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Infinite Dimensional Lie Theory

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Abstract. The workshop focussed on recent developments in infinite-dimensional Lie theory. The talks covered a broad range of topics, such as structure and classification theory of infinite-dimensional Lie algebras, geometry of infinite-dimensional Lie groups and homogeneous spaces and representations theory of infinite-dimensional Lie groups, Lie algebras and Lie-superalgebras.

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Introduction by the Organisers

The workshop *Infinite-dimensional Lie theory* was organized by Karl-Hermann Neeb (Erlangen), Arturo Pianzola (Edmonton), and Tudor Ratiu (Lausanne). Nowadays infinite-dimensional Lie theory is a core area of modern mathematics, covering a broad range of branches, such as the structure and classification theory of infinite-dimensional Lie algebras, geometry of infinite-dimensional Lie groups and their homogeneous spaces, the analytic representation theory of infinite-dimensional Lie groups, and the algebraic representation theory of infinite Lie algebras and Lie-superalgebras. The focus of this workshop was on recent developments in all these areas with a particular emphasis on connections with other branches of mathematics, such as algebraic groups and Galois cohomology.

The meeting was attended by 27 participants from many european countries, Canada, the USA, Japan and Australia.

The meeting was organized around a series of 20 lectures each of 50 minutes duration representing the major recent advances in the area. On Thursday, November 18, we organized common sessions with the parallel meeting on "Representation theory and harmonic analysis", organized by T. Kobayashi and B. Krötz. On this day we scheduled talks by J. Bernstein, Y. Neretin, S. Kumar and P. Littelmann whose subject matter was on the borderline of the two workshops.

We feel that the meeting was exciting and highly successful. The quality of the lectures, several of which provided surveys of recent developments, was outstanding. The exceptional atmosphere of the Oberwolfach Institute provided the optimal environment for bringing people working in algebraically, geometrically or analytically oriented areas of infinite-dimensional Lie theory together and to create an atmosphere of scientific interaction and cross-fertilization.

Without going too much into detail, let us mention some importent new developments. In the area of infinite-dimensional Lie algebras, the structure and representation theory of multiloop algebras is presently a hot topic. We had talks on the classification of finite dimensional representations of these Lie algebras (Neher), their forms over rings of Laurent polynomials (Gille) and conjugacy of maximal abelian diagonalizable subalgebras (Chernousov). Another rapidly developing direction is the representation theory of direct limit groups, where the principal series representations are now much better understood (J. Wolf) and global structures on natural representation categories, such as the Koszul property, were recently discovered (Penkov, Serganova). This branch of representation theory is complemented by the discovery of interesting new representations of the Lie algebra of of vector fields on a torus (Billig).

In the analytic representation theory of infinite dimensional Lie groups, various classes of Banach–Lie groups seem to carry the most interesting classes of representations. A major step in the systematic development of this theory has recently been taken by the development of analytic extension techniques relating unitary group representations to representations of naturally arising semigroups (Merigon, Neeb). We also had a series of inspiring talks on representations of Banach–Lie groups with various geometric origins: Gerbes and corresponding gerbal representations (Mickelsson), energy representations of path groups coming from stochastic analysis (Gordina), representations coming from infinite dimensional versions of Weyl calculus (Beltita), structure of L^*-groups (Tumpach), and for Lie supergroups a systematic theory of unitary representations starts to evolve (Salmasian). Since convolution algebras do not exist for infinite dimensional groups, other links to C^*-algebras are presently being explored in various contexts (Grundling).

The representation theory of Lie groups has always been developed in close connection to geometric structures. On the geometric side we had talks on new methods to calculate homotopy groups of infinite dimensional Lie groups (Glöckner), a very recent proof of a long standing conjecture on the structure of central extensions of gauge groups (Janssens), visible actions of Lie groups on complex manifolds (Sasaki), symplectic Howe duality (Wurzbacher) and new integrable systems arising in the context of Banach–Lie–Poisson spaces (Odzijewicz, Ratiu).

More specific information is contained in the abstracts which follow in this volume.

Workshop: Infinite Dimensional Lie Theory

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Abstracts

Towards a Group Algebra for $\mathbb{R}^{(\mathbb{N})}$

Hendrik Grundling
(joint work with Karl-Hermann Neeb)

We consider the question of how to generalize the notion of a group algebra to topological groups which are not locally compact, hence have no Haar measure. Such a generalization, called a *full host algebra,* has been proposed in [2, 3]. Briefly, it is a C^*-algebra $\mathcal{L}$ whose multiplier algebra $M(\mathcal{L})$ admits a homomorphism $\eta : G \to U(M(\mathcal{L}))$, such that the (unique) extension of the representation theory of $\mathcal{L}$ to $M(\mathcal{L})$ pulls back via η to the continuous unitary representation theory of G, producing a bijection $\eta^* : \mathrm{Rep}(\mathcal{L}, \mathcal{H}) \to \mathrm{Rep}(G, \mathcal{H})$ for any Hilbert space $\mathcal{H}$. Thus, given a full host algebra $\mathcal{L}$, the continuous unitary representation theory of G can be analyzed on $\mathcal{L}$ with a large arsenal of C^*-algebraic tools. Such a host algebra need not exist for a general topological group because there are topological groups with faithful unitary representations but without non-trivial irreducible ones. One example of a full host algebra for a group which is not locally compact, has been constructed explicitly for the σ–representations of an infinite dimensional topological linear space S, considered as a group cf. [4].

Probably the simplest infinite dimensional group is $\mathbb{R}^{(\mathbb{N})}$ (the set of real-valued sequences with only finitely many nonzero entries) with the inductive limit topology w.r.t. the natural inclusions $\mathbb{R}^n \subset \mathbb{R}^{(\mathbb{N})}$. We want to define a (full) host algebra for $\mathbb{R}^{(\mathbb{N})}$. Recall that for the group C*-algebras we have:

$$C^*(\mathbb{R}^n) \otimes C^*(\mathbb{R}^m) \cong C^*(\mathbb{R}^{n+m})$$

and this suggests that for a host algebra of $\mathbb{R}^{(\mathbb{N})}$ we should try an infinite tensor product of $C^*(\mathbb{R})$. This is difficult to do, for two reasons:

- $C^*(\mathbb{R}) \cong C_0(\mathbb{R})$ is nonunital, and the standard infinite tensor products of C*-algebras require unital algebras.
- There is a definition for an infinite tensor product of nonunital algebras developed by Blackadar cf. [1], but this requires the algebras to have nonzero projections, and the construction depends on the choice of projections. However, $C^*(\mathbb{R}) \cong C_0(\mathbb{R})$ has no nonzero projections, so this method will not work.

In the light of these difficulties, we will develop an infinite tensor product of $C_0(\mathbb{R})$ relative to a choice of approximate identity in each entry, to replace the choice of projections in Blackadar's approach. As expected, the construction will depend on the choice of approximate identities, though it still produces for each choice an algebra with strong host algebra properties.

Fix a reference sequence $\mathbf{b} = (b_1, b_2, \ldots) \in \prod_{n=1}^{\infty} C_0(\mathbb{R})_+$ of bump functions, and in the algebraic tensor product $\bigotimes_{n=1}^{\infty} C_0(\mathbb{R})$ define the linear space

$$[\![\mathbf{b}]\!] := \mathrm{Span}\Big\{ \mathop{\otimes}_{n=1}^{\infty} x_n \;\Big|\; \mathbf{x} \text{ differs from } \mathbf{b} \text{ in only finitely many entries}\Big\},$$

then we consider the *-algebra generated by it in the algebraic tensor product. We define $\mathcal{L}[\![\mathbf{b}]\!]$ to be the C*-completion of $\pi_u\big(*\text{-alg}([\![\mathbf{b}]\!])\big)$ where the representation $\pi_u : *\text{-alg}([\![\mathbf{b}]\!]) \to \mathcal{B}(\mathcal{H}_u)$ is constructed as follows. Let $\pi_u : \mathbb{R}^{(\mathbb{N})} \to \mathcal{U}(\mathcal{H}_u)$ be the universal representation of $\mathbb{R}^{(\mathbb{N})}$, and let $\pi_u^k : \mathbb{R} \to \mathcal{U}(\mathcal{H}_u)$ be the representation obtained from it by restriction to the k^{th} component. This defines a representation on the group algebra of $\mathbb{R}$ so we obtain a representation $\pi_u^k : C_0(\mathbb{R}) \to \mathcal{B}(\mathcal{H}_u)$. We finally define $\pi_u : *\text{-alg}([\![\mathbf{b}]\!]) \to \mathcal{B}(\mathcal{H}_u)$ by

$$\pi_u\big(L_1 \otimes L_2 \otimes \cdots\big) := \operatorname*{s-lim}_{n\to\infty} \pi_u^1(L_1)\cdots\pi_u^n(L_n) \in \mathcal{B}(\mathcal{H}_u)\,.$$

To make $\mathcal{L}[\![\mathbf{b}]\!]$ into a host algebra for $\mathbb{R}^{(\mathbb{N})}$, define the unitary embedding $\eta : \mathbb{R}^{(\mathbb{N})} \to U(M(\mathcal{L}[\![\mathbf{b}]\!]))$ by applying the usual embedding $\eta_0 : \mathbb{R} \to M(C^*(\mathbb{R}))$ componentwise. For any Hilbert space $\mathcal{H}$ we have therefore a map

$$\eta^* : \mathrm{Rep}\big(\mathcal{L}[\![\mathbf{b}]\!], \mathcal{H}\big) \to \mathrm{Rep}\big(\mathbb{R}^{(\mathbb{N})}, \mathcal{H}\big)$$

by the usual unique extensions of representations. We obtain the following results:

- For any $\pi \in \mathrm{Rep}\big(\mathcal{L}[\![\mathbf{b}]\!], \mathcal{H}\big)$ define its *excess* operator by
$$\begin{aligned} Q \;&:=\; \operatorname*{s-lim}_{n\to\infty} \pi\big(\mathbf{1}\otimes\mathbf{1}\cdots\otimes\mathbf{1}\otimes b_n\otimes b_{n+1}\otimes\cdots\big) \\ &=\; \operatorname*{s-lim}_{n\to\infty}\Big(\operatorname*{s-lim}_{k\to\infty}\pi\big(E_k^{(n)}\otimes b_n\otimes b_{n+1}\otimes\cdots\big)\Big)\end{aligned}$$
where $\{E_k^{(n)}\}_{k\in\mathbb{N}}$ is any approximate identity of $\bigotimes_{j=1}^{n-1} C_0(\mathbb{R})$. The first line has to be interpreted in $M(\mathcal{L}[\![\mathbf{b}]\!])$ as $C_0(\mathbb{R})$ is nonunital. Then $0 < Q \leq \mathbf{1}$, and π is normal w.r.t. π_u iff $Q = \mathbf{1}$. Let $\mathrm{Rep}_0\big(\mathcal{L}[\![\mathbf{b}]\!], \mathcal{H}\big)$ denote those representations on $\mathcal{H}$ with $Q = \mathbf{1}$.
- η^* is injective on $\mathrm{Rep}_0\big(\mathcal{L}[\![\mathbf{b}]\!], \mathcal{H}\big)$, and
$$\eta^*\left(\mathrm{Rep}_0\big(\mathcal{L}[\![\mathbf{b}]\!], \mathcal{H}\big)\right) = \eta^*\left(\mathrm{Rep}\big(\mathcal{L}[\![\mathbf{b}]\!], \mathcal{H}\big)\right) \subset \mathrm{Rep}\big(\mathbb{R}^{(\mathbb{N})}, \mathcal{H}\big),$$
and η^* takes irreducibles to irreducibles.

Unfortunately η^* is not surjective, and its range depends on $\mathbf{b}$. To remedy this defect, we choose a set S of reference sequences $\mathbf{b}$ such the projection to any entry produces an approximate identity. We construct $*\text{-alg}\{[\![\mathbf{b}]\!] \mid \mathbf{b} \in S\} \subset \bigotimes_{n=1}^{\infty} C_0(\mathbb{R})$ and take the C*-algebra $\mathcal{L}$ generated by it in π_u. Let $\mathrm{Rep}_0\big(\mathcal{L}, \mathcal{H}\big)$ denote the set of those $\pi \in \mathrm{Rep}\big(\mathcal{L}, \mathcal{H}\big)$ for which the excess operator $Q_{\mathbf{b}}(\pi)$ of each restriction $\pi \restriction \mathcal{L}[\![\mathbf{b}]\!]$ is a projection. Then

- η^* is injective on $\mathrm{Rep}_0(\mathcal{L},\mathcal{H})$, and
$$\eta^*\left(\mathrm{Rep}_0(\mathcal{L},\mathcal{H})\right)=\eta^*\left(\mathrm{Rep}(\mathcal{L},\mathcal{H})\right)=\mathrm{Rep}(\mathbb{R}^{(\mathbb{N})},\mathcal{H}),$$
so we have the desired surjectivity.

Since η is not injective on the full set $\mathrm{Rep}(\mathcal{L},\mathcal{H})$, $\mathcal{L}$ is not a host algebra in the usual sense. The only obstruction for $\mathcal{L}$ to be a full host for $\mathbb{R}^{(\mathbb{N})}$ is the additional part $\mathrm{Rep}(\mathcal{L},\mathcal{H})\backslash\mathrm{Rep}_0(\mathcal{L},\mathcal{H})$, which corresponds to the fact that we used bump functions to construct the infinite tensor products. To characterize this part, let $\mathcal{Q}:=\{Q_{\mathbf{b}}(\pi_{\mathcal{L}})\mid\mathbf{b}\in S\}$ where $\pi_{\mathcal{L}}$ is the universal representation of $\mathcal{L}$. This is a multiplicative semigroup. Then

- Each $\pi\in\mathrm{Rep}(\mathcal{L},\mathcal{H})$ is of the form $\pi(A)=\pi_0(A)\gamma(Q_{\mathbf{b}}(\pi_{\mathcal{L}}))$ for $A\in[\![\mathbf{b}]\!]$, $\pi_0\in\mathrm{Rep}_0(\mathcal{L},\mathcal{H})$ and $\gamma\in\mathrm{Rep}(\mathcal{Q},\mathcal{H})$.

If we write $\mathcal{L}\cong C_0(X)$ using the fact that it is a commutative C*-algebra, then the important question is whether the subset $X_0\subset X$ corresponding to those characters in $\mathrm{Rep}_0(\mathcal{L},\mathbb{C})$ is locally compact or not w.r.t. the relative topology. This is not known, and it is a crucial question for the existence of a full host for $\mathbb{R}^{(\mathbb{N})}$.

A bonus of this construction is that we obtain the Bochner–Minlos theorem for $\mathbb{R}^{(\mathbb{N})}$ from state decompositions:

- For each characteristic function $\omega:\mathbb{R}^{(\mathbb{N})}\to\mathbb{C}$ (i.e. a continuous, positive definite and normalized map) there is a $\mathbf{b}\in S$, and a unique state ω_0 on $\mathcal{L}[\![\mathbf{b}]\!]$ with GNS–representation with excess $Q=\mathbf{1}$, such that $\eta^*(\omega_0)=\omega$.
- There is a unique probability measure ν concentrated on the pure states $\mathfrak{S}_P$ of $\mathcal{L}[\![\mathbf{b}]\!]$ such that
$$\omega_0(A)=\int_{\mathfrak{S}_P}\varphi(A)\,d\nu(\varphi,\ A\in\mathcal{L}[\![\mathbf{b}]\!].$$
- There is a map $\xi:\mathfrak{S}_P\to\mathbb{R}^{\mathbb{N}}$ such that $\eta^*(\varphi)(\mathbf{x})=\exp\left(i\langle\mathbf{x},\xi(\varphi)\rangle\right)$ where $\langle\cdot,\cdot\rangle$ is the natural evaluation between $\mathbb{R}^{(\mathbb{N})}$ and $(\mathbb{R}^{(\mathbb{N})})^*=\mathbb{R}^{\mathbb{N}}$.
- There is a bijection between characteristic functions ω of $\mathbb{R}^{(\mathbb{N})}$ and regular Borel probability measures μ on $\mathbb{R}^{\mathbb{N}}$ (with product topology) given by
$$\omega(\mathbf{x})=\int_{\mathbb{R}^{\mathbb{N}}}e^{i\langle\mathbf{x},\mathbf{y}\rangle}d\mu(\mathbf{y})\,,\quad \mathbf{x}\in\mathbb{R}^{(\mathbb{N})}\,.$$

In terms of the notation above, the Bochner-Minlos measure is $\mu=\nu\circ\xi^{-1}$.

Thus we obtain the Bochner–Minlos theorem from state space decompositions of states on (partial) host algebras $\mathcal{L}[\![\mathbf{b}]\!]$ in terms of pure states. A natural question is whether this formalism can be extended from $\mathbb{R}^{(\mathbb{N})}$ to separable nuclear spaces.

This talk was extracted from reference [5].

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Finite-dimensional representations of equivariant map algebras

Erhard Neher
(joint work with Alistair Savage and Prasad Senesi)

In this abstract, all algebras are assumed to be defined over an algebraically closed field of characteristic 0. To define an equivariant map algebra we use

- a finite-dimensional Lie algebra $\mathfrak{g}$,
- a commutative associative unital k-algebra A, and
- a finite (not necessarily abelian) group Γ acting on A and on $\mathfrak{g}$ by automorphisms.

The associated *equivariant map algebra* is the fixed point algebra of the canonical action of Γ on $\mathfrak{g} \otimes A$,

$$\mathfrak{M} := M(A, \mathfrak{g})^\Gamma := \{m \in \mathfrak{g} \otimes A \mid \gamma \cdot m = m \text{ for all } \gamma \in \Gamma\}.$$

This is a subalgebra of the Lie algebra $\mathfrak{g} \otimes A$, whose Lie algebra product is given by $[u_1 \otimes a_1,\, u_2 \otimes a_2] = [u_1, u_2] \otimes a_1 a_2$ for $u_i \in \mathfrak{g}$ and $a_i \in A$. There is an equivalent, more geometric definition which also explains the name "*map algebra*". Namely, let $X = \operatorname{Spec}(A)$ be the affine scheme corresponding to A. By functoriality, Γ acts on X, and viewing $\mathfrak{g}$ as an affine scheme it makes sense to consider regular maps from X to $\mathfrak{g}$. Then

$$\mathfrak{M} \cong \{f : X \to \mathfrak{g} \mid f \text{ regular and } \Gamma\text{-equivariant}\}.$$

Conversely, instead of using A, one can start with an affine scheme X with a Γ-action, and let $A = \mathcal{O}(X)$ be its ring of global regular functions. Then the Lie algebra of equivariant maps from X to $\mathfrak{g}$ under pointwise multiplication is isomorphic to the equivariant map algebra $M(A, \mathfrak{g})^\Gamma$. These two points of view, one more algebraic, the other more geometric, complement each other nicely.

The notion of an equivariant map algebra was introduced in [NSS]. Several important classes of Lie algebras can be viewed as equivariant map algebras.

Examples: (0) For $\Gamma = \{1\}$ the equivariant map algebra $\mathfrak{M} = \mathfrak{g} \otimes A$ is called the *(generalized) current algebra*.

(1) Let $\mathfrak{g}$ be simple, $\Gamma = \langle \sigma \rangle$ a group of diagram automorphisms of $\mathfrak{g}$, $X = k^\times = k \setminus \{0\}$, and let σ act on $X^\times$ by $\sigma \cdot x = \zeta x$ for ζ a $|\sigma|$-th primitive root of 1. The corresponding equivariant map algebra is the twisted ($\Gamma \neq \{1\}$) or untwisted ($\Gamma = \{1\}$) *loop algebra* of $\mathfrak{g}$.

(2) Let $\Gamma = \langle \sigma_1, \dots, \sigma_n \rangle$ be abelian, acting on a simple $\mathfrak{g}$ by automorphisms, let $A = k[t_1^{\pm}, \dots, t_n^{\pm 1}]$ be the Laurent polynomial ring in n variables, and assume

that $\sigma_i \cdot t_j = t_j$ for $j \neq i$ and $\sigma_i \cdot t_i = \zeta_i t_i$ for ζ_i as in (1). The corresponding equivariant map algebra is called a *multiloop algebra*.

(3) Let $\Gamma = \{1, \sigma\}$, a group of order 2, acting on a simple $\mathfrak{g}$ by some involution and on $X = k^\times$ by $\sigma \cdot x = x^{-1}$. The corresponding map algebra $\mathfrak{M}$ is called a *generalized Onsager algebra*. It is the *Onsager algebra* in case $\mathfrak{g} = \mathfrak{sl}_2$ and σ acts by the Chevalley involution.

We are interested in classifying the finite-dimensional irreducible representations of an equivariant map algebra $\mathfrak{M}$. We start by defining some examples. Let $\mathfrak{M} = M(X, \mathfrak{g})^\Gamma$ be an equivariant map algebra, and let $X_{\mathrm{rat}} = \{x \in X : \kappa(x) \cong k\}$ be the set of k-rational points of X (if X is an algebraic variety as in the examples above, then $X = X_{\mathrm{rat}}$). For every $x \in X_{\mathrm{rat}}$ we have an *evaluation map* $\mathrm{ev}_x : X \to \mathfrak{g}$, $\mathrm{ev}_x(m) = m(x)$, where we view m as a regular map $X \to \mathfrak{g}$. In general ev_x is not surjective. Rather we have

$$\mathrm{ev}_x(\mathfrak{g}) = \mathfrak{g}^x := \{u \in \mathfrak{g} : \gamma \cdot u = u \text{ for all } \gamma \in \Gamma \text{ with } \gamma \cdot x = x\}.$$

An *evaluation representation* of $\mathfrak{M}$ is a representation that arises as follows. We are given a finite set $\mathbf{x} \subseteq X_{\mathrm{rat}}$ of Γ-inequivalent points and a finite-dimensional representation $\rho_x : \mathfrak{g}^x \to \mathfrak{gl}(V_x)$ for every $x \in \mathbf{x}$. This allows to construct the evaluation representation $\mathrm{ev}(\rho_{\mathbf{x}}) : \mathfrak{M} \to \mathfrak{gl}(\otimes_{x\in\mathbf{x}} V_x)$ as the composition of maps

$$\mathrm{ev}(\rho_{\mathbf{x}}) : \mathfrak{M} \xrightarrow{\boxplus\, \mathrm{ev}_x} \boxplus_{x\in\mathbf{x}}\, \mathfrak{g}^x \xrightarrow{\otimes \rho_x} \mathfrak{gl}(\otimes_{x\in\mathbf{x}} V_x)$$

where $\boxplus$ denotes the direct product of algebras. It is easily seen that $\mathrm{ev}(\rho_x)_{x\in\mathbf{x}}$ is irreducible iff all ρ_x, $x \in \mathbf{x}$, are so.

Since $\mathfrak{M}$ is in general not perfect (for example this is so for the Onsager algebra), we also may have nontrivial 1-dimensional representations. Recall that the 1-dimensional representations of $\mathfrak{M}$ are given by linear forms $\lambda \in \mathfrak{M}^*$ with $\lambda([\mathfrak{M}, \mathfrak{M}]) = 0$.

Theorem 1 ([NSS]) *The finite-dimensional irreducible representations of* $\mathfrak{M}$ *are tensor products of a* 1*-dimensional representation and an irreducible evaluation representation.*

We denote by $\mathcal{S}$ the category of finite-dimensional irreducible (=simple) representations of $\mathfrak{M}$, and by $\mathcal{E} \subset \mathcal{S}$ the subcategory of evaluation representations. The theorem has an obvious corollary:

Corollary ([NSS]) *If* $\mathfrak{M}$ *is prefect, i.e.,* $\mathfrak{M} = [\mathfrak{M}, \mathfrak{M}]$*, the finite-dimensional irreducible representations of* $\mathfrak{M}$ *are precisely the irreducible evaluation representations, i.e.,* $\mathcal{S} = \mathcal{E}$.

Examples for $\mathfrak{M} = [\mathfrak{M}, \mathfrak{M}]$: (0) Current algebras ($\Gamma = \{1\}$) with $\mathfrak{g}$ semisimple. In this case, the finite-dimensional irreducible were determined in [CFK] using the theory of Weyl modules – a technique that is (so far) not available for arbitrary equivariant map algebras.

(1) and (2): (Multi)loop algebras: In this case, the finite-dimensional irreducible representations were determined in a sequence of papers, starting with the very influential paper by Chari [C] and ending with the recent paper of Lau [L].

It is important to note that $\mathcal{S} = \mathcal{E}$ may be true, even if $\mathfrak{M}$ is not perfect. The reason for this is our new definition of evaluation representations that allows us to view some 1-dimensional representations as evaluation representation. Rather than formulating the general result to this effect, contained in [NSS], we mention the following special case: *If $\mathfrak{M}$ is a generalized Onsager algebra, then $\mathcal{S} = \mathcal{E}$.* The interest in this example comes from the fact that a generalized Onsager algebra is in general not perfect. On the other hand, $\mathcal{E} \neq \mathcal{S}$ in general:

Theorem 2 ([NSS]) *If $|\{x \in X_{\mathrm{rat}} : \mathfrak{g}^x \text{ not perfect }\}| = \infty$ then $\mathcal{E} \subsetneq \mathcal{S}$.*

The remaining part of this abstract is based on joint work with Alistair Savage ([NS]). It is known that the category $\mathcal{F}$ of finite-dimensional representations of an equivariant map algebra $\mathfrak{M}$ is not semisimple. It is therefore of interest to study the extensions between the irreducible modules in $\mathcal{F}$, i.e., to determine the Ext-group $\mathrm{Ext}^1_{\mathfrak{M}}(V_1, V_2) \cong \mathrm{H}^1(\mathfrak{M}, \mathrm{Hom}_k(V_1, V_2))$ for irreducible modules $V_1, V_2 \in \mathcal{F}$.

Theorem 3 ([NS]) *Let V and V' be irreducible evaluation modules of $\mathfrak{M} = (\mathfrak{g} \otimes A)^\Gamma$. Thus, after possibly adding trivial 1-dimensional representations, we can write $V = \mathrm{ev}_{\mathbf{x}}(V_x)_{x\in\mathbf{x}}$ and $V' = \mathrm{ev}_{\mathbf{x}}(V'_x)_{x\in\mathbf{x}}$. Suppose A is Noetherian, $\mathfrak{g}$ is semisimple and $\mathrm{Ext}^1(V, V') \neq 0$. Further suppose that all $\mathfrak{g}^x$, $x \in \mathbf{x}$, are semisimple.*

Then there exists at most one point $x_0 \in \mathbf{x}$ with $V_{x_0} \not\cong V'_{x_0}$. Moreover,

(a) *if such an x_0 exists then $\mathrm{Ext}^1_{\mathfrak{M}}(V, V') \cong \mathrm{Ext}^1_{\mathfrak{M}}(V_{x_0}, V'_{x_0})$, while*

(b) *if such an x_0 does not exist then $\mathrm{Ext}^1_{\mathfrak{M}}(V, V') \cong \bigoplus_{x\in\mathbf{x}} \mathrm{Ext}^1_{\mathfrak{M}}(V_x, V'_x)$.*

The assumption on the subalgebras $\mathfrak{g}^x$, $x \in \mathbf{x}$, can be replaced by the more natural assumption that all $\mathfrak{g}^x$, $x \in \mathbf{x}$, are reductive, but the formulation of the theorem is more complicated. The theorem reduces the study of extensions of arbitrary irreducible evaluation modules to modules "concentrated" in one point. For these we have the following result.

Theorem 4 ([NS]) *Suppose $\mathfrak{g}$ and $\mathfrak{g}^x$ are semisimple and V_x and V'_x are simple finite-dimensional representations of $\mathfrak{g}^x$. Put $\mathfrak{K} = \mathrm{Ker}(\mathrm{ev}_x) \lhd \mathfrak{M}$. Then $\mathrm{Ext}^1_{\mathfrak{M}}(V_x, V'_x) \cong \mathrm{Hom}_{\mathfrak{g}^x}(\mathfrak{M}/[\mathfrak{K}, \mathfrak{K}], V^*_x \otimes V'_x)$. In particular, if $\mathfrak{g}^x = \mathfrak{g}$, then $\mathrm{Ext}^1_{\mathfrak{M}}(V_x, V'_x) \cong \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, V^*_x \otimes V'_x)\otimes)(I/I^2)^\Gamma$ for $I = \{f \in A : f(x) = 0\}$.*

One can use the last two theorems to describe the blocks of the category $\mathcal{F}$ for the Lie algebras in the examples above (current algebra, multiloop algebras and Onsager algebra). In particular, we recover, with new proofs, results of Chari-Moura [CM], Kodera [Ko] and Senesi [S].

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Weyl calculus for infinite-dimensional Lie groups

Daniel Beltiţă
(joint work with Ingrid Beltiţă)

There exists nowadays a quite extensive literature devoted to the differential equations and differential operators in infinitely many variables; see for instance [10], [12], [11], [1], [18], [13], and the references therein. The problems related to this circle of ideas have been hitherto treated mainly by methods borrowed from probability theory and stochastics.

We here wish to point out (following [4], [8], and [9]) that the above mentioned area can be alternatively addressed by using methods from representation theory of infinite-dimensional Lie groups. The advantage of this idea consists in taking into account the symmetry groups of the structures involved in various problems of the infinite-dimensional analysis, which allows one to state and address these problems in a systematic manner.

Our approach goes back to the remarkable paper [17] by N.V. Pedersen, where the orbit method was greatly enhanced by constructing the Weyl correspondence for unitary irreducible representations of finite-dimensional nilpotent Lie groups. That correspondence is highly significant from the perspective of quantum physics as well: It can be interpreted as a quantization procedure on the level of variables, inasmuch as it takes distributions on a coadjoint orbit ("observables on a classical phase space") to unbounded operators in the Hilbert space of an irreducible representation constructed by geometric quantization ("observables on a quantum phase space"). We have recently established in [5] continuity properties of the unbounded operators obtained in this way for symbols in suitable modulation spaces. In the special case of the Heisenberg group, this approach reveals the representation theoretic background of several basic properties of the pseudo-differential Weyl-Hörmander calculus on $\mathbb{R}^n$.

On the other hand, the so-called magnetic Weyl calculus has been rather recently developed in a hard-analysis fashion in a series of papers including [15] and [14] with motivation coming from quantum mechanics. We have later shown in [2], [3], [6], and [7] that this magnetic pseudo-differential calculus can also be approached within the framework of representation theory. The relevant symmetry groups in this situation are certain infinite-dimensional Lie groups whose coadjoint orbits are however finite dimensional, and their Weyl quantization is therefore easier to construct and describe.

Our purpose here is to provide a version of the localized Weyl calculus of [2] and [6], which is general enough for dealing with Weyl quantizations of some infinite-dimensional coadjoint orbits. In particular, this allows one to derive results like the ones of [1] and [13] in a systematic manner, by using the representation theory of certain infinite-dimensional Heisenberg groups. This approach also suggests many challenging questions related to extensions of these results to other classes of infinite-dimensional nilpotent Lie groups, which in turn provides consistent motivation for developing the representation theory of these groups.

We now provide some details of the method of constructing a Weyl calculus for representations of infinite-dimensional Lie groups. Let M be a locally convex Lie group with Lie algebra $\mathfrak{m}$ and smooth exponential map $\exp_M\colon \mathfrak{m} \to M$ (see [16]), and $\pi\colon M \to \mathcal{B}(\mathcal{Y})$ a continuous unitary representation on the complex Hilbert space $\mathcal{Y}$. We shall think of the dual space $\mathfrak{m}^*$ as a locally convex space with respect to the weak*-topology. Let $\mathcal{UC}_b(\mathfrak{m}^*)$ be the commutative unital C^*-algebra of uniformly continuous bounded functions on the locally convex space $\mathfrak{m}^*$ and for every $\mu \in \mathcal{UC}_b(\mathfrak{m}^*)^*$ define the function

$$\widehat{\mu}\colon \mathfrak{m} \to \mathbb{C}, \quad \widehat{\mu}(X) = \langle \mu, \mathrm{e}^{\mathrm{i}\langle \cdot, X\rangle}\rangle,$$

where either of the duality pairings $\mathfrak{m}^* \times \mathfrak{m} \to \mathbb{R}$ and $\mathcal{UC}_b(\mathfrak{m}^*)^* \times \mathcal{UC}_b(\mathfrak{m}^*) \to \mathbb{C}$ is denoted by $\langle \cdot, \cdot \rangle$. Assume the setting defined by the following data:

- a locally convex real vector space Ξ and a Borel measurable map $\theta\colon \Xi \to \mathfrak{m}$,
- a locally convex space (of "measures") $\Gamma \hookrightarrow \mathcal{UC}_b(\mathfrak{m}^*)^*$ with continuous inclusion map, where $\mathcal{UC}_b(\mathfrak{m}^*)^*$ is endowed with the weak*-topology,
- a locally convex space (of "smooth vectors") $\mathcal{Y}_{\Xi,\infty} \hookrightarrow \mathcal{Y}$ with continuous inclusion map,

subject to the following conditions:

(1) The linear mapping $\mathcal{F}_\Xi\colon \Gamma \to \mathcal{UC}_b(\Xi)$, $\quad \mu \mapsto \widehat{\mu} \circ \theta$ is well defined and injective. Denote $\mathcal{Q}_\Xi := \mathcal{F}_\Xi(\Gamma) \hookrightarrow \mathcal{UC}_b(\Xi)$ and endow it with the topology which makes the *Fourier transform* $\mathcal{F}_\Xi\colon \Gamma \to \mathcal{Q}_\Xi$ into a linear topological isomorphism. We also have the linear topological isomorphism $(\mathcal{F}_\Xi^*)^{-1}\colon \Gamma^* \to \mathcal{Q}_\Xi^*$.

(2) We have the well-defined continuous sesquilinear functional

$$\mathcal{Y}_{\Xi,\infty} \times \mathcal{Y}_{\Xi,\infty} \to \mathcal{Q}_\Xi, \quad (\phi, \psi) \mapsto (\pi(\exp_M(\theta(\cdot)))\phi \mid \psi).$$

Definition (cf. [4]). In the above framework, the *quasi-localized Weyl calculus for π along θ* is the linear map $\mathrm{Op}\colon \Gamma^* \to \mathcal{L}(\mathcal{Y}_{\Xi,\infty}, \overline{\mathcal{Y}}^*_{\Xi,\infty})$ defined by

$$(\mathrm{Op}(a)\phi \mid \psi) = \langle (\mathcal{F}_\Xi^*)^{-1}(a), (\pi(\exp_M(\theta(\cdot)))\phi \mid \psi)\rangle$$

for an arbitrary *symbol* $a \in \Gamma^*$ and $\phi, \psi \in \mathcal{Y}_{\Xi,\infty}$, where $\overline{\mathcal{Y}}^*_{\Xi,\infty}$ denotes the space of antilinear continuous functionals on $\mathcal{Y}_{\Xi,\infty}$. In the right-hand side of the above formula we use the duality pairing $\langle \cdot, \cdot \rangle\colon \mathcal{Q}_\Xi^* \times \mathcal{Q}_\Xi \to \mathbb{C}$.

Remark. If $a \in \Gamma^*$ and the functional $(\mathcal{F}^*_\Xi)^{-1}(a) \in \mathcal{Q}^*_\Xi$ is defined by a complex Borel measure on Ξ denoted in the same way, then for all $\phi, \psi \in \mathcal{Y}_{\Xi,\infty}$ we get

$$(\mathrm{Op}(a)\phi \mid \psi) = \int_\Xi (\pi(\exp_M(\theta(\cdot)))\phi \mid \psi)\mathrm{d}(\mathcal{F}^*_\Xi)^{-1}(a)$$

which is very similar to the definition of the Weyl-Pedersen calculus for irreducible representations of finite-dimensional nilpotent Lie groups with the locally convex space Ξ in the role of a predual of the coadjoint orbit under consideration (see [17] or [9]). We also get $\mathrm{Op}(a) \in \mathcal{B}(\mathcal{Y})$ and $\|\mathrm{Op}(a)\| \leq \|(\mathcal{F}^*_\Xi)^{-1}(a)\|$, where $\|(\mathcal{F}^*_\Xi)^{-1}(a)\|$ denotes the norm of the measure $(\mathcal{F}^*_\Xi)^{-1}(a)$ as an element of the dual Banach space $\mathcal{UC}_b(\Xi)^*$.

We also note that, due to the continuous inclusion map $\Gamma \hookrightarrow \mathcal{UC}_b(\mathfrak{m}^*)^*$, every function $f \in \mathcal{UC}_b(\mathfrak{m}^*)$ gives rise to a functional $a_f \in \Gamma^*$, $a_f(\gamma) = \langle \gamma, f\rangle$ for every $\gamma \in \Gamma$. Furthermore, if the function $f \in \mathcal{UC}_b(\mathfrak{m}^*)$ is the Fourier transform along θ of a Radon measure $\mu \in \mathcal{M}_{\mathrm{t}}(\Xi)$, in the sense that $f(\cdot) = \int_\Xi \mathrm{e}^{\mathrm{i}\langle \cdot, \theta(X)\rangle}\mathrm{d}\mu(X)$, then $\mathcal{F}^*_\Xi(\mu) = a_f$, hence $\mathrm{Op}(a_f) = \int_\Xi \pi(\exp_M(\theta(\cdot)))\mathrm{d}\mu$ and $\mathrm{Op}(a_f) \in \mathcal{B}(\mathcal{Y})$.

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Forms of Lie algebras over Laurent polynomial rings

Philippe Gille
(joint work with Arturo Pianzola)

1. Introduction

Let S be a scheme. In algebraic geometry, the term "form" or S-form of an object over S is used to describe another object over S that "locally look the same" to the given one, in the sense that the two objects become isomorphic after applying a suitable change to the base S.

This leads to Galois cohomology and more generally to étale cohomology. In this talk we shall discuss mainly the so-called isotrivial situation, namely that of objects V over an affine scheme $S = \mathrm{Spec}(R)$ such that there exists a finite étale covering $S' = \mathrm{Spec}(R')$ which makes the objects isomorphic after base change S'/S.

A perfect example for us is that of the punctured affine line $\mathbb{G}_m = \mathrm{Spec}(\mathbb{C}[t^{\pm 1}])$ which affords standard Kummer coverings of degree d, namely $\mathbb{G}_m \to \mathbb{G}_m$, $t = t^d$. We deal also with the analogous n-variables version of this example, namely $R_n = \mathbb{C}[t_1^{\pm 1}, .., t_n^{\pm 1}]$ and its covering $R_{n,m} = \mathbb{C}[t_1^{\frac{\pm 1}{m}}, .., t_n^{\frac{\pm 1}{m}}]$.

We are interested in classifying semisimple and reductive group schemes over $\mathrm{Spec}(R_n)$ and also their Lie algebras. There is a strong motivation for doing this coming from the theory of extended affine Lie algebras (EALAs for short. See [AABFP]). These are infinite dimensional complex Lie algebras defined by a set of axioms. In nullity one (i.e. when $n = 1$) EALAs are nothing but the affine Kac-Moody Lie algebras. Neher has shown how to construct EALAs out of the centreless cores. The most interesting class of EALAs has the remarkable property their centreless cores are known to be (multi)loop Lie algebras over R_n.

2. Loop algebras

2.1. **Definition.** Let $\mathfrak{g}$ be a semisimple complex Lie algebra. If $\sigma = (\sigma_1, ..., \sigma_n)$ is a family of finite order commuting automorphisms of $\mathfrak{g}$ whose orders divide an integer m, then we can define the Lie algebra

$$\mathcal{L}(\mathfrak{g},\sigma) = \bigoplus_{(i_1,\dots,i_n)\in\mathbb{Z}^n} t_1^{\frac{i_1}{m}} \cdots t_n^{\frac{i_n}{m}}\, \mathfrak{g}_{i_1,\dots,i_n} \subset \mathfrak{g}\otimes R_{n,m}$$

where

$$\mathfrak{g}_{i_1,\dots,i_n} = \Big\{ X\in\mathfrak{g} \mid \sigma_j(X) = \zeta_m^{i_j} X \quad \forall j=1,...,n \Big\}$$

stands for the eigenspace attached to the common diagonalization of the σ_j, where $\zeta_m = e^{\frac{2i\pi}{m}}$.

Since the eigenspaces are m-periodic in each coordinate, $\mathcal{L}(\mathfrak{g},\sigma)$ is a R_n-module. The relations $[\mathfrak{g}_{i_1,\dots,i_n}, \mathfrak{g}_{i'_1,\dots,i'_n}] \subset \mathfrak{g}_{i_1+i'_1,\dots,i_n+i'_n}$ provides $\mathcal{L}(\mathfrak{g},\sigma)$ with an R_n-Lie algebra structure. This Lie algebra is called the (multi)loop algebra of the pair $(\mathfrak{g},\sigma)$. Note that it is independent of the choice of It is easy to see that

$$\mathcal{L}(\sigma,\mathfrak{g})\otimes_{R_n} R_{n,m} \xrightarrow{\sim} \mathfrak{g}\otimes_{R_n} R_{n,m},$$

as $R_{n,m}$-Lie algebras. Thus $\mathcal{L}(\sigma,\mathfrak{g})$ is an R_n-form of $\mathfrak{g}\otimes R_n$ (or simply of $\mathfrak{g}$, for simplicity of terminology).

A natural question is to classify all $R-n$-forms of $\mathfrak{g}$, and in particular classify and characterize multiloop algebras, among all forms. We should note that Lie theorists are interested in classifying these objects over $\mathbb{C}$. However there is a "rigidity" result (called the centroid trick) which shows that two R_n-forms $\mathcal{L}$ and $\mathcal{L}'$ of $\mathfrak{g}$ are $\mathbb{C}$-isomorphic if and only if there exists $f \in \mathrm{GL}_n(\mathbb{Z}) = \mathrm{Aut}_{\mathbb{C}}(R_n)$ such that $f^*\mathcal{L}$ is R_n-isomorphic to $\mathcal{L}'$ [GP1]. We should concentrate then on the classification/characterization question over R_n.

In nullity one this program was carried out in [P]. The cohomological approach yields a new proof of the classification of the affine Kac-Moody algebras. In this case, all forms are loop algebras.

In higher nullity $n \geq 2$ the authors tried hard to show that it is also the case, but it is not (see the Margaux algebra [GP1]) ! A possible way to construct exotic objects over R_n would be by relaxing the splitting condition, namely to look at R_n-forms $\mathcal{L}$ of $\mathfrak{g}$ which are trivialized by a general faithfully flat base change R'/R_n. But we have shown (Isotriviality Theorem, [GP2]) that this approach is futile: all relevant objects are trivialized by some generalized Kummer covering $R_{n,m}/R_n$.

In practice the construction of counterexamples lead to technical complications because one needs to isolate the class of loop algebras. In the two dimensional case, we have conjectured that the only counterexamples are Margaux-like, so that they are can be described by an invertible module over a 2-loop Azumaya algebra which is rationally a division algebra.

For classical types A, B, C and D, Alexander Steinmetz has shown that the conjecture is true with the possible exception of small dimension cases [SZ]. This uses work of Parimala and also cancellation theorems (Bass, Suslin, Knus, Bertuccioni,...).

2.2. **Internal characterization of loop algebras.** The first characterization is given by grading considerations. We have proven in [GP2] that an R_n-form $\mathcal{L}$ of $\mathfrak{g}$ is a loop algebra if and only we there exists a $\mathbb{Z}^n$-grading on $\mathcal{L}$ together with a trivialization $\mathcal{L} \otimes_{R_n} R_{n,m} \cong \mathfrak{g} \otimes R_{n,m}$. which is a graded isomorphism.

This explains somehow why exotic algebras were not considered (gradings are an essential ingredient of EALAs, and the Margaux example is constructed in such a way as to "break" the grading). The previous criterion is of external nature (since it appeals to gradings that are put on the Lie algebras under consideration). We have another internal characterization of loop algebras which is much more useful in practice.

Theorem 2.1. *[GP3] Let $\mathcal{L}$ be a R_n-form of $\mathfrak{g}$. The following are equivalent:*
(i) $\mathcal{L}$ is a loop algebra.
(ii) $\mathcal{L}$ carries a maximal Cartan algebra, that is a subalgebra $\mathcal{C}$ which is locally (for the Zariski topology) a direct summand of $\mathcal{L}$ and whose geometric fibers are Cartan algebras in the usual sense.

According to [SGA3, XIV.4], (ii) is equivalent to the fact that the semisimple adjoint R_n-group scheme $\mathbf{Aut}(\mathcal{L})^0$ is "toral', i.e. it admits a maximal R_n-torus. The hard implication is $(ii) \Longrightarrow (i)$. That $(i) \Longrightarrow (ii)$ is a consequence of Borel-Mostow's theorem [BM]. If $\mathcal{L} = \mathcal{L}(\mathfrak{g}, \sigma)$, then since the σ_i generate an abelian subgroup of $\mathrm{Aut}(\mathfrak{g})$, we know that there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which is stable under the σ_i. Then $\mathcal{L}(\mathfrak{h}, \sigma)$ is a Cartan subalbebra of $\mathcal{L}$.

3. The main results

We denote by $F_n = \mathbb{C}((t_1))...((t_n))$ the iterated Laurent series field in n-variables. An important fact is that $\pi_1(R_n, .) \cong \mathcal{G}al(F_n) \xrightarrow{\sim} (\hat{\mathbb{Z}})^n$. This implies that R_n and F_n have the "same" finite étale coverings.

Theorem 3.1. *The tensor product $\otimes_{R_n} F_n$ induces a one to one correspondence: between isomorphisms of loop R_n-forms of $\mathfrak{g}$ and F_n-forms of $\mathfrak{g}$.*

As in Tits classification ([T] over F_n), the problem reduces to that of "anisotropic objects". The proof of the main theorem proceeds by several delicate steps, and by looking closely at the abelian subgroups of $\mathbf{Aut}(\mathfrak{g})(\mathbb{C})$. A crucial fact, based on Bruhat-Tits theory, is the following:

Theorem 3.2. *Let σ be an anisotropic n-tuple of commuting automorphims of* $\mathrm{Aut}(\mathfrak{g})$ *of finite order (which amounts to the common centralizer of all the σ_i in* $\mathrm{Aut}(\mathfrak{g})$ *being finite). Let σ' be another n-uple. Then the following are equivalent :*

(1) σ and σ' are conjugated under $\mathrm{Aut}(\mathfrak{g})(\mathbb{C})$*;*

(2) $\mathcal{L}(\mathfrak{g}, \sigma) \cong \mathcal{L}(\mathfrak{g}, \sigma')$ as R_n-Lie algebras.

Essentially, the classification of finite abelian subgroups of $\mathrm{Aut}(\mathfrak{g})(\mathbb{C})$ provides the classification of loop algebras. But it is not easy to classify these subgroups! The only general result is about p-elementary abelian subgroups due to Griess. This is sufficient to provide many interesting loop algebras, specially for the exceptional groups G_2, F_4, E_8. The remarkable fact is that we can go the other way around; indeed one knows quite well semisimple F_n-Lie algebras and groups.

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Irreducible representations of Lie algebra of vector fields on a torus and chiral de Rham complex

Yuly Billig

(joint work with Vyacheslav Futorny)

In this talk we discuss representation theory of a classical infinite-dimensional Lie algebra – the Lie algebra $\mathrm{Vect}(\mathbb{T}^N)$ of vector fields on a torus,

$$\mathrm{Vect}(\mathbb{T}^N) = \mathrm{Der}\mathbb{C}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}] = \bigoplus_{p=1}^{N} \mathbb{C}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}] \frac{\partial}{\partial t_p}. \tag{1.1}$$

This algebra has a class of representations of a geometric nature – tensor modules, since vector fields act on tensor fields of any given type via Lie derivative. Tensor modules are parametrized by finite-dimensional representations of gl_N, with the fiber of a tensor bundle being a gl_N-module W:

$$T = \mathbb{C}[q_1^{\pm 1}, \ldots, q_N^{\pm 1}] \otimes W \tag{1.2}$$

with the action given by

$$t^r d_a(q^m \otimes w) = m_a q^{m+r} \otimes w + \sum_{p=1}^{N} r_p q^{m+r} \otimes E_{pa} w, \tag{1.3}$$

where $r, m \in \mathbb{Z}^N$, $a = 1, \ldots, N$, $d_a = t_a \frac{\partial}{\partial t_a}$ and E_{pa} is the matrix with 1 in (p, a)-position and zeros elsewhere.

Irreducible gl_N-modules yield tensor modules that are irreducible over $\mathrm{Vect}(\mathbb{T}^N)$, with exception of the modules of differential k-forms. In the latter case, the gl_N-module is irreducible, yet the modules of k-forms are reducible, which follows from the fact that the differential of the de Rham complex is a homomorphism of $\mathrm{Vect}(\mathbb{T}^N)$-modules. In the talk we present a vertex algebra analogue of this result.

There is another class of irreducible modules with finite-dimensional weight spaces for the Lie algebra of vector fields on a torus. These are bounded modules, which are generalizations of the highest weight modules and constructed using the technique developed in [1].

In our constructions one of the coordinates will play a special role. From now on, we will be working with the $N+1$-dimensional torus and will index our coordianates as $t_0, t_1, \ldots, t_N$, where t_0 is the "special variable". We would like to construct modules for the Lie algebra $\mathcal{D} = \mathrm{Vect}(\mathbb{T}^{N+1})$ in which the "energy operator" $-d_0$ has spectrum bounded from below.

Let us consider a $\mathbb{Z}$-grading of $\mathcal{D}$ by degrees in t_0. This $\mathbb{Z}$-grading induces a decomposition

$$\mathcal{D} = \mathcal{D}_- \oplus \mathcal{D}_0 \oplus \mathcal{D}_+ \tag{1.4}$$

into subalgebras of positive, zero and negative degrees in t_0. The degree zero part is

$$\mathcal{D}_0 = \bigoplus_{p=0}^{N} \mathbb{C}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}] d_p. \tag{1.5}$$

In particular, $\mathcal{D}_0$ is a semi-direct product of the Lie algebra of vector fields on $\mathbb{T}^N$ with an abelian ideal $\mathbb{C}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}] d_0$.

We begin the construction of a bounded module by taking a tensor module for $\mathcal{D}_0$. Fix a finite-dimensional irreducible gl_N-module W and $\beta \in \mathbb{C}$. We define a $\mathcal{D}_0$-module T as a space

$$T = \mathbb{C}[q_1^{\pm 1}, \ldots, q_N^{\pm 1}] \otimes W \tag{1.6}$$

with the tensor module action of the subalgebra $\mathrm{Vect}(\mathbb{T}^N) \subset \mathcal{D}_0$ and with $\mathbb{C}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}] d_0$ acting by shifts

$$t^r d_0(q^m \otimes w) = \beta\, q^{m+r} \otimes w. \tag{1.7}$$

Next we let $\mathcal{D}_+$ act on T trivially and define $M(T)$ as the induced module

$$M(T) = \mathrm{Ind}_{\mathcal{D}_0 \oplus \mathcal{D}_+}^{\mathcal{D}} T \cong U(\mathcal{D}_-) \otimes T. \tag{1.8}$$

The module $M(T)$ has a weight decomposition with respect to the Cartan subalgebra $\langle d_0, \ldots, d_N \rangle$ and the (real part of) spectrum of $-d_0$ on $M(T)$ is bounded from below. However the weight spaces of $M(T)$ that lie below T are all infinite-dimensional.

It turns out that the situation improves dramatically when we pass to the irreducible quotient of $M(T)$. One can immediately see that the Lie algebra $\mathcal{D}$ belongs to the class of Lie algebras with polynomial multiplication (as defined in [1]), whereas tensor modules belong to the class of modules with polynomial action. A general theorem of [1] yields in this particular situation the following

Theorem 1.9. *[1] (i) The module $M(T)$ has a unique maximal submodule M^{rad}.*
(ii) The irreducible quotient $L(T) = M(T)/M^{rad}$ has finite-dimensional weight spaces.

This leads to the following natural questions:
Problem 1. Determine the character of $L(T)$.
Problem 2. Find a realization of $L(T)$.

It turns out that these problems may be solved with the help of the results of [2] on toroidal Lie algebras. Let us briefly review these here.

Let $\mathcal{K}$ be the quotient space of 1-forms by differentials of functions,

$$\mathcal{K} = \Omega^1(\mathbb{T}^{N+1})/d\Omega^0(\mathbb{T}^{N+1}). \tag{1.10}$$

The Lie algebra $\mathcal{D} = \mathrm{Vect}(\mathbb{T}^{N+1})$ acts on $\mathcal{K}$ via Lie derivative and we can form a semi-direct product $\mathcal{D} \ltimes \mathcal{K}$. A category of bounded modules for$\mathcal{D} \ltimes \mathcal{K}$ is studied in [2] and realizations of irreducible modules in this category are given.

Our approach is to look at the representations of this semidirect product, constructed in [2], and to study their reductions to the subalgebra $\mathcal{D}$ of vector fields on $\mathbb{T}^{N+1}$. Surprisingly, as we will see below, most of the irreducible modules for $\mathcal{D} \ltimes \mathcal{K}$ remain irreducible when restricted to $\mathcal{D}$.

Representation theory of $\mathcal{D} \ltimes \mathcal{K}$ is controlled by a tensor product of three vertex operator algebras: vertex subalgebra V^+_{Hyp} of a rank $2N$ hyperbolic lattice vertex algebra, level 1 affine $\widehat{gl}_N$ vertex algebra V_{gl_N} and the Virasoro vertex algebra V_{Vir} with zero central charge.

Theorem 1.11. *[2]. Let M_{Hyp}, M_{gl_N}, M_{Vir} be irreducible modules for V^+_{Hyp}, V_{gl_N} and V_{Vir} respectively. Then the tensor product*

$$M_{Hyp} \otimes M_{gl_N} \otimes M_{Vir} \tag{1.12}$$

is an irreducible module for the Lie algebra $\mathcal{D} \ltimes \mathcal{K}$.

The main result of this talk is the following

Theorem 1.13. *The module $M_{Hyp} \otimes M_{gl_N} \otimes M_{Vir}$ remains irreducible when restricted to the subalgebra* $\mathrm{Vect}(\mathbb{T}^{N+1})$*, unless it appears in the chiral de Rham complex.*

Chiral de Rham complex was introduced by Malikov et al. in [3]. In case a of torus $\mathbb{T}^N$ the space of this differential complex is a tensor product of two vertex (super) algebras

$$V^+_{Hyp} \otimes V_{\mathbb{Z}^N}. \tag{1.14}$$

Here $V_{\mathbb{Z}^N}$ is the lattice vertex superalgebra of the standard euclidean lattice $\mathbb{Z}^N$. The vertex superalgebra $V_{\mathbb{Z}^N}$ has a fermionic $\mathbb{Z}$-grading We have a decomposition

$$V_{\mathbb{Z}^N} = \bigoplus_{k\in\mathbb{Z}} V^k_{\mathbb{Z}^N}. \tag{1.15}$$

and the homogeneous components $V^k_{\mathbb{Z}^N}$ of this grading are irreducible level 1 highest weight modules for the affine $\widehat{gl}_N$. Thus the spaces $V^+_{Hyp} \otimes V^k_{\mathbb{Z}^N}$ admit the structure of irreducible $\mathcal{D} \ltimes \mathcal{K}$-modules (with a trivial 1-dimensional module M_{Vir}).

The differential in the chiral de Rham complex is a map

$$d: \ V^+_{Hyp} \otimes V^k_{\mathbb{Z}^N} \to V^+_{Hyp} \otimes V^{k+1}_{\mathbb{Z}^N}. \tag{1.16}$$

We prove that this map is a homomorphism of $\mathrm{Vect}(\mathbb{T}^{N+1})$-modules, thus the kernels and images of d are $\mathrm{Vect}(\mathbb{T}^{N+1})$-submodules. This shows that the $\mathrm{Vect}(\mathbb{T}^{N+1}) \ltimes \mathcal{K}$-modules that appear in the chiral de Rham complex are exceptional and become reducible when restricted to the Lie algebra of vector fields. This is a vertex analogue of the classical result on irreducibility of tensor modules.

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Energy representations of path groups

Maria Gordina

(joint work with Sergio Albeverio, Bruce K. Driver, A.M. Vershik)

The results described below concern two unitary representations of the groups of paths in a compact semi-simple Lie group such as SU (2). These representations are the Brownian representation and the energy representation, and we describe them below. One of the results presented is that these representations are equivalent. Note that these infinite-dimensional groups are not locally compact, so we can not use standard representation theory techniques.

After establishing the unitary equivalence of these two representations, we talk about the structure of the energy representation. Our research has been motivated by [1–4], but we employ stochastic analysis more significantly than was possible at the time when these papers have been written.

1. Notation and some results

Let G be a compact connected Lie group. Without loss of generality we can assume that G is a Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$. The identity of G is denoted by e, and the dimension of the Lie algebra of G, $\mathfrak{g}$, by d. We assume that the Lie algebra $\mathfrak{g}$ of G is identified with the tangent space at e, and $\mathfrak{g}$ is equipped with an Ad_G-invariant inner product $\langle \cdot, \cdot \rangle$, which is equal to the negative of the Killing form.

Notation 1.1. *Suppose* $0 < T < \infty$. *Let us introduce the Wiener and Cameron-Martin (finite energy) spaces, and the corresponding probability measures.*

(1) $W(G) = W([0,T], G) = \{g_t \in C([0,T], G), t \in [0,T], g_0 = e\}$ *is the space of all continuous paths in* G *with the* sup *norm and the pointwise multiplication as the group operation;*

(2) $H(G) = \{h \in W(G), h$ *is absolutely continuous and the norm*

$$\|h\|_H^2 = \int_0^T |h(s)^{-1}h'(s)|^2 ds < \infty\}$$

is the Cameron-Martin (finite energy) subgroup of $W(G)$. *Here* $|\cdot|$ *is the norm induced by the inner product on the Lie algebra* $\mathfrak{g}$;

(3) μ *is the Wiener measure on* $W(G)$ *determined by its finite dimensional distributions;*

(4) the corresponding spaces of paths with values in the Lie algebra $\mathfrak{g}$ *and starting at* 0 *are denoted by* $W(\mathfrak{g})$ *and* $H(\mathfrak{g})$, *and the Wiener measure on* $W(\mathfrak{g})$ *is denoted by* ν;

(5) let $\mathbb{T}$ *be a maximal torus of* G, *and* $\mathfrak{h}$ *be its Lie algebra with* $\mathfrak{m} = \mathfrak{h}^{\perp}$, *then the corresponding Wiener and Cameron-Martin spaces are denoted by* $W(\mathbb{T})$, $H(\mathbb{T})$, $W(\mathfrak{h})$, $H(\mathfrak{h})$, *and the Wiener measures by* $\mu_{\mathbb{T}}$ *and* $\nu_{\mathfrak{h}}$. *We will also need the spaces* $W(\mathfrak{m})$, $H(\mathfrak{m})$ *with the measure* $\nu_{\mathfrak{m}}$.

We fix a complete probability space $(\Omega, \mathcal{F}, P)$ with a right continuous increasing family of σ-fields of $\mathcal{F}$, $\{\mathcal{F}_t\}_{t \geqslant 0}$. We assume that $\mathcal{F}_0$ contains all P-null sets.

It is known that the measure μ on the Wiener measure on $W(G)$ is quasi-invariant under the right multiplication by elements in $H(G)$. Let $R_\varphi g := g\varphi$ for any $\varphi \in H(G)$ and $g \in W(G)$. Then the right Radon-Nikodym density for μ is in $L^1(W(G), \mu)$ and is given by the following formula

$$D^R(\varphi)(g_t) = \frac{d\left(R_\varphi\right)_* \mu}{d\mu}(g_t) = \frac{d\left(\mu \circ R_\varphi^{-1}\right)}{d\mu}(g_t) = \exp\left(-\int_0^t \langle \varphi' \varphi^{-1}(s), \delta w(g_s)\rangle - \frac{1}{2}\|\varphi\|_{H,t}^2\right). \tag{1.1}$$

for any $g_t \in W(G), \varphi \in H(G)$. Similarly the Wiener measure μ is quasi-invariant under the left multiplication by elements in $H(G)$, and the left Radon-Nikodym density for μ is in $L^1(W(G),\mu)$ and is denoted by

$$D^L(\varphi)(g_t) = \frac{d(L_\varphi)_* \mu}{d\mu}(g_t) = \frac{d\left(\mu \circ L_\varphi^{-1}\right)}{d\mu}(g_t), \tag{1.2}$$

where $L_\varphi g := \varphi^{-1} g$ for any $g_t \in W(G), \varphi \in H(G)$.

$$D^R(\varphi)(g_t) = \exp\left(-\int_0^t \langle \varphi' \varphi^{-1}(s), \delta w^R(g_s)\rangle - \frac{1}{2}\|\varphi\|^2_{H,t}\right).$$

Theorem 1.2. *[Cyclicity of* **1***.] Suppose that G is a compact connected simply connected Lie group, then*

$$H_G := \operatorname{Span}\left\{h_\varphi(g_t) = \left(D^R(\varphi)(g_t)\right)^{1/2},\ \varphi \in H(G)\right\}$$

is dense in $L^2(W(G),\mu)$.

The unitary representation of $H(G)$ on $L^2(W(G))$ we define in this section is induced by the quasi-invariance of the Wiener measure μ.

Definition 1.3. Let $W(G)$ and $H(G)$ be as before.

(1) The **right Brownian measure representation** U^R of $H(G)$ on $L^2(W(G),\mu)$ is defined by

$$\left(U^R_\varphi f\right)(g_t) := \left(U^R(\varphi) f\right)(g_t) = f(g_t\varphi)\left(D^R(\varphi)(g_t)\right)^{1/2}$$

for any $f \in L^2(W(G),\mu)$, $\varphi \in H(G)$, $g_t \in W(G)$;

(2) the **left Brownian measure representation** U^L on $L^2(W(G),\mu)$ is defined by

$$\left(U^L_\varphi f\right)(g_t) := \left(U^L(\varphi) f\right)(g_t) = f\left(\varphi^{-1} g_t\right)\left(D^L(\varphi)(g_t)\right)^{1/2}$$

for any $f \in L^2(W(G),\mu)$, $\varphi \in H(G)$, $g_t \in W(G)$.

The energy representation is a unitary representation of $H(G)$ on the space $L^2(W(\mathfrak{g}),\nu)$, where ν is the Gaussian measure.

Definition 1.4. For any $\varphi \in H(G)$

$$(E_\varphi f)(w_t) := e^{i\langle \varphi^{-1} d\varphi, \delta w_t\rangle} f\left(\operatorname{Ad}_{\varphi^{-1}} \delta w_t\right).$$

for any $f \in L^2(W(\mathfrak{g}))$. E_φ is called the **energy representation** of $H(G)$.

Theorem 1.5. *Both U^R and U^L are unitarily equivalent to the energy representation E.*

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ as before, and $\mathfrak{m} = \mathfrak{h}^\perp$, then $H(\mathfrak{g}) = H(\mathfrak{h}) \oplus H(\mathfrak{m})$. Denote by $\nu_\mathfrak{h}$ and $\nu_\mathfrak{m}$ the corresponding Gaussian measures on $W(\mathfrak{h})$ and $W(\mathfrak{m})$. Then $L^2(W(\mathfrak{g}),\nu) = L^2(W(\mathfrak{h}),\nu_\mathfrak{h}) \otimes L^2(W(\mathfrak{m}),\nu_\mathfrak{m})$ which follows from the isomorphism with the corresponding Fock spaces. Define

$$\mathcal{W}\left(e^{h}\right) := E_{e^{h}}, h \in H\left(\mathfrak{h}\right).$$

Then $\mathcal{W}$ is a unitary representation of $H(\mathbb{T})$ in $L^2(W(\mathfrak{g}), \nu)$. For any $w_t \in W(\mathfrak{g})$ we can write it as $a_t + m_t$ with $a_t \in W(\mathfrak{h})$ and $m_t \in W(\mathfrak{m})$, and so

$$\left(\mathcal{W}\left(e^{h}\right) f\right)(w_t) = \left(\mathcal{W}\left(e^{h}\right) f\right)(a_t, m_t) = e^{i\langle dh, \delta a_t\rangle} f\left(a_t, \mathrm{Ad}_{e^{-h}}\, \delta m_t\right). \tag{1.3}$$

Equation (1.3) allows us to decompose the representation W as follows

$$\begin{aligned} &\mathcal{W} = \mathcal{W}_{\mathfrak{h}} \otimes \mathcal{W}_{\mathfrak{m}}, \\ &\mathcal{W}_{\mathfrak{h}}\left(e^{h}\right) f\left(a_t\right) := e^{i\langle dh, \delta a_t\rangle} f\left(a_t\right),\ f \in L^2\left(W\left(\mathfrak{h}\right), \nu_{\mathfrak{h}}\right), \\ &\mathcal{W}_{\mathfrak{m}}\left(e^{h}\right) f\left(m_t\right) := f\left(\mathrm{Ad}_{e^{-h}}\, \delta m_t\right),\ f \in L^2\left(W\left(\mathfrak{m}\right), \nu_{\mathfrak{m}}\right). \end{aligned} \tag{1.4}$$

The next step (that goes beyond what we can explain here) is to find the spectral type of the von Neumann algebra generated by the unitary representation $\mathcal{W}$ described by (1.3). The main difficulty is that we would like to determine the spectral type of unitary representations of an Abelian topological group which might not be locally compact.

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Principal Series Representations of Direct Limit Lie Groups

Joseph A. Wolf

We start with the three classical simple locally finite countable–dimensional Lie algebras $\mathfrak{g}_{\mathbb{C}} = \varinjlim \mathfrak{g}_{n,\mathbb{C}}$, and their real forms $\mathfrak{g}_{\mathbb{R}}$. The Lie algebras $\mathfrak{g}_{\mathbb{C}}$ are the classical direct limits, $\mathfrak{sl}(\infty, \mathbb{C}) = \varinjlim \mathfrak{sl}(n; \mathbb{C})$, $\mathfrak{so}(\infty, \mathbb{C}) = \varinjlim \mathfrak{so}(2n; \mathbb{C}) = \varinjlim \mathfrak{so}(2n+1; \mathbb{C})$, and $\mathfrak{sp}(\infty, \mathbb{C}) = \varinjlim \mathfrak{sp}(n; \mathbb{C})$, where the direct systems are given by the inclusions of the form $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$. We often consider the locally reductive algebra $\mathfrak{gl}(\infty; \mathbb{C}) = \varinjlim \mathfrak{gl}(n; \mathbb{C})$ along with $\mathfrak{sl}(\infty; \mathbb{C})$.

The real forms of the classical simple locally finite countable–dimensional complex Lie algebras $\mathfrak{g}_{\mathbb{C}}$ are:

If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(\infty; \mathbb{C})$, then $\mathfrak{g}_{\mathbb{R}}$ is one of $\mathfrak{sl}(\infty; \mathbb{R}) = \varinjlim \mathfrak{sl}(n; \mathbb{R})$, the real special linear Lie algebra; $\mathfrak{sl}(\infty; \mathbb{H}) = \varinjlim \mathfrak{sl}(n; \mathbb{H})$, the quaternionic special linear Lie algebra, given by $\mathfrak{sl}(n; \mathbb{H}) := \mathfrak{gl}(n; \mathbb{H}) \cap \mathfrak{sl}(2n; \mathbb{C})$; $\mathfrak{su}(p, \infty) = \varinjlim \mathfrak{su}(p, n)$, the complex special unitary Lie algebra of real rank p; or $\mathfrak{su}(\infty,\infty) = \varinjlim \mathfrak{su}(p,q)$, complex special unitary algebra of infinite real rank.

If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(\infty;\mathbb{C})$, then $\mathfrak{g}_{\mathbb{R}}$ is one of $\mathfrak{so}(p,\infty) = \varinjlim \mathfrak{so}(p,n)$, the real orthogonal Lie algebra of finite real rank p; $\mathfrak{so}(\infty,\infty) = \varinjlim \mathfrak{so}(p,q)$, the real orthogonal Lie algebra of infinite real rank; or $\mathfrak{so}^*(2\infty) = \varinjlim \mathfrak{so}^*(2n)$

If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sp}(\infty;\mathbb{C})$, then $\mathfrak{g}_{\mathbb{R}}$ is one of $\mathfrak{sp}(\infty;\mathbb{R}) = \varinjlim \mathfrak{sp}(n;\mathbb{R})$, the real symplectic Lie algebra; $\mathfrak{sp}(p,\infty) = \varinjlim \mathfrak{sp}(p,n)$, the quaternionic unitary Lie algebra of real rank p; or $\mathfrak{sp}(\infty,\infty) = \varinjlim \mathfrak{sp}(p,q)$, quaternionic unitary Lie algebra of infinite real rank.

If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(\infty;\mathbb{C})$, then $\mathfrak{g}_{\mathbb{R}}$ is one $\mathfrak{gl}(\infty;\mathbb{R}) = \varinjlim \mathfrak{gl}(n;\mathbb{R})$, the real general linear Lie algebra; $\mathfrak{gl}(\infty;\mathbb{H}) = \varinjlim \mathfrak{gl}(n;\mathbb{H})$, the quaternionic general linear Lie algebra; $\mathfrak{u}(p,\infty) = \varinjlim \mathfrak{u}(p,n)$, the complex unitary Lie algebra of finite real rank p; or $\mathfrak{u}(\infty,\infty) = \varinjlim \mathfrak{u}(p,q)$, the complex unitary Lie algebra of infinite real rank.

The structure of parabolic subalgebras of these algebras was worked out in [3] in the complex cases, in [5] in general. In this report we indicate just which real parabolics are minimal parabolic subalgebras and we use that information to construct the principal series representations of the corresponding infinite dimensional real Lie groups. This extends the considerations of [**?**] for the cases where the relevant minimal parabolic is itself a direct limit of minimal parabolics of finite dimensional Lie groups.

Parabolic Subalgebras

Let $\mathfrak{g}_{\mathbb{C}}$ be one of $\mathfrak{gl}(\infty,\mathbb{C})$, $\mathfrak{sl}(\infty,\mathbb{C})$, $\mathfrak{so}(\infty,\mathbb{C})$, and $\mathfrak{sp}(\infty,\mathbb{C})$. For our purposes they should be described as follows. V and W are paired countable dimensional complex vector spaces, $\mathfrak{gl}(\infty,\mathbb{C}) = \mathfrak{gl}(V,W) := V \otimes W$ consists of all finite linear combinations of the $v \otimes w : x \mapsto \langle w,x\rangle v$, and $\mathfrak{sl}(\infty,\mathbb{C}) = \mathfrak{sl}(V,V_*)$ is the traceless part of $\mathfrak{gl}(\infty,\mathbb{C})$. In the orthogonal and symplectic cases we identify V and W by the bilinear form. Then $\mathfrak{so}(\infty,\mathbb{C}) = \Lambda\mathfrak{gl}(V,V)$ is the image of $\Lambda : v \otimes w \mapsto v \otimes w - w \otimes v$ and $\mathfrak{sp}(V,V) = S\mathfrak{gl}(V,V)$ is the image of $S : v \otimes w \mapsto v \otimes w + w \otimes v$.

A *Borel subalgebra* of $\mathfrak{g}_{\mathbb{C}}$ is a maximal locally solvable subalgebra. A *parabolic subalgebra* of $\mathfrak{g}_{\mathbb{C}}$ is a subalgebra that contains a Borel.

Certain parabolic subalgebras of $\mathfrak{g}_{\mathbb{C}}$ are the $\mathfrak{g}_{\mathbb{C}}$–stabilizers of a flag $\mathcal{F}$ in V and a flag $\mathcal{G}$ in W that satisfy some technical conditions: they are increasing families of subspaces that are *generalized flags* $\{F_i\}_{i\in I}$ (I can be any countable well–ordered set) in the sense that $F_i \subset F_j$ for $i \leqq j$, each F_i belongs to an *immediate predecessor–successor pair* (IPS) $\{F_i', F_i''\}$ and the double annihilator $F_i^{\perp\perp}$ belongs to $\{F_i', F_i''\}$ (and the same conditions for $\mathcal{G}$), and they form a *taut couple* in the sense that if $F \in \mathcal{F}$ then its annihilator $F^\perp$ is invariant by the $\mathfrak{gl}$–stabilizer of $\mathcal{G}$ and if $G \in \mathcal{G}$ then $G^\perp$ is invariant by the $\mathfrak{gl}$–stabilizer of $\mathcal{F}$. In the $\mathfrak{so}$ and $\mathfrak{sp}$ cases one can take $V = W$ and $\mathcal{F} = \mathcal{G}$, and the subspaces should be isotropic or co-isotropic.

There is a complication here: $\mathfrak{sl}(\infty)$ contains a Borel subalgebra of $\mathfrak{gl}(\infty)$. See Example 4 on page 8 of [7]. So a general parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ means a parabolic as just described, cut down by finite linear combinations of trace conditions on the $\mathfrak{gl}(\infty)$ summands of its Levi components.

Let $\mathfrak{g}_\mathbb{R}$ be a real form of $\mathfrak{g}_\mathbb{C}$. Let $G_\mathbb{R}$ be the corresponding connected real subgroup of $G_\mathbb{C}$. When $\mathfrak{g}_\mathbb{R}$ has two inequivalent defining representations, we denote them by $V_\mathbb{R}$ and $W_\mathbb{R}$, and when $\mathfrak{g}_\mathbb{R}$ has only one defining representation, we denote it by $V_\mathbb{R}$.

Let $\mathbb{D}$ denote the algebra of $\mathfrak{g}_\mathbb{R}$–endomorphisms of $V_\mathbb{R}$. Then $\mathbb{D} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and either $\mathfrak{g}_\mathbb{R}$ consists of the $\mathbb{D}$–linear transformations of $V_\mathbb{R}$ (or those of trace 0), or $\mathfrak{g}_\mathbb{R}$ is specified by a nondegenerate $\mathbb{D}$–bilinear or $\mathbb{D}$–sesquilinear form ω on $V_\mathbb{R}$. In the first cases we write $\omega = 0$.

The parabolic subalgebras of $\mathfrak{g}_\mathbb{R}$ are described in [5] as follows. 1. The parabolic subalgebras $\mathfrak{p} \subset \mathfrak{g}_\mathbb{R}$ are the subalgebras whose complexification $\mathfrak{p}_\mathbb{C}$ is parabolic in $\mathfrak{g}_\mathbb{C}$. Then the corresponding flag(s) for $\mathfrak{p}$ complexify to the corresponding flag(s) for $\mathfrak{p}_\mathbb{C}$.

2. If $\mathfrak{g}_\mathbb{R}$ has two inequivalent defining representations then a subalgebra of $\mathfrak{g}_\mathbb{R}$ (resp. subgroup of $G_\mathbb{R}$) is parabolic if and only if it is defined by infinite trace conditions (resp. infinite determinant conditions) on the $\mathfrak{g}_\mathbb{R}$–stabilizer (resp. $G_\mathbb{R}$–stabilizer) of a taut couple of $\mathbb{D}$–generalized flags $\mathcal{F}$ in $V_\mathbb{R}$ and $\mathcal{G}$ in $W_\mathbb{R}$.

3. If $\mathfrak{g}_\mathbb{R}$ has only one defining representation then a subalgebra of $\mathfrak{g}_\mathbb{R}$ (resp. subgroup) of $G_\mathbb{R}$ is parabolic if and only if it is defined by infinite trace conditions (resp. infinite determinant conditions) on the $\mathfrak{g}_\mathbb{R}$–stabilizer (resp. $G_\mathbb{R}$–stabilizer) of a self–taut $\mathbb{D}$–generalized flag $\mathcal{F}$ in $V_\mathbb{R}$.

Levi Components

Definition ([2, Def 4.1]): Let $\mathfrak{p}$ be a locally finite Lie algebra and $\mathfrak{r}$ its locally solvable radical. A subalgebra $\mathfrak{l} \subset \mathfrak{p}$ is a *Levi component* if $[\mathfrak{p}, \mathfrak{p}]$ is the semidirect sum $(\mathfrak{r} \cap [\mathfrak{p}, \mathfrak{p}]) \Subset \mathfrak{l}$. Every finitary Lie algebra has a Levi component [2]. In general a Levi component of a parabolic subalgebra $\mathfrak{p}$ in $\mathfrak{gl}(\infty; \mathbb{C})$ is a maximal locally semisimple subalgebra (by the definition). In contrast to the finite dimensional situation, a maximal locally semisimple subalgebra of a parabolic subalgebra $\mathfrak{p}$ in $\mathfrak{gl}(\infty; \mathbb{C})$ need not be a Levi component of $\mathfrak{p}$ [1].

Let $X \subset V$ and $Y \subset W$ be nondegenerately paired subspaces, isotropic in the orthogonal and symplectic cases. The subalgebras $\mathfrak{gl}(X, Y) \subset \mathfrak{gl}(V, W)$, $\mathfrak{sl}(X, Y) \subset \mathfrak{sl}(V, W)$, $\Lambda\mathfrak{gl}(X, Y) \subset \Lambda\mathfrak{gl}(V, V)$ and $S\mathfrak{gl}(X, Y) \subset S\mathfrak{gl}(V, V)$ are called *standard*. In [3] it is shown that a subalgebra $\mathfrak{l} \subset \mathfrak{g}_\mathbb{C}$ is the Levi component $\mathfrak{l}$ of a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}_\mathbb{C}$ if and only if it is the direct sum of standard special linear subalgebras and a subalgebra $\Lambda\mathfrak{gl}(X, Y)$ in the $\mathfrak{so}$ case, $S\mathfrak{gl}(X, Y)$ in the $\mathfrak{sp}$ case.

The possible Levi components $\mathfrak{l}$ of complex parabolic subalgebras $\mathfrak{p}$ are described in [3]. If $\mathfrak{g}_\mathbb{C}$ is $\mathfrak{sl}(V, W)$ or $\mathfrak{gl}(V, W)$, then the Levi components of parabolic subalgebras of $\mathfrak{g}_\mathbb{C}$ are direct sums of standard $\mathfrak{sl}$ subalgebras, and any such direct sum is the Levi component of a parabolic. If $\mathfrak{g}_\mathbb{C}$ is $\mathfrak{so}(V)$ (resp. $\mathfrak{sp}(V)$) then the Levi components of parabolic subalgebras of $\mathfrak{g}_\mathbb{C}$ are are direct sums of standard $\mathfrak{sl}$ subalgebras plus at most one standard $\Lambda\mathfrak{gl}(X, Y)$ (resp. $S\mathfrak{gl}(X, Y)$). The arguments constructing $\mathfrak{p}$ from $\mathfrak{l}$ in [3] show that if $\mathfrak{l}_1 \subsetneqq \mathfrak{l}_2$ one constructs $\mathfrak{p}_1 \subsetneqq \mathfrak{p}_2$.

The Levi components of real parabolic subalgebras are real forms of these. Thus the Levi components of minimal real parabolics cannot be broken down further. This means that the summands of a Levi component $\mathfrak{l}_{\mathbb{R}}$ do not themselves contain proper parabolic subalgebras. In other words $\mathfrak{l}_{\mathbb{R}}$ is a direct sum of simple Lie algebras $\mathfrak{su}(p)$, $\mathfrak{so}(p)$ or $\mathfrak{sp}(p)$ of compact groups; or Lie algebras $\mathfrak{su}(\infty)$, $\mathfrak{so}(\infty)$ or $\mathfrak{sp}(\infty)$ of lim–compact groups; or can be viewed as a Lie algebra $\mathfrak{sl}(1;\mathbb{H}) = \mathfrak{su}(2)$. These are the compact real forms, in the sense of [8], of the complex $\mathfrak{sl}$, $\mathfrak{so}$ and $\mathfrak{sp}$.

Now we are dealing with a minimal Levi component $\mathfrak{l}_{\mathbb{R}} = \bigoplus_{i \in I} \mathfrak{l}_i$. Let $X_{\mathbb{R}}$ denote the sum of the corresponding subspaces $(X_i)_{\mathbb{R}} \subset V_{\mathbb{R}}$ and $Y_{\mathbb{R}}$ the analogous sum of the $(Y_i)_{\mathbb{R}} \subset W_{\mathbb{R}}$. Then $X_{\mathbb{R}}$ and $Y_{\mathbb{R}}$ are nondegenerately paired. Of course they may be small, even zero. In any case, $V_{\mathbb{R}} = X_{\mathbb{R}} \oplus Y_{\mathbb{R}}^{\perp}$ and $W_{\mathbb{R}} = Y_{\mathbb{R}} \oplus X_{\mathbb{R}}^{\perp}$, and $X_{\mathbb{R}}^{\perp}$ and $Y_{\mathbb{R}}^{\perp}$ are nondegenerately paired. When $\mathfrak{g}_{\mathbb{R}}$ is defined by a hermitian or bilinear form f, which we use to identify $V_{\mathbb{R}}$ and $W_{\mathbb{R}}$, these direct sum decompositions become $V_{\mathbb{R}} = (X_{\mathbb{R}} \oplus Y_{\mathbb{R}}) \oplus (X_{\mathbb{R}} \oplus Y_{\mathbb{R}})^{\perp}$ and f is nondegenerate on $(X_{\mathbb{R}} \oplus Y_{\mathbb{R}})^{\perp}$. Let $X'_{\mathbb{R}}$ and $Y'_{\mathbb{R}}$ be paired maximal isotropic subspaces of $(X_{\mathbb{R}} \oplus Y_{\mathbb{R}})^{\perp}$. Let $Z'_{\mathbb{R}} := (X'_{\mathbb{R}} \oplus Y'_{\mathbb{R}})^{\perp} \cap (X_{\mathbb{R}} \oplus Y_{\mathbb{R}})^{\perp}$. Then $V_{\mathbb{R}} = (X_{\mathbb{R}} \oplus Y_{\mathbb{R}}) \oplus (X'_{\mathbb{R}} \oplus Y'_{\mathbb{R}}) \oplus Z'_{\mathbb{R}}$.

The subalgebra of $\mathfrak{g}_{\mathbb{R}}$ that is zero on $(X_{\mathbb{R}} \oplus Y_{\mathbb{R}})$ and $Z'_{\mathbb{R}}$ has a maximal toral subalgebra $\mathfrak{a}_{\mathbb{R}}$ in which every element has all eigenvalues real. It is obtained as a sum of the standard $\mathfrak{gl}(x'_j\mathbb{R}, y'_j\mathbb{R})$ as the x_j run over a basis of $X'_{\mathbb{R}}$ and y_j in the dual basis of $Y'_{\mathbb{R}}$ is paired to x_j. The subalgebra of $\mathfrak{g}_{\mathbb{R}}$ that is zero on $(X_{\mathbb{R}} \oplus Y_{\mathbb{R}})$ and $(X'_{\mathbb{R}} \oplus Y'_{\mathbb{R}})$ has a maximal toral subalgebra $\mathfrak{t}'_{\mathbb{R}}$ in which every eigenvalue is pure imaginary, because f is definite on $Z'_{\mathbb{R}}$. If $\mathfrak{l}_i = \mathfrak{su}(*)$ define $\widetilde{\mathfrak{l}_i} = \mathfrak{u}(*)$; otherwise let $\widetilde{\mathfrak{l}_i} = \mathfrak{l}_i$. Now let $\widetilde{\mathfrak{l}_{\mathbb{R}}} = (\bigoplus \widetilde{\mathfrak{l}_i}) \cap \mathfrak{g}_{\mathbb{R}}$.

Define $\mathfrak{m}_{\mathbb{R}} = \widetilde{\mathfrak{l}_{\mathbb{R}}} + \mathfrak{t}'_{\mathbb{R}}$. Then $\mathfrak{m}_{\mathbb{R}}$ and $\mathfrak{a}_{\mathbb{R}}$ are the analogs of compact and the $\mathbb{R}$–split components in the minimal parabolic $\mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ of the finite dimensional reductive Lie group setting.

The analogous construction goes through for the other cases of $\mathfrak{g}_{\mathbb{R}}$.

Principal Series

It is work in progress to relate parabolic subalgebras $\mathfrak{p}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ with locally reductive component $\mathfrak{m}_{\mathbb{R}} + \mathfrak{a}_{\mathbb{R}}$, to positive $\mathfrak{a}_{\mathbb{R}}$–root systems. Given such a $\mathfrak{p}_{\mathbb{R}}$ we define $\rho : \mathfrak{a}_{\mathbb{R}} \to \mathbb{R}$ (analog of half the sum of the positive roots) by its inner product with the simple positive $\mathfrak{a}_{\mathbb{R}}$–roots. Given an irreducible unitary representation μ of $M_{\mathbb{R}}$ and a linear map $\sigma : \mathfrak{a}_{\mathbb{R}} \to \mathbb{R}$ we have the representation

$$\eta_{\mu,\sigma} : man \mapsto \mu(m)\exp((i\sigma + \rho)(\log a))$$

of the parabolic $P_{\mathbb{R}}$ on the space $\mathcal{H}_\mu$ of μ. Then the unitarily induced representation $\pi_{\mu,\sigma}$ is the left action of $G_{\mathbb{R}}$ on an appropriate space of functions

$$h : G_{\mathbb{R}} \to \mathcal{H}_\mu \text{ such that } h(gman) = \eta_{\mu,\sigma}(man)^{-1}h(g).$$

Now the problem is to pin down a good class of representations μ of $M_{\mathbb{R}}$ and a correspondingly appropriate space of functions $h : G_{\mathbb{R}} \to \mathcal{H}_\mu$ such that $h(gman) = \eta_{\mu,\sigma}(man)^{-1}h(g)$.

In the finite dimensional setting, $M_{\mathbb{R}}$ is compact, so μ is unitary and is specified modulo some technicalities on the finite extension

$$1 \to M_{\mathbb{R}}^0 \to M_{\mathbb{R}} \to M_{\mathbb{R}}/M_{\mathbb{R}}^0 \to 1$$

by its highest weight, and we can take h to be in $L^2(K_{\mathbb{R}}/M_{\mathbb{R}})$.

In our infinite dimensional setting, "finite dimensional" has been replaced by "finitary". Then μ is an $(M_{\mathbb{R}}/M_{\mathbb{R}}^0)$–extension of the (possibly infinite) tensor product of a unitary character on the toral factor of $M_{\mathbb{R}}^0$, irreducible highest weight representations of the finite dimensional factors of $M_{\mathbb{R}}^0$, and representations of a class to be decided on the $SU(\infty)$, $SO(\infty)$ and $Sp(\infty)$ factors. See [6] for a discussion of a class of representations of $U(\infty)$ and $SU(\infty)$ that is appropriate for analytic reasons[1]. Further, since $L^2(K_{\mathbb{R}}/M_{\mathbb{R}})$ no longer has a good meaning, h must be regular in the sense of [10], essentially meaning polynomial, but this must be done in a manner consonant with the tensor factors of μ along the infinite dimensional factors of $M_{\mathbb{R}}^0$. These matters are not yet finalized.

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Symplectic Howe pairs

Tilmann Wurzbacher
(joint work with Carsten Balleier)

The results presented in the this talk can be found with more details and proofs in the article [1]: *On the Geometry and Quantization of Symplectic Howe Pairs.*

1. Introduction

The philosophy of "geometric quantization in the presence of symmetries" can be traced back to the early days of quantum mechanics, e.g. to the correspondence principle of Niels Bohr (see [2]), and notably to the question of how to quantize constrained classical systems. It is rooted in the study of canonical quantization

[1] Probably a similar analysis can be made for $SO(\infty)$ and $Sp(\infty)$.

of phase spaces (i.e. cotangent bundles of configuration spaces Q) and point-transformation symmetries (i.e. diffeomorphisms of Q) on the one hand and in the orbit method of Kostant, Kirillov and Souriau on the other hand (compare, e.g., [3] for the latter subject). Many rigorous results were proven in this area starting with important groundbreaking work of Guillemin and Sternberg in the early 80's.

Roughly speaking, this philosophy connects Hamiltonian Lie group actions on symplectic manifolds with the subject of continuous linear representations of Lie groups via "quantization" and "dequantization". Prominent examples of the dequantization aspect are the following:

- transitive Hamiltonian actions are coverings of coadjoint orbits and should correspond to irreducible linear representations
- Hamiltonian actions such that the Poisson-algebra of invariant functions is commutative, are called multiplicity-free and should correspond to multiplicity-free linear representations (see, e.g., [4] and [5]).

In these examples, the crucial point is the analogy between smooth functions on a symplectic manifold M (classical observables) and bounded linear operators on its quantization given by a topological vector space $Q(M)$ (quantum observables), both being considered as modules of a given Lie group or Lie algebra of symmetries.

If a product $G_1 \times G_2$ of two Lie groups is linearly represented on a complex vector space U ($\rho : G_1 \times G_2 \to GL(U)$), we say that the representation satisfies the *Howe condition* or is equipped with a *Howe duality* if there is a subset $\mathcal{D}$ of $\widehat{G_1}$, the set of equivalence classes of irreducible complex representations of G_1, and an injective map $\Lambda : \mathcal{D} \to \widehat{G_2}$ such that

$$(*) \qquad U \cong \bigoplus_{\alpha \in \mathcal{D}} V_\alpha \otimes W_{\Lambda(\alpha)},$$

where V_α represents a class α in $\mathcal{D}$ and $W_{\Lambda(\alpha)}$ represents the class $\Lambda(\alpha)$. Denoting for $k = 1, 2$ the restriction of the representation ρ to G_k by ρ_k (as well as the corresponding Lie algebra representation), condition $(*)$ is -under certain conditions on the Lie groups and the representation- equivalent to

$$(**) \qquad Z_{\mathcal{B}(U)}(\rho_i(\mathcal{U}\mathfrak{g}_i)) = \rho_j((\mathcal{U}\mathfrak{g}_j) \qquad \text{for } i \neq j,$$

where $\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra (over $\mathbb{C}$) of a Lie algebra $\mathfrak{g}$, and for a set $S \subset \mathcal{B}(U)$, $Z_{\mathcal{B}(U)}(S)$ is the centralizer of S in the bounded endomorphisms of U.

In this talk we report on the study of commuting hamiltonian actions of two Lie groups on a symplectic manifold M with equivariant moment maps Φ_i for $i = 1, 2$. We define in this situation a *symplectic Howe condition*, naturally corresponding to the above condition $(**)$, namely

$$(***) \qquad Z_{C^\infty(M)}(\Phi_i^* C^\infty(\mathfrak{g}_i^*)) = \Phi_j^* C^\infty(\mathfrak{g}_j^*) \qquad \text{for } i \neq j.$$

Here and in the sequel, the Lie algebra of a Lie group G will always be denoted by $\mathfrak{g}$ and, for $A \subset C^\infty(M)$, $Z_{C^\infty(M)}(A) = \{f \in C^\infty(M) \mid \{f, a\} = 0 \, \forall a \in A\}$.

From the point of view of observables, the conditions $(**)$ resp. $(***)$ say that the centralizer of the collective observables coming from one action are the collective observables coming from the other action, on the quantum resp. the classical level.

Let us pause to note that a “Howe pair” is often defined as a pair (G_1, G_2) of closed subgroups of a finite dimensional Lie group G mutually centralizing each other (inside G, not inside a representation), i.e. fulfilling: $Z_G(G_i) = G_j$ for $i \neq j$. Here for a subset $T \subset G$, $Z_G(T) = \{g \in G \mid gt = tg \text{ for all } t \in T\}$. The classical case is that G equals the symplectic automorphisms of a finite dimensional real symplectic vector space, $G \cong Sp(2n, \mathbb{R})$. From this point of view, $(***)$ should be replaced by a pair of Lie subgroups mutually centralizing each other inside an appropriate version of the group of all symplectic diffeomorphisms of a symplectic manifold. The study of the classical analogue of a Howe pair in $Sp(2n, \mathbb{R})$ is then intrinsically a problem of infinite dimensional Lie theory, but of course $(***)$ is already an identity of infinite dimensional Lie algebras!

2. Results

For proper actions of general Lie groups we can elucidate the orbit structure as follows:

Lemma (Moment levels are orbits). *Let commuting Hamiltonian proper actions of the connected Lie groups G_1 and G_2 with equivariant moment maps Φ_1 and Φ_2 be given on the symplectic manifold (M, ω). Then the symplectic Howe condition implies that*

$$\forall z \in M \text{ holds } \Phi_i^{-1}(\Phi_i(z)) = G_j \cdot z \qquad \text{for } i \neq j,$$

i.e., the level sets of the moment maps of one action are the orbits of the other one. Notably, all level sets of both moment maps are connected.

Theorem (Symplectic Howe correspondence or Orbit correspondence). *In the situation of the preceding lemma, we have* *(i) a bijection ("orbit correspondence")*

$$\Lambda : \Phi_1(M)/G_1 \to \Phi_2(M)/G_2,\ \Lambda(\mathcal{O}_{\alpha_1}) := \Phi_2(\Phi_1^{-1}(\mathcal{O}_{\alpha_1})),\ \textit{and}$$

(ii) G_2-equivariant symplectomorphisms ("reduced spaces are coadjoint orbits"):

$$M_{\alpha_1} \cong \Lambda(\mathcal{O}_{\alpha_1}) \text{ and analogously for 1 and 2 exchanged.}$$

Let now G be a compact connected Lie group and $\alpha \in \mathfrak{g}^*$. We call α *integral* if there exists $\chi_\alpha : G_\alpha \to U(1)$ such that $(\chi_\alpha)_{*e} = 2\pi i\alpha$. If α is integral, we call $\mathcal{O}_\alpha$ an *integral orbit*. We here have the following useful information concerning the "pre-quantization" of the actions:

Proposition (Preservation of integrality). Let $\mathcal{O}_{\alpha_1}$ and $\mathcal{O}_{\alpha_2}$ be two coadjoint orbits in correspondence as above in point (i) and assume furthermore that (M, ω) is prequantizable and that G_1 and G_2 are compact and connected. Then $\mathcal{O}_{\alpha_1}$ is integral if and only if $\mathcal{O}_{\alpha_2}$ is integral.

As an example of a quantization result, going from classical to quantum, we consider the Kähler case:

Theorem (Symplectic Howe duality implies representation-theoretic Howe duality). *Let G_1 and G_2 be compact connected Lie groups acting by holomorphic transformations on an integral Kähler manifold M such that the actions extend to actions of the respective complexified groups. Suppose that the actions of G_1 and G_2 commute and are Hamiltonian with equivariant moment maps Φ_1 and Φ_2. Assume the symplectic Howe condition to be satisfied. Then we have the following isomorphy of $G_1 \times G_2$-representations:*

$$Q_{hol}(M) \cong \widehat{\bigoplus}_{\alpha_1 \in (\Phi_1(M)/G_1) \cap \widehat{G_1}} V_{\alpha_1} \otimes W_{\alpha_2}\,,$$

where Λ is the orbit correspondence map, $\Lambda(\mathcal{O}_{\alpha_1}) = \mathcal{O}_{\alpha_2}$ and $\widehat{G_1}$ is the set of equivalence classes of irreps of G_1, realized as (orbits of) integral points in $\mathfrak{g}_1^*$ and V_{α_1} resp. W_{α_2} are the irreducible G_1- resp. G_2-representations associated to $\mathcal{O}_{\alpha_1}$ resp. to $\mathcal{O}_{\alpha_2}$. The map

$$\mathcal{D} := (\Phi_1(M)/G_1) \cap \widehat{G_1} \to \widehat{G_2},\ V_{\alpha_1} \mapsto W_{\alpha_2}$$

is then here the representation-theoretic Howe duality map.

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One-skeleton galleries, the path model and a generalization of Macdonald's formula for Hall-Littlewood polynomials

Peter Littelmann
(joint work with Stéphane Gaussent)

The aim of the project is to give a direct geometric interpretation of the path model for a representation and the associated combinatorics (see for example [7–9, 11–14]), which leads, among other things, to a generalization of Macdonald's formula for Hall-Littlewood polynomials. As a side effect, this point of view provides also a geometric connection to the work done by Kapovich, Leeb and Millson [2–5].

Since we work in the setting of affine buildings, some restrictions to the paths have to be imposed. The paths we will consider have to be compatible with the structure of the building, we consider only paths in the one skeleton of the building. In the language of buildings these paths will be called galleries.

Let us start with the standard apartment. Examples for this class are paths which run along the edges (one dimensional face) in the direction of (Weyl group conjugates of) fundamental weights. To give a more explicit example let $\omega_1, \ldots, \omega_n$ be an enumeration of the fundamental weights, and let $\lambda = a_1\omega_1 + \ldots a_n\omega_n$ be a dominant weight. Take the path joining the first origin with $a_1\omega_1$ by a straight line, then $a_1\omega_1$ with $a_1\omega_1 + a_2\omega_2$ by a straight line etc., and then $a_1\omega_1 + \ldots + a_{n-1}\omega_{n-1}$ with λ by a straight line. By the construction of the associated path model, all paths obtained from the given one by applying the folding operators are in the class of paths running along the edges in the direction of Weyl group conjugates of fundamental weights.

In this setting the paths in the apartment are very close to the original formulation using generalized Young tableaux in the approach by Lakshmibai, Musili and Seshadri [7, 8]. In fact, the combinatorics developed in these papers to define a generalization of Young tableaux for other types and which may look a little ad hoc at a first glance, gets in this setting a very natural geometric interpretation. Gallery, path and tableau in the sense of Lakshmibai, Musili and Seshadri become synonymous. The condition to be a *semi-standard tableau* in the sense of Lakshmibai, Musili and Seshadri, which they define via certain integrality conditions and the existence of a (combinatorially heavy looking) *defining chain*, (these conditions show also up in the general formulation of the path model), these condition turn out to be exactly the combinatorial description for the *geometric* fact that this possibly folded path (or gallery or tableau) can be unfolded in the affine building to a *minimal path*, and the cell obtained by all unfoldings has certain dimension. In this dictionary unfolding to a minimal path corresponds to the existence of the defining chain, the integrality condition correspond to the right dimension of the cell.

To be more precise, let G be a semisimple algebraic group defined over $\mathbb{C}$, fix a Borel subgroup B and a maximal torus T. Let U^- be the unipotent radical of the opposite Borel subgroup. Let $\mathcal{O} = \mathbb{C}[[t]]$ be the ring of complex formal power series and let $\mathcal{K} = \mathbb{C}((t))$ be the quotient field. For a dominant coweight λ and an arbitrary coweight μ consider the following intersection in the affine Grassmannian $G(\mathcal{K})/G(\mathcal{O})$:

$$Z_{\lambda,\mu} = G(\mathcal{O}).\lambda \cap U^-(\mathcal{K}).\mu.$$

Let $\mathbb{F}_q$ be the finite field with q elements and replace the field of complex numbers by the algebraic closure K of $\mathbb{F}_q$. Assume that all groups are defined and split over $\mathbb{F}_q$. Replace $\mathcal{K}$ by $\mathcal{K}_q = \mathbb{F}_q((t))$ and $\mathcal{O}$ by $\mathcal{O}_q = \mathbb{F}_q[[t]]$; the Laurent polynomials $L_{\lambda,\mu}$ defined by $L_{\lambda,\mu}(q) = |Z^q_{\lambda,\mu}|$ show up as coefficients in the Hall-Littlewood polynomial: $P_\lambda = \sum_{\mu \in X^\vee_+} q^{-\langle \rho, \lambda+\mu\rangle} L_{\lambda,\mu} m_\mu$.

Based on the description of $Z_{\lambda,\mu}$ in [1], Schwer gives a decomposition $Z^q_{\lambda,\mu} = \bigcup S_\delta$, where the δ are certain galleries of alcoves in the standard apartment of the associated affine building. The structure of the S_δ is quite simple and hence $|S_\delta|$ is easy to compute, but the decomposition has the disadvantage that the sum $|Z^q_{\lambda,\mu}| = \sum |S_\delta|$ has many terms.

For G of type $\mathtt{A}_n$, there are other formulas, for example one can specialize the Haglund-Haiman-Loehr formula for Macdonald polynomials to get a formula for the Hall-Littlewood polynomials. By analyzing the combinatorics involved in the formulas, Lenart [10] has shown that in type $\mathtt{A}_n$ certain terms in Schwer's formula can be naturally grouped together such that the resulting formula coincides with the specialization of the Haglung-Haiman-Loehr formula, he calls this the compression phenomenon. Another formula for Hall-Littlewood polynomials in type $\mathtt{A}_n$ is the one due to Macdonald, see [15].

Our approach to "compression" is geometric and independent of the type of the group. We replace the desingularization of the Schubert variety X_λ in [1] by a Bott-Samelson type variety Σ which is a fibred space having as factors varieties of the form H/R, where H is a semisimple algebraic group and R is a maximal parabolic subgroup. In terms of the affine building, a point in this variety is a sequence $(P_0 = G(\mathcal{O}), Q_0, P_1, Q_1, \ldots, P_r, Q_r, P_{r+1})$ of parahoric subgroups of $G(\mathcal{K})$ reciprocative contained in each other, i.e. $G(\mathcal{O}) \supset Q_0 \subset P_1 \supset Q_1 \subset \ldots \supset Q_r \subset P_{r+1}$.

More precisely, in terms of the faces of the building, a point in Σ is a sequence of closed one-dimensional faces (corresponding to the parahoric subgroups $Q_0, \ldots, Q_r$), where successive faces have (at least) a common zero-dimensional face (i.e. a vertex corresponding to one of the maximal parahoric subgroups $P_0, \ldots, P_{r+1}$). So if the sequence is contained in an apartment, then the point in Σ corresponds to a piecewise linear path in the apartment joining the origin with a special vertex.

We introduce the notion of a minimal one-skeleton gallery (which always lies in some apartment) and of a positively folded combinatorial gallery in the one-skeleton. The points in Σ corresponding to the points in the open orbit $G(\mathcal{O}).\lambda \subset X_\lambda$ are exactly the minimal galleries, we identify those two sets. By choosing a generic one parameter subgroup of T in the anti-dominant Weyl chamber, we get a Białynicki-Birula decomposition of Σ, the centers δ of the cells C_δ correspond to combinatorial one-skeleton galleries δ (i.e. the galleries lying in the standard apartment). We show that $C_\delta \cap G(\mathcal{O}).\lambda \neq \emptyset$ if and only if δ is positively folded.

The Białynicki-Birula decomposition of Σ can be used to define a decomposition $Z_{\lambda,\mu} = \bigcup_\delta Z_{\lambda,\mu} \cap C_\delta$, the indexing set of the strata are positively folded one-skeleton galleries. To see the *geometric compression* compared to the decomposition in [1], consider the case for G of type $\mathtt{A}_n$. It is known that $Z_{\lambda,\mu}$ has at least $\dim V(\lambda)_\mu$ irreducible components. Now in the $\mathtt{A}_n$-case the galleries can be translated into the language of Young tableaux, and the positively folded galleries ending in μ correspond exactly to the semi-standard Young tableaux of shape λ and weight μ. In this sense the new decomposition can be viewed as the optimal geometric decomposition for type $\mathtt{A}_n$. The general feature of the new approach is that there are much less non-LS-galleries (see below) than in the old approach. For example in the case of type $\mathtt{A}_n$, all positively folded galleries are LS-galleries.

To investigate the intersection $Z_{\lambda,\mu} \cap C_\delta$ we need to *unfold* the (possibly) folded gallery δ. As a consequence of the unfolding procedure we present the formula for the coefficients of the Hall-Littlewood polynomials, the summands below counting the number of points in the intersection of $Z^q_{\lambda,\mu} \cap C_\delta$ for δ being positively folded and ending in μ:

Theorem.
$L_{\lambda,\mu}(q) = \sum_{\delta\in\Gamma^+(\gamma_\lambda,\mu)} q^{\ell(w_{D_0})} \left(\prod_{j=1}^{r} \sum_{\mathbf{c}\in\Gamma^+_{s^j_{V_j}}(\mathbf{i}_j,op)} q^{t(\mathbf{c})}(q-1)^{r(\mathbf{c})} \right).$

To get a rough idea of what this formula means without getting drowned by the technical details, let us consider the case where G of type $\mathtt{A}_n$. We identify the positively folded galleries with the semi-standard Young tableaux of shape λ having weight μ. We use the convention that the entries in the tableau are weakly increasing in the rows and strictly increasing in the columns, so the one dimensional faces of the gallery correspond to the columns of the tableau. We enumerate the columns such that the right most column is the first one. Given such a tableau δ, let $E_0, \ldots, E_r$ be the columns. We want to investigate the set of all minimal galleries in C_δ lying in $Z_{\lambda,\mu}$. We show that this set has a product structure

$$B^- w_{D_0} Q^-_{E_0}/Q^-_{E_0} \times \prod_{j=1}^{r} Min(E_{j-1}, E_j) \ ,$$

which explains the product structure for each summand. To get a minimal gallery in the building that lifts δ, i.e. is an element of C_δ and lies in $Z_{\lambda,\mu}$, the possibilities for the first column E_0 form a Schubert cell leading to the term $q^{\ell(w_{D_0})}$. For $j \geq 1$, the possibilities for lifts of E_j depend on the column E_{j-1} before. It can be shown that Macdonald's algorithm (see [15]) can be expressed also column-wise. More precisely, Klostermann [6] has shown in the framework of her thesis that the structure of the second sum in the formula above in Theorem can be simplified in the $\mathtt{A}_n$-case so that, in terms of Young tableaux, the resulting algorithm is exactly the same as Macdonald's algorithm.

The positively folded one-skeleton galleries having $q^{\langle\lambda+\mu,\rho\rangle}$ as a leading term in the counting formula for $|Z^q_{\lambda,\mu} \cap C_\delta|$, are called *LS-galleries*; this is an abbreviation for Lakshmibai-Seshadri galleries. We discuss the special role of the

LS-galleries and the connection with the indexing system by generalized Young tableaux introduced by Lakshmibai, Musili and Seshadri in a series of papers, see for example [7–9]. Recall that these papers were the background for the path model theory started in [11]. An important notion introduced in the theory of standard monomials is the defining chain ([7,8,11]), which was a breakthrough on the way for the definition of standard monomials and generalized Young tableaux. In the context of the crystal structure of the path theory this notion again turned up to be an important combinatorial tool to check whether a concatenation of paths is in the Cartan component or not. Still, the definition had the air of an ad hoc combinatorial tool. But in the context of Białynicki-Birula cells, the folding of a minimal gallery by the action of the torus occurs naturally: during the limit process (going to the center of the cell) the direction attached to a minimal gallery (or minimal path in the path language) is transformed into the weakly decreasing sequence of Weyl group elements, the defining chain for the positively folded one-skeleton gallery in the center of the cell.

The connection between the path model theory and the one-skeleton galleries is summarized in the following corollary. For a fundamental coweight ω let π_{ω_i} : $[0,1] \to X^\vee_{\mathbb{R}}$, $t \mapsto t\omega$ be the path which is just the straight line joining $\mathfrak{o}$ with ω and let γ_ω be the one-skeleton gallery obtained as the sequence of edges and vertices lying on the path.

Corollary 1. *Write a dominant coweight* $\lambda = \omega_{i_1} + \ldots + \omega_{i_r}$ *as a sum of fundamental coweights, write* $\underline{\lambda}$ *for this ordered decomposition. Let* $\mathcal{P}_{\underline{\lambda}}$ *be the associated path model of LS-paths of shape* $\underline{\lambda}$ *defined in [11] having as starting path the concatenation* $\pi_{\omega_{i_1}} * \ldots * \pi_{\omega_{i_r}}$. *For a path* π *in the path model denote by* γ_π *the associated gallery in the one-skeleton of* $\mathbb{A}$ *obtained as the sequence of edges and vertices lying on the path. The one-skeleton galleries* γ_π *obtained in this way are precisely the LS-galleries of the same type as* $\gamma_{\omega_{i_1}} * \ldots * \gamma_{\omega_{i_r}}$.

In fact, the notion of a *defining chain for LS-paths* coincides in this case with the notion of a defining chain for the associated gallery.

Since the number of the LS-galleries is the coefficient of the leading term of $L_{\lambda,\mu}$, and since $P_\lambda \to s_\lambda$ for $q \to \infty$, we get as an immediate consequence of Theorem the following character formula. In combination with Corollary 1, this provides a geometric proof of the path character formula, first conjectured by Lakshmibai (see for example [9]) and proved in [11]:

Corollary 2. Char $V(\lambda) = \sum_\delta e^{target(\delta)}$, *where the sum runs over all LS-galleries of the same type as* γ_λ.

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Root theory of L^*-algebras and applications

Alice Barbara Tumpach

The topic of our research is to do (some) geometry on infinite-dimensional manifolds like the restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$, or the restricted Siegel disc $\mathrm{D}_{\mathrm{res}}$. The restricted Grassmannian is related to loop groups ([10], [14]), hierarchies of equations of KdV-type ([13]), and Fermionic second quantization ([21]). The restricted Siegel disc contains in a natural way the universal Teichmüller space $T_0(1)$ (with the Hilbert manifold structure constructed in [16]) as well as the Teichmüller space of any compact Riemann surface.

In recent work ([17]) we construct a hyperkähler metric on the cotangent space of any infinite-dimensional Hermitian-symmetric affine coadjoint orbit of an L^*-group of compact type. An example of such an orbit is the restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}$. This metric is natural in the following sense: it is invariant under the L^*-group under consideration, restricts to the Kähler metric of the orbit and is compatible with the complex symplectic form of the cotangent space. In the case of the restricted Grassmannian, we obtain a hyperkähler metric on the cotangent space $T^*\mathrm{Gr}_{\mathrm{res}}$, which is invariant under the group U_2 of unitary operators which differ from the identity by Hilbert-Schmidt operators, restricts to the natural Kähler metric of the restricted Grassmannian and is compatible with the complex-symplectic structure of the cotangent space. This result is a generalization of finite-dimensional results in [4], [5] and [6] to the infinite-dimensional setting where the root theory of L^*-algebras is extensively used. The proof goes by identifying the cotangent space of a Hermitian-symmetric affine coadjoint orbit of an L^*-group of compact type with the orbit of the complexified L^*-group, where a potential can be computed. For orbits of non-compact type (also known as symmetric Hilbert domains) this identification does not hold anymore and it is still work in progress (with T. Ratiu and F. Gay-Balmaz) to construct a natural hyperkähler metric on the cotangent space of any symmetric Hilbert domain such as the restricted Siegel disc $\mathrm{D}_{\mathrm{res}}$ for instance.

The background for this research is the root theory of simple separable L^*-algebras developed by Schue in 1960-1961. The objects of study are infinite-dimensional Hermitian-symmetric affine coadjoint orbits of L^*-groups, which one may want to classify. The classification of the irreducible orbits can be made as in the finite-dimensional case ([20]) using the notion of roots of non-compact type ([18]). Finite-dimensional examples include the 2-sphere and the hyperbolic space. The tool for the construction of the hyperkähler structures mentioned above is the existence of maximal sets of strongly orthogonal roots on one hand, and the existence of Cartan subalgebras adapted to a given Cartan decomposition on the other ([8]). Looking back at the 50 year old theory of L^*-algebras, some questions arise: given that even for simple L^*-algebras some automorphisms are not inner, what are the conjugacy classes of Cartan subalgebras by inner automorphisms? Is any Cartan subalgebra of a simple L^*-algebra the centralizer of one of its elements? The conjugacy classes of Cartan subalgebras of simple L^*-algebras under the whole group of automorphisms are characterized for complex L^*-algebras in [2] and for real simple L^*-algebras in [1] (see also [9] and [3] for similar questions on other infinite-dimensional algebras).

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Examples of integrable Hamiltonian systems on Banach Lie–Poisson spaces

Anatol Odzijewicz

(joint work with Alina Dobrogowska)

A Banach Lie-Poisson space $(\mathfrak{b}, \{\cdot,\cdot\})$ is a real or complex Banach Poisson space $\mathfrak{b}$ such that its dual $\mathfrak{b}^* \subset C^\infty(\mathfrak{b})$ is a Banach Lie algebra under the Poisson bracket operation with condition

$$\mathrm{ad}_x^* \mathfrak{b} \subset \mathfrak{b} \subset \mathfrak{b}^{**}, \qquad x \in \mathfrak{b}^*,$$

where the adjoint map $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ is given by $\mathrm{ad}_x g := [x, y]$.

In the paper [4] the following statement was proved

Proposition 1. *The Banach space* $\mathfrak{b}$ *is a Banach Lie-Poisson space* $(\mathfrak{b}, \{\cdot,\cdot\})$ *if and only if it is predual* $\mathfrak{b}^* = \mathfrak{g}$ *of some Banach Lie algebra* $(\mathfrak{g}, [\cdot,\cdot])$ *satisfying*

$\operatorname{ad}^*_x \mathfrak{b} \subset \mathfrak{b} \subset \mathfrak{g}^*$ *for all* $x \in \mathfrak{g}$. *The Poisson bracket of* $f, g \in C^\infty(\mathfrak{b})$ *is given by*

$$\{f, g\}(b) = \langle [Df(b), Dg(b)] \; ; \; b\rangle, \tag{1.1}$$

where $b \in \mathfrak{b}$.

In that case the Hamilton equations assume the form

$$\frac{d}{dt} b = - \operatorname{ad}^*_{Dh(b)} b, \qquad b \in \mathfrak{b}, \tag{1.2}$$

where $h \in C^\infty(\mathfrak{b})$ is Hamiltonian of a system.

Let $\mathcal{L}^\infty$ denote the Banach algebra of the bounded operators acting in a real Hilbert space $\mathbb{H}$. By $\mathcal{L}^p$ we denote the operators of Schatten class:

$$\mathcal{L}^p := \left\{ \rho \in \mathcal{L}^\infty : \|\rho\|_p := (\operatorname{Tr}(\sqrt{\rho^* \rho})^p)^{\frac{1}{p}} < \infty \right\} \tag{1.3}$$

One has the following dualities

$$\mathcal{L}^p = (\mathcal{L}^q)^* \qquad \text{for } \frac{1}{q} + \frac{1}{p} = 1 \tag{1.4}$$

given by the pairing

$$\langle X \; ; \; \rho \rangle := \operatorname{Tr}(X\rho). \tag{1.5}$$

For $p = 1$ we get ideal of trace-class operators $\mathcal{L}^1$ and for $p = 2$ — ideal of Hilbert–Schmidt operators $\mathcal{L}^2$.

The space $\mathcal{L}^p$ is Banach Lie-Poisson space with Poisson bracket

$$\{f, g\}_p(\rho) = Tr(\rho[Df(\rho), Dg(\rho)]) \tag{1.6}$$

of $f, g \in C^\infty(\mathcal{L}^p)$

Let us decompose the element $\rho \in \mathcal{L}^2$

$$\rho = \sum_{n,m=0}^{\infty} \rho_{nm} \, |n\rangle \, \langle m| \, , \tag{1.7}$$

where $\{|n\rangle\}_{n=0}^\infty$ is an orthonormal basis in $\mathbb{H}$. We define the splitting $\mathcal{L}^2 = \mathcal{L}^2_- \oplus \mathcal{L}^2_0 \oplus \mathcal{L}^2_+$, where $\mathcal{L}^2_-$, $\mathcal{L}^2_0$ and $\mathcal{L}^2_0$ are Hilbert subspaces of strictly lower triangular operators, diagonal operators and strictly upper triangular operators respectively.

For $x_+ \in \mathcal{L}^2_+$ we define the map $\alpha : \mathcal{L}^2_+ \to \mathcal{L}^2_+$ as follows

$$\alpha(x_+) := \sum_{0 \le i < j} \alpha_{ij} x_{ij} \, |i\rangle \, \langle j| \, , \tag{1.8}$$

where $\alpha_{ij} \in \mathbb{R}$ satisfy the condition

$$\alpha_{ij}\alpha_{jk} = \alpha_{ik}. \tag{1.9}$$

Since of (1.9) the map (1.8) is a morphism of associative algebra $\mathcal{L}^2_+$, i.e.

$$\alpha(x_+ y_+) = \alpha(x_+)\alpha(y_+),$$

Thus

$$\mathcal{A}^2_\alpha := \{x_- - \alpha(x_-^\top) : x_- \in \mathcal{L}^2_-\} \tag{1.10}$$

is Banach Lie algebra with $\mathcal{L}_+^2$ as its predual. Therefore $\mathcal{L}_+^2$ is a Banach Lie–Poisson space with the bracket $\{\cdot,\cdot\}_{+,\alpha}$ given by

(1.11)
$$\{f,g\}_{+,\alpha}(\rho_+) = Tr\left\{\rho_+\left[(Df(\rho_+))^\top - \alpha\,(Df(\rho_+))\,,(Dg(\rho_+))^\top - \alpha\,(Dg(\rho_+))\right]\right\},$$

where $Df(\rho_+)$ denotes the Fréchet derivative at $\rho_+ \in \mathcal{L}_+^2$.

From Magri method (see [2, 3]) we get the following integrals of motion

$$I_\epsilon^k(\rho_+) := Tr\left((1+\epsilon)\,(\alpha_{0\infty} + \epsilon\beta_{0\infty})\,\rho_+^2 - - \rho_+\,(\eta_\alpha + \epsilon\eta_\beta)\,\rho_+^\top\,(\delta_\alpha + \epsilon\delta_\beta) - \right.$$
$$\left. - (\eta_\alpha + \epsilon\eta_\beta)\,\rho_+^\top\,(\delta_\alpha + \epsilon\delta_\beta)\,\rho_+ + (\eta_\alpha + \epsilon\eta_\beta)\left(\rho_+^\top\right)^2(\delta_\alpha + \epsilon\delta_\beta)\right)^k \tag{1.12}$$

in involution, i.e. $\{I_\epsilon^k, I_{\epsilon'}^l\} = 0$, where

$$\eta_\alpha := \sum_{i=0}^\infty \alpha_{i\infty}|i\rangle\langle i|, \quad \delta_\alpha := \sum_{i=0}^\infty \alpha_{0i}|i\rangle\langle i|, \quad \alpha_{i\infty} := \prod_{j=i}^\infty a_j \tag{1.13}$$

and $a_i := \alpha_{i,i+1}$, $b_i := \beta_{i,i+1}$ satisfy

$$(a_i \dots a_{j-1} - b_i \dots b_{j-1})\,(a_j - b_j) = 0. \tag{1.14}$$

For the special case when $a_i = 1$ for $i \in \mathbb{N}\cup\{0\}$, $b_1 = b$ and $b_i = 1$ for $i \neq 1$ we use for ρ_+ the block notation

$$\rho_+ = \left(\begin{array}{cc|c} 0 & a & x^\top \\ 0 & 0 & y^\top \\ \hline \mathbf{0} & \mathbf{0} & \delta \end{array}\right) \in \mathcal{L}_+^2, \tag{1.15}$$

where $a \in \mathbb{R}$, $x, y \in l^2$, $N \in \mathbb{N}\cup\{\infty\}$, and

$$\delta = \begin{pmatrix} 0 & \delta_{12} & \delta_{13} & \dots \\ 0 & 0 & \delta_{23} & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \delta_{ij} \in \mathbb{R}. \tag{1.16}$$

The Hamilton equation (1.2) for the Banach Lie–Poisson space $(\mathcal{L}_+^2, \{\,\cdot\,,\,\cdot\,\}_{+,\alpha})$ in the complex vector variable $z := x + iy \in (l^2)^{\mathbb{C}}$ assumes the form

$$\frac{dz}{dt} = \left(\left(\frac{\partial h}{\partial \boldsymbol{\delta}}\right)^\top - \frac{\partial h}{\partial \boldsymbol{\delta}} + i\frac{\partial h}{\partial a}\mathbf{1}\right) z + 2\left(\boldsymbol{\delta} - \boldsymbol{\delta}^\top - ia\mathbf{1}\right)\frac{\partial h}{\partial \bar{z}}. \tag{1.17}$$

For the following Hamiltonian

$$h := \frac{1}{2}\left((\bar{z}^\top z)^2 - |z^\top z|^2 + Tr(\boldsymbol{\delta} - \boldsymbol{\delta}^\top)^2 - 2a^2\right) \tag{1.18}$$

we get the equation of motion

$$\frac{1}{2}\frac{dz}{dt} = (1 + \bar{z}^\top z)\left(\boldsymbol{\delta} - \boldsymbol{\delta}^\top\right) z - ia\left(1 + \bar{z}^\top z\right) z - z^\top z(\boldsymbol{\delta} - \boldsymbol{\delta}^\top - ia\mathbf{1})\bar{z} \tag{1.19}$$

Using the integrals of motion (1.12) We can solve the non-linear equation (1.19) in quadratures for the cases when $\dim\mathbb{H} - 2 = 2, 3, 4$, see [3]. Solution for the case $\dim\mathbb{H} = 4$ is the following

$$z(t) = \frac{1}{\sqrt{2}}\sqrt{\varrho\cos(\omega_1 t + \varphi_0) + c^2}\begin{pmatrix} \cos\alpha_1(t) \\ \sin\alpha_1(t) \end{pmatrix} + \tag{1.20}$$
$$+\frac{i}{\sqrt{2}}\sqrt{-\varrho\cos(\omega_1 t + \varphi_0) + c^2}\begin{pmatrix} \cos\beta_1(t) \\ \sin\beta_1(t) \end{pmatrix},$$

where

$$\alpha_1(t) = 2(c_1 - \lambda_1(1+c^2))t + 4\frac{\lambda_1\varrho - \frac{c_1c^2}{\varrho}}{\omega_1\sqrt{\frac{c^4}{\varrho^2} - 1}} arctg\frac{\left(\frac{c^2}{\varrho} - 1\right) tg\frac{\omega_1 t + \varphi_0}{2}}{\sqrt{\frac{c^4}{\varrho^2} - 1}}, \tag{1.21}$$

$$\beta_1(t) = 2(c_1 - \lambda_1(1+c^2))t - 4\frac{\lambda_1\varrho + \frac{c_1c^2}{\varrho}}{\omega_1\sqrt{\frac{c^4}{\varrho^2} - 1}} arctg\frac{\left(-\frac{c^2}{\varrho} - 1\right) tg\frac{\omega_1 t + \varphi_0}{2}}{\sqrt{\frac{c^4}{\varrho^2} - 1}}. \tag{1.22}$$

In the infinite-dimensional case $\dim\mathbb{H} = \infty$ the Hamiltonian system (1.19) can be considered as an itegrable integro-differential nonlinear system. We suppose it can have application to models of quantum physics. Other examples of integrable systems on Banach Lie–Poisson spaces can be found in [1] and [5].

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A Koszul tensor category of integrable representations for $\mathfrak{g} = sl(\infty), o(\infty), sp(\infty)$

Elizabeth Dan-Cohen, Ivan Penkov and Vera Serganova

The ground field is $\mathbb{C}$. By $sl(\infty), o(\infty), sp(\infty)$ we denote the *simple finitary infinite-dimensional Lie algebras*: $sl(\infty) = \varinjlim sl(n), o(\infty) = \varinjlim o(n), sp(\infty) = \varinjlim sp(2n)$, where in each direct limit the inclusions can be chosen as "left upper corner" inclusions.

We study integrable $\mathfrak{g}$-modules M for $\mathfrak{g} \simeq sl(\infty), o(\infty), sp(\infty)$. By definition, a $\mathfrak{g}$-module M is *integrable* if $\dim\{m, g\cdot m, g^2\cdot m, ...\} < \infty$ for any $g \in \mathfrak{g}, m \in M$. In the recent paper [2] several categories of integrable $\mathfrak{g}$-modules have been introduced and studied. In this report we announce some results in progress on yet another interesting category of integrable $\mathfrak{g}$-modules.

Let G denote the connected component of Aut$\mathfrak{g}$. If $\gamma \in G$ and M is a $\mathfrak{g}$-module, M^γ stands for the $\mathfrak{g}$-module M twisted by the automorphism γ. By $\mathfrak{h} \subset \mathfrak{g}$ we denote a *splitting Cartan subalgebra* of $\mathfrak{g}$, i.e. a maximal toral subalgebra of $\mathfrak{g}$ which yields a root decomposition of $\mathfrak{g}$, see [3].

Theorem 1. *The following conditions on a $\mathfrak{g}$-module M of finite length are equivalent.*

- *(i) M is a weight module for any splitting Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, i.e. for any $\mathfrak{h}$, $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ where $M_\lambda = \{m \in M | h \cdot m = \lambda(h)m \; \forall h \in \mathfrak{h}\}$.*
- *(ii) M is an integrable $\mathfrak{g}$-module satisfying* (i)*.*
- *(iii) M is G-invariant, i.e. for any $\gamma \in G$ there is a $\mathfrak{g}$-isomorphism $M^\gamma \simeq M$, and M is a weight module for some splitting Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.*
- *(iv) M is integrable and $\forall m \in M$, $\mathrm{Ann}_{\mathfrak{g}} m$ contains the commutator subalgebra of the centralizer in $\mathfrak{g}$ of a finite-dimensional subalgebra of $\mathfrak{g}$.*

By $T_{\mathfrak{g}}$ we denote the full subcategory of the category of integrable $\mathfrak{g}$-modules consisting of modules satisfying the equivalent conditions of Theorem 1. Then $T_{\mathfrak{g}}$ is a tensor category with respect to the usual tensor product of $\mathfrak{g}$-modules.

By V and V_* we denote respectively the natural and conatural $\mathfrak{g}$-module, [4]. For $\mathfrak{g} = o(\infty), sp(\infty)$, V is isomorphic to V_*. By $Te^\bullet$ we denote the tensor algebra of the module $V \oplus V_*$ for $\mathfrak{g} = sl(\infty)$, and the tensor algebra of the module V for $\mathfrak{g} = o(\infty), sp(\infty)$.

Theorem 2. *Any simple $\mathfrak{g}$-module in $T_{\mathfrak{g}}$ is isomorphic to a simple $\mathfrak{g}$-submodule of $Te^\bullet$.*

The simple $\mathfrak{g}$-submodules of $Te^\bullet$ are described in [5], and each such simple submodule M is realized as the socle of a unique (up to isomorphism) direct summand $\widetilde{M}$ of the $\mathfrak{g}$-module $Te^\bullet$.

Theorem 3. *For every simple $\mathfrak{g}$-module M in $T_{\mathfrak{g}}$, $\widetilde{M}$ is an injective hull of M in $T_{\mathfrak{g}}$.*

Let $T_{\mathfrak{g}}^r$ be the full subcategory of $T_{\mathfrak{g}}$ such that all simple objects of $T_{\mathfrak{g}}^r$ are submodules of $Te^{\leq r}$. Then $T_{\mathfrak{g}} = \varinjlim T_{\mathfrak{g}}^r$. Moreover, $I^r := Te^{\leq r}$ is an injective generator of $T_{\mathfrak{g}}^r$. Consider the finite-dimensional algebra $\mathcal{A}_{\mathfrak{g}}^r := \mathrm{End}_{\mathfrak{g}} I^r$. Then $\mathcal{A}_{\mathfrak{g}}^r$ is $\mathbb{Z}_{\geq 0}$-graded: $(\mathcal{A}_{\mathfrak{g}}^r)_i := \oplus_{j \leq r} \mathrm{Hom}_{\mathfrak{g}}(Te^j, Te^{j-2i})$.

Theorem 4. *The $\mathbb{Z}_{\geq 0}$-graded algebra $\mathcal{A}_{\mathfrak{g}}^r$ is a Koszul ring, [1]. The category $T_{\mathfrak{g}}^r$ is canonically antiequivalent to the category of unitary finite-dimensional $\mathcal{A}_{\mathfrak{g}}^r$-modules.*

Theorem 5. *Fon any $r \geq 0$, there is an isomorphism $\mathcal{A}_{sl(\infty)}^r \simeq (\mathcal{A}_{sl(\infty)}^r)^!$ (the definition of the dual Koszul ring $(\ )^!$ see in [1]).*

Note that there are natural injective homomorphisms $\mathcal{A}_{\mathfrak{g}}^r \to \mathcal{A}_{\mathfrak{g}}^{r+1}$ for $r \geq 0$.

Theorem 6. *The category $T_{\mathfrak{g}}$ is canonically antiequivalent to the category of locally unitary finite-dimensional $\mathcal{A}_{\mathfrak{g}}$-modules, where $\mathcal{A}_{\mathfrak{g}} = \varinjlim \mathcal{A}^r_{\mathfrak{g}}$.*

Theorem 7. *The algebras $\mathcal{A}_{o(\infty)}$ and $\mathcal{A}_{sp(\infty)}$ are isomorphic, therefore the categories $T_{o(\infty)}$ and $T_{sp(\infty)}$ are equivalent.*

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On conjugacy of MADs in *k*-loop algebras

Vladimir Chernousov

(joint work with P. Gille, A. Pianzola)

Throughout k will denote a field of characteristic 0. For integers $n \geq 0$ we set $R = R_n = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. We let $\dot{\mathfrak{g}}$ denote a split simple finite dimensional Lie algebra over k and $\dot{\mathbf{G}}$ the corresponding simple simply connected algebraic group over k. Recall that a Lie algebra $\mathfrak{g}$ over R is called a form of $\dot{\mathfrak{g}} \otimes_k R$ if there exists a faithfully flat R-algebra S such that

$$\mathfrak{g} \otimes_R S \simeq (\dot{\mathfrak{g}} \otimes_k R) \otimes_R S \simeq \dot{\mathfrak{g}} \otimes_k S.$$

Our form $\mathfrak{g}$ can also be viewed as a Lie algebra over k (which is infinite dimensional if $n \geq 1$). In the theory of affine Kac-Moody algebras, or more generally for extended affine Lie algebras, the emphasis is in viewing the relevant objects over k (and not R). We now introduce the most relevant k-objects attached to $\mathfrak{g}$ in this work.

A subalgebra $\mathfrak{m}$ of the k-Lie algebra $\mathfrak{g}$ is called an AD subalgebra if $\mathfrak{g}$ admits a k-basis consisting of simultaneous eigenvectors of $\mathfrak{m}$, i.e. there exists a family (λ_i) of functionals $\lambda_i \in \mathfrak{m}^*$, and a k-basis $\{ v_i \}_{i \in I}$ of $\mathfrak{g}$ such that

$$[h, v_i] = \langle \lambda_i, h \rangle \, v_i \quad \forall h \in \mathfrak{m}.$$

It is not difficult to see that any such $\mathfrak{m}$ is necessarily abelian, so AD can be thought as shorthand for abelian k-diagonalizable or ad-k-diagonalizable. An AD subalgebra which is maximal (in the sense that it is not properly included in another AD) is called a MAD.

In infinite dimensional Lie theory $\mathfrak{m}$ plays the role which a Cartan subalgebra $\mathfrak{h}$ plays for $\dot{\mathfrak{g}}$ in the classical theory. One of the central theorems of classical

Lie theory is that all split Cartan subalgebras of $\dot{\mathfrak{g}}$ are conjugate under $\dot{\mathbf{G}}(k)$, a theorem of Chevalley. This result yields the most elegant proof that the type of the root system of $(\dot{\mathfrak{g}}, \dot{\mathfrak{h}})$ is an invariant of $\dot{\mathfrak{g}}$. The main thrust of our work is to investigate the question of conjugacy of MADs of $\mathfrak{g}$. Our first result says that the conjugacy of MADs is equivalent to conjugacy of maximal split tori in a simple simply connected group scheme over R corresponding to $\mathfrak{g}$.

Theorem. Let $\mathbf{G}$ be a simple simply connected group scheme over R and

$$\mathfrak{g} = \mathrm{Lie}\,(\mathbf{G}) = \underline{\mathrm{Lie}}\,(\mathbf{G})(R).$$

(1) If $\mathbf{S}$ is a maximal split torus of $\mathbf{G}$ then its Lie algebra $\mathrm{Lie}\,(\mathbf{S}) \subset \mathfrak{g}$ contains a unique MAD $\mathfrak{m} = \mathfrak{m}(\mathbf{S})$ of $\mathfrak{g}$.

(2) Let $\mathfrak{m}$ be a MAD of $\mathfrak{g}$. Then $Z_{\mathbf{G}}(R\mathfrak{m}) := H$ is a reductive R-group. Its radical contains a unique maximal split torus $\mathbf{S}\,(\mathfrak{m})$ of $\mathbf{G}$.

(3) The process $\mathfrak{m} \to \mathbf{S}\,(\mathfrak{m})$ and $\mathbf{S} \to \mathfrak{m}\,(\mathbf{S})$ described above gives a bijection between the set of MADs of $\mathfrak{g}$ and the set of maximal split tori of $\mathbf{G}$.

From the way we constructed the above bijective correspondence it follows that the conjugacy of two maximal k-diagonalizable subalgebras in $\mathfrak{g}$ is equivalent to conjugacy of the corresponding maximal R-split tori in $\mathbf{G}$. The following example shows that in general case maximal R-split tori are not necessary conjugate.

Example. Let D be a quaternion algebra over $R = k[t_1^{\pm 1}, t_2^{\pm 1}]$ with generators T_1, T_2 and relations $T_1^2 = t_1,\ T_2^2 = t_2$ and $T_2T_1 = -T_1T_2$ and let $A = M_2(D)$. We may view A as a D-endomorphism algebra of a 2-dimensional space $V = D \oplus D$ over D where D acts on V on the right. Let $\mathbf{G} = \mathrm{SL}\,(1, A)$. It contains an R-split torus $\mathbf{S}_1$ whose R-points are matrices of the form

$$\begin{pmatrix} x & o \\ 0 & x^{-1} \end{pmatrix}$$

where $x \in R^{\times}$. Let $K = k(t_1, \ldots, t_n)$. Since K-rank of $\mathbf{G}$ is equal to 1, the torus $\mathbf{S}_1$ is a maximal split in $\mathbf{G}$.

Consider now a D-linear map $f : V = D \oplus D \to D$ given by

$$(u, v) \to (1 + T_1)u - (1 + T_2)v.$$

Let $\mathcal{L}$ be its kernel. One can show that $\mathcal{L}$ is a projective D-module which is not free. Since f is split, we have another decomposition $V \simeq \mathcal{L} \oplus D$. Let $\mathbf{S}_2$ be an R-split torus in $\mathbf{G}$ consisting of linear transformations acting on the first summand $\mathcal{L}$ by multiplication $x \in R^{\times}$ and on the second summand by x^{-1}. As above, $\mathbf{S}_2$ is a maximal R-split torus in $\mathbf{G}$. We claim that $\mathbf{S}_1$ and $\mathbf{S}_2$ are not conjugate. To see this we note that given $\mathbf{S}_1$ we can restore two summands in the decomposition $V = D \oplus D$ as two subspaces in V consisting of eigenvectors of elements $s \in \mathbf{S}_1(R)$. Similarly, we can uniquely restore two summands in the decompositions $V = \mathcal{L} \oplus D$ out of $\mathbf{S}_2$. Assuming now that $\mathbf{S}_1$ and $\mathbf{S}_2$ are conjugate by an element in $\mathbf{G}(R)$ we obtained immediately that the subspace $\mathcal{L}$ in V is isomorphic to one of the components of $V = D \oplus D$, in particular $\mathcal{L}$ is free – a contradiction.

Thus for twisted forms of a split group scheme over R the conjugacy fails in general case. However for a large class of Lie algebras called multiloop algebras and for the corresponding group schemes we do have conjugacy. We recall that a group scheme $\mathbf{G}$ and its Lie algebra $\mathfrak{g}$ are called multiloop if $\mathbf{G}$ is a twisted form of a split $\dot{\mathbf{G}}$ by a cocycle with coefficients in $\mathrm{Aut}\,(\dot{\mathbf{G}})(\bar{k})$.

Theorem. Let $\mathbf{G}$ be a simple simply connected multiloop R-group scheme. Then two maximal R-split tori $\mathbf{S}_1$, $\mathbf{S}_2$ in $\mathbf{G}$ such that their centralizers in $C_{\mathbf{G}}(S_i)$ are multiloop are conjugate by an element in $\mathbf{G}(R)$.

Remark. The above counter-example shows that the assumptions that $\mathbf{G}$ and $C_{\mathbf{G}}(S_i)$ are multiloop cannot be dropped in general case.

The main ingredient of the proof of conjugacy is the following result on torsors over R which provides us with the classification of multiloop group schemes and their Lie algebras.

Theorem. Let G be an algebraic group over k. Let $F = k((t_1))\cdots((t_n))$. Then a canonical mapping $H^1_{loop}(R, G) \to H^1(F, G)$ is bijective.

Gerbes, gerbal representations and 3-cocycles

Jouko Mickelsson

In this talk I will explain relations between on one hand the recent discussion on 3-cocycles and categorical aspects of representation theory, [FZ], and on the other hand gauge anomalies, gauge group extensions and 3-cocycles in quantum field theory, [CGRS].

The set up for categorical representation theory consists of an abelian category C, a group G, and a map F which associates to each $g \in G$ a functor F_g in the category C such that for any pair $g, h \in G$ there is an isomorphism

$$i_{g,h} : F_g \circ F_h \to F_{gh}.$$

For a triple $g, h, k \in G$ we have a pair of isomorphisms $i_{g,hk} \circ i_{h,k}$ and $i_{gh,k} \circ i_{g,h}$ from $F_g \circ F_h \circ F_k$ to F_{ghk} :

They are not necessarily equal; one can have a *central extension* (with values in an abelian group)

$$i_{g,hk} \circ i_{h,k} = \alpha(g, h, k) i_{gh,k} \circ i_{g,h}$$

with $\alpha(g, h, k) \in \mathbf{C}^\times$ a 3-cocycle.

The smooth loop group LG (G compact,simple) has a central extension defined by a (local) 2-cocycle. According to Frenkel and Zhu, increase the cohomoligal degree by one unit by going to the double loop group $L(LG)$. They do this algebraically, utilizing the idea of A. Pressley and G. Segal by embedding the loop group LG (actually, its Lie algebra) to an appropriate universal group $U(\infty)$ (or its Lie algebra). The point of this talk is to show how this is done in the smooth setting, globally, and connecting to the old discussion of QFT anomalies in the 1980's.

Following [ML], let $\mathcal{B}$ be an associative algebra and G a group. Assume that we have a group homomorphism $s : G \to \mathrm{Out}(\mathcal{B})$ where $\mathrm{Out}(\mathcal{B})$ is the group of outer automorphims of $\mathcal{B}$, that is, $\mathrm{Out}(\mathcal{B}) = \mathrm{Aut}(\mathcal{B})/\mathrm{In}(\mathcal{B})$, all automorphims modulo the normal subgroup of inner automorphisms.

If one chooses any lift $\tilde{s} : G \to \mathrm{Aut}(\mathcal{B})$ then we can write

$$\tilde{s}(g)\tilde{s}(g') = \sigma(g, g') \cdot \tilde{s}(gg')$$

for some $\sigma(g, g') \in \mathrm{In}(\mathcal{B})$. From the definition follows immediately the cocycle property

$$\sigma(g, g')\sigma(gg', g'') = [\tilde{s}(g)\sigma(g', g'')\tilde{s}(g)^{-1}]\sigma(g, g'g'')$$

Prolongation by central extension

Let next H be any central extension of $\mathrm{In}(\mathcal{B})$ by an abelian group a. That is, we have an exact sequence of groups,

$$1 \to a \to H \to \mathrm{In}(\mathcal{B}) \to 1.$$

Let $\hat{\sigma}$ be a lift of the map $\sigma : G \times G \to \mathrm{In}(\mathcal{B})$ to a map $\hat{\sigma} : G \times G \to H$ (by a choice of section $\mathrm{In}(\mathcal{B}) \to H$). We have then

$$\hat{\sigma}(g, g')\hat{\sigma}(gg', g'') = [\tilde{s}(g)\hat{\sigma}(g', g'')\tilde{s}(g)^{-1}]\hat{\sigma}(g, g'g'') \cdot \alpha(g, g', g'') \text{ for all } g, g', g'' \in G$$

where $\alpha : G \times G \times G \to a$.

Here the action of the outer automorphism $s(g)$ on $\hat{\sigma}(*)$ is defined by $s(g)\hat{\sigma}(*)s(g)^{-1}$ = the lift of $s(g)\sigma(*)s(g)^{-1} \in \mathrm{In}(\mathcal{B})$ to an element in H. One can show that α is a 3-cocycle

$$\alpha(g_2, g_3, g_4)\alpha(g_1g_2, g_3, g_4)^{-1}\alpha(g_1, g_2g_3, g_4) \\ \times\alpha(g_1, g_2, g_3g_4)^{-1}\alpha(g_1, g_2, g_3) = 1.$$

Remark If we work in the category of topological groups (or Lie groups) the lifts above are in general discontinuous; normally, we can require continuity (or smoothness) only in an open neighborhood of the unit element.

The above situation appears in gauge theory. The algebra $\mathcal{B}$ is realized as the C^* algebra of fermionic anticommutation relations for fermions on a circle and in the simplest case the outer automorphism as the group of functions on an interval with values in a compact Lie group, the inner automorphisms as the loop group LG (elements of which are implemented up to projective factor as operators in the fermionic Fock space). The central extension comes automatically when lifting the 1-particle operators to operators in the Fock space. The group 3-cocycle can be computed but is complicated. Instead, the corresponding Lie algebra 3-cocycle is simple and equal to

$$\frac{1}{4\pi i}\operatorname{tr} X[Y,Z],$$

where X, Y, Z are elements of the Lie algebra of G and the trace is computed in an appropriate representation of G.

This construction can be generalized to gauge theory in higher dimensions. The loop group LG is then replaced by a group $Map(M, G)$ of G-valued functions on a compact space G and the central extension by an abelian extension induced by renormalization effects in quantum field theory, [M1]. For further details see [M2].

Back to the double loop group $L(LG)$

Next we can replace the group G by $\mathcal{G} = L(LG)$. Assuming G connected, simply connected, the group $\mathcal{G}$ is connected and we can again go through the same steps as in the case of G earlier, except that now for $L\mathcal{G}$ the representation has to be understood in the sense of groupoid central extension or in other words, as Hilbert cocycle. The groupoid here is actually the natural transformation groupoid coming from the gauge action of $L\mathcal{G}$ on gauge connections A on a 3-torus. The cocycle is then a function of the parameter A.

As before, one can compute the 3-cocycle for the double loop group. The corresponding Lie algebra 3-cocycle is obtained by transgression from the Lie algebra 2-cocycle for $L\mathcal{G}$, [M1,2]. Explicit expressions are given as

$$c_2 = const. \int_{T^3} \operatorname{tr} A[dX, dY]$$

with $X, Y : T^3 \to \mathfrak{g}$, transgressing to

$$c_3 = const. \int_{T^2} \operatorname{tr} X[dY, dZ]$$

with now $X, Y, Z : T^2 \to \mathfrak{g}$.

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Universal central extensions of gauge groups

Bas Janssens
(joint work with Christoph Wockel)

We indicate how to calculate the universal central extension of the gauge algebra $\Gamma(\mathrm{ad}(P))$, and how to obtain from this the corresponding universal central extension of the gauge group $\Gamma(\mathrm{Ad}(P))$.

Gauge groups occur as vertical symmetries of gauge theories, in which fields are connections on a principal G-bundle $P \to M$, and the action is invariant under vertical automorphisms of P. If we set $\mathrm{Ad}(P) := P \times_{\mathrm{Ad}} G$ and similarly $\mathrm{ad}(P) := P \times_{\mathrm{ad}} \mathfrak{g}$ (with $\mathfrak{g}$ the Lie algebra of G), we can identify the group of vertical automorphisms with $\Gamma(\mathrm{Ad}(P))$, and its Lie algebra with $\Gamma(\mathrm{ad}(P))$.

In the case that P admits a *flat* equivariant connection, these gauge algebras closely resemble equivariant map algebras and (twisted multi) loop algebras. Using the flat connection, one finds a cover $N \to M$, a monodromy group $\Delta < \pi_1(M)$, and a homomorphism $\Delta \to G$ such that $P = N \times_\Delta G$. The adjoint bundle then takes the shape $\mathrm{ad}(P) = N \times_\Delta \mathfrak{g}$, so that the gauge algebra is just $\Gamma(\mathrm{ad}(P)) = (C^\infty(N, \mathbb{R}) \otimes_{\mathbb{R}} \mathfrak{g})^\Delta$, the Lie algebra of smooth equivariant maps from N to $\mathfrak{g}$.

If X is an affine variety over $\mathbb{R}$ with an action of a discrete group Δ, and Δ acts by automorphisms on a real Lie algebra $\mathfrak{g}$, then the equivariant map algebra $(\mathbb{C}[X] \otimes_{\mathbb{R}} \mathfrak{g})^\Delta$ is the Lie algebra of equivariant regular maps from X to $\mathfrak{g}_{\mathbb{C}}$.

The set $X^{\mathrm{reg}}_{\mathbb{R}}$ of regular real points constitutes a smooth manifold, and under suitable conditions (see prop. 3), the homomorphism $\mathbb{C}[X] \to C^\infty(X^{\mathrm{reg}}_{\mathbb{R}}, \mathbb{C})$ is injective with dense image. If the action of Δ restricts to $X^{\mathrm{reg}}_{\mathbb{R}}$, then we obtain an inclusion $(\mathbb{C}[X] \otimes_{\mathbb{R}} \mathfrak{g})^\Delta \hookrightarrow (C^\infty(X^{\mathrm{reg}}_{\mathbb{R}}, \mathbb{C}) \otimes_{\mathbb{R}} \mathfrak{g})^\Delta$ of Lie algebras with dense image. If moreover $X^{\mathrm{reg}}_{\mathbb{R}}/\Delta$ is a manifold, then we have realised the equivariant map algebra as a dense subalgebra of the complexification of the gauge algebra $\Gamma(\mathrm{ad}(P))$, with $P \to X^{\mathrm{reg}}_{\mathbb{R}}/\Delta$ the principal $\mathrm{Aut}(\mathfrak{g})$-bundle $P = X^{\mathrm{reg}}_{\mathbb{R}} \times_\Delta \mathrm{Aut}(\mathfrak{g})$.

For example, let X be $T^n = \{(\vec{z}, \vec{w}) \in \mathbb{C}^{2n} \,|\, z_k^2 + w_k^2 = 1, 1 \leq k \leq n\}$, the complex n-torus. In this case, $\mathbb{C}[T^n] \hookrightarrow C^\infty(T^n_{\mathbb{R}}, \mathbb{C})$ is injective with dense image by Fourier theory. We now look for a regular group action on T^n that restricts to $T^n_{\mathbb{R}}$ and such that $M = T^n_{\mathbb{R}}/\Delta$ is a manifold. Although the Bieberbach groups spring to mind, the choice that is studied most is $\Delta = \prod_{k=1}^n \mathbb{Z}/r_k\mathbb{Z}$, with δ : $(z_k \pm iw_k) \mapsto e^{\pm 2\pi i \delta_k / r_k}(z_k \pm iw_k)$. In this case, $M = T^n_{\mathbb{R}}/\Delta$ is again a torus. For any homomorphism $\Delta \to \mathrm{Aut}(\mathfrak{g})$, the twisted multiloop algebra $(\mathbb{C}[T^n] \otimes_{\mathbb{R}} \mathfrak{g})^\Delta$ forms a dense subalgebra of the complexification of $\Gamma(\mathrm{ad}(P))$, where P is the principal $\mathrm{Aut}(\mathfrak{g})$-bundle $P = T^n_{\mathbb{R}} \times_\Delta \mathrm{Aut}(\mathfrak{g})$ over $T^n_{\mathbb{R}}$.

The case of the circle is special in that every principal G-bundle over $M = T^1_{\mathbb{R}}$ is given by a twist $g \in G$ upon a full rotation, and therefore admits a flat connection. A smooth path connecting g to g' yields an isomorphism of the corresponding bundles, so principal G-bundles are classified by $\pi_0(G)$. Consequently, the complexified adjoint bundles are classified by $\pi_0(\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}}))$, which for simple $\mathfrak{g}_{\mathbb{C}}$ amounts to diagram automorphisms of order 1, 2 or 3. Complexified gauge algebras over $T^1_{\mathbb{R}}$ are thus precisely the closures of twisted loop algebras.

We return to the case of smooth principal fibre bundles which do not necessarily have a flat connection, and sketch the universal 2-cocycle for the compactly supported gauge algebra $\Gamma_{\mathrm{c}}(\mathrm{ad}(P))$. We refer the interested reader to [1] for details.

For any Lie algebra $\mathfrak{g}$, set $V(\mathfrak{g}) := (\mathfrak{g} \otimes_s \mathfrak{g})/\operatorname{der}(\mathfrak{g}) \cdot (\mathfrak{g} \otimes_s \mathfrak{g})$, and denote by $\kappa\colon \mathfrak{g} \times \mathfrak{g} \to V(\mathfrak{g})\,;\,(x, y) \mapsto [x \otimes_s y]$ the universal $\operatorname{der}(\mathfrak{g})$-invariant bilinear form on $\mathfrak{g}$. Any Lie connection ∇ on $\mathrm{ad}(P)$ induces a flat connection $\mathtt{d}$ on the vector bundle $V(\mathrm{ad}(P)) \to M$, which does not depend on ∇ as any two Lie connections differ by a pointwise derivation, which acts trivially on $V(\mathrm{ad}(P))$. Using the identities $\mathtt{d}\kappa(\xi, \eta) = \kappa(\nabla\xi, \eta) + \kappa(\xi, \nabla\eta)$ and $\nabla[\xi, \eta] = [\nabla\xi, \eta] + [\xi, \nabla\eta]$ for all sections $\xi, \eta \in \Gamma_{\mathrm{c}}(\mathrm{ad}(P))$, one checks that

$$\omega_\nabla\colon \wedge^2 \Gamma_{\mathrm{c}}(\mathrm{ad}(P)) \to \overline{\Omega}{}^1_{\mathrm{c}}(M, V(\mathrm{ad}(P)) \quad \xi \wedge \eta \mapsto [\kappa(\xi, \nabla\eta)] \tag{1.1}$$

defines a Lie algebra cocycle, where the subscript c denotes compact support, and we set $\overline{\Omega}{}^1_{\mathrm{c}} := \Omega^1_{\mathrm{c}}/\mathtt{d}\Omega^0_{\mathrm{c}}$. If $\mathfrak{g}$ is semisimple, then the cohomology class $[\omega_\nabla]$ does not depend on ∇. We equip our spaces of smooth forms and sections with the usual LF-topology, in terms of which the universality result is formulated as follows.

Proposition 1. *If $\mathfrak{g}$ is semisimple, then $[\omega_\nabla]$ is universal; every continuous 2-cocycle ψ with values in a trivial real topological module W can be written up to coboundary as $\psi = \phi \circ \omega_\nabla$, for some continuous $\mathbb{R}$-linear $\phi : \overline{\Omega}{}^1_{\mathrm{c}}(M, V(\mathrm{ad}(P))) \to W$.*

In [1], this is proved by noting that 2-cocycles are automatically diagonal, so that the second cohomology in fact constitutes a sheaf. The result can then be reduced to the well known local one, described e.g. in [2]. Using the results of [3–5], this can be used (see [1]) to prove the following theorem.

Theorem 2. *Let $P \to M$ be a principal fibre bundle with compact connected base, and with a semisimple structure group with finitely many connected components. Then the cocycle (1.1) integrates to a central extension of $\Gamma(\mathrm{Ad}(P))$ that is universal for abelian Lie groups modelled on Mackey-complete locally convex spaces.*

Although the application of differential geometric techniques in an algebraic context has intrinsic drawbacks, it is perhaps worth while to briefly explore the ramifications of proposition 1 to equivariant map algebras. We start by substantiating our claim as to the injectivity and denseness of $\mathbb{C}[X] \to C^\infty(X^{\mathrm{reg}}_{\mathbb{R}}, \mathbb{C})$.

Proposition 3. *Let X be an affine variety over $\mathbb{R}$ such that every connected component of X^{an} possesses a nonsingular real point. Then the ring homomorphism $\mathbb{C}[X] \hookrightarrow C^\infty(X^{\mathrm{reg}}_{\mathbb{R}}, \mathbb{C})$ is injective, with dense image in the topology of uniform convergence of derivatives on compact subsets.*

Proof. First, we prove that the image is dense. As every smooth function on $X^{\mathrm{reg}}_{\mathbb{R}}$ can be approximated by compactly supported smooth functions, and every compactly supported (in $X^{\mathrm{reg}}_{\mathbb{R}}$) smooth function on $X^{\mathrm{reg}}_{\mathbb{R}}$ extends to a compactly supported (in $\mathbb{R}^n$) smooth function on $\mathbb{R}^n$, it is enough to show that every smooth function f on a compact subset K of $\mathbb{R}^n$ can be approximated by polynomials. Now by Weierstraß' theorem, there exist, for any multi-index $\vec{\mu}$, polynomials p with $\sup_K |\partial_{\vec{\mu}} f - p|$ arbitrarily small. By integrating these, one can produce polynomials

p such that $\sup_K |\partial_{\vec{\nu}} f - \partial_{\vec{\nu}} p|$ is arbitrarily small for all $\vec{\nu} < \vec{\mu}$. A sequence p_k of such polynomials for $\vec{\mu}_k \to \infty$ (in the sense that for every fixed $\vec{\nu}$, we eventually have $\vec{\nu} < \vec{\mu}_k$) will converge to f uniformly on K for every derivative.

Next, we prove injectivity. Denote by $\mathcal{O}_Y^{\mathrm{an}}$ and C_Y^∞ the sheaves of analytic and smooth functions on Y. Choose a nonsingular real point x_i in each connected component (in the analytic topology) of X^{an}, so that $\mathbb{C}[X] \to \bigoplus_i \mathcal{O}^{\mathrm{an}}_{X,x_i}$ is injective. Using the inverse function theorem, we find analytic charts $\phi_i : \mathbb{C}^d \supset U_i \to V_i \subset X^{\mathrm{an}}$ around x_i in which $U_i \cap \mathbb{R}^d$ corresponds to $V_i \cap X_{\mathbb{R}}^{\mathrm{reg}}$. In those coordinates, the map $\mathcal{O}^{\mathrm{an}}_{X,x_i} \to C^\infty_{X_{\mathbb{R}}^{\mathrm{reg}},x_i}$ corresponds to the injective map $\mathcal{O}^{\mathrm{an}}_{\mathbb{C}^d,0} \to C^\infty_{\mathbb{R}^d,0}$, and is therefore injective. Since the injective map $\mathbb{C}[X] \to \bigoplus_i C^\infty_{X_{\mathbb{R}}^{\mathrm{reg}},x_i}$ factors through $\mathbb{C}[X] \to C^\infty(X_{\mathbb{R}}^{\mathrm{reg}}, \mathbb{C})$, the latter must be injective itself. □

Consider X, $\mathfrak{g}$ and Δ as before, but now with $X_{\mathbb{R}}^{\mathrm{reg}}/\Delta$ a *compact* manifold, and $\mathfrak{g}$ *semisimple*. Since $\iota\colon (\mathbb{C}[X] \otimes_{\mathbb{R}} \mathfrak{g})^\Delta \hookrightarrow \Gamma(\mathrm{ad}(P)_{\mathbb{C}})$ is a dense inclusion, we conclude with [7, Lem. 2] that $\iota^*\colon H^2_{\mathrm{ct}}(\Gamma(\mathrm{ad}(P))_{\mathbb{C}}, W) \to H^2_{\mathrm{ct}}((\mathbb{C}[X] \otimes_{\mathbb{R}} \mathfrak{g})^\Delta, W)$ is an isomorphism, where W is a complex Fréchet space considered as a trivial module, and continuity is in the C^∞-topology on *both* sides. Restricted to the equivariant map algebra, our canonical cocycle takes values in the space $\overline{\Omega}^1_{\mathrm{alg}}(\mathbb{C}[X])$ of Kähler differentials modulo closed ones, and can be written

$$\omega_{\mathrm{alg}} : \wedge^2(\mathbb{C}[X] \otimes_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}})^\Delta \to (\overline{\Omega}^1_{\mathrm{alg}}(\mathbb{C}[X]) \otimes_{\mathbb{C}} V(\mathfrak{g}_{\mathbb{C}}))^\Delta \quad : \quad \xi \wedge \eta \mapsto [\kappa(\xi, d\eta)]\,. \tag{1.2}$$

It is universal in the sense that every *continuous* $\mathbb{C}$-valued cocycle τ on the equivariant map algebra can be written up to coboundary as $\tau = \phi \circ \omega_{\mathrm{alg}}$ for some continuous $\mathbb{C}$-linear functional ϕ on $(\overline{\Omega}^1_{\mathrm{alg}}(\mathbb{C}[X]) \otimes_{\mathbb{C}} V(\mathfrak{g}_{\mathbb{C}}))^\Delta$.

In the case of twisted multiloop algebras, a cocycle is continuous if it is of polynomial growth in the $\mathbb{Z}^n$-grading of $\mathbb{C}[T^n]$. If $\mathfrak{g}_{\mathbb{C}}$ is simple, then κ is just the $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$-invariant Killing form, so that $V(\mathfrak{g}_{\mathbb{C}}) \simeq \mathbb{C}$ is a trivial Δ-representation. The universal cocycle thus takes values in $\overline{\Omega}^1_{\mathrm{alg}}(\mathbb{C}[T^n])^\Delta$, in agreement with the purely algebraic result [6]. It might not be overly optimistic to hope for universality of (1.2) for equivariant map algebras with semisimple $\mathfrak{g}_{\mathbb{C}}$ in a more general context.

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Invariant cones and unitary representations of Lie supergroups

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(joint work with Karl-Hermann Neeb)

The study of unitary representations of Lie supergroups was motivated by its connections to supersymmetry, e.g., the classification of free relativistic particles in SUSY quantum mechanics. The simplest approach to study unitary representations of Lie supergroups is to forget the global structure, i.e., to think of a representation as a unitarizable module for a Lie superalgebra, i.e., a module which is endowed with a (suitably defined) contravariant positive definite Hermitian form. However, this Naïve approach is sometimes not quite satisfactory, for at least two reasons:

1. At the Lie superalgebra level, certain standard tools of representation theory (e.g., Mackey theory, systems of imprimitivity, etc.) may not be available.
2. When one is interested in the case of non-reductive Lie superalgebras, there are no general results which guarantee the existence of a correspondence between unitary representations of a Lie supergroup on a super Hilbert space and unitarizable modules of the corresponding Lie superalgebra.

A global approach to the study of unitary representations of Lie supergroups, which addresses the above issues, has been introduced in [3], where a classification of irreducible unitary representations of super Poicaré groups is also obtained.

The aim of this report is to explain a connection between the classification of irreducible unitary representations (in the global sense explained above) of Lie supergroups and the theory of invariant cones in Lie algebras [5]. The idea of using methods of invariant cones to study unitary representations has proved quite powerful in the case of holomorphic representations of Lie groups and their extensions to complex semigroups [4]. From this point of view, it seems quite reasonable to expect this idea to play an important role in harmonic analysis over homogeneous superspaces.

Throughout this report, all Lie superalgebras are over $\mathbb{R}$. Let $\mathcal{G} = (G, \mathfrak{g})$ be a Harish–Chandra pair (see [1, §3.8] for a definition). We say $\mathcal{G}$ is *$\star$–reduced* iff for every nonzero $X \in \mathfrak{g}$ there exists a unitary representation $(\pi, \rho^\pi, \mathscr{H})$ of $\mathcal{G}$ such that $\rho^\pi(X) \neq 0$. Note that if $\mathfrak{g}$ is a simple Lie superalgebra, then $\mathcal{G}$ is $\star$–reduced iff it has a nontrivial unitary representation.

Assume that $\mathcal{G}$ is $\star$-reduced. Consider the closed, convex, G-invariant cone $\mathrm{Cone}(\mathcal{G}) \subseteq \mathfrak{g}_0$ generated by elements of the form $[X, X]$, $X \in \mathfrak{g}_1$. One can prove that $\mathrm{Cone}(\mathcal{G})$ is pointed, i.e., it does not contain any affine lines [5]. Moreover, if $\mathrm{Cone}(\mathcal{G})$ is generating (i.e., as a subset of $\mathfrak{g}_0$ it has nonempty interior), then there exists a Cartan subsuperalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ such that the group generated by $\{\exp(X) \mid X \in \mathfrak{h}_0\}$ is a relatively compact subgroup of $\mathrm{Aut}(\mathfrak{g})$. The even part of such a Cartan subsuperalgebra, which is called *compactly embedded in* $\mathfrak{g}$, acts semisimply on $\mathfrak{g}$, and therefore its action on $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is diagonalizable. Moreover, there is a natural positive definite conjugate-linear pairing on every $\mathfrak{h}_0$–root space.

These necessary conditions turn out to be sufficiently strong to determine exactly which of the Lie supergroups that correspond to real simple Lie superalgebras are $\star$–reduced. The reader is referred to [5, Section 6.2] for more details. In particular, real forms of Lie superalgebras of strange or Cartan type essentially do not have any interesting unitary representations. This is in stark contrast with the case of basic classical Lie superalgebras, for which there are many interesting irreducible unitary representations [2].

It turns out that for an arbitrary $\star$–reduced Lie supergroup $\mathcal{G}$, if $\mathrm{Cone}(\mathcal{G})$ has nonempty interior then in fact every irreducible representation should be a generalized highest weight module. More precisely, let $\mathcal{G} = (G, \mathfrak{g})$ be $\star$–reduced such that $\mathrm{Cone}(\mathcal{G})$ has nonempty interior. Then there exists a compactly embedded Cartan subsuperalgebra of $\mathfrak{h} \subseteq \mathfrak{g}$ and a corresponding triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

such that for every irreducible unitary representation $(\pi, \rho^\pi, \mathscr{H})$ of $\mathcal{G}$, we have

$$\mathscr{H}^{\mathfrak{h}} = U(\mathfrak{g}) \otimes_{U(\mathfrak{n}^-)} \mathscr{V}_\lambda.$$

Here $\lambda \in \mathfrak{h}_0^*$ and $\mathscr{V}_\lambda$ is a (finite dimensional) irreducible representation of $\mathfrak{h}$ on which $\mathfrak{h}_0$ acts via weight λ and $U(\mathfrak{n}^+)\mathscr{V}_\lambda = \{0\}$, and $\mathscr{H}^{\mathfrak{h}}$ denotes the space of $\mathfrak{h}$-finite smooth vectors in $\mathscr{H}$. (One can prove that $\mathscr{H}^{\mathfrak{h}}$ is a dense subspace of $\mathscr{H}$.) Moreover, two irreducible unitary representations $(\pi, \rho^\pi, \mathscr{H})$ and $(\pi', \rho^{\pi'}, \mathscr{H}')$ are unitarily equivalent iff their corresponding $\mathfrak{h}$-modules $\mathscr{V}(\lambda)$ and $\mathscr{V}(\lambda')$ are isomorphic.

The proof of the above result, which is given in [5, Section 7], is based on a delicate analysis of spaces of smooth and analytic vectors in unitary representations. Although it is easy to see that the symmetric operators $e^{-\frac{\pi i}{4}}\rho^\pi(X),\ X \in \mathfrak{h}_0$, are nonpositive definite, what is much harder to prove is that in fact there exists a weight vector for the action of $\mathfrak{h}_0$ on $\mathscr{H}$, which is also an analytic vector for the representation $(\pi, \mathscr{H})$ of the Lie group G. The latter statement can be proved by means of functional calculus on certain Fréchet spaces of analytic vectors.

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Visible actions on multiplicity-free spaces

Atsumu Sasaki

The notion of (strongly) visible actions was introduced by T. Kobayashi [5, 6] for the biholomorphic action on a complex manifold with (possibly) infinitely many orbits.

Suppose that a Lie group G acts holomorphically on a connected complex manifold D. Then, we say that this action is *strongly visible* if there exist a real submanifold S in D (called a *slice*) and an anti-holomorphic diffeomorphism σ on D such that the following conditions are satisfied ([6, Definition 3.3.1]):

(V.1) $$D' := G \cdot S \text{ is open in } D,$$

(S.1) $$\sigma|_S = \mathrm{id}_S,$$

(S.2) $$\sigma \text{ preserves each } G\text{-orbit in } D.$$

The concept of strongly visible actions is the key geometric setting for propagation theorem of multiplicity-free property including both finite-dimensional case and infinite-dimensional case (see [6, 7]). We are interested in the classification problem of strongly visible actions, and this is our object.

First, we will explain the classification of strongly visible linear actions.

Let $G_{\mathbb{C}}$ be a connected complex reductive Lie group. Suppose that we are given a holomorphic representation of $G_{\mathbb{C}}$ on a complex vector space V. We say that $(G_{\mathbb{C}}, V)$ is a *multiplicity-free space* if the induced representation on the polynomial ring $\mathbb{C}[V]$ defined by $f(v) \mapsto (\pi(g)f)(v) = f(g^{-1} \cdot v)$ is multiplicity-free. Multiplicity-free spaces were classified by Kac [3], Benson–Ratcliff [1], and Leahy [8].

By the Cartan–Weyl highest weight theory, irreducible finite-dimensional holomorphic representations of $G_{\mathbb{C}}$ are parametrized by highest weights. We denote by ρ_λ those representations having λ as their highest weights.

Let $(G_{\mathbb{C}}, V)$ be a multiplicity-free space. By definition, the polynomial ring $\mathbb{C}[V]$ is decomposed into the multiplicity-free sum of irreducible representations of $G_{\mathbb{C}}$. We write $\mathbb{C}[V] \simeq \bigoplus_{\lambda \in \Lambda} \rho_\lambda$. It is known that the parameter set λ is a finitely generated semigroup. We define the *rank* of a multiplicity-free space $(G_{\mathbb{C}}, V)$ to be the rank of Λ, namely, the number of generators of the semigroup Λ. The rank of multiplicity-free spaces and explicit generators of Λ were found by Howe–Umeda [2] and Knop [4].

Let G_u be a compact real form of $G_{\mathbb{C}}$. Our main theorem is stated as follows:

Theorem 1 ([10, 11]). *The following two conditions about a linear action on V are equivalent:*

(i) $(G_{\mathbb{C}}, V)$ *is a multiplicity-free space.*
(ii) The G_u*-action on* V *is strongly visible.*

As a consequence, the classification of strongly visible linear actions coincides with that of multiplicity-free spaces.

The implication (ii) $\Rightarrow$ (i) is a special case of propagation theorem of multiplicity-free property (see [6, 7]). We prove the converse implication (i) $\Rightarrow$ (ii) by finding a concrete description of S and σ for each multiplicity-free space $(G_{\mathbb{C}}, V)$.

Furthermore, we have the following theorem:

Theorem 2 ([10–12]). *For any strongly visible G_u-action on a multiplicity-free space $(G_{\mathbb{C}}, V)$, one can find a slice S and an anti-holomorphic diffeomorphism σ satisfying the following conditions:*

(a) The dimension of S equals the rank of the multiplicity-free space $(G_{\mathbb{C}}, V)$.
(b) σ is involutive, that is, $\sigma^2 = \mathrm{id}$.
(c) There exists an anti-holomorphic involution $\sigma_\sharp$ on $G_{\mathbb{C}}$ such that the Lie algebra of the fixed point group $G_{\mathbb{C}}^{\sigma_\sharp}$ is a normal real form of $\mathrm{Lie}(G_{\mathbb{C}})$ and $\sigma_\sharp$ is a compatible automorphism for σ, namely,

$$\sigma(g \cdot v) = \sigma_\sharp(g) \cdot \sigma(v) \quad (\forall g \in G_{\mathbb{C}},\ \forall v \in V).$$

Next, we consider an application of the classification of strongly visible linear actions to nilpotent orbits.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $G_{\mathbb{C}}$ be a connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$. The Lie group $G_{\mathbb{C}}$ acts on $\mathfrak{g}$ as adjoint representations. Let X be a non-zero nilpotent element of $\mathfrak{g}$. We denote by $\mathcal{N}_X$ the nilpotent $G_{\mathbb{C}}$-orbit $\mathrm{Ad}(G_{\mathbb{C}})X$.

We take a compact real form G_u of $G_{\mathbb{C}}$. Then, we have:

Theorem 3 ([13]). *For a nilpotent orbit $\mathcal{N}_X$, the following four conditions are equivalent:*

(i) $\mathcal{N}_X$ is spherical.
(ii) The height $\mathrm{ht}(\mathcal{N}_X)$ of $\mathcal{N}_X$ is two or three.
(iii) The space $\mathcal{O}(\mathcal{N}_X)$ of holomorphic functions on $\mathcal{N}_X$ is multiplicity-free as a representation of $G_{\mathbb{C}}$.
(iv) The G_u-action on $\mathcal{N}_X$ is strongly visible.

Here, a nilpotent orbit $\mathcal{N}_X$ is spherical if a Borel subgroup of $G_{\mathbb{C}}$ has an open orbit in $\mathcal{N}_X$. Further, we define $\mathrm{ht}(\mathcal{N}_X)$ to be the maximum of eigenvalues of $\mathrm{ad}(H) \in \mathrm{End}(\mathfrak{g})$, where $\{H, X, Y\}$ forms an $\mathfrak{sl}_2$-triple and H is a semisimple element of $\mathfrak{g}$. This definition does not depend on the choice of $\mathfrak{sl}_2$-triples containing X.

The equivalence between (i) and (ii) was proved by Panyushev [9]. Due to the highest weight theory, $\mathcal{O}(\mathcal{N}_X)$ is multiplicity-free if $\mathcal{N}_X$ is spherical. The converse is also true by Vinberg [14]. Moreover, the implication (iv) $\Rightarrow$ (iii) is a special case of propagation theorem of multiplicity-free property (see [6, 7]). The implication (ii) $\Rightarrow$ (iv) is new [13].

Let us see our machinery of the proof for the implication (ii) $\Rightarrow$ (iv). We set $\mathfrak{g}(m) := \{Z \in \mathfrak{g} : \mathrm{ad}(H)Z = mZ\}$ $(m \in \mathbb{Z})$. Then, $\mathfrak{g}$ is written as $\mathfrak{g} = \bigoplus_{m\in\mathbb{Z}} \mathfrak{g}(m)$. We set a complex reductive Lie algebra $\mathfrak{l} := \mathfrak{g}(0)$, and a nilpotent subalgebra $\mathfrak{n}$ of

$\mathfrak{g}$ as

$$\mathfrak{n} := \bigoplus_{m\geq 2} \mathfrak{g}(m).$$

Let $L_{\mathbb{C}}$ be a connected closed subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{l}$, and L_u a compact real form of $L_{\mathbb{C}}$. In this setting, we have:

Lemma 4. *If the L_u-action on $\mathfrak{n}$ is strongly visible, then the G_u-action on $\mathcal{N}_X$ is strongly visible.*

We point out that the strong visibility for linear case induces the strong visibility for noon-linear case of nilpotent orbits.

Thanks to Lemma 4, it is sufficient to consider the L_u-action on $\mathfrak{n}$. Since this action is linear, we apply the result Theorem 1 for the linear case. Then, we have:

Theorem 5 ([13]). *If* $\mathrm{ht}(\mathcal{N}_X) \leq 3$, *then the L_u-action on $\mathfrak{n}$ is strongly visible.*

Hence, the implication (ii) $\Rightarrow$ (iv) follows from Theorem 5 and Lemma 4.

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A generalization of Fulton's conjecture for arbitrary groups

Shrawan Kumar
(joint work with Prakash Belkale and Nicolas Ressayre)

In this talk we prove a generalization of Fulton's conjecture which relates intersection theory on an arbitrary flag variety to invariant theory. Let L be a connected reductive complex algebraic group with a Borel subgroup B_L and maximal torus $H \subset B_L$. The isomorphism classes of finite dimensional irreducible representations of L are parametrized by the set $X(H)^+$ of L-dominant characters of H via the highest weight. For $\lambda \in X(H)^+$, let $V(\lambda) = V_L(\lambda)$ be the corresponding irreducible representation of L with highest weight λ. Define the *Littlewood-Richardson coefficients* $c^{\nu}_{\lambda,\mu}$ by:

$$V(\lambda) \otimes V(\mu) = \sum_{\nu} c^{\nu}_{\lambda,\mu} V(\nu).$$

The following result was conjectured by Fulton and proved by Knutson-Tao-Woodward. (Subsequently, geometric proofs were given by Belkale and Ressayre.)

Theorem 1.1. *Let $L = \mathrm{GL}(r)$ and let $\lambda, \mu, \nu \in X(H)^+$. Then, if $c^{\nu}_{\lambda,\mu} = 1$, we have $c^{n\nu}_{n\lambda,n\mu} = 1$ for every positive integer n.*

(Conversely, if $c^{n\nu}_{n\lambda,n\mu} = 1$ for some positive integer n, then $c^{\nu}_{\lambda,\mu} = 1$. This follows from the saturation theorem of Knutson-Tao.)

For $\lambda, \mu, \nu \in X(H)^+$, observe that the space of $\mathrm{SL}(r)$-invariants $[V(\lambda) \otimes V(\mu) \otimes V(\nu)]^{\mathrm{SL}(r)}$ is isomorphic with the space of $\mathrm{GL}(r)$-invariants $[V(\lambda) \otimes V(\mu) \otimes V(\nu + d\epsilon)]^{\mathrm{GL}(r)}$, for some $d \in \mathbf{Z}$ (where ϵ is the determinant character: $\epsilon(h) = \det(h)$, for $h \in H$). Moreover, if $[V(\lambda) \otimes V(\mu) \otimes V(\nu)]^{\mathrm{GL}(r)}$ is nonzero (for $\lambda, \mu, \nu \in X(H)^+$), then this space coincides with $[V(\lambda) \otimes V(\mu) \otimes V(\nu)]^{\mathrm{SL}(r)}$. Thus, replacing $V(\nu)$ by the dual $V(\nu)^*$, the above theorem is equivalent to the following:

Theorem 1.2. *Let $L = \mathrm{GL}(r)$ and let $\lambda, \mu, \nu \in X(H)^+$. Then, if the dimension $\dim\big([V(\lambda) \otimes V(\mu) \otimes V(\nu)]^{\mathrm{SL}(r)}\big) = 1$, we have $\dim\big([V(n\lambda) \otimes V(n\mu) \otimes V(n\nu)]^{\mathrm{SL}(r)}\big) = 1$, for every positive integer n.*

The direct generalization of the above theorem for an arbitrary reductive L is false. It is also known that the saturation theorem fails for arbitrary reductive groups. It is a challenge to find an appropriate version of the above result for $\mathrm{GL}(r)$ which holds in the setting of general reductive groups.

The aim of this work is to achieve one such generalization. This generalization is a relationship between the intersection theory of homogeneous spaces and the invariant theory. To obtain this generalization, we must first reinterpret the above result for $\mathrm{GL}(r)$ as follows.

Without loss of generality, we only consider the irreducible polynomial representations of $\mathrm{GL}(r)$. These are parametrized by the sequences $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0)$, where we view any such λ as the dominant character $\mathrm{diag}(t_1, \dots, t_r) \mapsto t_1^{\lambda_1} \dots t_r^{\lambda_r}$ of the standard maximal torus consisting of the diagonal matrices in $\mathrm{GL}(r)$. Let $\mathfrak{P}(r)$ be the set of such sequences (also called Young diagrams or partitions) $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0)$ and let $\mathfrak{P}_k(r)$ be the subset of $\mathfrak{P}(r)$ consisting of those partitions λ such that $\lambda_1 \leq k$. Then, the Schubert cells in the Grassmannian $\mathrm{Gr}(r, r+k)$ of r-planes in $\mathbf{C}^{r+k}$ are parametrized by $\mathfrak{P}_k(r)$. For $\lambda \in \mathfrak{P}_k(r)$, let σ_λ be the corresponding Schubert cell and $\bar{\sigma}_\lambda$ its closure. By a classical theorem, the structure constants for the intersection product in $H^*(\mathrm{Gr}(r, r+k), \mathbf{Z})$ in the basis $[\bar{\sigma}_\lambda]$ coincide with the corresponding Littlewood-Richardson coefficients for the representations of $\mathrm{GL}(r)$. Thus, the above theorem can be reinterpreted as follows:

Theorem 1.3. *Let $L = \mathrm{GL}(r)$ and let $\lambda, \mu, \nu \in \mathfrak{P}_k(r)$ (for some $k \geq 1$) be such that the intersection product*

$$[\bar{\sigma}_\lambda] \cdot [\bar{\sigma}_\mu] \cdot [\bar{\sigma}_\nu] = [\bar{\sigma}_{\lambda^o}] \text{ in } H^*(\mathrm{Gr}(r, r+k), \mathbf{Z}),$$

where $\lambda^o := (k \geq \cdots \geq k)$ (r copies of k). Then, $\dim\big([V(n\lambda) \otimes V(n\mu) \otimes V(n\nu)]^{\mathrm{SL}(r)}\big) = 1$, for every positive integer n.

1.1. **Generalization for arbitrary groups.** Our generalization of Fulton's conjecture to an arbitrary reductive group is by considering its equivalent formulation in Theorem 1.3. Moreover, the generalization replaces the intersection theory of the Grassmannians by the deformed product $\odot_0$ in the cohomology of G/P introduced by Belkale-Kumar. The role of the representation theory of $\mathrm{SL}(r)$ is replaced by the representation theory of the semisimple part L^{ss} of the Levi subgroup L of P.

To be more precise, let G be a connected reductive complex algebraic group with a Borel subgroup B and a maximal torus $H \subset B$. Let $P \supseteq B$ be a (standard) parabolic subgroup of G. Let $L \supset H$ be the Levi subgroup of P, B_L the Borel subgroup of L and $L^{ss} = [L, L]$ the semisimple part of L. Let W be the Weyl group of G, W_P the Weyl group of P, and let W^P be the set of minimal length coset representatives in W/W_P. For any $w \in W^P$, let X_w be the corresponding Schubert variety and $[X_w] \in H^{2(\dim G/P - \ell(w))}(G/P, \mathbf{Z})$ the corresponding Poincaré dual class. The following is our main theorem.

Theorem 1.4. *Let G be any connected reductive group and let P be any standard parabolic subgroup. Then, for any $w_1, \ldots, w_s \in W^P$ such that*

$$[X_{w_1}] \odot_0 \cdots \odot_0 [X_{w_s}] = [X_e] \in H^*(G/P), \tag{1.1}$$

we have, for every positive integer n,

$$\dim\Big(\big[V_L(n\chi_{w_1}) \otimes \cdots \otimes V_L(n\chi_{w_s})\big]^{L^{ss}}\Big) = 1, \tag{1.2}$$

where $V_L(\lambda)$ is the irreducible representation of L with highest weight λ and $\chi_w := \rho - 2\rho^L + w^{-1}\rho$ (ρ and ρ^L being the half sum of positive roots of G and L respectively).

Remark 1.5. Let $\mathcal{M}$ be the GIT quotient of $(L/B_L)^s$ by the diagonal action of L^{ss} linearized by $\mathcal{L}(\chi_{w_1}) \boxtimes \cdots \boxtimes \mathcal{L}(\chi_{w_s})$. Then, the conclusion of Theorem 1.4 is equivalent to the rigidity statement that $\mathcal{M} =$ point. Theorem 1.4 can therefore be interpeted as the statement "multiplicity one in intersection theory leads to rigidity in representation theory".

We recall the following proposition due to Belkale-Kumar.

Proposition 1.6. *Let $w_1, \dots, w_s \in W^P$ be such that*

$$[X_{w_1}] \odot_0 \cdots \odot_0 [X_{w_s}] = d[X_e] \in H^*(G/P), \text{ for some } d \neq 0.$$

Then, $m := \dim\Big(H^0\big((L/B_L)^s, \big(\mathcal{L}_P(\chi_{w_1}) \boxtimes \cdots \boxtimes \mathcal{L}_P(\chi_{w_s})\big)_{|(L/B_L)^s}\big)^{L^{ss}}\Big) \neq 0.$

Note that, by the Borel-Weil theorem, for any $w \in W^P$,

$$H^0(L/B_L, \mathcal{L}_P(\chi_w)_{|(L/B_L)}) = V_L(\chi_w)^*.$$

The condition (1.1) can be translated into the statement that a certain map of parameter spaces $X \to Y = (G/B)^s$ appearing in Kleiman's transversality theorem is birational. Here X is the "universal intersection" of Schubert varieties. It is well known that, for any birational map $X \to Y$ between smooth projective varieties, no multiple of the ramification divisor R in X can move even infinitesimally (i.e., the corresponding Hilbert scheme is reduced, and of dimension 0 at nR for every positive integer n). We may therefore conclude that $h^0(X, \mathcal{O}(nR)) = 1$ for every positive integer n. In our situation, X is not smooth, and moreover $H^0(X, \mathcal{O}(nR))$ needs to be connected to the invariant theory. We overcome these difficulties by taking a closer look at the codimension one Schubert cells inside the Schubert varieties.

The proof also brings into focus the largest (standard) parabolic subgroup Q_w acting on a Schubert variety $X_w \subseteq G/P$ (where $w \in W^P$), the open Q_w orbit $Y_w \subseteq X_w$ and the smooth locus $Z_w \subseteq X_w$. The difference $X_w \backslash Z_w$ is of codimension at least two in X_w (since X_w is normal) and can effectively be ignored.

The varieties Y_w give us the link to invariant theory. The difference $Z_w \setminus Y_w$ turns out to be quite subtle. A key result in this work is the intersection $\cap_i g_i Z_{w_i}$ of translates is *non-transverse* 'essentially' at any point which lies in $\big(\cap_{i \neq j} g_i Z_{w_i}\big) \cap g_j(Z_{w_j} \setminus Y_{w_j})$ for some j. This reveals the significance of Q_w in the intersection theory of G/P and, in particular, to the deformed product $\odot_0$. The "complexity" of the varieties $Z_w \backslash Y_w$ can therefore be expected to relate to the deformed product $\odot_0$. Note that by a result of Brion-Polo, if P is a cominuscule maximal parabolic subgroup (in particular, for the Grassmannians $\mathrm{Gr}(r, r+k)$), then $Y_w = Z_w$, and in this case the deformed cohomology product $\odot_0$ coincides with the standard intersection product as well.

In the case of $G = \mathrm{GL}(r+k)$ and $G/P = \mathrm{Gr}(r, r+k)$, if we specialize Theorem 1.4 to $G = \mathrm{GL}(r+k)$, we get Theorem 1.3.

Observe that in the case $G = \mathrm{GL}(r+k)$ and $G/P = \mathrm{Gr}(r, r+k)$, we get the stronger relation $m = d^2$. In general, however, there are no known numerical relations between m and d.

We remark that if we replace the condition (1.1) in Theorem 1.4 by the standard cohomology product, then the conclusion of the theorem is false in general. Also, the converse to Theorem 1.4 is not true in general.

Lüscher-Mack Theory for symmetric Banach-Lie groups

Stéphane Merigon
(joint work with Karl-Hermann Neeb)

The Lüscher-Mack Theorem [LM75] was proved for finite dimensional Lie groups as a tool in Conformal Field Theory. Its generalisation to the context of Banach–Lie groups is motivated by the study of the representation theory of infinite dimensional classical groups.

In the following (G, θ) is a *symmetric Banach–Lie group*, i.e., G is a Banach–Lie group and $\theta \in \mathrm{Aut}(G)$ is an involution. We denote by $(\mathfrak{g}, \theta)$ the corresponding symmetric Lie algebra, write

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$$

for the associated decomposition of $\mathfrak{g}$ and let H denote the identity component of G^θ. We set $g^* = \theta(g)^{-1}$ and assume that G contains a non-empty open $*$-semigroup S which has a polar decomposition

$$S = H \exp W,$$

where W is a open $\mathrm{Ad}(H)$-invariant convex cone in $\mathfrak{q}$. We also assume that the map $H \times W \to S,\ (h, x) \to h \exp x$ is an analytic diffeomorphism. We are interested in non-degenerate strongly continuous $*$-representations of S in a (separable) Hilbert space $\mathcal{H}$. First we observe that, although H is not contained in S, any non-degenerate representation of S extends in a unique fashion to a $*$-representation of the semigroup $S \cup H$. We call a representation *smooth* if its space of smooth vectors is dense.

Proposition 1. *For every non-degenerate $*$-representation $(\rho, \mathcal{H})$ of S there exists a unique unitary representation $\rho_H \colon H \to U(\mathcal{H})$ satisfying*

$$\rho(hs) = \rho_H(h)\rho(s) \quad \text{for} \quad h \in H, s \in S.$$

Moreover $(\rho_H, \mathcal{H})$ is smooth if $(\rho, \mathcal{H})$ is.

Let $(\rho, \mathcal{H})$ be a non-degenerate strongly continuous $*$-representation of S. We obtain for every $x \in W$ a selfadjoint operator

$$\overline{\mathrm{d}\rho}(x)\xi := \frac{d}{dt}\Big|_{t=0} \rho(\exp tx)\xi,$$

the generator of the symmetric one-parameter semigroup ρ_x. It is defined on the subspace $\mathcal{D}(\overline{\mathrm{d}\rho}(x))$ where the derivative exists (Hille–Yosida Theorem). For $x \in \mathfrak{h}$, we likewise write $\overline{\mathrm{d}\rho}(x)$ for the generator of the corresponding strongly continuous unitary one-parameter group $\rho_x(t) := \rho_H(\exp tx)$ (Proposition 1; Stone's Theorem). Our main result is the following:

Theorem 2. *Assume that G_c is a simply connected Lie group with Lie algebra*

$$\mathfrak{g}_c = \mathfrak{h} \oplus i\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}.$$

Let $\rho : S \to B(\mathcal{H})$ be a non-degenerate strongly continuous smooth $*$-representation of S. Then there exists a smooth unitary representation $(\pi, \mathcal{H})$ of G_c on $\mathcal{H}$ whose *space of smooth vectors is contained in* $\mathcal{D}(\overline{\mathrm{d}\rho}(x))$ *for every* $x \in \mathfrak{h} \cup W$*, and whose derived representation satisfies*

$$\mathrm{d}\pi(x+iy) = \overline{\mathrm{d}\rho}(x) + i\,\overline{\mathrm{d}\rho}(y) \text{ for } x \in \mathfrak{h} \text{ and } y \in W. \tag{0.1}$$

The strategy is to define a representation of the Lie algebra $\mathfrak{g}_c$ on a dense domain so that (0.1) is satisfies and then verify that we can use the results in [Mer10] to show that this representation of $\mathfrak{g}_c$ integrates to a representation of G_c.

The representation π of G_c obtained from the Lüscher-Mack Theorem has a remarquable property: the spectrum of the operator $\frac{1}{i}\overline{\mathrm{d}\pi}(x)$ is bounded from above for $x \in iW$. We next give a theorem which at the same time gives a converse to the Lüscher-Mack Theorem (which is new even in the finite dimensional setting) and the existence of Holomorphic extension for a classe of *semibounded representations.*

Let L be a Banach–Lie group with Lie algebra $\mathfrak{l}$. For a smooth unitary representation π of L we consider the map

$$s_\pi : \mathfrak{l} \to \mathbb{R} \cup \{\infty\},\ s_\pi(x) := \sup\left(\mathrm{Spec}(i\overline{\mathrm{d}\pi}(x))\right).$$

Definition 3. (a) The representation π is called semibounded if s_π is bounded on a non-empty open subset of $\mathfrak{l}$. Then

$$W_\pi := \mathrm{int}\{x \in \mathfrak{l} : s_\pi(x) < \infty\}$$

is an open $\mathrm{Ad}(L)$-invariant convex cone in $\mathfrak{l}$ on which the function s_π, being convex, is continuous, in particular locally bounded.

(b) More generaly we say that π is semibounded on the cone $W \subseteq \mathfrak{l}$ if s_π is locally bounded on W.

Our theorem applies to a certain class of cones:

Definition 4. Let W be a cone in $\mathfrak{l}$ which is open in $\mathfrak{q} := \overline{W - W}$ and let $\mathfrak{h}$ be a closed subalegebra of $\mathfrak{g}$.

(a) The cone W is said to be $\mathfrak{h}$-compatible if $[W, W] \subseteq \mathfrak{h}$ and $e^{\mathrm{ad}\,\mathfrak{h}}W \subseteq W$. Then $\mathfrak{g}_c := \mathfrak{h} \oplus i\mathfrak{q}$ is turned in a symmetric Lie algebra by defining the involution to be Id on $\mathfrak{h}$ and $-\mathrm{Id}$ on $i\mathfrak{q}$.

(b) The cone W is then said to be integrable if there exists a symmetric Banach–Lie group (G, θ) with symmetric Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus i\mathfrak{q}$, which contains, setting $H_1 = (G^\theta)_0$ and $g^* = \theta(g)^{-1}$, an open $*$-semigroup S with (analytic) polar decompostion $S = H_1 \exp iW$.
In this case we have by elementary covering theory a Banach–Olshanski semigroup $S_H(iW) = H\,\mathrm{Exp}\,iW$ for each connected Lie group H with Lie algebra $\mathfrak{h}$

Theorem 5. *Let* L *be a Banach Lie group with Lie algebra* $\mathfrak{l}$*. Let* $\mathfrak{h}$ *be a closed complemented Lie subalgebra of* $\mathfrak{l}$ *and* H *be the corresponding integral subgroup in* L*. Let* π *be a smooth representation of* L *which is semibounded on the integrable* $\mathfrak{h}$*-compatible cone* W*. Then the formula*

$$\rho(h\,\mathrm{Exp}\,ix) := \pi(h)e^{i\overline{\mathrm{d}\pi}(x)} \text{ for } h \in H \text{ and } x \in W,$$

defines a strongly continuous smooth $*$*-representation* ρ *of* $S_H(iW) = H\,\mathrm{Exp}\,iW$*.*

Applying the preceding theorem to $\mathfrak{l} = \mathfrak{g}_c = \mathfrak{h} + i\mathfrak{q}$ and $-iW \subseteq i\mathfrak{q}$ we obtain the following converse to the Lüscher-Mack Theorem:

Corollary 6. *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be a symmetric Banach-Lie algebra and W be an integrable $e^{\mathrm{ad}\,\mathfrak{h}}$-invariant open convexe cone in $\mathfrak{q}$. Let G_c be a Banach Lie group with Lie algebra $\mathfrak{g}_c = \mathfrak{h} + i\mathfrak{q} \subseteq \mathfrak{g}_{\mathbb{C}}$ and let H_c be its integral subgroup with Lie algebra $\mathfrak{h}$. Let π be a smooth representation of G_c which is semibounded on iW. Then the formula*

$$\rho(h \operatorname{Exp} x) := \pi(h) e^{-\overline{\mathrm{d}\pi}(x)} \text{ for } h \in H_c \text{ and } x \in W,$$

defines a strongly continuous smooth $*$-representation ρ of $S_{H_c}(W)$.

Let us now assume that the smooth unitary representation $\pi : L \to \mathcal{U}(\mathcal{H})$ is semibounded. Then W_π is open in $\mathfrak{l}$, and $\mathrm{Ad}(H) = e^{\mathrm{ad}\,\mathfrak{l}}$-compatible. The theorem applies to any integrable $\mathrm{Ad}(L)$-invariant open convex cone $W \subseteq W_\pi$. For such a cone the trivialisation $S_L(iW) \times \mathfrak{l}_{\mathbb{C}} \simeq TS_L(iW)$ shows that $S_L(iW)$ carries the structure of a involutive complex Banach semigroup.

Corollary 7. *Let L be a Banach-Lie group with Lie algebra $\mathfrak{l}$. Let π be a smooth semibounded representation of L, and let $W \subseteq W_\pi$ be an integrable $\mathrm{Ad}(L)$-invariant open convex cone. Then the formula*

$$\rho(l \operatorname{Exp} ix) := \pi(l) e^{i\overline{\mathrm{d}\pi}(x)} \text{ for } l \in L \text{ and } x \in W,$$

defines a holomorphic $*$-representation ρ of $S_L(iW) = L \exp iW$. In particular the vectors in $\rho(S_L(iW))\mathcal{H}$ are analytic for π.

This theorem generalises Olshanski's Holomorphic Extension Theorem for unitary highest weight representations to the Banach-Lie setting. We should note here that the proof in the finite dimensional case relies on the existence of analytic vectors for the representation: One proves first that ρ is holomorphic, and then the multiplicativity of ρ is proved by analytic continuation. In our approach the multiplicativity of ρ follows from the (assumed) existence of smooth vectors and then the its analyticity follows as a bonus from its multiplicativity.

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Homotopy groups of topological spaces containing a dense directed union of manifolds

Helge Glöckner

Consider a Hausdorff topological space M that is a union $M = \bigcup_{n=1}^{\infty} M_n$ of topological spaces $M_1 \subseteq M_2 \subseteq \cdots$, such that all inclusion maps $M_n \to M_{n+1}$ and $M_n \to M$ are continuous. We are interested in conditions ensuring that the homotopy groups $\pi_k(M, p)$ of M at $p \in M$ (for $k = 1, 2, \ldots$) can be expressed as the direct limit of those of the spaces M_n (as $n \to \infty$), i.e.,

$$\pi_k(M, p) = \lim_{\rightarrow} \pi_k(M_n, p)\,. \tag{0.1}$$

If M is *compactly regular* with respect to the sequence $M_1 \subseteq M_2 \subseteq \cdots$ in the sense that each compact subset K of M is contained in some M_n and M_n induces the same topology on K as M, then it is easy to see that (0.1) holds. In the following, alternative conditions will be described which are independent of compact regularity. Some of them will even work if $\bigcup_{n=1}^{\infty} M_n$ is merely dense in M, or if a (possibly uncountable) directed set of topological spaces is considered instead of an ascending sequence.

In the case of ascending unions, assume now that M and each M_n is a topological manifold in the sense that each point x has an open neighbourhood U homeomorphic to an open subset V of some topological vector space (as usual, such homeomorphisms $\phi\colon U \to V$ will be called charts around x).

So far, our setting is still too wide (for example, M might be non-discrete but each M_n discrete), and it is clearly necessary to assume stronger links between the topologies on M and the M_n. To this end, let us call a chart $\phi\colon U \to V \subseteq E$ of M a *weak direct limit chart* with respect to $M_1 \subseteq M_2 \subseteq \cdots$ (in the sense of topological manifolds) if there exists a positive integer n_0 and charts $\phi_n\colon U_n \to V_n \subseteq E_n$ of M_n for $n \geq n_0$ such that $U = \bigcup_{n=n_0}^{\infty} U_n$, $U_n \subseteq U_{n+1}$, $E_n \subseteq E$ and $E_n \subseteq E_{n+1}$, $\phi|_{U_n} = \phi_n$, and all inclusion maps $E_n \to E$ and $E_n \to E_{n+1}$ are continuous linear. Then the following holds (see [5]):

Theorem. *If $M = \bigcup_{n=1}^{\infty} M_n$ admits weak direct limit charts around each point, then* (0.1) *holds for each $p \in M$ and natural number k.*

We mention that many infinite-dimensional Lie groups G of interest are ascending unions $G = \bigcup_{n=1}^{\infty} G_n$ of (finite- or infinite-dimensional) Lie groups G_n, such that all inclusion maps $G_n \to G$ and $G_n \to G_{n+1}$ are smooth homomorphisms. In this case, G admits weak direct limit charts around each point if and only if it admits a weak direct limit chart around the neutral element $1 \in G$ (see [5]). This condition is often easily checked, e.g. for the group $\mathrm{Diff}_c(M)$ of compactly supported smooth diffeomorphisms of a σ-compact finite-dimensional manifold M and its subgroups $\mathrm{Diff}_{K_n}(M)$ of diffeomorphisms supported in K_n, for an exhaustion $K_1 \subseteq K_2 \subseteq \cdots$ of M by compact sets. Likewise for the group $C^\infty_c(M, H) = \bigcup_{n=1}^{\infty} C^\infty_{K_n}(M, H)$ of compactly supported smooth maps from M to a Lie group H; for weak direct products $\bigoplus_{n=1}^{\infty} H_n := \bigcup_{n=1}^{\infty} H_1 \times \cdots \times H_n$ of Lie groups H_n; and for all other prime examples of ascending unions usually considered (see [5]). By construction of the differentiable structure, also the direct limit Lie groups $G = \bigcup_{n=1}^{\infty} G_n$ to ascending sequences $G_1 \subseteq G_2 \subseteq \cdots$ of finite-dimensional Lie groups (as constructed in [6] and [3]) always admit direct limit charts. The same applies to the Lie group structures on ascending unions of Banach-Lie groups recently constructed in [2].

We mention that, in the setting of infinite-dimensional Lie groups, weak direct limit charts (in the sense of smooth manifolds) and a certain stronger concept of 'direct limit charts' were introduced in [4] and used there to study the direct limit properties of Lie groups G which are an ascending union $G = \bigcup_{n=1}^{\infty} G_n$ of Lie groups.

A model for farther-reaching results is the following classical fact concerning the homotopy groups of open subsets of locally convex spaces (cf. [8]).

Theorem. *Let E be a locally convex topological vector space, $U \subseteq E$ be an open subset, $E_\infty \subseteq E$ be a dense vector subspace, $p \in E_\infty$ and $\mathcal{F}$ be the set of all finite-dimensional vector subspaces F of E_∞ such that $p \in F$. Then $\pi_k(U,p) = \varinjlim \pi_k(U \cap F, p)$ for each $k = 1, 2, \ldots$, for F ranging through the directed set $(\mathcal{F}, \subseteq)$.*

Consider a directed set $(A, \leq)$ and topological manifolds M and M_α for $\alpha \in A$ (modelled on topological vector spaces) such that $M_\alpha \subseteq M$, $M_\alpha \subseteq M_\beta$ if $\alpha \leq \beta \in A$, all inclusion maps $M_\alpha \to M$ and $M_\alpha \to M_\beta$ are continuous, and the union $M_\infty := \bigcup_{\alpha \in A} M_\alpha$ is dense in M. In [5] one finds the definition of a certain substitute for 'weak direct limit charts' in the current wider setting, the so-called 'well-filled' charts. The definition is too complicated to be repeated here. But special cases of well-filled charts are charts $\phi\colon U \to V \subseteq E$ of M such that, for suitable charts $\phi_\alpha\colon U_\alpha \to V_\alpha \subseteq E_\alpha$ for $\alpha \in A$ with $\alpha \geq \alpha_0$, we have that $E_\alpha \subseteq E$, the inclusion maps $E_\alpha \to E$ and $E_\alpha \to E_\beta$ are continuous linear (for $\alpha \leq \beta$), $U \cap M_\alpha = U_\alpha$, $V \cap E_\alpha = V_\alpha$ and $\phi_\alpha = \phi|_{U_\alpha}$. The following non-linear analogue of Palais' Theorem was obtained in [5]:

Theorem. *Assume that the directed union $M_\infty = \bigcup_{\alpha \in A} M_\alpha$ is dense in M. If M admits well-filled charts around each point, then $\pi_k(M,p) = \varinjlim \pi_k(M_\alpha, p)$ for each $p \in M_\infty$ and $k = 1, 2, \ldots$.*

Let m be a positive integer and H be a Lie group modelled on a locally convex space E. Then a Lie group $\mathcal{S}(\mathbb{R}^m, H)$ of 'rapidly decreasing' smooth H-valued maps can be defined, which is modelled on the Schwartz space $\mathcal{S}(\mathbb{R}^m, E)$ of rapidly decreasing smooth E-valued maps on $\mathbb{R}^m$ (see [1], [9]). Using the preceding theorem, on finds that $\pi_k \mathcal{S}(\mathbb{R}^m, H) = \pi_{k+m}(H)$ for $k = 1, 2, \ldots$ (see [5]). This had been conjectured in [1] and was open since 1981.

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