gamma polymer

The 1+1 dimensional log-gamma polymer is a directed path from the origin $(0, 0)$ to a fixed endpoint (m, n). Here are the basic notational elements of the model.

- The set of admissible paths is $\Pi_{m,n} = \{\text{up-right paths } x_\cdot = (x_k)_{k=0}^{m+n} \text{ from } (0,0) \text{ to } (m,n)\}$
- The environment $(Y_{i,\,j} : (i,j) \in \mathbb{Z}_+^2)$ consists of independent weights $Y_{i,j}$ with joint distribution $\mathbb{P}$.
- The quenched polymer measure on paths and the partition function are given by

$$Q_{m,n}(x_\cdot) = \frac{1}{Z_{m,n}} \prod_{k=1}^{m+n} Y_{x_k} \quad \text{and} \quad Z_{m,n} = \sum_{x_\cdot \in \Pi_{m,n}} \prod_{k=1}^{m+n} Y_{x_k}.$$

- The averaged measure is $P_{m,n}(x_\cdot) = \mathbb{E}Q_{m,n}(x_\cdot)$.

The key assumption is on the distributions of the weights. Fix two parameters $0 < \theta < \mu < \infty$. Then the weights are reciprocals of gamma distributed random variables, with these parameters for $i, j \ge 1$: $Y_{i,0}^{-1} \sim \text{Gamma}(\theta)$, $Y_{0,j}^{-1} \sim \text{Gamma}(\mu - \theta)$, and $Y_{i,j}^{-1} \sim \text{Gamma}(\mu)$.

For the asymptotic results the endpoint (m, n) of the polymer is taken to infinity in a particular characteristic direction as $N \nearrow \infty$:

$$|\, m - N\Psi_1(\mu - \theta)\,| \le \gamma N^{2/3} \quad \text{and} \quad |\, n - N\Psi_1(\theta)\,| \le \gamma N^{2/3}$$

for some fixed constant γ. Under these assumptions we have the following fluctuation bounds that verify the conjectured values for the exponents.

Theorem 1. *There exist constants $0 < C_1, C_2 < \infty$ such that, for $N \geq 1$,*

$$C_1 N^{2/3} \leq \mathbb{V}\mathrm{ar}(\log Z_{m,n}) \leq C_2 N^{2/3}.$$

Let $v_0(j)$ and $v_1(j)$ denote the left- and rightmost points of the path on the horizontal line with ordinate j:

$$v_0(j) = \min\{i \in \{0, \ldots, m\} : \exists k \text{ such that } x_k = (i, j)\}$$

and

$$v_1(j) = \max\{i \in \{0, \ldots, m\} : \exists k \text{ such that } x_k = (i, j)\}.$$

Theorem 2. *Let $0 < \tau < 1$. Then there exist constants $C_1, C_2 < \infty$ such that for $N \geq 1$ and $b \geq C_1$,*

$$P\big\{v_0(\lfloor \tau n \rfloor) < \tau m - bN^{2/3} \;\; \textit{or} \;\; v_1(\lfloor \tau n \rfloor) > \tau m + bN^{2/3}\big\} \leq C_2 b^{-3}.$$

Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\overline{\lim_{N\to\infty}} \, P\{ \; \exists k \textit{ such that } |\, x_k - (\tau m, \tau n)| \leq \delta N^{2/3} \; \} \leq \varepsilon.$$

Further details can be found in [5].

2. Polymer in a Brownian environment

The results for the polymer in a Brownian environment are analogous to those for the log-gamma polymer. We only indicate what that model is like by giving the partition function. (The model was introduced and first studied by [4].) The environment is given by independent Brownian motions $B_1, B_2, \ldots, B_n$. The partition function (without special boundary conditions) is defined by

$$Z_{n,t}(\beta) = \int_{0<s_1<\cdots<s_{n-1}<t} \exp\big[\beta\big(B_1(s_1) \; + \; B_2(s_2) - B_2(s_1) \; + \\ + \; B_3(s_3) - B_3(s_2) \; + \; \cdots + \; B_n(t) - B_n(s_{n-1})\big)\big] \, ds_{1,n-1}.$$

Further details can be found in [6].

3. Hopf-Cole solution of the Kardar-Parisi-Zhang equation

The Kardar-Parisi-Zhang (KPZ) equation for the height function $h(t, x)$ of a 1+1 dimensional interface is the formal stochastic partial differential equation

$$h_t \; = \; \tfrac{1}{2}\, h_{xx} \; - \; \tfrac{1}{2}\, (h_x)^2 \; + \; \dot{W}$$

where $\dot{W}$ is Gaussian space-time white noise. We take the initial height $h(0, x)$ as a two-sided Brownian motion for $x \in \mathbb{R}$. Giving the KPZ equation a rigorous meaning has been a challenge. The formal transform $Z = \exp(-h)$ satisfies a stochastic heat equation $Z_t \; = \; \frac{1}{2}\, Z_{xx} \; - \; Z\, \dot{W}$ that can be solved rigorously and uniquely in a class of functions that includes the Brownian initial condition $Z(0, x) = e^{-B(x)}$. It is also a fact that $Z(t, x) > 0$, so one can define the Hopf-Cole solution of the KPZ equation by $h \; = \; -\log Z$. For this solution of KPZ we have these variance bounds:

Theorem 3. *There exist constants $0 < c, C_1, C_2 < \infty$ such that, for $t \geq c$,*

$$C_1 t^{2/3} \leq \ \mathit{Var}(h(t,0)) \leq C_2 t^{2/3}.$$

The key points behind this theorem are (i) that the KPZ height function h can be realized as the weak limit of the scaled height of a weakly asymmetric simple exclusion process (WASEP) [3] and (ii) that analogous variance bounds can be established for the exclusion process [2]. Here is a description of the WASEP height function $\zeta_\varepsilon(t,x)$. The increments satisfy $\zeta_\varepsilon(x+1) - \zeta_\varepsilon(x) = \pm 1$ and the jump rates are

$$\zeta_\varepsilon(x) \longrightarrow \begin{cases} \zeta_\varepsilon(x)+2 & \text{with rate } \frac{1}{2} + \sqrt{\varepsilon} \ \text{ if } \zeta_\varepsilon(x) \text{ is a local minimum} \\ \zeta_\varepsilon(x)-2 & \text{with rate } \frac{1}{2} \qquad\quad \text{if } \zeta_\varepsilon(x) \text{ is a local maximum.} \end{cases}$$

The initial height function is taken as $\zeta_\varepsilon(0,x+1) - \zeta_\varepsilon(0,x) = \pm 1$ with probability $1/2$. The scaled height is

$$h_\varepsilon(t,x) \ = \ \varepsilon^{1/2} \left(\zeta_\varepsilon(\varepsilon^{-2}t, [\varepsilon^{-1}x]) \ - \ v_\varepsilon t \right).$$

Theorem 4. [3] *As $\varepsilon \searrow 0$, $h_\varepsilon \Rightarrow h$ in the path space of height processes.*

From coupling arguments for WASEP we can deduce these variance bounds:

Theorem 5. *There exist constants $0 < c, C_1, C_2 < \infty$ such that, for $t \geq c$ and all $\varepsilon \in (0, 1/4)$,*

$$C_1 t^{2/3} \ \leq \ \ \mathit{Var}(h_\varepsilon(t,0)) \ \leq \ C_2 t^{2/3}.$$

These ingredients and some analysis lead to the bounds for h. Further details in [1].

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How the KPZ equation fits into the KPZ universality class

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(joint work with Gideon Amir, Jeremy Quastel)

We consider the solution to the stochastic heat equation $\partial_T \mathcal{Z} = \frac{1}{2}\partial_X^2 \mathcal{Z} - \mathcal{Z}\dot{\mathcal{W}}$ with delta function initial data whose logarithm, with appropriate normalizations, is the free energy of the continuum directed polymer, or the solution of the Kardar-Parisi-Zhang (KPZ) equation with narrow wedge initial conditions. We obtain explicit formulas for the one-dimensional marginal distributions – the *crossover distributions* – which interpolate between a standard Gaussian distribution (small time) and the GUE Tracy-Widom distribution (large time). The proof is via a rigorous steepest descent analysis of the Tracy-Widom formula for the asymmetric simple exclusion with anti-shock initial data, which is shown to converge to the continuum equations in an appropriate weakly asymmetric limit. The limit also describes the crossover behavior between the symmetric and asymmetric exclusion processes and hence demonstrates that the KPZ equation is, in fact, a crossover equation between two universality classes – the KPZ class and the Edwards Wilkinson class.

The KPZ and EW universality classes are believed to describe the large-scale/long-time behavior and fluctuations of a wide array of stochastic models in one space and one time $(1 + 1)$ dimension. These models include: (1) Growth models, among which the such as the corner growth or polynuclear growth model; (2) Interacting particle systems such as the exclusion process; (3) Polymer models such as last passage percolation, directed polymers in random environments, or the continuum directed random polymer; (4) Stochastic PDEs such as the KPZ equation, stochastic heat equation or stochastic Burgers equation. A few of these models benefit from also being exactly solvable which means that one can write down formulas for many of the natural observables. Using this solvability we are able to solve for the exact distribution of the solutions to these continuum stochastic PDEs.

1. Growth models

There exist a variety of models which have been created to simulate the behavior of interfaces growing stochastically according to local rules. Depending on the nature of these rules an interface will fluctuate around its limiting profile in different ways. It was predicted in [13] that the fluctuations for a variety of different such models would all behave (i.e., scale) in the same way. These models were then said to be in the KPZ universality class. Underlying this predication was a continuum object – the stochastic PDE now known of as the KPZ equation. The physical predictions only suggested the scaling exponents for these KPZ class models, and it wasn't until mathematicians took up the problem that the exact scaling distributions and processes were uncovered.

Much of my work has revolved around developing our understanding of this universality class as well as its relationship to the KPZ equation. The growth

model I have focused on is the corner growth model in which local valleys $\vee$ are replaced with local hills $\wedge$ at rate q and the reverse occurs at rate $p = 1 - q$.

Different geometries, or the existence of certain external growth sources can affect both the limit shape as well as the fluctuations of the interface height function. In [5] we proved a conjecture of Prähofer and Spohn [14] which provides a complete characterization of the effect of different geometries and external sources on the one-point fluctuations of the height function. In [8] we expand this characterization from one-point fluctuations to multi-point fluctuations. The results of these two papers expand on the ground-breaking work of [12] and [15] and contribute to an understanding of how geometry and external sources affect growth behavior.

In [9] we prove a general phenomena for growth processes – that there exist certain space-time directions (characteristics) along which height function fluctuations are carried to a much longer scale than any other direction. With this scaling in mind, the results of [8] give a spatial limit process for the height function fluctuations, and a full non-trivial space-time limit remains open.

The above results were shown for the totally asymmetric case of the corner growth model ($q = 1$ and $p = 0$). Some of the one-point function results have since been generalized to $q > p$ by Tracy and Widom [22, 23, 24, 25]. The case of symmetry $q = p = 1/2$ has a different flavor with fluctuations no longer described by the KPZ universality class, but rather the EW class.

In [2, 10] we discover that there exists a very interesting crossover between these two classes which is achieved by scaling the asymmetry $q - p$ to zero with the other model parameters. In this weakly asymmetric scaling we are able to describe the entire space-time limit and to prove an exact formula for the one-point function (for two different geometries), which we call the *crossover distributions*. We also prove that the process limit is described in terms of the KPZ equation and thus demonstrate that the KPZ equation, originally conjured up as a continuum model to predict scaling exponents, is actually a crossover equation between two universality classes. [17, 18, 19] also determine the crossover distribution formula in a similar manner (independently and in parallel to [2]).

It would be very interesting to expand the scope of growth models for which the above can be done rigorously beyond the corner growth model – for instance the ballistic deposition model, or models with growth rules which are more than just nearest neighbor. Additionally, there remain certain geometries, such as the equilibrium or flat geometries for which the above analysis is not complete (see [11, 4, 3] for progress in equilibrium), especially in the case of weak asymmetry.

2. Interacting particle systems

Particles moving stochastically and interacting according to a specified rule system are effective ways of simulating real world systems and gaining key insight into complex phenomena. We consider particle systems where individual particles attempt to orchestrate random walks with the caveat that they are influenced by their local environment (of other particles). It is believed that for a wide range

of mechanisms for this influence, the limiting fluctuations in integrated particle density are universally and governed by the KPZ and EW classes.

The simple exclusion process is the poster child for all such particle systems since it is simple to state and exactly solvable (in many cases), yet still seems to contain all of the expected complexities and phenomena of a general system. Particle's attempt continuous time simple random walks, jumping left at rate q and right at rate p with the caveat that jumps are suppressed if the destination site is already inhabited. The exclusion process can be coupled to the corner growth model by replacing up-steps by particles and down-steps by holes. On account of this, the results stated above also apply here and imply KPZ/EW class fluctuations for a variety of observables.

It remains an important problem to extend the scope of rigorous universality beyond the nearest neighbor exclusion process (see [16] for some results for finite range exclusion).

3. Directed polymers and last passage percolation

Directed polymers are an important class of models which were originally introduced as simplifications of self-avoiding polymers, but have come to be studied and applied in their own right. The basic model is of a (discrete) path stretched in one (time) dimension, but allowed to wiggle in the other dimension (space). Without any effects of environment the polymer would configure itself randomly as a random walk path (due to entropic/elastic energy). However, when one introduces a disordered environment the path must now take into account the combine entropic/elastic energy and the disordered energy (the sum of disorder energy along the path). The relative strength of the competition of entropy and disorder energy is determined by the temperature of the system and in the limit of zero temperature the disorder energy becomes of singular importance and the model reduces to directed last/first passage percolation.

An important statistical observable in these polymer models is the free energy which is essentially the logarithm of the quenched partition function. It is predicted that as long as the disorder energy distribution has sufficiently many moments, the fluctuations of the free energy should be in the KPZ universality class. There are, however, only a few cases in which this has been proved and finding a more general approach remains an important problem.

My work on polymers has been focused on two extremes – the zero temperature last passage percolation model and the high temperature polymer limit [1] of the continuum polymer. Last passage percolation can be mapped into a version of the corner growth process, and in the case of exponentially distributed disorder the results of [5, 8, 9] apply to give a complete characterization of the free energy multi-point fluctuations for a family of models with boundary pinning.
Rigorously, very little is known for distributions other than exponential/geometric (for positive temperature polymers, the log-gamma and Brownian cases have been shown to have the right KPZ scaling [20, 21]). In fact the connection between

polymers with exponential/geometric disorder seems to break down at positive temperature and so the above results are only valid for last passage percolation.

Surprisingly there is a second, very high temperature regime at which polymers and growth models become rigorously related again [1]. Scaling temperature along with the other model parameters one shows convergence to the free energy of a model called the continuum directed random polymer. This continuum object is related to the continuum limit of the weakly asymmetric corner growth model, i.e., the KPZ equation, via the Hopf-Cole transform and a version of the Feynman Kacs formula [6]. In [2, 10] we prove formulas for the exact probability distribution for the one-point fluctuations of the free energy of this continuum polymer in a space-time white-noise disorder (in the point-to-point case and also a point-to-line model). Our work confirms the physical prediction that the long-time scaling of the free energy of the continuum polymer is given by the KPZ universality class. We also determine the short-time scaling which is given by the EW class.

4. Continuum limits and KPZ

Continuum models in statistical physics often arise at critical phase transition as fine scaling limits of systems and display a high level of universality with respect to perturbations of the discrete models being considered. My work in [2, 10] has shed some light on the collection of continuum models associated with $1+1$ dimensional growth models, particle systems and polymers.

The continuum models which we consider are the KPZ equation, the stochastic heat and Burgers equation and the continuuum directed random polymer. All are connected via various transforms. Our contributions are two fold: We prove exact formulas for one-point functions for the solution to these equations; and we prove convergence of critically scaled models to these continuum objects. A punch-line of our work is the rigorous understanding that the KPZ equation represents a crossover between the KPZ and EW universality classes.

There are a host of problems left in this area: Analyzing these continuum models in geometries such as equilibrium (see [3]) or flat; computing multipoint functions; determining whether the integrable structure of the Tracy-Widom distributions generalizes to these crossover distributions; and proving convergence to these continuum models in greater generality. Also, if one could use weakly asymmetric convergence too these continuum objects to rigorously prove positive asymmetry/temperature results, this could prove a very powerful technique towards proving universality.

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Interacting particle systems and random matrices

Patrik L. Ferrari

TASEP. We consider the totally asymmetric simple exclusion process on $\mathbb{Z}$ (in continuous time). Particles jump independently to their right neighboring site with rate 1, provided the site is empty. Denote by $\eta_t(j) \in \{0,1\}$ the occupation variable of site j at time t (with 1 meaning occupied). The standard representation as height function h (at position x and time t) is given by

$$h(x,t) = \begin{cases} 2N_t + \sum_{y=1}^{x}(1-2\eta_y(t)), & \text{for } x \geq 1, \\ 2N_t, & \text{for } x = 0, \\ 2N_t - \sum_{y=x+1}^{0}(1-2\eta_y(t)), & \text{for } x \leq -1, \end{cases} \tag{1}$$

where N_t is the number of particles which have crossed the bond 0 to 1 during the time span $[0,t]$. This model belongs to the Kardar-Parisi-Zhang [15] universality class of growth models in $1+1$ dimensions.

Universality is expected in the long time limit, i.e., the asymptotic height fluctuations should be independent of the particular model used to derive them. Unlike in the equilibrium statistical mechanics, the scaling exponents are not enough to single out the large time statistics: *initial conditions matter!* One still have to distinguish between (a) curved limit shape, (b) flat limit shape obtained from non-random initial fluctuations, (c) flat limit shape coming from stationary (random) initial conditions.

From KPZ scaling, the correlation length scales as $t^{2/3}$ and height fluctuations as $t^{1/3}$ [11, 20]. Therefore, given the limit shape

$$h_{\mathrm{ma}}(\xi) := \lim_{t\to\infty} \frac{h(\xi t, t)}{t}, \tag{2}$$

the scaling limit to be considered is

$$h_t^{\mathrm{resc}}(u) = \frac{h(\xi t + u t^{2/3}, t) - t h_{\mathrm{ma}}(\xi + u t^{-1/3})}{t^{1/3}}, \tag{3}$$

with of course a freedom in the choice of scaling coefficients (independent of t) for horizontal and vertical scaling.

TASEP with step Initial Conditions. Consider first step initial condition, $\eta_j(0) = 1$ for $j \leq 0$ and $\eta_j(0) = 0$ for $j > 0$, i.e., $h(x,0) = |x|$. *The limit shape is curved*: $\frac{1}{2}(1+\xi^2)$ for $|\xi| \leq 1$. Let us focus around $\xi = 0$, i.e., consider

$$h_t^{\mathrm{resc}}(u) := \frac{h(2u(t/2)^{2/3}, t) - \big(t/2 + u^2(t/2)^{1/3}\big)}{-(t/2)^{1/3}}. \tag{4}$$

For the one-point distribution it is proven [12] that

$$\lim_{t\to\infty} \mathbb{P}\left(h(0,t) \geq t/2 - s(t/2)^{1/3}\right) = F_2(s), \tag{5}$$

where F_2 is known as the GUE Tracy-Widom distribution, first discovered in random matrices [18]. Moreover, concerning the joint distributions, it is proven [6,

4, 13] that

$$\lim_{t\to\infty} h_t^{\mathrm{resc}}(u) = \mathcal{A}_2(u), \tag{6}$$

where $\mathcal{A}_2$ is called the Airy$_2$ process, first discovered in the PNG model by Prähofer and Spohn [16].

GUE matrices. The distribution function F_2 and the Airy$_2$ process describe also the statistics of the largest eigenvalue in the Gaussian Unitary Ensemble of random matrices. Consider $N \times N$ hermitian matrices distributed according to the probability measure

$$\text{const} \exp\left(-\mathrm{Tr}(H^2)/2N\right) dH, \tag{7}$$

where $dH = \prod_{i=1}^N dH_{i,i} \prod_{1\le i<j\le N} d\Re(H_{i,j}) d\Im(H_{i,j})$ is the reference measure. Denote by $\lambda_{N,\max}^{\mathrm{GUE}}$ the largest eigenvalue of a $N \times N$ GUE matrix. Then Tracy and Widom [18] showed that fluctuations of $\lambda_{N,\max}^{\mathrm{GUE}}$ are asymptotically F_2-distributed:

$$\lim_{N\to\infty} \mathbb{P}\left(\lambda_{N,\max}^{\mathrm{GUE}} \le 2N + sN^{1/3}\right) = F_2(s). \tag{8}$$

The parallel between GUE and TASEP with step initial condition goes even further. Dyson's Brownian Motion (DBM) is a matrix-valued Ornstein-Uhlenbeck process introduced by Dyson in 1962 [7]. More precisely, the GUE DBM is the stationary process on matrices $H(t)$ whose evolution is governed by

$$dH(t) = -\frac{1}{2N} H(t)dt + dB(t) \tag{9}$$

where $dB(t)$ is a (hermitian) matrix-valued Brownian motion. More precisely, the entries $B_{i,i}(t)$, $1 \le i \le N$, $\Re(B_{i,j})(t)$ and $\Im(B_{i,j})(t)$, $1 \le i < j \le N$, perform independent Brownian motions with variance t for diagonal terms and $t/2$ for the remaining entries. Denote by $\lambda_{N,\max}^{\mathrm{GUE}}(t)$ the largest eigenvalue at time t (when started from the stationary measure (7)). Its evolution is, in the large N limit, governed by the Airy$_2$ process:

$$\lim_{N\to\infty} \frac{\lambda_{N,\max}^{\mathrm{GUE}}(2uN^{2/3}) - 2N}{N^{1/3}} = \mathcal{A}_2(u). \tag{10}$$

TASEP with step Alternating Conditions. The alternating initial condition is the following: $\eta_j(0) = 0$ for odd j and $\eta_j(0) = 1$ for even j. *The limit shape flat, not curved:* $h_{\mathrm{ma}}(\xi) = 1/2$. Thus, the rescaled height function becomes

$$h_t^{\mathrm{resc}}(u) := \frac{h(2ut^{2/3}, t) - t/2}{-t^{1/3}}. \tag{11}$$

In the large time limit

$$\lim_{t\to\infty} \mathbb{P}\left(h(0,t) \ge t/2 - st^{1/3}\right) = F_1(2s), \tag{12}$$

where F_1 is known as the GOE Tracy-Widom distribution, first discovered in random matrices [19]. Moreover, as a process, it was discovered by Sasamoto,

see [17, 5], it holds

$$\lim_{t\to\infty} h_t^{\mathrm{resc}}(u) = \mathcal{A}_1(u), \tag{13}$$

where $\mathcal{A}_1$ is called the Airy_1 process.

GOE matrices. The Gaussian Orthogonal Ensemble (GOE) of random matrices has density on $N \times N$ symmetric matrices

$$\text{const}\, \exp\left(-\mathrm{Tr}(H^2)/4N\right) dH, \tag{14}$$

where $dH = \prod_{1\le i\le j\le N} dH_{i,j}$ is the reference measure. Denote by $\lambda_{N,\max}^{\mathrm{GOE}}$ its largest eigenvalue. The asymptotic distribution of the largest eigenvalue is F_1 [19]:

$$\lim_{N\to\infty} \mathbb{P}\left(\lambda_{N,\max}^{\mathrm{GOE}} - 2N \le sN^{1/3}\right) = F_1(s). \tag{15}$$

Also DBM is defined for symmetric matrices by

$$dH(t) = -\frac{1}{4N} H(t)dt + dB(t) \tag{16}$$

where $dB(t)$ is a symmetric matrix-valued Brownian motion (as before without imaginary parts). However, numerical evidence shows [2] that, the limit process of a properly rescaled $\lambda_{N,\max}^{\mathrm{GOE}}(t)$ is not the Airy_1 process:

$$\lim_{N\to\infty} \frac{\lambda_{N,\max}^{\mathrm{GOE}}(8uN^{2/3}) - 2N}{2N^{1/3}} =: \mathcal{B}_1(u) \ne \mathcal{A}_1(u). \tag{17}$$

The process $\mathcal{B}_1$ is yet unknown.

Conclusion. First of all, *initial conditions matter* for the long time statistics of interfaces in the KPZ class. Secondly, there are partial connections with random matrices. The parallel between TASEP with step initial conditions and GUE random matrices holds for joint distributions. To understand this connection, one way is to compare the interlacing structure on the GUE minors [14] and the interlacing structure on an extension of TASEP to a $2+1$ interacting particle system introduced in [3], see the lecture notes [8] for details. Note that the connection extends *partially* to the evolution of minors as shown in [10, 1]. On the other hand, the parallel between TASEP with alternating initial conditions and GOE random matrices stops at the level of one-point distributions. For a more extended discussion, see the recent review paper [9].

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Directed Polymers in a Heavy Tailed Environment

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Originally introduced in [8], Directed Polymers in Random Environment is a model for an interaction between a polymer chain and a medium with microscopic impurities. In one variant of this model, the environment σ is a signed measure on $([0,n] \cap Z) \times Z^d$ and the polymer $s : [0,n] \to \mathbb{Z}^d$ is a random walk trajectory whose measure conditioned on the environment is

$$\mu^{\sigma}_{n,\beta}(s) = \frac{1}{Q^{\sigma}_{n,\beta}} \exp\left(-\beta \mathcal{H}^{\sigma}(s)\right) \quad ; \qquad \mathcal{H}^{\sigma}(s) = -\sigma(\text{graph}(s))$$

where $\mathcal{H}^{\sigma}(s)$ is the Hamiltonian or *energy* of s, the parameter $\beta \in [0,\infty)$ represents the overall strength of the interaction (the inverse temperature) and $Q^{\sigma}_{n,\beta}$ is a normalizing constant.

It is expected (yet far from being proved) that in $d \leq 2$ and any finite temperature, or $d > 2$ and low enough (but not necessarily zero, at least for d small enough) temperature, the behavior of the polymer path is super-diffusive with transversal fluctuations of order n^{ψ} for some $\psi > 0$, while in infinite ($d \leq 2$) or high enough

($d > 2$) temperature the polymer is diffusive with fluctuations of order $n^{\frac{1}{2}}$. This should be universal with respect to a large class of environment distributions, certainly including the case where σ are i.i.d. with marginal distribution whose tail decays sufficiently fast. For partial results see [3, 4, 9, 15].

At zero temperature ($\beta \to \infty$), the polymer measure is uniform over all H-minimizing paths and the model coincides with the well known model of directed last passage percolation (LPP). Here universal super-diffusive behavior is expected in all dimensions, but very little is proved (see [10, 12, 14]).

In the paper "Directed polymers in random environment with heavy tails" [1] we address the case where $\sigma(\{z\})$ are i.i.d. but their marginal distribution has a (right) tail which is heavy enough to fall outside the universality classes discussed above. The zero temperature case was treated in [7] and our work can be seen as an extension to all temperatures. Among the results in the paper, we show that for any finite temperature $\beta > 0$ the polymer is localized to a region of diameter $o(n)$ around the last passage (energy minimizing) path and if the temperature is scaled correctly $\beta(n) = \beta n^{\frac{2}{\alpha}} L_0(t)$ (where $L_0(t)$ is another slowly varying function), then localization to a region of diameter $o(n)$ around a β-dependent optimal energy-entropy balancing curve is observed. We identify this β-optimal path and show that its distribution admits a scaling limit on a linear order, thus we prove that fluctuations are of order n. The scaling limits form two-parameters (α, β) family of distributions on 1-Lipschitz curves on $[0, 1]$ that interpolates the cases $\beta = 0$ to $\beta = \infty$ and we discuss some of its properties. The proof uses tools from large deviations, extreme value theory, random walks and functional analysis. It is joint work with O. Louidor (UCLA).

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Quasi-potential for conservation laws with boundary conditions

Christophe Bahadoran

Setting. In [10, 11, 12], non-local free energy functionals were derived for stationary exclusion processes with open boundaries coupled to reservoirs. In symmetric and weakly asymmetric cases, these functionals were recovered ([5, 6, 7]) as a quasi-potential:

$$V[r(.)] := \inf\{I[\rho(.,.)] : \rho(-\infty,.) \in \mathcal{S},\ \rho(0,.) = r(.)\} \tag{1}$$

where $r(.) = (r(x),\ x \in [0,1])$ is a stationary fluctuation of the spatial density profile, $\rho(.,.) = (\rho(t,x),\ t \le 0,\ x \in [0,1])$, a dynamic fluctuation of the space-time profile, $I[\rho(.,.)]$ the dynamic large deviation functional, and $\mathcal{S}$ the set of stationary solutions of the hydrodynamic equation. In [2], we undertake this derivation starting from a dynamic large deviation functional relevant to asymmetric systems. Here, the hydrodynamic equation is of hyperbolic type:

$$\partial_t \rho(t,x) + \partial_x f[\rho(t,x)] = 0 \tag{2}$$

where $\rho(t,x) \in [0,1]$. For such PDE's, entropy conditions inside, and boundary conditions in the sense of Bardos et al. ([3]), are essential differences compared to previously studied cases (where a diffusion term was present). They induce boundary-driven phase transition (i.e. $\mathcal{S}$ may not reduce to a singleton) and deeply modify the structure of the relevant dynamic functional.

Stationary solutions and dynamic functional. These are constructed from the following input: (i) an exclusion-like flux function f in (2), i.e. f is uniformly concave on $[0,1]$ with $f(0) = f(1) = 0$ and a unique maximum at $\rho^* \in (0,1)$; (ii) a strictly convex entropy $h(\rho)$; iii) left/right reservoir densities $\rho_l, \rho_r \in [0,1]$.

The set $\mathcal{S}$ consists of three phases ([13, 1]) : low-density (LD) $\rho_l < \rho_r$, $f(\rho_l) < f(\rho_r)$ or $\rho_r \le \rho_l < \rho^*$, with bulk density ρ_l; high-density (HD) $\rho_l < \rho_r$, $f(\rho_l) > f(\rho_r)$ or $\rho^* < \rho_r \le \rho_l$ with bulk density ρ_r; maximal current (MC) $\rho_l \ge \rho^* \ge \rho_r$,

with bulk density ρ^*. On the phase transition line $\rho_l < \rho_r$, $f(\rho_l) = f(\rho_r)$, $\mathcal{S}$ consists of arbitrarily located shocks connecting ρ_l and ρ_r. For ASEP,

$$f(\rho) = \rho(1-\rho), \quad h(\rho) = \rho \log \rho + (1-\rho)\log(1-\rho) \tag{3}$$

The dynamic functional is decomposed as $I = I^0 + I^l + I^r$, where I^0 measures deviation from (2) including entropy conditions, and I^l, I^r measure deviation from boundary conditions. A natural bulk functional $I^0[\rho(.,.)]$ ([16, 18]) is the total positive mass of entropy production $\partial_t h(\rho) + \partial_x g(\rho)$ if $\rho(.,.)$ is a weak solution of (2), $+\infty$ otherwise. We consider instead (see below) a closely related functional J^0 introduced in [4, 17]. Boundary functionals write

$$I^l[\rho(.,.)] = \int_{-\infty}^{0} i^l[\rho(t,0)|\rho_l]dt, \quad I^r[\rho(.,.)] = \int_{-\infty}^{0} i^r[\rho(t,1)|\rho_r]dt \tag{4}$$

for local cost functions $i^l(\rho|\rho_l)$ and $i^r(\rho|\rho_r)$. These functions were introduced in [8] in the case (3).

Results. Let $\varphi(\rho)$ be the unique decreasing function such that $f \circ \varphi = \varphi$, e.g. $\varphi(\rho) = 1-\rho$ for ASEP. Under the symmetry assumption $h''[\varphi(\rho)]\varphi'(\rho) = -h''(\rho)$, we obtain a closed expression for (1) that is a generalization of the functional in [6]. We determine all fluctuation paths $\rho(.,.)$, i.e. minimizers of (1). Contrary to the diffusive case, these paths relax from $r(.)$ to $\mathcal{S}$ in finite time, except in the MC phase. In the regime $\rho_l > \rho_r$, the unique fluctuation path is the space-time reversal of an entropy solution of (2) with time-dependent boundary conditions different from ρ_l, ρ_r. In the regime $\rho_l < \rho_r$ there may be several paths. Each of them evolves in two steps. First, it is the space-time reversal of a weak solution of (2) with a single antishock and boundary data ρ_l, ρ_r. The antishock starts at a position $y \in [0,1]$ determined from $r(.)$ as a minimizer of some function. For each such minimizing position there is a unique fluctuation path. When the antishock reaches a boundary, the solution turns entropic inside and the driving boundary condition is also imposed at the opposite boundary.

Open questions.

1. Weakly asymmetric systems with vanishing viscosity. The bulk part J^0 of the dynamic functional ([17]) governs large deviations for a vanishing visosity stochastic perturbation of (2). Can we add boundary conditions to this stochastic equation and obtain the boundary terms for the dynamic large deviations ? Combined with [1], this would yield stationary large deviations similar to [6]. Weakly asymmetric exclusion-like models with vanishing viscosity ([14]) are formally similar. Perhaps the same could be done for such systems.

2. Extension to Jensen-Varadhan functional. Do we have similar results with the functional I^0 of [16, 18], which governs TASEP large deviations ? The two functionals are very close for stricly concave flux: $J^0 \geq I^0$, and $J^0 = I^0$ at least for BV functions. But J^0 provides control over all entropies of (2), while I^0 only does for a single entropy. To deal with I^0 we need would need an extension of [9]: that if entropy production is a measure for a single stricly convex entropy, then it is for all convex entropies. The statement of [9] is with "negative measure" only.

3. Dynamic large deviations for TASEP. Assuming the above extension holds, a

dynamic large deviation principle for open TASEP combined with [1] would imply stationary large deviations. However it is not clear how to rigorously obtain the boundary large deviation terms for genuinely asymmetric systems. One reason seems to be that boundary conditions for (2) really use many entropies, while for [16, 18] only the microscopic entropy is available.

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Euler hydrodynamics of one-dimensional attractive particle systems in random environment

Ellen Saada
(joint work with C. Bahadoran, H. Guiol, K. Ravishankar)

We have been analyzing Euler hydrodynamics of conservative attractive interacting particle systems on $\mathbb{Z}$ (see [12] for definition and properties of attractiveness) in a series of papers described below. We consider models with at most K particles per site ($K \in \mathbb{N}$), of 'misanthrope type' (see [9]), that is, particles' jump rates depend on the configuration only through the occupation numbers on the departure and target sites of the jump. However the validity of our results is not restricted to this condition, they apply for instance to k-step exclusion processes (see [10]).

Alternative to previous methods (see [14], [11], [15], [19], [20]), we have developed a constructive method to derive that the hydrodynamic limit under Euler time scaling is given by the entropy solution (see e.g. [21]) to the scalar conservation law

$$\partial_t u(t,x) + \partial_x G(u(t,x)) = 0 \tag{1}$$

where the macroscopic flux function G is Lipschitz-continuous. Our idea was to generalize the proof done in [1], [2] by Andjel & Vares, of Riemann hydrodynamics for misanthropes processes with convex flux function. Our approach relies on: (i) a variational formula for entropy solutions (to the hydrodynamic equation) without assuming flux convexity, and (ii) an approximation method (inspired by Glimm's scheme) to prove that the hydrodynamic limit for initial Riemann profiles implies the hydrodynamic limit for general initial profiles.

In [3], we studied systems with irreducible and finite range jumps, and product invariant measures. In [4], we generalized the previous work to processes without explicit invariant measures. Papers [3] and [4] are summarized in [18] (see also [17]).

In [5], under the same hypotheses on the model, we obtained strong hydrodynamics, that is an almost sure limit, with respect to the probability distribution underlying the graphical representation of the process (indeed, in the aforementioned works, hydrodynamic limits referred to weak convergence of empirical measures). We did not use the subadditive ergodic theorem (unsufficient in our setting), which was the core of the strong hydrodynamic results obtained up to now (see for instance the seminal work [16]); our key idea was to replace it by large deviation estimates.

A crucial tool in those proofs was 'macroscopic stability' of the models, which had been derived by Bramson & Mountford in [8] for exclusion processes with finite range jump kernels. In [13], we got rid of the finite range assumption on the jump rates to derive macroscopic stability for misanthrope type models on compact state spaces.

Quenched hydrodynamic limits in random environment under Euler time scaling have been derived in [7] (for asymmetric zero-range process with site disorder on

$\mathbb{Z}^d$) and in [19] (for the totally asymmetric nearest-neighbor K-exclusion process on $\mathbb{Z}$ with i.i.d. site disorder).

We are now able, in [6], to prove a quenched hydrodynamic limit under hyperbolic time scaling for bounded attractive particle systems on $\mathbb{Z}$ in random ergodic environment. Our result is again a strong law of large numbers. An essential difficulty for the disordered system is the simultaneous loss of translation invariance *and* absence of explicit invariant measures. A central idea is to introduce a joint disorder-particle process that is translation invariant, to characterize its invariant measures, and then (thanks to the tools developed in [5]), to derive its strong hydrodynamic limit; the latter induces our quenched result. The flux G is defined as an integral of the microscopic flux independent of the chosen disorder. Our method is quite robust with respect to the model (so long as it is attractive) and to the form of the disorder: We apply it to generalizations of misanthropes process (including site or bond disorders), of k-step exclusion, and to a traffic model in various cases of random environment.

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Hydrodynamic limit for 2D and 3D Young diagrams

Tadahisa Funaki
(joint work with Makiko Sasada)

The asymptotic shapes of two-dimensional random Young diagrams with large size were studied by Vershik [3] under several types of statistics including the uniform and restricted uniform statistics, which were also called the Bose and Fermi statistics, respectively. To each partition $p = \{p_1 \geq p_2 \geq \cdots \geq p_j \geq 1\}$ of a positive integer n by positive integers $\{p_i\}_{i=1}^j$ (i.e., $n = \sum_{i=1}^j p_i$), the height function of the Young diagram is defined by

$$\psi_p(u) = \sum_{i=1}^{j} 1_{\{u<p_i\}}, \quad u \geq 0.$$

For each fixed n, the uniform statistics (U-statistics in short) μ_U^n assigns an equal probability to each of possible partitions p of n, i.e., to the Young diagrams of area n. The restricted uniform statistics (RU-statistics in short) μ_R^n also assigns an equal probability, but restricting to the distinct partitions satisfying $q = \{q_1 > q_2 > \cdots > q_j \geq 1\}$. These probabilities are called canonical ensembles. Grandcanonical ensembles μ_U^ε and μ_R^ε with parameter $0 < \varepsilon < 1$ are defined by superposing the canonical ensembles in a similar manner known in statistical physics. Vershik [3] proved that, under the canonical U- and RU-statistics $\mu_U^{N^2}$ and $\mu_R^{N^2}$ (with $n = N^2$), the law of large numbers holds as $N \to \infty$ for the scaled height variable

$$\tilde{\psi}_p^N(u) := \frac{1}{N}\psi_p(Nu), \quad u \geq 0,$$

of the Young diagrams $\psi_p(u)$ with area N^2 and for $\tilde{\psi}_q^N(u)$ defined similarly, and the limit shapes ψ_U and ψ_R are given by

$$(1) \quad \psi_U(u) = -\frac{1}{\alpha}\log\left(1 - e^{-\alpha u}\right) \quad \text{and} \quad \psi_R(u) = \frac{1}{\beta}\log\left(1 + e^{-\beta u}\right), \quad u \geq 0,$$

with $\alpha = \pi/\sqrt{6}$ and $\beta = \pi/\sqrt{12}$, respectively. These results can be extended to the corresponding grandcanonical ensembles μ_U^ε and μ_R^ε, if the averaged size of the diagrams is N^2 under these measures.

Our goal is to study these results from a dynamical point of view. Allowing a creation and an annihilation of a unit square on the surface of the Young diagram, to the grandcanonical U- and RU-statistics, one can associate a weakly asymmetric zero-range process p_t respectively a weakly asymmetric simple exclusion process q_t on a set of positive integers with a stochastic reservoir at the boundary site $\{0\}$ in both processes as natural time evolutions of the Young diagrams or those of the gradients of their height functions. Then, under the diffusive scaling in space and time and choosing the parameter $\varepsilon = \varepsilon(N)$ of the grandcanonical ensembles such that the averaged size of the Young diagrams is N^2, one can derive non-linear partial differential equations in the limit and show that the Vershik curves defined by (1) are actually unique stationary solutions to the deived PDEs in both cases.

We state our main results more precisely. Define

$$\tilde{\psi}_p^N(t,u) := \tilde{\psi}_{p_{N^2t}}^N(u) \equiv \frac{1}{N}\psi_{p_{N^2t}}(Nu), \quad u > 0,$$

from p_t in U-case and define $\tilde{\psi}_q^N(t,u)$ similarly from q_t in RU-case.

Theorem 1. (1) *If* $\tilde{\psi}_p^N(0,u) \underset{N\to\infty}{\longrightarrow} \psi_0(u)$, *then* $\tilde{\psi}_p^N(t,u) \underset{N\to\infty}{\longrightarrow} \psi_U(t,u)$ *in probability. The limit* $\psi_U(t,u)$ *is a solution of nonlinear PDE:*

$$\begin{aligned}
&\partial_t\psi = \partial_u\left(\frac{\partial_u\psi}{1-\partial_u\psi}\right) + \alpha\frac{\partial_u\psi}{1-\partial_u\psi}, \quad u > 0,\\
&\psi(0,\cdot) = \psi_0(\cdot),\\
&\psi(t,0+) = \infty,\ \psi(t,\infty) = 0,
\end{aligned}$$

where $\partial_t\psi = \partial\psi/\partial t, \partial_u\psi = \partial\psi/\partial u\ (< 0)$.
(2) *If* $\tilde{\psi}_q^N(0,u) \underset{N\to\infty}{\longrightarrow} \psi_0(u)$, *then* $\tilde{\psi}_q^N(t,u) \underset{N\to\infty}{\longrightarrow} \psi_R(t,u)$ *in probability. The limit* $\psi_R(t,u)$ *is a solution of nonlinear PDE:*

$$\begin{aligned}
&\partial_t\psi = \partial_u^2\psi + \beta\partial_u\psi(1+\partial_u\psi), \quad u > 0,\\
&\psi(0,\cdot) = \psi_0(\cdot),\\
&\partial_u\psi(t,0+) = -\frac{1}{2},\ \psi(t,\infty) = 0.
\end{aligned}$$

I discussed the conservative dynamics associated with the canonical ensemble. Under the scaling $t \mapsto N^4t$, one can expect to derive a Cahn-Hilliard type nonlinear PDE of fourth order in the limit. This is a model for the so-called surface diffusion.

The extension to 3D situation was studied by Cerf and Kenyon [1] under uniform statistics and the limit surface called the Wulff shape is characterized by a certain variational formula. I also stated a conjecture for the hydrodynamic limit for the associated dynamics of the 3D Young diagrams (confined on a torus). Our model can be viewed as describing a motion of (decreasing) interfaces, called SOS dynamics.

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Superdiffusive bounds on self-repellent precesses in $d = 2$

Bálint Tóth

(joint work with Benedek Valkó)

We study the long time asymptotics of the *self repelling Brownian polymer process (SRBP)* in $\mathbb{R}^d$ defined by the SDE

$$dX_t = dB_t\big(\eta_0 - \operatorname{grad} V * l_t\big)(X(t))dt, \tag{1}$$

where B_t is standard Brownian motion in $\mathbb{R}^d$, $\eta_0 : \mathbb{R}^d \to \mathbb{R}^d$ is a gradient vector field with sufficient regularity,

$$l_t(A) := |\{s \in [0,t] : X_s \in A\}|$$

is the occupation time measure of the process X_t and $V : \mathbb{R}^d \to [0,\infty)$ is a C^∞, spherically symmetric approximate identity with sufficiently fast decay at infinity. It is assumed that $V(\cdot)$ is positive definite:

$$\hat{V}(p) = \int_{\mathbb{R}^2} e^{ip\cdot x} V(x)dx \geq 0.$$

The process is pushed by the negative gradient of its own local time, mollified by convoluting with V. The process was introduced by Durrett and Rogers in [2] and further investigated in a series of probability papers. For a survey and complete list of references see [6]. It is phenomenologically similar to the so-called *true self-avoiding random walk (TSAW)* which arose in the physics literature initiated by Amit, Parisi and Peliti in [1] and further investigated in a series of physics papers. For a survey and complete list of references see [10].

Conjectures based on scaling and renormalization group arguments regarding this family of models are the following (see e.g. [1]):

(1) In $d = 1$: $X(t) \sim t^{2/3}$ with intricate, non-Gausssian scaling limit.
(2) In $d = 2$: $X(t) \sim t^{1/2}(\log t)^{1/4}$ and Gaussian (that is Wiener) scaling limit expected.
(3) In $d \geq 3$: $X(t) \sim t^{1/2}$ with Gaussian (i.e. Wiener) scaling limit expected.

Some if these conjectures had been proven or at least partially settled. For results in $d = 1$ see [9], [13], [12], [8] and the survey [10]. For results in $d \geq 3$ see [3]. The present lecture concentrates on recent results on the $d = 2$ case. The complete results and proofs will appear in [11].

The natural framework of formulation of the problem and results is *the environment seen by the random walker*. Let

$$\eta_t(x) := \big(\eta_0 - \operatorname{grad} V * l_t\big)(X(t)).$$

Then $t \mapsto \eta_t$ is a Markov process with continuous sample paths in the function space

$$\Omega = \big\{\omega \in C^\infty(\mathbb{R}^2 \to \mathbb{R}^2) \,:\, \operatorname{rot}\omega \equiv 0, \| \omega \|_{k,m,r} < \infty\big\}$$

where $\| \omega \|_{k,m,r}$ are the seminorms

$$\| \omega \|_{k,m,r} = \sup_{x\in\mathbb{R}^d} \big(1 + |\, x \,|\big)^{-1/r} \left| \partial^{|\, m \,|}_{m_1,\dots,m_d} \omega_k(x) \right|$$

defined for $k = 1, 2$, multiindices $m = (m_1, \dots, m_d)$, $m_j \geq 0$, and $r \geq 1$.

It was proved in [8] (for $d = 1$) and [3] (for arbitrary d) that the Gaussian measure π on Ω defined by the covariances

$$\int_\Omega \omega_k(x) d\pi(\omega) = 0, \qquad K_{kl}(x - y), \qquad \text{with} \qquad \hat{K}_{kl}(p) = \frac{p_k p_l}{|\, p \,|^2} \hat{V}(p)$$

is stationary and ergodic for the Markov process η_t.

The random vector field ω distributed according to π the gradient of the massless free Gaussian field smeared out by convolution with U, where $U * U = V$.

Let

$$\hat{E}(\lambda) := \int_0^\infty e^{-\lambda t} \mathbf{E}\left(|\, X_t \,|^2\right) dt.$$

The main result of [11], reported in this lecture is the following theorem.

Theorem 1. *Consider the process defined by the stochastic differential equation* (1) *in* $\mathbb{R}^2$ *and let the initial vector field* η_0 *be sampled from the stationary distribution* π. *Then there exist constants* $0 < C_1, C_2 < \infty$ *such that for* $0 < \lambda < 1$ *the following bounds hold*

$$C_1 \log |\log \lambda\,| \leq \lambda^2 \hat{E}(\lambda) \leq C_2 |\log \lambda\,|$$

Remarks: (1) Modulo Tauberian inversion, these bounds mean in real time

$$C_3 t \log\log t \leq \mathbf{E}\left(|\, X(t) \,|^2\right) \leq C_4 t \log t,$$

with $0 < C_3, C_4 < \infty$ and for t sufficiently large.
(2) The upper bound is straightforward, it follows directly form estimates on diffusion in random scenery. The superdiffusive lower bound is the main result.

The proof of Theorem 1 follows the main lines of [5]. See also [4]. However on the computational level there are notable differences. For similar recent results referring to second class particle motion in various asymmetric exclusion processes see also [7].

Full proofs are available in [11].

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Hydrodynamic limit for exclusion processes with velocity

Makiko Sasada

We introduce a new class of one-dimensional exclusion processes with velocities featuring random flips of the sign of velocities. We consider a system of particles with velocity $+1$ or -1 where particles interact with each other via hard-core exclusion potential. Each particle can only hop in the direction of its velocity. The sign of the velocity of each particle flips at rate $\gamma > 0$. We prove the hydrodynamic limit for this nonreversible and nongradient system under the diffusive space-time scaling. The hydrodynamic equation is a certain nonlinear diffusion equation and its diffusion coefficient is characterized by a variational formula. We also obtain the asymptotic behavior of the diffusion coefficient as γ goes to ∞ or 0.

The model we consider is a system of particles with velocity on the one-dimensional discrete lattice $\mathbb{Z}$ under the constraint that at most one particle can occupy each site. A set of possible velocities is $\{1, -1\}$ and the state space of the process is $\{1, 0, -1\}^{\mathbb{Z}}$. Its elements (called configurations) are denoted by $\omega = (\omega(x), x \in \mathbb{Z})$ with $\omega(x) = 0$ or 1 or -1 depending on whether $x \in \mathbb{Z}$ is empty or occupied by

a particle with velocity 1 or a particle with velocity -1, respectively. A particle at site x with velocity 1 (resp. -1) waits for an exponential time at rate one and then jumps to $x+1$ (resp. $x-1$) provided the site is not occupied. If the site is occupied by a particle with velocity 1 (resp. -1), then the jump is suppressed. On the other hand, if the site is occupied by a particle with velocity -1 (resp. 1), then the particle at x collides to the particle at $x+1$ (resp. $x-1$), namely the particle cannot jump to $x+1$ but instead the velocities of these two particles are exchanged. The changes of velocities also happen due to random external factors. The sign of velocity of each particle flips (from 1 to -1 or from -1 to 1) with exponential law with a positive constant rate γ. Flips of the sign of each particle's velocity happen independently of each other. Moreover, jumps or collisions of each particle and flips of the sign of each particle's velocity are all independent.

It is easy to see that the number of particles is a unique conserved quantity for such process. We prove the hydrodynamic limit for the density of particles and obtain a variational formula for the diffusion coefficient of the hydrodynamic equation, assuming the continuity of the diffusion coefficient.

The hydrodynamic limit for a nonreversible and nongradient system is first considered in [8] where the hydrodynamic behavior of a one-dimensional mean-zero zero-range process is studied. Later, Komoriya studied the hydrodynamic behavior of a mean-zero exclusion process in [1]. A crucial step for extending the entropy method first developed for reversible systems to nonreversible systems consists in controlling the asymmetric part of the generator by the symmetric one. This is related to the so-called sector condition. In [8], [7] and [6], some versions of the sector condition are proved using the idea called *loop decomposition* first introduced in [8]. The idea depends deeply on the mean-zero property of random walks considered there. In our proof, we do not use *loop decomposition*, and instead we use the parity of the system to show a version of the sector condition. This new method can be applied to Hamiltonian systems.

The model defined above can be considered as an intermediate between the totally asymmetric exclusion process (TASEP) and the simple symmetric exclusion process (SSEP). If we consider the case with $\gamma = 0$ and the initial condition satisfying that the velocities of all particles are same (e.g. 1), the system evolves as same as TASEP. In this situation, since there is a transport in the system, we have to consider the hyperbolic space-time scaling instead of the diffusive scaling to know the time evolution of the density of particles. On the other hand, if we consider the case with $\gamma = \infty$ heuristically, each particle jumps to right or left with probability $1/2$ under the exclusive constraint after exponential waiting time with rate one. Therefore the system evolves as same as SSEP where the density of particles evolves according to the heat equation with a constant diffusion coefficient $1/2$ under the diffusive scaling. From this point of view, we study the asymptotic behavior of the diffusion coefficient of our model as γ goes to 0 or ∞ and obtain some results which show that the asymptotic behavior of the diffusion coefficient is consistent with the asymptotic behavior of the evolution of the model. Specifically, we show that the diffusion coefficient, denoted by $D^\gamma(\rho)$ as a function

of the density of particles ρ, is strictly bigger that $1/2$ for all positive γ and $\rho \in [0,1]$, and it converges to $1/2$ as γ goes to ∞ for all $\rho \in [0,1]$. We also show that $D^\gamma(\rho) = O(\frac{1}{\gamma})$ as γ goes to 0 for $\rho \in [0,1)$. On the other hand, for $\rho = 1$, we show that $D^\gamma(\rho) \leq O(\frac{1}{\sqrt{\gamma}})$. The difference of the order between $\rho \in [0,1)$ and $\rho = 1$ implies that $\rho = 1$ has some special property. In [5], we study a high-dimensional version of the model we consider here and show that in the case $d \geq 3$, the diffusion coefficient $D^\gamma(\rho)$ of the hydrodynamic equation goes to ∞ for $\rho \in [0,1)$ but remains finite for $\rho = 1$ as γ goes to 0. Especially, we conjecture that $\lim_{\gamma \downarrow 0} D^\gamma(1)$ relates to the diffusion coefficient for TASEP, which diverges for $d = 1, 2$ but remains finite for $d \geq 3$ (see e.g. [2], [3]) .

In the talk, I also stated two following conjectures implied from numerical simulations: (i) the diffusion coefficient $D^\gamma(\rho)$ is decreasing as a function of ρ, (ii) the diffusion coefficient $D^\gamma(\rho)$ is decreasing as a function of γ. The first statement means that the lower the density is, the faster the diffusion is. This property itself is usual and natural from the physical point of view. The significant thing is even though there is no such a structure in the microscopic system of our model, we can see this property at the macroscopic level. This is not the case for SSEP where the diffusion coefficient is a constant.

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Thermal transport in one-dimensional systems: Some numerical results

Gabriel Stoltz

(joint work with Alessandra Iacobucci, Michele Lazzeri, Frédéric Legoll, Francesco Mauri, Natalio Mingo and Stefano Olla)

Thermal properties are usually investigated using Fourier's law. For nonequilibrium steady states where the system is put in contact with two reservoirs at different temperatures, there is a net energy flow from the hotter to the colder reservoir. The heat current density $\mathbf{J}$ is proportional to the temperature gradient

$$\mathbf{J} = \kappa \nabla T, \tag{1}$$

κ being the thermal conductivity (a tensor, in general). Denoting by ΔT the temperature difference between the reservoirs, and by L the system size in the direction of the temperature gradient,

$$\kappa = \frac{|\mathbf{J}|L}{\Delta T},$$

provided the temperature profile is linear. For usual three dimensional materials, the thermal conductivity does not depend on the system size, and so, it is a well-defined thermodynamical quantity. The situation is different for one-dimensional (1D) systems. For defect free periodic one-dimensional harmonic systems, there is no scattering mechanism, and these systems can sustain a current which does not depend on the system's length. The thermal conductivity therefore diverges as L, and is not well-defined. In general, one dimensional (1D) systems in which scattering processes can take place should exhibit an intermediate scaling $|\mathbf{J}|/|\nabla T| \sim L^{\alpha}$ with $0 < \alpha < 1$, in which case the thermal conductivity is again not well-defined.

We here study the thermal conductivity of one dimensional systems in two cases:

- the Toda lattice perturbed by a stochastic dynamics preserving energy and momentum;
- carbon nanotubes with harmonic interactions and mass disorder.

Thermal transport in the Toda chain with a noise preserving energy and momentum. Numerical evidence shows that *non-integrability* (whatever the definition of this concept one considers) is not a sufficient condition for normal conductivity, in particular for anharmonic chains of unpinned oscillators like the FPU model. Stochastic perturbations of the dynamics have been introduced in order to understand these phenomena. Stochastic perturbation conserving both the energy *and* the momentum, as introduced in [1], model the chaotic effect of nonlinearities. These systems may then be seen as *completely non-integrable*, since the only conserved quantities left are the energy and the momentum. For anharmonic interactions, rigorous upper bounds on the thermal conductivity can be established. In the one dimensional case, this leads to $\kappa_L \leq C\sqrt{L}$ (see [2]).

These rigorous results motivated us to analyze the effect of stochastic perturbations on another completely integrable system, the Toda chain. In contrast with the harmonic case where many computations can be performed analytically, the nonlinear dynamics considered here has to be solved numerically. We considered a chain in the nonequilibrium steady state setting, with two Langevin thermostats at different temperatures attached to its boundaries. We chose the simplest possible stochastic perturbation conserving both momentum and energy: Each couple of nearest neighbor particles exchange their momentum at random times distributed according to an exponential law of parameter $\gamma > 0$. The corresponding numerical results are presented in [3].

Our main results are the following:

(1) As soon as some noise is present, *i.e.* $\gamma > 0$, the ballistic transport is immediately destroyed (as in the harmonic case) and energy superdiffuses, with $\kappa_L \sim L^\alpha$ for $0 < \alpha \leq 1/2$;
(2) the exponent α seems to depend on the noise strength γ, in a monotonically increasing way.

If 1 was somehow expected, 2 is quite surprising. It may be explained by the fact that the noise destroys some diffusive phenomena due to nonlinearities, like localized breathers, with the result that current-current correlation decays slower when more noise is present Besides, 2 suggests that any theory claiming the existence of a universal parameter α has to be properly circumstanced.

Quantum thermal transport in harmonic carbon nanotubes with mass disorder. We present in [4, 5] a study of the phononic thermal conductivity of isotopically disordered carbon nanotubes. In particular, the behavior of the thermal conductivity as a function of the system length is investigated, using Green's function techniques to compute the transmission across the system. The method is implemented using linear scaling algorithms, which allows us to reach systems of lengths up to $L = 2.5$ μm (with up to 200,000 atoms). As for 1D systems, it is observed that the conductivity diverges with the system size L. We also consider the agreement of the model with a reduced description based on Boltzmann's equation in [4], and study the influence of the mass disorder patterns in [5].

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Numerical methods for free energy calculations

Tony Lelièvre
(joint work with M. Rousset and G. Stoltz)

The sampling of multimodal measures is a central problem in Monte Carlo approaches, in many contexts like in Bayesian statistics or in molecular dynamics. In molecular dynamics, the question is how to efficiently sample a Boltzmann-Gibbs probability measure of the form

$$d\mu(x) = Z^{-1} \exp(-\beta V(x))\, dx$$

where $Z = \int \exp(-\beta V(x))\, dx$, β is proportional to the inverse of the temperature, and $V : \mathbb{R}^{3N} \to \mathbb{R}$ associates to a given position of N particles, an energy. To sample μ, ergodic dynamics with respect to μ are used, typically of the form

$$dX_t = -\nabla V(X_t)\, dt + \sqrt{2\beta^{-1}} dW_t.$$

The multimodality of the measure μ comes from the fact that the potential V exhibits many wells which are separated by energy barriers. This means that X_t is typically a metastable process, and that using trajectorial averages along X_t to sample μ may be difficult in practice. The main question is: how to cure this difficulty ?

The main assumption which is behind most of the methods used in molecular dynamics is that most of the metastability of the process X_t is actually "contained" in very few degrees of freedom. A precise mathematical meaning in terms of logarithmic Sobolev inequality constants is given in [10, 7]. More precisely, one introduces a so-called reaction coordinate (or order parameter)

$$\xi : \mathbb{R}^{3N} \to \mathbb{T}$$

where $\mathbb{T}$ denotes the one-dimensional torus, which is typically such that $\xi(X_t)$ is metastable. The range of ξ is here the torus for simplicity, but the techniques apply as soon as it is a low-dimensional manifold. One quantity of particular interest for applications in molecular dynamics, and that will be useful to circumvent the difficulties due to the metastability of X_t, is the free energy which is defined as

$$A(z) = -\beta^{-1} \ln \int_{\Sigma(z)} \exp(-\beta V) \delta_{\xi(x)-z}(dx),$$

where $\Sigma(z) = \{x, \xi(x) = z\}$ and $\delta_{\xi(x)-z}(dx)$ is a measure with support $\Sigma(z)$ defined by the conditioning formula $dx = \delta_{\xi(x)-z}(dx)\, dz$. Otherwise stated, $\exp(-\beta A(z))dz$ is the image of the measure μ by ξ. The free energy may be seen as an "effective energy" along ξ.

Computing the free energy (or more precisely free energy differences, since this is a quantity defined up to an additive constant) is a fundamental problem in molecular dynamics [3]. In particular, using the naive algorithm which would consist in using $\xi(X_t)$ to sample $\exp(-\beta A(z))\, dz$ does not work in practice due to the metastability of $\xi(X_t)$.

Traditional methods to compute free energy differences may be divided into four classes:

- Perturbation and histogram methods (based on classical sampling techniques),
- Thermodynamic integration (based on homogeneous constrained Markov processes to sample conditional probability measures),
- Out-of-equilibrium dynamics and fluctuation identities (based on Feynman-Kac formulae and non-homogeneous Markov processes),
- Adaptive methods (based on adaptive importance sampling techniques, and non-homogeneous and non-linear Markov processes).

Let us focus on adaptive methods. We refer to [11] for a general introduction to all these techniques.

The bottom line of adaptive methods is the following: for a well-chosen function ξ, the dynamics associated to the biased potential $V - A \circ \xi$ is less metastable than the original dynamics, associated to the potential V. This is intuitively sensible, since the image of the measure $\exp(-\beta(V - A \circ \xi))$ by ξ is the uniform law on the torus, which is of course very easy to sample, and not multimodal. The technique will thus work if the essential metastability of X_t is along ξ.

Of course, A is not known in general, so that using the biased potential $V - A \circ \xi$ is not feasible in practice. The idea is then to use a dynamics associated to a time-dependent biased potential $V - A_t \circ \xi$, where A_t is at time t an approximation of A. One idea to update A_t is to start from the formula (which is a consequence of the co-area formula):

$$A'(z) = \frac{\int f \exp(-\beta V) \delta_{\xi(x)-z}(dx)}{\int \exp(-\beta V) \delta_{\xi(x)-z}(dx)}$$

where $f = \frac{\nabla V \cdot \nabla \xi}{|\nabla \xi|^2} - \beta^{-1} \mathrm{div}\left(\frac{\nabla \xi}{|\nabla \xi|^2}\right)$. An equivalent formulation in terms of conditional expectation is

$$A'(z) = \mathbb{E}_\mu(f(Y) | \xi(Y) = z).$$

Then, a natural Adaptive Biasing Force method [5, 6] is the following:

$$\begin{cases} dY_t = -\nabla(V - A_t \circ \xi)(Y_t)\, dt + \sqrt{2\beta^{-1}} dW_t, \\ A_t'(z) = \mathbb{E}(f(Y_t) | \xi(Y_t) = z). \end{cases}$$

Notice in particular that, if a stationary state is reached, then necessarily $A_\infty' = A'$: the free energy is recovered in the longtime limit. Of course, the natural question is then: what did we gain compared to the original dynamics on X_t ?

We have shown in a series of work [10, 7, 2, 8] that the dynamics on Y_t goes to equilibrium essentially as fast as the dynamics using the biased potential $V - A \circ \xi$. In other words, the adaptive potential method is as efficient as a biasing by the free energy. This technique can be used in other contexts than molecular dynamics to enhance the sampling of multimodal measure [4], and admits various refinements and extensions [9, 2].

Let us conclude by two current research directions which are interesting in molecular dynamics. First, there is a lack of efficient techniques to *sample non-equilibrium steady-states*, namely probability distributions which are defined as the stationary measure of a stochastic dynamics. In such a context, classical techniques such as those mentioned above (constrained sampling, importance sampling) are not applicable. How to efficiently deal with metastability in such a context is mostly an open problem. This is related to theoretical questions on the rate of convergence to equilibrium for Fokker-Planck equation with a non-gradient drift. Second, a very important question concerns the *sampling of metastable dynamics* (and not only multimodal measures). Here, the question is how to generate efficiently equilibrium trajectories which leave one well of the potential V to reach another well. Techniques exist [1], but the situation is not so clear as for the sampling of multimodal measures. Mathematics should help to classify the methods, and improve them.

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Effective dynamics using conditional expectations

Frédéric Legoll
(joint work with Tony Lelièvre)

Consider the overdamped Langevin dynamics

$$dX_t = -\nabla V(X_t)\,dt + \sqrt{2\beta^{-1}}\,dW_t, \tag{1}$$

where $X_t \in \mathbb{R}^N$ represents the state of the system at time t, $V : \mathbb{R}^N \to \mathbb{R}$ is a given potential energy and W_t is a N-dimensional brownian motion. This dynamics is often used in molecular simulation to sample the Gibbs measure

$$d\mu = Z^{-1} \exp(-\beta V(x))\,dx. \tag{2}$$

Consider now a coarse-grained variable (supposed in this work to be scalar),

$$\xi : \mathbb{R}^N \to \mathbb{R},$$

which represents a quantity of interest (for instance, X describes the position of all atoms of a molecule, while $\xi(X)$ is a particular angle between three atoms of the molecule). The question we consider is to build a process $t \mapsto z_t$ that approximates the process $t \mapsto \xi(X_t)$.

We proceed as follows [2, 3]. First, from the dynamics on X_t, we obtain

$$d\left[\xi(X_t)\right] = \left(-\nabla V \cdot \nabla\xi + \beta^{-1}\Delta\xi\right)(X_t)\,dt + \sqrt{2\beta^{-1}}\;|\nabla\xi|(X_t)\;dB_t,$$

where B_t is a one-dimensional brownian motion. We then take conditional expectations of the drift and diffusion coefficients with respect to the Gibbs equilibrium measure (2):

$$\begin{aligned} b(z) &:= \mathbb{E}_{\text{Gibbs}}\left[\left(-\nabla V \cdot \nabla\xi + \beta^{-1}\Delta\xi\right)(X) \mid \xi(X) = z\right], \\ \sigma^2(z) &:= \mathbb{E}_{\text{Gibbs}}\left[|\nabla\xi|^2(X) \mid \xi(X) = z\right], \end{aligned}$$

and consider the effective dynamics

$$dz_t = b(z_t)\,dt + \sqrt{2\beta^{-1}}\,\sigma(z_t)\,dB_t, \quad z_0 = \xi(X_0). \tag{3}$$

This dynamics is ergodic (and reversible) for the measure $\xi \star \mu$, which is the equilibrium law of $\xi(X)$ when X is distributed according to (2).

We have assessed the accuracy of the effective dynamics (3) in a two-fold manner:

- we have analytically derived an upper-bound on the relative entropy

$$E(t) = H(\psi_{\text{exact}}|\phi_{\text{eff}}) = \int_{\mathbb{R}} \ln\left(\frac{\psi_{\text{exact}}(t,z)}{\phi_{\text{eff}}(t,z)}\right)\phi_{\text{eff}}(t,z)\;dz$$

  where ψ_{exact} is the law of $\xi(X_t)$ and ϕ_{eff} is the law of z_t.

  This bound is not an asymptotic result (valid in the limit of a small parameter going to 0). In the case when an appropriate time-scale separation is present (an assumption that we formalize in the language of logarithmic Sobolev inequalities), we infer from this bound that $E(t)$ is small, and thus the effective dynamics (3) is accurate.

- a quantity of paramount importance in molecular simulation is the average residence time that a system spends in a potential energy well (that is, close to a specific conformation) before hopping to another one, when its dynamics is given by (1). Most often, conformations can be characterized by the value of $\xi(X)$, that is the system is in a given conformation if and only if $\xi(X)$ is in a given interval. Knowing the evolution of $\xi(X_t)$ (or an approximation of it) is thus sufficient to compute residence times. On several test cases, we numerically checked that our effective dynamics provides accurate residence times, in comparison to the reference ones.

We have recently extended the numerical strategy to the case of the Langevin equation

$$\begin{aligned} dX_t &= p_t\,dt, \\ dp_t &= -\nabla V(X_t)\,dt - \gamma p_t\,dt + \sqrt{2\gamma\beta^{-1}}\,dW_t, \end{aligned}$$

where $\gamma > 0$ is a parameter. In this case, we again numerically observe that the obtained effective dynamics accurately reproduces the residence times in the metastable domains [1].

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Rigorous scaling law for the heat current in a disordered harmonic chain

François Huveneers

(joint work with Oskari Ajanki)

In this talk, I present a new result about the energy current in a model of heat conduction, first considered in detail by Casher and Lebowitz [4]. The model consists of a one-dimensional disordered harmonic chain of n i.i.d. random masses, connected to their nearest neighbors via identical springs, and coupled at the boundaries to Langevin heat baths, with respective temperatures T_1 and T_n. Let $\mathsf{E}(J_n)$ be the steady-state energy current across the chain, averaged over the masses. We prove that $\mathsf{E}(J_n) \sim (T_1 - T_n)n^{-3/2}$ in the limit $n \to \infty$, as has been conjectured by various authors over the time. Our proof relies on a new explicit representation for the elements of the product of associated transfer matrices.

Let us see this in some more details. In a bulk of material, Fourier's law is said to hold if the flux of energy J is proportional to the gradient of temperature, i.e.,

$$J = -\kappa\nabla T\,, \tag{1}$$

where κ is called the conductivity of the material. This phenomenological law has been widely verified in practice. Nevertheless, the mathematical understanding of thermal conductivity starting from a microscopic model is still a challenging question [3] [6] (see also [9] for a historical perspective).

Since the work of Peierls [13], it has been understood that anharmonic interactions between atoms should play a crucial role in the derivation of Fourier's law for perfect crystals. It has been known for a long time that the conductivity of perfect harmonic crystals is infinite. Indeed, in this case, phonons travel ballistically without any interaction. This yields a wave like transport of energy across the system, which is qualitatively different than the diffusion predicted by the Fourier's law (1). For example, in [14], it is shown that the energy current in a one-dimensional perfect harmonic crystal, connected at each end to heat baths, is proportional to the difference of temperature between these baths, and not to the temperature gradient.

In addition to the non-linear interactions, also the presence of impurities causes scattering of phonons and may therefore strongly affect the thermal conductivity of the crystal. Thus, while avoiding formidable technical difficulties associated to anharmonic potentials, by studying disordered harmonic systems one can learn about the role of disorder in the heat conduction. Moreover, many problems arising with harmonic systems can be stated in terms of random matrix theory, or can be reinterpreted in the context of disordered quantum systems.

Indeed, in [5], Dhar considered a one-dimensional harmonic chain of n oscillators connected to their nearest neighbors via identical springs and coupled at the boundaries to rather general heat baths parametrized by a function $\mu : \mathbb{R} \to \mathbb{C}$ and the temperatures T_1 and T_n of the left and right bath, respectively. Dhar expressed the steady state heat current $J_n^{(\mu)}$ as the integral over oscillation frequency w of the modes:

$$J_n^{(\mu)} \;=\; (T_1 - T_n) \int_{\mathbb{R}} |v_{\mu,n}^t(w) A_n(w) \cdots A_1(w) v_{\mu,1}(w)|^{-2} \mathrm{d}w \,. \tag{2}$$

Here $A_k(w) \in \mathbb{R}^{2\times 2}$ is the random transfer matrix corresponding the mass of the k^{th} oscillator, while $v_{\mu,1}(w)$ and $v_{\mu,n}(w)$ are $\mathbb{C}^2$-vectors determined by the bath function μ and the masses of the left and the right most oscillators, respectively. Thanks to the work of furstenberg [7] and Matsuda and Ishii [10], it is known that, asymptotically, the norm of $Q_n(w) := A_n(w) \cdots A_1(w)$ grows almost surely like $\mathrm{e}^{\gamma(w)n}$ where the non-random function $\gamma(w) \geq 0$ is the associated Lyapunov exponent. In the context of heat conduction this corresponds the localization of the eigenmodes of one-dimensional chains while in disordered quantum systems one speaks about the one-dimensional Anderson localization [2].

However, in the absence of an external potential (pinning), the Lyapunov exponent scales like w^2, when w approaches zero, and this makes the scaling behavior of (2) non-trivial as well as highly dependent on the properties of the bath. Indeed, only those modes for which the localization length $1/\gamma(w) \sim w^2$ [10] is of equal or higher order than the length of the chain, n, do have a non-exponentially vanishing contribution in (2). Thus the heat conductance of the chain depends

crucially on how the bath vectors $v_{\mu,1}(w), v_{\mu,n}(w)$ weight the critical frequency range $w^2 n = \mathcal{O}(1)$. In other words, explaining the scaling of the heat current in disordered harmonic chains reduces to understanding the limiting behavior of the matrix product $Q_n(w)$ when $w \leq n^{-1/2+\epsilon}$ for some $\epsilon > 0$.

The evolution of $n \mapsto Q_n(w)$ reaches stationarity only when $w^2 n \sim 1$ while the components of $Q_n(w)$ oscillate in the scale $wn \sim 1$ with a typical amplitude of $w^{-1}\mathrm{e}^{\gamma_0 w^2 n}$ as observed numerically in [5]. Thus the challenge when working in this small frequencies regime is that the analysis does fall back neither to classical asymptotic estimates for large n, nor to the estimate of the Lyapunov exponent for small w.

Of course, the difficulty of this analysis depends also on the exact form of the vectors $u_{\mu,k}$ in (2), i.e., on the choice of the heat baths. Besides some rather recent developments, most of the studies so far have concentrated on two particular models. In the first model, introduced by Rubin and Greer [15], the heat baths themselves are semi-infinite ordered harmonic chains distributed according to Gibbs equilibrium measures of temperatures T_1 and T_n, respectively. Rubin and Greer were able to show that $\mathsf{E} J_n^{\mathrm{RG}} \geq \mathrm{C}\, n^{-1/2}$ with $\mathsf{E}(\cdot)$ denoting the expectation over the masses. Later Verheggen [16] proved that $\mathsf{E} J_n^{\mathrm{RG}} \sim n^{-1/2}$.

In the second model the heat baths are modeled by adding stochastic Ornstein-Uhlenbeck terms to the Hamiltonian equations of the chain. This model, first analyzed by Casher and Lebowitz [4] in the context of heat conduction, was conjectured by Visscher (see ref. 9 in [4]) to satisfy $\mathsf{E}(J_n^{\mathrm{CL}}) \sim n^{-3/2}$. In [1], we established this result rigorously. Moreover, our method allowed us to treat also some other cases introduced in [5]. All our results, however, only concern the expectation of the current, and it should be highly desirable to know whether almost sure bounds are valid as well.

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Thermodynamic Limit for the Invariant Measures of Zero Range Processes at the Critical Density

Michail Loulakis
(joint work with Inés Armendáriz, Stefan Großkinsky)

We consider zero range processes with decreasing jump rates

$$g(n) \sim 1 + \frac{b}{n^\lambda} \quad \text{as } n \to \infty \ .$$

For $\lambda \in (0,1)$, $b > 0$ and for $\lambda = 1$, $b > 2$ the following phase transition is established: If the particle density ρ exceeds a critical value ρ_c, the system phase separates into a homogeneous background with density ρ_c and a single randomly located condensate that contains a macroscopic fraction of all the particles [5, 2, 4].

To understand the onset of the condensate formation at the critical density ρ_c, we consider processes with N particles on L sites in the thermodynamic limit with $N/L \to \rho_c$, but where the excess mass $N - \rho_c L$ tends to infinity on a scale $o(L)$. We identify the exact critical scale, below which the excess mass is distributed among all lattice sites, and above which it concentrates on a single one. We also characterize the behaviour at the critical scale, where both scenarios can occur with positive probability. Our results provide a rather complete picture of the transition from a homogeneous subcritical to condensed supercritical behaviour, including the change of the extreme value statistics of the maximum. They also provide a detailed understanding of finite-size effects, which are important in applications [3].

Let $\mu_{L,N}$ denote the unique invariant measure of a zero range process with N particles on L sites. These measures can be constructed as the grand-canonical measures at the critical density ν_{ϕ_c} conditioned on the total number of particles,

$$\mu_{L,N}\big[\,\cdot\,\big] = \nu_{\phi_c}\Big[\ \cdot\ \Big|\sum_{x=1}^{L}\eta_x = N\Big]$$

We state here the results for $\lambda = 1,\ b > 3$, in which case ν_{ϕ_c} have polynomial tails and a finite variance, σ^2. We refer the reader to [1] for the corresponding results when $\lambda \in (0,1)$.

Theorem 1 (upside deviations)**.** *Assume that $N \geq \rho_c L$ and define $\gamma_L \in \mathbb{R}$ by*

$$N = \rho_c L + \sigma\sqrt{(b-3)L\log L}\left(1 + \frac{b}{2(b-3)}\frac{\log\log L}{\log L} + \frac{\gamma_L}{\log L}\right). \tag{1}$$

a) If $\gamma_L \to -\infty$ the distribution under $\mu_{L,N}$ of the maximum M_L is asymptotically equivalent to its distribution under ν_{ϕ_c}. Precisely, there exists a $u > 0$ such that for all $x > 0$ we have

$$\lim_{L\to\infty} \mu_{L,N}\left[\frac{M_L}{L^{\frac{1}{b-1}}} \leq x\right] = \lim_{L\to\infty} \nu_{\phi_c}\left[\frac{M_L}{L^{\frac{1}{b-1}}} \leq x\right] = e^{-ux^{1-b}}\ . \tag{2}$$

$$\text{In particular, if } \ N - \rho_c L \gg L^{\frac{1}{b-1}} \ \text{ then } \qquad \frac{M_L}{N-\rho_c L} \xrightarrow{\mu_{L,N}} 0.$$

b) If $\gamma_L \to +\infty$ the normalized fluctuations of the maximum around the excess mass under $\mu_{L,N}$ converge in distribution to a normal r.v.,

$$\frac{M_L - (N-\rho_c L)}{\sqrt{L\sigma^2}} \xrightarrow{d} N(0,1)\ . \tag{3}$$

$$\text{In particular,} \qquad \frac{M_L}{N-\rho_c L} \xrightarrow{\mu_{L,N}} 1.$$

c) If $\gamma_L \to \gamma \in \mathbb{R}$ we have convergence in distribution to a Bernoulli random variable,

$$\frac{M_L}{N-\rho_c L} \xrightarrow{d} Be(p_\gamma)\ ,$$

where $p_\gamma \in (0,1)$ is an explicit constant, such that $p_\gamma \to 0$ (1), as $\gamma \to -\infty$ $(+\infty)$.

In view of (1) the critical scale above which the excess mass concentrates on the maximum and forms a condensate is of order $N - \rho_c L \sim \sqrt{L\log L}$. On that scale the transition appears suddenly at $\Delta_L = \sigma\sqrt{(b-3)L\log L}$, with the condensate forming out of particles taken from the bulk, and the correlations switch from being entirely absorbed by the bulk (2) to being entirely absorbed by the maximum (Theorem 1b in [2]). It is also interesting to note that the excess mass $N - \rho_c L$ is always either distributed among all sites, or concentrates on a single site only, and that both phases can exist with positive probability at the critical point.

The following theorem together with (3), describes how the fluctuations of the maximum switch from Gumbel at subcritical densities, to Gaussian at supercritical densities, via Fréchet at criticality.

Theorem 2 (downside deviations)**.** *Define $\omega_L \geq 0$ by $N = \rho_c L - \omega_L \sigma^2 L^{\frac{b-2}{b-1}}$.*
a) If $\omega_L \to 0$ then (2) holds, thus the distribution under $\mu_{L,N}$ of the maximum M_L is asymptotically equivalent to its distribution under ν_{ϕ_c}.
b) If $\omega_L \to \omega > 0$ then there exists a positive constant A such that for all $x > 0$

$$\lim_{L\to\infty} \mu_{L,N}\left[\frac{M_L}{L^{\frac{1}{b-1}}} \leq x\right] = \exp\left\{-A\int_x^\infty e^{-\omega t}\frac{dt}{t^b}\right\}.$$

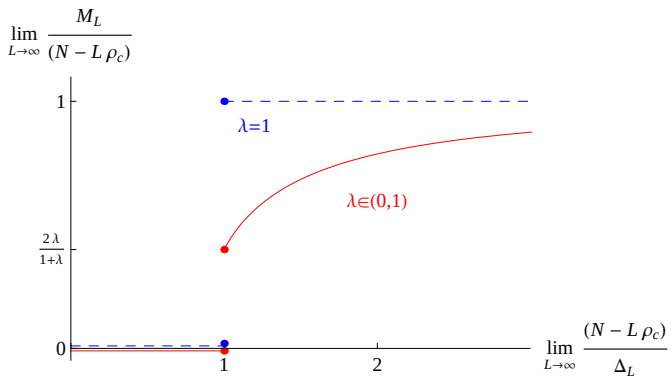

FIGURE 1. The law of large numbers for the excess mass fraction $\frac{M_L}{N-\rho_c L}$ in the condensate at the critical scale.

c) If $\omega_L \to \infty$ then there exist sequences $B_L \to \infty$ and $s_L = \frac{\rho_c L - N}{\sigma^2 L}(1+o(1))$ with $B_L s_L \to \infty$, such that for all $x \in \mathbb{R}$

$$\lim_{L\to\infty} \mu_{L,N}\left[M_L \le B_L + x\frac{1}{s_L}\right] = \exp\{-e^{-x}\}.$$

The next result complements Corollary 3 in [2], which describes the supercritical fluctuations of the bulk.

Theorem 3 (subcritical fluctuations of the bulk)**.** *If $\gamma_L \to -\infty$ in (1), then the distribution under $\mu_{L,N}$ of the bulk fluctuation process, converges to a Brownian Bridge conditioned to return to the origin at time 1, that is*

$$X_t^L = \frac{1}{\sigma\sqrt{L}}\sum_{x=1}^{[tL]}(\eta_x - \frac{N}{L}) \overset{\mu_{L,N}}{\Longrightarrow} BB_t \ .$$

When $\lambda \in (0,1)$ (the tails of ν_{ϕ_c} are stretched exponential) the results are analogous, with one notable exception. The critical scale above which the excess mass forms a condensate is of order $N-\rho_c L \sim L^{\frac{1}{1+\lambda}}$. On that scale a transition appears at $\Delta_L = c_\lambda(\sigma^2 L)^{\frac{1}{1+\lambda}}$, for some explicitly known $c_\lambda > 0$, when the maximum jumps to a positive fraction of $N-\rho_c L$. However, the excess mass is shared between the maximum and the bulk at this scale, as Figure 1 illustrates.

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Finite size effects and condensation in zero-range processes

Stefan Grosskinsky
(joint work with Paul Chleboun)

We study zero-range processes with jump rates

$$g(n) = \begin{cases} 1 + \frac{b}{n^\gamma} & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases} \tag{1}$$

with $\gamma \in (0,1)$ and $b > 0$, which are known to exhibit a condensation transition [1]. If the particle density exceeds a finite critical value ρ_c, a non-zero fraction of all particles accumulates on a single lattice site. This phenomenon has been a subject of major research interest in theoretical physics and applied probability, where most results focus on the thermodynamic limit. The system also exhibits large finite size effects, which have been addressed in several recent studies [2, 3, 4, 5]. We observe a switching between metastable fluid and condensed phases and establish a rigorous scaling limit explaining this discontinuous behaviour near criticality. Our results are published in [6] and constitute an interesting example where the thermodynamic limit fails to capture essential parts of the dynamics. This is particularly relevant in applications with moderate system sizes such as traffic flow [7] or granular clustering (see [8] and references therein).

Consider a one dimensional lattice $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$ of size L with periodic boundary conditions and denote configurations by $\boldsymbol{\eta} = (\eta_x)_{x\in\Lambda_L}$, where $\eta_x \geq 0$ is the number of particles on site $x \in \Lambda_L$. The zero-range process is given by the generator

$$(\mathcal{L}f)(\boldsymbol{\eta}) = \sum_{x=1}^{L} g(\eta_x)\left(f(\boldsymbol{\eta}^{x,x+1}) - f(\boldsymbol{\eta})\right) \ , \tag{2}$$

where $\eta_z^{x,x+1} = \eta_z - \delta(z,x) + \delta(z,x+1)$ denotes the configuration change due to a particle hopping. For simplicity of presentation we focus on totally asymmetric nearest neighbour jumps in one dimension, but our results hold in a more general setting (see [6] for details). The stationary measures are known to be of product form, and for a fixed number N of particles they are given by

$$\pi_{L,N}[\boldsymbol{\eta}] = \frac{1}{Z(L,N)} \prod_{x\in\Lambda_L} w(\eta_x)\, \delta\Big(\sum_x \eta_x, N\Big) \ , \tag{3}$$

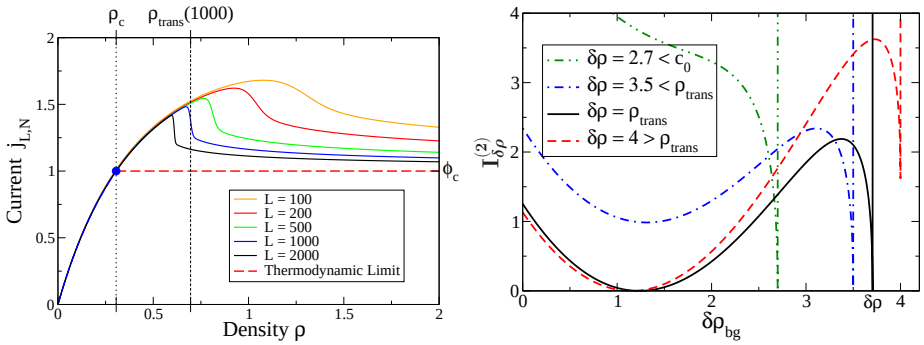

FIGURE 1. Finite size effects for $\gamma = 0.5$ and $b = 4$. **Left:** Current overshoot for various system sizes. **Right:** Scaling limit for the rate function of the excess mass distribution for various values of $\delta\rho$.

with weights $w(n) = \prod_{k=1}^{n} \frac{1}{g(k)} \sim e^{-\frac{b}{1-\gamma} n^{1-\gamma}}$ asymptotically decaying as a stretched exponential. The normalizing partition function $Z(L, N)$ determines the stationary particle current by

$$j_{L,N} := \mathbb{E}_{\pi_{L,N}} g(\eta_x) = \frac{Z(L, N-1)}{Z(L, N)} \ , \tag{4}$$

which can be computed numerically by making use of the recursion

$$Z(L, N) = \sum_{k=0}^{N} w(k) Z(L-1, N-k) \ . \tag{5}$$

The current as a function of density $\rho = N/L$ shows a large overshoot compared to the thermodynamic limit behaviour (see Fig. 1). For large L a sudden drop in the current occurs at $\rho_{trans}(L) > \rho_c$, and near that density Monte-Carlo simulations show a metastable switching between a condensed phase, where the excess mass concentrates on a single lattice site, and a fluid phase, where it is distributed homogeneously (see also [4]). These metastable phases correspond to the upper and lower branch of the current curve which can be extended and characterized using a current matching argument and the grand-canonical distributions under a change of measure with cut-off [6]. To describe the switching, we rigorously derive an effective double-well potential as a large-deviation rate function for the maximal occupation number $M_L = \max_{x \in \Lambda_L} \eta_x$. In a particular scaling limit around the critical point we parameterise the excess density of particles by $\delta\rho > 0$ and the part of $\delta\rho$ that is outside the maximum by $\delta\rho_{bg} \in (0, \delta\rho)$.

Theorem 1. *Let $\sigma_c^2 < \infty$ be the variance of η_x at the critical point. In the limit $L, N, m \to \infty$ with $N = \rho_c L + \delta\rho L^{\frac{1}{1+\gamma}}$ and $m = (\delta\rho - \delta\rho_{bg})L^{\frac{1}{1+\gamma}}$ we have*

$$I_{\delta\rho}^{(2)}(\delta\rho_{bg}) := \lim_{L\to\infty} L^{-\frac{1-\gamma}{1+\gamma}} \log \pi_{L,N}[M_L = m]$$
$$= \frac{\delta\rho_{bg}^2}{2\sigma_c^2} + \frac{b}{1-\gamma}(\delta\rho - \delta\rho_{bg})^{1-\gamma} - \inf_{r\in(0,\delta\rho)} \left\{ \frac{r^2}{2\sigma_c^2} + \frac{b}{1-\gamma}(\delta\rho - r)^{1-\gamma} \right\}$$

provided $\frac{1}{\sigma_c^2}\delta\rho_{bg} < \frac{b}{(1-\gamma)(\delta\rho-\delta\rho_{bg})^\gamma}$.

The double-well structure of $I_{\delta\rho}^{(2)}$ is illustrated in Fig. 1 (right), where the right boundary minimum corresponds to the fluid phase and the left minimum to the condensed phase, which exists only for large enough excess mass $\delta\rho$. From the explicit form of the rate function we can derive $\rho_{trans}(L)$ and the typical size of the maximum in accordance with [4], and further get predictions for the lifetimes of the metastable phases which are supported by simulation results.

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Multitype Contact Process on $\mathbb{Z}$: Extinction and Interface

Daniel Valesin

This talk covers the contents of the following works:
[9] Multitype Contact Process on $\mathbb{Z}$: Extinction and Interface,
Daniel Valesin,
[2] Tightness for the Interface of One-Dimensional Contact Process,
Enrique Andjel, Thomas Mountford, Leandro P. R. Pimentel, Daniel Valesin.

The Contact Process is a model for the spread of an infection in a population and has been extensively studied since its introduction in 1974 ([7]). In my presentation, I talked about modified versions of the Contact Process that allow for more than one type of occupant. They are the Multitype Contact Process and the

Grass-Bushes-Trees model, both models for the competition between different biological species. Each vertex, or site, of the underlying graph is a geographic region, and its state at a given time indicates the type of its occupant (or the absence of an occupant) at that time. The transitions of the dynamics are interpreted as natural deaths of individuals and generation of descendants.

1. Multitype Contact Process

The Multitype Contact Process was introduced in [8]. Its definition depends on a symmetric probability kernel $p(\cdot)$ in $\mathbb{Z}^d$ and infection rates $\lambda_1, \lambda_2 \in (0, \infty)$. The state space is $\{0, 1, 2\}^{\mathbb{Z}^d}$; given a configuration ξ, the transitions at a site $x \in \mathbb{Z}$ are, for each $i \in \{1, 2\}$,

$$i \to 0 \quad \text{with rate } 1;$$
$$0 \to i \quad \text{with rate } \lambda_i \textstyle\sum_y I_{\{\xi(y)=i\}} \cdot p(x-y);$$

I denotes the indicator function. Let us describe the dynamics in words. An individual of type i occupying a site waits an exponential time of parameter 1, after which he dies, but also waits an exponential time of parameter λ_i, after which he chooses a site to which he tries to send a descendant. The choice is made according to the kernel p, and the new birth is forbidden if the chosen site is already occupied. Given an initial configuration ξ_0, ξ_t will denote the configuration at time $t \geq 0$ obtained from the above dynamics.
In [9], we have taken $d = 1$, p to be the uniform measure on $\{-R, \ldots, -1, 1, \ldots, R\}$ (R is the *range* of the process) and $\lambda_1 = \lambda_2 > \lambda_c$, where λ_c is the critical infection rate for the original, one-type Contact Process. The first theorem of [9] gives necessary and sufficient conditions for one of the two types becoming extinct. It is a generalization of Theorem 1.1 in [1].

Theorem 1. *Assume at least one site is occupied by a 1 in ξ_0. The 1's become extinct with probability one if and only if there exists $L > 0$ such that*
(A) $\xi_0(x) \neq 1 \; \forall x \notin [-L, L]$ and
(B) $\#\{x \in (-\infty, -L] : \xi_0(x) = 2\} = \#\{x \in [L, \infty) : \xi_0(x) = 2\} = \infty$.

Another result, a "coexistence theorem", adds some information to the above theorem:

Theorem 2. *Assume that $0 < \#\{x : \xi_0(x) = 1\}, \#\{x : \xi_0(x) = 2\} < \infty$. Then, with positive probability, for all $t \geq 0$ there exist $x, y \in \mathbb{Z}$ such that $\xi_t(x) = 1, \xi_t(y) = 2$.*

Now consider $\xi^h = I_{(-\infty,0]} + 2 \cdot I_{(0,\infty)}$, the configuration containing 1's to the left and 2's to the right of the origin. Let ξ_t^h be the process started from $\xi_0^h = \xi^h$. For time $t \geq 0$, let $r_t = \sup\{x : \xi_t^h(x) = 1\}$, $l_t = \inf\{x : \xi_t^h(x) = 2\}$. The interval delimited by l_t and r_t is called the *interface area* at time t, and $|r_t - l_t|$ is called the *interface size* at time t. We now ask: if t is large, is it reasonable to expect a large interface? A negative answer is given by

Theorem 3. *The stochastic process* $(|r_t - l_t|)_{t\geq 0}$ *is tight; that is, for any* $\epsilon > 0$*, there exists* $L > 0$ *such that* $\mathbb{P}(|r_t - l_t| > L) < \epsilon$ *for every* $t \geq 0$*.*

2. Grass-Bushes-Trees

This model was first studied in [5] and [6]. It is also a modification of the Contact Process and has state space $\{0, 1, 2\}^{\mathbb{Z}^d}$. Fix a symmetric kernel p and infection rates $\lambda_1, \lambda_2 \in (0, \infty)$. Given a configuration η, transitions at a site $x \in \mathbb{Z}$ are

$$\begin{aligned} i &\to 0 \quad \text{with rate 1, for } i = 1, 2; \\ 0 &\to i \quad \text{with rate } \lambda_i \textstyle\sum_y I_{\{\eta(y)=i\}} \cdot p(x-y); \\ 2 &\to 1 \quad \text{with rate } \lambda_1 \textstyle\sum_y I_{\{\eta(y)=1\}} \cdot p(x-y). \end{aligned}$$

Here, in contrast with the previous model, the 1's, interpreted as trees, are allowed to displace the 2's, interpreted as bushes. The empty state 0 is interpreted as grass.

A convenient feature of the Grass-Bushes-Trees model is that, from the point of view of the 1's, the presence of the 2's is irrelevant, so the set of 1's evolves as an ordinary one-type Contact Process with rate λ_1. The dynamics from the point of view of the 2's is also interesting. Letting (η_t) denote the process started from a deterministic initial configuration η_0, define $\Pi_1 = \{(x, t) \in \mathbb{Z} \times [0, \infty) : \eta_t(x) = 1\}$. Conditioned on Π_1, the set occupied by the 2's behaves as a one-type Contact Process with rate λ_2 that is not allowed to enter space-time points in Π_1. This can be seen as a Contact Process in a random environment given by the area not corroded by the evolution of another Contact Process.

In [2], we have considered the Grass-Bushes-Trees model in dimension 1 with p equal to uniform measure on $\{-R, \ldots, -1, 1, \ldots, R\}$ (again, R is the range), and $\lambda_1 = \lambda_2 > \lambda_c$. Define η^h, r_t and l_t and the interface size $|r_t - l_t|$ as in the previous section. It was a conjecture from Cox and Durrett ([3]) that the process $(|r_t - l_t|)_{t\geq 0}$ is tight, and we have obtained this result in [2]:

Theorem 4. *The stochastic process* $(|r_t - l_t|)_{t\geq 0}$ *is tight.*

As explained above, the evolution of the 1's is simply the evolution of a one-type Contact Process started from the configuration where only sites in $(-\infty, 0]$ are infected. We then see the 2's, which initially occupy $(0, \infty)$, as attempting to invade the area that is left vacant by the 1's. The above theorem shows that this attempt is unsuccessful, since their conquered space does not grow in distribution as time passes.

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Quasi-polynomial mixing of the 2D stochastic Ising model with + boundary up to criticality

Fabio Martinelli
(joint work with E. Lubetzky, A. Sly, F. L. Toninelli)

We consider the heat bah dynamics of the Ising model at low temperature $\beta > \beta_c$ in a two-dimensional box Λ_L of side L, with uniform "+" boundary conditions, or random but "mostly +" boundary conditions. Each spin is updated with rate one, by sampling its value from the equilibrium distribution conditional on the instantaneous value of its neighboring spins. A celebrated conjecture [1] states that the equilibration time T_{mix} is of order L^2: if the system is started from the all "−" initial condition, the bubble of "minus" phase should shrink via a suitable version of motion by mean curvature. Until recently, the best known upper bound for T_{mix} was $T_{mix} \le \exp(c(\beta,\epsilon)L^{1/2+\epsilon})$ for each $\epsilon > 0$ and for large enough β [3]. In [4] we considerably improve such result by showing that, always for β large and $\epsilon > 0$, $T_{mix} \le \exp(c(\beta,\epsilon)L^{\epsilon})$. Finally, in [2] we extend the result to all $\beta > \beta_c$ and we improve the bound to a quasi-polynomial one: $T_{mix} \le L^{c(\beta)\log L}$. The key ingredients of our method are i) an iterative procedure where the equilibration time in a rectangle of sides L and $L^{1/2+\epsilon}$ and boundary conditions which are − on three sides and + on one side is estimated via the same quantity for a similar rectangle where L is replaced by $L/2$ ii) the so-called "censoring inequalities" proved by Y. Peres and P. Winkler [5]. The major progress in [2] with respect to [4] is given by a refinement of the recursive procedure and to the use of duality/random line representation tools (instead of cluster expansion techniques) to prove some key equilibrium estimates for Peierls contours.

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Zero temperature 3D Ising dynamics and dimer covering fluctuations

Fabio Toninelli
(joint work with P. Caputo, F. Martinelli, F. Simenhaus)

We consider the zero-temperature heat bath dynamics of the three-dimensional Ising model with homogeneous, say +, boundary conditions. At each site $i \in \mathbb{Z}^3$ there is a spin $\sigma_i = \pm 1$. Spins outside a cube C_L of side L are fixed to the value $+1$ for all times, while spins inside C_L are given at time $t = 0$ the value -1. The dynamics is defined as follows: to each $i \in C_L$ is associated an i.i.d. exponential clock of mean 1. When the clock at site i rings, then: a) if the spin σ_i has a majority of neighboring + (resp. $-$) spins, then we set $\sigma_i = +$ (resp. $\sigma_i = -$) b) if there are three + and three $-$ neighbors, then we set $\sigma_i = \pm 1$ with equal probabilities. We are interested in the first random time τ_+ at which all spins take the value +. It is immediate to see that τ_+ is almost surely finite as long as L is finite. Our main result in [1] is that

$$\mathbf{P}\left(\frac{L^2}{c\log L} \le \tau_+ \le L^2(\log L)^c\right) = 1 - o(1) \tag{1}$$

when $L \to \infty$, for a suitable constant c, where $\mathbf{P}$ is the law of the dynamics. This is to be compared with the previously known bound $cL \le \tau_+ \le c'L^3$, proven in [2]. The conjectured correct behavior is $\tau_+ \sim cL^2$ (based on a "motion by mean curvature heuristics" [4]), i.e. both our upper and lower bounds are off by logarithmic factors. The proof of our upper bound $\tau_+ \le L^2(\log L)^c$ involves the analysis of equilibration time and of equilibrium fluctuations of discrete monotone interfaces (i.e. 3D Young diagrams, i.e. dimer coverings of finite domains of the honeycomb lattice, see [3]). In few words, we show that the domain of "$-$" spins is contained with high probability in a domain which evolves via a deterministic motion by mean curvature.

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Asymmetric exclusion process with long-range interaction

Gunter M. Schütz
(joint work with V. Popkov and D. Simon)

We consider a totally asymmetric exclusion process with N particles on a periodic chain of L sites where each site can be occupied by at most one particle. A configuration $\eta \in \{0,1\}^L$ changes to a configuration η' only due to nearest neighbour hopping. Jumps occur independently after an exponentially distributed random time. The hopping rate for a particle jump depends on the position of all other particles on the chain. For a jump of particle k at site r_k to $r_k + 1$ (modulo L) the rate is given by

$$W_{\eta'\eta} = \prod_{l \neq k} \frac{\sin(\pi(r_k + 1 - r_l)/L)}{\sin(\pi(r_k - r_l)/L)} \tag{1}$$

where the product is over all particles $l \neq k$. The generator of this process can be written in quantum Hamiltonian form as

$$H = \Delta^{-1} H_{ff} \Delta - E_0. \tag{2}$$

The quantity E_0 is the lowest eigenvalue of the non-hermitian quantum Hamiltonian

$$H_{ff} = -\sum_{k=1}^{L} s_k^+ s_{k+1}^-. \tag{3}$$

with the spin-1/2 lowering and raising operators $s_k^\pm$ of the Lie algebra $SU(2)$ acting on sites k of the chain. The matrix Δ is diagonal and has the components of the lowest left eigenvector of H_{ff} on its diagonal. Remarkably, from a probabilistic perspective, all eigenvectors and eigenvalues of H_{ff} can be computed in explicit form using the Jordan Wigner transformation [1], which turns H_{ff} into a bilinear Hamiltonian for non-interacting fermions hopping on a lattice. A Fourier transformation then diagonalizes the free fermion Hamiltonian.

The symmetric version of this model was first considered by Spohn [2]. Its asymmetric counterpart (3) has recently emerged independently as describing the evolution of the usual ASEP (without long-range interaction) in the quasi-stationary regime of very large current [3]. In such a conditioned ensemble all histories up to time T have a weight $\exp(sJ(T))$ where $J(T)$ is the integrated particle current along this history. By conditioning on an atypical, very large positive current the histories with jumps to the left do not contribute to the ensemble. Hence only the left hopping part of the generator of the ASEP appears in (3). The appearance of the matrix Δ and of the lowest eigenvalue E_0 come from requiring the large deviation to persist for a very long time interval. It has been observed [4] that such extreme-current histories are realized by reaching from any given initial configuration an ensemble of configurations which allow for a large current to flow, staying there for a long time $t_2 - t_1 \propto T$ (thus described by a quasi stationary ensemble) and ultimately leaving such a regime to a more typical final state at

time T. The diagonal similarity transformation encodes this observation by projectiing the initial configuration onto the quasistationary distribution given by the lowest eigenvector $|\Delta\rangle$ and then projecting from the quasistationary distribution onto the final configuration. Our rigorous analysis of this exclusion process with long range interaction yields the following results [3, 5].

(i) Invariant measure: The invariant measure can be expressed by an explicit formula for the stationary probability of the particle configuration η with particles located at coordinates r_k given by

$$P_L(\eta) = \frac{2^{N(N-1)}}{L^N} \prod_{\substack{k,l \\ 1\le k<l\le N}} \sin^2\left(\pi \frac{r_k - r_l}{L}\right) \tag{4}$$

From this one can recover the well-known expression [1] for the static two-point density correlation for the particle occupation numbers $\eta_r \in \{0, 1\}$

$$S(n, 0) := \langle \eta_{r+n}\eta_r\rangle - \rho^2 = -\frac{\sin^2 n\pi\rho}{n^2\pi^2}. \tag{5}$$

For the stationary current we find

$$j = -E_0/L = \sin(\pi\rho)/(L\sin(\pi/L)) \tag{6}$$

Notice that finite-size corrections are of order $1/L^2$ rather than of order $1/L$ which is expected from systems with short-range interactions.

(ii) Spectral gap: Using the free fermion structure of the process the complete relaxation spectrum can be computed. The real part of the spectral gap, i.e., the inverse of the longest relaxation time τ_L in a finite system of size L is given by

$$1/\tau_L = 2\sin\left(\frac{\pi}{L}\right)\sin\pi\rho \tag{7}$$

where $\rho = N/L$ is the particle density. We see that for large L the gap is inversely proportional to the system size. Note that in the unconstrained ASEP the real part of the spectrum gap scales as $O(1/L^{3/2})$. We conclude that the long-range interaction of the effective process leads to a marked reduction of the longest relaxation time (which is the inverse spectral gap). The long range interaction brings about a change of the dynamical universality class.

(iii) Transition probabilities: We pick two arbitrary configurations η_0 and η of our system containing N particles at the positions n_i and m_i respectively, i.e., $|\eta_0\rangle = |n_1 n_2 ....n_N\rangle$, $|\eta\rangle = |m_1 m_2 ....m_N\rangle$. We assume both sets to be ordered, $n_i < n_{i+1}$ and $m_i < m_{i+1}$. The probability to find the system in configuration η at time t, provided it has been in configuration η_0 at time $t = 0$ is given by

$$P_L(\eta, t; \eta_0, 0) = \mathrm{e}^{E_0 t}\sqrt{\frac{P_L(\eta)}{P_L(\eta_0)}}\det[g_L(m_j - n_i, t)] \tag{8}$$

Here

$$g_L(d,t) = \sum_{\kappa=0}^{\infty} [(-1)^\kappa \mathrm{sign}(d)]^{N+1} \frac{t^{d_L+\kappa L}}{(d_L+\kappa L)!}, \tag{9}$$

where d is an integer ranging from $-L+1$ to $L-1$ and $d_L = d$ for $d > 0$ and $d_L = d + L$ for $d < 0$. The function sign(d) $= 1$ for $d \geq 0$ and sign(d) $= -1$ for $d < 0$. As pointed out by Spohn [2] the symmetric case can be interpreted as a system of non-intersecting random walks. The determinantal structure of (8) with the the stationary prefactors and the propagator of the totally asymmetric random walk (9) suggest an interpretation of the long-range TASEP as a totally asymmetric Dyson random walk, i.e., a lattice model for Dyson's Brownian motion (DBM) driven by an external field. This is in agreement also with the form of the hopping rates (1) which can be understood as deriving from an interaction potential that is a discrete version of the interaction in DBM.

(iv) Dynamic Structure factor: The dynamic structure factor is defined as the Fourier transform of the time-dependent stationary correlation function $h_L(n,t) = \langle \eta_{r+n}(t)\eta_r(0)\rangle - \rho^2$. Taking the thermodynamic limit we find for the Fourier modes $p \in [-\pi, \pi]$ in the range $\rho \leq 1/2$

$$\hat{S}(p,t) = \begin{cases} \frac{1}{2\pi}\int_{\rho\pi}^{\rho\pi+p} dx\, \mathrm{e}^{t(1-\mathrm{e}^{ip})\mathrm{e}^{-ix}} & p \in [0, 2\rho\pi] \\ \frac{1}{2\pi}\int_{-\rho\pi+p}^{\rho\pi+p} dx\, \mathrm{e}^{t(1-\mathrm{e}^{ip})\mathrm{e}^{-ix}} & p \in [-\pi, -2\rho\pi] \cup [2\rho\pi, \pi] \\ \frac{1}{2\pi}\int_{-\rho\pi+p}^{-\rho\pi} dx\, \mathrm{e}^{t(1-\mathrm{e}^{ip})\mathrm{e}^{-ix}} & p \in [-2\rho\pi, 0] \end{cases} \tag{10}$$

This, along with the symmetry relations $\hat{S}(1-\rho, p, t) = \hat{S}(\rho, -p, t)$ and $\hat{S}(p,t) = \hat{S}(p + 2\pi, t)$ provides an exact integral presentation valid for all densities ρ, momenta p and times t.

We are particularly interested in the large scale behaviour as expressed in the scaling limit of small p and large t of the form $p^z t = u$ where z is the dynamical exponent and u is the scaling variable. From (10) we conclude that there is non-trivial scaling behaviour for $z = 1$. In this scaling we have $t(1 - \mathrm{e}^{ip}) = -iut$ and therefore

$$\hat{S}(u) = \frac{|u|}{2\pi t} \mathrm{e}^{-iu\cos\rho\pi - |u|\sin\rho\pi} \tag{11}$$

which is valid for all $\rho \in [0, 1]$. We read off the collective velocity $v_c = \cos\rho\pi$ of the lattice gas.

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Universality in one dimensional hierarchical coalescence processes

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(joint work with Fabio Martinelli, Cyril Roberto, Cristina Toninelli)

There are several situations arising in one dimensional physics in which the non-equilibrium evolution of the system is dominated by the coalescence of certain domains or droplets, which leads to interesting coarsening phenomena. Examples are given by the coalescence of large vapor droplets in breath figures and of ordered domains in Ising and Potts models at zero temperature. Computer simulations have revealed the existence of scaling limits and, assuming the convergence to a limiting state, physicists have derived some non trivial limiting distributions for the relevant quantities in basic stochastic models (see e.g. [1], [2], [3], [6] and [7]). Surprisingly very different models present the same limiting distributions. Two questions arise naturally: How can one explain the existence of scaling limits? Why do very different models exhibit the same limiting distributions?

Motivated by the above questions we consider a large class of "hierarchical coalescence processes" (HCP). An HCP consists of an infinite sequence of coalescence processes $\{\xi^{(n)}(\cdot)\}_{n\geq 1}$: each process occurs in a different "epoch" (indexed by n) and evolves for an infinite time, while the evolution in subsequent epochs are linked in such a way that the initial distribution of $\xi^{(n+1)}$ coincides with the final distribution of $\xi^{(n)}$. The dynamics is characterized by a diverging sequence of positive numbers $d^{(1)} < d^{(2)} < \cdots$ such that $2d^{(n)} \geq d^{(n+1)}$, and by a sequence of function $\lambda_\ell^{(n)}, \lambda_r^{(n)} : [d^{(n)}, d^{(n+1)}) \to [0,1]$, $n \geq 1$. Inside each epoch the process, described by a suitable simple point process representing the boundaries between adjacent intervals (domains), evolves as follows. Only domains whose length belongs to the interval $[d^{(n)}, d^{(n+1)})$ are *active*, i.e. they can incorporate their left or right neighboring domain. This happens with probability rate $\lambda_\ell^{(n)}(d), \lambda_r^{(n)}(d)$ respectively, where d denotes the length of the incorporating domain. Inactive domains cannot incorporate their neighbors and can increase their length only if they are incorporated by active neighbors. Due to the assumption $2d^{(n)} \geq d^{(n+1)}$, after a merging step the newly produced domain always becomes inactive for that epoch but active for some future epoch. Moreover, starting at the first epoch with domains of length at least $d^{(1)}$ (as we always assume), at the end of the n^{th}–epoch, and therefore also at the beginning of the $(n+1)^{th}$–epoch, only domains of length at least $d^{(n+1)}$ are present.

In [4] we show that: (i) if the initial distribution describes a renewal process then such a property is preserved at all later times and all future epochs; (ii) the

distribution of certain rescaled variables, e.g. the domain length, has a well defined and universal limiting behavior as $n \to \infty$ independent of the details of the process (i.e. from the parameters $d^{(n)}$, $\lambda_\ell^{(n)}(d)$, $\lambda_r^{(n)}(d)$). This last result explains the universality in the limiting behavior of several very different physical systems (e.g. the East model of glassy dynamics or the Past-all-model) which was observed in several simulations and analyzed in many physical papers. More precisely, calling $g(s)$ the Laplace transform of the domain length in the initial distribution, we prove that if the limit $c_0 := \lim_{s\downarrow 0}\big[-sg'(s)/(1-g(s))\big]$ exists, then $c_0 \in [0,1]$ and moreover the domain length at the beginning of the n^{th} epoch, rescaled by $d^{(n)}$, weakly converges to a random variable $X_\infty^{(c_0)}$ with Laplace transform

$$g_\infty^{(c_0)}(s) = 1 - \exp\Big\{ - c_0 \int_1^\infty \frac{e^{-sx}}{x} dx \Big\}.$$

In addition, we give very simple and general conditions assuring the existence of the limit c_0: (i) if the domain length in the initial distribution has finite mean, then the limit c_0 exists and equals 1, (ii) if the domain length in the initial distribution belongs to the domain of attraction of an α–stable law ($0 < \alpha < 1$), then the limit c_0 exists and equals α. As explained in [4], the above results can be extended to the case that the initial distribution is exchangeable, i.e. it is left invariant by any finite permutation of the domains.

Although not explicitly stated, several coalescing systems present a hierarchical structure. Let us consider for example the Past–all-model, whose dynamics is described in [3] as follows: at each step one searches for the shortest domain which is pasted as a whole to either one of its neighbors, with equal probability. When working with infinite domains, the above procedure is not well defined since there can be infinite domains with shortest length. A rigorous definition, in the case of domains having integer length, can be obtained by means of the HCP taking $d^{(n)} = n$, $\lambda_\ell^{(n)} = \lambda_r^{(n)} = 1/2$ on $[n, n+1)$, $n \geq 1$. In the first epoch only domains of length 1 (the smallest length) are active and incorporate left and right neighbors with equal probability. The same happens at the 2^{nd} epoch, in which the active domains are the domains of length 2 (the smallest one in that epoch). And so on.

An interesting and highly non trivial example of HCP is represented by the high density (or low temperature) non-equilibrium dynamics of the East model after a deep quench from a normal density state (see also the abstract of C. Toninelli on this same volume concerning [5]). The East model is a well known example of kinetically constrained stochastic particle system with site exclusion which evolves according to a Glauber dynamics submitted to the following constraint: the 0/1 occupancy variable at a given site $x \in \mathbb{Z}$ can change only if the site $x+1$ is empty. In this case, if a domain represents a maximal sequence of consecutive occupied sites and if the particle density is very high, then the characteristic range of the length of active domains for the n^{th}-epoch is $[2^{n-1}, 2^n)$ and active domains can only merge with their left neighbor. As already mentioned, the East model and the Paste–all–model present the same limiting distributions, despite the fact that their dynamics are very different.

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Kinetically constrained models: non-equilibrium coarsening dynamics

Cristina Toninelli

(joint work with Alessandra Faggionato, Fabio Martinelli, Cyril Roberto)

We consider the non-equilibrium dynamics of the East model, a special example of unidimensional kinetically constrained spin model which has been introduced in physics literature in [1] and much analyzed in the context of the study of the liquid/glass transition and more generally of glassy dynamics [2, 3, 4].

The East model can be informally described as follows. A configuration is defined by assigning to each site of the lattice, $x \in \mathbb{Z}$, its occupation variable $\eta_x \in \{0,1\}$ corresponding to an empty and occupied site respectively. Each site x waits an independent, mean one, exponential time and then, provided its right nearest neighbor is empty (i.e. iff $\eta_{x+1} = 0$), it refreshes its occupation variable to a filled or empty state with probability $1-q$ or q respectively.

Since the constraint required to perform the elementary move on x does not involve η_x it can be easily verified that the Bernoulli$(1-q)$ product measure is an invariant reversible measure for the process. Furthermore the model is known to be ergodic for any $q \neq 0, 1$ with a positive spectral gap [5, 6] and relaxation to the equilibrium reversible measure occurs exponentially fast when evolution is started from e.g. any non-trivial product measure [7]. However, as $q \downarrow 0$, the relaxation time $T_{\text{relax}}(q)$ diverges very fast as $O\left(\exp(\lambda \log(1/q)^2)\right)$ with a sharp constant λ (see [6]). A key issue, both from the mathematical and the physical point of view, is therefore that of describing accurately the evolution at $q \ll 1$ when the initial distribution is different from the reversible one and for time scales which are large but still much smaller than $T_{\text{relax}}(q)$ (when the exponential relaxation to the reversible measure takes over). This setting is usually referred to in physics literature as "performing a quench to low temperature" and it is relevant to analyze the slow dynamics which follows a sudden quench from the liquid to the glass phase.

A rough picture of the non-equilibrium dynamics after the temperature quench to $q \downarrow 0$ is the following. Since the equilibrium vacancy density is $q \ll 1$, most of the non-equilibrium evolution will try to remove the excess of vacancies present in the initial distribution and will thus be dominated by the coalescence of domains corresponding to the intervals separating two consecutive vacancies. Of course this process must necessarily occur in a kind of cooperative way since, because of the constraint, in order to remove a vacancy other vacancies must be created nearby to its right. Since the creation of vacancies requires the overcoming of an *energy barrier*, in a first approximation the non-equilibrium dynamics of the East model for $q \ll 1$ is driven by a non-trivial energy landscape. In order to better explain the structure of this landscape suppose that we start from a configuration with only two vacancies located at the sites a and $a + \ell$, with $\ell \in [2^{n-1} + 1, \dots, 2^n]$. Then, a nice combinatorial argument (see [8] and also [4]) shows that, in order to remove the vacancy at a within time t, there must exists $s \leq t$ such that the number of vacancies inside the interval $(a, a + \ell)$ at time s is at least n. It is rather easy to show that at any given time s the probability of observing n vacancies in $(a, a + \ell)$ is $O(q^n)$ so that, in order to have a non negligible probability of observing the disappearance of the vacancy at a, we need to wait an *activation time* $t_n = O(1/q^n)$. In a more physical language the energy barrier which the system must overcome is $n = O(\log_2 \ell)$. Moreover one can also show that, once the system decides to overcome the barrier and kill the vacancy, it does it in a time scale $t_{n-1} \ll t_n$. These arguments indicates the following heuristic picture.
(i) A *hierarchical structure* of the activation times $t_n = 1/q^n$ (and of the energy landscape) well separated one from the other for $q \ll 1$.
(ii) A kind of metastable behavior of the dynamics which removes vacancies in a hierarchical fashion.
(iii) Since the characteristic time scales t_n are well separated one from the other, the evolution should show *active* and *stalling* periods. During the n^{th}-active period, identified with e.g. the interval $[t_n^{1-\epsilon}, t_n^{1+\epsilon}]$, $\epsilon \ll 1$, only the vacancies with another vacancy to their right at distance less than 2^n can be removed. At the end of an active period no vacancies with distance less than $2^n + 1$ are present anymore as well as no extra (i.e. not present at time $t = 0$) vacancies. During the n^{th}-stalling period $[t_n^{1+\epsilon}, t_{n+1}^{1-\epsilon}]$ nothing interesting happens in the sense that none of the vacancies present at the beginning of the period are destroyed and no new vacancies are created at the end of the period.
(iv) Because of the presence of active and stalling periods one time quantities like the vacancy density or the persistence function (i.e. the probability that a site has been always vacant up to the chosen time) should exhibit a *staircase* behavior. Furthermore two-time quantities like the density-density autocorrelation function should exhibit *aging*, namely the autocorrelation should not be asymptotically a function of the difference of the two observation times, at variance with what occurs for the equilibrium dynamics.

Such a general picture was suggested in two interesting physics papers [3, 4] and some of the conclusions (properties (iv) above) were indeed observed in numerical simulations. Furthermore the true East dynamics was replaced with that of a certain *hierarchical coalescence process* (HCP) mimicking the features (i)–(iii) described above. In turn, under the assumption that the interval (domain) between two consecutive vacancies in the n-th stalling period rescaled by 2^n has a well defined limiting distribution as $n \to \infty$, the form of this limiting distribution when the initial distribution is a Bernoulli product measure has been computed for the coalescence model thus implying the knowledge of the limiting distribution for the East domains provided the above scaling assumption for HCP holds and provided the East dynamics can indeed be approximated by HCP. Partly motivated by this work and partly by other coalescence models which are relevant in statistical physics, the present authors introduced in [9] (see also the abstract by A.Faggionato on this same volume) a large class of HCP which includes the one considered in [3, 4] and: (1) proved the existence of a scaling limit under very general assumptions, (2) proved the universality of the scaling limit depending only on general features of the initial distribution and not on the details of the model.

Concerning the East model our main result is the mathematical derivation of the above mentioned heuristic picture for the non-equilibrium dynamics, in particular of the aging and staircase behavior. Furthermore we prove that, with probability tending to one as $q \downarrow 0$, when the initial distribution is such that the vacancies form a renewal process the dynamics is well approximated (in variation distance) by an HCP with rates depending on suitable large deviation probabilities of the East model. As a consequence, via the asymptotic results in [9], we prove a scaling limit for: i) the inter-vacancy distance and ii) the position of the first vacancy for the model on the positive half line. In particular we prove that this scaling limit is universal if the initial renewal process has finite mean and coincides with the scaling limit which had been conjectured in [3, 4]. If instead the initial distribution is the domain of attraction of an α-stable law, $\alpha \in (0, 1)$, the scaling limit is different and falls in another universality class depending on α.

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Tunneling of reversible condensed zero range processes on a finite set

Johel Beltrán
(joint work with Claudio Landim)

A zero range process is an interacting particle system in which particles leave any given site at a rate $g(k)$ that only depends on the number k of particles present at the site. When the rate $g(\cdot)$ decreases with the number of particles, a mechanism of effective attraction between the particles is observed. Under some additional conditions, this can lead to a phenomenon called condensation in the physics literature.

Let us set the following specific sequence of zero range processes. Fix a real number $\alpha > 1$. Let $g : \mathbb{N} \to \mathbb{R}$ be given by

$$g(0) = 0\,, \quad g(1) = 1\,, \quad \text{and} \quad g(n) \;=\; \Big(\frac{n}{n-1}\Big)^{\alpha}\,,\; n \geq 2\,,$$

so that $\prod_{i=1}^{n} g(i) = n^{\alpha}$, $n \geq 1$. Consider the zero range process on $S = \{1, 2, \ldots, \kappa\}$, $\kappa \geq 2$ in which a particle jumps from a site x, occupied by k particles, to a site y at rate $g(k)r(x, y)$. The total number of particles is conserved by the dynamics, and for each fixed integer $N \geq 1$ the process restricted to the set of configurations with N particles, denoted by E_N, is irreducible. Let μ_N be the unique stationary probability measure on E_N. This measure exhibits a structure of condensation in the following sense. Fix a sequence $\epsilon_N \downarrow 0$. The majority of particles is said to be at site $x \in S$ if the number of particles at this site is larger than $N(1 - \epsilon_N)$. Let us denote by Δ_N the set of configurations for which such site does not exist. Then, provided $N\epsilon_N \to \infty$, we observe that

$$\mu_N(\Delta_N) \to 0\,, \quad \text{as } N \uparrow \infty.$$

In other words, the stationary measure concentrates on configurations which have the majority of particles located at one single site.

Several aspects of the condensation phenomenon for zero range dynamics have been examined. Let the condensate be the site with the maximal occupancy. Precise estimates on the number of particles at the condensated, as well as its fluctuations, have been obtained in [6, 5, 3]. The equivalence of ensembles has been proved by Großkinsky, Schütz and Spohn [5]. Ferrari, Landim and Sisko [4] proved that if the number of sites is kept fixed, as the total number of particles $N \uparrow \infty$, the distribution of particles outside the condensated converges to the grand canonical distribution with critical density. Armendariz and Loulakis [1] generalized this result showing that if the number of sites κ grows with the number of particles N in such a way that the density N/κ converges to a value greater

than the critical density, the distribution of the particles outside the condensate converges to the grand canonical distribution with critical density.

In this talk we discuss the dynamical aspects of the condensation phenomenon. Fix an initial configuration in Δ_N^c. Now, denote by X_t^N the position of the majority of particles at time $t \geq 0$. In case at time t the process is located in Δ_N, X_t^N remains in the last position. The process $\{X_t^N : t \geq 0\}$ then evolves randomly on S according to some non-Markovian dynamics. The main result of our work states that if the sequence ϵ_N is suitably chosen then, in the time scale $N^{1+\alpha}$, the process $\{X_t^N : t \geq 0\}$ evolves asymptotically according to a Markov process.

In order to give a more precise information about this Markov process limit, le us recall the notion of capacity. Fix an irreducible continuous time Markov process on S which jumps from x to y at some rate $r(x, y)$. Assume that this dynamics is reversible with respect to some probability measure m on S: $m(x)r(x, y) = m(y)r(y, x)$, x, $y \in S$. Denote by cap_S the capacity associated to this Markov process: For two disjoint proper subsets A, B of S,

$$\mathrm{cap}_S(A, B) \;=\; \inf_{f \in \mathcal{B}(A,B)} \frac{1}{2} \sum_{x,y \in S} m(x)\, r(x, y)\, \{f(y) - f(x)\}^2 \,,$$

where $\mathcal{B}(A, B)$ stands for the set of functions $f : S \to \mathbb{R}$ equal to 1 at A and equal to 0 at B. When $A = \{x\}$, $B = \{y\}$, we represent $\mathrm{cap}_S(A, B)$ by $\mathrm{cap}_S(x, y)$.

Let $M_\star$ be the maximum value of the probability measure m: $M_\star = \max\{m(x) : x \in S\}$ and denote by $S_\star$ the sites where m attains its maximum value: $S_\star = \{x \in S : m(x) = M_\star\}$. Of course, in the symmetric, nearest–neighbor case, where $r(x, y) = 1$ if $y = x \pm 1$, modulo κ, and $r(x, y) = 0$ otherwise, m is constant and $S_\star$ and S coincide.

Then, our main result states that, in the time scale $N^{1+\alpha}$, the process $\{X_t^N : t \geq 0\}$ converges in law to the Markov process $\{X_t : t \geq 0\}$ on $S_\star$ which jumps from x to y at a rate proportional to the capacity $\mathrm{cap}_S(x, y)$. In the terminology of [2], we are proving that the condensate exhibits a tunneling behavior in the time scale $N^{1+\alpha}$.

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Harmonic deformation of Delaunay triangulations

Pablo Augusto Ferrari
(joint work with Rafael Grisi and Pablo Groisman)

Let $\tilde{S}$ be a homogeneous intensity-1 Poisson process on $\mathbb{R}^d$ and let $S = \tilde{S} \cup \{0\}$ its Palm version. Call $\mathbb{P}$ and $\mathbb{E}$ the probability and expectation associated to S. The Voronoi cell of a point s in S is the set of sites in $\mathbb{R}^d$ that are closer to s than to any other point in S. Two points are *neighbors* if the intersection of the closure of the respective Voronoi cells has dimension $d-1$. The graph $\mathcal{G}$ with vertices S and edges given by pairs of neighbors is called the Delaunay triangulation of S. Our main result is the construction of a function $H : S \to \mathbb{R}^d$ such that the graph with vertices $H(S)$ and edges $\{(H(s), H(s')),\ s \text{ and } s' \text{ are neighbors}\}$ has the following properties: (a) each vertex $H(s)$ is in the barycenter of its neighbors and (b) sublinearity along lines, that is, $|H(s) - s|/|s|$ vanishes as $|s|$ grows to infinity along any straight line. If such an H exists, the resulting graph $\mathcal{H}$ is called a *harmonic deformation* of the Delaunay triangulation of S. This problem has been solved in the graph induced by the supercritical percolation cluster in $\mathbb{Z}^d$ by Biskup and Berger [3] and Matthieu and Piatnitski [10]. The function $H(s) - s$ is called *corrector*. The motivation of those works is to show a quenched invariance principle for the random walk in the percolation cluster. The general strategy is to use the fact that the random walk in the graph $\mathcal{H}$ is a martingale and then use the sublinearity of H to export the invariant principle to the original graph $\mathcal{G}$. Related works on the quenched invariance principle in $\mathbb{Z}^d$ are Biskup and Prescott [4], Barlow [1] and Barlow and Deuschel [2]. Caputo, Faggionato and Prescott [5] worked on a percolation-type graph in point processes on $\mathbb{R}^d$.

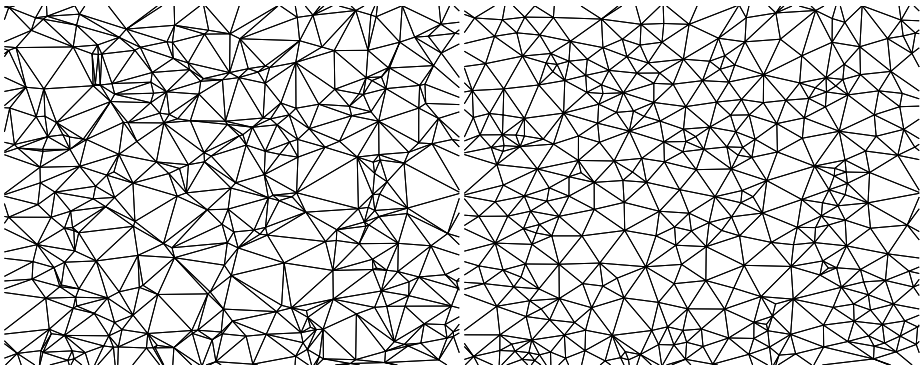

Figure 1. Delaunay triangulation of a Poisson process and its harmonic deformation in a finite box with fixed boundary points (not shown).

The coordinates $h_1, \dots, h_d$ of H are *harmonic functions* from S to $\mathbb{R}$; that is $h_i(s)$ is the mean of $\{h_i(s'),\ s' \text{ neighbor of } s\}$, $i = 1, \dots, d$. A function $f : \mathbb{R}^d \to \mathbb{R}$ has *inclination* α *in the direction* u (a unit vector) if $(f(Ku) - K\alpha)/K$ converges to zero as K goes to infinity. The corrector is sublinear if and only if each h_i has *inclination* 1 in the direction of the i-th canonical vector.

Fixing a direction u, we construct a harmonic function $h : S \to \mathbb{R}$ with inclination one in the direction u as the limit and a fixed point of a noiseless harness process, a dynamics introduced by Hammersley [9, 7]. The process is easily described by associating to each point s of S a one-dimensional homogeneous Poisson process of rate 1. Fix an initial surface $\eta_0 : S \to \mathbb{R}$ and at the epochs τ of the Poisson process associated to s update $\eta_\tau(s)$ to the average of the heights $\{\eta_{\tau-}(s'),\ s' \text{ neighbor of } s\}$. It is clear that if h is harmonic, then h is invariant for this dynamics. We start the harness process with the plane $\eta_0(s) = s_1$, where s_i is the i-th coordinate of s and show that $\eta_t(\cdot) - \eta_t(0)$ converges to h in $L_2(\mathbb{P} \times P)$, where $\mathbb{P}$ is the law of the point configuration S and P is the law of the dynamics.

We prove that the inclination is invariant for the harness process for each t and in the limit when $t \to \infty$. In a finite graph the average of the square of the height differences of neighbors is decreasing with time for the harness process. Since essentially the same happens in infinite volume, the gradients of the surface converge under the harness dynamics. To conclude we show that (1) the limit of the gradients is a gradient field and (2) the limit is harmonic. Both statements follow from almost sure convergence along subsequences.

A key ingredient of the approach is the expression of the inclination of a surface as the inner product of the gradient of the surface with a specific field. This together with the time-invariance of the inclination imply that the limiting surface has the same inclination as the initial one.

A paper with these results has been uploaded to ArXiv [6]. Part of this work and the applications to random walk in the Delaunay triangulation of a Poisson process can be found in the Phd Thesis of Rafael Grisi [8].

We thank Marek Biskup for motivating us to pursue this approach.

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Large deviations for the current and a tagged particle in 1D symmetric simple exclusion

Sunder Sethuraman
(joint work with SRS Varadhan)

Informally, the one dimensional symmetric simple exclusion process follows a collection of continuous time symmetric nearest-neighbor random walks on $\mathbb{Z}$ whose jumps to occupied vertices are suppressed. Recently, laws of large numbers, starting from certain nonequilibrium configuration measures, have been shown recently for the integrated current of particles across a bond, and a tagged, or distinguished particle in this system (Jara-Landim [3]). In this talk, we discuss corresponding large deviation principles, and evaluate the rate functions, showing different growth behaviors near and far from their zeroes which connect with work of Derrida-Gerschenfeld.

Specifically, (1) we show that the current and tagged particle large deviation principles can be written as a certain contraction of the bulk empirical density large deviations given in Kipnis-Olla-Varadhan [4]. Next, (2) we compute that the rate functions near their zeroes are quadratic (as should be expected given that central limit theorems should hold), and also evaluate the diffusivity coefficient when the initial measure corresponds to a flat profile. Finally, (3) we show that the rates for large exceedences are third order which, with respect to the current, in part echoes results in Derrida-Gerschenfeld [1], [2].

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Stationary behavior of interacting stochastic systems

Márton Balázs

Modelling flocks and prices: jumping particles with an attractive interaction (work in progress)**.** With Miklós Zoltán Rácz we investigated a model of finitely many particles which form a continuous-time Markov jump process. We have heard about this model from Bálint Tóth. Particles move on $\mathbb{R}$, and the evolution of their positions is governed by

- their jump rates that depend on their positions *relative to the center of mass of the particles*,
- an independent random variable drawn from a jump-length distribution.

Denoting the positions of the n particles by $x_1, x_2, \ldots, x_n$, the center of mass is located at

$$m_n = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

Given the positions x_i of the particles, each particle waits an exponential time of parameter $w(x_i - m_n)$, and then adds a random variable, independent of everything, of distribution φ to its position x_i (that is, jumps a random length). Attraction of particles is ensured by choosing a positive decreasing rate function w. It is also natural to pick a φ that is concentrated on positive reals, and is not heavy-tailed. The effect of these choices is that those particles left behind (relative to the center of mass) have higher jump rates while those in front move slower. This effect acts as an attraction between the particles, and the system as seen from the center of mass has a stationary distribution.

We addressed the following questions.

- The two-particle case is, of course, tractable. We computed the stationary distribution of the gap between the two particles for several choices of φ. However, we could not handle the situation even for $n = 3$ particles (the process is not reversible).
- If the number of particles $n \to \infty$, we expect that the motion of the center of mass converges to a deterministic one. This motion, together with the distribution of the particles, satisfy a deterministic differential equation in this so-called *fluid limit*. We call the resulting equation the mean field equation. We wrote up this equation by heuristic arguments and, in some cases, we solved it for a travelling wave form, i.e. the stationary solution as seen by the center of mass. Even finding the travelling wave in this mean field equation is non trivial.
- Some travelling wave solutions lead us to Gumbel distributions, which naturally posed the question whether extreme value interpretation can be given to the process. We could find such connections (an idea by Attila Rákos) in some examples.
- Under restrictive assumptions on the rates w, we prove that indeed the evolution of the process in the fluid limit $n \to \infty$ satisfies the deterministic

differential equation we formulated by heuristic arguments, this is work in progress. We use weak convergence methods and uniqueness results on the differential equation. Since m_n is not a bounded function of the particle positions, standard weak convergence techniques need to be extended to our situation.

Random walking shocks with second class particles. With György Farkas, Péter Kovács and Attila Rákos we connected two types of results, shown for both asymmetric exclusion and the exponential bricklayers process:

- Derrida, Lebowitz and Speer [6], and Balázs [1] showed that some particular shock-distributions are stationary as seen by a second class particle in the two aforementioned models.
- Belitsky and Schütz [5], and Balázs [2] showed that exactly the above shock-distributions "perform simple random walks" that is, the shock measures evolve into linear combinations of the same structure, with coefficients given as random walk transitions. In case of multiple shocks, the walkers attract and, in the exclusion case, also exclude each other.

The natural question emerges, whether the second class particle itself, in an annealed sense, behaves as a random walker in the middle of the shock measure. We give an affirmative answer [3], and show a result of similar flavor in a branching-coalescing random walk model.

$t^{1/3}$ order current fluctuations. With Júlia Komjáthy and Timo Seppäläinen we built up a general framework for proving $t^{1/3}$ scaling of current fluctuations through the characteristics and $t^{2/3}$ scaling for the fluctuations of the second class particle in the stationary evolution of a family of one dimensional nearest neighbor interacting systems [4]. The assumption we need is the validity of a complicated coupling construction we call *microscopic concavity/convexity*. We have verified all the requirements for asymmetric exclusion, the constant rate zero range process, a zero range process with bounded and concave enough jump rates. We are typing up the argument for the exponential bricklayers process. Interestingly, its convex jump rates make it necessary to use random walking shock results of the previous paragraph. With Júlia Komjáthy we started to investigate whether we can make the arguments work for an extended exclusion process with product binomial stationary distribution.

Irreversible Markov chains and electric networks (work in progress). With Áron Folly we looked at the well-known analogy between reversible Markov chains and electric networks (see e.g. [7]). The electric network is built up of simple resistors with which the arguments only work for reversible Markov chains. We found the electric component the network of which extends the analogy from reversible Markov chains to the case of irreversible ones. The new part we use is an amplifier with a rather simple characteristic. Currently we investigate how much monotonicity arguments generalize from resistor networks to ones built of the new

amplifiers, and whether some of the recurrence/transience results for reversible chains that use the electric network analogy can be saved for irreversible chains.

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On the fragmentation of a torus by random walk

Augusto Teixeira

We discussed in this talk some problems on the geometry of the set of sites not visited by a random walk on a discrete torus. More precisely, we consider a simple random walk on the discrete torus $\mathbb{T}^d = (\mathbb{Z}/N\mathbb{Z})^d$ run up to time uN^d, where u is a fixed positive constant. In this talk we discuss the components of the set $\mathcal{V}^u = \mathbb{T}^d \setminus \{X_0, X_1, \ldots, X_{uN^d}\}$. As we showed, this set undergoes a phase transition in the following sense: if u is small enouth, then for any dimensions $d \geq 3$ then $\mathcal{V}^u$ contains a giant component with volume of order N^d. On the other hand, if u is chosen large, then the set $\mathcal{V}^u$ is composed solely of small components of volume no larger than $\log^\lambda N$. This talk is based in a recent article joint with David Windisch. Let us give a more precise description of this work.

Consider a simple random walk on the d-dimensional torus $\mathbb{T}_N = (\mathbb{Z}/N\mathbb{Z})^d$ with large side length N and fixed dimension $d \geq 3$. The aim of this talk is to discuss the percolative properties of the set of vertices not visited by the random walk until time uN^d, where the parameter $u > 0$ remains fixed and N tends to infinity. We refer to this set as the vacant set. The vacant set occupies a proportion of vertices bounded away from 0 and 1 as N tends to infinity, so it is natural to study the sizes of its components. At this point, the main results on the vacant set are the ones of Benjamini and Sznitman [1], showing that for high dimensions d and small parameters $u > 0$, there is a component of the vacant set with cardinality of order N^d with high probability. As is pointed out in [1], this result raises several questions, such as:

(1) Do similar results hold for any dimension $d \geq 3$?
(2) For small parameters $u > 0$, does the second largest component have a volume of order less than N^d?

(3) Provided $u > 0$ is chosen large enough, do all components of the vacant set have volumes of order less than N^d?

The results presented in this talk give positive answers to these questions, and thereby confirm observations made in computer simulations (see Figure 1). We thus prove the existence of distinct regimes for the vacant set as u varies, similar to the ones exhibited by Bernoulli percolation on the torus and other random graph models.

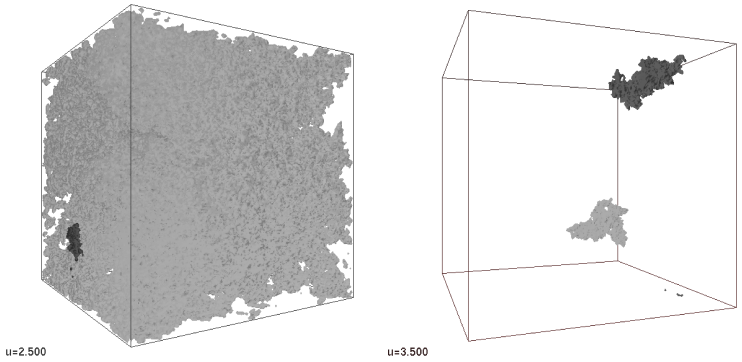

FIGURE 1. A computer simulation of the largest component (light gray) and second largest component (dark gray) of the vacant set left by a random walk on $(\mathbb{Z}/N\mathbb{Z})^3$ after $[uN^3]$ steps, for $N = 200$. The picture on the left-hand side corresponds to $u = 2.5$, the right-hand side to $u = 3.5$.

Our answers are closely linked to Sznitman's model of random interlacements. This relation was already established in [3], and all the results presented here come from an improvement of this results in the following sense. In [3], David Windisch proved that random interlacements appear as the local picture of the vacant set $\mathcal{V}^u$ as N grows in the sense of weak convergence. In this talk we discussed a new technique, which was able to prove this relation between the local picture of $\mathcal{V}^u$ and random interlacements in a quantitative point of view. We establish a domination of the vacant set both from above and from below by interlacements, also providing a good estimate on the error made in this approximation.

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Mixing of the environment viewed by the particle, with applications

Jean-Christophe Mourrat

Let us say that two points in $\mathbb{Z}^d$ are neighbours when the distance from one to the other is equal to 1. This turns $\mathbb{Z}^d$ into a graph, and we write $\mathbb{B}$ for its set of (non-oriented) edges. We call *environment* a family $\omega = (\omega_e)_{e\in\mathbb{B}}$, and we refer to $\omega_e > 0$ as the *conductance* of the edge e. For a given environment ω, let $(Z_t)_{t\geq 0}$ be the continuous-time Markov chain such that the jump rate from x to a neighbour y is given by $\omega_{x,y}$. We write $\mathbf{P}_0^\omega$ for the law of Z_t starting from 0, and $\mathbf{E}_0^\omega$ for the associated expectation.

A crucial property of this random walk is the symmetry of its jump rates: $\omega_{x,y} = \omega_{y,x}$. It makes the counting measure on $\mathbb{Z}^d$ reversible.

The environment itself is chosen at random. We assume that the $(\omega_e)_{e\in\mathbb{B}}$ are independent and identically distributed random variables, whose law we write $\mathbb{P}$ (expectation $\mathbb{E}$). We also assume that the conductances are uniformly bounded from below, say $\omega_e \geq 1$.

The first natural question one can ask concerns the scaling limit of the random walk.

Theorem 1. *There exists $\sigma > 0$ such that for almost every environment, the law of $\sqrt{\varepsilon} Z_{\varepsilon^{-1}t}$ under $\mathbf{P}_0^\omega$ converges, as ε tends to 0, to the law of a Brownian motion of variance σ^2.*

Under our present assumption that $\omega_e \geq 1$, this result is due to [BD10]. Many people contributed to this result under different assumptions, and we refer to J.-D. Deuschel's abstract in this volume for more precision.

The proof of this theorem relies on the ergodicity of an auxiliary process, called "the environment viewed by the particle". Naturally, the ergodic theorem only provides with asymptotic statements, without any indication about the speed of convergence, and we would like to know more about this aspect.

Let $(\theta_x)_{x\in\mathbb{Z}^d}$ be the translations acting on the set of environments. The *environment viewed by the particle* is defined to be

$$\omega(t) = \theta_{Z_t}\ \omega.$$

It is a Markov process, and has $\mathbb{P}$ as a reversible and ergodic measure. We would like to be able to say in some quantitative way that, for large t,

$$f_t(\omega) := \mathbf{E}_0^\omega[f(\omega(t))] \simeq \mathbb{E}[f].$$

More precisely, we would like to show that the following holds

$$(*) \qquad \mathbb{V}\mathrm{ar}(f_t) \leq \frac{V(f)}{t^\alpha},$$

for some functional V and some exponent $\alpha > 0$ as large as possible. Here is a partial result I obtained in [Mo10].

Theorem 2. *Inequality $(*)$ is true for $\alpha = 1/2$ in dimension 1, $\alpha = 1\vee(d/2-2)$ in larger dimension, and some functional V that is finite on bounded local functions.*

One may suspect however that the correct exponent is $\alpha = d/2$, a simple heuristic being that at time t, the random walk should be more or less uniformly spread in a ball of size $\sqrt{t}$. The proof of Theorem 2 goes through the establishment of a Nash inequality with exponent $d/2$. Unfortunately, the semi-norm that appears in the Nash inequality obtained is not contractive, and the method used to overcome this problem is not sufficient to lead to the expected exponent.

I find it very worthwhile to try to improve Theorem 2, and in order to justify this view, I now present two cases were such a result have proved useful.

For the first instance, detailed in [GM10], we assume furthermore that the conductances are uniformly bounded from above. The (positive) generator $\mathcal{L}$ of the environment viewed by the particle is self-adjoint on $L^2(\mathbb{P})$. Assume that $\mathbb{E}[f] = 0$ for simplicity, and let e_f be the spectral measure of $\mathcal{L}$ projected on the function f. From the observation that

$$\mathbb{V}\mathrm{ar}(f_t) = \mathbb{E}[(f_t)^2] = \int e^{-2\lambda t} \, \mathrm{d}e_f(\lambda),$$

one can easily see that

$$(*) \quad \Leftrightarrow \quad \exists C > 0 : \int_0^\mu \mathrm{d}e_f(\lambda) \leq C\mu^\alpha. \tag{1}$$

We would like to find a numerical method to compute the asymptotic variance σ^2 appearing in Theorem 1. One convenient way to express this parameter is to introduce the function $g(\omega) = \omega_{0,e_1} - \omega_{0,-e_1}$, where e_1 is the first vector of the canonical basis of $\mathbb{Z}^d$. One can construct the so-called corrector ϕ, which is such that $\mathcal{L}\phi = g$, and enables to write the identity

$$\frac{\sigma^2}{2} = \mathbb{E}[\omega_e] - \langle \mathcal{L}\phi, \phi \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(\mathbb{P})$. The problem is that it is not possible to compute ϕ in practice, so one could instead introduce a small parameter $\mu > 0$, and solve $(\mu + \mathcal{L})\phi_\mu = g$ (which one can do in a box of size $\mu^{-1/2}$). The problem is then to evaluate the difference between the desired $\langle \mathcal{L}\phi, \phi \rangle$ and the computable $\langle \mathcal{L}\phi_\mu, \phi_\mu \rangle$. A spectral computation yields

$$\langle \mathcal{L}\phi_\mu, \phi_\mu \rangle = \int \frac{\lambda}{(\mu + \lambda)^2} \, \mathrm{d}e_g(\lambda), \tag{2}$$

while $\langle \mathcal{L}\phi, \phi \rangle = \int \lambda^{-1} \, \mathrm{d}e_g(\lambda)$. From these observations, it is easy to relate the difference between $\langle \mathcal{L}\phi, \phi \rangle$ and $\langle \mathcal{L}\phi_\mu, \phi_\mu \rangle$ with the behaviour of the spectral measure close to 0. This behaviour is in turn related to inequality $(*)$ for the function g, as we see from the equivalence (1). In [GM10], we obtain the following result.

Theorem 3. *For the function g, the inequality $(*)$ is valid with $\alpha = (d/2+1) \wedge 4$.*

Although we have seen before that one should expect a decay of order $d/2$ in general, the particular form of the function g, that can be written as a gradient, enables to obtain a stronger exponent. For this function, we expect that the correct exponent is $d/2 + 1$ in any dimension, but the method we use cannot go

beyond exponent 4. Note that Theorem 2 still ensures that the exponent can also be taken to be equal to $d/2-2$, which is a better exponent than the one given by Theorem 3 when $d > 12$.

In fact, the use of spectral representations of the form of (2) enables us to devise new methods for the computation of σ^2, for which we can estimate the error using Theorem 3.

For the second instance where Theorem 2 have been useful, we drop the assumption that the conductances are bounded from above. Let $(X_t)_{t\geq 0}$ be the random walk that follows the path of $(Z_t)_{t\geq 0}$, but waits a random exponential time of mean 1 before making each jump. When the random walk meets an edge with a very large conductance, it will perform a large number of jumps accross it, and thus can act as a trap for the walk X. Let $n(\omega)$ be the total jump rate for Z at the origin: $n(\omega) = \sum_{|z|=1} \omega_{0,z}$, and

$$A(t) = \int_0^t n(\omega(s))\, \mathrm{d}s.$$

One can see that $X_{A(t)} = Z_t$, and as a consequence, if one whishes to understand the scaling limit of X, it seems reasonable to focus on the asymptotic behaviour of $A(t)$ as t goes to infinity. When $n(\omega)$ is integrable, the ergodicity of $(\omega(s))$ ensures that $A(t)$ grows linearly with t, a fact which, together with Theorem 1, ensures convergence to a Brownian motion for X after a diffusive scaling. The question is less clear when n becomes non-integrable. Assume that there exists some $a \in (0,1)$ such that $\mathbb{P}[\omega_e \geq y] \sim y^{-a}$. Under this assumption and for $d \geq 3$, the subdiffusive scaling limit has been identified in [BČ10]. They prove in particular that $\varepsilon^{1/a} A(\varepsilon^{-1}t)$ converges in distribution to an a-stable subordinator. The proof uses delicate estimates on the transition probabilities of the walk Z, in a coarse-graining procedure. For $d \geq 5$, I proposed in [Mo10] an alternative proof ! of this result, that is based instead on the mixing properties of the environment viewed by the particle given by Theorem 2.

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