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Topological and Geometric Combinatorics

Organised by
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Abstract. The 2011 Oberwolfach meeting "Topological and Geometric Combinatorics" was organized by Anders Björner (Stockholm), Gil Kalai (Jerusalem), Isabella Novik (Seattle), and Günter M. Ziegler (Berlin). It covered a wide variety of aspects of Discrete Geometry, Topological Combinatorics, and Geometric Topology. Some of the highlights of the conference included (1) counterexamples to the Hirsch conjecture, (2) the latest results around the colored Tverberg theorem, and (3) recent developments on the complexity of the simplex algorithm.

Mathematics Subject Classification (2000): 05xx, 52xx, 55xx, 57xx.

Introduction by the Organisers

The 2011 Oberwolfach meeting "Topological and Geometric Combinatorics" was organized by Anders Björner (KTH and Mittag-Leffler Institute, Stockholm), Gil Kalai (Hebrew University, Jerusalem), Isabella Novik (University of Washington, Seattle), and Günter M. Ziegler (Technical University, Berlin). The conference consisted of three one-hour lectures by Francisco Santos, Pavle Blagojević, and Thomas Hansen on outstanding recent developments in the field, as well as twenty-eight talks ranging from half-hour to 45-minutes presentations, a problem session (led by Gil Kalai), and many more informal sessions, group discussions, and a great variety of small group and pairwise discussions. It was a very productive and enjoyable week.

The conference treated a broad spectrum of topics from Discrete Geometry (such as polytopes, epsilon nets, rigidity, complexity, etc.), Topological Combinatorics (such as problems surrounding Tverberg's theorem, topological representations of matroids, poset topology, etc.), and Geometric Topology (triangulated manifolds, embeddings of polyhedra, homology of random complexes, etc.). It is impossible to summarize in one-page report the richness and depth of the work and presentations. Instead we will concentrate here on some of the highlights.

The very first lecture on Monday was given by Francisco Santos and was devoted to describing his recent counterexamples to the Hirsch conjecture. This 53-year-old conjecture posits that the diameter of a convex polytope with n facets in dimension d is at most $n-d$. Aside from its mathematical interest, this conjecture is important because the simplex method for solving a linear program walks along a path on the surface of a polytope (often in dimensions as large as $d = 10000$). Thus the diameter of the polytope provides a lower bound on its worst-case complexity, and establishing an upper bound on the diameter raises that complexity lower bound. In his talk Santos also announced the explicit polymake computation of an example (joint with Christophe Weibel): it is a 20-dimensional simple polytope with 40 facets and 36'442 vertices, of diameter 21.

Spectacular recent developments on the complexity of the simplex algorithm were presented in the talk by Thomas Dueholm Hansen describing his joint work with Oliver Friedmann and Uri Zwick. The three of them have managed to prove subexponential lower bounds of the form 2^{n^α} for two basic randomized pivot rules for the simplex algorithm. This is the first result of its kind and deciding if this is possible was an open problem for several decades.

A very impressive account on how topological methods were used in a very recent series of works by Pavle Blagojević, Benjamin Matschke, and Günter M. Ziegler to solve several long-standing problems surrounding Tverberg's theorem was given by Pavle Blagojević.

Janos Pach talked about his recent paper with Gabor Tardos describing a breakthrough in geometric constructions for ϵ-nets. It follows from the general theory of VC-dimension that *epsilon*-nets of size $O(\epsilon \log(1/\epsilon))$ can be constructed for various classes of geometric objects such as half spaces in Euclidean spaces. Pach and Tardos showed that this estimate is sharp for very simple geometric objects such as half spaces in four dimensions.

Patricia Hersh described new topological methods to study stratified spaces leading to a proof of a conjecture by Fomin and Shapiro on certain complexes arising from Coxeter groups. Mark Noy's lecture outlined the remarkable recent understanding of diameter of random planar triangulations with n vertices which in agreement with old mysterious conjectures from physics behaves like $n^{1/4}$.

We also cannot avoid mentioning a lively and incredible problem session: a large number of the problems/questions raised were answered on spot.

The collection of abstracts below presents an overview of the official program of the conference. It does not cover all the additional smaller presentations, group discussions and blackboard meetings, nor the lively interactions that occurred during the week. However, it does convey the manifold connections between the themes of the conference, refinements of well-established bridges, completely new links between seemingly distant themes, problems, methods, and theories, as well as demonstrates substantial progress on older problems. In short, it shows that the area is very much alive!

We are extremely grateful to the Oberwolfach institute, its director and to all of its staff for providing a perfect setting for an inspiring, intensive week of "Topological and Geometric Combinatorics".

Anders Björner, Gil Kalai, Isabella Novik, Günter M. Ziegler
Stockholm/Jerusalem/Seattle/Berlin, March 2011

Workshop: Topological and Geometric Combinatorics

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Abstracts

Counter-example(s) to the Hirsch Conjecture

Francisco Santos

The Hirsch Conjecture, posed by Warren M. Hirsch in 1957 in a personal communication to George Dantzig [1, p. 25], stated that the graph of a polytope of dimension d with n facets could not exceed $n - d$. The conjecture was posed and is relevant in the context of the simplex method for linear programming[1], invented by Dantzig some ten years earlier. In this talk I went through the main ideas behind my recent construction of counter-examples to the Hirsch Conjecture.

From the existence of a single non-Hirsch polytope, with somehow standard glueing techniques (similar to those of [2]) one can get:

Theorem 1. *There is an infinite family of polytopes in a fixed dimension d with diameter growing as $(1+\epsilon)(n-d)$, for a positive $\epsilon = \epsilon(d)$.*

The values of ϵ we can obtain so far are $\epsilon(40) = 1/40$ and $\lim_{d\to\infty} \epsilon(d) = 1/20$. They follow from the existence of a polytope of dimension 20 with 40 facets and of diameter 21 (constructed jointly with Christophe Weibel).

All known constructions of non-Hirsch polytopes are based on the combination of two results: A "strong d-step Theorem for spindles and prismatoids" and the construction of prismatoids of large width. Details can be found in [5].

The strong d-step Theorem

The d-step Theorem of Klee and Walkup [3] implies that to prove or disprove the Hirsch Conjecture one only needs to look at the case $n = 2d$:

Lemma 2 ([3])**.** *Let P be a polytope of dimension d, with n facets and diameter δ. Then, the wedge of P on any of its facets has dimension $d+1$, $n+1$ facets, and diameter at least δ.*

Corollary 3 (d-step Theorem [3])**.** *The maximum diameter among all polytopes with k facets more then their dimension (but with varying dimensions) is achieved for polytopes of dimension k.*

The *Strong d-step Theorem* is a variant of this which applies to the following class of polytopes:

Definition 4. *A polytope P with two distinguished vertices u and v is a* spindle *if every facet of it contains either u or v, but not both. The* length *of a spindle is the distance from u to v along the graph of P.*

[1] In fact, the original conjecture was not for polytopes but for (perhaps unbounded) polyhedra; that is, for feasibility regions of linear programs defined by n inequalities in d variables. But a counter-example to the unbounded case was found by Klee and Walkup [3] so the conjecture remained only for the bounded case.

Lemma 5 ([5]). *Let P be a spindle of dimension d, with n facets and diameter δ. Suppose that $n > 2d$. Then, a certain perturbation of the wedge of P on one of its facets is a spindle of dimension $d+1$ with $n+1$ facets and diameter at least $\delta+1$.*

Corollary 6 (Strong d-step Theorem). *If a spindle P with n facets and dimension d has length greater than d then there is another spindle P' of dimension $n-d$ with $2(n-d)$ facets and with length $\delta+(n-2d) > n-d$. Hence, P' is a counter-example to the Hirsch Conjecture.*

Prismatoids and pairs of maps

Instead of looking at the length of spindles we find it easier to work in the dual setting, where this translates to the width of prismatoids:

Definition 7. *A polytope Q with two distinguished parallel facets Q^+ and Q^- is a* prismatoid *if every vertex of it lies either in Q^+ or Q^- (but not both, since the facets are parallel). The* width *of a prismatoid is the distance from Q^+ to Q^- along the dual graph of Q. (The dual graph has the facets of Q as nodes and the ridges as edges).*

The combinatorics of a d-prismatoid Q can be deduced from that of an intermediate slice $Q \cap H$, where H is a hyperplane parallel to and in between the facets Q^+ and Q^-. This slice, in turn, is a weighted Minkowski sum of Q^+ and Q^-, whose normal fan is the superposition (or *common refinement*) of those of Q^+ and Q^-. To reduce the dimension of the problem by one more unit, we look at the normal fans of Q^+ and Q^- as lying in the unit sphere of dimension $d-2$. This suggests the following definition and result:

Definition 8. *A* geodesic map *on the sphere S^{d-2} is a regular cell decomposition of it into convex polyhedral pieces (that is, the intersection of S^{d-2} with a complete fan in $\mathbb{R}^{d-1}$). Given two geodesic maps G^+ and G^- in S^{d-2} the* width *of the pair (G^+, G^-) is the minimum graph distance from a vertex of G^+ to a vertex of G^- along the graph of the common refinement of G^+ and G^-.*

Lemma 9. *The width of a prismatoid $Q \subset \mathbb{R}^d$ equals two plus the width of its corresponding pair of maps (G^+, G^-) in S^{d-2}.*

So, the construction of spindles of length greater than their dimension is equivalent to the construction of prismatoids of width greater than their dimension, and to the construction of pairs of maps of width greater than the dimension of their ambient sphere.

The width of prismatoids of dimensions 4 and 5

The initial counter-example to the Hirsch Conjecture [5] was based on a prismatoid of dimension 5 and width 6 with 48 vertices, constructed as a pair of geodesic maps in the 3-sphere, each map having 24 facets. (This yields a non-Hirsch polytope of dimension 43 and with 86 facets). The construction technique is based on the Hopf decomposition of the 3-sphere or, more precisely, on placing the vertices of the two maps G^+ and G^- in two geodesic tori parallel to one another and sufficiently far apart, so that the "interesting part" of the pair of maps, the region between the two tori, has the same intersection pattern as the two maps would have if drawn on the same flat torus.

More recently, together with B. Matschke and C. Weibel, similar techniques have been used to improve this construction in the following two senses:

- There are 5-prismatoids of width six with only 25 vertices, which implies there are non-Hirsch polytopes of dimension 20.
- There are 5-prismatoids with an arbitrarily large number n of vertices and width growing as $\Omega(\sqrt{n})$.

But the following results limit the diameter of the counter-examples that can be constructed with these ideas:

- No 4-prismatoid has width larger than 4 [6, 7].
- No 5-prismatoid has width larger than $n/2 + 3$, where n is its number of vertices.
- No d-prismatoid has width larger than $2^{d-3}n$.

The second statement implies that no polytope constructed via Corollary 6 from a 5-prismatoid will violate the Hirsch Conjecture by more than 50%. As a conclusion, although the counter-examples constructed so far brake a "psychological barrier", in the context of the simplex method they are somehow irrelevant. Much more important would be having an answer to any of the following questions:

Questions 10. *(1) Is there a polynomial upper bound to the maximum diameter of a d-polytope with n-facets? (Equivalentely, of a d-polytope with* $2d$ *facets).*

(2) Is there a family of polytopes whose diameter grows superlinearly with the number of facets? (If so, it is known that the dimension has to grow too).

We finished the talk with the following variation of a Conjecture of Nikolai Hähnle (see the problem session in this same Oberwolfach report for more details on this):

Conjecture 11. *The diameter of a d-polytope with n facets cannot exceed* $d(n-d)$.

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The Bárány–Larman conjecture and the colored Tverberg theorems

Pavle Blagojević
(joint work with Benjamin Matschke, Günter Ziegler)

1. Topological Tverberg theorem

In 1966, solving a problem by Bryan John Birch, Helge Tverberg in the paper [9] proved the following coincidence theorem.

Theorem 1. *Let $d \geq 1$, $r \geq 2$ be integers, and $N := (d+1)(r-1)$. For every affine map $f : \Delta_N \to \mathbb{R}^d$ of the standard N-simplex Δ_N there are r disjoint faces $F_1, \ldots, F_r$ of Δ_N whose images under f intersect, that is,*

$$f(F_1) \cap f(F_2) \cap \cdots \cap f(F_r) \neq \emptyset.$$

Weakening the assumption that f is an affine map by asking only for continuity, Bárány, Shlosman and Szűcs in the paper [3] from 1981 invited topology to the realm of discrete and convex geometry. They conjectured the so called Topological Tverberg theorem.

Conjecture 2. *Let $d \geq 1$, $r \geq 2$ be integers, and $N := (d+1)(r-1)$. For every continuous map $f : \Delta_N \to \mathbb{R}^d$ there are r disjoint faces $F_1, \ldots, F_r$ of Δ_N whose images under f intersect, that is,*

$$f(F_1) \cap f(F_2) \cap \cdots \cap f(F_r) \neq \emptyset.$$

Surprisingly, the conjecture essentially depends on the number of disjoint faces we ask for. In the line of proving the conjecture for prime r's, Bárány, Shlosman and Szűcs established the sufficient condition for the Topological Tverberg conjecture to hold:

If there is no Σ_r-equivariant map $\mathrm{E}_r\Sigma_r \to S\left(W_r^{\oplus d}\right)$, then the conjecture is true for $r \geq 2$.

Here Σ_r denotes the symmetric group on r letters, $\mathrm{E}_r\Sigma_r$ stands for an r-dimensional, $(r-1)$-connected, free Σ_r space and $W_r = \{(x_1, \ldots, x_r) \in \mathbb{R}^r \ : \ x_1 + \cdots + x_r = 0\}$ for Σ_r-representation given by coordinate permutation. Sarkaria also established a sufficient conditions for the Topological Tverberg conjecture to hold via the so called deleted join construction:

If there is no Σ_r-equivariant map $[r]^{*(N+1)} \to S\left(W_r^{\oplus(d+1)}\right)$, then the conjecture is true for $r \geq 2$.

By $[r]$ we assume the 0-dimensional simplicial complex with r vertices.

Different equivariant topology methods were applied:

- for r a prime: Bárány, Shlosman and Szűcs, relaying on connectivity of $\mathrm{E}_r\Sigma_r$ and free $\mathbb{Z}_r \subseteq \Sigma_r$ action on the sphere $S\left(W_r^{\oplus d}\right)$, used Dold's theorem to prove the **non-existence** of a $\mathbb{Z}_r \subseteq \Sigma_r$ equivariant map $\mathrm{E}_r\Sigma_r \to S\left(W_r^{\oplus d}\right)$;

- for $r = p^n$ a prime power: Özaydin, in the paper [7], using the connectivity of $\mathrm{E}_r\Sigma_r$ and fixed point free action of $(\mathbb{Z}_p)^n \subseteq \Sigma_r$ on the sphere $S\left(W_r^{\oplus d}\right)$ applied equivariant cohomology of elementary abelian groups to prove the **non-existence** of a $(\mathbb{Z}_p)^n \subseteq \Sigma_r$ equivariant map $\mathrm{E}_r\Sigma_r \to S\left(W_r^{\oplus d}\right)$; and finally
- for r not a prime power: Özaydin, in the same paper [7], used equivariant obstruction theory to prove the **existence** of a Σ_r equivariant map $\mathrm{E}_r\Sigma_r \to S\left(W_r^{\oplus d}\right)$.

Thus, the topological Tverberg conjecture is wide open for r non a prime power and presents one of the most challenging and resistent problems in the area.

2. Colored Tverberg Theorem of Bárány and Larman

In their 1990 study of halving lines and halving planes, Bárány, Füredi and Lovász, in the paper [1], observed the need for a colored version of Tverberg's theorem. They provided a first case of three triangles in the plane. In response to this, Bárány and Larman in the paper [2] from 1992 formulated the following general problem and proved the first group of results for it.

The colored Tverberg problem: Let $d \geq 1$, $r \geq 2$ be integers. Determine the smallest number $n = n(d, r)$ such that for every affine map $f : \Delta_{n-1} \to \mathbb{R}^d$ and every partition of the vertex set $\mathrm{vert}(\Delta_{n-1}) = C_0 \uplus \cdots \uplus C_d$ into $d+1$ colors, with the property that $|C_i| \geq r$ for every $i \in \{0, \ldots, d\}$, there are r disjoint "rainbow" faces $F_1, \ldots, F_r$ of the simplex Δ_{n-1}, i.e.

$$(\forall\ i \in \{0, \ldots, d\}, j \in \{1, \ldots, r\})\ \ |C_i \cap F_j| \leq 1,$$

such that

$$f(F_1) \cap f(F_2) \cap \cdots \cap f(F_r) \neq \emptyset.$$

A trivial lower bound for the function $n(d, r)$ is $(d + 1)r$. Bárány and Larman proved that the trivial lower bound is tight in the cases $N(r, 1) = 2r$ and $N(r, 2) = 3r$, and presented a proof by Lovász for $N(2, d) = 2(d + 1)$. They conjectured the following equality.

The Bárány–Larman Conjecture: *$n(d, r) = (d + 1)r$ holds for all $r \geq 2$ and $d \geq 1$.*

Still in 1992, Vrećica and Živaljević, in the paper [10], proposed an alternative colored Tverberg problem.

The alternative colored Tverberg problem: Let $d \geq 1$, $r \geq 2$ be integers. Determine the smallest number $t = t(d, r)$ such that for every affine map $f : \Delta_{(d+1)t} \to \mathbb{R}^d$ and every partition of the vertex set $\mathrm{vert}(\Delta_{(d+1)t}) = C_0 \uplus \cdots \uplus C_d$ into $d+1$ colors, with the property that $|C_i| = t$ for every $i \in \{0, \ldots, d\}$, there are r disjoint "rainbow" faces $F_1, \ldots, F_r$ of the simplex $\Delta_{(d+1)t}$, i.e.

$$(\forall\ i \in \{0, \ldots, d\}, j \in \{1, \ldots, r\})\ \ |C_i \cap F_j| \leq 1,$$

such that

$$f(F_1) \cap f(F_2) \cap \cdots \cap f(F_r) \neq \emptyset.$$

In language of the function $t(d,r)$ the Bárány–Larman Conjecture claims that $t(d,r) = r$ for all $r \geq 2$ and $d \geq 1$. Vrećica and Živaljević established the following sufficient condition for the bounding function $t(d,r)$:

If there is no Σ_r-equivariant map $\Delta_{t,r}^{*(d+1)} \to S\left(W_r^{\oplus(d+1)}\right)$, then $t(d,r) \leq t$.

Here $\Delta_{m,n}$ stands for $m \times n$ chessboard complex.

Using the connectivity result of Björner, Lovász, Vrećica and Živaljević:

$$\mathrm{conn}(\Delta_{m,n}) = \min\left\{m, n, \left\lfloor \tfrac{m+n+1}{3} \right\rfloor\right\} - 2$$

for the chessboards, and Dold's theorem when r is a prime, Vrećica and Živaljević gave the upper bound $t(d,r) \leq 2r-1$ for r prime. This bound yields the bound $t(d,r) \leq 4r-3$ for all r's. The method of Özaydin applied on topological Tverberg theorem for prime powers allowed Živaljević in 1998 to extend the bound $t(d,r) \leq 2r-1$ to prime power r's.

A surprising observation is that bounds for the function $t(d,r)$ do not imply any bound on the function $n(d,r)$.

3. Tight colored Tverberg theorem

In 2009, Blagojević, Matschke and Ziegler extended the results of Bárány, Larman, Živaljević and Vrećica in many new directions. The restrain of connectivity of chessboard complex was overcome, new instances of Bárány–Larman conjecture were proved, new optimal bounds for the function $t(d,r)$ were established and a new natural tight colored Tverberg theorem emerged.

The first result of the paper [5, Proposition 4.1] establishes the existence of Σ_r-equivariant maps $\Delta_{r,r}^{*(d+1)} \to S\left(W_r^{\oplus(d+1)}\right)$ and consequently demonstrates the failure of sufficiency condition by Živaljević and Vrećica in all cases. This indicates the necessity of radically new approaches.

The surprising and a unique step comes with the introduction of the new sufficiency condition:

If there is no Σ_r-equivariant maps $\Delta_{r-1,r}^{*d+1} * [r] \to S\left(W_r^{\oplus(d+1)}\right)$, then $n(d,r-1) = (d+1)(r-1) \Leftrightarrow t(d,r-1) = r-1$.

Using the equivariant obstruction theory, along with the key technical result of Shareshian and Wachs [8], the complete answer to the existence of relevant Σ_r-equivariant maps was established in [5, Proposition 4.2]:

Σ_r-equivariant map $\Delta_{r-1,r}^{*d+1} * [r] \to S\left(W_r^{\oplus(d+1)}\right)$ exists ***if and only if*** $r \mid (r-1)!^d$.

The obstruction theory proof easily implied a partial degree based proof of almost all cases of the non-existence direction. The case of nonexistance for $r = 4$ and $d = 1$ can only be obtained via obstruction theory.

This result implied a lot of new and improved results:

- new instances of Bárány–Larman Conjecture: for $r+1$ a prime, $t(d,r) = r$,

- new optimal bounds for the function $t(d,r)$: for all $d \geq 1$ and $r \geq 2$, $t(d,r) \leq 2r-2$,
- first upper bound for the function $n(d,r)$: for r a prime, $n(d,r) \leq 2(d+1)(r-1)+1$,
- exact values of the Fadell–Husseini index for chessboards: for r a prime,

$$\mathrm{Index}_{\mathbb{Z}_r,\mathbb{F}_r}\Delta_{i,r} = \begin{cases} H^{\geq i}(\mathbb{Z}_r,\mathbb{F}_r), & \text{for } 1 \leq i \leq r-1,\ [6] \\ H^{\geq i-1}(\mathbb{Z}_r,\mathbb{F}_r), & \text{for } i = r, \text{ due to Carsten Schultz} \\ H^{\geq r}(\mathbb{Z}_r,\mathbb{F}_r), & \text{for } i \geq 2r-1,\ [10] \end{cases}$$

The colored Tverberg problem originally arose as a tool to obtain complexity bounds in computational geometry. As a consequence, these results can be applied to improve these bounds. Note that in some of these results $t(d,d+1)^d$ appears in the exponent, so even slightly improved estimates on $t(d,d+1)$ have considerable effect. For details consult [5, Corollary 2.7, 2.8 and 2.9].

Before stating the new tight colored Tverberg theorem let us point out that, unlike in the case of embedding problems, the existence of Σ_r-equivariant maps $\Delta_{r,r}^{*(d+1)} \to S\left(W_r^{\oplus(d+1)}\right)$ can not give rise to a counter example for the Bárány–Larman Conjecture.

Finally, we state the general tight colored Tverberg theorem than emerged as one of the central new results in this breakthrough.

Theorem 3. *Let $r \geq 2$ be prime, $d \geq 1$, and $N := (r-1)(d+1)$. Let Δ_N be an N-dimensional simplex with a partition of the vertex set into $m+1$ parts ("color classes")*

$$\mathbb{C} \quad = \quad C_0 \uplus \cdots \uplus C_m,$$

with $|C_i| \leq r-1$ for all i. Then for every continuous map $f : \Delta_N \to \mathbb{R}^d$, there are r disjoint "rainbow" faces $F_1,\ldots,F_r$ of Δ_N whose images under f intersect, that is,

$$|C_i \cap F_j| \leq 1 \text{ for every } i \in \{0,\ldots,m\},\ j \in \{1,\ldots,r\}, \text{ and } f(F_1)\cap\cdots\cap f(F_r) \neq \emptyset.$$

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New Constructions of Epsilon-Nets

János Pach
(joint work with Gábor Tardos)

We construct a set of n points in $\mathbb{R}^4$ with the property that for any $\frac{1}{8}\frac{1}{\varepsilon}\log_2\frac{1}{\varepsilon}$-element subset there is a half-space disjoint from it that contains $\geq \epsilon n$ points. Apart from the value of the constant ($\frac{1}{8}$), this bound cannot be improved, according to a well known result of Haussler and Welzl (1987). In $\mathbb{R}^3$, $O(\frac{1}{\varepsilon})$ elements are always sufficient to "hit" all half-spaces that cover $\geq \varepsilon n$ points. We also construct a set of n points in the plane such that none of its $c\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}$-element subsets hits every axis-parallel rectangle that contains $\geq \varepsilon n$ points. (Here $c > 0$ is an absolute constant.) This result is also optimal, apart from the value of c, according to a theorem of Aronov, Ezra, and Sharir (2010).

A conjecture on polyominoes, with consequences for Toeplitz' "square on a Jordan curve" problem (1911)

Helge Tverberg

1. The conjecture

Let P be a polyomino with boundary a closed Jordan curve J. We assume that P is built from unit squares with vertices at lattice points. Then the conjecture says:

Conjecture 1. *Let the greatest open square OS, with horizontal and vertical sides, contained in the bounded component of R^2 /J, have sidelength s (clearly an integer). Then J contains four lattice points forming the vertices of a square S of sidelength $\geq s/\sqrt{2}$ if s is even, $\sqrt{m^2+(m+1)^2}$ if s =2m+1.*

Having checked the conjecture for all polyominoes within the square $[0,6]$x$[0,6]$ and many others, by hand, I plan to go a bit further by using a computer.

2. Toeplitz' problem

In 1911 Otto Toeplitz [1] posed the following problem, which he could solve only in the case when J is convex:

Problem 2. *Let J be any closed Jordan curve in R^2. Are there always four points on J which form the vertices of a square?*

During the past century the answer YES has been given to this question for many classes of curves and in particular for polygons. If one lets a sequence of polygons, $P_1, P_2, \dots$ converge towards a given J, one knows that every P_i contains the vertices of a square S_i. It is clear that some subsequence of the sequence $S_1, S_2, \dots$ is convergent to a limit S. If S is a square, its vertices will be on J, but if S is a point one is stuck. The latter case will be avoided if there is a positive number δ so that diameter(S)$\geq \delta$ for all i.

One way of finding a δ as described is to choose the polygons as $J_1, J_2/2, J_3/3, \dots$ where J_i is the boundary of a suitably chosen polyomino. Assume the conjecture to be true. Then, if the bounded component of $R^2 - J$,where J is any given closed Jordan curve, contains an open square of sidelength s, J will clearly contain the vertices of some square of sidelength $\geq s/\sqrt{2}$. A point to be noticed is that a priori one does not know that the square vertices one finds on J_i are lattice points. But using the geometry of the square it is easy to see that near the found square there is one with vertices lattice points on J_i and of a size which differs from that by 0 or by an amount which is negligible for the present purpose.

The value of the polyomino form of the conjecture is that it is then better suited for computer experiments. It may also make Toeplitz' conjecture more interesting to mathematicians with a combinatorial taste. Note also that the lower bound conjectured is not canonical, but it seemed reasonable after my consideration of thousands of particular polyominoes.

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Homology of Random Complexes

Roy Meshulam

(joint work with L. Aronshtam, N. Linial, T. Łuczak, N. Wallach)

Let Δ_{n-1} denote the $(n-1)$-dimensional simplex. Let $Y_k(n,p)$ denote the probability space of k-dimensional subcomplexes of Δ_{n-1} obtained by starting with the full $(k-1)$-dimensional skeleton $\Delta_{n-1}^{(k-1)}$ of Δ_{n-1} and then adding each k-simplex independently with probability p. We study some homological aspects of random complexes in $Y_k(n,p)$.

The following result determines the sharp threshold for the homological connectivity of a random k-dimensional complex in $Y_k(n,p)$ with coefficients in a fixed finite abelian group R. The 2-dimensional case with $\mathbb{F}_2$ coefficients was obtained in [2]. The general case is in [3].

Theorem 1 ([2, 3])**.** *Fix $k \geq 1$ and a finite abelian group R. For any function $\omega(n)$ that tends to infinity*

$$\lim_{n\to\infty} \Pr\left[Y \in Y_k(n,p) : \tilde{H}_{k-1}(Y;R) = 0\right] = \begin{cases} 0 & p = \frac{k\log n - \omega(n)}{n} \\ 1 & p = \frac{k\log n + \omega(n)}{n} \end{cases} .$$

A key ingredient in the proof of Theorem 1 is the notion of homological expansion of a $(k-1)$-cochain. Let $\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}$ and let $X(k)$ be the set of k-simplices in X. For $\phi \in C^{k-1}(\Delta_{n-1};R)$ let

$$b_X(\phi) = |\{\tau \in X(k) \ : \ d_{k-1}\phi(\tau) \neq 0\}| \ .$$

Let $\text{supp}(\phi) = \{\sigma \in \Delta_{n-1}(k-1) : \phi(\sigma) \neq 0\}$. The *weight* of ϕ is defined by

$$w(\phi) = \min \left\{ \ |\text{supp}(\phi + d_{k-2}\psi)| \ : \ \psi \in C^{k-2}(\Delta_{n-1};R) \ \right\}.$$

The *homological expansion* of $\phi \in C^{k-1}(\Delta_{n-1};R) - B^{k-1}(\Delta_{n-1};R)$ with respect to X is defined by

$$e_X(\phi) = \frac{b_X(\phi)}{w(\phi)}.$$

The following result was obtained in a somewhat weaker form in [2] for the case $(k,R) = (2,\mathbb{F}_2)$. The general case is in [3].

Theorem 2 ([2, 3])**.** *For any $\phi \in C^{k-1}(\Delta_{n-1};R) - B^{k-1}(\Delta_{n-1};R)$*

$$e_{\Delta_{n-1}}(\phi) \geq \frac{n}{k+1}.$$

We now consider the threshold for the vanishing of the top homology of a random complex in $Y_k(n,p)$. Let $c_k \leq (k+1)(1-e^{-k-1})$ be the unique positive solution of the equation

$$(k+1)(x+1)e^{-x} + x(1-e^{-x})^{k+1} = k+1.$$

Theorem 3 ([1])**.** *For any fixed $c > c_k$*

$$\lim_{n\to\infty} \Pr\left[Y \in Y_k(n,\frac{c}{n}) : \ H_k(Y;\mathbb{Z}) \neq 0\right] = 1 \ .$$

Remark: In the two dimensional case, Theorem 3 implies that if $c > c_2 \simeq 2.783$ then $Y \in Y_2(n,\frac{c}{n})$ a.a.s. satisfies $\mathrm{H}_2(Y;\mathbb{Z}) \neq 0$. Simulations indicate that this already happens for $c > 2.74$.

We next give an lower bound on the threshold for non-triviality of $\mathrm{H}_k(Y;\mathbb{F}_2)$. Let $\mathcal{F}_{n,k}$ denote the family of all $\Delta_{n-1}^{(k-1)} \subset Y \subset \Delta_{n-1}^{(k)}$ that do not contain the boundary of a $(k+1)$-simplex. Clearly, if $\mathrm{H}_k(Y;\mathbb{F}_2) = 0$ then $Y \in \mathcal{F}_{n,k}$.

Theorem 4 ([1])**.** *Let $k \geq 2$ and $0 < c < 1$ be fixed. Then in the probability space $Y_k(n, \frac{c}{n})$*

$$\lim_{n\to\infty} \Pr\ [H_k(Y;\mathbb{F}_2) = 0 \mid Y \in \mathcal{F}_{n,k}] = 1\ .$$

The number of $(k+1)$-simplices in Δ_{n-1} whose boundary is contained in $Y \in Y_k(n, \frac{c}{n})$ tends to a Poisson distribution with parameter

$$\lambda_k = \lim_{n\to\infty} \binom{n}{k+2} (\frac{c}{n})^{k+2} = \frac{c^{k+2}}{(k+2)!}.$$

Theorem 4 therefore implies

Corollary 5 ([1])**.** *Under the assumptions of Theorem 4*

$$\lim_{n\to\infty} \Pr\ [H_k(Y;\mathbb{F}_2) = 0] = \lim_{n\to\infty} \Pr[\mathcal{F}_{n,k}] = \exp(-\lambda_k) = \exp(-\frac{c^{k+2}}{(k+2)!})\ .$$

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Crossing numbers in $\mathbb{R}^3$

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(joint work with Alfredo Hubard)

The *crossing number* of a graph $G = (V,E)$ is the minimum number of crossings between edges of G among all the ways to draw G in the plane. It is denoted $\text{cr}(G)$. The edges in a drawing of G need not be line segments, they are allowed to be arbitrary continuous curves. If one restricts to the straight-line drawings, then one obtains the *rectilinear crossing number* lin-cr(G). It is clear that $\text{cr}(G) \leq \text{lin-cr}(G)$, and there are examples where $\text{cr}(G) = 4$, but lin-cr(G) is unbounded [BD93]. The principal result about crossing numbers is the crossing lemma of Ajtai–Chvátal–Newborn–Szemerédi and Leighton [ACNS82, Lei84] which states that

$$\text{(1)} \qquad \text{cr}(G) \geq c\frac{|E|^3}{|V|^2} \qquad \text{whenever } |E| \geq C|V|.$$

The inequality is sharp apart from the values of c and C (see [PT97] for the best known estimate on c). The most famous applications of the crossing lemma are short and elegant proofs by Székely [Szé97] of Szemerédi–Trotter theorem on point-line incidences and of Spencer–Szemerédi–Trotter theorem on the unit distances. Another remarkable application is the bound on the number of halving lines by

Dey[Dey98]. In this paper we propose an extension of the crossing number to $\mathbb{R}^3$, in such a way that the corresponding "space crossing lemma" (Theorem 2 below) implies (1) (up to a logarithmic factor).

A *spatial drawing* of a graph G is representation of vertices of G by points in $\mathbb{R}^3$, and edges of G by continuous curves. A *space crossing* consists of a quadruple of vertex-disjoint edges $(e_1, \ldots, e_4)$ and a line l that meets these four edges. The *space crossing number* of G, denoted $\mathrm{cr}_4(G)$ is the least number of crossings in any spatial drawing of G. As in the planar case, the *spatial rectilinear crossing number* $\text{lin-cr}_4(G)$ is obtained by restricting to straight-line spatial drawings.

For a graph G pick a drawing of G in the plane with the fewest crossings. By perturbing the drawing slightly, we may assume that there are no points where three vertex-disjoint edges meet. The drawing can be lifted to a drawing G on a large sphere without changing any of the crossings. Since no line meets the sphere in more than two points, every space crossings in the resulting spatial drawing comes from a pair of crossings in the planar drawing. Thus,

$$\mathrm{cr}_4(G) \leq \binom{\mathrm{cr}(G)}{2}. \tag{2}$$

Let us note that the space crossing number is not the usual crossing number in disguise, for the inequality in the reverse direction does not hold:

Proposition 1. *For every natural number n there is a graph G with $\mathrm{cr}_4(G) = 0$ and $\mathrm{cr}(G) \geq n$.*

The principal result that justifies the introduction of the space crossing number is the following generalization of the crossing lemma.

Theorem 2. *Let $G = (V, E)$ be an arbitrary graph, then*

$$\mathrm{cr}_4(G) \geq \frac{|E|^6}{4^{178}|V|^4 \log_2^2|V|}$$

whenever $|E| \geq 4^{41}|V|$.

Since (1) is sharp, in the light of the argument that led to (2) there are graphs on the sphere for which the bound in Theorem 2 is tight up to the logarithmic factor. In the drawings of the these graphs, the edges are of course not straight. It turns out that there are also straight-line spatial drawings for which Theorem 2 is tight.

Theorem 3. *For all positive integers m and n satisfying $m \leq \binom{n}{2}$ there is a graph G with n vertices and m edges, and the rectilinear space crossing number at most $6720m^6/n^4$.*

The construction in the proof of Theorem 3 uses the idea of stair-convexity introduced in [BMN]. We shall briefly review the necessary background before the proof of Theorem 3.

Our final result is the lower bound on the space crossing number of (possibly sparse) pseudo-random graphs.

Theorem 4. *There is an absolute constant $\varepsilon > 0$ such that the following holds. Let $G = (V, E)$ be a graph such that whenever V_1, V_2 are any two subsets of V of size $\varepsilon|V|$, the number of edges between V_1 and V_2 is at least N. Then* $\text{lin-cr}_4(G) \geq N^4$.

The condition of the theorem holds for several models of random graphs, as well as for (n, d, λ)-graph (see for example [KS06, Theorem 2.11]).

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Bin Packing via Discrepancy of Permutations

Friedrich Eisenbrand

(joint work with Dömötör Pálvölgyi, Thomas Rothvoß)

The *bin packing* problem is the following. Given n items of size $s_1, \dots, s_n \in [0, 1]$ respectively, the goal is to pack these items in as few bins of capacity one as possible. Bin packing is a fundamental problem in Computer Science with numerous applications in theory and practice.

The development of heuristics for bin packing with better and better performance guarantee is an important success story in the field of *Approximation Algorithms.* In 1982, Karmarkar and Karp [7] proposed an approximation algorithm for bin packing that can be analyzed to yield a solution using at most $OPT + O(\log^2 n)$ bins. This seminal procedure is based on the *Gilmore Gomory LP relaxation* [5, 3]:

$$\text{(LP)}\qquad \begin{array}{lll} \min \sum_{p\in\mathcal{P}} x_p & & \\ \sum_{p\in\mathcal{P}} p \cdot x_p & \geq & \mathbf{1} \\ x_p & \geq & 0 \quad \forall p \in \mathcal{P} \end{array}$$

Here $\mathbf{1} = (1, \dots, 1)^T$ denotes the all ones vector and $\mathcal{P} = \{p \in \{0,1\}^n \colon s^T p \leq 1\}$ is the set of all feasible *patterns*, i.e. every vector in $\mathcal{P}$ denotes a feasible way to pack one bin. Let OPT and OPT_f be the value of the best integer and fractional solution respectively. The linear program (LP) has an exponential number of variables but still one can compute a basic solution x with $\mathbf{1}^T x \leq OPT_f + \delta$ in time polynomial in n and $1/\delta$ [7] using the Grötschel-Lovász-Schrijver variant of the Ellipsoid method [6].

The procedure of Karmarkar and Karp [7] yields an *additive integrality gap* of $O(\log^2 n)$, i.e. $OPT \leq OPT_f + O(\log^2 n)$, see also [14]. This corresponds to an asymptotic FPTAS for bin packing. The authors in [9] conjecture that even $OPT \leq \lceil OPT_f \rceil + 1$ holds and this even if one replaces the right-hand-side $\mathbf{1}$ by any other positive integral vector b. This *Modified Integer Round-up Conjecture* was proven by Sebő and Shmonin [10] if the number of different item sizes is at most 7.

Much of the hardness of bin packing seems to appear already in the special case of *3-partition*, where $3n$ items of size $\frac{1}{4} < s_i < \frac{1}{2}$ with $\sum_{i=1}^{3n} s_i = n$ have to be packed. It is strongly **NP**-hard to distinguish between $OPT \leq n$ and $OPT \geq n+1$ [4]. No stronger hardness result is known for general bin packing. A closer look into [7] reveals that, with the restriction $s_i > \frac{1}{4}$, the Karmarkar-Karp algorithm uses $OPT_f + O(\log n)$ bins.

Let $[n] := \{1, \dots, n\}$ and consider a set system $\mathcal{S} \subseteq 2^{[n]}$ over the ground set $[n]$. A *coloring* is a mapping $\chi : [n] \to \{\pm 1\}$. In *discrepancy theory*, one aims at finding colorings for which the difference of "red" and "blue" elements in different sets is as small as possible. Formally, the *discrepancy* of a set system $\mathcal{S}$ is defined as

$$\mathrm{disc}(\mathcal{S}) = \min_{\chi:[n]\to\{\pm 1\}} \max_{S\in\mathcal{S}} |\chi(S)|.$$

where $\chi(S) = \sum_{i\in S} \chi(i)$. A random coloring provides an easy bound of $\mathrm{disc}(\mathcal{S}) \leq O(\sqrt{n \log |\mathcal{S}|})$ [8]. The famous "*Six Standard Deviations suffice*" result of Spencer [11] improves this to $\mathrm{disc}(\mathcal{S}) \leq O(\sqrt{n \ln(2|\mathcal{S}|/n)})$.

The following conjecture is coined *three-permutations-conjecture* or simply *Beck's conjecture* (see Problem 1.9 in [1]):

> Given any 3 permutations on n symbols, one can color the symbols with red and blue, such that in any interval of any of those permutations, the number of red and blue symbols differs by $O(1)$.

A set of permutations $\pi_1, \dots, \pi_k : [n] \to [n]$ induces a set-system

$$\mathcal{S} = \{\{\pi_i(1), \dots, \pi_i(j)\} : j = 1, \dots, n;\ i = 1, \dots, k\}.$$

If we denote the maximum discrepancy of such a set-system induced by k permutations over n symbols as $D_k^{\mathrm{perm}}(n)$, then Beck's conjecture can be rephrased as $D_3^{\mathrm{perm}}(n) = O(1)$.

So far the best known bound on $D_3^{\mathrm{perm}}(n)$ is $O(\log n)$ and more generally $D_k^{\mathrm{perm}}(n)$ can be bounded by $O(k \log n)$ [2] and by $O(\sqrt{k} \log n)$ [13, 12] using the so-called *entropy method.*

The *main result* of our paper is a proof of the following theorem.

Theorem 1. *If Beck's conjecture holds, then the integrality gap of the linear program* (LP) *restricted to 3-partition instances is bounded by an additive constant.*

This result is constructive in the following sense. If one can find a constant discrepancy coloring for any three permutations in polynomial time, then there is an $OPT + c$ approximation algorithm for 3-partition for a constant c.

The proof of Theorem 1 itself is via two steps.

i) We show that the additive integrality gap of (LP) is at most twice the maximum *linear discrepancy* of a k*-monotone matrix* if all item sizes are larger than $1/(k+1)$. This step is based on matching techniques and *Hall's theorem.*

ii) We then show that the linear discrepancy of a k-monotone matrix is at most k times the discrepancy of k permutations. This result uses a theorem of Lovász, Spencer and Vesztergombi.

The theorem then follows by setting k equal to 3 in the above steps. Furthermore, we show that the discrepancy of k permutations is at most 4 times the linear discrepancy of a k-monotone matrix. And finally, we provide a $5k \cdot \log_2(\min\{m, n\})$ upper bound on the linear discrepancy of a k-monotone $n \times m$-matrix.

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Centrally symmetric manifolds with few vertices

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(joint work with Isabella Novik)

The study of vertex-minimal triangulations of manifolds has motivated a great deal of research in topological combinatorics, beginning with the work of Ringel and Youngs [7, 8] and Walkup [11]. One may add the additional requirement that these triangulations are centrally symmetric (abbreviated cs). It is easy to see that a centrally symmetric triangulation of a closed $(d-2)$-manifold requires at least $2d$ vertices. We will be interested in studying vertex-minimal cs triangulations of sphere products $\mathbb{S}^i \times \mathbb{S}^{d-i-2}$.

The first, and most natural question one may ask is whether or not such triangulations exist. Kühnel and Lassmann [3] gave cs triangulations of $\mathbb{S}^1 \times \mathbb{S}^{d-3}$ on $2d$ vertices for all $d \geq 3$. This seems to be the only infinite family of cs triangulations of products of spheres (on $2d$ vertices) that was known until now. In his thesis, Sparla [9] gave a cs triangulation of $\mathbb{S}^2 \times \mathbb{S}^2$ on 12 vertices, and conjectured that cs triangulations of $\mathbb{S}^{k-1} \times \mathbb{S}^{k-1}$ on $4k$ vertices exist for all k. Lutz [6] used the computer programs MANIFOLD_VT and BISTELLAR to construct many cs $2d$ vertex triangulations of $\mathbb{S}^i \times \mathbb{S}^{d-i-2}$ when $d \leq 10$ (in particular, answering Sparla's conjecture in the affirmative when $k \leq 5$. More recently, Effenberger [1] constructed a family of cs simplicial complexes with $4k$ vertices that were conjectured to triangulate $\mathbb{S}^{k-1} \times \mathbb{S}^{k-1}$. Using the software package SIMPCOMP, he was able to verify this conjecture for $k \leq 12$.

The primary technique used in all of the above constructions was to impose an extra requirement that the required triangulations admit a vertex-transitive action by a dihedral group. Lutz [6] refined Sparla's conjecture and proposed that there exists a cs triangulation of $\mathbb{S}^{\lfloor \frac{d}{2} \rfloor} \times \mathbb{S}^{\lceil \frac{d}{2} \rceil}$ on $2d$ vertices admitting a vertex-transitive action by $\mathcal{D}_{2d}$, the dihedral group of order $4d$. Lutz [6] also showed that no cs triangulation of $\mathbb{S}^2 \times \mathbb{S}^4$ on 16 vertices admits a vertex-transitive action by a cyclic group of order 16, and that no cs triangulation of $\mathbb{S}^2 \times \mathbb{S}^6$ on 20 vertices admits a vertex-transitive action by $\mathcal{D}_{20}$. In particular, it should not be expected that cs triangulations of $\mathbb{S}^i \times \mathbb{S}^{d-i-2}$ on $2d$ vertices admitting a vertex-transitive action by $\mathcal{D}_{2d}$ exist for all i and all d.

Our main result provides a cs $2d$-vertex triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-i-2}$ for *all* nonnegative integers $0 \leq i \leq d-2$, and in particular settles Sparla's conjecture in full generality. Moreover, we show that these triangulations admit a vertex-transitive action by a group of order $4d$; and that in many cases, this group is in fact the dihedral group of order $4d$.

Theorem 1. *For all pairs of integers (i, d) with $0 \le i \le d-2$, there exists a centrally symmetric $2d$-vertex triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-i-2}$. This triangulation admits a vertex-transitive action by the dihedral group of order $4d$, $\mathcal{D}_{2d}$, if at least one of the numbers i and $d-i$ is odd, and by the group $\mathbb{Z}_2 \times \mathcal{D}_d$ otherwise.*

The crux of the proof of Theorem 1 is the construction of a certain simplicial complex $\mathcal{B}(i, d)$ (for all $0 \le i \le d-1$) that is rather easy to analyze. This complex is constructed as a pure, full-dimensional subcomplex of the boundary complex of a d-cross polytope. We summarize the main properties of this complex here.

Theorem 2. *For $0 \le i < d-1$, the complex $\mathcal{B}(i, d)$ satisfies the following:*

(a) $\mathcal{B}(i, d)$ contains the entire i-skeleton of the d-dimensional cross polytope as a subcomplex.

(b) $\mathcal{B}(i, d)$ is centrally symmetric. Moreover, it admits a vertex-transitive action of $\mathbb{Z}_2 \times \mathcal{D}_d$ if i is even and of $\mathcal{D}_{2d}$ if i is odd.

(c) The complement of $\mathcal{B}(i, d)$ in the boundary complex of the d-dimensional cross polytope, denoted $\mathbb{C}(i, d)$, (that is, the complex generated by the facets of the cross polytope that are not in $\mathcal{B}(i, d)$) is simplicially isomorphic to $\mathcal{B}(d-i-2, d)$.

(d) $\mathcal{B}(i, d)$ is a combinatorial manifold (with boundary) whose integral (co) homology groups coincide with those of $\mathbb{S}^i$.

(e) The boundary of $\mathcal{B}(i, d)$ is homeomorphic to $\mathbb{S}^i \times \mathbb{S}^{d-i-2}$.

In the remainder of this note, we explicitly describe the construction of the complex $\mathcal{B}(i, d)$, and provide some brief notes on our general proof techniques. Let $\mathbb{C}_d^*$ denote the boundary complex of the d-dimensional cross polytope on vertex set $\{x_1, \ldots, x_d, y_1, \ldots, y_d\}$ where the labeling is such that x_j and y_j are antipodal vertices of $\mathbb{C}_d^*$ for all j. Each facet τ of $\mathbb{C}_d^*$ can be identified with a word $w(\tau) = w_1 w_2 \cdots w_d$ in the alphabet $\{x, y\}$ by setting $w_j = x$ if $x_j \in \tau$ and $w_j = y$ otherwise. For example, the word $xxyyxy$ encodes the facet $\tau = \{x_1, x_2, y_3, y_4, x_5, y_6\}$ in $\mathbb{C}_6^*$. For each word $w = w_1 \cdots w_d$ in the $\{x, y\}$ alphabet, we define the *switch set* of w, denoted $\mathcal{S}(w)$, to be the collection of all indices $1 \le j \le d-1$ for which $w_j \ne w_{j+1}$. In our previous example, $\mathcal{S}(xxyyxy) = \{2, 4, 5\}$. We define $\mathcal{B}(i, d)$ to be the pure subcomplex of $\mathbb{C}_d^*$ generated by all facets encoded by words with *at most* i switches. Thus $\mathcal{B}(0, d)$ is generated by the two facets $\{x_1, \ldots, x_d\}$ and $\{y_1, \ldots, y_d\}$, and $\mathcal{B}(d-1, d)$ is the entire complex $\mathbb{C}_d^*$.

From this construction, it is clear not only that $\mathcal{B}(i-1, d) \subset \mathcal{B}(i, d)$, but also that $\mathcal{B}(i, d-1) \subset \mathcal{B}(i, d)$. In fact, more can be said. Specifically, the closed stars of x_d and y_d in $\mathcal{B}(i, d)$ are shellable $(d-1)$-dimensional balls that intersect along $\mathcal{B}(i-1, d-1)$. This observation is key in establishing that $\mathcal{B}(i, d)$ is a combinatorial manifold whose homology groups are isomorphic to those of $\mathbb{S}^i$. In order to establish that $\partial \mathcal{B}(i, d)$ is a combinatorial triangulation of $\mathbb{S}^i \times \mathbb{S}^{d-i-2}$, we employ the following theorem of Matthias Kreck [4].

Theorem 3. *Let M be a simply connected codimension-one submanifold of $\mathbb{S}^{d-1}$ with $d \ge 6$. If M has the homology of $\mathbb{S}^i \times \mathbb{S}^{d-i-2}$ and $1 < i \le \frac{d}{2} - 1$, then M is homeomorphic to $\mathbb{S}^i \times \mathbb{S}^{d-i-2}$.*

Let $j := \min\{i, d-i-2\}$, and observe that $\partial\mathcal{B}(i,d) = \mathcal{B}(i,d) \cap \mathbb{C}(i,d)$ contains the entire j-skeleton of $\mathbb{C}_d^*$. This establishes that $\partial\mathcal{B}(i,d)$ is simply connected when $1 < i < \frac{d}{2} - 1$, and the Poincaré-Lefschetz Duality Theorem is used to show that $\partial\mathcal{B}(i,d)$ has the same integral homology groups as $\mathbb{S}^i \times \mathbb{S}^{d-i-2}$.

We conclude with a discussion of the group actions admitted by $\mathcal{B}(i,d)$. We define permutations, D, E, R, and R', on the vertex set of $\mathbb{C}_d^*$.

- D maps x_j to y_j and y_j to x_j; this permutation has order 2.
- E maps x_j to x_{d-j+1} and y_j to y_{d-j+1}; this permutation has order 2.
- R maps x_j to x_{j+1} and y_j to y_{j+1}, where the addition is taken modulo d; this permutation has order d.
- R' maps x_j to x_{j+1} and y_j to y_{j+1} when $1 \leq j \leq d-1$, it maps x_d to y_1 and y_d to x_1; this permutation has order $2d$.

First we examine the case that i is even. It is clear that each of the maps D, E, R induces a simplicial automorphism of $\mathbb{C}_d^*$, and we must check that these maps indeed induce simplicial automorphisms on $\mathcal{B}(i,d)$. It is clear that $|\mathcal{S}(\tau)| = |\mathcal{S}(D(\tau))| = |\mathcal{S}(E(\tau))|$ for any facet τ of $\mathbb{C}_d^*$. Moreover, if i is even and τ is a facet of $\mathcal{B}(i,d)$ with i switches, the first letter of $w(\tau)$ is the same as the last letter of $w(\tau)$, and hence $|\mathcal{S}(R(\tau))| \leq i$. Thus D, E, and R all act as permutations on the facets of $\mathcal{B}(i,d)$. Since $ERE = R^{-1}$ and D commutes with both E and R, it follows that D, E and R generate the group $\mathbb{Z}_2 \times \mathcal{D}_d$.

In the case that i is odd, it is possible that $|\mathcal{S}(R(\tau))| > |\mathcal{S}(\tau)|$ for some facet τ in $\mathcal{B}(i,d)$. We see, however, that $|\mathcal{S}(R'(\tau))| \leq |\mathcal{S}(\tau)| + 1$, and moreover, since i is odd, $|\mathcal{S}(R'(\tau))| \leq i$ if $|\mathcal{S}(\tau)| = i$. Again, this establishes that R' acts as a permutation on the facets of $\mathcal{B}(i,d)$ when i is odd. Once again, E and R generate the dihedral group $\mathcal{D}_{2d}$ of order $4d$.

Of course, one very natural (yet strangely elusive) question remains.

Question 4. *Is $\mathcal{B}(i,d)$ a combinatorial triangulation of $\mathbb{S}^i \times \mathbb{B}^{d-i-1}$?*

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Dvoretzky type theorems for multivariate polynomials

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(joint work with Vladimir Dol'nikov)

1. Introduction

The following theorem was conjectured in [15] (see also [16]), it is known as the Gromov–Milman conjecture. This theorem resembles the famous theorem of Dvoretzky [7] on near-elliptical sections of convex bodies. It considers polynomials instead of convex bodies, and unlike the Dvoretzky theorem, it gives strict "roundness" rather than approximate "roundness".

Theorem 1. *For an even positive integer d and a positive integer k there exists $n(d, k)$ such that for any homogeneous polynomial f of degree d on $\mathbb{R}^n$, where $n \geq n(d, k)$, there exists a linear k-subspace $V \subseteq \mathbb{R}^n$ such that $f|_V$ is proportional to the $d/2$-th power of the standard quadratic form*

$$Q = x_1^2 + x_2^2 + \cdots + x_n^2.$$

Remark 2. *The conjecture in* [15] *was stated in a bit different way: the restriction $f|_V$ was required to be proportional to the $d/2$-th power of* some *quadratic form. But these two statements are equivalent modulo the precise values of $n(d, k)$.*

Besides the trivial case $d = 2$, there were other partial results in this conjecture. Theorem 1 was proved in [15, 13, 14] (the essential idea goes back to M. Gromov) for $k = 2$ by topological methods with good bounds for $n(d, 2)$. Actually, the stronger Conjecture 5 (see below) was proved for $k = 2$. In case of special polynomials of the form $f = x_1^d + x_2^d + \cdots + x_n^d$ this theorem was proved in [17], see also [15] for a short proof with the averaging trick.

We combine the topological technique with the averaging trick of [15] to prove Theorem 1. Let us state a more general conjecture, that would imply Theorem 1, if it were true.

Definition 3. *Denote G_n^k the Grassmannian of linear k-subspaces in $\mathbb{R}^n$, denote by $\gamma_n^k : E(\gamma_n^k) \to G_n^k$ its canonical bundle.*

Definition 4. *For a vector bundle $\xi : E(\xi) \to X$ denote $\Sigma^d(\xi)$ its fiberwise symmetric d-th power. We consider every vector bundle ξ along with some Riemannian metric on its fibers, i.e. a nonzero section $Q(\xi)$ of $\Sigma^2(\xi)$.*

False Conjecture 5. *Suppose d and k are even positive integers. Then there exists $n(d,k)$ such that for every section of the bundle $\Sigma^d(\gamma_n^k)$ over G_n^k with $n \geq n(d,k)$, there exists a subspace $V \in G_n^k$ such that this section is a multiple of $(Q(\gamma_n^k))^{d/2}$ over V.*

This conjecture would imply Theorem 1, because every polynomial of degree d defines a section of $\Sigma^d(\gamma_n^k)$ tautologically. Unfortunately, there already exist some negative results on Conjecture 5. It is shown in [10, Ch. IV, § 1 (A)] (with reference to [9]) that this conjecture fails for odd k. In [4] a counterexample to Conjecture 5 is given for $k = 4$ and $d \geq 4$. As it was noted, Conjecture 5 is known to be true for $k = 2$, see [15, 13, 14]. We also prove that it is true for $d = 2$ and $d = 2$ and $k = 2p^\alpha$ for a prime p. In fact, a stronger statement is true: arbitrary number of sections can be made "round". Consider the case of odd d in Theorem 1. In [3] it was shown to be true in an (obviously) stronger from, i.e. $f = 0$ on a k-dimensional subspace. In [1] the bound on $n(d,k)$ was improved. We use cohomology computations to prove the best known bounds for odd d (or complex polynomials and any d), and for several polynomials simultaneously. In fact we prove Conjecture 5 for odd d or complex bundles over the complex Grassmannian.

2. Knaster's conjecture

Conjecture 6. *There exists $n = n(\ell)$ such that for any ℓ points $X = \{x_1, \dots, x_\ell\}$ on the unit sphere S^{n-1} and any continuous function $f : S^{n-1} \to \mathbb{R}$ there exists a rotation $\rho \in O(n)$ such that*

$$f(\rho x_1) = f(\rho x_2) = \cdots = f(\rho x_\ell).$$

Originally Knaster conjectured in [12] that $n(\ell) = \ell$, but counterexamples to his conjecture were found in [11]. In [8] it was proved that $n(3) = 3$, but already the value $n(4)$ seems to be unknown and not shown to be finite. Known results in this conjecture either consider sets X, distributed along a two-dimensional vector subspace of $\mathbb{R}^n$ (this is similar to the case $k = 2$ of the Gromov–Milman conjecture), or require very specific symmetry conditions, e.g. require X to be an (almost) orthogonal frame. In [15] it was shown that the original Knaster conjecture ($n(\ell) = \ell$) would imply the Dvoretzky theorem with good estimates on $n(k, \varepsilon)$, and it would also imply Theorem 1.

The weak form of the Knaster conjecture presented here would also give some bounds in the Dvoretzky theorem, as well as explicit bounds in Theorem 1. Note that in order to deduce Dvoretzky type results from the Knaster conjecture we have to consider sets X distributed densely enough in a sphere S^{k-1} of given dimension.

3. A few words about the proof

The results for odd d or complex polynomials are proved by showing that the Euler class of the symmetric power $\Sigma^d(\gamma_n^k)$ is nonzero over G_n^k for large enough n. In this case it is enough to use mod 2 cohomology in the real case and rational cohomology in the complex case. Theorem 1 is proved using the following fact (the Borsuk—Ulam theorem for p-groups). See [6] for this particular statement and [5, 2] for more general information about Borsuk–Ulam type results for finite groups.

Lemma 7. 1) *Suppose G is a finite p-group and Y is a finite polyhedron with piecewise linear action of G. Then the image of any G-equivariant map $f : EG \to Y$ intersects Y^G (the G-fixed points).* 2) *The above claim has a quantitative analogue: There exists $n(G, Y)$ such that if a free G-space X is $(n-1)$-connected and $n \geq n(G, Y)$ then the image of an equivariant map $f : X \to Y$ intersects Y^G.*

Unfortunately, the explicit bounds for $n(G, Y)$ are known only for p-tori (groups $(\mathbb{Z}_p)^\alpha$) from the cohomology computations and for cyclic groups $\mathbb{Z}_{p^\alpha}$ from the K-theory computations. In our proof $G = \Sigma_m^{(2)}$ is the 2-Sylow subgroup of the permutation group and Y is its certain linear representation. In this case no lower bounds on $n(G, Y)$ are known.

Then we use the following strategy to prove Theorem 1. Choose large enough m (depending on k and d explicitly). By the Borsuk–Ulam theorem for p-groups we may find $n = n(m, d)$ such that any homogeneous polynomial of degree d in n variables becomes $\Sigma_m^{(2)}$-symmetric after restriction to some linear image of $\mathbb{R}^m$ in $\mathbb{R}^n$. Then we study a $\Sigma_m^{(2)}$-symmetric homogeneous polynomial f of degree d and show (using some combinatorics and the averaging trick of V. Milman) that we may restrict f to some k-dimensional linear subspace $L \subset \mathbb{R}^m$ and take its $d/2$-th root after this. In fact, the space L does not depend on f; it makes any $\Sigma_m^{(2)}$-symmetric homogeneous polynomial of degree d "round".

For more info: The full version is available at `http://arxiv.org/abs/1009.0392`.

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Computing bounded subcomplexes of unbounded polyhedra

Michael Joswig
(joint work with Sven Herrmann, Marc E. Pfetsch)

Let P be a *convex polyhedron* in $\mathbb{R}^d$, that is, the intersection of finitely many affine halfspaces. To keep this exposition concise we assume that P is unbounded and that it does not contain any affine line. If it satisfies the latter property we call P *pointed*. Those faces of P which are bounded form the *bounded subcomplex* $\mathcal{B}(P)$. Since P does not contain an affine line $\mathcal{B}(P)$ is not empty, and since P is unbounded $\mathcal{B}(P)$ is contractible. Relevant examples for bounded subcomplexes of unbounded polyhedra include tight spans of finite metric spaces [1] and tropical polytopes [2].

For each pointed unbounded polyhedron P there is an admissible projective transformation which maps P onto a polytope $\overline{P}$ minus one proper face F_∞. The combinatorial type of the pair $(\overline{P}, F_\infty)$ is uniquely determined by P. The polytope $\overline{P}$ is called a *projective closure* of P. Studying the combinatorics of pointed unbounded polyhedra is equivalent to studying the combinatorics of polytopes with a distinguished proper face.

We describe two algorithms for computing $\mathcal{B}(P)$. The first one uses as input the incidence information between the vertices and the facets of a projective closure $\overline{P}$ of P. Its time complexity is $\Theta(\overline{n} \cdot \alpha \cdot \phi')$ time, where $\overline{n}$ is the number of vertices of the polytope $\overline{P}$ and α is the number of its vertex-facet incidences. This algorithm is output-sensitive since its running time also depends on ϕ', the number of bounded faces. Since this dependence on ϕ' is only linear this algorithm can be considered optimal with respect to output-sensitivity. Our method relies on a variation of the technique for enumerating all the faces of a bounded polytope due to Kaibel and Pfetsch [4].

Our second algorithm uses less information. It only reads the vertex-facet incidences of the polyhedron P; the behavior at infinity is ignored. It has time complexity $\Theta(\max(n \cdot \phi', \beta) \cdot n^2 \cdot \phi')$. Here β denotes the number of vertex-facet incidences of P. Since this second algorithm depends quadratically on ϕ' (which typically is the dominating parameter) it will be inferior very often. However, there are examples of unbounded polyhedra for which this algorithm is, in fact, superior. The reason is that β is much smaller than α in these cases, small enough to compensate for an additional factor of ϕ'. The key idea leading to the second algorithm is to compute the Möbius function of the partially ordered set of vertex sets of faces of P. The correctness then follows from a known characterization of boundedness in terms of the vertex-facet incidences of not necessarily bounded polyhedra [3].

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A Combinatorial Topological Toolkit for Stratified Spaces

Patricia Hersh

A new criterion is given for deciding if a finite CW complex is regular with respect to a choice of characteristic maps. As an application, the Bruhat stratification of the neighborhood of the origin in the totally nonnegative part of a matrix Schubert variety is proven to be a regular CW complex homeomorphic to a ball.

This regularity criterion involves an interplay of combinatorics with codimension one topology, and is as follows.

Theorem 1. *Let K be a finite CW complex with characteristic maps $f_\alpha : B^{\dim e_\alpha} \to \overline{e_\alpha}$. Then K is regular with respect to the characteristic maps $\{f_\alpha\}$ if and only if the following conditions hold:*

(1) For each α, $f_\alpha(B^{\dim e_\alpha})$ is a union of open cells.

(2) For each f_α, the preimages of open cells of dimension exactly one less than e_α form a dense subset of the boundary of $B^{\dim e_\alpha}$.

(3) The closure poset of K is thin. Additionally, each open interval (u, v) with $rk(v) - rk(u) > 2$ is connected.

(4) For each α, the restriction of f_α to the preimages of the open cells of dimension exactly one less than e_α is an injective map.

(5) For each $e_\sigma \subseteq \overline{e_\alpha}$, f_σ factors as an embedding $\iota : B^{\dim e_\sigma} \to B^{\dim e_\alpha}$ followed by f_α.

Conditions 1 and 2 imply that the closure poset is graded by cell dimension. Theorem 1 seem to capture how the combinatorics (encoded in Condition 3) substantially reduces what one must check topologically. Notably absent is the requirement that f_α acts injectiviely on the entire boundary of $B^{\dim \alpha}$.

In [2], the Bruhat order was proven to be thin and shellable. This implies by results of [1] combined with work of [3] that it is the closure poset of a regular CW complex. Joseph Bernstein asked for naturally arising regular CW complexes coming from representation theory having the lower intervals of Bruhat order as their closure posets. This question appears in [1]. Fomin and Shapiro conjectured the following solution in [4].

Conjecture 2. *The neighborhood of the origin in the Bruhat stratification of the totally nonnegative part of the unipotent radical of a Borel in a semisimple, simply connected group over $\mathbb{C}$ split over $\mathbb{R}$ is a regular CW complex homeomorphic to a ball.*

Using the above regularity criterion, we prove this conjecture in [5]. An interesting special case is as follows:

Theorem 3. *The neighborhood of the origin in the Bruhat stratification of the space of upper triangular real matrices with 1's on the diagonal, all of whose minors are nonnegative, is a regular CW complex homeomorphic to a ball.*

The proof also involves a combinatorial topological toolkit for performing a series of collapses on the boundary of a convex polytope, with each collapse preserving homeomorphism type and regularity, though not polytopality. Details may be found in [5].

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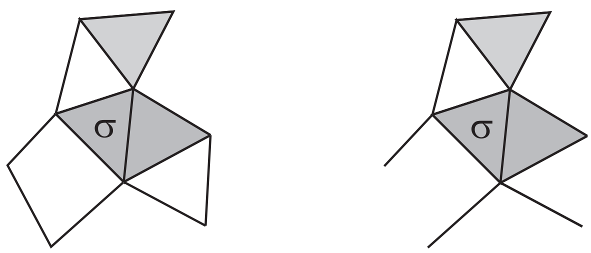
Star clusters in independence complexes of graphs

Jonathan A. Barmak

The independence complex I_G of a finite simple graph G is the simplicial complex whose simplices are the independent sets of vertices of G. We study the relationship between combinatorial properties of a graph and topological properties of the associated complex. Some well-known applications in this direction include a criterion for recognizing Tverberg graphs [8], a sufficient condition for the existence of independent systems of representatives [1] and the connection with Hom complexes [2].

We introduce the notion of *star cluster* of a simplex in a simplicial complex which seems to be very useful to study independence complexes. The definition of this concept was motivated by a question formulated by Engström and by a result of Jonsson. Engström asked in [7] whether it is possible to have torsion in the homology groups of the independence complex of a triangle-free graph. This question was answered by Jonsson who proved that the homotopy types of independence complexes of bipartite graphs are exactly those of suspensions of finite complexes [9]. We use star clusters to prove that independence complexes of triangle-free graphs are also suspensions up to homotopy. This is just the first application of this notion. Many of the results on independence complexes that can be found in the literature, use classical tools such as discrete Morse theory, arguments of collapsibility in general, the Nerve theorem, etc. On the other hand we will see that star clusters give a more conceptual approach to many problems, providing shorter and more elementary proofs, and allowing us to prove also interesting new results.

The *star cluster* $SC_K(\sigma)$ of a simplex σ in a simplicial complex K is the union $\bigcup\limits_{v\in\sigma} st_K(v)$ of the simplicial stars of the vertices of σ.



A simplex σ in a complex at the left
and the star cluster of σ at the right.

Recall that a finite simplicial complex K is said to be *clique* (or *flag*) if every set of vertices which pairwise constitute a 1-simplex is a simplex of K. The basic result that we will repeatedly use, is the following

Theorem 1. If K is a finite clique complex, then the star cluster of every simplex of K is contractible.

We use this theorem to give an alternative proof of Jonsson's result on the realizability of each suspension as the independence complex of a bipartite graph and to prove the classification theorem for homotopy types of independence complexes of triangle-free graphs.

Theorem 2. Homotopy types of independence complexes of triangle-free graphs are the same as homotopy types of suspensions.

In fact, the independence complex of a graph containing a vertex which is included in no triangle, also has the homotopy type of a suspension. Using star clusters we give alternative proofs to the following results of Kozlov [10], Eherenborg and Hetyei [6] and Csorba [5].

Proposition 3 (Kozlov)**.** If $G = C_n$ is a cycle with n vertices, I_G is homotopy equivalent to S^{k-1} if $n = 3k \pm 1$ and to $S^{k-1} \vee S^{k-1}$ if $n = 3k$.

Proposition 4 (Ehrenborg-Hetyei)**.** The independence complex of a forest is contractible or homotopy equivalent to a sphere.

Proposition 5 (Csorba)**.** The independence complex of G' is homotopy equivalent to the suspension of the Alexander dual of I_G. Here G' denotes the graph obtained from G by subdividing each edge adding a new vertex in the middle.

We also prove the following proposition, which mirrors a result of Braun [4] for stable Kneser graphs.

Proposition 6. The independence complex of the Kneser graph $KG_{2,k}$ is homotopy equivalent to a wedge of two-dimensional spheres.

Some other applications include lower bounds for the connectivity of independence complexes, in particular of claw-free graphs and therefore of matching complexes. The basic idea that we use is that if the star cluster $SC_K(\sigma)$ of a simplex σ contains every simplex of dimension at most k, then K is k-connected.

The last application we present is related to chromatic numbers and a homotopy invariant called the strong Lusternik-Schnirelmann category. The *strong category* $Cat(X)$ of a topological space X is the minimum number n such that there exists a CW-complex Y homotopy equivalent to X which can be covered by $n + 1$ contractible subcomplexes. This invariant related to the cup-length is connected with the chromatic number $\chi(G)$ of a graph in the following way:

Proposition 7. $\chi(G) \geq Cat(I_G) + 1$.

A sharper version of this inequality and further applications of star clusters appear in the main article [3].

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On stellated spheres and a tightness criterion for combinatorial manifolds

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(joint work with Bhaskar Bagchi)

This talk is based on the work in [2].

We introduce the class $\Sigma_k(d)$ of k-stellated triangulated spheres of dimension d, so that $\Sigma_0(d) \subseteq \Sigma_1(d) \subseteq \cdots \subseteq \Sigma_d(d) \subseteq \Sigma_{d+1}(d)$ is a filtration of the class of combinatorial d-spheres. We compare these classes with the classes $\mathcal{S}_k(d)$ of k-stacked d-spheres. Again, we have the filtration $\mathcal{S}_0(d) \subseteq \mathcal{S}_1(d) \subseteq \cdots \subseteq \mathcal{S}_d(d)$ of the class of all triangulated d-spheres, and the easy inclusion $\Sigma_k(d) \subseteq \mathcal{S}_k(d)$ with equality for $k \leq 1$. But, for each $k \geq 2$, there are k-stacked spheres which are not k-stellated.

In analogy with the (generalized) Walkup classes $\mathcal{K}_k(d)$ of triangulated d-manifolds all whose vertex-links are k-stacked spheres, we consider the class $\mathcal{W}_k(d)$ of combinatorial d-manifolds whose vertex-links are k-stellated spheres, so that $\mathcal{W}_k(d) \subseteq \mathcal{K}_k(d)$. For $d \geq 2k+2$ (respectively $d \geq 2k$), every member of $\mathcal{W}_k(d)$ (respectively, of $\Sigma_k(d)$) is the boundary of a canonically defined $(d+1)$-manifold (respectively $(d+1)$-ball).

Finally, we introduce the subclass $\mathcal{W}^*_k(d)$ of $\mathcal{W}_k(d)$ consisting of all $(k+1)$-neighbourly members of the latter class. We prove that, for $d \neq 2k+1$, all members of $\mathcal{W}^*_k(d)$ are tight (with respect to any field if $k \geq 2$). Also, the case $d = 2k+1$ is a genuine exception to this result since it is shown that all the cyclic spheres of dimension $2k+1$ are in the class $\mathcal{W}^*_k(2k+1)$. These results partially answer a recent question of Effenberger. We show that when d is even and $d \geq 2k+2$, any member of $\mathcal{W}^*_k(d)$ has the same homology with $\mathbb{Z}$-coefficients as the connected sum of finitely many copies of $S^k \times S^{d-k}$. We also conjecture a new lower bound on the number of vertices of an even dimensional triangulated manifold in terms of its dimension, connectivity and Euler characteristic. This is a common generalization of the Brehm-Kühnel lower bound on triangulated manifolds which are not k-connected, and Kühnel's lower bound conjecture (now a theorem of Novik and Swartz). Conjecturally, the members of $\mathcal{W}^*_k(d)$, d even, provide the only cases of equality in the new conjecture.

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Topological representations of matroids from diagrams of spaces

Alexander Engström

Swartz [7] proved that any matroid can be realized as the intersection lattice of an arrangement of codimension one homotopy spheres on a sphere. This was an unexpected extension from the oriented matroid case [4], but unfortunately the construction is not explicit. Anderson [1] later provided an explicit construction, but had to use cell complexes of high dimensions that are homotopy equivalent to lower dimensional spheres.

Using diagrams of spaces we give an explicit construction of arrangements in the right dimensions. Swartz asked if it is possible to arrange spheres of codimension two, and we provide a construction for any codimension. We also show that all matroids, and not only tropical oriented matroids, have a pseudo-tropical representation.

We determine the homotopy type of all the constructed arrangements.

In recent work by Stamps [6] this construction is proven to behave very well in the perspective of weak and strong maps of matroids. This provide new tools to understand the matroid grassmannian. There is also a version of the construction with an Orlik-Solomon algebra interpretation [3].

A real spherical representation, as treated by Swartz and Anderson, is the X-arrangement with $X = S^0$. The complex spherical representation is $X = S^1$, and the tropical ones follows from X being more than two disjoint points.

Definition 1. *An* X-arrangement *is a CW complex* Y *and a finite set* $\mathbf{A}$ *of subcomplexes of* Y *such that:*

1. *The complex* Y *is homotopy equivalent to* X^{*d} *for some* d*, and* $\dim(Y) = \dim(X^{*d})$.
2. *Each complex* A *in* $\mathbf{A}$ *is homotopy equivalent to* $X^{*(d-1)}$ *and* $\dim(A) = \dim(X^{*(d-1)})$.
3. *Each intersection* B *of complexes in* $\mathbf{A}$ *is homotopy equivalent to some* X^{*e}*, and* $\dim(B) = \dim(X^{*e})$.
4. *If there is a free group action of* Γ *on* X*, then it induces a free* Γ*-action on* Y *and every intersection of complexes in* $\mathbf{A}$.
5. *If* $B \simeq X^{*e}$ *is an intersection of complexes in* $\mathbf{A}$*, the complex* A *is in* $\mathbf{A}$*, and* $A \not\supseteq B$*, then* $A \cap B \simeq X^{*(e-1)}$.

Building on the excellent introductions to diagrams of spaces for combinatorics, by Ziegler, Živaljević and Welker [8, 9], in an integral combination with the discrete Morse theory by Forman [5], we prove this new representation theorem.

Theorem 2 (The Representation Theorem of Matroids)**.** *Let M be a rank r simple matroid given by its geometric lattice, and let ℓ be a rank- and order-reversing poset map from M to a boolean lattice on $\{1,2, \ldots, r\}$. Let X be a finite regular CW complex and define*

$$D_p = *_{i=1}^{r} \begin{cases} X & \text{if } i \in \ell(p) \\ \emptyset & \text{if } i \notin \ell(p) \end{cases}$$

to get an M-diagram $\mathcal{D}$ with inclusion morphisms.

Then

$$(Y, \mathbf{A}) = (\textit{hocolim } \mathcal{D}, \{\textit{hocolim } \mathcal{D}_{\geq a} \mid a \textit{ is an atom of } M\})$$

is an X–arrangement of M (that is, the intersection lattice of the X–arrangement is the lattice of M) and

$$\bigcup_{A \in \mathbf{A}} A \simeq \bigvee_{p \in M \setminus \hat{0}} \left(X^{*(d-rank(p))} * \bigvee^{|\mu(\hat{0},p)|} S^{rank(p)-2} \right).$$

The full paper is on the arXiv [2].

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How to construct a flag complex with a given face vector

Andrew Frohmader

A flag complex is a simplicial complex for which every minimal non-face is a two element set. Equivalently, a flag complex is the clique complex of its 1-skeleton, taking as a graph. The face vector, also called the f-vector, of a simplicial complex lists its numbers of faces of each dimension. The question we would like to address is which integer vectors are the face vector of some flag complex.

The Kruskal-Katona theorem [5, 4] characterizes the face vectors of all simplicial complexes, and asserts there is a simplicial complex with a given integer vector as its face vector if and only if the entries in the vector satisfy certain numerical conditions. If there isn't any simplicial complex at all with a proposed face vector, then there surely can't be a flag complex with this face vector. This gives us a way to say that a particular integer vector does not correspond to any flag complex.

However, this doesn't give us a way to say that the integer vector is the face vector of a flag complex. The Kruskal-Katona theorem may say that it is the face vector of some simplicial complex, but often it is not a flag complex. For example, $(1, 3, 3)$ is the face vector of a a triangle, without the two-dimensional face, so it is the face vector of a simplicial complex. It is not the face vector of a flag complex, however, as the set of all three vertices is a minimal non-face.

There are also other theorems that can assert that particular integer vectors are not the face vector of a flag complex. Frankl, Füredi, and Kalai [1] characterized the face vectors of simplicial complexes of at most a given chromatic number. Their theorem was similar to the Kruskal-Katona theorem, as it that a given integer vector is the face vector of a simplicial complex of at most a given number if and only if the entries in the integer vector satisfy certain bounds.

Kalai and Eckhoff independently conjectured that flag complexes satisfied these same numerical bounds, except using the maximum number of vertices in a face instead of the chromatic number. The author [2] later proved their conjecture. This put stronger restrictions on what could be the face vector of a flag complex than the chromatic number alone, as the chromatic number is at least as large as the maximum number of vertices in a face, but often strictly larger. For example, a pentagon has no face with more than two vertices, but has chromatic number three.

The author also proved other bounds [3] on what integer vectors could be the face vectors of flag complexes. This also showed that the face vectors of flag complexes can behave rather strangely. For example, there is a flag complex with 70 faces of dimension two and 85 of dimension three, and there is a flag complex with 85 faces of dimension three and 62 of dimension four. But there is no flag complex with exactly 70 faces of dimension two and 62 of dimension four, regardless of how many faces of dimension three the complex has.

All of these are negative results, however, and can only say that various integer vectors are not the face vector of any flag complex. None give any way to say that any integer vector is the face vector of a flag complex, as would be necessary to have real progress toward characterizing the face vectors of flag complexes.

We give a construction that attempts to take a given integer vector and produce a flag complex with that vector as its face vector. Of course, this cannot always be done, so sometimes the construction fails. But if there is any flag complex with the proposed face vector, it often succeeds.

The idea of the construction is to start by adding vertices and edges arranged to give the prescribed number of faces of the top dimension, while using relatively few faces of all smaller dimensions. Next, we add some additional vertices and edges to give the complex the prescribed number of faces of the next highest dimension, without adding any more faces of the highest dimension, and while still adding relatively few faces of all smaller dimensions. This continues, adding the faces of one dimension at a time, until the complex has the prescribed number of faces of all dimensions.

If the complex has used up more faces of a given dimension than allowed before we try to add faces of that particular dimension, then the construction fails. For example, if the top dimension is one (in which case the problem is trivial) and we want 25 edges, then the construction uses 10 vertices. If we only allow 8 vertices, then the construction fails. In this case, the construction fails for good reason, as other theorems show that there is no flag complex with a face vector of $(1, 8, 25)$.

Conversely, if every time we go to add faces of a given dimension, the number of faces of that dimension that we have already used is no greater than the prescribed number, then the construction succeeds. This gives us a way to say that if the entries in the proposed integer vector do not grow too quickly, then it does correspond to the face vector of a flag complex.

As an added bonus, the complexes given by the construction are not only flag, but also balanced. This means that with a slight tweak, we can use nearly the same construction to build a Cohen-Macaulay flag complex with a given h-vector.

The paper on which this talk is based is still in progress, and there is not yet a preprint available.

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The Haefliger-Wu Invariant for embeddings of polyhedra and piecewise linear manifolds

Arkadij Skopenkov

According to Zeeman, the classical problems of topology are the following.

(1) *The Homeomorphism Problem:* When are two given spaces homeomorphic?
(2) *The Embedding Problem:* When does a given space embed into $\mathbb{R}^m$?
(3) *The Knotting Problem:* When are two given embeddings isotopic?

This talk is on the Knotting Problem (and partly on the closely related Embedding Problem). For recent surveys see [5, 8, 2].

Recall that a *polyhedron* is a body of a simplicial complex.

(This terminology does not agree with that from discrete geometry: the complex need not be homeomorphic to the n-sphere.)

E.g. 1-dimensional polyhedron is a graph.

A map from a polyhedron to $\mathbb{R}^m$ is *piecewise-linear (PL)* if it is linear on each simplex of some triangulation of the polyhedron.

In this text we work in the PL category and so omit 'PL'.

A *(PL) embedding* of a polyhedron in $\mathbb{R}^m$ is an injective (PL) map.

Two embeddings $f, g : N \to \mathbb{R}^m$ are said to be *(ambient) isotopic*, if there exists a homeomorphism onto (an *isotopy*) $F : \mathbb{R}^m \times I \to \mathbb{R}^m \times I$ such that
- $F(y, 0) = (y, 0)$ for each $y \in \mathbb{R}^m$,
- $F(f(x), 1) = (g(x), 1)$ for each $x \in N$, and
- $F(\mathbb{R}^m \times \{t\}) = \mathbb{R}^m \times \{t\}$ for each $t \in I$.

We shorten 'n-dimensional' to just 'n-'.

General Position Theorem 1'. *Every n-polyhedron embeds into $\mathbb{R}^{2n+1}$.*

Here the number $2n + 1$ is the least possible:
for each n there exists an n-polyhedron, non-embeddable in $\mathbb{R}^{2n}$.

As an example one can take
- the n-th power of a non-planar graph (conjectured by Menger in 1929, proved by Ummel and M. Skopenkov [9, 3]);
- the n-skeleton of a $(2n + 2)$-simplex [4, 1];
- the $(n + 1)$-th join power of the three-point set [4, 1].

General Position Theorem 1. *Every two embeddings of an n-polyhedron into $\mathbb{R}^m$ are isotopic for $m \geq 2n + 2$.*

Here the restriction $m \geq 2n + 2$ is sharp as the Hopf linking $S^n \sqcup S^n \to \mathbb{R}^{2n+1}$ shows.

There is a cohomological van Kampen invariant of embeddings of an n-polyhedron into $\mathbb{R}^m$. When $m = 2n + 1 \geq 5$, this invariant is bijective. When it is incomplete (e.g. when $m = 2n$) there is a secondary invariant and so on. All these invariants are generalized to the following *Haefliger-Wu invariant*.

The *deleted product* $\widetilde{N}$ of a polyhedron N is the product of N with itself, minus the diagonal:

$$\widetilde{N} = \{(x, y) \in N \times N \mid x \neq y\}.$$

This is the configuration space of ordered pairs of distinct points.

For a triangulation T of a polyhedron N the polyhedron

$$\widetilde{T} = \cup\{\sigma \times \tau \in T \times T \mid \sigma \cap \tau = \emptyset\}$$

is called the *simplicial deleted product* of N.

This is 'the same' as $\widetilde{N}$ (i.e. the equivariant homotopy type of T depends only on N), so we write $\widetilde{N}$ instead of $\widetilde{T}$.

Example 1.
- $\widetilde{S^1} \cong S^1$;
- $\widetilde{S^n} \cong S^n = \{(x, -x)\} \subset S^n \times S^n$;
- $\widetilde{S^1 \sqcup S^1} \cong S^1 \sqcup S^1 \sqcup S^1 \times S^1 \sqcup S^1 \times S^1$;
- *the same for $S^n \sqcup S^n$;*
- $\widetilde{K_5}$ *is the sphere with 4 handles;*
- $\widetilde{K_{3,3}}$ *is the sphere with 6 handles.*

For an embedding $f : N \to \mathbb{R}^m$ define the map

$$\widetilde{f} : \widetilde{N} \to S^{m-1} \quad \text{by} \quad \widetilde{f}(x,y) := \frac{f(x)-f(y)}{|f(x)-f(y)|}.$$

This map is equivariant with respect to the 'exchanging factors' involution $t(x,y) = (y,x)$ on N and the antipodal involution on S^{m-1}.

For an embedding $f : N \to \mathbb{R}^m$ the *Haefliger-Wu* invariant is the equivariant homotopy class of the above-defined map $\widetilde{f} : \widetilde{N} \to S^{m-1}$.

Example 2. *The set of equivariant homotopy classes $\widetilde{N} \to S^{m-1}$ is in 1-1 correspondence with*

- $\emptyset$ *for* $m = 2$ *and* $N = K_5$*;*
- $\{1\}$ *if* $N = S^1$*;*
- $\mathbb{Z}$ *if* $N = S^1 \sqcup S^1$*;*
- π^S_{2n-m+1} *if* $N = S^n \sqcup S^n$*.*

It is important that using algebraic topology methods the range of the Haefliger-Wu invariant can sometimes be calculated.

Theorem 3. [10] *The Haefliger-Wu invariant is bijective for an n-polyhedron N and $2m \geq 3n+4$.*

Observe that the injectivity of the Haefliger-Wu invariant means that *if the Haefliger-Wu invariants of two embeddings are equal, then the embeddings are isotopic.*

E.g. the number of isotopy classes of embeddings $S^1 \times S^q \to \mathbb{R}^{2q+s}$ is given by the following table for $q + 2s \geq 6$.

s	2	1	0	-1	-2	-3
#, q even	∞	2	2^2	2^2	24	0
#, q odd	2	$\infty \times 2$	4	2×24	2	0

Example 4. *The Haefliger-Wu invariant is not injective for*

- $m = n+2$*, as knots* $S^1 \to \mathbb{R}^3$ *and* $S^n \to \mathbb{R}^{n+2}$ *show;*
- *most pairs* (m,n) *such that* $2m < 3n+4$ *and* $N = S^n \sqcup S^n$*;*
- *each pair* (m,n) *such that* $2m < 3n+4$ *and* $N = S^n \sqcup S^n \sqcup S^{2m-2n-3}$*.*

Theorem 5. [6] *The Haefliger-Wu invariant is bijective for a connected n-manifold N and $2m = 3n+3$.*

Example 6. [7] *The Haefliger-Wu invariant is not injective for $N = S^p \times S^{4k-1}$, $m = 6k$ and each p,k such that $0 < p < k$.*

So for $N = S^2 \times S^7$ the Haefliger-Wu invariant is injective when $m = 15$ and is not injective when $m = 12$.

The following conjecture was 'in the air' since 1960's. I learned it from A.N. Dranishnikov, E.V. Schepin, A. Szücs and O.Ya. Viro.

Conjecture 7. *The multiple Haefliger-Wu invariant is bijective for any connected n-manifold N and $m \geq n+3$.*

For a polyhedron N let

$$\widetilde{N}_p = \{(x_1, \dots, x_p) \in N^p \mid x_i \neq x_j \text{ for each } i, j\}.$$

The group S_p of permutations of p elements obviously acts on $\widetilde{N}$.

For an embedding $f : N \to \mathbb{R}^m$ define the map

$$\widetilde{f}_p : \widetilde{N}_p \to \widetilde{\mathbb{R}^m}_p \quad \text{by} \quad \widetilde{f}(x_1, \dots, x_p) = (fx_1, \dots, fx_p).$$

Clearly, the map $\widetilde{f}_p$ is S_p-equivariant. Define the *multiple Haefliger-Wu invariant* to be the equivariant homotopy class of $\widetilde{f}_p$.

The main result of this talk is disproof of this conjecture (a recent joint work with D. Crowley).

Example 8. *There are two non-isotopic embeddings $f, g : S^1 \times S^3 \to \mathbb{R}^7$ whose Haefliger-Wu invariants coincide.*

Construction of the example. Embedding f is the standard embedding. It is defined as the composition

$$S^1 \times S^3 \subset S^3 \times S^3 \subset \mathbb{R}^7$$

of standard inclusions.

Embedding g is defined as the composition

$$S^1 \times S^3 \overset{pr_2 \times t}{\to} S^3 \times S^3 \subset \mathbb{R}^7,$$

where pr_2 is the projection onto the second factor. $\subset$ is the standard inclusion.

Let us define $t : S^1 \times S^3 \to S^3$. Identify S^3 with the set of unit length quaternions. Define the Hopf map

$$\eta : S^3 \to S^2 = \mathbb{CP}^1 \quad \text{by} \quad \eta(a + bi + cj + dk) = (a + bi : c + di).$$

Identify S^2 with the set of 'purely imaginary' unit length quaternions of the form $ai + bj + ck$. Let

$$t(e^{i\theta}, y) := \eta(y) \cos\theta + \sin\theta.$$

Since $t|_{S^1 \times y}$ is an embedding for each $y \in S^3$, the map g is indeed an embedding.

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Subexponential lower bounds for randomized pivoting rules for the simplex algorithm

Thomas Dueholm Hansen
(joint work with Oliver Friedmann, Uri Zwick)

Linear programming is one of the most important computational problems studied by researchers in computer science, mathematics and operations research. The *simplex method*, developed by Dantzig in 1947 (see [2]), and its many variants, are still among the most widely used algorithms for solving linear programs. One of the most important characteristics of a simplex algorithm is the *pivoting rule* it employs. (The pivoting rule determines which non-basic variable is to enter the basis at each iteration of the algorithm). Although simplex-based algorithms perform very well in practice, essentially all *deterministic* pivoting rules are known to lead to an *exponential* number of pivoting steps on some LPs. This was first established for Dantzig's original pivoting rule by Klee and Minty [12]. For a unified view of lower bound constructions for deterministic pivoting rules see Amenta and Ziegler [1].

It is not known whether there exists a pivoting rule that requires a polynomial number of pivoting steps on *any* linear program. This is, perhaps, the most important open problem in the field of linear programming. The existence of such a polynomial pivoting rule would imply, of course, that the *diameter* of the edge-vertex graph of any polytope is polynomial in the number of facets defining it. The Hirsch conjecture, which states that the diameter of the graph defined by an n-facet d-dimensional polytope is at most $n - d$ has recently been refuted by Santos [17]. It remains open whether the diameter can be superpolynomial, or even superlinear, in n and d, however.

Kalai [10, 11] and Matoušek, Sharir and Welzl [14] devised *randomized* pivoting rules that never require more than an expected *subexponential* number of pivoting steps to solve any linear program. More specifically, the expected number of steps performed by their algorithms is at most $2^{\tilde{O}(\sqrt{n})}$, where n is the number of constraints in the linear program. Their algorithms can, in fact, be used to solve a more general class of problems known as *LP-type* problems. In this more general setting, Matoušek [13] constructed *Acyclic Unique Sink Orientations* (AUSOs) of combinatorial cubes, which form a subfamily of LP-type problems, on which the algorithm of [14] may require an almost matching expected subexponential number of iterations to find the *sink*. (The sink in this abstract setting corresponds to the optimal vertex of a linear program.) It was not known, however, whether such subexponential behavior may also occur on linear programs.

The pivoting rules of Kalai [10, 11] and of Matoušek *et al.* [14] are in a sense dual to each other (see Goldwasser [8]). We focus here on the pivoting rule of Matoušek *et al.* [14] and refer to it as the Random-Facet rule.

Perhaps the most natural randomized pivoting rule is Random-Edge, which among all improving edges from the current vertex chooses one uniformly at random. The upper bounds currently known for Random-Edge are still exponential (see Gärtner and Kaibel [7]). Random-Edge is also applicable in a much wider abstract setting. Matoušek and Szabó [15] showed that it can be subexponential on AUSOs. It was again not known whether such subexponential behavior could also occur on actual linear programs.

We show that both Random-Edge and Random-Facet may lead to an expected subexponential number of iterations on actual linear programs. More specifically, we construct concrete linear programs on which the expected number of iterations performed by Random-Edge is $2^{\Omega(n^{1/4})}$, where n is the number of variables, and (different) linear programs on which the expected number of iterations performed by Random-Facet is $2^{\Omega(\sqrt{n}/\log^c n)}$, for some fixed $c > 0$.

The linear programs on which Random-Edge and Random-Facet perform an expected subexponential number of iterations are obtained using the close relation between simplex-type algorithms for solving linear programs and *policy iteration* (also known as *strategy improvement*) algorithms for solving certain 2-player and 1-player games.

Friedmann [4] started the line of work pursued here by showing that the standard strategy iteration algorithm, which performs all improving switches *simultaneously*, may require an exponential number of iterations to solve certain *parity games* (PGs). Parity games form an intriguing family of deterministic 2-player games whose solution is equivalent to the solution of important problems in automatic verification and automata theory.

Fearnley [3] adapted Friedmann's construction to work for *Markov Decision Processes* (MDPs), an extremely important and well studied family of stochastic 1-player games. (For more on MDPs, see Howard [9] and Puterman [16].)

In [5], we recently constructed PGs on which the Random-Facet algorithm performs an expected subexponential number of iterations. Here, we use Fearnley's technique to transform these PGs into MDPs. The problem of solving an MDP, i.e., finding the optimal control *policy* and the optimal *values* of all states of the MDP, can be cast as a linear program. Furthermore, the improving switches performed by the (abstract) Random-Facet algorithm applied to an MDP correspond directly to the steps performed by the Random-Facet pivoting rule on the corresponding linear program. (Assuming, of course, that the same random choices are made by both algorithms.) The linear programs corresponding to our MDPs supply, therefore, concrete linear programs on which following the Random-Facet pivoting rule leads to an expected subexponential number of iterations.

To obtain concrete linear programs on which the simplex algorithm with the Random-Edge pivoting rule performs an expected subexponential number of pivoting steps, we follow a similar path. We start by constructing PGs on which the (abstract) Random-Edge algorithm performs an expected subexponential number of iterations. We convert these PGs into MDPs, and then to concrete linear programs. Although the conceptual path followed is similar, the concrete constructions used for Random-Edge are completely different, and somewhat more complicated, than the ones used here and in [5] for Random-Facet.

In high level terms, our MDPs, and the linear programs corresponding to them, are constructions of 'fault tolerant' *randomized counters*. The challenge in designing such counters is making sure that they count 'correctly' under most sequences of random choices made by the Random-Facet and Random-Edge pivoting rules. Our constructions are very different from the constructions used to obtain lower bounds for deterministic pivoting rules.

Random-Edge is perhaps the most natural randomized pivoting rule. The theoretically fastest pivoting rule currently known is Random-Facet. Prior to this work there were no non-polynomial lower bounds on their performance. We show that both rules may require an expected subexponential number of iterations on some linear programs, resolving a major open problem.

The full version of the paper is available at [6].

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On a colored Tverberg-Vrećica type problem

Benjamin Matschke
(joint work with Pavle V. M. Blagojević, Günter M. Ziegler)

1. Previous results

In 2009 we have shown the following colored Version of Tverberg's theorem.

Theorem 1 (Blagojević, M, Ziegler 2009, [4])**.** *Let $r \geq 2$ be prime and $d \geq 1$. Assume we are given $(r-1)(d+1)+1$ points in $\mathbb{R}^d$ that are colored with m colors such that every color class is of size at most $r-1$. Then we can partition these points into r rainbow set (that is, every part contains every color at most once) whose convex hulls intersect.*

Figure 1 shows an example.

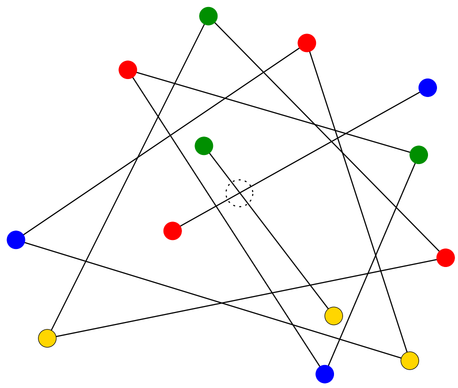

Figure 1. An example of Theorem 1 for $d = 2$ and $r = 5$.

In their 1993 paper [8] H. Tverberg and S. Vrećica presented a conjectured common generalization of some Tverberg type theorems, some ham sandwich type theorems (see Section 3) and intermediate results. See [12] for a further collection of implications.

Conjecture 2 (Tverberg–Vrećica Conjecture)**.** *Let $0 \leq k \leq d$ and let $\mathbb{C}^0, \dots, \mathbb{C}^k$ be finite point sets in $\mathbb{R}^d$ of cardinality $|\mathbb{C}^\ell| = (r_\ell - 1)(d-k+1)+1$. Then one can partition each $\mathbb{C}^\ell$ into r_ℓ sets $F_1^\ell, \dots, F_{r_\ell}^\ell$ such that there is a k-plane P in $\mathbb{R}^d$ that intersects all the convex hulls $\mathrm{conv}(F_j^\ell)$, $0 \leq \ell \leq k$, $1 \leq j \leq r_\ell$.*

The Tverberg–Srećica Conjecture has been verified for the following special cases:

- $k = d$ (trivial),
- $k = 0$ (Tverberg's theorem [7]),
- $k = d - 1$ (Tverberg & Vrećica [8]),
- for $k = d - 2$ a weaker version was shown in [8] (one requires two more points for each $\mathbb{C}^\ell$),
- k and d are odd, and $r_0 = \cdots = r_k$ is an odd prime (Živaljević [12]),
- $r_0 = \cdots = r_k = 2$ (Vrećica [10]), and
- $r_\ell = p^{a_\ell}$, $a_\ell \geq 0$, for some prime p, and $p(d-k)$ is even or $k = 0$ (Karasev [3]).

2. Main Theorem

In the talk we presented the following colorful version of the Tverberg–Vrećica conjecture.

Theorem 3 (Main Theorem; BMZ 2009, [5])**.** *Let r be prime and $0 \leq k \leq d$ such that $r(d-k)$ is even or $k = 0$. Let $\mathbb{C}^\ell$ $(\ell = 0, \ldots, k)$ be subsets of $\mathbb{R}^d$ of cardinality $|\mathbb{C}^\ell| = (r-1)(d-k+1)+1$. Let the $\mathbb{C}^\ell$ be colored,*

$$\mathbb{C}^\ell \;=\; \biguplus C_i^\ell,$$

such that no color class is too large, $|C_i^\ell| \leq r-1$. Then we can partition each $\mathbb{C}^\ell$ into colorful sets $F_1^\ell, \ldots, F_r^\ell$ (that is, $|F_j^\ell \cap C_i^\ell| \leq 1$) and find a k-plane P that intersects all the convex hulls $\mathrm{conv}(F_j^\ell)$.

See Figure 2 for an example. This theorem is tight in the sense that it becomes false if one single color class C_i^ℓ has r elements and all the other ones are singletons. Our proof uses equivariant algebraic topology, hence it has a natural topological extension, which we omit here.

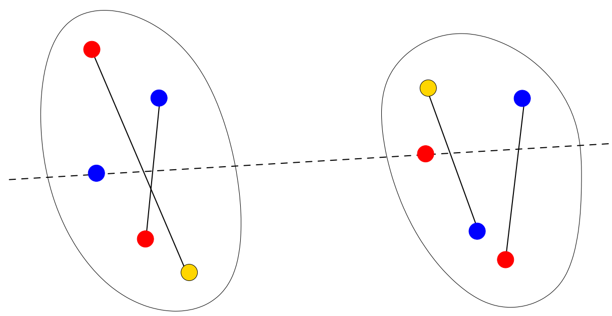

Figure 2. An example of Theorem 3 for $d = 2$, $r = 3$, $k = 1$, and $N = 4$.

3. Corollaries

The Tverberg–Vrećica Conjecture 2 as well as our Main Theorem 3 implies easily the Ham Sandwich Theorem.

Corollary 4 (Ham Sandwich Theorem; Banach 1938, [6])**.** *Any d masses in $\mathbb{R}^d$ can be bisected simultaneously by a hyperplane.*

To show this, approximate the given d masses by points in general position, apply the Main Theorem 3 for $k = d-1$, and observe that for any ℓ all but one of the sets F_j^ℓ will have cardinality two. The end-points of those F_j^ℓ lie on opposite sides of the hyperplane P. That is, P bisects the approximating mass. A limit argument finishes the proof.

More generally, for arbitrary k we get as a corollary the Center Transversal Theorem.

Corollary 5 (Center Transversal Theorem; Vrećica–Živaljević 1990, [11], and Dol'nikov 1992, [2])**.** *For any $(k+1)$ masses in $\mathbb{R}^d$, there exists a k-plane P, such that every hyperplane H containing P has at least $\frac{1}{d-k+1}$ from each mass on each side.*

4. Proof method

The proof of our Main Theorem 3 is based on a configuration space/test map scheme for vector bundles. Such a proof scheme was already used in [1], [2], [12], [10] and [3]. In our situation calculations are more involved. The major two new proof ingredients are

(1) a new Borsuk–Ulam type theorem for $\mathbb{Z}_p^m$-equivariant vector bundles, which generalizes results of Volovikov [9] and Živaljević [12], and
(2) a calculation of the Fadell–Husseini index of a join of chessboard complexes.

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Face enumeration in flag spheres

Eran Nevo
(joint work with Satoshi Murai, Kyle Petersen, Bridget Tenner)

Problem 1. *What are the possible face numbers of flag simplicial spheres?*

Let Δ be a finite simplicial complex. If all the minimal non-faces of Δ have size two, then Δ is *flag*. If its geometric realization $||\Delta||$ is homeomorphic to a sphere, Δ is a *simplicial sphere*. There is no conjecture as to what the answer to Problem 1 should be. However, there are some necessary conditions which are conjectured to hold, even in the generality of flag homology spheres.

Let $f(\Delta)$ denote the f-vector of Δ, $g(\Delta)$ its g-vector and $\gamma(\Delta)$ its γ-vector (they all carry the same information).

Conjecture 2. *(g-conjecture* [4, 8]*) If Δ is a homology sphere then $g(\Delta)$ is the f-vector of a multicomplex, i.e. an M-sequence.*

The following conjecture of Gal implies the Charney-Davis conjecture:

Conjecture 3. *(Gal's conjecture* [3]*) If Δ is a flag homology sphere then $\gamma(\Delta)$ is nonnegative.*

With Kyle Petersen we conjectured that stronger conditions hold:

Conjecture 4. *(*[6]*) If Δ is a flag homology sphere then $\gamma(\Delta)$ is the f-vector of a balanced simplicial complex.*

If true, Conjecture 4 implies Gal's conjecture, as well as that $g(\Delta)$ is the f-vector of a simplicial complex, which strengthens the g-conjecture in the flag case. We remark that the f-vectors of balanced simplicial complexes were characterized in [2].

Conjecture 4 holds in the following special cases.

Theorem 5. *For each of the following flag homology spheres Δ, $\gamma(\Delta)$ is the f-vector of a balanced simplicial complex:*

1. *Δ is the barycentric subdivision of a homology sphere. (*[7]*)*
2. *Δ is a Coxeter complex. (*[6]*)*
3. *Δ is a $(d-1)$-dimensional flag homology sphere with at most $2d+3$ vertices. (*[6]*)*
4. *Δ is the simplicial complex dual to an associahedron or a cyclohedron. (*[6]*)*

Very recently, with Satoshi Murai we proved the following special case of Conjecture 4, by using a variant of S-shellability. The proof is based on a careful analysis of Stanley's formula for the effect of S-shelling on the **cd**-index [9] and the monotonicity result of Ehrenborg-Karu w.r.t. subdivisions [1].

Theorem 6. *([5]) If Δ is the barycentric subdivision of the boundary complex of a polytope then $\gamma(\Delta)$ is the f-vector of a balanced simplicial complex.*

We conjecture that the entire **cd**-index of a Gorenstein* poset can be viewed as the *flag* f-vector of a colored simplicial complex, in a certain sense, that in particular implies the above theorem. ($f_S(\Delta)$ counts the number of faces in Δ with color set S, w.r.t. a vertex coloring with $\dim(\Delta)+1$ colors.)

Problem 7. *Let Δ be a flag homology sphere. Find an algebraic structure $A(\Delta)$ whose Hilbert series equals $\gamma(\Delta)$ that will prove Conjecture 4.*

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Face vectors of simplicial cell decompositions of manifolds

Satoshi Murai

The study of face numbers is one of the central topics in combinatorics. A goal of the study is to obtain characterizations of face vectors of certain combinatorial objects. In this paper, we study face vectors of simplicial posets, particularly those whose geometric realizations are manifolds.

A *simplicial poset* is a finite poset P with a minimal element $\hat{0}$ such that every interval $[\hat{0},\sigma]$ for $\sigma \in P$ is a Boolean algebra. For example, the face poset of a simplicial complex is a simplicial poset. But not all simplicial posets come from simplicial complexes. Let P be a simplicial poset. We say that an element $\sigma \in P$ has *rank* i, denoted $\mathrm{rank}\sigma = i$, if $[\hat{0},\sigma]$ is a Boolean algebra of rank $i+1$. The *dimension* of P is

$$\dim P = \max\{\mathrm{rank}\sigma : \sigma \in P\} - 1.$$

Let $f_i = f_i(P)$ be the number of elements $\sigma \in P$ having rank $i+1$ and $d = \dim P + 1$. The vector $f(P) = (f_{-1}, f_0, \ldots, f_{d-1})$ is called the *f-vector of P*. To study f-vectors, it is often convenient to consider the *h-vector* $h(P) = (h_0, h_1, \ldots, h_d)$ *of P* defined by

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i}.$$

It is easy to see that knowing $f(P)$ is equivalent to knowing $h(P)$.

It is known that simplicial posets are CW posets. Thus, for any simplicial poset P, there is a regular CW-complex $\Gamma(P)$ whose face poset is equal to P. A *simplicial cell sphere* is a simplicial poset P such that $\Gamma(P)$ is homeomorphic to a sphere. One of the most important results on face vectors of simplicial posets is the next result due to Stanley [St] and Masuda [Ma], which characterizes all possible h-vectors of simplicial cell spheres.

Theorem 1 (Stanley, Masuda)**.** *Let $h = (h_0, h_1, \ldots, h_d) \in \mathbb{Z}^{d+1}$. Then h is the h-vector of a $(d-1)$-dimensional simplicial cell sphere if and only if it satisfies the following conditions:*

(1) $h_0 = h_d = 1$ and $h_i = h_{d-i}$ for all i.
(2) $h_i \geq 0$ for all i.
(3) if $h_i = 0$ for some $1 \leq i \leq d-1$ then $h_0 + h_1 + \cdots + h_d$ is even.

Theorem 1 characterizes h-vectors of simplicial cell spheres, and therefore characterizes all possible their f-vectors. We say that a poset P is a *simplicial cell decomposition of a topological manifold M* if P is a simplicial poset such that $\Gamma(P)$ is homeomorphic to M. From topological and combinatorial viewpoints, it is natural to ask a characterization of face vectors of simplicial cell decompositions of a given topological manifold M. To study this problem it is convenient to consider h''-vectors introduced by Novik [No].

For a simplicial poset P, let

$$\beta_i = \beta_i(P) = \dim_{\mathbb{Z}_2} \tilde{H}_i(\Gamma(P); \mathbb{Z}_2)$$

be the ith *Betti number of P*, where $\tilde{H}_i(\Gamma(P); \mathbb{Z}_2)$ is the ith reduced homology group of $\Gamma(P)$ over $\mathbb{Z}_2$. The *h''-vector $h''(P) = (h''_0, h''_1, \ldots, h''_d)$ of P (over $\mathbb{Z}_2$)* is defined by

$$h''_k(P) = \begin{cases} 1, & \text{if } k = 0, \\ h_k - \binom{d}{k} \left\{ \sum_{\ell=1}^{k} (-1)^{\ell-k} \beta_{\ell-1} \right\}, & \text{if } 1 \leq k \leq d-1, \\ h_d - \sum_{\ell=1}^{d-1} (-1)^{\ell-d} \beta_{\ell-1} = \beta_{d-1}, & \text{if } k = d. \end{cases}$$

If one knows Betti numbers, then knowing $h(P)$ is equivalent to knowing $h''(P)$. (Though we use $\mathbb{Z}_2$, we can consider any field. See [NS].) The following is known.

Theorem 2. *Let P be a simplicial poset such that $\Gamma(P)$ is a connected $(d-1)$-manifold without boundary and let $h''(P) = (h''_0, h''_1, \ldots, h''_d)$. Then*

(1) (Novik [No]) $h''_0 = h''_d = 1$ and $h''_i = h''_{d-i}$ for all i.

(2) (Novik-Swartz [NS]*)* $h''_i \geq 0$ *for all* i.
(3) ([Mu]*) if* $h''_i = 0$ *for some* $1 \leq i \leq d-1$ *then* $h''_0 + h''_1 + \cdots + h''_d$ *is even.*

The above result gives a strong necessary conditions on face vectors of simplicial cell decompositions of manifolds. I recently proved the next results.

Theorem 3. *Fix integers* $n, m \geq 1$. *Let* $d = n+m+1$ *and* $h'' = (h''_0, h''_1, \ldots, h''_d) \in \mathbb{Z}^{d+1}$. *Then* h'' *is the* h''*-vector of a simplicial cell decomposition of* $S^n \times S^m$ *if and only if* h'' *satisfies the conditions (1), (2) and (3) in Theorem 2.*

Theorem 4. *Let* $d > 0$ *and* $h'' = (h''_0, h''_1, \ldots, h''_d) \in \mathbb{Z}^{d+1}$. *Then* h'' *is the* h''*-vector of a simplicial cell decomposition of* $\mathbb{R}P^{d-1}$ *if and only if* h'' *satisfies the conditions (1), (2) and (3) in Theorem 2.*

Note that the above theorem also holds for connected sums of those manifolds. It would be interesting to find characterizations of face vectors of simplicial cell decompositions of several types of manifolds. Another interesting manifold for which conditions (1)–(3) characterize all possible face vectors of its simplicial cell decompositions is $\mathbb{C}P^2$. On the other hand, condition (1)–(3) do not characterize all possible face vectors of any manifolds. For example, there are no simplicial cell decompositions P of 3-dimensional torus $T^3 = S^1 \times S^1 \times S^1$ with $h''(P) = (1, 0, 0, 0, 1)$ (see [Li, p. 29]). There are many questions on this subject. Below we list a few of them.

Problem 5. *Do conditions (1)–(3) characterize all possible face vectors of simplicial cell decompositions of* $\mathbb{C}P^3$ *(and more generally* $\mathbb{C}P^n$*)?*

Problem 6. *Prove that if* P *is a simplicial cell decomposition of a* 3*-dimensional torus* $T^3 = S^1 \times S^1 \times S^1$ *then* $h_2(P) \geq 22$.

Note that the solution of Problem 6 yields the complete characterization of face vectors of simplicial cell decomposition of T^3. For d-dimensional torus, we do not have a good conjecture of characterization of face vectors. On the other hand, there is a beautiful construction of simplicial cell decomposition of a torus, called *Steinberg torus* [DPS]. The construction gives a simplicial cell decomposition of d-dimensional torus with $(d+1)!$ facets. We ask:

Problem 7. *Is it true that if* P *is a simplicial cell decomposition of a* d*-dimensional torus* $T^d = S^1 \times \cdots \times S^1$ *then* P *has at least* $(d+1)!$ *facets?*

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Inversion arrangements and Bruhat intervals

Axel Hultman

Let n be a positive integer. Given indices $1 \leq i < j \leq n$, define a hyperplane

$$H_{i,j} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i = x_j\}.$$

The arrangement of all such hyperplanes

$$\mathcal{A}_n = \{H_{i,j} \mid 1 \leq i < j \leq n\}$$

is known as the *braid arrangement*. The orthogonal reflections in the hyperplanes $H_{i,j}$ generate a finite reflection group isomorphic to the symmetric group $\mathfrak{S}_n$; a natural isomorphism is given by associating a reflection through $H_{i,j}$ with the transposition $(i, j) \in \mathfrak{S}_n$.

Given a permutation $w \in \mathfrak{S}_n$, we define its *inversion arrangement* as the following subarrangement of $\mathcal{A}_n$:

$$\mathcal{A}_w = \{H_{i,j} \mid 1 \leq i < j \leq n,\, w(i) > w(j)\}.$$

In particular, $\mathcal{A}_{w_0} = \mathcal{A}_n$, where $w_0 \in \mathfrak{S}_n$ is the reverse permutation $i \mapsto n+1-i$.

The inversion arrangement cuts the ambient space into a set $\mathrm{re}(w)$ of *regions*, a region being a connected component of the complement $\mathbb{R}^n \setminus \cup\mathcal{A}_w$.

Let $[\cdot,\cdot]$ denote closed intervals in the Bruhat order on $\mathfrak{S}_n$. Postnikov [3] discovered a numerical relationship between $\mathrm{re}(w)$ and the Bruhat order ideal $[e, w]$, where $e \in \mathfrak{S}_n$ is the identity permutation. When w is a Grassmannian permutation, he proved that the sets are equinumerous; both are in 1-1 correspondence with certain cells in a CW decomposition of the totally nonnegative Grassmannian. For arbitrary w, he conjectured the following results that were subsequently proven in [1]:

(A) For all $w \in \mathfrak{S}_n$, $\#\mathrm{re}(w) \leq \#[e, w]$.
(B) Equality holds in (A) if and only if w avoids the patterns 4231, 35142, 42513 and 351624.

We have just defined $\mathcal{A}_w$ using $\mathfrak{S}_n$-specific language. It is, however, completely natural to replace $\mathfrak{S}_n$ by an arbitrary finite reflection group W and consider $\mathcal{A}_w$, $\mathrm{re}(w)$ and $[e, w]$ for any $w \in W$. In fact, it was not (A) but the following result which was established in [1]:

(A′) Given a finite reflection group W and any $w \in W$, $\#\mathrm{re}(w) \leq \#[e, w]$.

This generalises (A),[1] but notice that there is no statement (B′). Indeed, the problem of how to characterise those $w \in W$ for which equality holds in (A′)

[1] An explanation of the implication (A′) $\Rightarrow$ (A) can be found in [1].

was posed as [1, Open problem 10.3]. Such a characterisation is the main result reported on here:

Theorem 1. *Equality holds in (A′) if and only if the following property is satisfied for every $u \leq w$: among the shortest paths from u to w in the Cayley graph of W with edges generated by reflections, at least one visits vertices in order of increasing Coxeter length.*

A number of consequences can be derived from the main result:

First, we may conclude that the characterising property is poset-theoretic. That is, whether or not equality holds in (A′) can be determined by merely looking at the Bruhat interval $[e, w]$ as an abstract poset.

Second, a new proof of the difficult direction of (B) can be obtained. In [1], (A′) was proven by exhibiting an injective map ϕ from (essentially) re(w) to $[e, w]$. Thus, proving (B) amounts to characterising surjectivity of ϕ in terms of pattern avoidance when $W = \mathfrak{S}_n$. That surjectivity implies the appropriate pattern avoidance is a reasonably straightforward consequence of the construction of ϕ; see [1, Section 4]. Contrastingly, the proof of the converse statement given in [1, Section 5] is a direct, fairly involved, counting argument which does not use ϕ at all. In light of our main result, surjectivity of ϕ can now, however, be related to pattern avoidance in a rather straightforward way.

Third, when W is a Weyl group, each element $w \in W$ corresponds to a Schubert variety $X(w)$. We derive from the above theorem that equality holds in (A′) whenever $X(w)$ is rationally smooth. It is to be noted that Oh and Yoo [2] recently derived a stronger q-analogue of equality in (A′) for rationally smooth $X(w)$.

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Finite motions from periodic frameworks with added symmetry

Bernd Schulze

(joint work with Elissa Ross and Walter Whiteley)

The theory of rigidity of periodic frameworks has undergone rapid and extensive development in the last four years [2, 7, 8]. We now have necessary conditions (called *Maxwell type counts*) for such frameworks to be rigid, either with a fixed lattice of translations or with a flexible lattice of translations. Underlying much of the recent work are finite ‘lattice rigidity matrices’ for the equivalence classes of vertices and edges under the infinite group of translations $\mathbb{Z}^d$ in d-space. With the corresponding count of periodicity-preserving trivial motions under these constraints (typically d translations), the number of rows, e, and columns, $dv + l$ (where l is the number of lattice parameters) of these 'orbit matrices' lead to necessary Maxwell type counts for a framework to be infinitesimally rigid [7, 8]: $e \geq dv + l - d$.

The theory of rigidity of finite symmetric frameworks has also experienced some breakout results, building on a decade or more of initial Maxwell-type necessary conditions for frameworks with various symmetry groups [5, 4, 9]. In some key cases, these symmetry conditions predict finite (i.e., continuous or equivalently, analytic) motions for frameworks which are realized generically within the symmetry constraints, but whose underlying graphs are generically rigid without symmetry [3, 12]. Recently, key results of this work have been expressed in terms of 'orbit rigidity matrices' for the equivalence classes of vertices and edges under the group of symmetry operations $\mathcal{S}$ [10, 11]. With modified counts for the symmetry-preserving trivial motions $triv_{\mathcal{S}}$, and with e_0 and v_0 denoting the number of edge orbits and vertex orbits under the group action of $\mathcal{S}$, respectively, these matrices lead to Maxwell type necessary counts for frameworks to be infinitesimally rigid: $e_0 \geq dv_0 - triv_{\mathcal{S}}$.

Given that many crystal structures combine both periodic structure and symmetry within the unit cells, it is natural to investigate the interactions of these two types of group operations. So we consider frameworks with 'combined symmetry groups' of the form $\mathbb{Z}^d \rtimes \mathcal{S}$, where $\mathbb{Z}^d$ is the group of translations of the framework, $\mathcal{S}$ is the group of additional symmetries of the framework, and $\rtimes$ denotes the semi-direct product of $\mathcal{S}$ acting on $\mathbb{Z}^d$. To study motions of symmetric periodic frameworks which preserve both the periodicity and the added symmetry of the structure, we introduce 'combined orbit rigidity matrices' for the groups $\mathbb{Z}^d \rtimes \mathcal{S}$. An analysis of the basic structure of these matrices provides extended Maxwell type necessary counts for infinitesimal rigidity. In this setting we:

(1) count the rows of the combined orbit matrix: one row per orbit of edges $r = e_0$;
(2) count the columns of the combined orbit matrix: one vector column per orbit of vertices plus columns for symmetry-preserving lattice deformations: $c = dv_0 + \ell_{\mathcal{S}}$;
(3) the dimension of the space of trivial motions (translations) left by symmetries: $t_{\mathcal{S}}$.

The minimum dimension of the space of non-trivial symmetry-preserving infinitesimal periodic motions of the periodic structure is:

$$m = c - t_{\mathcal{S}} \quad - r \quad \text{or} \quad m = dv_0 + \ell_{\mathcal{S}} - t_{\mathcal{S}} \quad - e_0.$$

This is compared with the corresponding count on the graph without symmetry, where with orbits of size $k_{\mathcal{S}}$ and no fixed edges or vertices, for the fully flexible lattice, we would anticipate:

$$m = d(k_{\mathcal{S}} v_0) + \binom{d+1}{2} - d \quad - (k_{\mathcal{S}} e_0).$$

In addition, if we choose the positions of the vertices generically within the symmetry (i.e., make one generic choice for each orbit of vertices) then the predicted infinitesimal motions will be finite motions [10, 11, 1].

The results are a surprise – adding symmetry can sometimes cause additional flexibility beyond what the original graph without symmetry would exhibit in the periodic lattice. These more flexible examples include symmetries such as inversive symmetry, or half-turn symmetry with a mirror, found in a number of crystals, such as zeolites. Recent studies have confirmed that flexibility is a feature of natural zeolites and contributes to their physical and chemical properties [6]. In turn, this suggests that predicted flexibility in a computer designed theoretical 'zeolite' would be a criterion for selecting which theoretical compounds should be synthesized for further testing.

When adding symmetry to a periodic lattice structure, we must consider the flexibility that this symmetry allows in the lattice structure. Inversive symmetry is a key example, since it fits all possible lattice deformations (it occurs in 'triclinic lattices'), and the addition of this symmetry to the framework generates non-trivial motions from frameworks that previously were minimally rigid, while preserving the full range of possible flexes of the lattice itself.

In contrast, only certain types of lattices leave open the addition of a half-turn symmetry in 3-space. A half-turn axis parallel to a side of the lattice requires that side to be perpendicular to the remaining parallelogram face. This leaves only four of the six possible flexes of the lattice ('monoclinic lattices'), but it does predict additional non-trivial motions. Similarly, mirrors of symmetry can fit parallel to faces of the lattice, and restrict the shapes to monoclinic lattices, with the variable angle now parallel to the mirror.

Larger symmetry groups with several generators can also be analyzed. Each group for the symmetric periodic structure and the associated crystal system requires some specific terms in the analysis. However, patterns emerge, and tables with the corresponding counts for these groups can easily be generated.

In the larger theory of rigidity of frameworks, infinitesimal motions of 'generic frameworks' transfer to finite motions, for appropriate versions of generic. This holds for generic frameworks without symmetry [1], for frameworks generic within the symmetry class [10], and for periodic frameworks with generic configurations within the unit cells [8]. This property extends to periodic frameworks with added symmetry, so our orbit counts can detect flexibility on a finite scale, at generic realizations for representatives of the orbits under the action of the group $\mathbb{Z}^d \rtimes \mathcal{S}$.

We note that frameworks with non-trivial symmetries may be regarded as graphs embedded on appropriately chosen orbifolds. This orbifold is defined by the original setting of the framework ($\mathbb{R}^d$) modulo the symmetry group. For example, periodic frameworks have symmetry group $\mathbb{Z}^d$, and may be viewed as graphs on the d-dimensional topological torus $\mathbb{R}^d/\mathbb{Z}^d$. Similarly, a framework with n-fold rotational symmetry in the plane can be regarded as a framework on a cone, with cone angle $2\pi/n$. So the orbit matrices also provide conditions for rigidity and flexibility of frameworks on these surfaces.

For periodic frameworks with additional symmetry, the underlying orbifold may be more exotic. For example, periodic frameworks with mirror symmetry in the plane or space correspond to frameworks on 2- or 3-spheres $\mathbb{S}^2$ and $\mathbb{S}^3$, but with a flat metric. Frameworks in 3-space with inversive symmetry have an orbifold with topology of $\mathbb{P}^3$, projective 3-space. Similar statements are possible for all frameworks which admit an orbit framework under the action of their symmetry group. Again, the periodic symmetric orbit matrices represent the rigidity matrices for frameworks actually living in these more exotic spaces, with flat metrics.

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Non-evasiveness, collapsibility and explicit knotted triangulations

Bruno Benedetti

(joint work with Frank H. Lutz)

Collapsibility is a combinatorial strengthening of the topological notion of contractibility. In the Sixties, Bing and his student Goodrick proved with knot-theoretic techniques that not all triangulated 3-balls are collapsible [4, 7]. Building on a construction by Lutz [10], we announce the finding of a first explicit example:

Theorem 1. *There exists a non-collapsible simplicial* 3*-ball* $B_{15,66}$ *with* 15 *vertices and* 66 *tetrahedra.*

Non-evasiveness is a further strengthening of collapsibility, emerged in theoretical computer science and later studied by Kahn, Saks and Sturtevant [9] and Welker [11]. A 0-dimensional complex is non-evasive if and only if it consists of a single point. Recursively, a d-dimensional simplicial complex ($d > 0$) is non-evasive if and only if there is some vertex v whose link and deletion are both non-evasive.

Every non-evasive complex is collapsible. The converse is false: Collapsibility is not maintained under taking links. In fact, there are elementary examples of collapsible 2-complexes all of whose vertex links are non-contractible. A first such example with only six vertices was found by Björner; for another example, see Barmak–Minian [1, Figure 7]. However, the difference between collapsibility and non-evasiveness does not simply depend on vertex links. In fact, we show that even a manifold can be collapsible and evasive:

Theorem 2. *There exists a collapsible and evasive simplicial* 3*-ball* $B_{12,38}$ *with* 12 *vertices and* 38 *tetrahedra.*

This triangulation $B_{12,38}$ is obtained via knot theory: It contains a trefoil knot in its 1-skeleton, realized with one interior edge plus four boundary edges. Note that the link of every boundary vertex of a 3-ball is a 2-ball and hence non-evasive.

For d-manifolds, there exists also a combinatorial strengthening of the topological notion of simply-connectedness, known as *local constructibility.* This property was introduced by Durhuus and Jonsson [5] and later studied by the speaker and Ziegler [3]. A 3-sphere is locally constructible if and only if it can be obtained from a "tree of tetrahedra" (i.e. a simplicial 3-ball whose dual graph is a tree) by repeatedly identifying two adjacent boundary triangles. With knot-theoretic arguments one can show that not all 3-spheres are locally constructible [3]. More precisely, a 3-sphere is locally constructible if and only if the removal of any tetrahedron would turn it into a collapsible ball [3, Corollary 2.11]. Here we present a first explicit non-example:

Theorem 3. *There exists a non-locally-constructible simplicial* 3*-sphere* $S_{18,125}$ *with* 18 *vertices and* 125 *tetrahedra.*

In the language of discrete Morse theory, $B_{15,66}$ and $S_{18,125}$ are triangulations on which no discrete Morse function is sharp in bounding the Betti numbers from above. We believe that these first, explicit examples at the level of manifolds can help in developing and testing algorithms to find "good" Morse matchings. (Compare Engström [6] and Joswig–Pfetsch [8].)

Details will appear in [2].

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Diameter of random combinatorial types of 3-polytopes

Marc Noy

(joint work with Guillaume Chapuy, Eric Fusy, Omer Giménez)

We prove that the diameter of a random 3-polytope with n vertices is of order $n^{1/4}$. More precisely:

Theorem 1. *Let G_n be a random graph of a 3-polytope with n vertices, and let $D(G_n)$ be its diameter. Then with high probability, for each $\varepsilon > 0$ small enough,*

$$n^{1/4-\varepsilon} < D(G_n) < n^{1/4+\varepsilon}.$$

The proof uses the following ideas.

(a) The diameter of a random quadrangulation is of order $n^{1/4}$. This is proved using the bijection between quadrangulations and well-labelled trees, due to Cori–Vaquelin and to Schaeffer.

(b) There is a bijection between quadrangulations and planar maps, and the diameter increases at most by a factor of $\log n$.

(c) Given a map M with n edges, with high probability there is a block of linear size, and the remaining blocks are of order $O(n^{2/3})$ (Gao and Wormald). This way we can transfer the results on maps to results on 2-connected maps. A similar argument proves that the diameter of 3-connected maps is of order $n^{1/4}$.

(d) Finally, Steinitz's theorem, which characterizes 3-connected planar maps as graphs of 3-polytopes, finishes the job.

Unique 3-colorability and universal rigidity

Igor Pak
(joint work with Danny Vilenchik)

We present sufficient eigenvalue conditions for unique 3-colorability of 3-partite graphs. This is used to prove that certain random and pseudorandom 3-partite graphs are uniquely 3-colorable. Other applications include a computation of vector chromatic numbers of random graphs and an explicit construction of large 4-regular Cayley graphs which are uniquely 3-colorable and have large (logarithmic) girth. The proof uses bar and joint framework realization of 3-partite graphs, and Connelly's sufficient condition on the framework's universal rigidity.

The central curve of linear programming

Jesús De Loera
(joint work with Bernd Sturmfels, Cynthia Vinzant)

This project explores the *central curve* of a linear program, the smallest algebraic variety that contains the central path. The key tools come from the geometric combinatorics of matroids and classical algebraic geometry. To state the results recall the linear programming problem in its *primal* and *dual* formulation:

$$\text{Maximize } \mathbf{c}^T\mathbf{x} \text{ subject to } A\mathbf{x} = \mathbf{b} \text{ and } \mathbf{x} \geq 0; \tag{1}$$

$$\text{Minimize } \mathbf{b}^T\mathbf{y} \text{ subject to } A^T\mathbf{y} - \mathbf{s} = \mathbf{c} \text{ and } \mathbf{s} \geq 0. \tag{2}$$

Here A is a fixed matrix of rank d having n columns, while the vectors $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{b} \in \text{image}(A)$ may vary. Recall the following well-known result:

Lemma 1 (Fundamental Lemma of Interior Point Methods)**.** *For all real $\lambda > 0$, the system of polynomial equations*

$$A\mathbf{x} = \mathbf{b}\,,\ A^T\mathbf{y} - \mathbf{s} = \mathbf{c},\ \text{ and } x_i s_i = \lambda \text{ for } i = 1, 2, \ldots, n, \tag{3}$$

has a unique real solution $(\mathbf{x}^*(\lambda), \mathbf{y}^*(\lambda), \mathbf{s}^*(\lambda))$ with the properties

- $\mathbf{x}^*(\lambda) > 0$ *and* $\mathbf{s}^*(\lambda) > 0$.
- *The point $\mathbf{x}^*(\lambda)$ is the optimal solution of*

$$\text{Maximize } \mathbf{c}^T\mathbf{x} + \lambda \sum_{i=1}^{n} \log x_i, \text{ subject to } A\mathbf{x} = \mathbf{b} \text{ and } \mathbf{x} \geq 0. \tag{4}$$

- *The limit point $(\mathbf{x}^*(0), \mathbf{y}^*(0), \mathbf{s}^*(0))$ of these solutions for $\lambda \to 0$ is the unique optimal solution of the LP.*

The primal central path *is the curve $\{\mathbf{x}^*(\lambda) \,|\, \lambda > 0\}$ inside the polytope P. There is an analogous* dual central path. *The central path connects the optimal solution of the linear program in question with its* analytic center.

The *central curve* of a linear program is the Zariski closure of the central path. The algebraic-geometric study of central paths was pioneered by Bayer and Lagarias [1, 2]. They observed (on pages 569-571 of [2]) that the central path defines an irreducible algebraic curve in $\mathbf{x}$-space or $\mathbf{y}$-space, and they identified a complete intersection that has the central curve as an irreducible component. The last sentence of [2, §11] states the open problem of identifying polynomials that cut out the central curve, without any extraneous components. We solve that problem here. We also determined its degree, and genus. These invariants, along with the degree of the Gauss image of the curve, are expressed in terms of a matroid of the input data, specifically h-vectors of broken circuit complexes. As an application we give an instance-specific bound of the total curvature of the central path, a quantity relevant for interior point methods.

The geometry of a central curve is intimately connected to that of the underlying arrangement of constraint lines. *Matroid theory* and in particular the study of h-vectors of *broken circuit complexes* are be crucial for stating and proving our results.

Define the *central sheet* $\mathfrak{L}_{A,\mathbf{c}}^{-1}$ to be the Zariski closure in the affine space $\mathbb{C}^n$ of the set

$$\left\{ \left(\frac{1}{u_1}, \frac{1}{u_2}, \ldots, \frac{1}{u_n}\right) \in \mathbb{C}^n \;:\; (u_1, u_2, \ldots, u_n) \in \mathfrak{L}_{A,\mathbf{c}} \,\text{and}\, u_i \neq 0 \;\text{ for }\; i = 1, \ldots, n \right\}.$$

The linear space $\{A\mathbf{x} = \mathbf{b}\}$ has dimension $n - d$, and we write $I_{A,\mathbf{b}}$ for its linear ideal.

Lemma 2. *The prime ideal of polynomials that vanish on the central curve $\mathcal{C}$ is $I_{A,\mathbf{b}} + J_{A,\mathbf{c}}$. The degree of both $\mathcal{C}$ and the central sheet $\mathfrak{L}_{A,\mathbf{c}}^{-1}$ coincides with the Möbius number $|\mu(A, \mathbf{c})|$.*

Proposition 3 (Proudfoot-Speyer [7])**.** *The degree of the central sheet $\mathfrak{L}_{A,\mathbf{c}}^{-1}$, regarded as a variety in complex projective space, coincides with the Möbius number $|\mu(A, \mathbf{c})|$. Its prime ideal $J_{A,\mathbf{c}}$ is generated by a universal Gröbner basis consisting of all homogeneous polynomials*

$$\sum_{i \in \mathrm{supp}(v)} v_i \cdot \prod_{j \in \mathrm{supp}(v) \setminus \{i\}} x_j, \tag{5}$$

where $\sum v_i x_i$ runs over non-zero linear forms of minimal support that vanish on $\mathfrak{L}_{A,\mathbf{c}}$.

The polynomials in (5) correspond to the circuits of the matroid $M_{A,\mathbf{c}}$. There is at most one circuit contained in each $(d+2)$-subset of $\{x_1, \ldots, x_n\}$, so their number is at most $\binom{n}{d+2}$. If the matrix A is generic then $M_{A,\mathbf{c}}$ is uniform and its Möbius number equals

$$|\mu(A, \mathbf{c})| \;=\; \binom{n-1}{d}.$$

For arbitrary matrices A, this binomial coefficient furnishes an upper bound on the Möbius number $|\mu(A, \mathbf{c})|$.

Theorem 4. *The degree of the primal central path of (1) is the Möbius number $|\mu(A, \mathbf{c})|$ and is hence at most $\binom{n-1}{d}$. The prime ideal of polynomials that vanish on the primal central path is generated by the circuit polynomials (5) and the d linear polynomials in $A\mathbf{x} - \mathbf{b}$.*

The total curvature of the central path is an important quantity for the estimation of the running time of interior point methods in linear programming [6, 9, 10]. We relate our algebraic framework to the problem of bounding the total curvature. The relevant geometry was pioneered by Dedieu, Malajovich and Shub [4].

Consider an arbitrary curve $[a, b] \to \mathbb{R}^n$, $t \mapsto f(t)$, whose parameterization is twice differentiable and whose derivative $\dot{f}(t)$ is a non-zero vector for all parameter values $t \in [a, b]$. This curve has an associated *Gauss map* into the unit sphere S^{m-1}, which is defined as $\gamma : [a, b] \to S^{m-1}$, $t \mapsto \frac{\dot{f}(t)}{||\dot{f}(t)||}$. The image $\gamma = \gamma([a, b])$ of the Gauss map in S^{m-1} is called the *Gauss curve* of the given curve f. We also consider the *projective Gauss curve*. Our Gauss curve is algebraic, with known defining polynomial equations. The *total curvature* K of our curve f is the arc length of its associated Gauss curve γ, see [4, §3]. The arc length of the Gauss curve is bounded by the degree of the projective Gauss curve times π.

Let $M_{A,\mathbf{c}}$ denote the matroid of rank $d + 1$ on the ground set $[n] = \{1, \dots, n\}$ associated with the matrix $\binom{A}{\mathbf{c}}$. We write $(h_0, h_1, ..., h_d)$ for the h-vector of the broken circuit complex of $M_{A,\mathbf{c}}$,

Theorem 5. *The degree of the projective Gauss curve of the primal central curve $\mathbb{C}$ satisfies*

$$\deg(\gamma(\mathbb{C})) \ \leq \ 2 \cdot \sum_{i=1}^{d} i \cdot h_i. \tag{6}$$

In particular, we have the following upper bound which is tight for generic matrices A:

$$\deg(\gamma(\mathbb{C})) \quad \leq \quad 2 \cdot (n - d - 1) \cdot \binom{n-1}{d-1}. \tag{7}$$

Finally strong linear programming duality [8] says that the optimal points of the pair of linear programs (1) and (2) are precisely the feasible points satisfying $\mathbf{b}^T\mathbf{y} - \mathbf{c}^T\mathbf{x} = 0$. One usually thinks of the analytic center of the polytope $P = \{A\mathbf{x} = \mathbf{b}, \mathbf{x} \ \geq \ 0\}$ as the unique point of P maximizing the concave function $\sum_{i=1}^{n} \log(x_i)$. These are just points of the curve:

Theorem 6. *The primal central curve in $\mathbf{x}$-space $\mathbb{R}^n$ passes through all vertices of the arrangement $\mathcal{H}$. In between these vertices, it passes through the analytic centers of the bounded regions in $\mathcal{H}$. Similarly, the dual central curve in $\mathbf{s}$-space passes through all vertices and analytic centers of $\mathcal{H}^*$. Vertices of $\mathcal{H}$ in the primal curve correspond to vertices of $\mathcal{H}^*$ in the dual curve. The analytic centers of bounded regions of $\mathcal{H}$ correspond to points on the dual curve in $\mathbf{s}$-space at the hyperplane $\{s_0 = 0\}$, and the analytic centers of bounded regions of $\mathcal{H}^*$ correspond to points on the primal curve in $\mathbf{x}$-space at the hyperplane $\{x_0 = 0\}$.*

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f-vectors and three-manifold complexity

Ed Swartz

The linear relations among the face numbers, or f-vector (f_0, f_1, f_2, f_3), of a simplicial triangulation of a closed 3-manifold are simple. The Euler characteristic, $f_0 - f_1 + f_2 - f_3$ is zero and, since every two-dimensional face is in exactly two tetrahedra, $f_2 = 2f_3$. These two equations show that f_0 and f_1 determine f_2 and f_3. Progress understanding inequalities really began with Walkup's 1970 paper [9]. To state Walkup's main results, let Δ be a simplicial complex whose geometric realization $|\Delta|$ is homeomorphic to M, a closed 3-manifold. Define $\gamma(\Delta) = f_1 - 4f_0 + 10$.

Theorem 1. *(Walkup,* [9]*)* $\gamma(\Delta) \geq 0$. *If* $\gamma(\Delta) = 0$, *then* Δ *is a stacked sphere.*

If view of this result, it is reasonable to define

$$\Gamma(M) = \min_{|\Delta|=M} \gamma(\Delta).$$

Theorem 2. *(Walkup,* [9]*)*

$\Gamma(S^3) = 0, \Gamma(S^2 \times S^1) = \Gamma(S^2 \propto S^1) = 10, \Gamma(\mathbb{R}P^3) = 17.$ *Otherwise* $\Gamma(M) > 17.$

Here, $S^2 \propto S^1$ is the nonorientable S^2-bundle over S^1.

Walkup also proved that for any closed 3-manifold M there exists $N(M)$ such that for all $n \geq N(M)$ there exist triangulations with n vertices and $\binom{n}{2}$ edges.

Thus the real question associated to f-vectors of closed three-manifolds is a lower bound problem. How are $\Gamma(M)$ and the topology of M related? Here is the type of result we are looking for.

Theorem 3. *(Novik, S. [6]) Let β_1 be the first Betti number of M (any coefficient field). Then $\Gamma(M) \geq 10\beta_1$*

This result was originally conjectured by Kalai [1]. It remains an open question exactly which manifolds, if any, other than multiple connected sums of $S^2 \times S^1$ and $S^2 \propto S^1$ satisfy equality in the above theorem.

In 1990 Matveev introduced a notion of 3-manifold complexity that has since been the subject of considerable research [3, 4]. This notion of complexity, which we denote by $c(M)$, is defined for 3-manifolds with nonempty boundary. For a closed 3-manifold the complexity of M is defined to be $c(M - B^3)$. Matveev's notion of complexity for 3-manifolds M with boundary has many pleasant properties.

Theorem 4. *(Matveev [3]*

(1) $c(M) \in \mathbb{Z}^{\geq 0}$.
(2) $c(M) = c(M - B^3)$
(3) $c(M\#N) = c(M) + c(N)$ *and* $c(M \amalg N) = c(M) + c(N)$.
(4) For all n there exist only finitely many closed orientable irreducible 3-manifolds M with $c(M) \leq n$.
(5) If M can be triangulated with n tetrahedra, then $c(M) \leq n$.
(6) If M is closed, orientable, irreducible and $c(M) > 0$, then $c(M)$ is the minimum number of tetrahedra needed in any triangulation of M.

$M \amalg N$ stands for boundary connected sum - glue two sufficiently small disks on the boundary of M and N together. It is important to note that in the last two items nonsimplicial triangulations are allowed. These triangulations are those obtained by starting with disjoint tetrahedra and glueing triangles in pairs by affine isomorphisms. For instance, those covered by the last item usually have only one vertex. Both (4) and (6) were extended to closed nonorientable, irreducible and $\mathbb{R}P^2$-irreducible 3-manifolds by Martelli and Petronio [5].

The following connection between $\Gamma(M)$ and $c(M)$ was first observed in [2].

Theorem 5. *There exist constants $C_1, C_2 > 0$ such that*

(1) For all closed irreducible and $\mathbb{R}P^2$-irreducible 3-manifolds other than $S^2 \times S^1, \mathbb{R}P^3, L(3,1)$ and $S^2 \propto S^1$,

$$\Gamma(M) \leq C_1 \ c(M).$$

(2) For all closed irreducible 3-manifolds

$$c(M) \leq C_2 \ \Gamma(M).$$

Note that irreducible closed orientable 3-manifolds are always $\mathbb{R}P^2$-irreducible. With this in hand it is possible to translate many of the numerous results concerning $c(M)$ to $\Gamma(M)$. Two examples. For the first, let $\{a_1, \ldots, a_m | r_1, \ldots, r_k\}$ be a set of generators and relations for a finitely presented group G and define the length of the presentation to be $|r_1| + \cdots + |r_k|$. Now set $l(G)$ to be the minimum length of all presentations of G.

Theorem 6. *There exists $C > 0$ such that for all closed, orientable and irreducible 3-manifolds, $\Gamma(M) \geq C\ l(\pi_1(M))$.*

See [4, Prop. 2.6.6] for the $c(M)$ version.

Theorem 7. *There exists $C > 0$ with the following property. Let M be a closed, irreducible and $\mathbb{R}P^2$-irreducible 3-manifold with Heegaard decomposition $H_1 \cup H_2$. Suppose the meridans of the handle body H_1 intersect the meridians of the handle body H_2 transversally at m points. Further assume that the closure of one of the components that meridians decompose the surface $\partial H_1 = \partial H_2$ into contains n points. If M is not $\mathbb{R}P^3, L(3,1), S^2 \times S^1$ or $S^2 \propto S^1$, then*

$$\Gamma(M) \leq C(m - n).$$

See [4, Prop 2.1.8] for the $c(M)$ version.

Given the close connection between $c(M)$ and $\Gamma(M)$ for closed 3-manifolds one might hope that there is a simple combinatorial invariant that acts like $c(M)$ for manifolds with boundary. For Δ a simplicial complex whose geometric realization is a 3-manifold with boundary define $\gamma(\Delta) = f_1 - 3f_0 + 6 - \#\{$ interior vertices $\}$. It is not difficult to show that $\gamma(\Delta) = -4f_3 + 3f_2 - f_1 - 2f_0 + 6$. Kalai showed that $\gamma(\Delta) \geq 0$ and equals zero if and only if Δ is a stacked ball (stacked sphere with a vertex removed) [1]. Later, Novik and Swartz proved $\gamma(\Delta) \geq 4\ \tilde{\beta}_0(\partial\Delta) + 3\ \beta_1(\partial\Delta)$ ($\mathbb{Z}/2\mathbb{Z}-$ coefficients) [7]. As before we define $\Gamma(M) = \min_{|\Delta|=M} \gamma(\Delta)$.

Theorem 8. *(S. '11)*

1. *If M is closed, then $\Gamma(M - B^3)$ equals the previously defined $\Gamma(M)$. If $\partial M \neq \emptyset$, then $\Gamma(M - B^3) = \Gamma(M) + 4$.*
2. *For all $n \geq 0$ there are only finitely many $M, \partial M \neq \emptyset$ such that $\Gamma(M) \leq n$.*
3. *There exists $C > 0$ such that for all M irreducible, boundary irreducible, and $\partial M \neq \emptyset$,*

$$c(M) \leq C\ \Gamma(M)$$

For closed 3-manifolds (2) was originally shown in [8]. In view of (2) and the fact that there are infinitely many manifolds M with boundary such that $c(M) = 0$, there is no hope of getting a result corresponding to Theorem 5 (1). Except for those M covered by previously mentioned results and the above, $\Gamma(M)$ is only known for all possible combinations of connected sums and boundary connected sums of the solid torus and the solid Klein bottle.

However, the above result is enough for the following.

Theorem 9. *There exists $C > 0$ such that for all hyperbolic 3-manifolds with nonempty boundary $\Gamma(M) \geq C\ vol\ (M)$.*

See [4, Lemma 2.6.7] for the $c(M)$ version.

Question: Is $\Gamma(M)$ additive under boundary connected sum (and hence connected sum)? It is easy to show that $\Gamma(M \amalg N) \leq \Gamma(M) + \Gamma(N)$.

What about higher dimensions? For d-manifolds a natural candidate for study is $f_1 - df_0 + \binom{d+1}{2} - \#\{$ interior vertices $\}$. This invariant is known to be nonnegative [1], subadditive with respect to boundary connected sum (easy), and for all n there are only finitely many closed d-manifolds for which this invariant is less than or equal to n [8]. Does the last also hold for d-manifolds with boundary?

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Understanding the statement of the Kuratowski graph planarity criterion and 6/7 of the statement of the Robertson–Seymour–Thomas intrinsic linking criterion

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By a cell complex we mean a CW complex whose attaching maps are PL embeddings and whose cells are determined by their vertices. We call a cell complex B *dichotomial* if for each cell A of B there exists another cell of B whose vertices are precisely the vertices of B that are not in A.

Theorem 1. *There exist precisely two dichotomial 3-complexes, with 1-skeleta the Kuratowski graphs K_5 and $K_{3,3}$, and precisely six dichotomial 4-complexes, whose 1-skeleta are 6 out of the 7 graphs of the Petersen family.*

On the polygonal peg problems

Siniša Vrećica

The cyclohedron W_n arises both as the polyhedral realization of the poset of all cyclic bracketings of the word $x_1x_2\dots x_n$ and as an essential part of the Fulton-MacPherson compactification of the configuration space of n distinct, labelled points on the circle S^1. The "polygonal pegs problem" asks whether every simple, closed curve in the plane or in the higher dimensional space admits an inscribed polygon of a given shape. We develop a new approach to the polygonal pegs problem based on the obstruction theory and Fulton-MacPherson compactification of the configuration space of (cyclically) ordered n-element subsets in S^1. The results obtained by this method include proof of Grünbaum's conjecture about affine regular hexagons inscribed in smooth Jordan curves and a new proof of the conjecture of Hadwiger about inscribed parallelograms in smooth, simple, closed curves in the 3-space (originally established by Makeev).

Computing all maps into a sphere

Jiří Matoušek
(joint work with Martin Čadek, Marek Krčál, Francis Sergeraert, Lukáš Vokřínek, Uli Wagner)

Let X, Y be topological spaces, say given as finite simplicial complexes. We would like to understand the computational complexity of algorithmic problems such as computing the set $[X, Y]$ of all *homotopy classes* of continuous maps[1] $X \to Y$.

Our primary motivation is the computation of the $\mathbb{Z}_2$*-index* (or *genus*) of a $\mathbb{Z}_2$-space[2] X, i.e., the smallest d such that X can be equivariantly mapped into S^d. This is a fundamental quantity in Borsuk–Ulam type applications such as Lovász' topological lower bound for the chromatic number (see, e.g., [7]), and it also provides a characterization of embeddability of simplicial complexes into $\mathbb{R}^d$ in a certain range of dimensions; see [8]. Indeed, some authors (e.g., Kozlov [6]) consider a weaker, cohomologically defined index, because of the supposed intractability of computing the $\mathbb{Z}_2$-index.

At the current stage of our project, we deal only with the (technically simpler) non-equivariant setting. We have the following result.

Theorem 1. *Let $d \geq 2$. Assuming that $Y = S^d$ or, more generally, that Y is $(d-1)$-connected,*[3] *and that $\dim X \leq 2d-2$, the set $[X, Y]$ is computable, in the*

[1] We recall that two maps $f, g\colon X \to Y$ are *homotopic* if f can be continuously deformed into g, i.e., if there is a continuous $F\colon X \times [0,1] \to Y$ such that $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$.

[2] A $\mathbb{Z}_2$*-space* is a topological space X with an action of the group $\mathbb{Z}_2$; the action is described by a homeomorphism $\nu\colon X \to X$ with $\nu \circ \nu = \mathrm{id}_X$. A primary example is a sphere S^d with the antipodal action $x \mapsto -x$. An *equivariant map* between $\mathbb{Z}_2$-spaces is a continuous map that commutes with the $\mathbb{Z}_2$ actions.

[3] A *k-connected* space Y is one whose first k homotopy groups vanish; in other words, every map $S^i \to Y$ can be extended to D^{i+1}, $0 \leq i \leq k$.

following sense: It is known that, under the above conditions on X and Y, $[X, Y]$ can be naturally endowed with a structure of a (finitely generated) abelian group, in an essentially unique way.[4] *The algorithm computes the structure of this group (i.e., expresses it as a direct product of cyclic groups).*

The algorithm works with certain implicit representations of the elements of $[X, Y]$, it can output a set of generators of the group in this representation, and it contains a subroutine implementing the group operation. However, converting these implicit representations into actual maps $X \to Y$ (given, say, as simplicial maps from a sufficiently fine subdivision of X into Y) looks problematic, and even if worked out, it seems unlikely to yield any reasonable bounds on the complexity of the resulting explicit maps.

In a work still in progress, we hope to strengthen Theorem 1 as follows.

- For every *fixed* Y, the algorithm should run in time polynomial in the number of simplices of X (while the dependence on Y is at least exponential).
- After changes and extensions in several components of the machinery used in the algorithm, a similar approach should also yield an algorithm for deciding the existence of an equivariant map between two $\mathbb{Z}_2$-spaces X, Y, under the same assumptions as in Theorem 1.

Related work. For Y path-connected, the set $[S^1, Y]$ is nontrivial exactly if the fundamental group $\pi_1(Y)$ is nontrivial, and the latter is well known to be algorithmically undecidable (see, e.g., the survey Soare [14]). On the other hand, the higher homotopy groups $\pi_k(Y) \cong [S^k, Y]$, $k \geq 2$, are computable for Y simply connected (i.e., with $\pi_1(Y) = 0$); this was shown already in 1957 by Brown [2]. In the same paper, he also shows computability of $[X, Y]$ under the assumption that $\pi_1(Y) = 0$ and all $\pi_k(Y)$, $2 \leq k \leq \dim X$, are *finite*.

Methods. Classifying maps up to homotopy is a fundamental problem in algebraic topology, and in the 1950s and 1960s, topologists developed a wealth of deep concepts and methods to address this question. However, powerful as these methods are, they are not algorithmic in general. Tools such as spectral sequences have yielded many amazing results in particular cases, as well as general theorems, but they are generally not suitable, without further work, for mechanical computations. (See, e.g., for a discussion of this issue, with concrete examples, in [10, Section 1].)

Conceptually, the basis of our algorithms is classical *obstruction theory* [4], which proceeds by constructing maps $X \to Y$ inductively on the i-dimensional *skeleta* of X, extending them one dimension at a time. In a nutshell, at each stage, the extendability of a map from the i-skeleton to the $(i+1)$-skeleton is characterized by vanishing of a certain *obstruction*, which can, more or less by known techniques, be evaluated algorithmically.

[4]In particular, the groups $[X, S^d]$ are known as the *cohomotopy groups* of X; see [5].

Textbooks expositions may make the impression that obstruction theory is a general algorithmic tool for testing the extendability of maps (this is actually what some of the topologists we consulted seemed to assume). However, the extension at each step is generally not unique, and extendability at higher stages may depend, in a nontrivial way, on the choices made earlier. Thus, in principle, one needs to search an infinitely branching tree of extensions. (Brown's result mentioned earlier, on computing $[X, Y]$ with the $\pi_k(Y)$'s finite, is based on a complete search of this tree, where the branching is guaranteed to be finite in this case.)

Here we use the group structure of the sets $[X, Y]$, as well as of some related ones, for a finite encoding of the set of all possible extensions at a given level. Technically, we work with maps into the *Postnikov tower* of Y, and our main data objects are *simplicial sets* (more precisely, finite fragments of infinite simplicial sets). To find the maps extendable to the next stage, the algorithm solves a system of linear Diophantine equations, which essentially amounts to computing the *Smith normal form*. The space Y enters the algorithm in the form of a black box for evaluating k_i, the *ith Postnikov factor* of Y, $d \leq i \leq 2d-2$; these evaluations constitute a "nonlinear" part of the computations.

For Y fixed, these black boxes can be hard-wired once and for all. In some particular cases, they are given by known explicit formulas (e.g., for $Y = S^d$, k_1 corresponds to the *Steenrod square*, and k_2 to *Adém's operation*). However, in the general case, the only way of evaluating the k_i we are aware of is using *objects with effective homology* of Rubio, Sergeraert, Dousson, and Romero; see, e.g., [12, 10, 9]. This is one of several projects of making classical methods of algebraic topology constructive (see [11, 13] for others), and for our purposes, some extensions of the existing machinery seem to be required, which we plan to address in a separate paper.

Concluding remarks. *Algorithmic* or *computational topology* has been a blooming discipline in recent years (see, e.g., [3, 15]). However, our work addresses issues different from those investigated in the current mainstream of this field. We rely on somewhat more advanced concepts and methods from topology which, as we believe, offer an exciting field for complexity-theoretic study. In a full version of this work, we will aim at accessibility to a general computer science audience with only a moderate topological background, in order to help bridging the current "cultural gap" between computer science and topology.

As for concrete open questions, it would be very interesting to find some *hardness results* for computing $[X, Y]$ or the $\mathbb{Z}_2$-index. The only result we are aware of is the classical uncomputability of $\pi_1(X)$.[5] We suspect that once the dimension vs. connectivity assumptions in Theorem 1 are weakened, the problem of deciding, say, the nontriviality of $[X, Y]$ may become intractable.

[5]There is a also result of Anick [1] on #P-hardness of computing the higher homotopy groups. However, the way he presents it, it is not immediately relevant for spaces given as simplicial complexes, since his reduction uses a very compact representation of the input space—roughly speaking, he needs to encode degrees of attaching maps as binary integers.

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The cyclotomic polynomial topologically

Victor Reiner

(joint work with Gregg Musiker; see `arXiv:1012.1844`)

We interpret, via simplicial homology, the integer coefficients in the cyclotomic polynomial $\Phi_n(x)$. Recall that if ζ denotes a primitve complex n^{th} root of unity, then

$$\Phi_n(x) = \prod_{j \in (\mathbb{Z}/n\mathbb{Z})^\times} (x - \zeta^j) = \sum_{j=0}^{\phi(n)} c_j x^j$$

is its monic irreducible polynomial in $\mathbb{Q}[x]$, with degree given by $\phi(n)$, the Euler phi-function. The coefficients c_j are easily seen to lie in $\mathbb{Z}$, and although well-studied, interpretations and explicit formulas for them remain elusive. An easy and well-known reduction shows that one need only consider the case where n is a squarefree product of primes. Explicit formulas for all the c_j are known only in the case where n has two prime factors [4, 5], and in this case they take on the values in $\{0, \pm 1\}$. In general, the c_j can grow arbitrarily large.

Our interpretation asserts that c_j computes the homology of a certain simplicial complex $K_{\{j\}}$ defined below. Interestingly, they are special cases of simplicial complexes that originally arose in the work of E. Bolker [2], reappeared in the work of G. Kalai [3] and R. Adin [1] on higher-dimensional matrix-tree theorems, and were shown to be connected with cyclotomic extensions in work [6] of J. Martin and the speaker.

Given a positive integer p, let K_p denote a 0-dimensional abstract simplicial complex having p vertices, which we will label by the residues

$$\{0 \bmod p,\ 1 \bmod p,\ \ldots,\ (p-1) \bmod p\}.$$

Given distinct primes $p_1, \ldots, p_d$, let $K_{p_1,\ldots,p_d} := K_{p_1} * \cdots * K_{p_d}$ denote the *simplicial join*, $K_{p_1}, \ldots, K_{p_d}$, pure $(d-1)$-dimensional abstract simplicial complex, that may be thought of as the *complete d-partite complex* on vertex sets K_{p_1} through K_{p_d} of sizes $p_1, \ldots, p_d$. The *facets* (maximal simplices) of $K_{p_1,\ldots,p_d}$ are labelled by sequences of residues $(j_1 \bmod p_1, \ldots, j_d \bmod p_d)$. Denoting the squarefree product $p_1 \cdots p_d$ by n, the Chinese Remainder Theorem isomorphism

$$\mathbb{Z}/p_1\mathbb{Z} \times \cdots \times \mathbb{Z}/p_d\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \tag{1}$$

allows one to label such a facet by a residue $j \bmod n$; call this facet $F_{j \bmod n}$. Then for any subset $A \subseteq \{0, 1, \ldots, \phi(n)\}$, let K_A denote the subcomplex of $K_{p_1,\ldots,p_d}$ which is generated by the facets $\{F_{j \bmod n}\}$ as j runs through the following set of residues:

$$A \cup \{\phi(n)+1,\ \phi(n)+2,\ \ldots,\ n-2,\ n-1\}.$$

It turns out that every subcomplex K_A contains the full $(d-2)$-skeleton of $K_{p_1,\ldots,p_d}$.

Theorem. *For a squarefree positive integer $n = p_1 \cdots p_d$, with cyclotomic polynomial $\Phi_n(x) = \sum_{j=0}^{\phi(n)} c_j x^j$, one has*

$$\tilde{H}_i(K_{\{j\}}; \mathbb{Z}) = \begin{cases} \mathbb{Z}/c_j\mathbb{Z} & \text{if } i = d-2, \\ \mathbb{Z} & \text{if both } i = d-1 \text{ and } c_j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this interprets c_j only up to sign. To interpret the signs, one can proceed as follows. Using oriented simplicial homology, orient the facet $F_{j \bmod n}$ having $j \equiv j \bmod p_i$ for $i = 1, 2, \ldots, d$ as

$$[F_j] = [F_{j \bmod n}] = [j_1 \bmod p_1,\ \ldots,\ j_d \bmod p_d]. \tag{2}$$

Let $[z_{j \bmod n}] := \partial[F_{j \bmod n}]$ denote the homology class within the $(d-2)$-homology of the subcomplex $K_\emptyset$ of the $(d-2)$-cycle which is the image of $[F_{j \bmod n}]$ under the simplicial boundary map ∂.

Theorem. *Let $n = p_1 \cdots p_d$ be squarefree. Then the simplicial complex $K_\emptyset$ has the same integral homology as a $(d-2)$-sphere, and the boundary cycle $[z_{\phi(n) \bmod n}]$ generates its $(d-2)$-homology group $\mathbb{Z}$. Furthermore, the coefficient c_j of x^j in*

$\Phi_n(x)$ gives the "attaching degree" for the boundary cycle $[z_{j \bmod n}]$ of F_j with respect to this choice of generator, that is,

$$[z_{j \bmod n}] = c_j [z_{\phi(n) \bmod n}]$$

within $\tilde{H}_{d-2}(K_\emptyset) \cong \mathbb{Z}$.

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