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The Renormalization Group

Organised by
Margherita Disertori, Rouen
Joel Feldman, Vancouver
Manfred Salmhofer, Heidelberg

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Abstract. Invented as a multiscale approach to the theory of critical phenomena, the renormalization group has become a powerful mathematical tool in the analysis of infinite-dimensional systems. Its applications range from classical and quantum statistical mechanics and quantum field theory to partial differential equations, operator theory, and probability theory. Deep connections of renormalization group flows to geometric flows exist.

Mathematics Subject Classification (2000): 81T15, 81T16, 81T17, 82B05, 82C10.

Introduction by the Organisers

The workshop focused on the developments in this area since the last Oberwolfach workshop on this topic in 2006. The participants are from mathematics and theoretical physics institutes throughout Europe and North America.

Mathematical proofs by RG are often technically elaborate and demanding, but the workshop did not merely focus on technical issues. Besides presentations that provided a survey of important recent developments and some that exposed technical novelties, there were several talks that did not directly concern the RG but closely related fields, as well as potential new areas of application.

The topics covered in the 17 one-hour talks are many-body systems of quantum statistical mechanics relevant for materials science, in particular graphene, the Kosterlitz-Thouless transition in the two-dimensional Coulomb gas, nonlinear elasticity, aspects of quantum field theory (operator product expansions, renormalization in the Euclidian and on globally hyperbolic Lorentzian space-times), RG in stochastic population models, group quantum field theory, and the application of the RG to long-time limits of dynamics and the emergence of irreversible behaviour. Two talks were devoted to geometric flows and their relation to RG flows.

The programme adhered to the Oberwolfach traditions, leaving plenty of room for discussions and joint work. The Oberwolfach atmosphere and the excellent service at the centre were most appreciated.

Workshop: The Renormalization Group

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Abstracts

A massless quantum field theory over the p-adics

Abdelmalek Abdesselam
(joint work with Ajay Chandra, Gianluca Guadagni)

This talk is in three parts. In Part 1, we briefly outine a general program for the rigorous study of scalar quantum field theory (QFT), in the continuum. We use a probabilistic framework in the spirit of Dobrushin [3]. In Part 2, we explain that everything in Part 1 makes perfect sense when spacetime $\mathbb{R}^d$ is replaced by $\mathbb{Q}_p^d$. Finally, in Part 3, we report on ongoing progress made, in collaboration with A. Chandra and G. Guadagni (research funded by U.Va. and the NSF under grant DMS#0907198), on the p-adic analog of the massless model studied by Brydges, Mitter and Scoppola.

1. Outline of a program for Euclidean QFT in the continuum

The goal is to develop a mathematical theory which is a rigorous version of the methods one finds in physics QFT textbooks (e.g., Ch. 8 and Ch. 10 of [7]). We put the emphasis on Symanzik-Nelson positivity rather than reflection positivity. The study of QFT thus becomes that of probability measures $\mathrm{d}\mu$ on the space of distributions $S'(\mathbb{R}^d)$ with the cylinder σ-algebra. We focus on measures which have finite moments and are invariant by translation (stationary processes) and by the orthogonal group $O(d)$. One can also require self-similarity. This program involves the following steps.

Step 0: It is to classify the self-similar Gaussian case. Let $\mathbf{S}_n(f_1,\ldots,f_n) = \langle\phi(f_1)\cdots\phi(f_n)\rangle$ denote the moments of the measure $\mathrm{d}\mu$ under consideration. $\mathbf{S}_n$ can also be thought of as an element of $S'(\mathbb{R}^{nd})$. In the (centered) Gaussian case, $\mathbf{S}_2$ contains all the information. By translation invariance, $\mathbf{S}_2(x,y) = S_2(x-y)$ where $S_2 \in S'(\mathbb{R}^d)$. The classification reduces to that of $O(d)$-invariant distributions S_2 homogeneous of degree $-2[\phi]$ where $[\phi]$ is the scaling dimension of the field. For any $[\phi] \in \mathbb{R}$, there is a 1-dimensional space of solutions. Adding the positive-type condition entails $[\phi] \geq 0$. We will restrict the discussion to the range $0 < [\phi] < \frac{d}{2}$. One then has a simple expression in both direct and Fourier space for the two-point function $\mathbf{S}_2(x,y) \sim |x-y|^{-2[\phi]}$, $\widehat{S}_2(k) \sim |k|^{2[\phi]-d}$. Note that this is only part of a bigger picture. One can handle zero-modes, e.g., by restricting $S(\mathbb{R}^d)$ using moment vanishing conditions [3]. See [5] for $d=2$, $[\phi]=0$ which is pertinent for conformal QFT. For work related to $d=1$, $[\phi]<0$, see the talk by J. Unterberger.

Step 1: Putting cut-offs. One replaces, e.g, the covariance $C = S_2$ by $C_r(x) \sim \int_{L^r}^{\infty} \frac{\mathrm{d}\rho}{\rho}\rho^{-2[\phi]}u(\frac{x}{\rho})$ for some nice function u. Here L is the renormalization group (RG) magnification ($L > 1$ is an integer). One also introduces a box Λ_s of side length L^s.

Step 2: Perturb the cut-off Gaussian $\mathrm{d}\mu_{C_r}(\phi)$ to get a new probability measure $\mathrm{d}\nu_{r,s}(\phi) = \frac{1}{Z}\exp(-\widetilde{V}_{r,s}(\phi))\mathrm{d}\mu_{C_r}(\phi)$ where $\widetilde{V}_{r,s}(\phi) = \int_{\Lambda_s} \mathrm{d}^d x \ \{ \ \widetilde{g}_r : \phi^4 :_{C_r} (x) + \widetilde{\mu}_r \ : \phi^2 :_{C_r} (x) + \cdots \}$. Given a bare ansatz, i.e., the germ at $-\infty$ of a sequence $(\widetilde{g}_r, \widetilde{\mu}_r, \ldots)_{r\in\mathbb{Z}}$, the key problem is to study the double limit $\mathrm{d}\nu_{r,s} \to \mathrm{d}\nu$ when $r \to -\infty$ and $s \to \infty$. One can either use the Bochner-Minlos Theorem or a Hamburger moment reconstruction theorem with $n!$ growth for $\mathbf{S}_n$ in order to recover the wanted QFT $\mathrm{d}\nu$. To be interesting, $\mathrm{d}\nu$ should not be Gaussian. In fact, we would like a stronger notion of nontriviality which begs the question: is there a good notion of Borchers class in this probabilistic setting? Also of interest is the massless situation. Note that, for $\mu > 0$, $(|k|^{d-2[\phi]} + \mu)^{-1}$ has large distance decay $|x|^{2[\phi]-2d}$ if $d - 2[\phi] \notin 2\mathbb{N}$. Thus, the appropriate definition of 'massless' for general $[\phi]$ is the requirement of non L^1 rather than power law decay for S_2.
Step 3: Composite fields and operator product expansion (OPE). Borrowing our notation from the talk by S. Hollands, we would like to define local field operators $\mathcal{O}_A[\phi](x)$, e.g., renormalized versions of $\phi(x)^n$. After smearing by $f \in S(\mathbb{R}^d)$, one would like $\phi \to \mathcal{O}_A[\phi](f)$ to be a function $S'(\mathbb{R}^d) \to \mathbb{C}$. This typically fails if $[\phi] > 0$. One should instead define $\phi \to \mathcal{O}_A[\phi](f)$ as a generalized function (or rather functional) in the spirit of Hida's white noise calculus [4]. One needs a space $\mathcal{D}(S'(\mathbb{R}^d))$ of test functionals F on $S'(\mathbb{R}^d)$ which should at least contain monomials of the form $\phi(f_1)\cdots\phi(f_n)$. Then $\mathcal{O}_A$ should be constructed as a linear map from $S(\mathbb{R}^d)$ to the dual space of generalized functionals $\mathcal{D}'(S'(\mathbb{R}^d))$. The duality pairing is that given by the QFT/measure $\mathrm{d}\nu$. Note that the correlations $\langle \mathcal{O}_A[\phi](f) \ F(\phi)\rangle = \langle \mathcal{O}_A[\phi](f) \ \phi(f_1)\cdots\phi(f_n)\rangle$ make sense, in the free case, even at coinciding points. Namely, this defines a distribution on all of $\mathbb{R}^{(n+1)d}$. The functional F corresponds to the spectator fields for the OPE. In the case of a single operator insertion, one can then follow the procedure explained in the talk by S. Hollands, in order to study the singularities on the diagonals and inductively define the operator products $\mathcal{O}_A$ from the corresponding short distance asymptotics. For the OPE with several operator insertions, one needs to define the mixed correlations at noncoinciding points, then repeat the procedure.
Step 4: Instead of perturbing, in Step 2, around a solution of Step 0, one can also consider similar perturbations of nontrivial RG fixed points along relevant directions.

2. The same over $\mathbb{Q}_p$

The message here is that everything in Part 1 works perfectly if one considers random fields $\phi : \mathbb{Q}_p^d \to \mathbb{R}$. Besides, the RG is much simpler and cleaner than in the real case. Indeed, it reduces to the hierarchical RG. For p a prime number, the field $\mathbb{Q}_p$ is defined as the completion of the field $\mathbb{Q}$ with respect to the p-adic norm/absolute value $|p^n \frac{a}{b}|_p = p^{-n}$, for $n, a, b \in \mathbb{Z}$ such that $b \neq 0$ and p does not divide ab. A p-adic number $x \in \mathbb{Q}_p$ has a unique convergent representation $\sum_{j\in\mathbb{Z}} a_j p^j$, with only finitely many negative powers of p, where the 'digits' a_j are in $\{0, 1, \ldots, p-1\}$. The polar part $\{x\}_p = \sum_{j<0} a_j p^j$ is a rational number. The valuation is given by $\mathrm{val}_p(x) = \min\{j, a_j \neq 0\}$. The extention of the previous norm is $|x|_p = p^{-\mathrm{val}_p(x)}$. The unit ball $\mathbb{Z}_p = \{x \in \mathbb{Q}_p, |x|_p \le 1\}$ is a compact additive subgroup. In dimension d, the norm of a point $x = (x_1, \ldots, x_d)$ in $\mathbb{Q}_p^d$ is defined by $|x| = \max |x_i|_p$. We take $L = p^l$ for the RG zooming ratio. The lattice of mesh L^r is given by $\mathbb{Q}_p^d/(L^{-r}\mathbb{Z}_p)^d$. The big volume is $\Lambda_s = (L^{-s}\mathbb{Z}_p)^d = \{x, |x| \le L^s\}$. For the space of test functions we take the Schwartz-Bruhat space $S(\mathbb{Q}_p^d)$ of locally constant functions $f : \mathbb{Q}_p^d \to \mathbb{R}$ of compact support, with the finest locally convex topology. The Fourier transform is $\widehat{f}(k) = \int f(x) \mathrm{e}^{-2i\pi\{x\cdot k\}_p} \mathrm{d}^d x$ where $x \cdot k = \sum x_i k_i$ and the additive Haar measure $\mathrm{d}^d x$ gives measure 1 to $\mathbb{Z}_p^d$. The analog of $O(d)$ is the maximal compact subgroup $GL_d(\mathbb{Z}_p)$ of $GL_d(\mathbb{Q}_p)$, defined by fixing the norm $|x|$. For Step 1, the cut-off covariance C_r is obtained from $C(x) \sim \sum_{j\in\mathbb{Z}} p^{-2j[\phi]} \mathbb{1}_{\mathbb{Z}_p^d}(p^j x)$ by imposing $j \ge rl$. The RG map corresponds to integrating over fluctuations with covariance $C_0 - C_1$. With these modifications, Part 1 works in the p-adic setting too. For other work on p-adic QFT see [6] and references therein.

3. The p-adic BMS model

The BMS model corresponds to $d = 3$ and $[\phi] = \frac{3-\epsilon}{4}$ for some small positive bifurcation parameter ϵ. The bare ansatz only contains ϕ^4 and ϕ^2 couplings $\widetilde{g}_r$, $\widetilde{\mu}_r$. We rescale to unit lattice $\widetilde{V}_{r,s} \to V_r^{(0)}$ and produce new bulk potentials $V_r^{(0)} \to V_r^{(1)} \to \cdots$ by iterating the RG map (we suppressed s in the notation). Constructing a QFT morally amounts to establishing the transverse convergence criterion (TCC): $\forall q \in \mathbb{Z}, \lim_{r\to-\infty} V_r^{(q-r)}$ exists (the effective theory at log-scale q). This produces an ideal RG trajectory $(P_q)_{q\in\mathbb{Z}}$. One conjectures that TCC $\Rightarrow \lim \mathrm{d}\nu_r = \mathrm{d}\nu$ exists. Together with A. Chandra and G. Guadagni we adapted the proofs in [2, 1] to the p-adic case and rigorously constructed (in suitable Banach spaces) the nontrivial infrared (IR) fixed point, together with its stable and unstable manifolds. We constructed ideal trajectories as well as established the TCC, starting from a bare ansatz, for two massless theories: one which should be self-similar, at the IR fixed point, and another one which joins the Gaussian and the IR fixed points. Modulo the previous conjecture, we completed all previous steps except Step 3. We are also making rapid progress towards proving this conjecture. This hinges on extending our RG tools to nonuniform local perturbations of the massless Gaussian. This should also help for Step 3.

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Strict convexity of the surface tension for non-convex interaction energies via renormalisation group methods

Stefan Adams
(joint work with R. Kotecký, Stefan Müller)

We consider an effective model with gradient interaction. The model describes a phase separation in $\mathbb{R}^{d+1}$, e.g. between the liquid and vapor phase. For simplicity we consider a discrete basis $\Lambda \subset \mathbb{Z}^d$, and real-valued height variables

$$x \in \Lambda \mapsto \varphi(x) \in \mathbb{R}.$$

This model ignores overhangs like in Ising models, but gives a good approximation in the vicinity of the phase separation. The distribution of the interface is given in terms of a Gibbs distribution with nearest neighbour interactions of gradient type, that is, the interaction between neighboring sites $x, x+\mathrm{e}_i$ depends only on the gradient $\nabla_i\varphi(x) = \varphi(x+\mathrm{e}_i) - \varphi(x), i = 1, \ldots, d$. More precisely, the Hamiltonian is of the form

$$H_\Lambda(\varphi) = \sum_{x\in\Lambda}\sum_{i=1}^{d} W(\nabla_i\varphi(x)),$$

where $W\colon \mathbb{R} \to \mathbb{R}$ is a perturbation of a quadratic functions, i.e.

$$W(\eta) = \frac{1}{2}\eta^2 + V(\eta) \qquad \text{with some perturbation } V\colon \mathbb{R} \to \mathbb{R}.$$

The Gibbs distribution for a given boundary condition $\Psi \in \mathbb{R}^{\partial\Lambda}$, where $\partial\Lambda = \{z \in \mathbb{Z}^d\colon |z-x| = 1 \text{ for some } x \in \Lambda\}$, at inverse temperature $\beta > 0$ is given by

$$\gamma_\Lambda^{\beta,\Psi}(\mathrm{d}\varphi) = \frac{1}{Z_\Lambda(\beta,\Psi)} \exp\big(-\beta H_\Lambda(\varphi)\big) \prod_{x\in\Lambda} \mathrm{d}\varphi(x) \prod_{x\in\partial\Lambda} \delta_{\Psi(x)}(\mathrm{d}\varphi(x)),$$

where the normalisation constant $Z_\Lambda(\beta,\Psi)$ is the integral of the density and is called the partition function. One is particularly interested in tilted boundary conditions

$$\Psi_u(x) = \langle x, u\rangle, \qquad \text{for some tilt } u \in \mathbb{R}^d.$$

An object of basic relevance in this context is the surface energy or free energy defined by the limit

$$\sigma(u) = \lim_{\Lambda\uparrow\mathbb{Z}^d} -\frac{1}{\beta|\Lambda|}\log Z_\Lambda(\beta,\Psi). \tag{1}$$

This surface tension $\sigma(u)$ can also be seen as the price to pay to tilt a totally flat interface. The existence of the above limit follows from a standard sub-additivity argument. In case of *strictly* convex potential, Funaki and Spohn showed in [3] that

σ is convex as a function of the tilt. The simplest strictly convex potential is the quadratic one with $V = 0$, which corresponds to a Gaussian model, also called the gradient free field or harmonic crystal. Strict convexity of the surface tension for strictly convex W with $0 < c_1 \leq W'' \leq c_2 < \infty$, was proved in [4]. In [2] Deuschel *et al* showed the strict convexity of the surface tension for non-convex potentials in the small β (high temperature) regime. The following questions for non-convex interaction potentials W are possible formulations of the so-called Cauchy-Born rule:

Questions:

(a) Is the surface tension $\sigma(u)$ strictly convex for large β and small tilt u?
(b) Is there uniqueness for gradient Gibbs measures for a given tilt?
(c) Is the scaling limit still the Gaussian free field?

We show (a), i.e., the strict convexity of the surface tension for large enough β (low temperatures) and sufficiently small tilt using multi-scale techniques.

Following an idea in [3] we work on a torus $T_N := (\mathbb{Z}/L^N\mathbb{Z}^d)^d$ for some $L > 0$. Hence we consider periodic height functions, and to kill the constant for the discrete gradient mapping we add a condition, i.e. we consider the space of configurations

$$\Omega_N = \Big\{\varphi\colon \mathbb{Z}^d \to \mathbb{R}; \varphi(x+k) = \varphi(x)\,\forall k \in (L^N\mathbb{Z})^d; \sum_{x\in T_N} \varphi(x) = 0\Big\}.$$

We use λ_N to denote the $(L^N - 1)$-dimensional Hausdorff measure on Ω_N. The partition function for given tilt $u \in \mathbb{R}$ is then

$$Z_N(\beta, u) = \int_{\Omega_N} \exp\big(-\beta H_N^u(\varphi)\big)\lambda_N(\mathrm{d}\varphi), \tag{2}$$

with Hamiltonian

$$H_N^u(\varphi) = \frac{1}{2}L^{Nd}|u|^2 + \sum_{x\in T_N}\sum_{i=1}^{d} \frac{1}{2}(\nabla_i\varphi(x))^2 + V(\nabla_i\varphi(x)). \tag{3}$$

To state our main result, we need a condition on the smallness of the perturbation V. We will state it in terms of the Mayer function $K_u\colon \mathbb{R}^d \to \mathbb{R}$ associated with the function $V\colon \mathbb{R} \to \mathbb{R}$ determining the Hamiltonian H_N^u. Namely, we take $K_u(z) = \exp\big\{-\beta\sum_{i=1}^d V\big(\frac{z_i}{\sqrt{\beta}} - u_i\big)\big\} - 1$. Given any $h > 0$, we then consider the Banach space E of functions $K\colon \mathbb{R}^d \to \mathbb{R}$ with the norm

$$\|K\|_h = \sup_{z\in\mathbb{R}^d} \sum_{|\alpha|\leq 14} h^{|\alpha|}\big|\partial_z^\alpha K(z)\big|\mathrm{e}^{-h^{-2}|z|^2}.$$

Here, the sum is over nonnegative integer multiindices $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \mathbb{N}, i = 1, \ldots, d$ with $|\alpha| = \sum_{i=1}^d \alpha_i \leq 14$, and $\partial^\alpha = \prod_{i=1}^d \partial_i^{\alpha_i}$.
Our main result is then:

Theorem. There exist $\delta_0 > 0$, $\epsilon_0 > 0$, $M_0 > 0$, and $h_0 > 0$ such that if the map $\mathbb{R}^d \supset B_\delta(0) \ni u \mapsto K_u \in E$ is C^3, $\|K_u\|_h \le \epsilon$, and

$$\sum_{i=1}^d \Big\| \frac{\partial}{\partial u_i} K_u \Big\|_h + \sum_{i,j=1}^d \Big\| \frac{\partial^2}{\partial u_i \partial u_j} K_u \Big\|_h + \sum_{i,j,k=1}^d \Big\| \frac{\partial^3}{\partial u_i \partial u_j \partial u_j} K_u \Big\|_h \le M$$

with $h \ge h_0$, $\epsilon \le \epsilon_0$, $\delta \le \delta_0$, $M < M_0$ and for any $u \in B_\delta(0)$, then the surface tension $\sigma(u)$ exists and it is strictly convex uniformly in $u \in B_\delta(0)$ and in N sufficiently large.

The proof employes a multi-scale analysis based on ideas outlined by Brydges in [5]. The main ingredient is a finite range decomposition of families of Gaussian measures in [1]. The Hamiltonian in (3) shows that the partition function is a Gaussian integral with measure μ having density given by the square of the gradients and normalisation Z_N which is independent on the tilt and β (after some rescaling), that is

$$(4)\quad Z_N(\beta, u) = Z_N \mathrm{e}^{-\frac{1}{2} L^{dN}|u|^2} \int_{\Omega_N} \exp\big(-\beta \sum_{x \in T_N} \sum_{i=1}^d V(\frac{1}{\sqrt{\beta}} \nabla_i \varphi(x) - u_i)\big\} \mu(\mathrm{d}\varphi).$$

The finite range decomposition in [1] of the Gaussian measure μ means that $\mu(\mathrm{d}\varphi) = \mu_1 * \cdots * \mu_{N+1}(\mathrm{d}\varphi)$ where $\mu_1, \ldots, \mu_{N+1}$ are Gaussian measures with a particular finite range property. Namely, the covariances of the measures $\mu_k, k = 1, \ldots, N+1$, vanish for $|x| \ge \frac{1}{2} L^k$. Hence it is possible to perform the Gaussian integration in (4) in steps according to length scales given by $L^k, k = 1, \ldots, N+1$. This defines roughly the renormalisation group mapping, eg. on scale L^k we get for an integrand F

$$(R_k F)(\varphi) = \int_{\Omega_N} F(\varphi + \xi) \mu_k(\mathrm{d}\xi).$$

These mappings generate a dynamical system with an expanding and a contracting direction and trivial fix point. The crucial step is to get the correct initial Gaussian such that after N steps the mapping is close to the fix point. This enables one to control the second derivative with respect to the tilt of the integral in (4). To get the correct initial Gaussian measure we employ a change of Gaussian measure at the beginning with some quadratic form defined by a symmetric matrix $q \in \mathbb{R}^{d \times d}$. An application of the implicit function theorem will ensure the existence of a unique matrix q_0 for which the systems reaches his fix point. However, the renormalisation mappings depend on these matrices and we have loss of regularity once we compute derivatives of Gaussian expectations with respect to this parameter. This can be resolved with some version of the implicit function theorem which allows a loss of regularity.

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Renormalisation Group Leftovers

David Brydges

Let Λ be a subset of the lattice $\mathbb{Z}^d$, let A be the $\Lambda \times \Lambda$ matrix that represents the finite difference Laplacian when acting on a function $\phi : \Lambda \to \mathbb{R}$ and let V be bounded below. Integrals of the form

$$\frac{1}{Z}\int_{\mathbb{R}^\Lambda} e^{-1/2\sum_{x,y\in\Lambda}\phi_x A_{x,y}\phi_y} e^{-g\sum_{x\in\Lambda}V(\phi_x,\nabla\phi_x)}\phi_a\phi_b d^\Lambda\phi, \tag{1}$$

are important in a wide range of problems. Z normalises the integral so that it would equal one in the absence of $\phi_a\phi_b$. In this conference we have seen expressions of the form (1) in the lectures by Pierluigi Falco and Stefan Adams and in [1] an extension involving differential forms is used to represent self-avoiding walk on $\mathbb{Z}^d$. The rigorous renormalisation group (RG) is a useful technique in all these cases. In sufficiently high dimensions and when V is small it can achieve the remarkable feat of evaluating the numerator and denominator in (1) so accurately that their $\exp\left[O(|\Lambda|)\right]$ growths are exactly cancelled and the dependence on $|a-b|$ is obtained uniformly in Λ. Following [2] I attempt a pedagogical review of some of the principles of proof by RG. In particular I introduce a useful class of norms [3] that play a critical role in these proofs and show how to use them to control the approximation of error terms in the RG by expressions of the same form as in the exponent of (1).

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Diffusion in Hamiltonian systems

Wojciech De Roeck
(joint work with A. Kupiainen)

The rigorous derivation of long-time diffusion from first principles of mechanics, be it quantum or classical, remains an inspiring challenge in mathematical physics. To our best knowledge, there are up to this date very few results of this type. Recently, in [4], a model was introduced which is quite tractable and for which diffusion was proven in dimension $d \geq 4$. It is a quantum system described by a Hamiltonian of the type

$$H = H_{\mathrm{S}} + H_{\mathrm{E}} + \lambda H_{\mathrm{SE}}, \qquad \lambda > 0 \tag{1}$$

where H_{S} (S for 'system') is the Hamiltonian of a free particle moving on the lattice and it consists of two parts $H_{\mathrm{S}} = H_{kin} + H_{spin}$ describing the translational degrees of freedom and a spin degree of freedom, respectively. The Hamiltonian H_{E} (E for 'environment') describes a free field of phonons or photons, and H_{SE} effectuates the coupling between both. The system is started with the phonons in a thermal state at inverse temperature β. Such models are a paradigm of open quantum systems. Let us list the properties that allow us to handle this model:

- The mass of the particle, or, since we are on a lattice, rather the inverse hopping strength, is chosen large. In (1) this is accomplished by choosing H_{kin} small. This allows a better control of a diagrammatic expansion in real space, since the particle needs a long time to explore a large volume on the lattice.
- Even though the mass is large, the 'mixing rate' that the particle degrees of freedom (except for the position) experience due to the interaction with the phonons is not small. This is possible because of the inclusion of a spin-degree of freedom
- By choosing the interaction Hamiltonian sufficiently smooth (in the momentum of the phonons), we ensure that the free space-time correlation functions of the phonon field decay at an integrable rate in time. We need them to decay at least as $O(t^{-(1+\alpha)})$ for large times t with $\alpha > 1/4$. To engineer this, it suffices to choose the dispersion relation of the bosons to be quadratic in the momentum for small momenta, and to cut off the interaction in small scales.
- By choosing the coupling constant λ small, we have a well controlled Markovian approximation (Lindblad equation) that describes the particle for times of $O(\lambda{-}2)$. This Markovian approximation serves as a first approximation to the true behaviour and we set up an expansion to control the deviations from it.

Under these assumptions we prove that the reduced dynamics of the particle is diffusive: Let X be the position operator of the particle and $\rho_{\mathrm{S},t}$ the density matrix of the particle at time t, obtained by taking partial trace over the environment.

Then we prove, for λ sufficiently small

$$\lim_{t\to\infty} \mathrm{Tr}[\rho_{\mathrm{S},t} e^{ik\frac{X}{\sqrt{t}}}] = e^{-Dk^2/2}, \qquad k \in R^d$$

for some diffusion constant $D = D(\lambda) > 0$. Our proof is based on a renormalization group (RG) method that was developed in [2, 3, 1] to prove diffusion for random walk in a random environment (RWRE). In the present context the random environment is provided by the phonon field. Unlike in the case of RWRE in the case at hand the particle influences the environment and the reduced dynamics is non-Markovian. However, the Markovian approximation mentioned above provides a starting point for the analysis where a Markovian dynamics is perturbed by a small non-Markovian noise. In units of the weak coupling time scale $O(\lambda^{-2})$ our model can then be viewed as a (quantum) random walk in a (quantum) random environment. The RG method consists of an iterative scheme to show that on successive larger temporal and spatial scales the random environment becomes smaller and smaller and the dynamics tends to a renormalized Markovian "fixed point". We show that the renormalized noise vanishes in this limit by showing that its (quantum) correlation functions tend to zero. Here we use a formalism developed earlier by us [5] for the confined case, i.e. the proof that the state of a confined quantum system interacting with a similar field as here tends to the equilibrium state.

The difference of the model considered in this paper and the one treated in [4] is that in the latter case an additional condition was imposed on the free boson correlation function that restricts the model to dimensions $d \geq 4$ and to a rather special class of analytic particle-phonon interaction terms. In the context of these models where the particle mass is chosen to scale as $O(\lambda^{-2})$ it still remains a challenge to treat more generic phonon or photon reservoirs where the temporal correlations decay as $O(t^{-1})$. To deal with these cases with our method one needs a more careful RG analysis. A much more difficult and interesting problem is to relax the large mass assumption. In this case the control of the corrections to the Markovian approximation seems still beyond current techniques.

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Kosterlitz-Thouless Transition Line for the 2D Coulomb Gas

Pierluigi Falco

The two-dimensional Coulomb Gas is the statistical system of point particles on a planar lattice, carrying a charge ± 1 and interacting through the two-dimensional electrostatic potential that, for large distances, is

$$V(x-y) \sim -\frac{1}{2\pi} \ln |x-y| \; . \tag{1}$$

This toy model acquired a great theoretical importance when Kosterlitz and Thouless, [21], found in it the solution of a puzzling dichotomy in the theory of the two dimensional *XY model*: the *spin-wave* approximation predicts absence of order (in agreement with Mermin-Wagner argument) and power law fall-off of the spin correlations; on the other hand, high temperature expansion clearly demonstrates exponential decay of the correlations. The two scenarios were merged together by the observation that the spin-wave picture doesn't take into account the spin configurations with *vortex excitations*, which interact through the same logarithmic potential of the Coulomb Gas.

Using RG ideas, Kosterlitz and Thouless, [21], [20], were able to describe the diagram of phases of the Coulomb Gas (and hence of the XY model) in the activity (z) vs. inverse temperature (β) space. At high temperatures, the gas is in the *plasma* - or *Debye screening* - phase: the correlation length is expected to be finite, and certain *screening sum rules* are conjectured. At low temperatures, on the contrary, the effective range of the interactions remains long, and the correlations length is infinite; this regime is the *dipole* - or *KT* - phase. In between the plasma and the dipole phases there is (at least) one transition curve, $\beta_c(z)$, called *KT transition line*, that was found to intersect the $z = 0$ axis at $\beta_c(0) = 8\pi$. Besides, approaching $\beta_c(z)$ from higher temperatures, the correlation length diverges as $\xi \sim e^{c(z)|\beta-\beta_c(z)|^{-\frac{1}{2}}}$: the *$KT$ phase transition* is then quite different from the more common *second order phase transition* of spin systems, in which case the correlation length diverges as $\xi \sim |\beta - \beta_c|^{-\nu}$ (or $\log |\beta - \beta_c|$).

Efforts of many other authors were addressed to the topic, in search of stronger evidence of the pioneering analysis of Kosterlitz and Thouless; the reader interested in theoretical physics works can find useful discussions and a good selection of references in [1], [18] and [11] - see also conjectures in [17].

The first rigorous result was the proof of Fröhlich and Park, [13], of the existence of the thermodynamic limit for pressure and correlations. Later on, Fröhlich and Spencer, [14], [15], proved, for β large enough, an upper and lower power-law bound for the decay of correlations of fractional charges; then, refinements of the same technique allowed Marchetti, Klain and Peres, [24], [22] [23], to cover increasing regions in dipole phase that eventually included the point $(\beta, z) = (8\pi, 0)$ - but not the rest of the KT transition line. Despite its fast improvement, Fröhlich-Spencer method seems to have some unavoidable limitations: it cannot provide the exact power of the correlation fall-off, nor can exclude logarithmic corrections (which actually are expected along the KT line); and it works for correlations of *fractional* charges, but does not provide any useful bound for correlations of *integer* charges. For this reasons, different authors started developing a Renormalization Group (RG) approach - at the beginning under some approximations: hierarchical metric, or order by order in perturbation theory, see [2], [25], [9], [26], [19], [3], [16]. Later, Dimock and Hurd, [10], provided a full-fledged RG construction of the pressure in a region of the dipole phase that included $(\beta, z) = (8\pi, 0)$ but not the rest of the KT transition line; they could not discuss correlations. Finally, the only rigorous result on the plasma phase is the one of Yang, [27], that extended to dimension two the proof of the dynamical mass generation for small β given in [5] and [6] for the higher dimensional case.

The objective of my work, [12], is to study the Coulomb Gas *along* the KT transition line, for small activity. Using the general RG approach of [8], [4], the covariances with compact supports of [7], and some estimates for the Coulomb gas in [10], I provide a constructive proof of the existence of the pressure. I do not discuss in this paper the critical exponents of the correlation functions, which are certainly the most exciting features of the model; but, in view of the bounds established this paper, that task should not be difficult and will be possibly addressed soon.

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Algebraic approach to gauge theories and gravity

Klaus Fredenhagen
(joint work with Katarzyna Rejzner)

A perturbative construction of quantum gravity around a given background might be performed in the sense of quantum field theory on curved spacetimes, as developed by Brunetti, Fredenhagen, Verch [4] and Hollands, Wald [10, 11] 10 years ago. There are, however, several obstructions. First, one has to take into account the gauge freedom as in gauge theories; the latter problem was solved recently by Hollands [9], based on the Batalin-Vilkovisky (BV) formalism [2, 3], and previous work of Dütsch et al. [6, 7].

In the locally covariant quantum field theory framework we associate with a physical system the configuration space $\mathfrak{E}(M)$ of all fields of the theory. In case of general relativity this is $\mathfrak{E}(M) = (T^*M)^{\otimes 2}$. In terms of category theory $\mathfrak{E}$ is a contravariant functor from the category of globally hyperbolic spacetimes **Loc** with causal isometric embedding as morphisms to the cateogry **Vec** of locally convex vector spaces. In contrast, the space of compactly supported fields $\mathfrak{E}_c(M)$ is assigned by a covariant functor from **Loc** to **Vec**, and $\mathfrak{D} : \mathbf{Loc} \to \mathbf{Vec}$ is a covariant functor that associates with M the space of test functions $\mathfrak{D}(M)$. Finally we define a functor $\mathfrak{F} : \mathbf{Loc} \to \mathbf{Vec}$ that assigns to M the space of smooth, compactly supported functionals on $\mathfrak{E}(M)$. We can also require some further regularity conditions on the functionals, like fulfilling the microlocal spectrum condition by each functional derivative at a given point (microcausality). The details on the functional-analytic and topological aspects of the structure are given in [8].

The dynamics of the theory is introduced by the so called *generalized Lagrangian* and the action $S(L)$ is a certain equivalence class of Lagrangians, c.f. [5]. Let g be the background metric, $h \in \mathfrak{E}(M)$ the infinitesimal perturbation and $\tilde{g} = g + h$. The Einstein-Hilbert generalized Lagrangian reads: $L_{(M,g)}(f)(h) \doteq \int R[\tilde{g}]\, f \,\mathrm{d}\,\mathrm{vol}_{(M,\tilde{g})}$.

The Euler-Lagrange derivative of S is a natural transformation $S' : \mathfrak{E} \to \mathfrak{D}'$ defined by: $\langle S'_M(\varphi), h\rangle = \left\langle L_M(f)^{(1)}(\varphi), h\right\rangle$ with $f \equiv 1$ on supp$h$. The field equation is: $S'_M(\varphi) = 0$ and we denote the space of functionals on the solution space by $\mathfrak{F}_S(M)$. On $\mathbb{E}(M)$, seen as a differentiable manifold, we can define the vector fields. We restrict ourselves to smooth microcausal vector fields X with compact support and with the image in $\mathfrak{E}_c(M)$. The space of such vector fields is denoted by $\mathfrak{V}(M)$. A vector field $X \in \mathfrak{V}(M)$ is called a symmetry of the action S if $\forall \varphi \in \mathfrak{E}(M)$ we have $0 = \langle S'_M(\varphi), X(\varphi)\rangle =: \delta_S(X)(\varphi)$. In concrete examples the set of symmetries can be characterized in a more explicite way. A symmetry X is called trivial if it vanishes on-shell. In general relativity all nontrivial symmetries can be written as elements of $\mathfrak{G}(M) := \mathcal{C}^\infty(\mathfrak{E}(M), \mathfrak{X}_c(M))$, where $\mathfrak{X}_c(M)$ is the space of compactly supported vector fields on M. This is the Lie algebra of the diffeomorphism group Diff(M).

The action ρ of $\mathfrak{G}(M)$ on $F \in \mathfrak{F}(M)$ is induced by the Lie derivative. The full BV algebra for a fixed background reads:

$$\mathfrak{BV}(M) = \bigwedge \mathfrak{V}(M) \widehat{\otimes} \bigwedge \mathfrak{X}'(M) \widehat{\otimes} S^{\bullet} \mathfrak{X}_c(M)\,,$$

where the elements of $\mathfrak{V}(M)$ are the antifields (with ghost number -1), $\mathfrak{X}'(M)$ is the space of ghosts (forms on $\mathfrak{X}$ with ghost number 1) and $S^{\bullet}\mathfrak{X}_c(M)$ of the antighosts (ghost number -2). All these objects have a geometrical interpretation in the framework presented in [8]. On the graded algebra $\mathfrak{BV}(M)$ one introduces a graded BV differential $s = s^{(-1)} + s^{(0)}$, where $s^{(-1)}$ is the Koszul-Tate differential providing the resolution of $\mathfrak{F}_S(M)$:

$$\ldots \to \Lambda^2 \mathfrak{V} \oplus \mathfrak{G} \xrightarrow{s^{(-1)}=\delta_S \oplus \rho} \mathfrak{V} \xrightarrow{s^{(-1)}=\delta_S} \mathfrak{F} \to 0$$

The other term $s^{(0)}$ is the Chevalley-Eilenberg differential, which describes the invariants under the action of $\mathfrak{X}(M)$. We obtain a double complex which encodes the algebra of gauge-invariant on-shell functionals as the 0-th cohomology: $H^0(\mathfrak{BV}(M), s) = \mathfrak{F}_S^{\text{inv}}(M)$. A direct application of this method to gravity however fails because of the absence of local observables in gravity. This renders the above defined cohomology trivial. One can instead lift the structure to the level of natural transformations. We define the extended algebra of fields (understood as natural transformations, c.f. [4]) as: $Fld = \bigoplus_{k=0}^{\infty} \text{Nat}(\mathfrak{E}_c^k, \mathfrak{BV})$. The action of symmetries on natural transformations $\Phi \in \text{Nat}(\mathfrak{E}_c, \mathfrak{F})$ is given by:

$$(\rho_M(X)\Phi_M)(f) := \partial_{\rho_M(X)}(\Phi_M(f)) + \Phi_M(\rho_M(X)f), \qquad X \in \mathfrak{X}(M)\,.$$

The set Fld becomes a graded algebra if we equip it with a graded product defined as:

$$(\Phi\Psi)_M(f_1, ..., f_{p+q}) = \frac{1}{p!q!} \sum_{\pi \in P_{p+q}} \Phi_M(f_{\pi(1)}, ..., f_{\pi(p)}) \Psi_M(f_{\pi(p+1)}, ..., f_{\pi(p+q)})\,.$$

The BV-differential on Fld is now:

$$(s\Phi)_M(f) := s_0(\Phi_M(f)) + (-1)^{|\Phi|} \Phi_M(\rho_M(.)f)\,,$$

where s_0 is the BV differential on the fixed background. The 0-cohomology of s is nontrivial, since it contains for example the Riemann tensor contracted with itself, smeared with a test function: $\Phi_{(M,g)}(f)(h) = \int_M R_{\mu\nu\alpha\beta}[\tilde{g}] R^{\mu\nu\alpha\beta}[\tilde{g}] f d\text{vol}_{(M,\tilde{g})}$.

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Universal conductivity in graphene

Alessandro Giuliani
(joint work with Vieri Mastropietro and Marcello Porta)

In condensed matter, there are very few phenomena, among which is the quantum Hall effect (QHE), displaying a very strong sort of universality; e.g., quantum Hall plateaus appear to be independent of the electron-electron interactions, thanks to an underlying topological invariance. Unfortunately, in the theory of the QHE, there is no first-principle derivation of this fact in any interacting many body system. In this talk I report the first rigorous proof of a similar universality phenomenon concerning the optical conductivity of graphene.

Indeed, recent optical measurements in graphene show that at half-filling and small temperatures, if the frequency is in a range between the temperature and the band-width, the conductivity is essentially constant and equal, up to a few percent, to $\sigma_0 = \frac{e^2}{h}\frac{\pi}{2}$, a universal value that does not depend on the material parameters, like the Fermi velocity. Such value coincides with the theoretical prediction in the free Fermi gas on the honeycomb lattice at half-filling [4]. Of course, interaction effects could produce modifications to this theoretical value: however, in the case of weak short range interactions and at half-filling, we *rigorously establish* that this is not the case: all the interaction corrections to the zero temperature and zero frequency conductivity cancel out exactly, as a consequence of exact lattice Ward Identities and of suitable regularity properties of the current-current response function.

Let me introduce the model and state our main results. We introduce creation and annihilation fermionic operators $\psi^{\pm}_{\vec{x},\sigma} = (a^{\pm}_{\vec{x},\sigma}, b^{\pm}_{\vec{x}+\vec{\delta}_1,\sigma}) = L^{-2}\sum_{\vec{k}}\psi^{\pm}_{\vec{k},\sigma}e^{\pm i\vec{k}\vec{x}}$ for electrons with spin index $\sigma =\uparrow\downarrow$ sitting at the sites of the two triangular sublattices Λ_A and Λ_B of a periodic honeycomb lattice of side L; we assume that $\Lambda_A = \Lambda$ has basis vectors $\vec{l}_{1,2} = \frac{1}{2}(3, \pm\sqrt{3})$ and that $\Lambda_B = \Lambda_A + \vec{\delta}_j$, with $\vec{\delta}_1 = (1, 0)$ and $\vec{\delta}_{2,3} = \frac{1}{2}(-1, \pm\sqrt{3})$ the nearest neighbor vectors; the sum over $\vec{k}$ runs over the first Brillouin zone associated to Λ. The Hamiltonian at half-filling is

$$\begin{aligned} H_\Lambda &= -t \sum_{\substack{\vec{x}\in\Lambda_A \\ j=1,2,3}} \sum_{\sigma=\uparrow\downarrow} (a^{+}_{\vec{x},\sigma} b^{-}_{\vec{x}+\vec{\delta}_j,\sigma} + b^{+}_{\vec{x}+\vec{\delta}_j,\sigma} a^{-}_{\vec{x},\sigma}) + \\ &\quad + U \sum_{\vec{x}\in\Lambda_A} \prod_{\sigma=\uparrow\downarrow} (a^{+}_{\vec{x},\sigma} a^{-}_{\vec{x},\sigma} - \frac{1}{2}) + \sum_{\vec{x}\in\Lambda_B} \prod_{\sigma=\uparrow\downarrow} (b^{+}_{\vec{x},\sigma} b^{-}_{\vec{x},\sigma} - \frac{1}{2}) . \end{aligned}$$

In the presence of an external vector potential $\vec{A}$, the *lattice current* is [4] $\vec{J}^{(A)}_{\vec{p}} = \vec{J}_{\vec{p}} + \int \frac{d\vec{q}}{(2\pi)^2} \hat{\Delta}_{\vec{p},\vec{q}} \vec{A}_{\vec{q}} + O(A^2)$, where, if $\eta^j_{\vec{p}} = \frac{1-e^{-i\vec{p}\vec{\delta}_j}}{i\vec{p}\vec{\delta}_j}$,

$$\vec{J}_{\vec{p}} = iet \sum_{\substack{\vec{x}\in\Lambda\\ \sigma,j}} e^{-i\vec{p}\vec{x}} \vec{\delta}_j \eta^j_{\vec{p}} \big(a^+_{\vec{x},\sigma} b^-_{\vec{x}+\vec{\delta}_j,\sigma} - b^+_{\vec{x}+\vec{\delta}_j,\sigma} a^-_{\vec{x},\sigma} \big) \ ,$$

is the *paramagnetic current* and

$$\big[\, \hat{\Delta}_{\vec{p},\vec{q}} \big]_{lm} = -e^2 t \sum_{\substack{\vec{x}\in\Lambda\\ j=1,2,3}} \sum_{\sigma} e^{-i(\vec{p}+\vec{q})\vec{x}} (\vec{\delta}_j)_l (\vec{\delta}_j)_m \eta^j_{\vec{p}} \eta^j_{\vec{q}} (a^+_{\vec{x},\sigma} b^-_{\vec{x}+\vec{\delta}_j,\sigma} + b^+_{\vec{x}+\vec{\delta}_j,\sigma} a^-_{\vec{x},\sigma}) \ ,$$

is the *diamagnetic tensor*. The current-current response function at Matsubara frequency $p_0 \in \frac{2\pi}{\beta}(\mathbb{Z} + \frac{1}{2})$ is defined as

$$\hat{K}^{\beta,L}_{ij}(p_0, \vec{p}) := \frac{1}{L^2} \int_0^\beta dx_0\, e^{-ip_0 x_0} \langle e^{x_0 H_\Lambda} J_{\vec{p},i} e^{-x_0 H_\Lambda} J_{-\vec{p},j} \rangle_{\beta,L}$$

where $\langle \cdot \rangle_{\beta,L} = \mathrm{Tr}\{e^{-\beta H_\Lambda} \cdot\}/\mathrm{Tr}\{e^{-\beta H_\Lambda}\}$. The *conductivity* is defined via Kubo formula as [4]:

$$\sigma^{\beta,L}_{ij}(p_0) = -\frac{2}{3\sqrt{3}} \frac{1}{p_0} \Big[\hat{K}^{\beta,L}_{ij}(p_0, \vec{0}) + \frac{1}{L^2} \langle [\hat{\Delta}_{\vec{0},\vec{0}}]_{ij} \rangle_{\beta,L} \Big] \ .$$

where $3\sqrt{3}/2$ is the area of the hexagonal cell of the honeycomb lattice.

Our main result is that the Hubbard interaction, while it analytically modifies several physical quantities, such as the Fermi velocity v_F and the quasi-particle weight Z^{-1} [1], leaves the zero frequency limit of the ground state conductivity exactly invariant. This is proved by a combination of Renormalization Group arguments, allowing us to prove the apriori existence and analyticity of the correlation functions, and lattice Ward Identities, allowing us to identify cancellations among the interaction corrections to the conductivity [2].

Theorem ([2]) *There exists a constant $U_0 > 0$ such that, for $|U| \le U_0$ and any fixed p_0, $\sigma^{\beta,L}_{ij}(p_0)$ is analytic in U uniformly in β, L as $\beta, L \to \infty$. Moreover,*

$$\sigma_{ij} = \lim_{p_0\to 0^+} \lim_{\beta\to\infty} \lim_{L\to\infty} \sigma^{\beta,L}_{ij}(p_0) = \frac{e^2}{h} \frac{\pi}{2} \delta_{ij} \ . \tag{1}$$

where we restored the dimensional constant $\hbar = h/(2\pi)$ in the final result.

Note that the definition of σ_{ij} involves a limiting procedure in which first the temperature and then the frequency are sent to zero; i.e., close to the limit we have $\beta^{-1} \ll p_0 \ll t$, which corresponds to the range of frequencies investigated with optical techniques in [3]. By taking the limits in the opposite order, we would get informations about the d.c. conductivity that, in the presence of disorder, also appears to have a universal value.

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All–order uniform momentum bounds for the massless ϕ^4 theory in four dimensional Euclidean space

Riccardo Guida
(joint work with Christoph Kopper)

A panoramic overview is given, of a theorem [1] establishing physical and uniform bounds on the Fourier–transformed Schwinger functions of a massless ϕ^4 theory in four Euclidean dimensions, at any loop order in perturbation theory.

The first step to set up the perturbative framework is to specify a free quantum theory describing a massless scalar field by fixing a centered Gaussian measure on $\mathcal{S}'(\mathbb{R}^4)$, $\mu_{\hbar C_R^{\Lambda,\Lambda_0}}$, whose covariance $\hbar C_R^{\Lambda,\Lambda_0}(x,y) := \hbar\chi_R(x)\,\chi_R(y)\,C^{\Lambda,\Lambda_0}(x-y)$ is assumed to be a distribution in $\mathcal{S}'(\mathbb{R}^8)$ acting as a positive bilinear form on test functions. $\hbar > 0$ denotes the variable of the formal perturbative series. The short–distance behavior (smoothness) of $C^{\Lambda,\Lambda_0}(x)$ as a function is controlled by $\Lambda_0 > 0$ (known as ultra–violet, UV, cutoff), while the long–distance regularity is controlled by $0 < \Lambda \le \Lambda_0$ (infra–red, IR, cutoff). C^{Λ_0,Λ_0} vanishes. When Λ_0 tends to infinity and Λ tends to zero, $C^{\Lambda,\Lambda_0}(x)$ approaches the standard free propagator $\langle x \,|\partial^{-2}|\, 0\rangle$. For any $R > 0$, the non–negative function $\chi_R \in \mathcal{C}_c^\infty\left(\mathbb{R}^4\right)$ satisfies the "finite–volume" constraint $\chi_R(x) = 1$ for any $|x| \le R$.

For any $\mathtt{N} \in \mathbb{N}$, and any $\mathtt{L} \in \mathbb{N}_0$ the Schwinger functions in momentum space are defined by

(1)
$$\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N},\mathtt{L}}(p_{[\mathtt{N}-1]}) := \lim_{R\to\infty}\Bigg[\left(\frac{1}{\mathtt{L}!}\frac{\partial^{\mathtt{L}}}{\partial\hbar^{\mathtt{L}}}\right)_{\hbar=0}\left(\frac{\delta}{\delta\varphi(0)}\prod_{e=1}^{\mathtt{N}-1}\int \mathrm{d}^4x_e\; e^{-\mathrm{i}x_e p_e}\frac{\delta}{\delta\varphi(x_e)}\right)_{\varphi=0} \left(-\hbar\log\left(\int \mathrm{d}\mu_{\hbar C_R^{\Lambda,\Lambda_0}}(\phi)e^{-\frac{1}{\hbar}S^{\mathrm{int}}(\phi+\varphi)}\Big/\int \mathrm{d}\mu_{\hbar C_R^{\Lambda,\Lambda_0}}(\phi)e^{-\frac{1}{\hbar}S^{\mathrm{int}}(\phi)}\right)\right)\Bigg],$$

where: $[a] := [1:b]$, $[a:b] := \{n\in\mathbb{Z}|a\le n\le b\}$, and $p_{[n]} := (p_1,\cdots,p_n)$. In (1), the interaction action $S^{\mathrm{int}}(\varphi)$ is defined by

$$S^{\mathrm{int}}(\varphi) := \int \mathrm{d}^4x\left(A(\hbar)\,\frac{(\partial\varphi(x))^2}{2} + B_2(\hbar)\frac{\varphi(x)^2}{2} + B_4(\hbar)\frac{\varphi(x)^4}{4!}\right), \qquad (2)$$

where A, B_2, B_4 are formal series in $\hbar$, whose coefficients are fixed order by order by appropriate renormalization conditions, in such a way that the "UV+IR limit" $\lim_{\Lambda_0\to\infty}\lim_{\Lambda\to 0^+}\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N},\mathtt{L}}$ exists in $\mathcal{S}'(\mathbb{R}^{4(\mathtt{N}-1)})$ for all $\mathtt{N},\mathtt{L}$. In particular, it turns out for a massless theory that A, B_2 are of order $O(\hbar)$, while $B_4 = g_0 + O(\hbar)$. From (1) and (2) it follows that $\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{2,0}$ and all $\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N},\mathtt{L}}$ with odd $\mathtt{N}$ vanish.

The UV+IR limit of $\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N},\mathtt{L}}$ is a regular function only at non–exceptional momenta, see e.g. [2]. (A collection of four vectors $p_{[\mathtt{N}-1]}$ is said *exceptional* iff it exists a non–empty $\mathbb{S} \subseteq [\mathtt{N}-1]$ such that $\sum_{e\in\mathbb{S}} p_e = 0$.)

Any Schwinger function $\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N},\mathtt{L}}$ defined in (1) can be computed from the standard weighted sum of all Feynman amplitudes proportional to $\hbar^{\mathtt{L}}$, obtained via Feynman rules from an appropriate set of connected amputated graphs with $\mathtt{N}$ external lines. Each such set includes all graphs with vertices of coordination number 4 and loop number $\mathtt{L}$. The word "amputated" means that Feynman rules do not associate any factor to the external lines.

Schwinger functions satisfy the "Polchinski" renormalization group (RG) flow equations, [3] (see [4] for an introduction), which in their perturbative form read:

$$(3)\quad \partial_\Lambda \hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N},\mathtt{L}}\left(p_{[\mathtt{N}-1]}\right) = \mathcal{F}^{\Lambda,\Lambda_0}_{\mathtt{N},\mathtt{L},w} := \Bigg(\frac{1}{2}\int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\,\partial_\Lambda \hat{C}^{\Lambda,\Lambda_0}(\ell)\;\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N}+2,\mathtt{L}-1}\left(p_{[\mathtt{N}-1]},-\ell,\ell\right) - \frac{1}{2}\sum_{\substack{\mathcal{E}'\uplus\mathcal{E}''=[0:\mathtt{N}-1]\\ \mathtt{L}'+\mathtt{L}''=\mathtt{L}}} \partial_\Lambda \hat{C}^{\Lambda,\Lambda_0}\left(\textstyle\sum_{e\in\mathcal{E}'} p_e\right)\;\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N}',\mathtt{L}'}\left(p_{\mathcal{E}'}\right)\;\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N}'',\mathtt{L}''}\left(p_{\mathcal{E}''}\right)\Bigg),$$

where $\mathtt{N}' := |\mathcal{E}'| + 1$, $\mathtt{N}'' := |\mathcal{E}''| + 1$, $p_0 := -\sum_{e\in[\mathtt{N}-1]} p_e$, and the sum on the r.h.s. of (3) runs over all disjoint (possibly empty) sets $\mathcal{E}', \mathcal{E}''$ whose union gives $[0:\mathtt{N}-1]$, as well as over all non–negative integers $\mathtt{L}', \mathtt{L}''$ whose sum gives $\mathtt{L}$.

When the field has a mass $m > 0$, it is not difficult to use the RG equations to bound Schwinger functions in momentum space (see e.g. [4]). Such bounds are simple but clearly unphysical because they depend polynomially on external momenta; moreover, they diverge when the mass vanishes and the IR limit is taken. More physical bounds have been proved in the massive case, [5].

The goal of the "existence and boundedness theorem" in [1] is to extend the ideas in [5] to obtain physical, uniform bounds for the massless case. The theorem assumes that the Fourier–transformed covariance $\hat{C}^{\Lambda,\Lambda_0}(p)$ is $O(4)$ invariant, smooth in some sense, and such that $\Lambda^3\partial_\Lambda \hat{C}^{\Lambda,\Lambda_0}(p)$ and $\Lambda_0^2\Lambda^2\partial_\Lambda\partial_{\Lambda_0}\hat{C}^{\Lambda,\Lambda_0}(p)$ (together will all necessary derivatives w.r.t. p) are exponentially decreasing when $|p|/\Lambda \to \infty$. The main result of the theorem is that for any $\mathtt{N}, \mathtt{L}$ and any multi–index $w \in \mathbb{N}_0^{4(\mathtt{N}-1)}$, there exist a polynomial $\mathcal{P}_\mathtt{L}$ of degree $\le \mathtt{L}$ and with non–negative coefficients, as well as a set of weighted trees $\mathcal{T}_{\mathtt{N},2\mathtt{L},w}$, such that (when e.g. $\mathtt{N} \ge 4$)

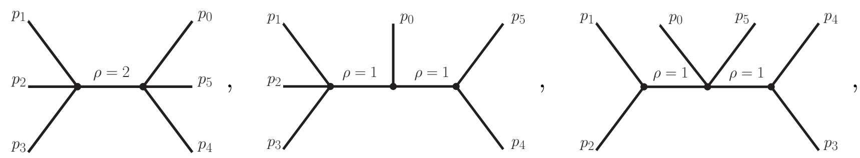
$$(4)\quad \left|\partial_p^w \hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N}\ge 4,\mathtt{L}}\left(p_{[\mathtt{N}-1]}\right)\right| \le \mathcal{P}_\mathtt{L}\left(\log_+\left(\frac{|p_{[\mathtt{N}-1]}|_\mu}{\kappa}\right), \log_+\frac{\Lambda}{\mu}\right) \sum_{T\in\mathcal{T}_{\mathtt{N},2\mathtt{L},w}} \prod_{i\in\mathcal{I}(T)} |k_i|_\Lambda^{-\theta(i)}$$

for any $\Lambda_0 > 0$, $0 < \Lambda \le \Lambda_0$ and $p_{[\mathtt{N}-1]} \in \mathbb{R}^{4(\mathtt{N}-1)}$. In (4), $\mu > 0$ is the renormalization scale; $|p_{[\mathtt{N}-1]}| := \sup_e |p_e|$; $|p|_\Lambda := \sup(\Lambda, |p|)$; $\log_+ x := \log\sup(1, x)$. $\kappa := \sup(\Lambda, \inf(\eta(p_{[\mathtt{N}-1]}), \mu)) > 0$ is defined in terms of a "dynamical IR cutoff" $\eta(p_{[\mathtt{N}-1]}) := \inf_{\emptyset\neq\mathbb{S}\subseteq[\mathtt{N}-1]} |\sum_{e\in\mathbb{S}} p_e|$ (positive for non–exceptional momenta). $\mathcal{I}(T)$

is the set of internal lines of the weighted tree T; k_i is the momentum flowing through the internal line i, and $\theta(i) > 0$ is the total weight associated to i.

The sets $\mathcal{T}_{\mathtt{N},\mathtt{R},w}$ ($\mathtt{R} \in \mathbb{N}_0$) satisfy two properties; *nestedness:* $\mathcal{T}_{\mathtt{N},\mathtt{R},w} \subseteq \mathcal{T}_{\mathtt{N},\mathtt{R}+1,w}$; *saturation:* $\mathcal{T}_{\mathtt{N},\mathtt{R},w} = \mathcal{T}_{\mathtt{N},3\mathtt{N}-2,w}$ for any $\mathtt{R} \geq 3\mathtt{N}-2$. The set $\mathcal{T}_{\mathtt{N},\mathtt{R},w=0}$ (corresponding to the absence of derivatives w.r.t. external momenta) is defined as the set of all $T = (\tau, \rho)$ in which τ is a tree and $\rho : \mathcal{I}(T) \to \{1, 2\}$ is a line weight, such that: *a)* τ has $\mathtt{N}$ external lines and vertices of coordination number in $\{3, 4\}$; *b)* the number of vertices with coordination 3 is $\leq \mathtt{R}$; *c)* $\sum_{i \in \mathcal{I}(T)} \rho(i) = \mathtt{N} - 4$; *d)* there is a bijection among the vertices of coordination number 3 and the internal lines with $\rho = 1$. In the case $w = 0$ one has $\theta(i) = \rho(i)$.

As an example, for any $\mathtt{L} > 0$ the set $\mathcal{T}_{\mathtt{N}=6,\mathtt{R}=2\mathtt{L},w=0}$ contains only the trees



and the trees derived from them by non–trivial permutations of the external momenta $p_{[0:5]}$. (Other trees with $\mathtt{N}$ external lines and vertices of coordination numbers $3, 4$ exist but do not satisfy to the defining conditions.) Correspondingly, in this case the bound (4) reads for any $\mathtt{L} > 0$

$$|\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{6,\mathtt{L}}(p_{[5]})| \leq \left(|p_1+p_2+p_3|_\Lambda^{-2}+|p_1+p_2+p_3|_\Lambda^{-1}|p_4+p_5|_\Lambda^{-1}+|p_1+p_2|_\Lambda^{-1}|p_3+p_4|_\Lambda^{-1}+\text{perms.}\right)\mathcal{P}_\mathtt{L},$$

where $\mathcal{P}_\mathtt{L}$ has been introduced in (4).

The proof of the theorem is based on the recursive structure of the perturbative RG equations (3) (see e.g. [4]). The main difficulty is to wisely deal with spurious exceptional momenta, in order to keep the bound finite in the IR limit.

In the flow $\mathcal{F}^{\Lambda,\Lambda_0}_{\mathtt{N},\mathtt{L},w}$, see (3), the term quadratic in Schwinger functions acts as a junction of the weighted trees T', T'' in the bounds, respectively, of $\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N}',\mathtt{L}'}$, $\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N}'',\mathtt{L}''}$. Now, the junction of two weighted trees happens to be a weighted tree of the appropriate class and the inductive bound for $\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N},\mathtt{L}}$ is then reproduced.

The linear term in $\mathcal{F}^{\Lambda,\Lambda_0}_{\mathtt{N},\mathtt{L},w}$ is more problematic, because it contains a loop integration which tends to destroy the tree structure of the bounds. The exponential fall–off in ℓ/Λ of the covariance allows to prove ([5],[1]) bounds of the form

$$\text{(5)} \qquad \int \mathrm{d}^4\ell \, |\partial_\Lambda \hat{C}^{\Lambda,\Lambda_0}(\ell)| \prod_{j=1}^{n} |\ell + k_j|_\Lambda^{-\theta_j} \leq c\,\Lambda \prod_{j=1}^{n} |k_j|_\Lambda^{-\theta_j},$$

which, roughly speaking, amount to "cut the loop" and to set $\ell = 0$ by deleting two external lines for each tree. This property makes the linear part of the flow more "tree friendly". The elimination of the unwanted Λ factor in (5) (using the bound $\Lambda \leq |k_{j'}|_\Lambda$ for some j'), and the integration over Λ (to recover Schwinger functions from the flow) are taken into account by eliminating the factors $|k_{j'}|_\Lambda^{-1}, |k_{j''}|_\Lambda^{-1}$ for each tree in the original bound of $\hat{\mathcal{L}}^{\Lambda,\Lambda_0}_{\mathtt{N}+2,\mathtt{L}-1}$, which amounts to consider a subtraction of two units in the original weights: this procedure can be consistently implemented as a mapping among our classes of weighted trees.

The logarithms in (4) originate from the Λ integration of the flow for marginal and irrelevant Schwinger functions, as well as from the integral interpolating marginal Schwinger functions from the renormalization point to a generic one.

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The operator product expansion converges in perturbative field theory

Stefan Hollands

All quantum field theories with well-behaved ultra violet behavior are believed to have an operator product expansion (OPE). This means that the product of any two local fields located at nearby points[1] x and y can be expanded in the form

$$\mathcal{O}_A(x)\mathcal{O}_B(y) \sim \sum_C \mathcal{C}_{AB}^C(x,y)\,\mathcal{C}(y), \tag{1}$$

where A, B, C are labels for the various local fields in the given theory (incorporating also their tensor character/spin), and where $\mathcal{C}_{AB}^C$ are certain numerical coefficient functions—or rather distributions[2]—that depend on the theory under consideration, the coupling constants, etc. The sign "$\sim$" indicates that this can be understood as an asymptotic expansion: If the sum on the right side is carried out to a sufficiently large but finite order, then the remainder goes to zero fast as $x \to y$ in the sense of operator insertions into a quantum state, or into a correlation function. In the talk, I reported on a forthcoming joint work with C. Kopper, which demonstrates in a specific model that the expansion is not only *asymptotic* in this sense, but even *converges*, to arbitrary orders in perturbative Euclidean quantum field theory.

Our result is not merely a technical footnote, but it furnishes an important insight into the general structure of relativistic/Euclidean quantum field theory. This is maybe best explained in the Minkowskian context. There, the analogue of our result would be that correlation functions such as the two-point function

[1] In the Minkowskian context (relativistic quantum field theory), the points should not be lightlike to each other.

[2] In a theory on Minkowski spacetime with translation invariance, these distributions only depend upon the difference $x - y$.

$\langle \mathcal{A}(x)\mathcal{B}(y)\rangle_\Psi$ in a state[3] ψ are entirely determined by the collection of OPE coefficients which are *state independent*, together with the 1-point functions $\langle \mathcal{C}(y)\rangle_\Psi$:

$$\langle \mathcal{A}(x)\mathcal{B}(y)\rangle_\Psi = \sum_C \mathcal{C}_{AB}^C(x-y)\ \langle \mathcal{C}(y)\rangle_\Psi\,, \tag{2}$$

where the infinite sum over "C" would be convergent, and $|x-y|$ would not necessarily have to be small. An analogous statement would apply to the higher n-point functions. Thus, the OPE coefficients capture the state-independent algebraic structure of QFT, while *all* the information about the quantum state is contained in the 1-point functions ("form factors"). It has been conjectured furthermore in [Zamolodchikov et al. 1994, Hollands & Wald 2008] that the OPE coefficients have a *convergent* expansion in the coupling constants near an UV fixed point, i.e. around a conformally invariant quantum field theory, provided that the basis of composite fields $\mathcal{A}$ is also chosen in a suitable manner. If this is the case, then all nonperturbative effects are encoded in this basis in the form factors.

In our recent work, we prove convergence of the OPE in the context of perturbative quantum field theory, to arbitrary but finite orders in perturbation theory. For simplicity, we work in a Euclidean formulation of the theory in flat 4-dimensional space, and we consider a scalar field with self-interaction $g\varphi^4$ and mass $m>0$. The composite fields $\mathcal{A}$ in this model are simply linear combinations of monomials in the basic field φ and its derivatives and are denoted by

$$\mathcal{A} = \partial^{w_1}\varphi\cdots\partial^{w_n}\varphi\,, \qquad A=\{n,w\}\,, \tag{3}$$

where each w_i is a 4-dimensional multi-index. We define the engineering dimension of such a field as usual by

$$[A] = n + \sum_i |w_i|\,. \tag{4}$$

Since the model is invariant under translations, the OPE-coefficient functions depend only on the difference variable and are consequently denoted as $\mathcal{C}_{AB}^C(x)\,,\ x\in \mathbb{R}^4\backslash\{0\}$. Each such coefficient is itself a formal power series in $\hbar$ ("loop expansion") or (equivalently) in the coupling constant g. As usual in perturbation theory, we are not concerned with the convergence of these expansions in g or $\hbar$. Instead, in our work, we are concerned with the convergence of the OPE (i.e. the expansion in "C") at fixed order in g or $\hbar$.

To analyze this issue, we must insert the left- and right sides into a Schwinger function containing suitable "spectator fields" which play the role of a quantum state in the Euclidean context. A simple and natural choice for the spectator fields is e.g.

$$\varphi(f_E) := \int_{p,x} \varphi(x)\hat{f}(p/E)\ \mathrm{e}^{ipx} \tag{5}$$

[3]The state should have a well-behaved high energy behavior. In the Minkowskian context, it should e.g. have bounded energy E, see below for an appropriate replacement in the Euclidean context.

where $\hat{f}(p)$ is a smoothed out version of the characteristic function of a unit ball in momentum space $\mathbb{R}^4$, or in fact any other smooth function of compact support in the unit ball. Thus, the spectator fields are, roughly speaking, given by the Fourier transformed field $\hat{\varphi}(p)$ integrated over a momentum space ball of radius E. Our main result is the following:

Theorem 1. *Let the sum* $\sum_C$ *in the operator product expansion* (1) *be over all* C *such that*

$$[C] - [A] - [B] \leq \delta \tag{6}$$

where δ *is some positive number. Then for each such* δ*, we have the following bound for the "remainder" in the OPE:*

$$\left| \Big\langle \mathcal{A}(x)\mathcal{B}(0)\, \varphi(f_E)\cdots\varphi(f_E) \Big\rangle - \sum_C \mathcal{C}^C_{AB}(x) \Big\langle \mathcal{C}(0)\, \varphi(f_E)\cdots\varphi(f_E) \Big\rangle \right| \tag{7}$$
$$\leq \sqrt{[A]![B]!}\ \tilde{K}^{[A]+[B]}\ (\sup|\hat{f}|)^n\ m^{[A]+[B]+n}\ \sup(1, \frac{E}{m})^{([A]+[B])(n/2+2l)+3n}$$
$$\times \sum_{\lambda=0}^{l} \frac{\log^\lambda \sup(1, \frac{E}{m})}{2^\lambda \lambda!} \frac{1}{\sqrt{\delta!}} \left(\tilde{K}\ m|x|\ \sup(1, \frac{E}{m})^{n/2+2l} \right)^\delta$$

Here, there are n *spectator fields,* $\langle\, .\, \rangle$ *denote Schwinger functions, and* $\tilde{K}$ *is a constant. The number* l *is the maximum number of loops, which is bounded by* $\frac{1}{2}([A] + [B] + 2r - n) + 1$ *when* r *is the order in* g *to within which all quantities are evaluated.*

This result establishes the convergence of the OPE, i.e. the sum over C, at each fixed order in perturbation theory, because the remainder evidently goes to zero as $\delta \to \infty$. There are no conditions on x, so the OPE converges even at arbitrarily large distances! But we note that such conditions would arise if we were to replace the spectator fields with sharp momentum cutoff E by ones that are averaged against a testfunction $\hat{f}(p)$ that is only decaying in momentum space, and not of compact support. This can be understood in a way by the fact that E gives a measure for the "typical energy" of the "state" in which we try to carry out the OPE. As the high energy behavior of the "state" becomes worse, so do the convergence properties of the OPE.

To prove the theorem, one first has to give a prescription for defining the Schwinger functions and OPE coefficients in renormalized perturbation theory. There are several options; in this paper we find it convenient to use the Wilson-Wegner-Polchinski flow equation method. In this method, one first introduces an infrared cutoff called Λ, and an ultraviolet cutoff called Λ_0. One then defines the quantities of interest for finite values of the cutoffs, and derives for them a flow equation as a function of Λ. This may be solved iteratively with appropriate boundary (= renormalization-) conditions, and one establishes certain bounds on the quantities of interest which are uniform in Λ_0. The last fact makes it possible to remove the

UV cutoff, and at the same time provides non-trivial bounds. In our case, we need bounds for the remainder in the OPE. Again, such bounds are verified iteratively.

While the general strategy is rather clear conceptually, it gets more involved in practice. This is because a relatively refined induction hypothesis is required to ensure that it replicates itself in the iteration process. The verification of the induction step is thus the main technical task of this paper.

A side result of our estimations which may be of some interest is that the "gradient expansion" of the effective action converges at each fixed number of loops; the precise statement may be found in our forthcoming paper.

Towards renormalisation of the hierarchically interacting heavy-tailed Λ-Cannings models

Anton Klimovsky

(joint work with Andreas Greven, Frank den Hollander, Sandra Kliem)

Initiated in a series of works by Dawson and Greven in 1993 (see, e.g., [4]), the renormalisation programme for analysis and identification of the *universal patterns* in population genetics models have been pursued since then by many authors, see [9, 10, 5], for reviews. So far, only the universal patterns for the *diffusive models* of population genetics have been analysed by the *renormalisation group* (RG) methods. In this work, we initiate the renormalisation analysis of *population genetics models with jumps*.

We introduce and study a class of dynamical stochastic models for genetics of spatially extended populations called *hierarchically interacting C^Λ-processes* (HIC^ΛP). The HIC^ΛP models the space-time evolution of the *allelic type distributions* in spatially subdivided populations. The *non-spatial version* of HIC^ΛP can be obtained as the continuous time-mass limit of the *Cannings model* ([2, 3]) in the heavy-tailed reproduction regime (dual to the Λ-coalescent, see, e.g., [1] for a review). In this model, a single individual is allowed to have the progeny of the size comparable with the size of the whole population.

The HIC^ΛP takes into account the effects of

- *migration* between geographically structured colonies of individuals;
- *local resampling*: haploid reproduction within colonies under the constrained amount of resources;
- *global resampling*: occasional correlated extinction-colonisation events that affect the whole patches of the geographical space.

An important feature of the HIC^ΛP is its *non-diffusive behaviour* characterised by the presence of *jumps*. The jumps reflect the substantial reproduction events and abrupt changes in the environment that lead to the large-scale extinction-colonisation effects.

In this work, the *geographical space* is assumed to have a hierarchical structure, cf. [11]. The individuals live in the colonies indexed by the *hierarchical group*:

$$\Omega_N = \Big\{ \eta = (\eta^{(k)})_{k\in\mathbb{N}} \in \{0, 1, \dots, N-1\}^{\mathbb{N}} \mid \sum_{k\in\mathbb{N}} \eta^{(k)} < \infty \Big\}, \quad N \in \mathbb{N}. \tag{1}$$

We endow the Ω_N with the following *ultrametric notion of distance*

$$d(\eta, \zeta) = \min\{k \in \mathbb{Z}_+ \mid \eta^{(l)} = \zeta^{(l)}, l > k\}, \quad \eta, \zeta \in \Omega_N. \tag{2}$$

Given $\eta \in \Omega_N$ and $k \in \mathbb{Z}_+$, denote the *k-vicinity* of the colony η by

$$B_k(\eta) := \{\zeta \in \Omega_N \mid d(\eta, \zeta) \le k\}. \tag{3}$$

We code the allelic types of individuals by the elements of some compact Polish space E. We assume that at each site $\eta \in \Omega_N$ there is a colony with allelic type distribution $X_\eta^{(N)}(t) \in \mathcal{M}_1(E)$ at time $t \in \mathbb{R}_+$. A similar setup for diffusive population genetics models without the global resampling was treated in [6].

Informally, the HIC^ΛP $X^{(N)} = (X_\eta^{(N)}(t))_{t\in\mathbb{R}_+, \eta\in\Omega_N}$ is the Markov process with the following evolution rules:

- *Initial distribution* is given by the distribution of types θ^{Ω_N}, where $\theta \in \mathcal{M}_1(E)$.
- *Migration* is parametrised by the sequence of *migration rates* between the colonies: $(c_k)_{k\in\mathbb{Z}_+} \in (0, +\infty)^{\mathbb{Z}_+}$. The migration is performed as follows. The individuals living in colonies labelled by Ω_N perform the *hierarchical random walk* (HRW, cf. [7]), i.e., each individual chooses the radius $k \in \mathbb{N}$ at rate c_k/N^k and jumps to a new location chosen uniformly at random within the k-vicinity around its current position.
- *Local/global resampling* is parametrised by the sequence of finite *resampling measures* $(\Lambda_k)_{k\in\mathbb{Z}_+} \in \mathcal{M}_{\text{finite}}((0, 1])^{\mathbb{Z}_+}$ that specify the rates at which the resampling of the given scale occurs. The resampling is performed according to the following algorithm:
  (1) For each colony $\eta \in \Omega_N$, the radius $k \in \mathbb{Z}_+$ is chosen at rate $1/N^{2k}$.
  (2) A *parent individual* is drawn uniformly form $B_k(\eta)$.
  (3) *Reshuffling:* each individual in $B_k(\eta)$ is relocated independently to a uniformly chosen location in $B_k(\eta)$.
  (4) A number $r \in (0; 1]$ is chosen at rate $\Lambda(\mathrm{d}r)/r^2$. The proportion r of uniformly chosen in $B_k(\eta)$ individuals are substituted with the *offspring* of the parent individual. The offspring receive the allelic type of the parent.

The *non-spatial single-colony C^Λ process* (NSC^ΛP) is a $\mathcal{M}_1(E)$-valued simplification of the HIC^ΛP for the geographical space that consists just from a single colony. Therefore, the dynamics of the NSC^ΛP has no migration part. The resampling part is defined as the HIC^ΛP one corresponding to $k = 0$ and is, hence, parametrised by a single resampling measure $\Lambda \in \mathcal{M}_1((0, 1])$.

Main results. We employ the RG type of multi-scale analysis as a tool to study the large space-time scale behaviour of the HIC$^\Lambda$P in the *hierarchical mean-field limit* $N \to +\infty$.

In particular, we consider the *spatial averages* of the HIC$^\Lambda$P

$$Y^{(N)}_{\eta,k}(t) = \frac{1}{N^k} \sum_{\zeta \in B_k(\eta)} X^{(N)}_\zeta(t), \quad \eta \in \Omega_N, \quad k \in \mathbb{Z}_+. \tag{4}$$

We identify the weak limit for (4) at *macroscopic times*, i.e., for $Y^{(N)}_{\eta,k}(tN^k)$, as $N \to +\infty$. It turns out that the weak limit of (4) is the sum of the following three components:

(1) the NSC$^\Lambda$P with the resampling measure Λ_k;
(2) the diffusive *Fleming-Viot process* (see, e.g., [8] for a review) with constant volatility d_k that can be specified in terms of the migration and resampling rates, cf. (6) below;
(3) *drift* of the speed c_k towards the initial distribution $\theta \in \mathcal{M}_1(E)$ of the HIC$^\Lambda$P.

We denote the above described three-component $\mathcal{M}_1(E)$-valued process by

$$Z^{c_k,d_k,\Lambda_k}_\theta = \left(Z^{c_k,d_k,\Lambda_k}_\theta(t) \right)_{t \in \mathbb{R}_+}. \tag{5}$$

To specify the volatility constants (d_k) of the Fleming-Viot process, define the sequence of the *total resampling rates* $\lambda_k := \Lambda_k(0,1]$, $k \in \mathbb{Z}_+$. Now, the *volatility constants* $d = (d_k)_{k \in \mathbb{Z}_+}$ are defined recursively as iterations of the *Möbius transformation*:

$$d_0 = 0, \quad d_{k+1} = \frac{c_k(\frac{1}{2}\lambda_k + d_k)}{c_k + (\frac{1}{2}\lambda_k + d_k)}, \quad k \in \mathbb{Z}_+. \tag{6}$$

Let $\mathcal{L}[\cdot]$ denote law of "$\cdot$" and "$\Longrightarrow$" the weak convergence on the Skorokhod path space $D(\mathbb{R}_+, \mathcal{M}_1(E))$. We arrive at the following result.

Theorem 2 (Macroscopic behaviour)**.** *For every $k \in \mathbb{N}$, uniformly in $\eta \in \Omega_N$, as $N \to +\infty$,*

$$\mathcal{L}\left[\left(Y^{(N)}_{\eta,k}(tN^k) \right)_{t \in \mathbb{R}_+} \right] \Longrightarrow \mathcal{L}\left[\left(Z^{c_k,d_k,\Lambda_k}_\theta(t) \right)_{t \in \mathbb{R}_+} \right]. \tag{7}$$

Analysing the recurrence (6), we show that, depending on the migration and the resampling rates, the ergodic behaviour of the HIC$^\Lambda$P displays either (1) *coexistence* of several allelic types within colonies, or (2) *clustering*, i.e., emergence of mono-type colonies. For each the above two ergodic behaviours, we identify the corresponding regimes of parameters that lead to them.

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Towards microscopic models of nonlinear elasticity

Roman Kotecky

I give an overview of nonlinear elasticity with the idea that this is a new region for application of the renormalization group methods. When discussing the links between microscopic and macroscopic theory I state some results including large deviation principle and the asymptotic behaviors in terms of Gradient Young-Gibbs measures.

Kinetic scaling limits, Boltzmann equations, and Bose condensates

Jani Lukkarinen

(joint work with Jogia Bandyopadhyay, Antti Kupiainen, and Herbert Spohn)

The first part of the talk concerns the results derived in [1] for the discrete nonlinear Schrödinger equation (dNLS). We study the evolution of a complex valued wave field $\psi_t(x)$, $x \in \Lambda$, satisfying $\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\psi_t(x) = \sum_{y\in\Lambda} \alpha(x-y)\psi_t(y) + \lambda|\psi_t(x)|^2\psi_t(x)$. The particles move on a finite periodic lattice Λ, with $\alpha(x)$ defining the hopping amplitudes, assumed to be real, symmetric, and sufficiently well localized. We also consider only the dispersive case $\lambda \geq 0$. Then standard methods guarantee the existence and uniqueness of a global solution for any initial data $\psi_0 : \Lambda \to \mathbb{C}$.

This system is Hamiltonian, with two conserved quantities: energy, $H_\Lambda(\psi) = \sum_{x,y\in\Lambda} \alpha(x-y)\psi(x)^*\psi(y) + \frac{1}{2}\lambda \sum_{x\in\Lambda} |\psi(x)|^4$, and norm $\|\psi\|^2 := \sum_{x\in\Lambda} |\psi(x)|^2$. The corresponding Gibbs ensemble has a weight function $\mathrm{e}^{-\beta(H_\Lambda(\psi)-\mu\|\psi\|^2)}/Z^\lambda_\Lambda$, where $\beta > 0$ is the inverse temperature and $\mu \in \mathbb{R}$ the chemical potential. We choose this as a random distribution of our initial data ψ_0, and let $\psi_t(x)$ be defined by the corresponding unique solution to the dNLS, thus making it a random variable. Physically, this means that the wave field is in thermal equilibrium.

We have to impose two types of assumptions. The first one requires that the initial measure is "ℓ_1-clustering": we assume that all of its cumulants are absolutely summable (with one of the arguments held fixed), and that this sum is uniformly bounded in $|\Lambda|$ and increases at most like a power of $n!$ in the degree n of the cumulant. In addition, we need that the corrections to the covariance are uniformly $O(\lambda)$. These properties are proven by Abdesselam, Procacci, and Scoppola in [2] for a large class of coupling functions α and β, μ. The second set of assumptions relates to the dispersivity of the free evolution (corresponding to $\lambda = 0$). The precise conditions can be found in the original publication, where we also prove that they hold for the standard nearest neighbor interactions whenever $d \geq 4$.

The main theorem concerns the field-field time-correlations, more precisely, the observable $Q_\Lambda^\lambda[g,f](t) = \mathbb{E}[\langle \widehat{f}, \widehat{\psi}_0\rangle^* \langle e^{-i\omega^\lambda t/\varepsilon}\widehat{g}, \widehat{\psi}_{t/\varepsilon}\rangle]$, where $\varepsilon = \lambda^2$, and f, g are arbitrary test functions with a finite support. In order to get a well-defined limit, it is necessary to cancel rapidly oscillating factors. The oscillations are produced by the free evolution and first order corrections, and the following choice suffices in the present case: set $\omega^\lambda(k) := \omega(k) + \lambda R_0$, where $\omega : \mathbb{T}^d \to \mathbb{R}$ denotes the free dispersion relation, defined by $\omega = \widehat{\alpha}$, and $R_0 = R_0(\lambda, \Lambda) = 2\mathbb{E}_\Lambda^\lambda[|\psi_0(0)|^2]$.

Theorem 3. *Under the previous assumptions, there is $t_0 > 0$ such that for $|t| < t_0$,*

$$\lim_{\lambda\to 0} \limsup_{\Lambda\to\infty} \left| Q_\Lambda^\lambda[g,f](t) - \int_{\mathbb{T}^d} dk\, \widehat{g}(k)^* \widehat{f}(k) W(k) e^{-\Gamma_1(k)|t| - it\Gamma_2(k)} \right| = 0\,.$$

Here $W(k) = \beta^{-1}(\omega(k) - \mu)^{-1}$ is the Fourier transform of the covariance of the free Gaussian measure, and $\Gamma_j(k)$ are real, with $\Gamma(k) = \Gamma_1(k) + i\Gamma_2(k)$ having an explicit integral representation which we do not reproduce here. In particular, $\Gamma_1(k) \geq 0$, and thus the corresponding factor provides exponential damping in time for those k with $\Gamma_1(k) > 0$. Heuristically, the theorem states that for all not too large $t = O(\lambda^{-2})$, we have $\mathbb{E}[\widehat{\psi}_0(k')^* \widehat{\psi}_t(k)] \approx \delta(k' - k) W(k) e^{-i\omega_{\text{ren}}^\lambda(k)t} e^{-|\lambda^2 t|\Gamma_1(k)}$, where $\omega_{\text{ren}}^\lambda(k) = \omega(k) + \lambda R_0 + \lambda^2\Gamma_2(k)$.

The proof of this result is based on perturbation and cluster expansions, followed by a classification of the resulting terms based on their "momentum integration" structure. This shows that only a small fraction of the terms contributes to the limit, and summing over these terms yields the explicit representation for $\Gamma(k)$ mentioned above. In the second part of the talk we consider an application of the above scheme for a weakly interacting quantum fluid, composed out of identical spinless bosons. Although an important part of the previous analysis is still missing in that case, namely an effective estimation of the remainder terms, most of the estimates carry over, provided ℓ_1-clustering is understood to hold in the corresponding quantum mechanical sense. In this manner, we obtain a conjecture not only for time-correlations but also for the time-evolution of the reduced density matrix, assuming that the initial state is translation invariant and ℓ_1-clustering. This leads to the standard Boltzmann-Nordheim (BN) equation, *cf.* [3].

However, at low temperatures such Bose fluids can exhibit *Bose condensation* where a macroscopic number of fluid particles form a condensate. If this happens, there is no reason to expect ℓ_1-clustering to hold. Thus it is not clear if the kinetic scaling limit will suffice to describe any condensation phenomena, since it might require processes (graphs) which become visible only after the kinetic time-scale, for $t = O(\lambda^{-p})$, $p > 2$. The main content of the second part of the talk is to present evidence which indicates that dynamical condensation can occur already on a finite kinetic time-scale. However, this evidence also indicates that great care must be taken in applying the BN equation when a condensate is present: the system will develop singularities and one cannot neglect the interaction between the normal fluid and the condensate.

For weakly interacting bosons, with translation invariant, quasi-free, and appropriately ℓ_1-clustering initial data, the kinetic conjecture states that the Fourier transform of the *reduced 1-particle density matrix* $\rho_1(r_1, r_2, t) =: g_t(r_1 - r_2)$ should have a convergent kinetic scaling limit $W(v, \tau) := \lim_{\lambda \to 0^+} \widehat{g}_{\lambda^{-2}\tau}(v)$. The limit should also satisfy the homogeneous BN equation $\partial_t W(v,t) = \mathcal{C}_4[W(\cdot,t)](v)$, with initial data $W(v, 0) = \widehat{g}_0(v)$. (This is a generalization of Conjecture 5.1 in [3] to a continuum setup.)

If $W(v, 0)$ is *isotropic* (depends only on $|v| = \sqrt{2x}$, x denoting the kinetic energy), then the above equation can be greatly simplified, and the remaining discussion only concerns this simpler isotropic case. When started with smooth, but supercritical, initial data, the numerical solution to this equation exhibits a blowup at $x = 0$ in a *finite* time [4]. It is not obvious what, if anything, should be done with this equation after the blowup. Mathematically, three different options present themselves for the continuation. **1) Pointwise:** One could continue using the previous evolution equation for $x > 0$ with initial data which is singular at $x = 0$. In the explicit case of $x^{-7/6}$ singularity, this has been shown to lead to a local existence of solutions which preserve the strength of the singularity, but not mass [5]. **2) Measure valued:** One could view the BN equation as a weak evolution equation for positive measures in x. This has been studied in [6] where it is shown that such solutions with many physically desirable properties, such as conservation of mass and energy, can be found. Possible non-uniqueness and non-constructive nature of these solutions presents problems for practical applications. **3) Physical ansatz:** In [4] and [7], an explicit measure valued ansatz in the form $f^{\mathrm{reg}}(x,t)\sqrt{x}\mathrm{d}x + n(t)\delta(x)\mathrm{d}x$ is studied. The coupled evolution equation for $f^{\mathrm{reg}}(x,t)$ and $n(t)$ can be solved numerically. Moreover, the thermal equilibrium states are then stationary. However, it was not shown if this set of equations is free from blowup, and there is no clear mechanism for generation of the condensate if initially $n(0) = 0$.

In [8], we consider a slight generalization of the last set of evolution equations. Imposing explicit conservation of mass we obtain a closed non-linear evolution equation for $f^{\mathrm{reg}}(x,t)$, of the form

$$\frac{d}{dt} f^{\mathrm{reg}}(x,t) = \mathcal{C}_4[f^{\mathrm{reg}}(\cdot,t)](x) + (m^{\mathrm{tot}}(0) - m[f^{\mathrm{reg}}(\cdot,t)])\mathcal{C}_3[f^{\mathrm{reg}}(\cdot,t)](x) \qquad (1)$$

where $\mathcal{C}_3$ is obtained from inserting the ansatz into $\mathcal{C}_4$. We study the existence and asymptotics of solutions to this nonlinear equation, assuming that the initial state $f^{\rm reg}(x,0)$ is a small perturbation (in a specific norm which is adapted to handle all singularities) of the stationary solution $f_{\beta,0}(x) := 1/(\mathrm{e}^{\beta x}-1)$, $\beta > 0$.

Although such problems are commonplace in the literature, due to marginal and competing singularities, the analysis in this case is not straightforward. By developing a strong control for the linearized evolution, we prove in [8] that, if the initial data contains a condensate and is a small perturbation of an equilibrium state, then there exists a solution $f^{\rm reg}(x,t)$, this solution conserves total energy and mass, and it converges exponentially fast to equilibrium: $f^{\rm reg} \to f_{\beta,0}$, and $n(t) \to m^{\rm tot}(0) - m[f_{\beta,0}]$ as $t \to \infty$. Moreover, the equations derived in [7] are satisfied and the corresponding family of measures provides a weak solution to the original BN equation, as considered in [6]. This gives some support to the validity of the evolution equation (1). However, two major open questions remain. First, in case a condensate is generated, does (1) have a unique solution which would describe the condensation? Secondly, is it possible to extend the perturbative scheme for initial data which are not ℓ_1-clustering, and this way to arrive to (1) instead of the standard BN equation?

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Time evolution of the external field problem in QED

Franz Merkl

(joint work with Dirk Deckert, Detlef Dürr, Martin Schottenloher)

Let $A \in C_c^\infty(\mathbb{R}^4,\mathbb{R}^4)$ be a classical, external vector potential, $U^A(t_1,t_0) : \mathcal{H} = L_2(\mathbb{R}^2,\mathbb{C}^4) \to \mathcal{H}$ be the Dirac time evolution from time t_0 to time t_1 in the time-dependent external potential A. Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be the splitting of $\mathcal{H}$ in the negative and positive spectral subspace of the free Dirac operator, $P_+ : \mathcal{H} \to \mathcal{H}_+$ and $P_- : \mathcal{H} \to \mathcal{H}_-$ and $\mathcal{F}$ the associated fermionic Fock spaces with field operators $\Psi, \Psi^+ : \mathcal{F} \times \mathcal{H} \to \mathcal{F}$. We say that a unitary map $\tilde{U} : \mathcal{F} \to \mathcal{F}$ lifts a unitary map $U : \mathcal{H} \to \mathcal{H}$ if $\Psi^{(+)}(\tilde{U}\phi, U\chi) = \tilde{U}\Psi^{(+)}(\phi,\chi)$ holds for all $\phi \in \mathcal{F}$ and $\chi \in \mathcal{H}$. The classical Shale-Stinespring criterion [9], [6] states the following: *For unitary $U : \mathcal{H} \to \mathcal{H}$, there is a unitary map $\tilde{U} : \mathcal{F} \to \mathcal{F}$ that lifts U if and only if the "non-diagonal parts" P_+UP_- and P_-UP_+ are Hilbert-Schmidt operators.*

The following has been observed already in the 1970's (see [1] and [7]; there are other versions by many people): *The maps $U^A(t_1,t_0)$ can be lifted to $\mathcal{F}$ for all times t_0 and t_1 if and only if $A_1 = A_2 = A_3 = 0$.*

Positive results. Langmann and Mickelsson [4], [5] constructed a family of unitary maps T_t, $t \in \mathbb{R}$, such that the "renormalized time evolution" $V^A(t_1,t_0) = T_{t_1} U^A(t_1,t_0) T_{t_0}^{-1}$ can be lifted to $\mathcal{F}$ for all times t_1, t_0. Their maps T_t depend on the vector potential $A(t)$ at time t and on finitely many time derivatives $\frac{\partial}{\partial t} A(t)$.

As a corollary, the S-operator can be lifted; this has also been proven by Scharf (see Chapter 2 in [8]) using a different technique than Langmann and Mickelsson.

There is quite some freedom in the choice of the renormalization map T_t. It seems natural to work with varying Fock spaces $\mathcal{F}_t$, $t \in \mathbb{R}$, corresponding to varying splittings (polarizations) $\mathcal{H} = \mathcal{H}_+(t) \oplus \mathcal{H}_-(t)$.

Definition: A *polarization* of $\mathcal{H}$ is a closed, linear subspace $V \subset \mathcal{H}$ such that V and $V^\perp$ have infinite dimension. Two polarisations V and W are called equivalent, $V \approx W$, if the orthogonal projections P_V and P_W to V and W differ only by a Hilbert-Schmidt operator. Let $\mathrm{Pol}(\mathcal{H})$ denote the set of all polarizations on $\mathcal{H}$.

The identity map on $\mathcal{H}$ lifts to a "Bogoliubov-transform" $B : \mathcal{F}(V, V^\perp) \to \mathcal{F}(W, W^\perp)$ between the Fock spaces $\mathcal{F}(V, V^\perp)$, $\mathcal{F}(W, W^\perp)$ associated to V and W if and only if $V \approx W$.

For $V \approx W$, the restriction $P_W|V \to W$ is a a Fredholm operator. Its Fredholm index $\mathrm{charge}(V, W)$ is called the *relative charge* between V and W. The finer relation $V \approx_0 W$, defined by $W \approx W$ and $\mathrm{charge}(V, W) = 0$ is also an equivalence relation. The finer relation $V \approx_0 W$ holds if and only if for any $c \in \mathbb{Z}$, the Bogoliubov transform B maps the charge c sector of $\mathcal{F}(V, V^\perp)$ to the charge c sector of $\mathcal{F}(W, W^\perp)$.

Theorem (see [2]): There is a map $C : C_c^\infty(\mathbb{R}^3, \mathbb{R}^4) \to \mathrm{Pol}(\mathcal{H})/{\approx_0}$, having the following properties:

(1) $C(0) = [\mathcal{H}_-]_{\approx_0}$.

(2) For all $A \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^4)$ and $t_0, t_1 \in \mathbb{R}$, one has $U^A(t_1,t_0)C(A(t_0) = C(A(t_1))$. In particular, for $V \in C(A(t_0)$ and $W \in C(A(t_1))$, the time evolution $U^A(t_1,t_0) : \mathcal{H} \to \mathcal{H}$ lifts to a charge-preserving unitary map

$$\tilde{U}^A(t_1,t_0); \mathcal{F}(V, V^\perp) \to \mathcal{F}(W, W^\perp).$$

(3) For $A, A' \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^4)$, the classes $C(A)$ and $C(A')$ are equal if and only if $A_\mu = A'_\mu$ holds for $\mu = 1, 2, 3$.

The author believes that for $A \neq 0$, there is no physically distinguished representative $V \in C(A)$. However, there are several "mathematically nice" representatives. Here are a few examples:

1. If T_t denotes again Langmann/Mickelsson's renormalization map, described in [4], then $T_t^{-1} \in C(A(t))$ holds for $t \in \mathbb{R}$. This implies that the the class $[T_t^{-1}\mathcal{H}_-]_{\approx_0}$ does *not* depend on time derivatives, although $T_t^{-1}\mathcal{H}_-$ may depend on time derivatives.

2. For $A \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^4)$, let $Z^A(p,q)$, $p, q \in \mathbb{R}^3$ denote the interaction term in the Dirac equation in momentum representation, $P_\pm(p)$ be the representation of the

projections $P_\pm : \mathcal{H} \to \mathcal{H}_\pm$ in momentum space as multiplication operators, and $Q^A : \mathcal{H} \to \mathcal{H}$ be the bounded operator having the following kernel in momentum representation:

$$\mathbb{R}^3 \times \mathbb{R}^3 \ni (p,q) \mapsto \frac{P_+(p)Z^A(p,q)P_-(q) - P_-(p)Z^A(p,q)P_-(q)}{i(E(p)+E(q))},$$

where $E(p) = \sqrt{p^2+m^2}$. Then one has $C(A) = [e^{Q^A}]_{\approx_0}$.
3. For $A \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^4)$, the essential spectrum of the Dirac operator H^A equals $(-\infty,-m] \cup [m,\infty)$. For $-m < E < m$, let V_E denote the spectral subspace belonging to the part $(-\infty, E] \cap \sigma(H^A)$ of the spectrum of the Dirac operator with static potential A. Then $V_E \approx V$ for all $V \in C(A)$. This observation was already made by Fierz and Scharf in [3], it is presented here in a different notation.

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Towards Renormalizing Group Field Theory

Vincent Rivasseau

This talk is based on the program [1].

The first part of the talk introduced various kinds of renormalization group which rely on different notions of scale decomposition and locality principles and lead to different power countings. Scalar field theories are a primary example of ordinary renormalization group. In condensed matter renormalization group the scales measure distance to the Fermi surface and power counting is independent of the dimension [2]. The renormalization group associated to the Grosse-Wulkenhaar model, a quantum field theory on the non commutative four dimensional Moyal space, is even more exotic. The slices mix ultraviolet and infrared, the locality principle is replaced by Moyality, and the power counting is that of a matrix model [3].

These examples point towards the possibility that quantum gravity might be also renormalizable quantum field theory after all, but in such an exotic sense. In the second part of the talk we reviewed briefly matrix models for 2d gravity and group field theories [4, 5]. Then we present the colored group field theories invented and developped by Razvan Gurau [6, 7] which are a promising formalism for summing over different space-time topologies in dimension 3 and 4. In particular these models admit a $1/N$ expansion in which the leading order graphs triangulate the sphere, in any dimension [8, 9, 10].

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From stochastic calculus to constructive field theory

Jérémie Unterberger

Let $B = (B_1, \ldots, B_d)$ be a fractional Brownian motion of Hurst index $\alpha \in (0, 1)$ with d independent, identically distributed components. The paths of this Gaussian process are continuous but very "rough", actually α-Hölder, or more precisely α^--Hölder for every $\alpha^- < \alpha$. This makes the very definition of stochastic integration along B or of solutions of stochastic differential equations driven by B a difficult problem, the solution of which is gradually emerging, with deep connections to sub-Riemannian geometry [4], combinatorial Hopf algebras of trees [21, 20, 2], and quantum field theory, more specifically renormalization [19]. Contrary to the case of usual Brownian motion (given by $\alpha = 1/2$), stochastic integrals may not be defined for small α by straightforward, e.g. piecewise linear approximations. Rough path theory [10, 11] shows that the key problem lies in a proper definition of *iterated integrals* of B of order $2, 3, \ldots, N$, with $N = \lfloor 1/\alpha \rfloor$, $\lfloor\ .\ \rfloor$=integer part, making up together what is called a *rough path over* B. The usual definition of rough paths is based on an axiomatization of the geometric properties of the iterated integrals of a smooth path, which may be considered (using Green-Riemann's formula) as (signed) areas and volumes generated by the path; these properties may be summarized by stating that the stack of iterated integrals are *regular* lifts of the path to sections of the trivial G_N-principal bundle over $\mathbb{R}$ (time coordinate), where G_N is a free nilpotent (or Carnot-Carathéodory) group of rank N. Regular lifts are unique in the smooth case, whereas Hölder-continuous lifts for Hölder continuous paths are non-canonical. Let us simply say here that a rough path over B is a limit in appropriate Hölder norms of iterated integrals of order $2, 3, \ldots, N$ of a sequence of approximations of B converging to B in α^--Hölder norm. Other more geometric or algebraic definitions exist, which are shown to be equivalent by using piecewise sub-Riemannian geodesic approximations, the natural (but far less explicit, especially for large N, due to the notorious difficulty of construction of geodesics in this setting) generalization of piecewise linear approximations. Despite an abstract (non constructive) proof of existence [12], and several recent investigations [21, 20, 2] yielding a sort of general classification of rough paths in the algebraic sense, the series of papers starting with [13] gives the first construction of a rough path over B for $\alpha \leq 1/4$ by means of an explicit sequence of approximations. The barrier at $\alpha = 1/4$ has been recognized by several authors using different approaches [1, 14, 17, 18], and shown to extend to other models as well [6].

Our solution relies on the previously mentioned algebraic investigations, which have brought to the light the crucial importance of the use of *skeleton integrals* instead of *iterated integrals* and of the concept of *Fourier normal ordering*, *and* – most essentially – on the reformulation of this problem in the language of *quantum field theory*. We shall concentrate here on the construction of second-order iterated integrals of fBm with $\alpha \in (1/8, 1/4)$ and $d = 2$. The *singular part* of the area of fBm is the sum of two terms, $\mathcal{A}^{\pm}(t) - \mathcal{A}^{\pm}(s)$, which are simply increments of two functions $\mathcal{A}^{\pm}$. These diverge in the *ultra-violet limit*. In other words, $\mathcal{A}^{\pm}$ diverges because of the contribution of highest frequency components of B. Precise statements may be given if one decomposes the "signal" B into its different "scales" by using a dyadic Fourier partition of unity. This is well-known to those acquainted either to Besov spaces, wavelets or quantum field theory. Replacing B with the *cut-off field* $B^{\to\rho} = \sum_{j=-\infty}^{\rho} B^j$, with $\mathcal{F}B^j$ supported on $[M^{j-1}, M^{j+1}] \cup [-M^{j+1}, -M^{j-1}]$ for some fixed base $M > 1$, one obtains cut-off functions $\mathcal{A}^{\to\rho}$, $\mathcal{A} = \mathcal{A}^{\pm}$, whose variance diverges like $M^{\rho(1-4\alpha)}$ when $\rho \to \infty$. This quantity may be expressed as an ultra-violet diverging Feynman diagram, see Fig. 1. Pursuing this reinterpretation, it is tempting to consider the entire bubble series instead of the single bubble diagram. By inserting thin lines with the correct scaling dimension between the bubbles, and considering vertices with an imaginary coupling constant iλ, one obtains a geometric series (see Fig. 2) which *formally* sums up to a finite quantity, with the correct degree of homogeneity.

Formally again, $1 - \lambda^2 \left(\frac{M^\rho}{|\xi|}\right)^{1-4\alpha} + \lambda^4 \left(\frac{M^\rho}{|\xi|}\right)^{2(1-4\alpha)} + \ldots = \frac{1}{1+\lambda^2 (M^\rho/|\xi|)^{1-4\alpha}}$, a very small quantity (for $\rho \to \infty$), measuring an almost insensitive interaction *but* sufficient to make the Lévy area converge. As explained in [13], section 3, this may be implemented (in theory at least) by multiplying the statistical weight of

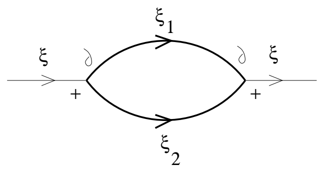

FIGURE 1. Bubble diagram. Boldface lines scale as $1/|\xi|^{1+2\alpha}$.

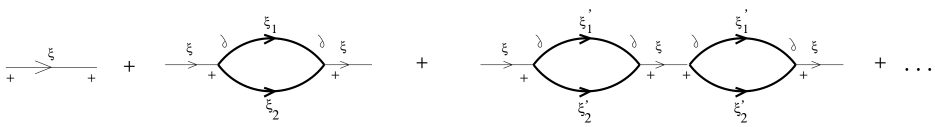

FIGURE 2. Bubble series. Thin lines scale as $1/|\xi|^{1-4\alpha}$.

the Gaussian paths by the exponential

$$e^{-\frac{1}{2}c'_\alpha\lambda^2\int\int dt_1dt_2|t_1-t_2|^{-4\alpha}\left(\partial\mathcal{A}^+(t_1)\partial\mathcal{A}^+(t_2)+\partial\mathcal{A}^-(t_1)\partial\mathcal{A}^-(t_2)\right)}. \tag{1}$$

Mathematically this sounds like a joke, since we are well beyond the radius of convergence of the series, even for small λ. But such summations may be performed rigorously scale after scale in a finite time horizon $V=[-T,T]$, going down from scale ρ to scale $-\infty$, uniformly in ρ and V. A quantum field theoretic model underlying this may be defined, yielding a sequence of Gibbs measures $\mathbb{P}_{\lambda,V,\rho}$ which converges weakly to a unique probability measure $\mathbb{P}_\lambda$ when $|V|,\rho\to+\infty$. The law of the process B under this measure is the same as its initial, Gaussian measure, *but* the cut-off singular quantities $\mathcal{A}^{\to\rho}$ in the interacting measures $\mathbb{P}_{\lambda,V,\rho}$ converge to give ultimately a finite rough path over the limit process B.

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Geometric flows as RG flows

Eric Woolgar

In the first of two talks, I give an introduction to geometric flows. Examples of such flows include the Ricci flow, the mean curvature flow, the Calabi flow, and others. I first review some of the basics of Riemannian geometry, and then discuss the naturality property under t-independent diffeomorphisms which makes these flows "geometric". Hamilton's Ricci flow [7] is not a parabolic system, but that a trick of DeTurck [2] produces a parabolic system which is related to Ricci flow via pullback by a t-dependent diffeomorphism. Friedan [4] showed that this flow arises as the 1-loop approximation to the RG flow of a 2-dimensional bosonic nonlinear sigma model. Where scales are such that multi-loop corrections can be ignored, ancient solutions (those defined for $t \in (-\infty, 0)$) of the flow correspond to cut-off removal. For two-dimensional target spaces at least, ancient solutions of the 1-loop are asymptotically free.

The C-theorem [15, 13] states that the RG flow has a monotonicity property. A stronger result follows for Ricci flow (and the 1-loop RG flow) with Perelman's special choice for the DeTurck term. This flow is in fact the gradient of Perelman's energy functional [12], a fact which was first noticed in the special case of 2-dimensional target space in [3]. Tseytlin [14] has argued that this implies that the full RG flow of the nonlinear sigma model is therefore gradient flow whenever the perturbative expansion is valid (but see also [10]). Remarkably, the gradient formula extends to the case of a nonlinear sigma model with antisymmetric B-field [10].

In the second talk, I concentrate on the Ricci flow of asymptotically flat manifolds and the behaviour of mass under Ricci flow, as described in [11]. This is motivated by a conjecture [5] for the evolution of closed bosonic strings. It is believed that a system of bosonic strings should decay from a false vacuum to a true vacuum, a phenomenon known as tachyon condensation. As this happens, closed string graviton excitations should carry away energy, so the total mass-energy of the system should decrease. As well, the evolution should be driven by the RG β-function and should be friction-dominated, so the evolution equation should be approximately Ricci flow. In [5] a Ricci soliton is found which describes this process for strings in a 2-dimensional target manifold. The soliton is asymptotic to a flat cone, whose cone angle does not change under the evolution, but which has flat space as its geometric limit as $t \to \infty$. Since asymptotic cone angle is the 2-dimensional mass, we see that the mass does not change during the flow, but instead decreases by jumping to zero in the limit of infinite time.

In [11], we show that this behaviour is seen in all dimensions. Specifically, for rotationally symmetric asymptotically flat initial metrics in all dimensions, if there is no minimal surface present in this initial data then no minimal surface ever forms during evolution by Ricci flow, so no neck-pinch singularity occurs, and indeed the flow exists for all time and converges to flat space. However, during the flow, the mass never varies from its initial value. We take this as confirmation of the string theory picture.

In the last part of the talk, I deal with an asymptotically flat Ricci flow system in which the rotational $\mathrm{SO}(d)$ symmetry is broken to $\mathbb{R} \times \mathrm{SO}(d-1)$. Fixed points of this flow correspond to solutions of the *static Einstein equations* [6], which play an important role in General Relativity. This flow system was introduced by List [8] and may be a useful way to address a long-standing conjecture by Bartnik [1]. This flow was studied on $\mathbb{R}^n$ in [6], where we were able to compute many estimates and show in the $d = 3$ case that the flow exists for $t \in [0, \infty)$ when no minimal surfaces are present initially (as with Ricci flow, then no minimal surface will form). The Bartnik conjecture concerns the class of asymptotically flat manifolds that have nonnegative scalar curvature, no closed minimal hypersurfaces, and an inner boundary B. It is conjectured that there are asymptotically flat solutions of the static Einstein equations on $M := \mathbb{R}^{d-1} \backslash K$, K a closed ball, obeying certain so-called *geometric boundary conditions* (see [1]) on $B = \partial M$, and these solutions minimize the mass functional. An approach to the conjecture via geometric flows will require an understanding of such flow on manifolds with boundary. We currently lack such understanding in all but a few simple cases.

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Reporter: Long Lu

Participants

Prof. Dr. Abdelmalek Abdesselam
Department of Mathematics
University of Virginia
Kerchof Hall
P.O.Box 400137
Charlottesville , VA 22904-4137
USA

Prof. Dr. Stefan Adams
Mathematics Institute
University of Warwick
Zeeman Building
GB-Coventry CV4 7AL

Prof. Dr. Volker Bach
Institut für Algebra u. Analysis
TU Braunschweig
Pockelsstr. 14
38106 Braunschweig

Dr. Miguel Ballesteros
Institut für Algebra u. Analysis
TU Braunschweig
Pockelsstr. 14
38106 Braunschweig

Roland Bauerschmidt
Department of Mathematics
University of British Columbia
121-1984 Mathematics Road
Vancouver BC V6T 1Z2
CANADA

Dr. Cindy Blois
Department of Mathematics
University of British Columbia
121-1984 Mathematics Road
Vancouver BC V6T 1Z2
CANADA

Dr. Jean-Bernard Bru
IKERBASQUE
Basque Foundation for Science
Plaza Bizkaia
E-48011 Bilbao

Prof. Dr. David C. Brydges
Department of Mathematics
University of British Columbia
121-1984 Mathematics Road
Vancouver BC V6T 1Z2
CANADA

Serena Cenatiempo
Dipartimento di Fisica
Universita degli Studi di Roma I
"La Sapienza"
Piazzale Aldo Moro, 5
I-00185 Roma

Ajay Chandra
Department of Mathematics
University of Virginia
Kerchof Hall
P.O.Box 400137
Charlottesville , VA 22904-4137
USA

Dr. Codina Cotar
Zentrum Mathematik
TU München
Boltzmannstr. 3
85748 Garching b. München

Dr. Wojciech de Roeck
Institut für theoretische Physik
Universität Heidelberg
Philosophenweg 19
69120 Heidelberg

Dr. Margherita Disertori
Laboratoire de Mathématiques
Raphael Salem, UMR-CNRS 6085
Université de Rouen
Avenue de l'Université, BP 12
F-76801 Saint Etienne de Rouvray

Dr. Michael Dütsch
Institut für Theoretische Physik
Universität Göttingen
Friedrich-Hund-Platz 1
37077 Göttingen

Dr. Pierluigi Falco
School of Mathematics
Institute for Advanced Study
1 Einstein Drive
Princeton , NJ 08540
USA

Prof. Dr. Klaus Fredenhagen
Institut II für Theoretische Physik
Universität Hamburg
Luruper Chaussee 149
22761 Hamburg

Prof. Dr. Alessandro Giuliani
Dipartimento di Matematica
Universita degli Studi Roma Tre
Largo S. L. Murialdo, 1
I-00146 Roma

Dr. Gianluca Guadagni
Department of Mathematics
University of Virginia
Kerchof Hall
P.O.Box 400137
Charlottesville , VA 22904-4137
USA

Dr. Riccardo Guida
Institut de Physique Theorique
CEA Saclay
F-91191 Gif-sur-Yvette Cedex

Prof. Dr. Stefan Hollands
School of Mathematics
Cardiff University
23, Senghennydd Road
GB-Cardiff CF24 4AG

Prof. Dr. John Imbrie
Institute for Advanced Study
School of Mathematics
Princeton , NJ 08540
USA

Dr. Sabine Jansen
Weierstraß-Institut für
Angewandte Analysis und Stochastik
im Forschungsverbund Berlin e.V.
Mohrenstr. 39
10117 Berlin

Dr. Anton Klimovsky
EURANDOM
Technical University Eindhoven
P.O. Box 513
NL-5600 MB Eindhoven

Prof. Dr. Horst Knörrer
Departement Mathematik
ETH-Zentrum
Rämistr. 101
CH-8092 Zürich

Prof. Dr. Wolfgang König
Weierstraß-Institut für
Angewandte Analysis und Stochastik
im Forschungsverbund Berlin e.V.
Mohrenstr. 39
10117 Berlin

Prof. Dr. Christoph Kopper
Centre de Physique Theorique
Ecole Polytechnique
Plateau de Palaiseau
F-91128 Palaiseau Cedex

Prof. Dr. Roman Kotecky
University of Warwick
Mathematics Institute
Zeeman Building
GB-Coventry CV4 7AL

Dr. Benjamin Leveque
Centre de Physique Theorique
Ecole Polytechnique
Plateau de Palaiseau
F-91128 Palaiseau Cedex

Martin Lohmann
Departement Mathematik
ETH-Zentrum
Rämistr. 101
CH-8092 Zürich

Long Lu
Institut für theoretische Physik
Universität Heidelberg
Philosophenweg 19
69120 Heidelberg

Dr. Jani Lukkarinen
Department of Mathematics
University of Helsinki
P.O. Box 68
FIN-00014 Helsinki

Prof. Dr. Jacques Magnen
Centre de Physique Theorique
Ecole Polytechnique
Plateau de Palaiseau
F-91128 Palaiseau Cedex

Peng Mei
Department of Mathematics
University of Helsinki
P.O. Box 68
FIN-00014 Helsinki

Prof. Dr. Franz Merkl
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Prof. Dr. Pronob K. Mitter
Laboratoire de Physique Theorique et
Astroparticules, UMR 5207 CNRS - UM2
Universite de Montpellier II
Pl. E. Bataillon
F-34095 Montpellier Cedex

Prof. Dr. Volkhard F. Müller
Fachbereich Physik
Universität Kaiserslautern
67653 Kaiserslautern

Prof. Dr. Felix Otto
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstr. 22 - 26
04103 Leipzig

Dr. Marcello Porta
Institut für Theoretische Physik
ETH Zürich
Hönggerberg
CH-8093 Zürich

Prof. Dr. Vincent Rivasseau
Laboratoire de Physique Theorique
Universite Paris XI
Batiment 210
F-91405 Orsay Cedex

Prof. Dr. Manfred Salmhofer
Institut für theoretische Physik
Universität Heidelberg
Philosophenweg 19
69120 Heidelberg

Prof. Dr. Benjamin Schlein
Institut für Angewandte Mathematik
Universität Bonn
Endenicher Allee 60
53115 Bonn

Zhiyi Tang
Fakultät für Mathematik & Physik
Universität Bayreuth
95440 Bayreuth

Prof. Dr. Jeremie Unterberger
Institut Elie Cartan
-Mathematiques-
Universite Henri Poincare, Nancy I
Boite Postale 239
F-54506 Vandoeuvre les Nancy Cedex

Prof. Dr. Fabien Vignes-Tourneret
Institut Camille Jordan
Universite Claude Bernard Lyon 1
43 blvd. du 11 novembre 1918
F-69622 Villeurbanne Cedex

Zhituo Wang
Laboratoire de Physique Theorique
Universite de Paris XI
Batiment 211
F-91405 Orsay Cedex

Matthias Westrich
Matematisk Institut
Aarhus Universitet
Ny Munkegade
DK-8000 Aarhus C

Prof. Dr. Eric Woolgar
Dept. of Mathematical Sciences
University of Alberta
Edmonton, Alberta T6G 2G1
CANADA

Dr. Mei Yin
Department of Mathematics
The University of Texas at Austin
1 University Station C1200
Austin , TX 78712-1082
USA

Prof. Dr. Martin Zirnbauer
Institut für Theoretische Physik
Universität Köln
Zülpicher Str. 77
50937 Köln