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Stochastic Analysis in Finance and Insurance

Organised by
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January 23rd – January 29th, 2011

Abstract. This workshop brought together leading experts and a large number of younger researchers in stochastic analysis and mathematical finance from all over the world. During a highly intense week, participants exchanged during talks and discussions many ideas and laid foundations for new collaborations and further developments in the field.

Mathematics Subject Classification (2000): 60Gxx, 60Hxx, 91Bxx.

Introduction by the Organisers

The workshop *Stochastic Analysis in Finance and Insurance*, organised by Dmitry Kramkov (Pittsburgh), Martin Schweizer (Zürich) and Nizar Touzi (Paris) was held January 23rd – January 29th, 2011. The meeting had a total of 53 participants from all over the world with a deliberately chosen mix of more experienced researchers and many younger participants.

During the five days, there were a total of 24 talks with many lively interactions and discussions. In addition, there were a historical lecture and two blocks of short communications, as will be explained below.

The topics presented in the talks covered a very wide spectrum. Major developments included a focus on new statistical problems, new mathematical and modelling issues arising out of and in connection with the recent financial crisis, and as always a number of foundational questions. To stimulate discussions and maximise interactions, talks were deliberately not organised into groups by major topics. A short overview of the talks given day by day looks as follows.

Philip Protter in the first talk of the workshop presented ideas on how one could discover financial bubbles in real time, combining ideas from local martingale modelling with statistical tools. *Marcel Nutz* presented new results on G-expectations in order to study markets with uncertainty about the volatility of assets. *Jean Jacod* gave an overview of recent developments in statistical problems for financial data and highlighted the difficulties coming from jumps in prices. *Christian Bender* introduced the concept of simple arbitrage with the goal of enlarging the class of feasible models by reducing arbitrage conditions to practically realistic assumptions. *Matheus Grasselli* presented a mathematical description of a model introduced by the economist Hyman Minsky in order to explain asset price bubbles from basic economic considerations. Finally, *Sergey Nadtochiy* explained the ideas behind forward performance processes to model optimal investment behaviour and showed in a class of examples how this leads to ill-posed Hamilton–Jacobi–Bellman equations.

Albert Shiryaev started the second day with an example of a non-classical testing problem for Brownian motion with drift, involving three instead of the usual two hypotheses. *Christoph Frei* gave examples of multidimensional quadratic backward stochastic differential equations having (in contrast to the one-dimensional case) no solution, and explained how these equations come up and can be used in connection with equilibrium problems in financial markets. *Peter Tankov* presented limit results for time-changed Lévy processes sampled at hitting times, instead of at fixed times, and showed how these can be used in a financial context. *Christoph Czichowsky* gave a new formulation for the classical Markowitz problem to overcome the well-known time-inconsistency problems associated with that criterion, and showed by relating discrete- and continuous-time theory that the new formulation is both natural and mathematically interesting. *Ronnie Sircar* used stochastic differential games and the associated Hamilton–Jacobi–Bellman equations to discuss the approaches by Bertrand and Cournot to study oligopolistic markets. At the end of the day, *Roger Lee* presented an effective mechanism to generate asymptotic expansions of arbitrarily high order for implied volatility.

On Wednesday, *David Hobson* presented new model-independent bounds for variance swaps with the help of Skorokhod embedding results. *Johannes Muhle-Karbe* gave new asymptotic results for portfolio optimisation with transaction costs by exploiting the recently developed idea of shadow prices. In addition, there were a number of short communications in a newly introduced format. Each presenter had 5 minutes to explain his result, which were then followed by 5 minutes of questions and discussion. This idea of explaining in a nutshell some current problems or results met with enormous success; the list of volunteers for giving a short presentation very quickly grew to a total of 17 names, and the corresponding talks were scheduled on Wednesday morning and Thursday morning. Wednesday afternoon was then reserved for the traditional excursion, which went to Oberwolfach Kirche instead of St. Roman because there was still quite a lot of snow and many tracks on the hills were very slippery.

Thursday started with *Luciano Campi* who presented a structural model for pricing and hedging derivatives in energy markets, a topic of increasing practical importance in recent years. *Jin Ma* used a system of interacting stochastic differential equations to describe possible defaults of correlated assets, and proved a law of large numbers for self-exciting dynamics via a fixed-point argument. A second block of short communications followed, leading again to intense discussions that continued into the afternoon and in the evenings. *Mete Soner* then gave new existence and uniqueness results for second order backward stochastic differential equations, a probabilistic analogue to a class of fully nonlinear partial differential equations. *Kasper Larsen* showed how a number of asset pricing puzzles from finance can be explained, via a clever construction, by equilibria in Brownian-driven but incomplete financial markets. Finally, *Albert Shiryaev* gave a historical talk in memory of the recently deceased Anatoli V. Skorokhod, one of the great Russian probabilists born in the 20th century.

On the last day, *Tahir Choulli* presented new ideas and results in connection with defaultable markets; in mathematical terms, this amounts to studying the behaviour of stochastic processes before and after a random time, and this leads to some quite challenging new problems. *Josef Teichmann* discussed affine processes and their applications in mathematical finance, focusing in particular on regularity and filtering questions. Complementing an earlier talk, *Peter Friz* derived new expansion results for the Heston model, one of the workhorses in practical applications of option pricing. *Jan Obłoj* studied the inverse problem of recovering the preferences of financial agents from their observed actions and showed that uniqueness as well as nonuniqueness can happen, depending on the setting. *Mihai Sîrbu* introduced a model for high-watermark fees in hedge fund investments and explained how to fruitfully use the Skorokhod equation in that context. Finally, *Freddy Delbaen* gave a new, more structural proof for the representation of the penalty function in time-consistent monetary utilities.

Like in the workshop three years before, there were an enormous number of discussions, interactions and exchanges. Everyone felt privileged to be able to spend a highly productive and creative week at the unique place that has been created in Oberwolfach and to profit from the excellent infrastructure, support and scientific environment. In particular, the younger participants and first-time visitors to Oberwolfach unanimously said that the actual experience of the workshop and the overall scientific atmosphere still exceeded their already high anticipations.

As organisers and on behalf of all participants, we want to express our gratitude to the Mathematisches Forschungsinstitut Oberwolfach for giving us the opportunity of having this very successful workshop, and we hope that we shall be able to come back at some time in the future.

Dmitry Kramkov
Martin Schweizer
Nizar Touzi

Workshop: Stochastic Analysis in Finance and Insurance

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Abstracts

Simple arbitrage

Christian Bender

We characterize absence of arbitrage with simple trading strategies in a discounted market with a constant bond and a stock. We suppose that a right-continuous adapted process X_t, $0 \leq t < \infty$, on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, P)$ (satisfying the usual conditions) models a discounted stock price. Recall that by applying a simple trading strategy, the portfolio is only changed at finitely many stopping times, i.e. a simple strategy is a stochastic process of the form

$$\Phi_t = \phi_0 \mathbf{1}_{\{0\}}(t) + \sum_{j=0}^{n-1} \phi_j \mathbf{1}_{(\tau_j, \tau_{j+1}]},$$

where $n \in \mathbb{N}$, $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n$ are a.s. finite stopping times with respect to $(\mathcal{F}_t)$ and the ϕ_j are $\mathcal{F}_{\tau_j}$-measurable real random variables. Φ_t represents the number of stocks held by an investor at time t. Given such a strategy Φ and zero initial endowment, the self-financing condition implies that the investor's wealth at time t is given by

$$V_t(\Phi) = \sum_{j=0}^{n-1} \Phi_{\tau_{j+1}} (X_{t\wedge\tau_{j+1}} - X_{t\wedge\tau_j}).$$

Φ is called a *simple arbitrage* if Φ is a simple strategy, $V_\infty(\Phi) := \lim_{t\to\infty} V_t(\Phi) \geq 0$ and $P(V_\infty(\Phi) > 0) > 0$. As usual, we do not impose the nds-admissibility on simple strategies, as doubling schemes cannot be implemented with finitely many trades only. It is well known that the existence of an equivalent local martingale measure is neither necessary nor sufficient for absence of simple arbitrage, see e.g. Delbaen and Schachermayer [1].

As a first result we show that existence of a simple arbitrage implies existence of one of two particularly favorable types of arbitrage: a 0-admissible arbitrage where the investor does not run into losses while waiting for a riskless gain, or an obvious arbitrage which promises a minimum riskless gain of some $\epsilon > 0$, if the investor trades at all.

More precisely, suppose X admits simple arbitrage. Then there are two a.s. finite stopping times $\sigma \leq \tau$ with $P(\sigma < \tau) > 0$ such that
(i) $\Phi_t = \mathbf{1}_{(\sigma,\tau]}$ or $\Phi_t = -\mathbf{1}_{(\sigma,\tau]}$ is an *obvious arbitrage*, i.e. there is an $\epsilon > 0$ such that

$$V_\infty(\Phi) \geq \epsilon \quad \text{on } \{\sigma < \tau\}$$

or
(ii) $\Phi_t = \mathbf{1}_{(\sigma,\tau]}$ or $\Phi_t = -\mathbf{1}_{(\sigma,\tau]}$ is a *0-admissible arbitrage*, i.e. it is an arbitrage and $V_t(\Phi) \geq 0$ for every $t \geq 0$. Moreover, in case (ii), σ and τ can be chosen as bounded stopping times.

For stock price processes X with continuous paths, a sufficient condition for absence of obvious arbitrage on a finite trading horizon is the well-studied property of conditional full support of the log-prices, see e.g. [2, 3, 4]. Processes which enjoy the conditional full support property include log-prices of many stochastic volatility models and local volatility models, but also fractional Brownian motion and mixed fractional Brownian motion.

Therefore we focus on the study of absence of 0-admissible simple arbitrage. We derive the following sufficient condition: Suppose $X = M + Y$, where M is a continuous $(\mathcal{F}_t)$-local martingale and Y is an $(\mathcal{F}_t)$-adapted processes which is locally 1/2-Hölder continuous with respect to the quadratic variation $\langle M \rangle$ of M, in the following sense: For every $K > 0$ there is a non-negative, a.s. finite random variable C_K such that

$$\forall_{0 \le t \le s \le K} \ |Y_s - Y_t| \le C_K |\langle M \rangle_s - \langle M \rangle_t|^{1/2}.$$

Then X does not admit a 0-admissible simple arbitrage.

The proof can be decomposed into three steps:

(1) A continuous process X does not admit 0-admissible simple arbitrage if and only if for every a.s. finite stopping time σ

$$\inf\{t \ge \sigma \mid X_t > X_\sigma\} = \inf\{t \ge \sigma \mid X_t < X_\sigma\},$$

i.e. whenever the stock price moves away from level X_σ, it crosses this level 'immediately'.

(2) The property in (1) is then proved for the case that the local martingale is a Brownian motion, making use of the law of the iterated logarithm.

(3) The general case can finally be derived by a time change argument applying the Dambis–Dubins–Schwarz theorem.

The results can then be combined to prove absence of simple arbitrage for many 'mixed' models on a finite trading horizon, i.e. some standard models (stochastic vol such as the Heston model, local vol), whose log-prices are perturbed by adding an independent continuous process which is 1/2-Hölder continuous on compacts. In particular, absence of simple arbitrage on a finite trading horizon holds for a mixed fractional Brownian motion with Hurst parameter $H > 1/2$, i.e. the sum of a Brownian motion and an independent fractional Brownian motion, which is known to be not a semimartingale if the Hurst parameter satisfies $H \in (1/2, 3/4]$.

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A structural risk-neutral model for pricing and hedging power derivatives

Luciano Campi
(joint work with R. Aïd, N. Langrené)

This talk aims to contribute to the development of an electricity price model that can provide explicit or semi-explicit formulae for European derivatives on electricity markets. Since the beginning of the liberalization process of electricity markets in the 90s in Europe and in the USA, there has been an important research effort devoted to electricity price modelling for pricing derivatives. Due to the non-storable nature of electricity, it was — and still is — a challenge to reach a completely satisfying methodology that would suit the needs of trading desks: a realistic and robust model, computational tractability of prices and Greeks, consistency with market data. Two main standard approaches have usually been used to face this problem. The first approach consists in modelling directly the forward curve dynamics and deducing the spot price as a futures with immediate delivery. Belonging to this approach are e.g. [12] and [7]. This approach is pragmatic in the sense that it models the prices of the available hedging instruments. However, it makes difficult to capture the right dependencies between fuels and electricity prices (without cointegration). The second approach starts from a spot price model to deduce futures price as the expectation of the spot under a risk-neutral probability. The main benefit of this approach is that it provides a consistent framework for all possible derivatives. This approach has been successfully applied to commodities in the seminal work of Schwartz [22]. Its main drawback is that it generally leads to complex computations for prices of electricity derivatives. Most of the authors following this approach use an exogenous dynamics for the electricity spot price [14, 5, 9, 18, 10, 6, 8, 16] and only a few try to deduce futures and option prices through an equilibrium model or through a model including a price formation mechanism [20, 11, 21, 19, 2].

The main contribution of this work is to provide analytical formulae for electricity futures and semi-explicit expressions for European options in an electricity spot price model that includes demand and capacities as well as fuel dynamics. Modeling the dependencies between fuels and electricity is of great importance for spread options evaluation. To our knowledge, this is the first attempt performed in that direction.

Concerning the use of an equilibrium model or a price mechanism for pricing electricity derivatives, the closest work to ours can be found in [20, 21, 11, 19]. It has been recognized that the mechanism leading to the electricity spot price was too complex to allow for a complete modelling that would fit the constraints of derivatives pricing. The simplest one is maybe Barlow's model [3] where the price is determined by the matching of a simple parametric offer curve and a random demand. Many authors have then derived a reduced equilibrium model for electricity prices in this spirit [17, 13]. In [20], electricity dependency on fuel prices is taken into account by modelling directly the dynamic of the marginal fuel.

The authors manage to provide the partial differential equation and its boundary conditions for the price of an European derivative. The approach followed by [11] and [19] is quite similar. Therein, the price is modelled as an exponential of a linear combination of demand and capacity. In general, it is difficult to introduce in the same framework the dependency of electricity spot price from fuels and at the same time its dependency on demand and capacity. Dependency among fuels is generally captured by a simple correlation among Ornstein-Uhlenbeck processes as in [15] or by cointegration method as in [4].

Here, we start from the marginal price model developed in [2] and enrich it substantially to take into account how the margin capacity uncertainty contributes to futures prices. In order to include the biggest price spikes in our model, we introduce a multiplying factor allowing the electricity spot price to deviate from the marginal fuel price when demand gets closer to the capacity limit. Since electricity is a non-storable commodity, this factor accounts directly for the scarcity of production capacity. Although such an additional feature makes the model more complex, we can still provide closed form formulae for futures prices. Under this model, any electricity futures contract behaves almost as a portfolio of futures contracts on fuels as long as the product is far from delivery. In contrast, near delivery, electricity futures prices are determined by the scarcity rent, i.e. demand and capacity uncertainties.

The talk is based on the joint work [1].

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New developments for defaultable markets

Tahir Choulli

This talk is based on works in progress, [1] and [3]. Precisely, we investigate some stochastic structures, mean variance hedging problems, and some non-arbitrage concepts for defaultable markets. From the large existing literature about the mean-variance hedging problem and the local-risk minimization problem, we can conclude that the solutions to these problems are based *essentially* on two main issues. The first issue is a sort of non-arbitrage condition on the market model and is called *structure conditions*, while the second issue is the Föllmer–Schweizer decomposition (FS decomposition hereafter). This decomposition is a natural extension of the Galtchouk–Kunita–Watanabe decomposition to the semimartingale framework. This explains our interest in these two problems (structure conditions and the FS decomposition) for defaultable markets. We start by illustrating these two problems on a simple market model with default for which the immersion property fails under any equivalent probability measure. Hence, for this example, the existing literature about the FS decomposition and/or the no free lunch with vanishing risk (NFLVR hereafter) for defaultable markets cannot be applied. This example motivates our investigation of the defaultable markets without assuming the immersion property. For all the problems (i.e. the FS decomposition, structure conditions, and non-arbitrage) that we address in [1] and [3], we proceed by distinguishing what happens before the default time and after the default time. We provide necessary and sufficient conditions on the default such that the structure conditions are preserved and/or the FS decomposition exists for the progressively enlarged filtration that makes the default time a stopping time. Furthermore, we describe the components of the FS decomposition under the enlarged filtration in terms of those obtained for the "public" filtration and vice versa.

We give sufficient conditions on the default such that NFLVR and/or non-arbitrage are preserved for the defaultable market. Here, again, we distinguish the case of before the default time and the case of after the default. The key ideas behind these results lie in investigating the variation of a number of stochastic tools with respect to a precise additional uncertainty represented by default time. In fact, we describe how stopping times, optional stochastic processes, local martingales, semimartingales, and local martingale orthogonality with respect to the enlarged filtration can be expressed in terms of the same stochastic concepts for the "public" filtration, respectively.

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Time-consistent mean-variance portfolio selection in discrete and continuous time

Christoph Czichowsky

Viewed as a family of conditional optimisation problems, *mean-variance portfolio selection (MVPS)* is time-inconsistent in the sense that it does not satisfy Bellman's optimality principle: If a strategy is optimal for the mean-variance criterion at the initial time optimised over the entire time interval, this strategy is no longer optimal for the conditional criterion on any remaining time interval. Therefore the usual dynamic programming approach fails to produce a time-consistent dynamic formulation of the optimisation problem. To overcome this, one has to use a weaker optimality criterion which consists of optimising the strategy only locally. This has recently been done in Markovian settings by Basak and Chabakauri [1] for MVPS and Björk and Murgoci [2] for generic time-inconsistent stochastic optimal control problems including MVPS. By exploiting that particular framework, they could characterise the local notion of optimality by a system of partial differential equations (PDEs).

In this talk (which is based on [3]), we develop such a local notion of optimality, called *local mean-variance efficiency*, for the conditional mean-variance problem in a general semimartingale setting where alternative characterisations in terms of PDEs are not available in general. We start in discrete time where this is straightforward, and then obtain the natural extension to continuous time which is similar to the notion of local risk minimisation in continuous time introduced by Schweizer in [6]. Our formulation in discrete as well as in continuous time embeds time-consistent mean-variance portfolio selection in a natural way into the already existing quadratic optimisation problems in mathematical finance, i.e. the Markowitz problem, mean-variance hedging, and local risk minimisation; compare [4] and [5]. Moreover, we obtain an alternative characterisation of the optimal strategy in terms of the structure condition and the Föllmer–Schweizer decomposition of the mean-variance tradeoff, which gives necessary and sufficient conditions for the existence of a solution. The link to the Föllmer–Schweizer decomposition allows us to exploit known results to give a recipe to obtain the solution in concrete models. Since the ingredients for this recipe can be obtained directly from the canonical decomposition of the asset price process, this can be seen as the analogue to the explicit solution in the one-period case. Additionally, this gives an intuitive interpretation of the optimal strategy as follows. On the one hand, the investor maximises the conditional mean-variance criterion in a myopic way one step ahead. In the multi-period setting, this generates a risk represented by the mean-variance tradeoff process which he then minimises on the other hand by local risk minimisation. Finally, using the alternative characterisation of the optimal strategy allows us to justify the continuous-time formulation by showing that it coincides with the continuous-time limit of the discrete-time formulation.

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The representation of the penalty function for a monetary utility function in a Brownian filtration: a functional analytic proof

Freddy Delbaen

Let u be a time consistent concave monetary utility function defined on L^∞ and based on the filtration generated by a d-dimensional Brownian motion W. The time interval is supposed to be finite, $[0,T]$ with $T < \infty$. We assume that for $\xi \in L^\infty$, the process $u(\xi)$ is the càdlàg version. Together with u we get the penalty function c which is defined for all probability measures that are absolutely continuous with respect to $\mathbb{P}$. We identify such a probability $\mathbb{Q}$ with its density function $L_t = \mathbb{E}_\mathbb{P}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t\right]$. This process can be written as a stochastic integral $L = \mathcal{E}(q \cdot W)$ where q is predictable. The admissibility sets are defined as follows: if $\sigma \le \tau \le T$ are stopping times, then $\mathcal{A}_{\sigma,\tau} = \{\xi \in L^\infty(\mathcal{F}_\tau), u_\sigma(\xi) \ge 0\}$. The penalty function or better process is defined as

$$c_{\sigma,\tau}(\mathbb{Q}) = \text{ess.sup}\{\mathbb{E}_\mathbb{Q}[-\xi \mid \mathcal{F}_\sigma], u_\tau(\xi) \ge 0, \xi \in L^\infty(\mathcal{F}_\tau)\}.$$

The process $(c_{t,T}(\mathbb{Q}))$ admits a càdlàg version. We assume that $c_0(\mathbb{Q}) = 0$. The time consistency is equivalent to either of the following conditions:

1) for all $\sigma \le \tau$ and all $\mathbb{Q} \ll \mathbb{P}$: $c_{\sigma,T}(\mathbb{Q}) = c_{\sigma,\tau}(\mathbb{Q}) + \mathbb{E}_\mathbb{Q}[c_{\tau,T}(\mathbb{Q}) \mid \mathcal{F}_\sigma]$, or
2) for all $\sigma \le \tau$ and all $\mathbb{Q} \ll \mathbb{P}$: $\mathcal{A}_{\sigma,T} = \mathcal{A}_{\sigma,\tau} + \mathcal{A}_{\tau,T}$.

These properties play a fundamental role in showing the following

Suppose that u_0 is Fatou and time consistent. Suppose that the filtration $\mathbb{F}$ is given by a d-dimensional Brownian motion W, defined on the bounded time interval $[0,T]$. Suppose that $c_0(\mathbb{P}) = 0$. Under these assumptions, there is a function

$$f\colon \mathbb{R}^d \times [0,T] \times \Omega \to \overline{\mathbb{R}_+},$$

such that

(1) *for each $(t,\omega) \in [0,T] \times \Omega$, the function $f(.,t,\omega)$ is convex on $\mathbb{R}^d$,*
(2) *for each $(t,\omega) \in [0,T] \times \Omega$, $f(0,t,\omega) = 0$,*
(3) *for each $x \in \mathbb{R}^d$, the function $f(x,.,.)$ is predictable,*
(4) *the function f is measurable for $\mathcal{B} \times \mathcal{P}$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^d$ and $\mathcal{P}$ is the predictable σ-algebra on $[0,T] \times \Omega$,*
(5) *for each $\mathbb{Q} \ll \mathbb{P}$ we have*

$$c_0(\mathbb{Q}) = \mathbb{E}_\mathbb{P}\left[\int_0^T f(q_t(.),t,.)\,dt\right].$$

The proof is done in different steps. We only need to prove it for $\mathbb{Q} \sim \mathbb{P}$. The first step is to show that the process $(c_{t,T}(\mathbb{Q}))$ is a $\mathbb{Q}$-supermartingale of class D. It is therefore represented by a $\mathbb{Q}$-potential α. The next step is to show that the measure $d\alpha$ is the supremum in the lattice of stochastic measures of the measures $qZ^\xi\,dt - dA^\xi_t$, where $du_t(\xi) = dA^\xi_t - Z^\xi\,dW_t$ is the Doob–Meyer decomposition of the submartingale $u(\xi)$. From this it already follows that the measure $d\alpha$ is absolutely continuous with respect to Lebesgue measure. The structure of the system $\mathcal{A}_{\sigma,\tau}$ is needed to show that the pointwise supremum of the measures is the same as the supremum calculated on the space $\Omega \times [0, T]$.

The present proof is more structural than the original proof of Delbaen, Peng and Rosazza-Gianin. That proof was based on a truncation argument, reducing the problem to the dominated case.

Equilibrium considerations in a financial market with interacting investors

Christoph Frei
(joint work with Gonçalo dos Reis)

While trading on a financial market, the agents we consider take the performance of their peers into account. In more detail, our model in [2] consists of n agents who can trade in the same market subject to some individual restrictions. Each agent measures her preferences by an exponential utility function and chooses a trading strategy that maximizes the expected utility of a weighted sum consisting of three components: an individual claim, the absolute performance and the relative performance compared to the other $n-1$ agents. The question is whether there exists a Nash equilibrium in the sense that there are individual optimal strategies simultaneously for all agents. We make the usual assumption that the financial market is big enough so that the trading of our investors does not affect the price of the assets.

A model similar to ours has been recently studied in the PhD thesis of Espinosa [1], but in the absence of individual claims and with assets modelled as Itô processes with deterministic coefficients. These assumptions crucially simplify the analysis and enable Espinosa [1] to show the existence of a Nash equilibrium. He also studies its form, while our focus is on existence questions in a more general setting and interpretations as well as possible alternatives in the absence of a Nash equilibrium. We obtain existence and uniqueness in a stochastic framework if all agents are faced with the same trading restrictions. Under different investment constraints, however, an agent may ruin another one by solely maximizing her individual utility. Different investment possibilities may allow an agent to follow a risky and beneficial strategy, and thereby negatively affect another agent who benchmarks her own strategy against the less restricted one. The bankruptcy of the agents can be avoided if agents with more investment possibilities are showing solidarity and willingness to waive some expected utility. This leads to the existence of an approximate equilibrium, in the sense that there exists an ϵ-equilibrium for every $\epsilon > 0$. In an ϵ-equilibrium, every agent uses a strategy whose outcome is at most ϵ away from that of the individual best response. Behind this well-known concept stands the idea that agents may not care about very small improvements. Our setting brings up the additional aspect of solidarity: by accepting a small deduction from the optimum, an agent can help to save the others from failure.

This financial interpretation goes along with an interesting mathematical basis, which is due to the correspondence between an equilibrium of the investment problem and a solution of a certain backward stochastic differential equation (BSDE). We present an illustrative counterexample which is easy to understand and shows that — and why — general multidimensional quadratic BSDEs do not have solutions despite bounded terminal conditions and in contrast to the one-dimensional case. This also gives a mathematical flavour for the absence of an equilibrium in the financial model, because there is a correspondence between existence of equilibria in our financial model and solutions to such a BSDE.

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On refined density and smile expansion in the Heston model

Peter Karl Friz
(joint work with Stefan Gerhold, Archil Gulisashvili, Stephan Sturm)

It is known that Heston's stochastic volatility model exhibits moment explosion, and that the critical moment can be obtained by solving (numerically) a simple equation (e.g. [2, 8]). This yields a leading order expansion for the implied volatility at large strikes, thanks to Roger Lee's moment formula [7]. Motivated by recent "tail-wing" refinements [1, 6] of this moment formula, we first derive a novel tail expansion for the Heston density, sharpening previous work of Dragulescu and Yakovenko [3], and then show the validity of a refined expansion where all constants are explicitly known as functions of the critical moment, the Heston model parameters, spot vol and time-to-maturity. In the case of the "zero-correlation" Heston model, such an expansion was derived by Gulisashvili and Stein [6]. Our methods and results may prove useful beyond the Heston model; the entire quantitative analysis is based on affine principles [8]. At no point do we need knowledge of the (explicit, but cumbersome) closed form expression of the Fourier transform of the log-price (equivalently: Mellin transform of the price); what matters is that these transforms satisfy ordinary differential equations of Riccati type, and our (saddle) point analysis makes essential use of higher order Euler estimates reminiscent of rough path analysis [4, 5]. Secondly, our analysis reveals a new parameter ("critical slope"), defined in a model free manner, which drives the second and higher order terms in tail- and implied volatility expansions.

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In search of the Minsky moment: credit dynamics, asset price bubbles and financial fragility

Matheus Grasselli
(joint work with Bernardo Costa Lima and Omneia Ismail)

Hyman Minsky's main contribution to economics – the financial instability hypothesis – links the expansion of credit for funding new investment to the increase in asset prices and the inherent fragility of an over-leveraged financial system [6]. In this talk I describe an attempt to mathematize his model.

I first briefly review the economic literature on asset price bubbles, starting with the theory of *rational bubbles* in discrete time, which arise naturally in the context of maximization of utility of consumption and satisfy

$$E_t[B_{t+1}] = \rho^{-1} B_t, \tag{1}$$

where $0 < \rho < 1$ is a discount factor. Among the immediate implications of the growth condition (1) are the facts that rational bubbles are always nonnegative and cannot be created after the first day of trade on an asset. More importantly, they cannot exist for an asset with finite maturity or in an economy with finitely many agents with fully dynamic rational expectations [8]. One alternative is to consider an economy growing at a rate bigger than ρ^{-1}, in which case rational bubbles are not just possible, but efficient instruments of wealth allocation between overlapping generations. Another alternative is to move beyond the rational expectations paradigm and allow for market inefficiencies to play a role in the formation of bubbles.

In a influential paper, Shiller [7] argued that introducing noise traders who react to fads and social dynamics alongside sophisticated investors who trade on the basis of rational expectations leads to prices that deviate from fundamentals while still preserving the degree of unpredictability confirmed by statistical tests on empirical data. A more detailed analysis of the effect of noise traders was presented in [2], where it was shown that not only prices can exhibit persistent deviations from fundamentals, but under certain regimes noise traders can earn higher returns than sophisticated investors and become dominant in the market. Another mechanism to generate prices deviating from fundamentals is the introduction of financial intermediation as suggested in [1], where it was shown that investors using borrowed funds push asset prices up by bidding more than they would if they had to use their own money.

Both noise traders and financial intermediation are essential ingredients in the Minsky story. While the existence of noise traders can be tacitly assumed (Larry Summers famously began a paper with the sentence "There are idiots – look around you!"), financial intermediation needs more justification. In the second part of the talk I describe an agent-based model for the emergence of a banking system in a society with random liquidity preferences. This uses the fundamental model for a bank as a provider of liquidity proposed in [3] and the adaptive learning framework proposed in [4]. Starting from the individual liquidity preferences of agents placed on a rectangular grid, we were able to numerically simulate the appearance of heterogeneous banks. The next step in this computationally intensive part of the project consists of letting the banks themselves act as agents seeking insurance from liquidity shocks by forming an interbank loan network, which can then be compared with existing empirical networks.

Finally in the third part of the talk, I discuss the following three-dimensional dynamical system for wages ω, employment rate λ and debt δ proposed in [5]:

$$\begin{aligned}
\frac{d\omega}{dt} &= \omega[F(\lambda) - \alpha], \\
\frac{d\lambda}{dt} &= \lambda\left[\frac{k(\pi_n)}{\nu} - \alpha - \gamma - \beta\right], \\
\frac{d\delta}{dt} &= k(\pi_n) - (1-\omega) - \delta\left[\frac{k(\pi_n)}{\nu} - \gamma\right],
\end{aligned} \tag{2}$$

where α, γ and β are the rates of increase in productivity, capital depreciation and population, respectively. The essence of this model is that changes in wages depend nonlinearly on the employment rate through a Phillips curve $F(\lambda)$, whereas for a given interest rate r, new investment, which is partially financed by new debt δ, depends nonlinearly on the net profit $\pi_n = 1 - \omega - r\delta$. Through a series of examples, I show that this system exhibits the cyclical behaviour associated with booms and crashes, as well as locally stable but globally unstable equilibria.

Put together, these three ingredients, namely (i) a mechanism for bubble formation depending on the availability of credit, (ii) an agent-based model for the establishment of a banking sector in the economy and (iii) a dynamic model for the expansion and contraction of credit, constitute a first pass at a comprehensive model for endogenous formation and crash of asset price bubbles.

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Model independent prices for variance swaps

David Hobson
(joint work with Martin Klimmek)

If $(X_t)_{t\geq 0}$ is a continuous stochastic process, then applying Itô's formula to $2\log X_t$ yields

$$\int_0^T \frac{d[X]_t}{X_t^2} = -2\log X_T + 2\log X_0 + 2\int_0^T \frac{dX_t}{X_t}. \tag{1}$$

If X is the forward price of an asset, then this has a clear interpretation in finance: the payoff from the floating leg of a variance swap contract can be expressed, pathwise, as the payoff of a European contract and the gains from trade from a dynamic position in the underlying. If call options with maturity T are traded, so that the European claim can be replicated with an option portfolio, then the variance swap can be replicated exactly with vanilla instruments and there is a model-independent price for the variance swap.

What if $(X_t)_{t\geq 0}$ is not continuous? Then (1) breaks down, and there is no perfect hedge. Moreover, the payoff of the variance swap depends on the fine structure of the definition of the payoff, for instance whether we use squared proportional returns or squared log returns to define the contract. Nonetheless we show that there is a cheapest possible superhedge and a most expensive subhedge. These hedges are associated with time-changed versions of Perkins' solution to the Skorokhod embedding problem.

The main idea is, given a bivariate function H, to find functions ψ and δ such that $H(x,y) \geq \psi(y) - \psi(x) + \delta(x)(y-x)$ for all $x, y \geq 0$. Then, for a partition $0 = t_0 < t_1 < \cdots < t_n = T$,

$$\sum_{k=0}^{n-1} H(X_{t_k}, X_{t_{k+1}}) \geq \psi(X_T) - \psi(X_0) + \sum_{k=0}^{n-1} \delta(X_{t_k})(X_{t_{k+1}} - X_{t_k}) \tag{2}$$

and we have a subreplicating portfolio. The optimal choice for ψ and δ will depend on the kernel H, e.g. $H(x,y) = (y/x-1)^2$ or $(\log(y/x))^2$, and also on the prices of calls. The model for which there is equality in (2) is related by a discontinuous time change to Perkins' solution of the Skorokhod problem.

The quadratic variation of an Itô semimartingale without Brownian part

Jean Jacod
(joint work with Viktor Todorov)

In the context of high frequency data, like financial data, one of the main objects of interest is the quadratic variation. When the underlying process X is an Itô semimartingale, one knows the rate of convergence of the "approximated" quadratic variation when it is computed on the basis of a regular sampling with mesh Δ_n going to 0. This rate is $1/\sqrt{\Delta_n}$, and the limit of the normalized error process involves the volatility in an essential way, as well as the jumps of the semimartingale.

When X has no Brownian part, equivalently when the volatility vanishes identically, the limit above also vanishes, meaning that the rate is not appropriate. However, in some cases it is still possible to reach a central limit theorem:

We suppose that X is the sum of a drift term plus an integral $\int_0^t \sigma_{s-} dZ_s$, where σ is itself an Itô semimartingale, and Z is a Lévy process whose Lévy measure G is such that the "tail" near 0, say $G([-x,x]^c)$, is equivalent to θ/x^β for some $\theta > 0$ and some $\beta \in (0,2)$ (as $x \downarrow 0$); this is the case of course when Z is stable with index β, or temperate stable, or in many other examples. We then have a contrasted behavior:

(1) When $\beta > 1$, or when $\beta < 1$ and $Z_t = \sum_{s \leq t} \Delta Z_s$ is the sum of its jumps, or if $\beta = 1$ and Z is symmetric, then the rate of convergence of the approximate quadratic variation is $1/(\Delta_n \log(1/\Delta_n))^{1/\beta}$. Moreover, the limit is a stochastic integral $\int_0^t \sigma_{s-}^2 \, dZ'_s$, where Z' is a stable process with index β and independent of X.
(2) When $\beta = 1$ and Z is not symmetric (for example it has a drift), the rate is $1/\Delta_n \,(\log(1/\Delta_n))^2$ and the convergence holds in probability.
(3) When $\beta < 1$ and the drift is not vanishing, the rate becomes $1/\Delta_n$, and the limit is a rather complicated process involving the jumps of X, its drift, and extra independent variables.

In a sense, the situation (3) is like the case where there is a Brownian part, with the driving Wiener process W being replaced by the "driving drift" t. The situations (1) and (2) can be viewed, in contrast, as radically different.

Of course, the setting as described above may be viewed as rather restrictive. Probably one can add to the "main term" $\int_0^t \sigma_{s-} dZ_s$ another pure jump term with a Blumenthal–Getoor index smaller than β. On the other hand, since the rate depends on β in an essential way, there seems to be no way of significantly relaxing the assumption on the tail behavior of G, except perhaps by adding a slowly varying function.

Asset pricing puzzles explained by incomplete Brownian equilibria

Kasper Larsen
(joint work with Peter Ove Christensen)

We present incomplete Brownian based models allowing us to explicitly quantify the impact that unspanned income and preference heterogeneity can have on the resulting equilibrium interest rate and risk premium. The finite number of investors can trade continuously on a finite time horizon, and they maximize expected exponential utility of intermediate consumption. We show that if the investors cannot consume continuously over time, unspanned income can lower the risk-free rate and raise the risk premium when compared to the standard complete Pareto efficient equilibrium. Subsequently, we consider the limiting case where investors can consume continuously over time, and in a model-free manner we show that unspanned income can affect the equilibrium risk-free rate but can never affect the equilibrium instantaneous risk premium relative to the complete Pareto efficient equilibrium. However, if risk premia are measured over finite time-intervals (as in empirical studies of asset pricing puzzles), our model with unspanned income and stochastic volatility can raise the equilibrium risk premium (and lower the equilibrium risk-free rate) relative to the Pareto efficient analogue.

The questions of existence and characterization of complete equilibria in continuous time and state models are well-studied. By means of the representative agent method, the search for a complete market equilibrium can be reduced to a finite-dimensional fixed point problem. To the best of our knowledge, only [2] and [6] consider the abstract existence of a non-Pareto efficient equilibrium in a continuous trading setting. We provide tractable incomplete models for which the equilibrium price processes can be computed explicitly and, consequently, we can quantify the impact of market incompleteness.

To obtain incompleteness effects on the equilibrium risk premium, we incorporate a stochastic volatility v à la Heston's model into the equilibrium stock price dynamics. In Heston's original model [3], the stock's relative volatility is v, whereas in this paper v will be the stock's absolute volatility. We explicitly derive expressions for the equilibrium risk-free rate and the risk premium in terms of the individual income dynamics as well as the absolute risk aversion coefficients. The resulting type of equilibrium equity premium has been widely used in various optimal investment models, see e.g. [1] and [4], whereas the resulting type of equilibrium interest rate is similar to the celebrated CIR term structure model.

Translation invariant models (such as the exponential model we consider) allow consumption to be negative. [5] show that this class of models is fairly tractable even when income is unspanned. We first conjecture the equilibrium form of the market price of risk process and then use the idea in [2] to rewrite the individual investor's problem as a problem with spanned income and heterogeneous beliefs.

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Asymptotics of implied volatility in extreme regimes

Roger Lee
(joint work with Kun Gao)

Asymptotic approximations of implied volatility reveal information contained in implied volatility observations, and provide guidance for extrapolating implied volatility to unobserved strikes and expiries. Indeed, explicit formulas for a given model can connect, on one hand, information about the model's parameters, and on the other hand, key features (such as level/slope/convexity with respect to strike/expiry) of the implied volatility skew/smile. This leads to an understanding of which specific parameters influence which specific smile features, and it facilitates numerical calibration of those parameters to implied volatility data. Moreover, asymptotic formulas suggest the proper functional forms to use for the purpose of parametrically extrapolating or interpolating a volatility skew.

Pursuant to these background motivations (and complementary to previous work on asymptotic regimes of SDE parameters, such as [3] or [6]), a growing body of research explores asymptotic regimes of strikes and expiries; a typical result focuses on either long expiries, or short expiries, or extreme strikes. Taking a broader view in this paper, we exploit the similarities of extreme-strike and extreme-expiry asymptotics, to introduce a general framework that *unifies* all three extreme strike/expiry regimes, including variants in which strike and expiry vary jointly.

Our approach encompasses not only general asymptotic regimes, but also general models. Our main results express the implied volatility V in a *model-free* way, not in terms of the parameters of any particular model, but rather in terms of L, the absolute log of the option price, and k, the log strike. This type of model-independent formula has precedents in the literature; the leading examples in each regime are as follows. Deferring precise definitions until the body of this paper, let us write L_- and L_+ for the absolute logs of the prices of, respectively, an out-of-the-money call, and a covered-call position (long one share, short one call); then the following asymptotics are known:

For short expiries with constant strike, Roper/Rutkowski [8] show that

$$V^2 \sim \frac{k^2}{2L_-} \tag{1}$$

For long expiries, Tehranchi [9] shows that

$$V^2 = 8L_+ - 4\log L_+ + 4k - 4\log \pi + o(1). \tag{2}$$

For large strikes with constant expiry, Gulisashvili [5] shows that

$$V = G_-\Big(k, L_- - \frac{1}{2}\log L_-\Big) + O(L_-^{-1/2}), \tag{3}$$

where

$$G_-(\kappa, u) := \sqrt{2}(\sqrt{u+\kappa} - \sqrt{u}),$$

and that (3) implies other model-free results including the moment formula (Lee [7]) and tail-wing formula (Benaim/Friz [1]).

We sharpen all of the above formulas to *arbitrarily high* order of accuracy, in the following sense: We generate, for any given $J > 0$, implied volatility and implied variance formulas with rigorous error estimates of the type $O(1/L^J)$ where $L \to 0$. Low-order special cases of our theorem suffice to refine each of the formulas cited above.

Our general results have immediate applications to specific models. Consider, for example, the Heston model at large strikes, Lévy models at short expiries, and Lévy models at long expiries. In all three of these cases, there exist expansions for L (according to asymptotics in, respectively, Friz/Gerhold/Gulisashvili/Sturm [4], Figueroa-Lopez/Forde [2], and a refined saddlepoint expansion in this paper) which approximate L in terms of the model's parameters. Inserting these L approximations into our main theorem then produces explicit parametric implied volatility formulas, again with rigorous error estimates showing that we sharpen the sharpest previously known implied volatility formulas for those models: Friz/Gerhold/Gulisashvili/Sturm [4] in the Heston case, Figueroa-Lopez/Forde [2] in the short-dated Lévy case, and Tehranchi [9] in the long-dated Lévy case.

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Law of large numbers for self-exciting correlated defaults

Jin Ma
(joint work with Jakša Cvitanić, Jianfeng Zhang)

Modelling of correlation between default probabilities of multiple "names" (individuals, firms, countries, etc.) has been one of the central issues in the theory and applications of managing and pricing credit risk in the last several years. There have been dozens of models in the literature. While each of these models has its own advantages and disadvantages, lax use of such models in practice could in part affect the understanding of the risk of the credit default and consequently contribute to the extent of a potential crisis in the market.

In this paper we propose a "bottom-up" model for correlated defaults within the standard "reduced form" framework. In particular, we assume that in a large collection of defaultable entities, the intensity of each individual default depends on factors specific to the individual entity, and on a common factor. The main novelty of our model is that we further allow a part of the common factor to take the form of an "average loss process", which includes the average number of defaults to date as a special case, and thus to have a *self-exciting* nature. Such a self-exciting feature allows us, in the limiting case, to analyze the impact of a "general health" index on the individual entities.

The self-exciting structure of our model can be thought of as an example of the so-called "contagion" feature, which has been investigated by many authors using various approaches (see, for example, [2, 3, 5, 6, 7, 10, 11, 12, 14, 15, 16, 17, 23], to mention just a few). None of these models, however, contains the circular nature presented in our model. In a recent work [13], a model similar to ours was considered, but with a more special structure so that some large deviation type results can be obtained, in addition to the law of large numbers type results that we focus on. The self-exciting feature is also presented in [9], in a "top-down" model. For an overview of standard default risk models, one can consult, among many others, the texts [8, 21, 22] and the references cited therein.

A more precise description of our problem is as follows. We consider n "names", which could be individual investors, financial firms, loans, etc. We denote their default times by $\tau_1, \dots, \tau_n$, as random variables defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. We associate to each name a "loss process" L_t^i, $t \geq 0$, so that the loss of the name due to default at any time t is given by $L^i_{\tau^i}\mathbf{1}_{\{\tau^i\le t\}}$. We define the "average loss" of all defaults at time t by

$$\bar{L}_t := \bar{L}^n_t := \frac{1}{n}\sum_{i=1}^n L^i_{\tau_i}\mathbf{1}_{\{\tau_i\le t\}}. \tag{1}$$

Clearly, one can have various interpretations for $\bar{L}$ by imposing various choices for L^i. In particular, if we set $L^i \equiv 1$, then $\bar{L}$ is the average number of defaults. Our main purpose is to investigate the limiting behavior of $\bar{L}^n$ as $n\to\infty$, namely,

$$\bar{L}^*_t := \lim_{n\to\infty}\bar{L}^n_t, \tag{2}$$

whenever the limit exists, and to characterize the limit $\bar{L}^*$.

Let us now assume that the probability space is rich enough to support a sequence of independent Brownian motions $(B^0, B^1, \dots, B^n, \dots)$ and a sequence of exponential random variables $(E^1, \dots, E^n, \dots)$, all with rate 1 and independent of the Brownian motions. We define the following sub-filtrations of $\mathbb{F}$ generated by the Brownian motions B^0 and (B^0, B^i), respectively:

$$\mathbb{F}^0 := \mathbb{F}^{B^0}, \quad \mathbb{F}^i := \mathbb{F}^{B^0,B^i}, \quad i = 1, 2, \dots, \tag{3}$$

all being augmented by the $\mathbb{P}$-null sets. Denote $\mathbb{F} = \bigvee_{i=1}^\infty \Big(\mathbb{F}^i \vee \sigma(E^i)\Big)$. Then each τ is a $\mathbb{F}$-stopping time, but not necessarily an $\mathbb{F}^i$-stopping time. Furthermore, for each fixed n and the loss processes L^i, $i = 1, \dots, n$, we define, as in reduced form models (see e.g. [1, 8, 18])

$$\tau_i := \inf\Big\{t\ge 0 : Y^i_t \ge E^i\Big\}, \tag{4}$$

where, for the process $\bar{L}$ defined by (1), the process Y^i denotes the "hazard process"

$$Y^i_t := \int_0^t \lambda_i(s, B^0_{\cdot\wedge s}, B^i_{\cdot\wedge s}, X^0_s, X^i_s, \bar{L}_s)ds, \tag{5}$$

and X^0, X^i, $i = 1, 2, \dots$, are factor processes defined by

$$X^0_t = x_0 + \int_0^t b_0(s, B^0_{\cdot\wedge s}, X^0_s, \bar{L}_s)ds + \int_0^t \sigma_0(s, B^0_{\cdot\wedge s}, X^0_s, \bar{L}_s)dB^0_s, \tag{6}$$

$$\begin{aligned} X^i_t = x_i &+ \int_0^t b_i(s, B^0_{\cdot\wedge s}, B^i_{\cdot\wedge s}, X^0_s, X^i_s, \bar{L}_s)ds \\ &+ \int_0^t \sigma_i(s, B^0_{\cdot\wedge s}, B^i_{\cdot\wedge s}, X^0_s, X^i_s, \bar{L}_s)dB^i_s. \end{aligned} \tag{7}$$

We remark here that if $b_0, \sigma_0, b_i, \sigma_i, \lambda_i$ do not depend on $\bar{L}$, then our model becomes a standard reduced form model where the defaults are conditionally independent, conditional on the common factor X^0, and it is straightforward to check that in this case λ_i is the $\mathbb{F}^i$-intensity of τ_i, in the sense that

$$\mathbb{E}\{\tau_i > t|\mathcal{F}^i_t\} = \exp\left\{-\int_0^t \lambda_i(s, X^0_s, X^i_s)ds\right\}, \quad t\ge 0$$

(see e.g. [1, 8]). But in the general case when λ_i depends on $\bar{L}$, λ_i is obviously no longer an $\mathbb{F}^i$-adapted process (hence cannot be an "$\mathbb{F}^i$-intensity" of τ_i in the aforementioned sense). Due to the self-exciting nature of our model, λ^i has to be understood as the conditional intensity of τ^i, conditionally on all the past defaults. We refer to [19, 20] for more on the construction of default times with given intensities.

Our first result concerns the well-posedness of the problem, and a justification of λ being the "intensity" in this special setting. Note that we shall omit all technical assumptions in the statements of the theorems to simplify presentation, and refer to [4] for details.

Theorem 1 (i) Under reasonable assumptions, for each $n \in \mathbb{N}$, the system (1), (4)–(7) admits a unique $\mathbb{F}$-adapted solution $(X^0, \{X^i, Y^i\}_{i=1}^n)$.

(ii) For each $n \in \mathbb{N}$, let $\tau_1^* < \cdots < \tau_n^*$ be the ordered statistics of $\tau_1 < \cdots < \tau_n$. Moreover, for $0 \le k \le n$, and $i_1, \ldots, i_k$, denote

$$D_k := \{\tau_1^* = \tau_{i_1}^k, \ldots, \tau_k^* = \tau_{i_k}^k\}, \quad \mathcal{G}_t^k := \Big(\bigvee_{\ell=1}^k \mathcal{F}^{i_\ell}_{\tau_k^*+t}\Big) \vee \Big(\bigvee_{j\neq i_1,\ldots,i_k} \mathcal{F}^j_{\tau_k^*}\Big).$$

Then, for $j \neq i_1, \ldots, i_k$ and $t \ge 0$, it holds that

$$\mathbb{P}\Big\{\tau_j^{k+1} > \tau_k^* + t \,\Big|\mathcal{G}_t^k, D_k\Big\} = \mathbb{E}\Big\{\exp(Y^{j,k+1}_{\tau_k^*} - Y^{j,k+1}_{\tau_k^*+t})\Big|\mathcal{G}_t^k, D_k\Big\} \quad \mathbb{P}\text{-a.s. on } D_k.$$

(iii) For each k, conditionally on $\mathcal{G}_t^k \vee \sigma(D_k)$, the random vectors $(X^{j,k+1}_{\tau_k^*+t}, Y^{j,k+1}_{\tau_k^*+t}, \mathbf{1}_{\{\tau_j^{k+1} > \tau_k^*+t\}})$, $j \neq i_1, \ldots, i_k$, are conditionally independent on D_k, such that $\mathbb{P}$-a.s. on D_k,

$$\mathbb{P}\Big\{\tau_{k+1}^* > \tau_k^* + t\Big|\mathcal{G}_t^k, D_k\Big\} = \mathbb{E}\Big\{\exp\Big(\sum_{j\neq i_1,\ldots,i_k}(Y^{j,k+1}_{\tau_k^*} - Y^{j,k+1}_{\tau_k^*+t})\Big)\Big|\mathcal{G}_t^k, D_k\Big\}.$$

Our main objective is to identify the possible limit the average default loss will converge to, in the sense of the law of large numbers, as the number of names tends to infinity. It turns out that the limit process $\bar{L}^*$ can be determined via a fixed point problem. Since $\bar{L}^*$, if it exists, should be $\mathbb{F}^0$-adapted, we consider the following system for any given $\mathbb{F}^0$-adapted process α:

$$\begin{aligned}
X_t^{0,\alpha} &= x_0 + \int_0^t b_0(s, X_s^{0,\alpha}, \alpha_s)ds + \int_0^t \sigma_0(s, X_s^{0,\alpha}, \alpha_s)dB_s^0;\\
X_t^{i,\alpha} &= x_i + \int_0^t b_i(s, X_s^{0,\alpha}, X_s^{i,\alpha}, \alpha_s)ds + \int_0^t \sigma_i(s, X_s^{0,\alpha}, X_s^{i,\alpha}, \alpha_s)dB_s^i,\\
Y_t^{i,\alpha} &= \int_0^t \lambda_i(s, X_s^{0,\alpha}, X_s^{i,\alpha}, \alpha_s)ds, \quad \tau_i^\alpha = \inf\Big\{t \ge 0 : Y_t^{i,\alpha} \ge E_i\Big\}, \quad i = 1, \ldots, n;\\
\bar{L}_t^\alpha &= \bar{L}^{n,\alpha} = \frac{1}{n}\sum_{i=1}^n L^i_{\tau_i^\alpha} 1_{\{\tau_i^\alpha \le t\}}.
\end{aligned}$$

Let us consider a simplified situation (for more general results, see [4]). Assume that $x_i = x$, $b_i = b$, $\sigma_i = \sigma$, $\lambda_i = \lambda$, and $L^i_t = \varphi(t, B^0_{\cdot\wedge t}, B^i_{\cdot\wedge t})$, $t \geq 0$, $i = 1, 2, \ldots$, where $\varphi : \mathbb{R}_+ \times C([0,\infty);\mathbb{R})^2 \to \mathbb{R}_+$ is a bounded measurable function. Assume further that b_0 is decreasing in α; b is increasing in x_0 and decreasing in α; λ is decreasing in x_0, x_i and increasing in α; and φ is decreasing in t. We have the following result.

Theorem 2 Under the assumptions of Theorem 1, for any $\mathbb{F}^0$-adapted process α such that $|\alpha_t| \leq K$, one has

(i) τ^α_i are conditionally i.i.d., conditionally on $\mathbb{F}^0$, and

$$\lim_{n\to\infty} \mathbb{E}\{|\bar{L}^{n,\alpha}_t - \Gamma_t(\alpha)|\} = 0,$$

where

$$\Gamma_t(\alpha) = \int_0^t \mathbb{E}\Big\{\varphi(s, B^0_{\cdot\wedge s}, B^1_{\cdot\wedge s})\lambda(s, B^0_{\cdot\wedge s}, B^1_{\cdot\wedge s}, X^{0,\alpha}_s, X^{1,\alpha}_s, \alpha_s)e^{-Y^{1,\alpha}_s}\Big|\mathcal{F}^0_s\Big\}ds.$$

(ii) The process $\Gamma(\alpha)$ is continuous and increasing in t, increasing in α, and satisfies $0 \leq \Gamma_t(\alpha) \leq K$, a.s.

The fixed point problem is to find an $\mathbb{F}^0$-adapted process α such that $\alpha = \Gamma(\alpha)$. Our final result is the following.

Theorem 3 Assume the assumptions of Theorem 2 are all in force. Then there exists an $\mathbb{F}^0$-adapted process α such that $\alpha = \Gamma(\alpha)$. Furthermore, for such an α the following "law of large numbers" holds:

$$\lim_{n\to\infty} \mathbb{E}\Big\{|\bar{L}^n_t - \alpha_t|\Big\} = \lim_{n\to\infty} \mathbb{E}\Big\{|\bar{L}^{n,\alpha}_t - \alpha_t|\Big\} = 0. \tag{8}$$

Under appropriate conditions, we can show that for the average numbers, the limiting process α solves an ordinary differential equation, while for the average loss, the limiting process α solves a more general and complex equation. It is worth remarking that these results, being of asymptotic nature, are not directly applicable to individual credit risk derivatives, because they require a large number of names to be involved in the limiting process. However, our results should be useful for the risk management at the level of an institution, or a country, with a large portfolio of defaultable claims, when the aim is to analyze potential total losses. For example, it has been stated that the next crisis might come from potentially numerous defaults of credit card holders. This paper provides a theoretical model which may prove useful for addressing such issues.

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Asymptotics and duality in portfolio optimization with transaction costs

Johannes Muhle-Karbe
(joint work with Stefan Gerhold, Paolo Guasoni, Walter Schachermayer)

We propose a tractable benchmark of portfolio choice under transaction costs. Our analysis is based on the model of Dumas and Luciano [2], which concentrates on long-run asymptotics to gain in tractability.

Consider a market with a safe rate r, and a risky asset, trading at ask (buying) price $\overline{S} = S/(1-\varepsilon)$ and at bid (selling) price $\underline{S} = S(1-\varepsilon)$. $S = \sqrt{\underline{S}\overline{S}}$ denotes the (geometric) mid price, which follows geometric Brownian motion

$$\frac{dS_t}{S_t} = (\mu + r)dt + \sigma dW_t,$$

where W is a standard Brownian motion, $\mu > 0$ is the expected excess return, and $\sigma > 0$ is the volatility. In this market, an investor chooses her trading strategy (φ^0, φ) so as to maximize the *certainty equivalent rate*

$$\liminf_{T\to\infty} \frac{1}{(1-\gamma)T} \log E\left[(\varphi_t^0 S_t^0 + \varphi_t^+ \underline{S}_t - \varphi_t^- \overline{S}_t)^{1-\gamma}\right],$$

i.e., the long-run growth rate of expected power utility. Then, for small transaction costs ε, we establish the following results:

i) (*Welfare*) The investor is indifferent between trading the risky asset with transaction costs, and trading a hypothetical frictionless asset with the same volatility σ, but with expected excess return $\sqrt{\mu^2 - \lambda^2}$. That is, both markets lead to the same certainty equivalent rate

$$\beta = r + \frac{\mu^2 - \lambda^2}{2\gamma\sigma^2},$$

and λ represents the *liquidity premium.*

ii) (*Portfolio*) It is optimal to keep the risky asset weight within the buy and sell boundaries

$$\pi_- = \frac{\mu - \lambda}{\gamma\sigma^2}, \qquad \pi_+ = \frac{\mu + \lambda}{\gamma\sigma^2},$$

where π_- and π_+ are evaluated, respectively, at the ask and bid prices.

iii) (*Liquidity premium*) λ is identified as the unique value for which the solution $w(\lambda, x)$ of the initial value problem

$$w'(x) + (1-\gamma)w(x)^2 + \left(\frac{2\mu}{\sigma^2} - 1\right) w(x) - \gamma \left(\frac{\mu - \lambda}{\gamma\sigma^2}\right)\left(\frac{\mu + \lambda}{\gamma\sigma^2}\right) = 0$$

$$w(0) = \frac{\mu - \lambda}{\gamma\sigma^2},$$

satisfies the terminal boundary condition

$$w(\log(u_\lambda/\ell_\lambda)) = \frac{\mu + \lambda}{\gamma\sigma^2} \quad \text{where} \quad \frac{u_\lambda}{\ell_\lambda} = \frac{1}{(1-\varepsilon)^2} \frac{(\mu + \lambda)(\mu - \lambda - \gamma\sigma^2)}{(\mu - \lambda)(\mu + \lambda - \gamma\sigma^2)}.$$

Since $w(\lambda, x)$ can be expressed in terms of trigonometric functions, this is a one-dimensional equation for λ.

iv) (*Trading volume*) Relative turnover, defined as the number $d\|\varphi\|_t$ of shares traded divided by the number $|\varphi_t|$ of shares held, has long-term average equal to

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{d\|\varphi\|_t}{|\varphi_t|} = \left(1 - \frac{\mu-\lambda}{\gamma\sigma^2}\right)\frac{\sigma^2}{2}\left(\frac{\frac{2\mu}{\sigma^2}-1}{(u/\ell)^{\frac{2\mu}{\sigma^2}-1}-1}\right) + \left(1 - \frac{\mu+\lambda}{\gamma\sigma^2}\right)\frac{\sigma^2}{2}\left(\frac{1-\frac{2\mu}{\sigma^2}}{(u/\ell)^{1-\frac{2\mu}{\sigma^2}}-1}\right).$$

v) (*Asymptotics*) As the bid-ask spread becomes small ($\varepsilon \downarrow 0$), the following asymptotic expansions hold:

a) Liquidity premium:

$$\lambda = \left(\frac{3\mu^2(\mu-\gamma\sigma^2)^2}{2\gamma^2\sigma^2}\right)^{1/3}\varepsilon^{1/3} + O(\varepsilon).$$

b) Certainty equivalent rate:

$$\beta = r + \frac{\mu^2}{2\gamma\sigma^2} - \left(\frac{3\mu^2(\mu-\gamma\sigma^2)^2}{2^{5/2}\gamma^{7/2}\sigma^5}\right)^{2/3}\varepsilon^{2/3} + O(\varepsilon^{4/3}).$$

c) Trading boundaries:

$$\pi_\pm = \frac{\mu}{\gamma\sigma^2} \pm \frac{1}{\gamma\sigma^2}\left(\frac{3\mu^2(\mu-\gamma\sigma^2)^2}{2\gamma^2\sigma^2}\right)^{1/3}\varepsilon^{1/3} + O(\varepsilon).$$

d) Long-term average trading volume:

$$\left(\frac{\mu(\mu-\gamma\sigma^2)^4}{12\gamma^4\sigma^4}\right)^{1/3}\varepsilon^{-1/3} + O(\varepsilon^{1/3}).$$

Higher-order terms can be algorithmically computed.

vi) (*Shadow price*) The investor is indifferent – both in terms of certainty equivalent rate *and* optimal trading policy – between trading the asset S with transaction costs, and trading a frictionless asset with *shadow price* $\tilde{S}$ that follows the dynamics

$$d\tilde{S}_t/\tilde{S}_t = \tilde{\mu}(Y_t)dt + \tilde{\sigma}(Y_t)dW_t,$$

for deterministic functions $\tilde{\mu}(\cdot)$ and $\tilde{\sigma}(\cdot)$ of Y, the (normalized) logarithm of the ratio of risky and safe weights, which follows reflected Brownian motion with drift in $[0, \log(u_\lambda/\ell_\lambda)]$. The shadow price $\tilde{S}_t$ always lies within the bid-ask spread, and coincides with the trading price at times of trading for the optimal policy.

The main message is that the optimal trading policy, its welfare, and the resulting trading volume are all simple functions of investment opportunities r, μ, σ, of risk aversion γ, and, crucially, of the liquidity premium λ. The liquidity premium does not admit an explicit formula in terms of the transaction cost parameter ε, but is determined through the implicit relation in iii), and has the asymptotic expansion in v), from which all other asymptotic expansions follow through the explicit formulas.

The result has several novel implications. First, trading boundaries are symmetric around the frictionless Merton proportion $\mu/\gamma\sigma^2$. At first glance, this result seems to contradict previous studies (cf., e.g., [7]), which emphasize how these boundaries are asymmetric, and may even fail to include the Merton proportion. This literature employs a common reference price (the average of the bid and ask prices) to evaluate both boundaries. By contrast, we use trading prices to express trading boundaries (i.e., the ask price for the buy boundary, and the bid price for the sell boundary). This simple convention unveils the natural symmetry of the optimal policy, and resolves the paradoxes of asymmetry as figments of notation. Of course, such symmetry hinges on the absence of intermediate consumption, thereby raising the question of comparing our trading boundaries with those obtained in the consumption model of Davis and Norman [1].

Indeed, a comparison of our asymptotics to those obtained by Janeček and Shreve [5] in the model of [1] reveals that they are equal, at least to the first order. Hence, while the traditional separation between consumption and investment — which holds in a frictionless model with constant investment opportunities — fails in the presence of proportional transaction costs, it does hold *at first order*.

Second, our results show that, unlike in the frictionless theory, with transaction costs leverage does matter. More precisely, when transaction costs are considered, an investor is not indifferent between two markets with identical Sharpe ratios. Indeed, note that λ/σ^2, the liquidity premium per unit variance, depends on μ and σ only through μ/σ^2, the expected return per unit variance, not on the Sharpe ratio μ/σ. The parameter μ/σ^2 is not leverage invariant, since multiplying μ and σ by a constant does not change μ/σ, but does change μ/σ^2. The intuition is that even if two markets have the same Sharpe ratios, one of them can be more attractive than the other, if it leads to wider trading boundaries, and hence lower trading costs. As an extreme case, in one market it may be optimal lo leave all wealth in the risky asset, thereby eliminating trading costs.

Third, our model yields the first continuous-time benchmark for trading volume, giving a closed-form expression for stationary turnover and its asymptotic expansion. Trading volume is an elusive quantity for frictionless models, because they typically imply that turnover is infinite in any time interval[1]. Our asymptotic formula implies that, for large values of risk aversion, trading volume converges to a finite value. More risk averse investors hold less risky assets (reducing volume), but also rebalance more frequently (increasing volume). The two effects balance each other, leading to a finite limit that increases in μ and σ.

[1]The empirical literature has long been aware of this theoretical vacuum. [3] reckon that *The intrinsic difficulties of specifying plausible, rigorous, and implementable models of volume and prices are the reasons for the informal modeling approaches commonly used.* Eight years later, [8] still note that *although most models of asset markets have focused on the behavior of returns [...] their implications for trading volume have received far less attention.*

A key idea in our results – and in their proof – is that a market with constant investment opportunities with transaction costs is equivalent to another market, without transaction costs, but with stochastic investment opportunities. The state variable is the logarithm of the ratio between the risky and the safe weights positions, and tracks the location of the portfolio within the trading boundaries, affecting both the volatility and the expected return of the shadow price. Such a shadow price has previously been determined for log-investors (cf. [6, 4]); here we also construct it for investors with power utilities.

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Forward performance process and an ill-posed HJB equation

Sergey Nadtochiy
(joint work with Thaleia Zariphopoulou)

This work is concerned with the *forward performance* approach to the *optimal investment* problem. The classical point of view on the optimal investment problem is based on the concept of *utility function*, representing the preferences of a typical investor at some fixed moment of time in the future. The optimal investment strategy is then obtained by maximizing the expected utility of the terminal wealth. A more recent alternative approach, developed by T. Zariphopoulou and M. Musiela (see [1], [2]) suggests that instead of considering a utility function, one starts with the *forward performance* function, representing the current "instantaneous") preferences of the investor, and models its evolution forward in time. This results in an investment performance criterion (and, consequently, an optimal investment strategy), which is consistent across all maturities and only relies on the "local" characteristics of the investor's preferences. The cornerstone of the forward performance theory is a stochastic partial differential equation, which is an analogue of the HJB equation in the classical utility maximization theory.

It is possible (as shown, for example, in [3]) to prove existence of a solution to the aforementioned SPDE, for any admissible structure of the volatility of the forward performance process. However, the description of admissible volatility processes available so far is rather implicit, and in particular requires the knowledge of the corresponding optimal investment strategy. We, on the contrary, attempt to provide a constructive existence result, which, despite some loss of generality, would allow to estimate (or calibrate) the volatility and give a clear interpretation of the resulting forward performance process, together with the optimal investment strategy (which we treat as an output, rather than input, of the model). In particular, we consider a general two-factor stochastic volatility model, and search for a forward performance process in the form of a function of the spatial variable, time and the stochastic factor (which is equivalent to assuming that the volatility of forward performance is of functional form). The corresponding SPDE, in the present case, turns into a deterministic Hamilton–Jacobi–Bellman equation. However, instead of a terminal condition at some time horizon T, which appears in the classical formulation of the problem, the solution is expected to satisfy an *initial* condition, which makes the problem "*ill-posed*". We show that in the case of a complete market (or for some specific choices of initial preferences), the HJB equation can be linearized, and the problem reduces to an ill-posed linear parabolic PDE, with space-dependent coefficients. The characterization of solutions to this equation, for the case of constant coefficients, is known as *Widder's theorem* (see [4]). We provide an explicit integral representation of solutions to an ill-posed linear parabolic equation with non-constant coefficients, and prove its sufficiency. The necessity of this representation, which can be viewed as a direct generalization of Widder's theorem, is a subject of ongoing research.

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Dynamic risk measures under volatility uncertainty

Marcel Nutz

(joint work with H. Mete Soner)

The starting point of this talk is Peng's G-expectation; cf. [3] for extensive references. The G-expectation is a sublinear operator defined on a class of random variables on the canonical space Ω, while G is a real function of the form $G(x) = (\overline{\sigma}^2 x^+ - \underline{\sigma}^2 x^-)/2$ for some constants $\overline{\sigma} \geq \underline{\sigma} \geq 0$. If $\mathcal{P}$ is the set of martingale laws on Ω under which the volatility of the canonical process stays between $\underline{\sigma}^2$

and $\overline{\sigma}^2$, the G-expectation at time $t = 0$ may be expressed as the upper expectation $\mathcal{E}_0^G(X) := \sup_{P \in \mathcal{P}} E^P[X]$. For positive times t, Peng constructed the conditional G-expectation $\mathcal{E}_t^G(X)$ by using the nonlinear heat equation $\partial_t u - G(u_{xx}) = 0$.

The first part of the talk provides an extension of the G-expectation to the case where the constant bounds $\underline{\sigma}, \overline{\sigma}$ are replaced by path-dependent ones, which corresponds to a random function G. This extension, called random G-expectation, is constructed using regular conditional probability distributions and dynamic programming techniques (cf. [1]).

In the second part of the talk, we consider an axiomatic setup for a dynamic risk measure $\mathcal{E}$ under volatility uncertainty. Given a suitable random variable X, we construct a càdlàg process $\mathcal{E}(X)$ which corresponds to the dynamic evaluation of X and which we call the $\mathcal{E}$-martingale associated with X. We provide an optional sampling theorem for $\mathcal{E}(X)$. Furthermore, we obtain a decomposition of $\mathcal{E}(X)$ into an integral of the canonical process and a decreasing process, similarly as in the classical optional decomposition for incomplete markets. In particular, the $\mathcal{E}$-martingale yields the dynamic superhedging price of the financial claim X and the integrand Z^X yields the superhedging strategy. We also provide a connection between $\mathcal{E}$-martingales and second order backward SDEs by characterizing $(\mathcal{E}(X), Z^X)$ as the minimal solution of such an equation (cf. [2]).

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Utility theory front to back: recovering agents' preferences from their choices

Jan Obłój
(joint work with A.M.G. Cox, David Hobson)

We pursue an inverse approach to utility theory and consumption and investment problems. Instead of specifying the agents' utility function and deriving their actions, we assume we observe their actions (i.e. the consumption and investment strategies) and aim to derive a utility function for which the observed behaviour is optimal. We work in continuous time both in a deterministic and stochastic setting.

In a deterministic setup, the agents choose a consumption policy $c^*(t, w)$, function of time and their remaining wealth, which is then applied to a given initial capital. We find that there are infinitely many utility functions u for which a given consumption pattern maximises the integral of utility of consumption over time. If y^* denotes the inverse of c^*, then u is specified via $u_c(t, c) = F(y^*(t, c))$, where F is an arbitrary non-negative decreasing absolutely continuous function with $F(\infty) = 0$. In particular, we show that the same consumption may arise from very different preferences, e.g. with decreasing and increasing absolute risk aversion.

In the stochastic setting of the Black–Scholes complete market, it turns out that the consumption and investment strategies c, π, assumed to be functions of time and wealth, have to satisfy a consistency condition (PDE) if they solve a classical utility maximisation problem. This PDE has been first discovered by Black (1968) and this inverse Merton problem was then studied by He and Huang (1994). Our main results states that c, π solve Black's PDE and satisfy integrability and budget constraints if and only if they achieve a finite maximum in the problem

$$\max_{C_t, \Pi_t \in \mathcal{A}} \mathbb{E}\left[\int_0^\infty u(t, C_t)dt\right]$$

for a (regular) utility u and where admissible pairs $\mathcal{A}$ induce a nonnegative wealth process of the agent. The (recovered) utility function u is then specified (essentially) uniquely. We further show that agents' important characteristics such as their attitude towards risk (e.g. DARA) can be directly deduced from their consumption/investment choices. Finally we prove a lemma which gives a set of sufficient conditions on c, π for our main theorem to hold. This yields large classes of new examples of optimal consumption and investment policies. In particular we can exhibit examples with prescribed convexity/concavity properties for c and π and with absolute risk aversion which is neither decreasing nor increasing.

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Detecting financial bubbles in real time

Philip Protter
(joint work with Robert Jarrow, Younes Kchia)

After the 2007 credit crisis, financial bubbles have once again emerged as a topic of current concern. An open problem is to determine in real time whether or not a given asset's price process exhibits a bubble. Due to recent progress in the characterization of asset price bubbles using the arbitrage-free martingale pricing technology (see for example [1],[2],[3]), we are able to propose a new methodology for answering this question based on the asset's price volatility. We limit ourselves to the special case of a risky asset's price being modelled by a Brownian driven stochastic differential equation. Such models are ubiquitous both in theory and in practice. Our methods use sophisticated volatility estimation techniques combined with the method of reproducing kernel Hilbert spaces. We illustrate these techniques using several stocks from the alleged internet dot-com episode of 1998–2001, where price bubbles were widely thought to have existed. Our results support these beliefs.

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Around the problem of testing 3 statistical hypotheses for Brownian motion with drift

Albert N. Shiryaev
(joint work with Mikhail V. Zhitlukhin)

On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$, we observe a process

$$X_t = \mu t + B_t, \qquad t \geq 0,$$

where $B = (B_t)_{t\geq 0}$ is a Brownian motion and μ takes one of the three values

$$\begin{aligned} &\mu = \mu^1, && && \text{(hypothesis } H^1\text{)},\\ &\mu = \mu^0, && \text{with } \mu^1 < \mu^0 < \mu^2 && \text{(hypothesis } H^0\text{)},\\ &\mu = \mu^2, && && \text{(hypothesis } H^2\text{)}. \end{aligned}$$

We consider the sequential Bayesian formulation of testing the 3 hypotheses H^1, H^0, H^2 with the sequential decision rule $\delta = (\tau, d)$, where τ is an $(\mathcal{F}_t^X)_{t\geq 0}$-stopping time, d is $\mathcal{F}_\tau^X$-measurable $(d = d^1, d^0, d^2)$ and the risk of the decision rule δ is given by

$$R_\delta(\pi) = \mathsf{E}_\pi(c\tau + w(\mu, d)),$$

where $\mathsf{P}_\pi = \pi^1\mathsf{P}^1 + \pi^0\mathsf{P}^0 + \pi^2\mathsf{P}^2$, $\mathsf{P}^i = \mathrm{Law}(X \mid \mu = \mu^i)$, $\pi^i = \mathsf{P}(\mu = \mu^i)$; μ and B are independent. We take the terminal risk $w(\mu, d)$ of the form

$$w(\mu^i, d^i) = 0, \qquad w(\mu^i, d^j) = a_{ij}, \quad i \neq j.$$

It is easy to see that if $\pi_t^i = \mathsf{P}(\mu = \mu^i \mid \mathcal{F}_t^X)$, then

$$\inf_{\delta=(\tau,d)} R_\delta(\pi) = \inf_\tau \mathsf{E}_\pi\{c\tau + G(\pi_\tau^1, \pi_\tau^0, \pi_\tau^2)\},$$

where

$$G(\pi^1, \pi^0, \pi^2) = \min\bigl\{a_{10}\pi^1 + a_{20}\pi^2,\ a_{01}\pi^0 + a_{21}\pi^2,\ a_{02}\pi^0 + a_{12}\pi^1\bigr\}.$$

For simplicity assume that

$$\pi^1 = \pi^0 = \pi^2 = 1/3,$$
$$a_{ij} = 1, \quad i \neq j; \qquad a_{ii} = 0, \qquad \text{(symmetric case)},$$
$$\mu^1 = -1, \quad \mu^0 = 0, \quad \mu^2 = 1.$$

By the innovation representation for $X_t = \mu t + B_t$, we have

$$dX_t = A(t, X_t)\, dt + d\overline{B}_t,$$

where

$$A(t,x) = \frac{e^{-t/2}(e^x - e^{-x})}{1 + e^{-t/2}(e^x + e^{-x})}$$

and $(\overline{B}_t)_{t\geq 0}$ is an innovation Brownian motion. In terms of (t,x), the function $G(\pi_1, \pi^0, \pi^2)$ takes the form

$$G(t,x) = \frac{\min\{1 + e^{-x-t/2},\ 1 + e^{x-t/2},\ e^{-x-t/2} + e^{x-t/2}\}}{1 + e^{-t/2}(e^{-x} + e^x)}.$$

For $x \neq \pm t/2$,

$$L_{(t,x)} G(t,x) = 0,$$

where

$$L_{(t,x)} = \frac{\partial}{\partial t} + A(t,x)\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2}.$$

Taking this into account and applying the generalized Itô formula to $G(t, X_t)$, we find that

$$\mathsf{E}G(\tau, X_\tau) = G(0, X_0) - \frac{1}{2}\mathsf{E}\int_0^\tau \frac{dL_s}{2 + e^{-s}},$$

where L_s is the local time of the process X on the rays $x = \pm s/2$. Because $1/3 < 1/(2 + e^{-s}) < 1/2$, we have

$$\frac{1}{4}\mathsf{E}(4c\tau - L_\tau) \leq \mathsf{E}(c\tau + G(\tau, X_\tau)) \leq \frac{1}{6}\mathsf{E}(6c\tau - L_\tau).$$

So to get lower and upper bounds for $\inf_\tau \mathsf{E}(c\tau + G(\tau, X_\tau))$, it is sufficient to solve for the local time the optimal stopping problem

$$\tau \mapsto \inf_\tau \mathsf{E}(c\tau - L_\tau)$$

for different $c > 0$.

We show then that there exist two continuous functions $f(t)$, $g(t)$ and $T_0 > 0$ such that the optimal set C^* of continuation of observations has the form

$$C^* = C_1^* \cup C_2^*,$$

where

$$C_1^* = \{(t,x),\ t \leq T_0 \colon\ -f(t) < x < f(t)\},$$
$$C_2^* = \{(t,x),\ t \geq T_0 \colon g(t) < x < f(t) \text{ or } -f(t) < x < -g(t)\}.$$

Theorem 1. *For large t*

$$f(t) = \frac{t}{2} + A + O(e^{-t}), \qquad g(t) = \frac{t}{2} - A + O(e^{-t}),$$

where A is the unique solution of the equation

$$e^A - e^{-A} + 2A = 2c^{-1}.$$

The constant T_0 is a root of the equation $g(T_0) = 0$.

Also, we obtain integral equations for the boundaries $f = f(t)$, $t \geq 0$, and $g = g(t)$, $t \geq T_0$.

Optimal investment with high-watermark fees

Mihai Sîrbu
(joint work with Karel Janeček, Gerard Brunick)

The effect of high-watermark fees on *fund managers* is well studied in the finance literature, and more recently in mathematical finance [2]. The main goal of the present project is to analyze the effect of such fees on the *investor*, in models of increasing generality.

Consider an investor who chooses as investment vehicle a risky fund (hedge fund) with share/unit price F_t at time t. We assume that the investor can freely move money in and out of the risky fund and therefore continuously rebalance her investment. If the investor chooses to hold θ_t capital in the fund at time t and no fees of any kind are imposed, then her *accumulated profit* from investing in the fund evolves as

$$\begin{cases} dP_t = \theta_t \frac{dF_t}{F_t}, \ 0 \leq t < \infty \\ P_0 = 0. \end{cases}$$

Assume now that a proportion $\lambda > 0$ of the profits achieved by investing in the fund is paid by the investor to the fund manager. The fee is a commission to the fund manager for offering an investment opportunity for the investor (usually with a positive expected return). The fund manager keeps track of the accumulated profit that the investor made by holding the fund shares. More precisely, the manager tracks the *high-watermark of the investor's achieved profit*

$$M_t := \sup_{0 \leq s \leq t} P_s.$$

Any time the high-watermark increases, a λ percentage of this increase is paid to the fund manager, i.e., $\lambda \Delta M_t = \lambda(M_{t+\Delta t} - M_t)$ is paid by the investor to the manager in the interval $[t, t + \Delta t]$. Therefore, the evolution of the profit P_t of the investor is given by

$$\text{(1)} \qquad \begin{cases} dP_t = \theta_t \frac{dF_t}{F_t} - \lambda dM_t, \quad P_0 = 0 \\ M_t = \sup_{0 \leq s \leq t} P_s. \end{cases}$$

We emphasize that the fund price process F is exogenous to the investor. Equation (1) represents our general model of profits from investing in the fund, and can also be interpreted as a model of capital gain taxation.

In [3], we assume that the investor starts with initial capital $x > 0$ and the only additional investment opportunity is the money market paying zero interest rate. We further assume that the investor consumes at a rate $\gamma_t > 0$ per unit of time. We denote by C the accumulated consumption process. Since the money market pays zero interest rate, the wealth X_t of the investor at time t is given as initial wealth plus profit from the fund minus accumulated consumption, i.e. $X_t = x + P_t - C_t$. Taking this into account, the high-watermark of investor's achieved profit can be computed by tracking her wealth and accumulated consumption. More precisely, the high-watermark can be represented as

$$M_t = \sup_{0\leq s\leq t} (X_s + C_s) - x.$$

An investment and consumption strategy is called admissible if the corresponding wealth remains positive. The goal of the investor is to maximize discounted expected utility from consumption rate over an infinite horizon, which means to find the admissible (θ, γ) that maximizes

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t} U(\gamma_t) dt\right],$$

for some utility function U and discount factor $\beta > 0$. In order to use dynamic programming arguments, an important task is to choose carefully the state processes. We note that fees are paid whenever $X_t + C_t = \sup_{0\leq s\leq t}(X_s + C_s)$, which is the same as $X = N$ for

$$N_t := \sup_{0\leq s\leq t} (X_s + C_s) - C_t.$$

We therefore choose as state process the two-dimensional process (X, N) which satisfies $X \leq N$ and is reflected whenever $X = N$.

We further assume that the utility function U has the particular form

$$U(\gamma) = \frac{\gamma^{1-p}}{1-p}, \quad \gamma > 0,$$

for some $p > 0$, $p \neq 1$, and the fund follows a geometric Brownian motion with excess return $\alpha > 0$ and volatility σ. With these assumptions, the equation describing the evolution of (X, N) is

$$\begin{cases} dX_t = \big(\theta_t\alpha - \gamma_t\big)dt + \theta_t\sigma dW_t - \lambda(dN_t + \gamma_t dt), \quad X_0 = x \\ N_t = \sup_{0\leq s\leq t}\left\{X_s + \int_0^s \gamma_u du\right\} - \int_0^t \gamma_u du. \end{cases}$$

To summarize, we model the optimal investment and consumption in a hedge fund as the optimal control of a two-dimensional reflected diffusion (X, N). The Hamilton–Jacobi–Bellman equation can actually be reduced to one dimension, using the scaling property of the power utility function. We solve the problem by showing that the HJB equation has a smooth solution and then performing a verification argument. The solution of the HJB equation is found analytically, using Perron's method to obtain a viscosity solution and then upgrading its regularity. Since the problem does not admit closed-form solutions, we analyze the quantitative impact of the high-watermark fees through some numerical examples.

In [1] we take up the task of analyzing a more general model, with non-zero interest rates and additional assets. The key modelling observation is to consider the distance to paying fees, namely the process $Y := M - P$, as a state process instead of N as above. The process Y satisfies the equation

$$\begin{cases} dY_t = -\theta_t \frac{dF_t}{F_t} + (1+\lambda) dM_t, \\ Y_0 = 0, \quad Y_t \geq 0 \end{cases}$$

where the the positive measure dM charges only the set of times when $Y = 0$. This is the famous Skorohod equation, explaining the pathwise solution obtained in [2] or [3] for the accumulated profit P. In addition, considering the two-dimensional reflected diffusion (X, Y) as state process allows for a full analysis of interest rates and additional investment opportunities. The two-dimensional HJB equation still reduces to one dimension by scaling. We place particular emphasis on the analytic expansion of the optimal strategies with respect to the small fee $\lambda > 0$.

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Stochastic differential games and oligopolies

Ronnie Sircar
(joint work with Andrew Ledvina)

We discuss Cournot and Bertrand models of oligopolies, first in the context of static games and then in dynamic models. The static games, involving firms with different costs, lead to questions of how many competitors actively participate in a Nash equilibrium and how many are sidelined or blockaded from entry. The dynamic games lead to systems of nonlinear partial differential equations for which we discuss asymptotic and numerical approximations. Applications include markets for substitutable consumer goods (Bertrand) or differentiated grades of oil (Cournot).

Oligopolistic competition has been studied extensively in the economic literature, beginning with Cournot [2] where firms compete with one another in a static setup by choosing quantities to supply of a homogeneous good. This was later criticized by Bertrand [1] who said firms actually compete by setting prices. We study price-setting and quantity-setting oligopolies in continuous time and where the goods are differentiated from one another. However, much of the intuition about what one expects in certain market types is still grounded in the original static models. For example, the original Bertrand model results in perfect competition in all cases besides monopoly, which is unrealistic in most settings, leading one to conclude that the correct setup leads to the wrong result. The Cournot model leads to more realistic outcomes, but as most firms seem to set their prices, not their quantities, many economists have argued that the Cournot model gives the right answer for the wrong reason.

Our objective is to study the effect of product differentiation on the outcomes in these two oligopoly models. This builds on an earlier analysis of nonzero-sum differential games of Bertrand type in [6] and Cournot type in [3]. Moreover, this work is complementary to our comparison of Bertrand and Cournot oligopolies in a *static* setting in [4]. Here we compare Bertrand and Cournot oligopolies in a *continuous-time* framework with *two* players and a *linear* demand structure. The inverse demand system, which forms the basis of Cournot competition, is given by

$$p_i(q) = \alpha - \beta\left(q_i + \varepsilon q_j\right), \qquad i = 1,2; j \neq i. \tag{1}$$

The parameter ε is positive to model substitute goods. For $\varepsilon < 1$, we can invert the system (1), to obtain the duopoly demand

$$q_i(p) = \left(\frac{\alpha}{\beta(1+\varepsilon)}\right) - \left(\frac{1}{\beta(1-\varepsilon^2)}\right)\left(p_i - \varepsilon p_j\right), \qquad i = 1,2; i \neq j, \tag{2}$$

which is the basis of Bertrand competition.

Denote the capacity of firm $i \in \{1,2\}$ by $X_t^{i,b}$, $X_t^{i,c}$ for the Bertrand and Cournot games, respectively. Throughout what follows a superscript b will indicate a variable related to the game of Bertrand type and a superscript c for Cournot. We look for Markov perfect equilibria; in other words, firms use Markovian strategies. In the Bertrand game, let the Markovian price strategy of firm i at time t be given by $p^i(X_t^{1,b}, X_t^{2,b})$, $i = 1,2$. Similarly, in the Cournot game, let the Markovian strategic rate of supply of firm i be given by $q^i(X_t^{1,c}, X_t^{2,c})$.

In the Bertrand game, the dynamics of the state processes are given by the controlled stochastic differential equations

$$dX_t^{i,b} = -q_i(p^1(X_t^{1,b}, X_t^{2,b}), p^2(X_t^{1,b}, X_t^{2,b}))dt + \sigma^i dW_t^i \tag{3}$$

for $i = 1,2$ and where (W_t^1) and (W_t^2) are correlated Brownian motions with $\mathbb{E}\left\{dW_t^1 \cdot dW_t^2\right\} = \rho dt$. These are the correct dynamics provided that $X_t^{1,b} > 0$, $X_t^{2,b} > 0$. If either one is strictly positive and the other is zero, then the first has a monopoly, and the other remains at zero forever.

The dynamics for the Cournot state variables are defined in a similar fashion by

$$dX_t^{i,c} = -q^i(X_t^{1,c}, X_t^{2,c})dt + \sigma^i dW_t^i, \tag{4}$$

where the Brownian motions are the same as those above. Again, these only hold for $X_t^{1,c} > 0$, $X_t^{2,c} > 0$, but here things are slightly more simple. If either state variable hits zero then the corresponding q^i is equal to zero. The Brownian motions in (3) and (4) could represent uncertainty of actual demand or of remaining reserves, depending on the context of actual application.

The objective of the firms is to maximize expected lifetime profit in an equilibrium sense to be made precise below. To this end, we define the profit functionals of the firms

$$J^{i,b}(p^1(x_1,x_2),p^2(x_1,x_2)) := \mathbb{E}_{x_1,x_2}\left\{\int_0^{\tau^{i,b}} e^{-rt}p^i q_i(p^1,p^2)dt\right\}, \tag{5}$$

$$J^{i,c}(q^1(x_1,x_2),q^2(x_1,x_2)) := \mathbb{E}_{x_1,x_2}\left\{\int_0^{\tau^{i,c}} e^{-rt}p_i(q^1,q^2)q^i dt\right\}, \tag{6}$$

where $\tau^{i,b} = \inf\{t > 0 : X_t^{i,b} = 0\}$, and similarly for $\tau^{i,c}$. A vector $p^\star$ is a **Markov perfect Nash equilibrium** of the dynamic Bertrand game if for all positive and suitably regular (for example Lipschitz) Markov controls p^1 we have $J^{1,b}(p^{1,\star},p^{2,\star}) \geq J^{1,b}(p^1,p^{2,\star})$, and for all such Markov controls p^2 we have $J^{2,b}(p^{1,\star},p^{2,\star}) \geq J^{2,b}(p^{1,\star},p^2)$. The concept is defined analogously for the dynamic Cournot game.

As the players employ Markovian strategies, we define the value functions of the players as a function of their capacities by

$$V^{i,b}(x_1,x_2) = \sup_{p^i} J^{i,b}(p^1,p^2), \qquad V^{i,c}(x_1,x_2) = \sup_{q^i} J^{i,c}(q^1,q^2).$$

Assuming sufficient regularity of the value functions, a sufficient condition for equilibrium can be found by solving the associated system of HJB partial differential equations

$$\mathcal{L}V^{i,b} + \sup_{p^i\geq 0}\left\{-V^{i,b}_{x_1}q_1(p^1,p^2) - V^{i,b}_{x_2}q_2(p^1,p^2) + p^i\cdot q_i(p^1,p^2)\right\} = rV^{i,b}, \tag{7}$$

$$\mathcal{L}V^{i,c} + \sup_{q^i\geq 0}\left\{-V^{i,c}_{x_1}q^1 - V^{i,c}_{x_2}q^2 + p_i(q^1,q^2)\cdot q^i\right\} = rV^{i,c}, \tag{8}$$

where $\mathcal{L} = \frac{1}{2}(\sigma^1)^2\frac{\partial^2}{\partial(x_1)^2} + \rho\sigma^1\sigma^2\frac{\partial^2}{\partial x_1\partial x_2} + \frac{1}{2}(\sigma^2)^2\frac{\partial^2}{\partial(x_2)^2}$. In order to complete the description of the PDE problem associated with these games, we also need to specify boundary conditions. One of these conditions is quite straightforward; we must have $V^{1,\cdot}(0,x_2) = V^{2,\cdot}(x_1,0) = 0$ as there is no profit possible once a firm exhausts their capacity. The other boundary is slightly more complicated. Let v_m^b be the value function of a monopolist in a Bertrand market, and v_m^c be the corresponding value function in a Cournot market. We then have the condition $V^{1,\cdot}(x_1,0) = v_m^{\cdot}(x_1)$ and $V^{2,\cdot}(0,x_2) = v_m^{\cdot}(x_2)$.

The **degree of product differentiation** is measured by the quantity $\varepsilon \in [0,1)$. If $\varepsilon = 0$ then the individual firm inverse demand and direct demand functions are equal to their monopoly counterparts, i.e. each firm has a monopoly in the market for their individual good which implies their behavior is independent of the other firm. In the case of no randomness, $\sigma^i = 0$, we provide a three-term asymptotic expansion for the value functions for Cournot and Bertrand markets in powers of ε. It turns out that the first two terms are identical, but the third $\mathcal{O}(\varepsilon^2)$ is of different sign: negative for Bertrand and positive for Cournot. It follows from these approximations that the game ends sooner in the Bertrand market. This again aligns with previous intuition that a Bertrand market is more competitive than a Cournot market. This increased level of competition leads to a faster rate of capacity depletion, ceteris paribus. However, the intuition that comes from the static game breaks down when we plot the resulting price paths. The static game intuition says that Bertrand should have lower prices and higher quantities. This is true at the beginning of the game when both firms have a large capacity. But, as the firms deplete their capacities, we see that the price in the Bertrand market increases until it finishes above that in the Cournot market. Likewise, the quantity begins higher in the Bertrand market, but eventually drops below the quantity in the Cournot market. The dynamic nature of the game and the dependency on firms' capacities leads to these counterintuitive results which are discussed in more detail, along with a numerical study of the stochastic game, in [5].

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Second order BSDEs: existence and uniqueness

H. M. Soner

(joint work with N. Touzi and J. Zhang)

This talk summarizes a recent joint paper of the author with Touzi and Zhang [3]. The paper provides a new formulation of second order stochastic target problems introduced in [2] by modifying the reference probability so as to allow for different scales. This new ingredient enables us to prove a dual formulation of the target problem as the supremum of the solutions of standard backward stochastic differential equations. In particular, in the Markov case, the dual problem is known to be connected to a fully nonlinear, parabolic partial differential equation and this connection can be viewed as a stochastic representation for all nonlinear, scalar, second order, parabolic equations with a convex Hessian dependence.

We continue with the description of the target problem. Let B be a Brownian motion under the probability measure $\mathbb{P}_0$ and $\{\mathcal{F}_t, t \geq 0\}$ the corresponding filtration. For a continuous semimartingale Z, we denote by Γ the density of its covariation with B. We then define the controlled process Y by

$$Y_t := y - \int_0^t H_s(Y_s, Z_s, \Gamma_s)ds + \int_0^t Z_s \circ dB_s, \qquad d\langle Z, B\rangle_t = \Gamma_t dt, \tag{1}$$

where $\circ$ denotes the Fisk–Stratonovich stochastic integration. We assume that the given random nonlinear function H satisfies the standard Lipschitz and measurability conditions. Then, for any reasonable process Z and an initial condition y, a unique solution, which is denoted by $Y^{y,Z}$, exists. We now fix a time horizon, say $T = 1$, and a class of admissible controls Z^0. Then, given an $\mathcal{F}_1$-measurable random variable ξ, [2] defines the second order stochastic target problem by

$$V^0 := \inf \left\{ y : \ Y_1^{y,Z} \geq \xi \ \mathbb{P}_0\text{-a.s. for some } Z \in Z^0 \right\}. \tag{2}$$

In this formulation, the structure of the set of admissible controls is crucial. In fact, if Z^0 is not properly defined, then the dependence of the problem on the variable Γ can be trivialized. We refer to [1] for a detailed discussion of this issue in a particular example of mathematical finance. One of the achievements of the new approach given below is to avoid this strong dependence on the control set and simply to work with standard spaces.

Here we only provide an intuitive description of our formulation. For this heuristic explanation we assume a Markov structure. Namely we assume that H in (1) and ξ in (2) are given by

$$H_t(y, z, \gamma) = h(t, X_t, y, z, \gamma), \quad \xi = g(X_T), \tag{3}$$

where X is the solution of a Markov stochastic differential equation and h, g are deterministic scalar functions. Let $V^0(t, x)$ be defined as in (2) with time origin at t and $X_t = x$. As it is usual, we assume that $\gamma \mapsto h(t, x, y, z, \gamma)$ is non-decreasing. Then, by an appropriate choice of admissible controls, it is shown in [2] that this problem is a viscosity solution of the corresponding dynamic programming equation,

$$-\frac{\partial u}{\partial t} - h\left(t, x, u(t, x), Du(t, x), D^2u(t, x)\right) = 0, \quad u(1, x) = g(x). \tag{4}$$

We further assume that $\gamma \mapsto h(t, x, r, p, \gamma)$ is convex. Then,

$$h(t, x, r, p, \gamma) = \sup_{a \geq 0} \left\{ \frac{1}{2} a\gamma - f(t, x, r, p, a) \right\}, \tag{5}$$

where f is the (partial) convex conjugate of h with respect to γ. Let D_f be the domain of f as a function of a. By the classical maximum principle of parabolic differential equations, we expect that for every $a \in D_f$, the solution $u \geq u^a$, where u solves (4) and u^a is defined as the solution of the semi-linear PDE

$$-\frac{\partial u}{\partial t} - \frac{1}{2} aD^2u(t, x) + f(t, x, u(t, x), Du(t, x), a) = 0, \quad u(1, x) = g(x). \tag{6}$$

In turn, by standard results, $u^a(t,x) = Y_t^a$, where, for $s \in [t,T]$,

$$X_s^a = x + \int_t^s a_r^{1/2} dB_r, \tag{7}$$

$$Y_s^a = g\left(X_T^a\right) - \int_s^T f\left(r, X_r^a, Y_r^a, Z_r^a, a\right) dr - \int_t^T Z_r^a a^{1/2} dB_s. \tag{8}$$

We have formally argued that $V^0(t,x) \geq Y_t^a$ for any $a \in D_f$. Let A^f be the collection of all processes with values in D_f. By extending (7), (8) to processes a, it is then natural to consider the problem

$$V_t := \sup_{a \in A^f} Y_t^a \tag{9}$$

as the dual of the primal stochastic target problem. Indeed, the optimization problem (9) corresponds to the dual formulation of the second order target problem in the Markov case. Such a duality relation was suggested in the specific example of gamma constraints and can be proved rigorously by showing that $v(t,x) := V_t$ is a viscosity solution of the fully nonlinear PDE (4). This, by uniqueness, implies that $v = V^0$. Of course, such an argument requires some technical conditions at least to guarantee that comparison of viscosity supersolutions and subsolutions holds true for the PDE (4).

The main object of this paper is to provide a purely probabilistic proof of this duality result. Moreover, our duality result does not require to restrict the problem to the Markov framework.

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Asymptotic results and statistical procedures for time-changed Lévy processes sampled at hitting times

Peter Tankov

(joint work with Mathieu Rosenbaum)

In this talk, based on the paper [5], we focus on time-changed Lévy models, that is, we assume that the process of interest Y is given by $Y_t = X_{S_t}$ where X is a one-dimensional Lévy process and S is a continuous increasing process (a time change), which plays the role of the integrated volatility in this setting. Time-changed Lévy models were introduced into the financial literature in [2] and their estimation from high frequency data with deterministic sampling was recently addressed in [3, 6].

In the context of ultra high-frequency financial data, the assumption of deterministic sampling times is arguably too restrictive. In this work we assume that the sampling times are given by first hitting times of symmetric barriers whose distance with respect to the starting point is equal to ε. More precisely, the process Y is observed at times $(T_i^\varepsilon)_{i\geq 0}$ with $T_0^\varepsilon = 0$ and $T_{i+1}^\varepsilon = \inf\{t > T_i^\varepsilon : |Y_t - Y_{T_i^\varepsilon}| \geq \varepsilon\}$ for $i \geq 1$. The parameter ε is the parameter driving the asymptotic and thus we will assume that ε goes to zero. This scheme is probably the most simple and common endogenous sampling scheme. Moreover, in the spirit of [4] it can be seen as a first step towards a model for ultra high frequency financial data including jump effects.

Convergence of the exit time and the overshoot We focus on the class of Lévy processes such that for a suitable $\alpha \in (0, 2]$, the rescaled process

$$(X_t^{\alpha,\varepsilon})_{t\geq 0} := (\varepsilon^{-1} X_{\varepsilon^\alpha t})_{t\geq 0}$$

converges in law to a strictly α-stable Lévy process X^* as ε goes to zero. This class turns out to be rather large, and contains in particular all Lévy processes with non-zero diffusion component, all finite variation Lévy processes with non-zero drift and also most parametric Lévy models found in the literature. We show that for such Lévy processes the moments of first exit times from intervals, and certain functionals of the overshoot converge to the corresponding functionals of the limiting stable process, which are often known explicitly.

More precisely, denote the first exit time of the rescaled process from the interval $(-1, 1)$ by $\tau_1^\varepsilon := \inf\{t \geq 0 : |X_t^{\varepsilon,\alpha}| \geq 1\}$ and the first exit time of the limiting process X^* from the interval $(-1, 1)$ by τ^*. We show that

(1) $(\tau_1^\varepsilon, X_{\tau_1^\varepsilon}^\varepsilon)$ converges in law to $(\tau_1^*, X_{\tau_1^*}^*)$ as $\varepsilon \downarrow 0$.

(2) $\lim_{\varepsilon\downarrow 0} E[(\tau_1^\varepsilon)^k f(X_{\tau_1^\varepsilon}^\varepsilon)] = E[(\tau_1^*)^k f(X_{\tau_1^*}^*)]$ for all $k \geq 1$.

Under additional assumptions on the process X, the rate of the above convergence can be quantified, namely we show that

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-\alpha/2}(E[\tau_1^\varepsilon] - E[\tau_1^*]) = 0$$

and for a bounded continuous function f,

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-\alpha/2}(E[f(X_{\tau_1^\varepsilon}^\varepsilon)] - E[f(X_{\tau_1^*}^*)]) = 0.$$

Statistical applications The above asymptotic results, which are of interest in their own right, allow us to prove the convergence of quantities of the form

$$V^\varepsilon(f)_t = \sum_{T_i^\varepsilon \leq t} f\big(\varepsilon^{-1}(Y_{T_i^\varepsilon} - Y_{T_{i-1}^\varepsilon})\big)$$

to known deterministic functionals of the limiting process X^* and the time change S. More precisely, let

$$m(f) = \frac{E[f(X_{\tau_1^*}^*)]}{E[\tau_1^*]}$$

and let f be a bounded continuous function on $\mathbb{R}$. Then

$$\lim_{\varepsilon\downarrow 0} \varepsilon^\alpha V^\varepsilon(f)_t = m(f) S_t \quad (1)$$

in probability, uniformly on compact sets in t (ucp).

This result can be in particular used to build estimators of relevant quantities such as the time change or the Blumenthal–Getoor index. The time change can be recovered simply from the times (T_i^ε) as $\varepsilon \to 0$, by taking $f = 1$, which gives,

$$S_t = \lim_{\varepsilon \downarrow 0} \varepsilon^\alpha V^\varepsilon(1)_t E[\tau_1^*].$$

In a model where the limiting process X^* is a symmetric α-stable process with $\alpha \in (1,2)$, such as for example the CGMY process [2], the Blumenthal–Getoor index of X, which coincides with the parameter α, can be recovered via

$$\alpha = 2 \lim_{\varepsilon \downarrow 0} \frac{V^\varepsilon(f)_t}{V^\varepsilon(1)_t}, \quad f(x) = \frac{1}{x^2} \wedge 1.$$

Assuming that the time change S defining Y is independent of the underlying Lévy process X, one can in some cases establish the rate of convergence and asymptotic normality of the renormalized error in (1). Define $R_t^\varepsilon = (R_{t,1}^\varepsilon, \dots, R_{t,d}^\varepsilon)$ with

$$R_{t,j}^\varepsilon = \varepsilon^{-\alpha/2}(\varepsilon^\alpha V^\varepsilon(f_j)_t - m(f_j)S_t).$$

Then, as ε goes to zero, R^ε converges in law to $B \circ S$, for the usual Skorohod topology, with B a continuous centered $\mathbb{R}^d$-valued Gaussian process with independent increments, independent of S, such that $E[B_{t,j}B_{t,k}] = (t/(E[\tau_1^*])C_{j,k}$ with

$$C_{j,k} = \mathrm{Cov}[f_j(X^*_{\tau_1^*}) - m(f_j)\tau_1^*, f_k(X^*_{\tau_1^*}) - m(f_k)\tau_1^*].$$

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Matrix-valued affine processes and their applications

Josef Teichmann

(joint work with Christa Cuchiero, Martin Keller-Ressel and Walter Schachermayer)

We present two new results on affine processes: first, we show that affine processes (in the sense that the Fourier–Laplace transform of a stochastically continuous Markov process on some subset of $\mathbb{R}^d$ is exponentially affine in the state variable) necessarily admit a semimartingale version with characteristics affine in the state variable. This is an important progress towards the final goal of classifying all affine processes on all possible state spaces, since now one can analyse the problem from the point of view of stochastic invariance for Markovian semimartingales. The result has been obtained jointly with Christa Cuchiero, Martin Keller-Ressel and Walter Schachermayer. Second, a result on filtering of affine processes is presented. We show that the Zakai equation related to the noisy (linear) observation of an affine process admits a deterministic high-order approximation scheme by affine methodology. We introduce for this purpose stochastic Riccati equations and show that their solutions lead to stochastic evolutions of (unnormalized) conditional density processes.

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