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Real Enumerative Questions in Complex and Tropical Geometry

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Abstract. The workshop *Real Enumerative Questions in Complex and Tropical Geometry* was devoted to a wide discussion and exchange of ideas between the best experts representing various points of view on the subject. Enumeration of real curves largely motivated the development of the tropical geometry and led to the discovery of new interesting geometric phenomena and deep links between this problematic and algebraic geometry, symplectic geometry, topology, and mathematical physics.

Mathematics Subject Classification (2000): Primary 14-xx (algebraic geometry), Secondary 81Txx (quantum field theory, related classical field theories), 53Dxx (symplectic geometry, contact geometry).

Introduction by the Organisers

The workshop *Real Enumerative Questions in Complex and Tropical Geometry*, organized by Grigory Mikahlkin (Genève), Eugenii Shustin (Tel Aviv), Johannes Walcher (Genève), and Jean-Yves Welschinger (Lyon), was held April 18th–April 23rd, 2011. This meeting was well attended by with about 50 participants from around the world. The program of the workshop consisted of 18 one-hour talks given by leading experts in the subject as well as by perspective young researchers. In addition, four informal discussions on open problems and on questions related to the main topics of the workshop were ran during this week. Extended abstracts of the talks and reports on the discussions follow these introductory notes. A special feature of the workshop was a dialog between mathematicians and physicists around the main subject.

The idea of the workshop was to put in the center the enumeration of real curves, which, on one side, appeared as a natural counterpart of the complex Gromov-Witten theory and which largely motivated the development of tropical geometry, and, on the other side, is deeply linked with counting pseudoholomorphic curves with Lagrangian boundary conditions in symplectic geometry and mathematical physics. Despite a number of striking results in this direction, serious problems remain open on the way to a systematic theory in real enumerative geometry. We shortly comment on these problems and on how they were reflected in the talks and discussions during the workshop.

About 15 years ago a huge breakthrough in Mathematics happened when Kontsevich suggested a way to enumerate complex rational curves in the framework of String Theory. Real algebraic geometry is almost always a much more delicate subject then its complex counterpart. Nevertheless, a significant progress in understanding real enumerative geometry was done recently. A breakthrough was provided by the discovery by J.-Y. Welschinger (one of the organizers) of a way to invariantly enumerate rational curves in two and three dimensions passing through point constraints. In the same time a technique of tropical enumeration was developed (G. Mikhalkin). It allows to enumerate real and complex curves simultaneously which eventually amounts in the enumeration of certain graphs matching given constraints and equipped with "complex" or "real" combinatorial weights. Among arising key problems we mention (i) the understanding of real tropical enumerative invariants and related "cycles" in moduli spaces of tropical curves, (ii) The lack of appropriate correspondence theorems for the Kontsevich WDVV equation, computation of ψ-classes and some other problems of the complex enumeration, which could indicate their possible real enumerative analogues, (iii) the search for real enumerative invariants matching recently discovered real tropical invariants of positive genus. Particularly, the last problem reduces to a rigorous definition of the correction term to the Welschinger count, which is one of the main problem of the area. Considerations coming from Physics support the approach that includes enumeration of both type I and type II curves. In turn, enumeration of type II curves lies outside of a well-established Symplectic geometry approach of enumerating holomorphic curves with boundary on Lagrangian submanifolds (open Gromov-Witten theory). From the physical point of view, the real and complex enumerative geometry appears in topological quantum field theories and topological string theory, mirror symmetry and open Gromov-Witten theory.

Four survey lectures opened the workshop and presented the state of the art in the topological aspects of real and complex enumerative geometry (O. Viro), Lagrangian Floer theory as a symplectic side of the story (K. Ono), (p, q)-branes in string theory with relations to tropical and enumerative geometry (A. Hanany), recent developments in the tropical enumerative geometry (I. Itenberg). The tropical theory has been in the focus of the talks by E. Brugallé, A. Gathmann, K. Shaw, I. Tyomkin. Among them we especially mention the contribution by E. Brugallé, where the use of tropical modifications allowed to extend the tropical techniques beyond the range of toric examples, and the results presented by A. Gathmann, which resolve the lack of local non-invariance of tropical Welschinger invariants for configurations with imaginary points. The latter talk surprisingly resembles the approach to a correct definition of relative open Gromov-Witten invariants presented by R. Rasdeaconu. A common idea, which potentially can shed light on the problem of counting real curves of higher genus (cf. the discussion led by G. Mikhalkin), is to combine enumeration of real, resp. tropical curves of different kinds which together constitute an invariant. Very interesting topological ideas in the enumerative geometry have appeared in the talk by M. Polyak and in the informal discussion led by M. Kazarian. The promising picture of physical predictions in counting real holomorphic curves in Calabi-Yau three-folds was developed in the talk by D. Krefl, which also presented a physical intuition behind the invariant count of real curves. The lectures by N. C. Leung and K. Fukaya were devoted to the mirror symmetry of toric Calabi-Yau varieties, where the counting of signed (pseudo)holomorphic discs with suitable Lagrangian boundary conditions naturally enters the story. In its turn, the talk by M. Mariño linked the mirror symmetry of toric Calabi-Yau with spectral curves and their tropical limits, which opens a perspective application of tropical geometry. The algebra-geometric theory of mirror symmetry was discussed in the talk by L. Katzarkov. The talk by Y.-H. He exposed a wide physical picture involving quantum field theories, quiver gauge theories, dimer models, leading to amoebas and the tropical curves as their limits. The superpotentials describing interactions in the considered there field theories appeared a subject of a special discussion led by J. Walcher. In its turn, the complex-analytic point of view on amoebas and co-amoebas showed up in the discussion led by M. Passare. The symplectic problematic in the lecture by V. Shevchishin shares techniques and ideas with main problems of the workshop.

We believe that a very intensive and substantial exchange of a broad spectrum of ideas during the workshop will really stimulate a further research in the main discussed problems, which still are far from being completely settled.

Workshop: Real Enumerative Questions in Complex and Tropical Geometry

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Abstracts

Enumerative Real Algebraic Topology and its Encomplexing

Oleg Viro

Many classical invariants studied in algebraic topology are involved in solutions of enumerative problems. For example, the linking number of two oriented circles in the 3-space can be interpreted as the number of lines passing through a fixed point and meeting the circles. The lines are counted with signs. In the talk many other similar invariants and enumerative problems were considered. In these problems the setup is mixed: on one hand, the initial objects are purely topological, like the circles in the example above, on the other hand, the objects that are to be counted (like the lines in the example above) belong to the algebraic geometry. If the original objects are real algebraic varieties, then new opportunities related to existence of their complexification emerge. This allows one to formulate new problems which have similar formulations but admit a more robust solution. For example, the interpretation of the Whitney number of a planar curve in terms of the complexification of the curve (possible, if the curve is real algebraic and its real part is zero homologous in its complexification) a C^1-invariant turns into a C^0-invariant. This transition from invariants originating in the algebraic topology to similar invariants of real algebraic varieties is called an encomplexing. In the talk many similar examples were presented.

Lagrangian Floer Theory for Compact Toric Manifolds

Kaoru Ono

I gave an introductory talk on Floer thoery for Lagrangian intersections and explained some of my joint work with Kenji Fukaya, Yong-Geun Oh and Hiroshi Ohta.

Firstly, we sketch Floer's original construction. Let (X, ω) be a closed symplectic manifold. For a pair (L_0, L_1) of Lagrangian submanifolds, which intersect transversally, consider the free module $C(L_1, L_0)$ generated by the intersection points. Define the operator $\delta : C(L_1, L_0) \to C(L_1, L_0)$, which shifts the degree by $+1$ by counting isolated pseudo-holomorphic strips $u : \mathbf{R} \times [0, 1] \to X$ joining two intersection points $p^\pm$ such that

$$u(\mathbf{R} \times \{0\}) \subset L_0, \quad u(\mathbf{R} \times \{1\}) \subset L_1,$$
$$\lim_{\tau \to \pm\infty} u(\tau, t) = p^\pm.$$

When L_0 and L_1 are Hamiltonian isotopic and $\pi_2(X, L_0) = 0$, Floer proved that $\delta \circ \delta = 0$ (with $\mathbf{Z}/2$-coefficients), hence constructed what is now called Floer cohomology $HF(L_1, L_0)$. Moreover, under the same assumption, he showed that Floer cohomology is invariant under deformations L_0 and L_1 by distinct Hamiltonian diffeomorphisms. In fact, $HF(L_1, L_0)$ is isomorphic to the ordinary cohomology of L_0 (with $\mathbf{Z}/2$-coefficients).

Oh extended the construction to wider situation:L_0 and L_1 are monotone Lagrangian submanifolds with minimal Maslov number ≥ 3. (Later, if, in addition, L_1 is Hamiltonian isotopic to L_0, he weakened the hypothesis that the minimal Maslov number ≥ 3 to ≥ 2.)

However, the statement that $\delta \circ \delta = 0$ fails to hold, in general. We presented an easy example, in which it fails, and observed that such a defect is caused by bubbling-off of pseudo-holomorphic discs with boundary on L_0 or L_1. Therefore we were forced to study all pseudo-holomorphic discs systematically. Before proceeding, we would like to mention that there are two kinds of bubbling-off phenomena. One is bubbling-off of pseudo-holomorphic spheres and the other is bubbling-off of pseudo-holomorphic discs. Using the virtual fundamental cycle/chain techinque, e.g., the theory of Kuranishi structures, we can exclude the former possibility by "perturbation by multi-valued sections", since such phenomena occur in (formal) real codimension 2. However, the bubbling-off phenomena of pseudo-holomorphic discs occur in (formal) real codimension 1. Thus the virtual fundamental cycle/chain technique cannot exclude such possibility, in general. When we use the technique of multi-valued perturbations, we need signs, i.e., orientations of the moduli spaces and works with $\mathbf{Q}$ (not $\mathbf{Z}/2$ as in previous Floer's and Oh's works).

We compactify the moduli space of pseudo-holomorphic discs in the spirit of stable maps due to Kontsevich. For each $\beta \in H_2(X, L; \mathbf{Z})$, we consider the moduli space $\mathcal{M}_{k+1}(\beta)$ of bordered stable maps of genus 0 with $k+1$-marked points on the boundary, which is required connected. Using the evaluation maps $ev_i : \mathcal{M}_{k+1}(\beta) \to L$, we consider the following operation on differential forms on L:

$$\mathfrak{m}_{k,\beta} : (\eta_1, \ldots, \eta_k) \mapsto ev_{0!}(ev_1^* \eta_1 \wedge \cdots \wedge ev_k^* \eta_k) \quad \text{for } (\beta, k) \neq (0,1).$$

The evaluation map ev_0 may not be a submersion, but weakly submersive in the sense of Kuranishi structure. Therefore the operation "integration along fibers" makes sense. When $(\beta, k) = (0,1)$, the moduli space $\mathcal{M}_{k+1}(\beta)$ is empty, since the automorphism group of a constant disc with two boundary marked points is not finite. So we define the operation in this case separately:

$$\mathfrak{m}_{1,0} : \eta \mapsto d\eta.$$

Let R be a commutative ring with the unit, e.g., $\mathbf{Q}$, $\mathbf{C}$. We introduce the universal Novikov ring with the ground ring R.

$$\Lambda_{nov} = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} e^{\mu_i} | a_i \in R,\ \lambda_i \in \mathbf{R}, n_i \in \mathbf{Z}, \lim_{i \to \infty} \lambda_i = \infty \right\}.$$

We set the degree of T, resp. e, to 0, resp. 2. If we require all exponents λ_i are non-negative, resp. positive, we obtain $\Lambda_{0,nov}$, resp. $\Lambda_{+,nov}$. Set $\mathfrak{m}_k = \sum_\beta \mathfrak{m}_{k,\beta} T^{\int_\beta \omega} e^{\mu(\beta)/2}$. We extend them to coderivations $\widehat{\mathfrak{m}}_k$ on the bar complex $B\Omega(L; \Lambda_{0,nov})[1]$ and write $\widehat{d} = \sum_{k=0}^{\infty} \widehat{\mathfrak{m}}_k$. The stable map compactification of the moduli spaces $\mathcal{M}_{k+1}(\beta)$ and (relative version of) Stokes' formula imply that $\widehat{d} \circ \widehat{d} = 0$, i.e., the filtered A_∞-relations.

If $\mathfrak{m}_0 \neq 0$, we may not have $\mathfrak{m}_1 \circ \mathfrak{m}_1 = 0$. If there exists Λ_0-valued form b of shifted degree 0 such that

$$\mathfrak{m}_*(e^b) = \mathfrak{m}_0(1) + \mathfrak{m}_1(b) + \mathfrak{m}_2(b,b) + \cdots = 0,$$

which is called the Maurer-Cartan equation, we can deform $\{\mathfrak{m}_k\}$ to $\{\mathfrak{m}_k^b\}$ such that $\mathfrak{m}_1^b \circ \mathfrak{m}_1^b = 0$. More generally, if there exists b such that

$$\mathfrak{m}_*(e^b) = \mathfrak{m}_0(1) + \mathfrak{m}_1(b) + \mathfrak{m}_2(b,b) + \cdots = C1_L,$$

where $C \in \Lambda_+$ and 1_L the constant function 1 on L, we also find that $\mathfrak{m}_1^b \circ \mathfrak{m}_1^b = 0$. On the space of such b's, weak solutions of the Maurer-Cartan equation, we consider the gauge equivalence relation and denote by $\mathcal{M}_{\text{weak}}(L)$. We define the potential function $\mathfrak{PO} : \mathcal{M}_{\text{weak}}(L) \to \Lambda_+$ by

$$\mathfrak{m}_*(e^b) = \mathfrak{PO}(b)1_L.$$

(It is more appropriate to write the RHS as $\mathfrak{PO}(b)e1_L$, where e is the formal generator of Λ_{nov}.)

We can further deform the filtered A_∞-algebra structure using a cycle $\mathfrak{b}$ in X with coefficients in Λ_+ (bulk deformations). For this purpose, we use the moduli space of bordered stable maps with interior marked points in addition to the boundary marked points and cut out the moduli space by the constraints that the interior marked points are mapped to $\mathfrak{b}$ (we need to take care of coefficient in Λ_+, when we cook up the operations $\mathfrak{m}_k^{\mathfrak{b}}$). We also consider the Maurer-Cartan equation on the filtered A_∞-algebra deformed by $\mathfrak{b}$ and obtain $\mathcal{M}_{\text{def,weak}}(L)$ and the potential function $\mathfrak{PO}$ on it.

We have the following result, which covers Oh's discovery that Floer complex is defined for monotone L and its Hamiltonian deformation if the minimal Maslov number of L is at least 2.

Theorem. Let (L_0, L_1) be a relative spin pair of Lagrangian submanifolds, which intersects cleanly, e.g., transversally. Suppose that there is a cycle $\mathfrak{b}$ in X with coefficients in Λ_+ such that $b_i \in \mathcal{M}_{\mathfrak{b},\text{weak}}(L_i)$, $i = 0, 1$ with $\mathfrak{PO}_{L_0}(\mathfrak{b}, b_0) = \mathfrak{PO}_{L_1}(\mathfrak{b}, b_1)$. Then Floer's operator δ is $\delta^{\mathfrak{b}}_{b_0,b_1}$ such that

$$\delta^{\mathfrak{b}}_{b_0,b_1} \circ \delta^{\mathfrak{b}}_{b_0,b_1} = 0.$$

The resulting cohomology group $HF_{\mathfrak{b}}((L_1, b_1), (L_0, b_0); \Lambda_{nov})$ is invariant under Hamiltonian deformations of L_0 and L_1.

When $L_0 = L_1$, there is a spectral sequence converging to $HF_{\mathfrak{b}}((L, b), (L, b); \Lambda_{0,nov})$ with E_2-term $H(L; \Lambda_{0,nov})$.

When $\mathfrak{PO}_{L_0}(\mathfrak{b}, b_0) \neq \mathfrak{PO}_{L_1}(\mathfrak{b}, b_1)$, we find that

$$\delta^{\mathfrak{b}}_{b_0,b_1} \circ \delta^{\mathfrak{b}}_{b_0,b_1} = \mathfrak{PO}_{L_0}(\mathfrak{b}, b_0) - \mathfrak{PO}_{L_1}(\mathfrak{b}b_1).$$

In general, computation is difficult. However, in the case of Lagrangian torus fibers, i,e,, the inverse image of a point by the moment map, in a compact toric manifold, we can obtain satisfactory information from the leading order terms of the potential function, which I planned to explain, but did not have time in the talk.

(p, q)-branes

Amihay Hanany

The significance of branes in string theory became very clear by the mid 90's. One can observe a significant change of attention in the line of research of string theorist aiming most of the focus on various aspects of brane physics. One natural direction was the study of bound states of branes. It is pointed out that strings in Type IIB superstring theory can form bound states of (p, q) type. The description goes as follows. There are two 2-form gauge fields in Type IIB, traditionally named NS 2-form and the RR 2-form. The names originate from the natural construction as a perturbative string theory but the difference between them is just an artifact of the construction. One should really think of these two 2-forms as forming a doublet of some underlying $SL(2)$ symmetry. Under each of such 2-form there is a conserved electric charge and a conserved magnetic charge. Strings are said to carry the electric charge, while 5 branes are said to carry the magnetic charge. The bound states are formed by combining branes of the same dimension in such a way that they carry charges under both 2-form gauge fields. We say that a (p, q) string carries an electric charge p under the NS 2-form and an electric charge q under the RR 2-form. Similarly, a (p, q) 5-brane carries a magnetic charge p under the RR 2-form and a magnetic charge q under the NS 2-form.

An important ingredient into the construction of (p, q) webs is played by the notion of "branes ending on branes". It is said that a brane A can end on brane B, if the boundary of brane A carries a charge with respect to the gauge field that lives on the world volume of brane B. A typical example of a brane ending on brane is the case of a Dp brane that ends on a NS5 brane. The Dp brane is a $p+1$-dimensional object, standing for p space and 1 time directions, and its boundary is a $(p-1)+1$-dimensional object. When it ends on a NS5 brane, the boundary becomes an object which spans $p-1$ space directions inside 5, and therefore it is a codimension $6-p$ object on the world volume of the NS5 brane, which is $5+1$ dimensional. For example, a D1 brane which ends on a NS5 brane has a boundary which is a codimension 5 object, namely a particle which propagates in $5+1$ dimensions. Similarly, a D5 brane which ends on a NS5 brane is a codimension 1 object. In physics, we tend to call such objects in different names, depending on the codimension. The names which are given refer typically to the first time such an object appears in physical phenomena. Concretely, one calls a domain wall, vortex, monopole, instanton to a codimension 1, 2, 3, 4 object, respectively. We are interested in the domain wall case, as this is going to form the (p, q) web.

Consider a D5 brane that ends on a NS5 brane. It forms a domain wall on the world volume of the NS5 brane. Being codimension 1, it divides the world volume of the NS5 brane into 2 regions. If we sketch the world volume of the NS5 brane as a vertical line, then the two regions are one above and one below this wall, hence the name "domain wall", as it divides into domains. As in a traditional electrostatic problem, there is an elecric field that jumps across the domain wall, and the value of the jump is equal to the charge carried by the wall. Say if the electric field below a single D5 brane is 0, then the electric field above it is 1.

It is convenient to think of a (p, q) 5-brane as a collection of q NS5 branes that carry an electric field which is equal to p. This is an alternative way of thinking about a (p, q) 5-brane. With this information at hand, we can form the simplest (p, q) web that consists of a D5 brane that ends on a NS5 brane. Below the end of the D5 brane we have a $(0, 1)$ brane and above it we have a $(1, 1)$ brane which corresponds to a NS5 brane that carries one unit of electric field. We can also view this configuration as an intersection of 3 types of branes: The D5 brane or a $(1, 0)$ brane, with the NS5 or $(0, 1)$ brane, together with the $(1, 1)$ brane. All meet such that the total charge at the vertex of intersection is equal to 0. We thus learn, that at the intersection of branes there is a charge conservation which states that if all orientations of the branes are chosen such that they are all incoming, say a collection of branes with charges (p_i, q_i), then the sum is equal to zero, $\sum_i p_i = sum_i q_i = 0$.

Supersymmetry plays a crucial role in string theory. It simplifies the discussion while keeps the results exact, protecting from possible quantum corrections and other complicated issues which arise upon quantization of the system. For the case at hand, supersymmetry implies that at a special value of the Type IIB scalar, $\tau = i$, the (p, q) branes are oriented in a slope which is proportional to $p : q$. Thus we arrive at the formulation of (p, q) webs as the collection of all possible (p, q) 5-branes that meet at vertices with charge conservation and are oriented in space according to the slope condition. This web is well known in tropical geometry as the tropical curve.

The first use of (p, q) webs of 5 branes was in the context of $(4+1)$-dimensional gauge theories as described in detail in [1, 2]. One can form a (p, q) web of genus g with e external legs, which corresponds to a gauge theory of rank g and a collection of parameters equal to $e - 3$. There are two types of deformations - those which preserve the form on the external legs and do not move them, and those which do move the external legs. They are called local and global deformations. In the gauge theory this distinction is important and corresponds to parameters that control the strength of interactions and kinetic terms in the lagrangian, or to vacuum expectation values of scalar fields.

The typical 5-dimension theory is described by an object called quiver, which turns us into the link between (p, q) webs and quiver gauge theories. The quiver is constructed out of nodes, say d of them, which are connected by lines. There are two types of nodes. They are called gauged and ungauged, or local and global. The difference between them refers to the length of the corresponding segment in the (p, q) web. If the length is finite, then the quiver node is gauged or local. If the length is infinite, then the corresponding node is global or ungauged. A typical problem in quiver gauge theory is to find the space of all invariants with respect to the local gauge groups, while keeping all objects which transform covariantly under the global groups. A quiver which seems very relevant to the count of curves on P^2 is the line quiver that starts with $d-1$ gauged nodes, with ranks that admit an increasing order starting from 1 up to $d - 1$, together with another ungauged node of rank d. It appears that singularities on the moduli space of this quiver reproduces the Gromov-Witten invariants on P^2. This statement requires further study and it is our hope to get back to it in the near future.

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Tropical aspects of enumerative geometry

Ilia Itenberg

The talk is an attempt to make a brief overview of the tropical approach to real and complex enumerative questions. The main accent is put on applications of tropical geometry in real enumerative geometry.

1. Mikhalkin's correspondence theorem. Passing to the tropical limit for counting holomorphic curves and holomorphic disks was suggested by M. Kontsevich and K. Fukaya in 2002. Then, the tropical approach to enumerative questions was started by G. Mikhalkin [22, 23] who proved in 2002 the correspondence theorem concerning enumeration of curves in toric surfaces. We formulate here Mikhalkin's correspondence theorem only in the case of the projective plane. For any positive integer d and any non-negative integer g, denote by $N_d(g)$ the number of irreducible curves of degree d and genus g which pass through $3d - 1 + g$ given points in general position in the complex projective plane. Mikhalkin's correspondence theorem states in particular that *the number of irreducible tropical curves* (*counted with appropriate multiplicities*) *of degree d and genus g which pass through $3d-1+g$ points in general position in the tropical projective plane is equal to $N_d(g)$*.

In the talk, we mention various versions of this theorem which are due to G. Mikhalkin [22, 23, 24, 26, 27], E. Shustin [32, 33, 34], T. Nishinou - B. Siebert [28], I. Tyomkin [35], and B. Parker [29]. Mikhalkin's correspondence theorem and its modifications allow one to calculate Gromov-Witten type invariants in many situations. Expanding the tropical correspondence is an active topic of the current research.

2. Tropical calculation of Welschinger invariants. The Welschinger invariants [36, 37] can be seen as real analogs of genus zero Gromov-Witten invariants and are designed to bound from below the number of real rational curves passing through a given generic real collection of points on a real rational surface. In some cases these invariants can be calculated using the tropical approach (Mikhalkin's correspondence theorem or its modifications). In certain situations (for example, in the case of generic collections of real points on a real toric Del Pezzo surface with non-empty real part and in the case of the plane blown up at 4, 5 or 6 real points in general position), the tropical approach leads to a proof of positivity of

Welschinger invariants and their logarithmic asymptotic equivalence with genus zero Gromov-Witten invariants [5, 15, 16, 17, 19, 20].

The tropical approach can be used as well to calculate Welschinger invariants of $\mathbb{P}^3$ (see [6]). Other applications in real enumerative geometry include, for example, the study of sharpness of known upper and lower bounds for the number of real solutions in various enumerative problems [30, 7, 3, 4].

3. Tropical enumerative problems. Tropical enumerative geometry was initiated by G. Mikhalkin [22, 23] who proposed the so-called *lattice paths algorithm* for enumeration of tropical curves. Other important steps in the development of tropical enumerative geometry are the results on

- tropical moduli spaces (G. Mikhalkin [24, 25, 26], A. Gathmann - M. Kerber - H. Markwig [10], M. Herold [14]),
- tropical intersection theory (G. Mikhalkin [24, 26], L. Allerman - J. Rau [1], K. Shaw [31], G. François - J. Rau [9]),
- recursive formulas for tropical invariants (A. Gathmann - H. Markwig [11, 12], I. Itenberg - V. Kharlamov - E. Shustin [18, 19], A. Arroyo - E. Brugallé - L. Lopez de Medrano [2]).

We discuss here in more details certain results concerning recursive formulas. Several recursive formulas known is the complex algebraic world (Kontsevich's formula [21] for genus zero Gromov-Witten invariants of $\mathbb{P}^2$ and Caporaso-Harris formula [8] for relative Gromov-Witten invariants of $\mathbb{P}^2$) were translated to the tropical language by A. Gathmann and H. Markwig (see [11, 12]). A real version of the Caporaso-Harris formula in the tropical setting was proposed in [18]. The formula involves a series of relative tropical Welschinger-type invariants that can be seen as real tropical analogs of relative Gromov-Witten invariants and gives a possibility to calculate purely real Welschinger invariants (*i.e.*, Welschinger invariants in the situation where all the chosen points are real) of toric Del Pezzo surfaces equipped the standard real structure. A similar formula was proved by A. Arroyo, E. Brugallé, and L. Lopez de Medrano [2] in the case of Welschinger invariants counting real rational curves through real configurations containing pairs of imaginary points. A combinatorial proof of the invariance of the tropical Welschinger numbers appearing in [2] was proposed by A. Gathmann, H. Markwig, and F. Schroeter [13] who introduced for this purpose the concept of *broccoli curves*.

The technique of *floor diagrams* for enumeration of tropical objects was developed by E. Brugallé and G. Mikhalkin [6]. This technique is very close to the Caporaso-Harris approach and is the most efficient known way to calculate Gromov-Witten type invariants and Welschinger invariants in the tropical world.

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SYZ transformations for toric varieties

Naichung Conan Leung

In physics, the string theory on a space X can be loosely regarded as the quantum mechanics on the space of loops in X. There are two different sectors of string theory, called the *A-model* and *B-model*. Mathematically, A- and B-models correspond to symplectic and complex geometries respectively. Mirror symmetry asserts that these two seemingly very different geometries are equivalent to each other, but on different manifolds. Kontsevich [3] formulates mirror symmetry as an equivalence between categories. More precisely, Kontsevich's *homological mirror symmetry* (HMS) conjecture says that the (derived) Fukaya category of Lagrangians in X is equivalent to the derived category of coherent sheaves on Y, and vice versa.

The important question remains as to why such an amazing duality exists. Is there an explicit transformation that interchanges these two kinds of geometries?

In 1996, Strominger-Yau-Zaslow (SYZ) [7] has a ground breaking proposal which says that mirror symmetry is a form of Fourier transformation, called *T-duality*. Namely, when X and Y are mirror Calabi-Yau manifolds, then

(i) X should admit a special Lagrangian T^n-fibration and Y is the dual torus fibration, and
(ii) there is a fiberwise Fourier transformation which interchanges the symplectic geometry (resp. complex geometry) of X with the complex geometry (resp. symplectic geometry) of Y.

In this talk, I will give a heuristic reasoning for the SYZ proposal. This proposal has been tested successfully [6] in the semi-flat setting. For instance when the symplectic manifold X is given as $X = T^*B/\Lambda^*$, where Λ is a fiberwise lattice in the tangent bundle TB, with canonical symplectic form $\omega_X = \sum_j dx^j \wedge dy_j$. Its mirror Y is given by $Y = TB/\Lambda$ with canonical complex coordinate $z^1, \cdots, z^n$ with $z^j = x^j + iy^j$ and holomorphic volume form $\Omega_Y = \prod_j \left(dx^j + idy^j\right)$. Then

$$\Omega_Y = \int_{T^*} e^{\omega_X} \cdot e^{i\sum_j dy^j \wedge dy_j},\ e^{\omega_X} = \int_T \Omega_Y \cdot e^{-i\sum_j dy^j \wedge dy_j}.$$

This can be naturally interpreted as a Fourier-Mukai transformation. We will combine this with the Fourier transformation to define the SYZ transformaton

$\mathcal{F}$ which transforms differential forms on $T \times \Lambda$ to those on $T^* \times \Lambda^*$ on each Lagrangian fiber. Notice that $T \times \Lambda$ is the space of geodesic (or affine) loops in T inside the loop space of T, i.e. $T \times \Lambda = \mathcal{L}_{\min} T \subset \mathcal{L}T$, and the loop space certainly plays important roles in string theory. We are going to describe how such a transformation $\mathcal{F}^{SYZ}$, called *SYZ transformation*, interchanges symplectic and complex geometries, with quantum corrections included, in the toric case.

When X_Δ is a Fano toric variety with the open orbit $X \simeq (\mathbb{C}^\times)^n$. Symplectically, $X = T^*B/\Lambda^*$ and we denote by $\tilde{X} = X \times \Lambda^* \subset \mathcal{L}X_\Delta$ the space of fiberwise geodesics/affine loops in X_Δ. On $\tilde{X}$, we consider $\tilde{\omega}_X = \omega_X + \Psi$, where Ψ is a generating function which counts holomorphic disks with boundaries lying on fiber Lagrangians. In [2], we show that (i) The SYZ transformation carries the corrected symplectic structure on $\tilde{X}$ to the holomorphic volume form of the pair (Y, W): $\mathcal{F}^{SYZ}\left(e^{\omega_X + \Psi}\right) = e^W \Omega$. (ii) $\mathcal{F}^{SYZ} : QH^*(X_\Delta) \to Jac(W)$ gives an isomorphism between the quantum cohomology of X_Δ with the Jacobian ring of the superpotential W. These are proven by describing holomorphic spheres as suitable gluing of holomorphic disks.

Next we look at toric Calabi-Yau manifolds, say $X_\Delta = K_{\mathbb{P}^{n-1}}$. Instead of the toric fibration, we consider another Lagrangian fibration on X_Δ which is constructed by Gross and Goldstein independently. The affine structure on the base B is the upper half space but there are interior singular points lying on a hyperplane called the *wall*. As a point moves across the wall, the (virtual) number of holomophic disks bounded by the corresponding Lagrangian fiber jumps. This is called the *wall-crossing phenomenon*, as have been studied by Auroux.

Because of the presence of the wall in B, one needs to apply SYZ dual fibration construction on each connected component in the complement of the wall, with quantum corrections included. Then the wall-crossing formula let us glue the resulting pieces together to obtain a complex manifold Y (see Chan-Lau-Leung [1] for details). For instance, in the case of $X = K_{\mathbb{P}^2}$, we have

$$Y = \left\{ (z, w, u, v) \in \left(\mathbb{C}^\times\right)^2 \times \mathbb{C}^2 : uv = h(q) + z + w + \frac{q}{zw} \right\},$$

which belongs to the mirror family as constructed by Hori-Iqbal-Vafa from physical considerations.

The SYZ construction naturally gives us a map from the symplectic moduli space of X to the complex moduli space of Y. We call this the *SYZ map* f^{SYZ}. In [1], we indicate that the SYZ map coincides with the mirror map f^{mirror} for certain toric Calabi-Yau threefolds of the form K_Z. This explains the mysterious integrality property of the mirror map: its coefficients come from the counting of holomorphic disks with Lagrangian fibers boundary conditions.

The main step in proving these results is to compute certain *open Gromov-Witten* invariants in an *obstructed* situation. In this talk, I will explain how to relate these open GW invariants with closed GW invariants and how to compute them via algebraic geometry.

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Enumeration of Tropical Curves in Tropical Surfaces

Erwan Brugallé

The goal of this talk is to discuss tropical enumerative geometry of non-singular tropical surfaces, and its relation to real and complex enumerative geometry.

Tropical enumerative geometry in $\mathbb{R}^2$ has been developed by G. Mikhalkin in [Mik05]. In this celebrated paper, Mikhalkin proved a Correspondence between tropical curves in $\mathbb{R}^2$ of a given genus and Newton polygon passing through a generic configuration of (the expected number of) points on one side, and complex and real algebraic curves in $(\mathbb{C}^*)^2$ of the same genus and Newton polygon passing through some special configuration of points on the other side.

In the enumeration of tropical curves in a general non-singular tropical surface X, one sees immediately that tropical (i.e. purely combinatorial) enumerative problems can have a space of solutions of the wrong dimension, even in very simple cases. The reason for that is that some combinatorial types of tropical curves in X have a dimension bigger than the expected one (more precisely, their image by the evaluation map has dimension bigger than the expected one). Hence, it is necessary to find combinatorial properties fulfilled by *approximable* tropical curves in X, in addition to the balancing condition. In collaboration with G. Mikhalkin ([BMb]) on one hand and K. Shaw ([BS]) on the other hand, I obtained some local combinatorial obstructions to such an approximation. The strategy to obtain these obstructions is to use tropical intersection theory and its relation to classical intersection theory.

These local obstructions are enough to reduce the dimension of the moduli space of tropical curves to the expected one in some simple cases. Namely, when X is $\mathbb{R}^2$ modified along a tropical non-singular rational curve. As an application, I managed to work out tropically the deformation of the Hirzebruch surface Σ_{n+2} to Σ_n and to relate their enumerative invariants (joint with H. Markwig, [BMa]), and to enumerate real and complex algebraic curves in $\mathbb{C}P^2$ by specializing the points of the configuration to a conic ([Bru]).

In this latter work, working in the standard tropical projective plane modified along a non-singular conic, I found again some results from [Wel] and I computed enumerative invariants of the projective plane blown up in 6 points lying on a conic. Using the work of R. Vakil ([Vak00]), I could then deduce Gromov-Witten and Welschinger invariants of $\mathbb{C}P^2$ blown up in 6 generic real points.

Example. Let us denote by S the projective plane blown up at 6 real points, by L the pullback of a line not passing through the blown up points, by $E_1, \ldots, E_6$ six disjoint (-1)-curves disjoint form L, and by $W_S(6,2,2,2,2,2,2)$ the purely real Welschinger invariant of S of class $6L - 2\sum_{i=1}^{6} E_i$. Then

$$W_S(6,2,2,2,2,2,2) = 1000.$$

Note that purely Weslchinger invariants (i.e. for configurations of real points) of the surface S were also computed independently and by other methods by I. Itenberg, V. Khalramov and E. Shustin.

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Spectral curves and tropical geometry

Marcos Mariño

There are two contexts in modern mathematical physics in which algebraic curves play an important role:

(1) In local mirror symmetry, the mirror to a toric Calabi–Yau geometry is encoded in an algebraic curve $H(U,V) = 0$ in $\mathbb{C}^* \times \mathbb{C}^*$.
(2) In random matrix theory, the large N limit is encoded in a curve $H(x,y) = 0$ usually called the *spectral curve*.

These two contexts are not unrelated, since the spectral curve of some matrix models is a mirror curve. For example, the Chern–Simons matrix model introduced in [3] has as its large N limit the mirror curve to the resolved conifold

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1.$$

In recent years, it has been found that some quantities in $U(N)$ supersymmetric quantum field theories reduce to matrix integrals. The large N limit of these models is encoded sometimes in spectral curves closely related to the mirror curves appearing in local mirror symmetry. For example, the spectral curve of the matrix integral describing the partition function of ABJM theory on $\mathbb{S}^3$ [1] (which a particular example of supersymmetric Chern–Simons-matter theories) is the mirror curve to local $\mathbb{P}^1 \times \mathbb{P}^1$.

A natural question is then what is the meaning of the *tropical limit* of these curves. In the case of local mirror symmetry, it seems to be a not very interesting limit, since strictly speaking the contribution of non-trivial holomorphic maps goes to zero. However, when the spectral curve is related to the large N limit of a Chern–Simons-matter theory, this limit is nothing but the *strong coupling limit*, and many results coming from tropical geometry can be used to derive interesting results for these theories [2].

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Relative Open Gromov-Witten Invariants

Rareş Răsdeaconu

(joint work with Jake Solomon)

We introduce a theory of relative open Gromov-Witten invariants. This theory counts $J-$holomorphic disks with Lagrangian boundary in symplectic 4-manifolds endowed with an anti-symplectic involution with non-empty fixed locus. The disks are subject to tangency conditions with an invariant smooth divisor, also with non-empty fixed locus.

The complex enumerative geometry benefitted in the 90's from the new perspective of the Gromov-Witten theory and many classical old problems have been elegantly solved. Meanwhile, in the real enumerative geometry many basic questions were wide open. For example, while the number of complex rational plane curves was found in arbitrary degree [KM94], the existence of a real rational plane quartic through 11 generic real points was still an open problem. In 2003, Welschinger [W05] introduced in the mainstream the notion of *real symplectic manifolds* and his numerical invariants in dimension four and six, counting real rational curves with signs. His theory developed in several directions. In one direction, the Welschinger invariants were interpreted in tropical geometry and the existence question of plane real rational curves was given a positive answer in arbitrary degree. A second approach, via the symplectic field theory, was undertaken by Welschinger [W07] who provided precise computations for some of his invariants. The second author [S06] interpreted the Welschinger invariants as open Gromov-Witten invariants, and extended their definition to six-dimensional real Calabi-Yau manifolds. Together with Pandharipande and Walcher, he proved mirror symmetry for the real quintic threefold [PSW08]. Also, the second author found an analog of the WDVV equation in open Gromov-Witten theory [S]. In particular, this equation leads to recursive formulae computing the Welschinger invariants for plane real rational curves.

Despite the recent advances in the study of the real enumerative invariants, many foundational aspects are still to be settled. However, the interpretation of the Welschinger's invariants as open Gromov-Witten invariants suggests a parallelism with the Gromov-Witten theory which nowadays is considerably more developed. We are trying to fill some of the gaps by developing a theory of *relative open Gromov-Witten invariants* for four-dimensional real symplectic manifolds, in analogy with the relative Gromov-Witten theory [IP03, LR01]. The goal is to extract numerical invariants responsible for the counting of pseudo-holomorphic disks subject to tangency conditions with respect to a smooth real symplectic divisor. The tropical analog of such invariants has already been found in the tropical setting [IKS09] together with an appropriate Caporaso-Harris type formula.

To introduce our results, let (X, ω, ϕ) be *real symplectic 4-manifold*. That is X is a closed differentiable $4-$manifold endowed with a non-degenerate 2-form ω with $d\omega = 0$, and an involution ϕ on X satisfying $\phi^*\omega = -\omega$. We assume $\mathbb{R}X := Fix(\phi) \neq \emptyset$. Let $V \subset X$ be a smooth symplectic divisor, invariant under the real structure ϕ, such $\mathbb{R}V := V \bigcap \mathbb{R}X \neq \emptyset$. Fix $d \in H_2(X, \mathbb{R}X; \mathbb{Z})$ such that $d = -\phi_* d$. We want to define open Gromov-Witten-type invariants of X with respect to V, counting (with signs) pseudo-holomorphic disks of degree d with boundary in $\mathbb{R}X$. These disks are subject to fixed/moving, boundary/interior tangency conditions (TC) with V. We impose that all of the boundary contact points of the disks with the divisor have odd multiplicities.

We define *the moduli space of relative disks* as a subspace of the moduli space of pseudo-holomorphic disk maps with lagrangian boundary conditions. This is done by imposing the vanishing of the normal jets at all of the contact points of the disks with V. The Gromov-compactification of this moduli space comes naturally equipped with a total evaluation map at all of the marked points, including the fixed tangency points. One would like to define relative numbers as the degree of this evaluation map, under a dimension condition.

There are two major difficulties to overcome in defining such invariants which do not occur in the more familiar Gromov-Witten theory. In general, the spaces involved are not orientable. The second big problem consists in overcoming the presence of the codimension one boundary strata introduced in the Gromov-compactification of the moduli spaces of relative stable maps. This issue occurs when trying to prove the independence of the numbers defined of the choices made.

The orientation issue is dealt with by providing a canonical *relative orientation* for the total evaluation map. Recall that a relative orientation of a map $f : M \to N$ is an isomorphism $TM \simeq f^*TN$. The *relative numbers* are defined as the degree of the evaluation map with respect to its relative orientation.

The independence of the numbers defined above of the choices made usually relies on Stokes' theorem. For moduli spaces of disks, this argument must be treated carefully due to the presence of codimension one strata. In the open Gromov-Witten theory, this issue is overcome by using the flipping procedure [S06] which sends one component of a multi-disk map to its conjugate and leaving the other components unchanged. This procedure shows that the codimension one strata of the moduli space come in pairs of opposite relative orientations. The *new phenomenon* which occurs in our relative setting is that the flipping procedure preserves the canonical relative orientation on some codimension one strata. Since the full cancelling of the codimension one phenomena does not occur, the signed counting *depends* on the choices made. A different approach to cancelling is necessary.

For each boundary surviving the flipping procedure, we identify a suitable cartesian product of moduli spaces of relative disks equipped with relative orientations with respect to an evaluation map (and hence an associated relative number) having the same boundary, but with the reversed relative orientation as one of the boundaries. This suggested a combinatorial approach based on gluing various moduli spaces along appropriate codimension one strata. We define a finite connected graph $\mathfrak{G}_{TC} = (\mathcal{V}, \mathcal{E})$ with a distinguished vertex $\mathbf{v_0} \in \mathcal{V}$. The root $\mathbf{v_0}$ represents the moduli space of relative maps with the tangency conditions (TC). The vertices $\mathbf{v} \in \mathcal{V}$ represent the cartesian products of moduli spaces discussed, and the edges $e \in \mathcal{E}$ represent the codimension one phenomenon cancelled. The graph $\mathfrak{G}_{TC}$ is inductively defined. Its main property is that the valence of each vertex is the number of the surviving codimension one strata of the corresponding moduli space. Moreover, to each vertex we can associate a *relative number* $n_v \in \mathbb{Z}$, and a weight $w(v) \in \mathbb{Q}$. The main result is

Theorem [RS10a]. *The numbers*

$$\mathcal{R}_{TC} = \sum_{v \in \mathcal{V}} w(v) n_v$$

are independent of the choices made. Moreover, $\mathcal{R}_{TC}$ *are integers.*
The proof relies on gluing results which will appear in our joint work [RS10b]. The result extends the relative invariants introduced by Welschinger [W06].

In our talk, we described how this graph is constructed for some interesting cases. The first one is the case when there are no tangency conditions imposed, and we recover the original Welschinger invariants. In this case, the graph is linear. In the second case, we impose a third order boundary moving tangency. We will use this example to illustrate the difficulties in constructing the graph in general.

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Degenerations and Wall Crossings

Ludmil Katzarkov
(joint work with C. Diemer and G. Kerr)

Homologocal Mirror Symmetry and its connection with Hodge theory has been studied by many researchers - see references in [KKP]. In this paper we bring this connection to a new prospective - the prospective of degenerations. It goes back to an approach by B. Moishezon and M. Teicher based on so called braid factorizations. A lot of work in this direction was done by Auroux, Donaldson, Yotov and the author - see [ADKY].

In this talk we connect braid factorizations with invariants of categories. Complete details will appear in [DKK] - we outline a program.

We employ the following principle. We start with a manifold X (Fano, CY, general type) embedded in a toric variety Y. All toric degenerations of Y are parameterized by stack Z with a discriminant loci D. Connecting two (or more) degeneration by a path or a simplex in Z we get a master space of degenerations. To such a master space we associate a "master space of categories". The end points of the corresponding simplex relate to categories obtained via birational (cluster) transformations. These "master spaces" produce relations in the group of autoequivalences and as a result ghost sequences and generators - see e.g. [BFK].

We summarize our findings in the following:

Theorem. The combinatorics of $Z \setminus D$ determines spectra of the category.

In particular this leads to the fact that for Riemann surface C

$$\mathrm{Spec}(Fuk(C)) = 1, ..., 4g \ .$$

We also formulate two conjectures:

Conjecture 1. The rational homotopy type of $Z \setminus D$ determines gap of spectra of the category.

Conjecture 2. The “big” loops in $\pi_1(Z \setminus D)$ determine wall-crossings for the category.

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Broccoli curves and the tropical invariance of Welschinger numbers

Andreas Gathmann

(joint work with Hannah Markwig and Franziska Schroeter)

Welschinger invariants of real toric unnodal Del Pezzo surfaces count real rational curves belonging to an ample linear system D and passing through a generic conjugation invariant set $\mathcal{P}$ of $-K_\Sigma \cdot D - 1$ points, weighted with ± 1, depending on the nodes of the curve. It was shown in [Wel03, Wel05] that these numbers are invariant, i.e. do not depend on the choice of $\mathcal{P}$. They can be computed via tropical geometry: one can define a certain count of tropical curves and prove a Correspondence Theorem stating that this tropical count equals the Welschinger invariant [Mik05, Shu06].

It follows from the Correspondence Theorem and the fact that Welschinger invariants are independent of the point conditions that the corresponding tropical count is also invariant of the chosen points.

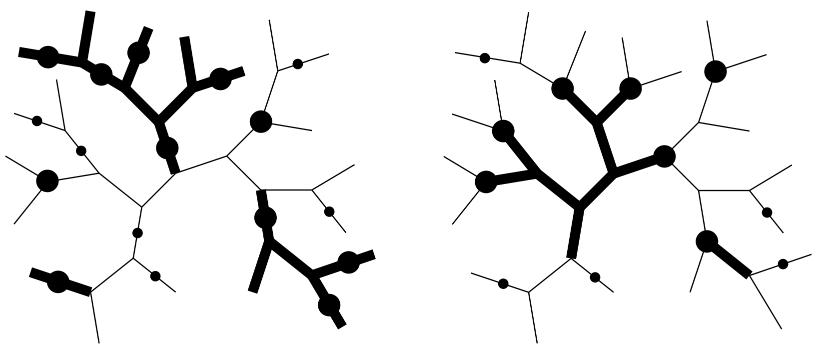
Still, it is interesting to find an argument within tropical geometry that proves the invariance of the tropical numbers. For the case when $\mathcal{P}$ consists of only real points, such a statement follows easily since the corresponding tropical count can be shown to be locally invariant, i.e. invariant around a codimension-1 cone of the corresponding moduli space of curves. In the general case however this is no longer true, and thus there was no known tropical proof for the (global) invariance of the tropical count. Even worse, if we try to generalize the tropical count to relative numbers, i.e. to curves with ends of higher weight, then these numbers are no longer invariant.

The goal of this talk is to introduce broccoli curves, a technique to prove the invariance of tropical Welschinger numbers for real and complex conjugate points entirely within tropical geometry. As an additional result this will allow us to construct corresponding tropical invariants in the relative setting (or more generally for any choice of directions for the ends of the curve). Using this result, we can then establish a Caporaso-Harris formula for rational curves in a much simpler way than in [ABLdM10].

The key idea to achieve this is to modify (and in fact also simplify) the class of tropical curves that we count in order to obtain the invariants. This modification is small enough so that the (weighted) number of these curves through given points remains the same in the toric Del Pezzo case, but big enough so that their count becomes locally invariant in the moduli space.

Let us explain this modification in more detail. For this it is important to distinguish between odd and even edges of a tropical curve, i.e. edges whose weight is odd resp. even. In the picture below we will draw odd edges as thin lines and even edges as thick lines. Moreover, we will draw real points as thin dots and complex points (i.e. those corresponding to a pair of complex conjugate points in the algebraic case) as thick dots.

The tropical curves that are usually counted to obtain the Welschinger invariants — we will call them Welschinger curves — then have the property that each connected component of even edges is connected to the rest of the curve at exactly one point (we can think of such a component as an end tree). Moreover, real points cannot lie on end trees, and each complex point is either on an end tree or on a vertex in the odd part of the curve [Shu06]. Below on the left we have drawn a typical (schematic) picture of such a Welschinger curve.



A Welschinger curve A broccoli curve

We now change this condition slightly to obtain a different class of curves that we call broccoli curves: each connected component of even edges can now be connected to the rest of the curve at several points, of which exactly one is a 3-valent vertex without marking as before (the "broccoli stem"), and the remaining ones are complex points (the "broccoli florets"). The even part of the curve (the "broccoli part") may not contain any points in its interior, whereas away from this part we can have real points on edges and complex points on vertices as before. The picture above on the right shows a typical schematic example of a broccoli curve. Note that, in contrast to Welschinger curves, complex points are always on vertices in broccoli curves.

Broccoli curves have the advantage that their count (with suitably defined multiplicities) is locally invariant in the moduli space, similarly to the situation mentioned above when we count complex curves or Welschinger curves through only real points. Hence counting these curves we obtain well-defined broccoli invariants — even for curves with directions of the ends for which the corresponding Welschinger count would not be invariant of the position of the points.

In addition, we show that in the toric Del Pezzo case broccoli invariants equal Welschinger numbers, thereby giving a new and entirely tropical proof of the invariance of Welschinger numbers. We prove this by constructing certain bridges between broccoli curves and Welschinger curves which show that their numbers must be equal.

It would certainly be very interesting to see if one could prove a Correspondence Theorem for broccoli curves that relates these tropical curves directly to certain real algebraic ones. So far there is no such statement known; in particular there is no algebraic counterpart to broccoli invariants for directions of the ends of the curves when the corresponding Welschinger number is not an invariant.

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Real curve counting via topological strings

Daniel Krefl

(joint work with Johannes Walcher)

As is by now well known, topological string theory constitutes a very powerful machinery to make predictions about enumerative questions in complex geometry. This is so because the closed topological string amplitudes $\mathcal{F}^{(g)}$ are, mathematically, generating functions for the number of (stable) holomorphic maps

$$f : \Sigma^{(g)} \to X \,,$$

where $\Sigma^{(g)}$ denotes a genus g Riemannian surface (the world-sheet of the string) and X the target space, a Calabi-Yau 3-fold. Hence, the (closed) topological string computes Gromov-Witten invariants of X. Physicists have developed powerful techniques to explicitly calculate the amplitudes $\mathcal{F}^{(g)}$ for given X, allowing them to make many non-trivial enumerative predictions. The most notably machineries are the holomorphic anomaly [1] and the topological vertex [2].

Similarly, the open topological string amplitudes $\mathcal{F}^{(g,h)}$ are generating functions for the number of maps

$$f : \Sigma^{(g,h)} \to (X, \mathcal{L}_i)\,,$$

of a genus g Riemannian surface $\Sigma^{(g,h)}$ with h boundary components into the target space X, where the boundaries of the surface map onto special Lagrangian 3-cycles $\mathcal{L}_i \subset X$ (for simplicitly, we only consider a single $\mathcal{L}$ in the following). Physically, such 3-cycles $\mathcal{L}_i$ are important because open strings can end on them, thereby defining non-perturbative objects called D-branes wrapped on $\mathcal{L}_i$. Along the known $\mathcal{L}$'s, there is one general class which shows intriguing features. Namely, the class of $\mathcal{L}$'s defined by the fixed-point locus of an anti-holomorphic involution $I : X \to X$. For example, for the quintic (at the Fermat point) we can find a $\mathcal{L} \cong \mathbb{RP}^3$, and ask (at tree-level, *i.e.*, $g = 0, h = 1$) the real enumerative question of how many maps of disks at given degree ending on the real locus exist. Based on an open string (or real) generalization of mirror symmetry, a first prediction has been put forward in [3] and latter verified and discussed in [4]. Similar predictions have been made for the other three 1-parameter Calabi-Yau hypersurfaces in (weighted) projective space in [5, 6].

An implicit result of these works was that disks of even degree do not contribute. An explanation was latter offered in [7], where the notion of 'topological tadpole cancellation' was introduced. Physically, tadpole cancellation refers to the fact that a D-brane carries a charge and the total charge with compact support has to cancel. The for the cancellation needed opposite charges are provided by O-planes, which arise due to an orientifold projection. Orientifold projection refers to modding out the theory by the simoultaneous action of an anti-holomorphic involution $\sigma : \Sigma^{(g)} \to \Sigma^{(g)}$ on the world-sheet and an anti-holomorphic space-time involution I. Hence, we have not only a D-brane on our $\mathcal{L}$ defined via I, but also an O-plane, which together form a chargeless object, sometimes called an O^0-plane. But this indicates that we should consider in our open topological string theory, and hence also in our counting of maps, all real structures, that is, we should count maps f which are equivariant with respect to the action of σ and I, and sum over the real structures of the world-sheet. Hence, the well-defined counting problem is actually the sum of maps

$$f_\chi : \Sigma^{(\chi)} \to (X/I, \mathcal{L})\,,$$

with fixed χ. Here $\Sigma^{(\chi)}$ denotes a Riemannian surface with Euler number χ. In particular, $\Sigma^{(\chi)}$ may be unoriented (that is, has cross-caps). As before, the boundaries are required to end on $\mathcal{L}$. At tree-level, this means that we have to take the sum of disks and cross-caps. Mathematically, this incorporates that as we vary the complex structure of X, we may loose disks and gain cross-caps, and vice versa. Hence, to have an invariant count, we have to add both. If we choose signs appropriately, this leads to the observed absence of even degree disks due to a cancellation with cross-caps. In physics, there are indications that this choice of sign is the only consistent one. Mainly, because for this choice of sign one can find a multi-covering formula yielding integer invariants, corresponding to the count of certain BPS states. However, mathematically it is not obvious that this sign choice is a necessity, as physics indicates, and it would be interesting to understand this better from a mathematical point of view.

Beyond tree-level, things become even more interesting. For example, at 1-loop we have now to consider the sum of the Annulus, Möbius-strip and Klein-bottle. Physicists have developed techniques to calculate the individual number of maps f_χ for general χ via a generalization of Kontsevich's work [8] to include the unoriented sector [7, 9], and also to directly calculate the sum over the real structures via generalizing the holomorphic anomaly [7] and via an extended (real) version of the topological vertex [9, 10]. There as the first approach and the real version of the topological vertex involve certain choice of signs to achieve cancellations similar as described at tree-level above, the holomorphic anomaly approach gives a basically unique result. However, what has been found is that the signs can be chosen in a way that all three calculations yield the same results, *i.e.*, the same enumerative predictions.

The upshot is, physics provides us with very efficient techniques to explicitly make real enumerative predictions. In particular, the most basic predictions we can infer from physics is that only the sum over all real structures gives a well-defined counting problem and that there appears to be a single consistent sign choice, perhaps leading to a mathematical notion of 'tadpole cancellation' in real enumerative geometry.

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Local obstructions to approximating tropical curves in surfaces

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(joint work with Erwan Brugall'e)

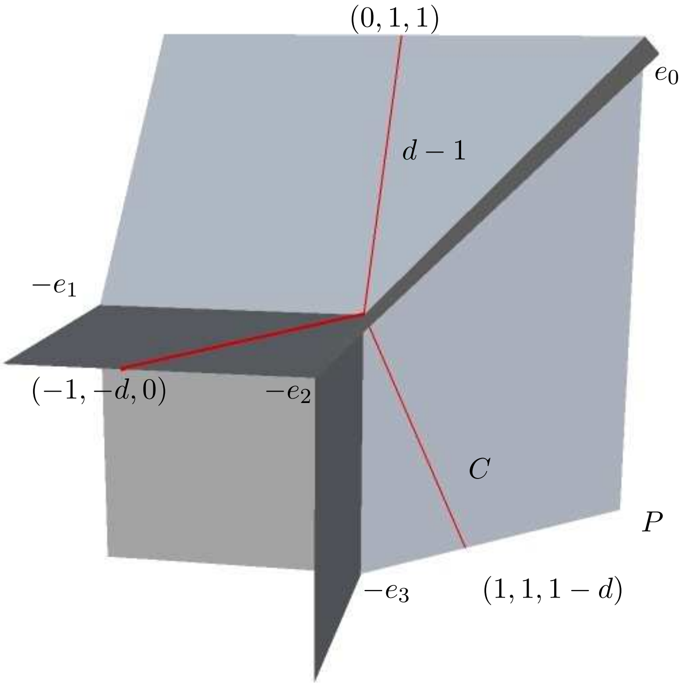

Figure 1. The tropical hyperplane $P \subset \mathbb{R}^3$ containing a trivalent fan tropical curve C of degree d.

For simplicity, we will only consider fan tropical curves in the standard tropical plane $P \subset \mathbb{R}^3$. This tropical plane, P is the tropical hypersurface defined by "$x + y + z + 1$" $= \max\{x, y, z, 0\}$ see Figure 1. In addition throughout we will fix $\mathcal{P} \subset (\mathbb{C}^*)^3$ the complex plane given by the equation $x + y + z + 1 = 0$. We remark that $\mathcal{P}$ is $\mathbb{C}P^2$ minus four lines. Then $\mathcal{P}$ *approximates* P in the sense that $\lim_{t\to\infty} \mathrm{Log}_t(\mathcal{P}) = P$. A fan tropical curve C contained in P is a tropical curve with a single vertex which is also the vertex of P. We are concerned with the following question:

Question. *Given a fan tropical curve $C \subset P$ does there exist an irreducible algebraic curve $\mathcal{C} \subset \mathcal{P}$ approximating C, i.e.*

$$\lim_{t\to\infty} Log_t(\mathcal{C}) = C$$

and for each edge of C the natural weight from Section 6 of [2] *is equal to the weight of the edge of the curve?*

It is known that the answer is not always positive. As a first example, Vigeland exhibited families of tropical lines on generic tropical surfaces in $\mathbb{R}^3$ of degree greater than two [6]. By an integer affine linear map these lines in surfaces can be transformed to fan curves in the plane P. From complex geometry it is known that these families of lines on surfaces cannot be approximated. However, until recently it was not known how to forbid these curves based only on the tropical data. In [1] the authors provide some necessary conditions to approximating tropical curves contained in P and an affine hyperplane which can rule out some, but not all, of Vigeland's forbidden curves. In [5] more examples of non-approximable curves in P are given, here the reasons for ruling out the curves come from tropical intersection theory. Moreover, conditions for approximating curves using the Riemann-Hurwitz formula have been given by Brugallé and Mikhalkin.

The above question is also of broader interest than just lifting curves in P. The standard tropical plane is one of the local models for tropical surfaces, for example, all smooth tropical surfaces in $\mathbb{R}^3$ are locally P up to a integer affine transformation. A curve which is not everywhere locally approximable in a surface cannot be globally approximated. However, local approximability still does not imply global approximability. The general conditions to approximating curves in P generalise to other local models of smooth tropical surfaces, in this talk we stick to the tropical plane P purely for simplicity.

To tackle the problem of local approximation we invoke two tools of complex geometry, intersection with the Hessian curve and the adjunction formula. The main tool allowing us to translate to the tropical world is tropical intersection theory from [5]. At the end we return to the case of fan tropical curves $C \subset P$ and contained in an affine plane as considered in [1].

Intersection with the Hessian curve.

Given a curve $\mathcal{C} \subset \mathbb{C}P^2$ defined by a homogeneous polynomial $P(x, y, z)$ of degree $d \geq 3$, the Hessian curve $\mathcal{H}_{\mathcal{C}}$ is the zero set of the degree $3(d-2)$ polynomial $\det(\text{Hess}(P))$. If $\mathcal{C}$ does not have a component which is a line then $\mathcal{C}$ and $\mathcal{H}_{\mathcal{C}}$ intersect in $3d(d-2)$ points counted with multiplicity. Suppose $[1 : 0 : 0] \in \mathcal{C} \cap \mathcal{H}_{\mathcal{C}}$, the Newton polytope for $\mathcal{C}$ with respect to coordinates x and y gives a lower bound for the multiplicity of the intersection at $[1 : 0 : 0]$. As there is a duality between the fan tropical curve and Newton polytope this bound can be expressed in terms of some unbounded rays of the tropical curve. As mentioned above, $\mathcal{P}$ can be viewed as $\mathbb{C}P^2$ minus four lines, the intersections of which yield exactly six points. For each of these six points we are able to extract from the tropical curve C a lower bound on the intersection of $\mathcal{C}$ and $\mathcal{H}_{\mathcal{C}}$ where $\mathcal{C}$ is a potential approximation of C. The sum of the six multiplicities must be less than $3d(d-2)$. We translate this condition to the level of the tropical curve to obtain an inequality with terms involving the degree and tropical self intersection of the curve along with the weights of edges, but the full formula is too technical to be included here. By applying this formula we are able to forbid all members of Vigeland's families of lines on surfaces of degree greater than two.

The adjunction formula.

If C is approximated by a smooth embedded curve $\mathcal{C}$ then we may also ask about the genus of a parameterisation of $\mathcal{C}$. Using the tropical intersection product given in [5] it is possible to translate the classical adjunction formula to the following formula, where C^2 is the tropical self intersection, $deg(C)$ is the projective degree of C and w_E denotes the weight of an edge $E \subset C$.

Theorem. *If $C \subset P$ is approximated by an embedded irreducible curve $\mathcal{C} \subset \mathcal{P}$ which is parameterised by $f : \mathcal{S} \longrightarrow \mathcal{P}$ then*

$$g(\mathcal{S}) \leq C^2 + deg(C) - \sum_{E_i \subset C} w_{E_i} + 2.$$

In particular, if the right hand side is negative then C is not approximated by any irreducible curve.

Classification of fan curves in $P \cap H$.

Now we return to the situation considered in [1]. Here H will denote an affine hyperplane, and the tropical fan curve C is will be contained in $P \cap H$. As mentioned above Bogart and Katz provide necessary conditions to approximating such curves. After strengthening their conditions and constructing some curves we obtain a complete classification of such tropical curves. Once again the main tool allowing us to extend the conditions is the tropical intersection product.

Theorem. *An tropical curve $C \subset P \cap H$ of degree d is approximable by a reduced and irreducible complex algebraic curve $\mathcal{C} \in \mathcal{P}$ if and only if C is one of the following:*

(1) *C is the stable intersection of P and H (see* [4] *or* [3]*);*
(2) *C is the curve depicted in Figure 1 up to symmetry of P.*

In case (2) when $d = 1$ the curve is an affine line (not trivalent) which bisects two faces of P. In general, the curve from case (2) is unique in each degree, moreover it is real, rational and has $d + 1$ punctures. Note that $C^2 = 0$ in case (1), and $C^2 = -1$ in case (2).

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Enumerating real rational curves with tangency conditions

Michael Polyak
(joint work with Sergei Lanzat)

A classical problem in enumerative geometry is the study of the number of algebraic curves of degree d and genus g, passing through some number of points in the affine or projective plane. The major difficulty in real enumerative problems is that such numbers usually depend on a configuration of geometrical objects. A natural way to overcome this difficulty is to count curves with signs and multiplicities so that the resulting algebraic numbers do not depend on a configuration. For rational curves such signs were proposed by J.-Y. Welschinger [1]. Welschinger's sign w_C of a real rational curve C is defined as $w_C = (-1)^{m(C)}$, where $m(C)$ is the number of solitary points of C. The next step is to ask about the number of algebraic curves passing through some number of points and tangent to some given curves. J.-Y. Welschinger [2] considered projective curves in $\mathbb{RP}^2$ passing through a generic set $\mathcal{P}$ of $3d-2$ points and tangent to a non-oriented smooth simple zero-homologous curve. In the present work we consider tangency with generic immersed oriented curves in $\mathbb{R}^2$. In contrast with [2], where the author used 4-dimensional symplectic geometry and hard-core techniques from the theory of moduli spaces of pseudo-holomorphic curves, we use down to earth classical tools of differential topology. In this way we also get a clear geometric interpretation of Welschinger's number w_C as the orientation of a certain surface in the manifold of oriented contact elements in $\mathbb{R}^2$, which parameterizes real rational curves passing through $\mathcal{P}$. We interpret the desired number of curves as a certain intersection number; the main claim follows from different ways of its calculation. Finally, we relate the dependence on a chosen configuration to the theory of finite type invariants. We count rational nodal curves with certain signs and add correction terms coming from degenerate cases of nodal, reducible and cuspidal curves. The desired number of curves is interpreted as a certain intersection number; the main claim follows from different ways of its calculation.

Main results. In what follows we only consider real rational algebraic curves of degree d in $\mathbb{R}^2$. Let $\mathcal{P} = \{p_1, \ldots, p_{3d-2}\}$ be a $(3d-2)$-tuple of points in $\mathbb{R}^2$ in general position. Define the following sets $\mathcal{C}_\mathcal{P}$ and $\mathcal{R}_\mathcal{P}$ in $\mathbb{R}^2 \setminus \mathcal{P}$ and multiplicities ι_p:

- (i) Let $p \in \mathcal{C}_\mathcal{P}$, if there exists an irreducible curve C, which passes through $\mathcal{P}$, and has a cusp at p; define $\iota_p = -w_C$.
- (ii) Let $p \in \mathcal{R}_\mathcal{P}$, if there exists a reducible curve $C = C_1 \cup C_2$ with $p \in C_1 \cap C_2$, which passes through $\mathcal{P}$; define $\iota_p = w_C$.

For $p \in \mathcal{P}$ denote by $\mathcal{D}(p)$ the set of all irreducible nodal curves, which pass through $\mathcal{P}$ and have a crossing node at p; for $p \in \mathcal{P}$ we define $\iota_p = -W_d + 2 \sum_{C \in \mathcal{D}(p)} w_C$.

Denote $\mathfrak{S} = \mathcal{P} \cup \mathcal{C}_\mathcal{P} \cup \mathcal{R}_\mathcal{P}$.

Let Γ be a generic immersed oriented curve in $\mathbb{R}^2$ in general position w.r.t. $\mathcal{P}$. Denote by $\mathcal{M}_d(\mathcal{P},\Gamma)$ the set of nodal curves passing through $\mathcal{P}$ and tangent to Γ. For each $C \in \mathcal{M}_d(\mathcal{P},\Gamma)$ we define the sign ε_C of C by $\varepsilon_C = w_C \cdot \tau_C$. Here w_C is the Welschinger's sign of C, and τ_C is a sign of tangency of C with Γ, defined as shown in Figure 1.

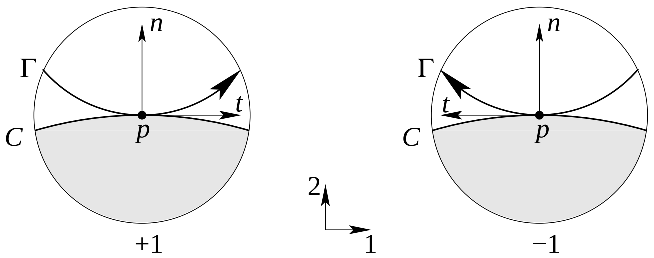

FIGURE 1. Signs of tangency τ_C.

Let $N_d(\mathcal{P},\Gamma)$ be the algebraic number $N_d(\mathcal{P},\Gamma) = \sum_{C\in\mathcal{M}_d(\mathcal{P},\Gamma)} \varepsilon_C$ of nodal curves passing through $\mathcal{P}$ and tangent to Γ.

Theorem 1. Let $\mathcal{P} = \{p_1,\dots,p_{3d-2}\} \subset \mathbb{R}^2$ and Γ be an immersed oriented curve in $\mathbb{R}^2$, all in general position. Then

$$N_d(\mathcal{P},\Gamma) = 2\left(W_d \cdot \operatorname{ind}(\Gamma) + \sum_{p\in\mathfrak{S}} \iota_p \cdot \operatorname{ind}_p(\Gamma)\right). \tag{1}$$

The idea of the proof. Consider a solid torus $M = \mathbb{D}^2 \times \mathbb{S}^1$, where $\mathbb{D}^2$ is a sufficiently large closed disk containing $\mathfrak{S}$. We show that the number $N_d(\mathcal{P},\Gamma)$ in Theorem 1 is the intersection number $I(L,\overline{\Sigma};M)$ of an oriented smooth curve L with a compactification $\overline{\Sigma}$ of an open two-dimensional surface Σ in M. The surface Σ is constructed as follows. For each $p \in \mathbb{D}^2 \setminus \mathfrak{S}$, we use a contact element (line) of curves passing through $\{p\} \cup \mathcal{P}$ to get Σ as a lift of $\mathbb{D}^2 \setminus \mathfrak{S}$ into M. Lifting Γ into M in a similar way we get L. Welschinger's sign w_C gives rise to the orientation on Σ, and the orientation of Γ defines the orientation of L.
In order to define the intersection number, we compactify Σ to get a compact surface $\overline{\Sigma}$ with boundary by blowing up punctures $\mathfrak{S}$ on $\mathbb{D}^2$. Due to generality of $(\mathcal{P},\Gamma)$, L transversally intersects $\overline{\Sigma}$ in a finite number of points. Each point $(p,\xi) \in L \pitchfork \overline{\Sigma}$ corresponds to a curve passing through $\mathcal{P}$ and tangent to Γ. We prove that the local intersection number $I_{(p,\xi)}(L,\overline{\Sigma};M)$ equals to $\tau_C \cdot w_C$, and thus $N_d(\mathcal{P},\Gamma) = I(L,\overline{\Sigma};M)$. Now to get the RHS of (1) we use the homological interpretation of the intersection number. We homotope Γ in the class of immersions to $\Gamma' = \operatorname{ind}(\Gamma)\cdot T$, where T is a small circle near $\partial\mathbb{D}^2$. Hence $[\Gamma]-[\Gamma'] = \partial K$ for some 2-chain K. Then

for the lifts L' and $\mathcal{K}$ of Γ' and K, respectively, into M we have $[L] - [L'] = \partial\mathcal{K}$ in $C_1(M;\mathbb{Z})$, and hence

$$I(L, \overline{\Sigma}; M) = I(L', \overline{\Sigma}; M) + I(\partial\mathcal{K}, \overline{\Sigma}; M).$$

It is easy to verify that $I(L', \overline{\Sigma}; M) = 2W_d \cdot \mathrm{ind}(\Gamma') = 2W_d \cdot \mathrm{ind}(\Gamma)$. Finally, to complete the proof we show that

$$I(\partial\mathcal{K}, \overline{\Sigma}; M) = I(\mathcal{K}, \partial\overline{\Sigma}; M) = 2 \sum_{p\in\mathfrak{S}} \iota_p \cdot \mathrm{ind}_p(\Gamma).$$

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The diffeotopy group of rational or ruled 4-manifolds

Vsevolod Shevchishin

A 4-manifold X is *rational* or *ruled* if it is diffeomorphic to a rational or resp. ruled complex surface, possibly blown-up several times. In particular, $\mathbb{C}P^2$ and $\mathbb{C}P^1 \times \mathbb{C}P^1 = S^2 \times S^2$ are rational, and the product $Y \times S^2$ of a Riemann surface Y with the sphere is ruled. Such manifolds can be characterized from the point of view of the symplectic geometry [5]: A compact symplectic 4-manifold (X, ω) is rational or ruled if and only if it contains a symplectic surface $\Sigma \subset X$ such that $c_1(X) \cdot [\Sigma] > 0$ and Σ is not an exceptional sphere. Further "symplectic" properties of rational or ruled manifolds are [1, 2, 4, 3]: For every symplectic form ω on such X there exists an integrable complex structure J such that ω is a Kähler form for J. For every pair of symplectic forms ω_1, ω_2 on such X with equal cohomology class $[\omega_1] = [\omega_2]$ there exists a diffeomorphism $F : X \to X$ with $F_*\omega_1 = \omega_2$.

The main result of my talk is [6]:

Theorem 1. Let (X, ω) be a rational symplectic 4-manifold and $F : X \to X$ a symplectomorphism which is *homotopically trivial*, ie., acts trivially on the homology group $\mathsf{H}_2(X, \mathbb{Z})$. Then F is isotopic to identity.

The meaning of the result is that the smooth isotopy class of a symplectomorphism of some rational complex surface is determined by its action in homology. It allows to give an almost complete description of the *diffeotopy group* $\Gamma = \Gamma(X)$ of rational 4-manifolds X, ie., the quotient group $\Gamma(X) := \mathcal{D}\!\mathit{iff}(X)/\mathcal{D}\!\mathit{iff}_0(X)$ of all diffeomorphisms of X by the group of isotopies.

Corollary 2. Let (X, ω) be a rational symplectic 4-manifold and Γ_0 the group of isotopy classes of homotopically trivial diffeomorphisms. Then Γ_0 acts *simply transitively* on the set of connected components of symplectic forms having given cohomology class $[\omega_0]$.

The latter result can be formulated as follows: On a rational complex surface there are as many mutually non-isotopic homotopically trivial diffeomorphisms as many mutually deformationally non-equivalent Kähler structures.

Theorem 3. The group Γ_0 remains unchanged under blow-ups. In particular, $\Gamma_0(\mathbb{C}P^2) = \Gamma_0(S^2{\times}S^2) = \Gamma_0(X)$ for every rational 4-manifold X.

The proof of the results is given in the preprint [6]. It contain also the description of the action of the diffeomorphism group $\mathcal{D}\!\mathit{iff}(X)$ on the homology. Here we recall that the intersection form on $\mathsf{H}_2(X,\mathbb{R})$ has Lorentz signature and therefore the set $\mathcal{K} := \{[A] \in \mathsf{H}_2(X,\mathbb{R}) : [A]^2 > 0\}$ is a quadratic cone, called the *positive cone* of X.

Theorem 4. Let X be a rational 4-manifold and Γ_{H_2} the image of the diffeomorphism group $\mathcal{D}\!\mathit{iff}(X)$ in the group $\mathsf{Aut}(\mathsf{H}_2(X,\mathbb{Z}))$. Further, let $\boldsymbol{L}; \boldsymbol{E}_1, \ldots, \boldsymbol{E}_\ell \in \mathsf{H}_2(X,\mathbb{Z})$ be the homology classes of the line and resp. the exceptional spheres with respect to some contraction map $\pi : X \to \mathbb{C}P^2$. Then Γ_{H_2} is generated by reflections with respect to hyperplanes in $\mathsf{H}_2(X,\mathbb{R})$ orthogonal to the classes $\boldsymbol{L} - (\boldsymbol{E}_1 + \boldsymbol{E}_2 + \boldsymbol{E}_3)$, $\boldsymbol{E}_i - \boldsymbol{E}_{i+1}$ with $i = 1, \ldots, \ell - 1$, and $\boldsymbol{E}_\ell$.

Moreover, the action of Γ_{H_2} on the positive cone $\mathcal{K}$ admit a fundamental domain consisting of those classes $[A] \in \mathcal{K}$ which have non-negative intersection with the classes $\boldsymbol{L} - (\boldsymbol{E}_1 + \boldsymbol{E}_2 + \boldsymbol{E}_3)$, $\boldsymbol{E}_i - \boldsymbol{E}_{i+1}$ with $i = 1, \ldots, \ell - 1$, and $\boldsymbol{E}_\ell$.

The meaning of the latter result is as follows. Let X be a rational complex surface and D an ample divisor on X. Then there exists a (holomorphic) contraction map $\pi : X \to \mathbb{C}P^2$ whose exceptional divisor E is the sum $E_1 + \cdots + E_l$ of rational curves with the homology classes $\boldsymbol{E}_1, \ldots, \boldsymbol{E}_\ell$ such that the divisors $\boldsymbol{L} - (\boldsymbol{E}_1 + \boldsymbol{E}_2 + \boldsymbol{E}_3)$, $\boldsymbol{E}_i - \boldsymbol{E}_{i+1}$ have positive intersection with D.

The preprint [6] contains also a description of the diffeotopy group $\Gamma(X) = \mathcal{D}\!\mathit{iff}(X)/\mathcal{D}\!\mathit{iff}_0(X)$ of irrational ruled 4-manifolds X. The main difference from the rational case is appearance of a new differential invariant, *secondary Stiefel-Whitney class* $\tilde{w}_2(F)$ of homotopically trivial diffeomorphisms. The main result is generalized in the following form:

Theorem 5. Let (X, ω) be a ruled symplectic 4-manifold and $F : X \to X$ a symplectomorphism. Then F is isotopic to the identity if and only if F is homotopically trivial, ie., acts trivially on the groups $\mathsf{H}_2(X,\mathbb{Z})$, $\pi_1(X)$, $\pi_2(X)$, and has trivial secondary Stiefel-Whitney class $\tilde{w}_2(F)$.

One also obtains the counterparts of Corollary 2 and Theorems 3, 4.

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Homological Mirror symmetry of toric manifolds

Kenji Fukaya
(joint work with M. Abouzaid, Y.-G. Oh, H. Ohta, and K. Ono)

For any relatively spin Lagrangian submanifold L of a compact symplectic manifold (M,ω) and $\mathfrak{b} \in H^{even}(M;\Lambda_0)$, we ([FOOO1]) associated a set

$$\mathcal{M}_{\text{weak}}(L;\Lambda_0) \subset H^{odd}(L;\Lambda_0)$$

together with a map

$$\mathfrak{PO} : \mathcal{M}_{\text{weak}}(L;\Lambda_0) \to \Lambda_+$$

such that if $b_i \in \mathcal{M}_{\text{weak}}(L_i;\Lambda_0)$ and L_1 is transversal to L_2 we have an operator ∂_{b_1,b_2}

$$\partial_{b_1,b_2} : CF(L_1,L_2;\Lambda) \to CF(L_1,L_2;\Lambda)$$

with

$$\partial^2_{b_1,b_2} = (\mathfrak{PO}(b_1) - \mathfrak{PO}(b_2))id.$$

In particular the homology of ∂_{b_1,b_2}, that is the Floer homology

$$HF((L_1,b_1),(L_2,b_2);\Lambda)$$

is defined if $\mathfrak{PO}(b_1) = \mathfrak{PO}(b_2)$.

Here $CF(L_1,L_2;\Lambda)$ is a Λ vector space of rank $\#L_1 \cap L_2$.

The universal Novikov ring Λ_0 is defined by

$$\Lambda_0 = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{\geq 0}, \lim_{i\to\infty} \lambda_i = +\infty \right\}.$$

The universal Novikov field Λ is its field of fractions. Namely :

$$\Lambda = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lim_{i\to\infty} \lambda_i = \infty \right\} \cong \Lambda_0[T^{-1}].$$

Λ_+ is the maximal ideal of Λ_0. Namely

$$\Lambda_+ = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{>0}, \lim_{i\to\infty} \lambda_i = +\infty \right\}.$$

We defined also a map

$$\mathfrak{p}_* : HF((L,b),(L,b);\Lambda) \to QH_{\mathfrak{b}}(M;\Lambda)$$

which is a deformation of the Gysin homomorphism. (Here the right hand side is $\mathfrak{b}$ deformed quantum cohomology ring.)

Theorem 1. If $a_i \in HF((L_i,b_i),(L_i,b_i);\Lambda)$ with

$$\langle \mathfrak{p}_*(a_1), \mathfrak{p}_*(a_2) \rangle \neq 0$$

then

$$\mathfrak{PO}(L_1, b_1) = \mathfrak{PO}(L_2, b_2).$$

Theorem 2. Let L_i, $b_i \in \mathcal{M}_{\text{weak}}(L_i; \Lambda_0)$. We assume that the union of

$$\mathrm{Im}\,(\mathfrak{p}_* : HF((L_i, b_i), (L_i, b_i); \Lambda) \to QH_{\mathfrak{b}}(M; \Lambda))$$

for $i = 1, \ldots, N$ generates $QH_{\mathfrak{b}}(M; \Lambda)$.

Then for any (L, b) with $HF((L, b), (L, b); \Lambda) \neq 0$ there exists (L_i, b_i), such that

$$\mathfrak{PO}(L, b) = \mathfrak{PO}(L_i, b_i). \tag{1}$$

Moreover $(L.b)$ is 'Floer theoretically equivalent' to a direct summand of the sum of copies of (L_i, b_i) satisfying (1).

There is a version using Hoshchild cohomology of $HF((L_i, b_i), (L_i, b_i); \Lambda_0)$ instead of $HF((L_i, b_i), (L_i, b_i); \Lambda_0)$ itself.

A similar result in the exact case was proved by M. Abouzaid [A].

Theorem 3. Let (X, ω) be a compact toric manifolds such that $QH_{\mathfrak{b}}(X; \Lambda)$ is semi-simple. Then there exists (L_i, b_i) $i = 1, \ldots, B$ (B is the Betti number of X) such that they satisfy the assumption of Theorem 2 and that L_i are toric fiber, $b_i \in H^1(L_i, \Lambda_i)$.

In the case $QH_{\mathfrak{b}}(X; \Lambda)$ is not semi-simple we can still find toric fibers which splits generates Fukaya category of (X, ω) using Hoshchild cohomology version.

Theorem 3 follows from the main result of [FOOO2].

We can use these theorems to show that the Fukaya category of a compact toric manifold (X, ω) is equivalent to the category of matrix factorization of $\mathfrak{PO}$ after taking split generation.

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Gauge Theories: Dessins of Quivering Amoebae

Yang-Hui He

Our beginning point is the class of quantum field theories, obviously of great physical interest, in 3+1-dimensions, with $\mathcal{N} = 1$ space-time supersymmetry and gauge group comprising of a product $\prod_i SU(N_i)$ of special unitary groups. Our ensuing discussions may also be readily generalized to other dimensions and supersymmetries. In particular, we focus on theories whose matter are *bi-fundamental fields*, Φ_{ij}, which are charged under two factors, say $SU(N_i)$ and $SU(N_j)$, and being fundamentals of one and anti-fundamentals of the other, as well as *adjoint fields*, Φ_{ii}, which are in the adjoint representation of a single gauge group factor. Our matter fields, therefore, can be thought of as $N_i \times N_j$ complex matrices (allowing for $i = j$). In addition, there will be a holomorphic and usually polymonomial function called the *superpotential*, W, in terms of the above fields, which governs the interactions in this field theory.

The above structure affords an elegant and graphical encoding: letting each gauge group be represented by a node with label N_i and each field, a directed arrow from the i-th to the j-th node; the critical values of W, obtained by partial derivatives with respective to the various fields, impose formal relations dubbed *F-terms* on the arrows. The resulting directed graph with n_G nodes and n_E edges, together with the F-terms, is called a *quiver gauge theory*. For convenience, we henceforth take all labels N_i to be 1 so that our gauge goup is $U(1)^{n_G}$ and our fields are simply complex numbers. The most fundamental quantity of a field theory is its vacuum. For our purposes, we will define the vacuum $\mathcal{M}$ (also known as a vacuum moduli space because of the generically continuous and non-zero dimensionality) as the *quiver variety* obtained by the (maximal spectrum of) the polynomial algebra formed by the space of loops in the quiver diagram quotiented by the ideal of F-terms.

In string theory, the aforementioned situation is prescribed by the AdS/CFT correspondence, wherein a D3-brane has, on its world-volume, the quiver gauge theory, and $\mathcal{M}$ is an affine Calabi-Yau variety of complex dimension 3 (which is itself a real cone over some real Sasaki-Einstein 5-fold). The prototypical example is, of course, when $\mathcal{M} = \mathbb{C}^3$, where the quiver is the "clover", consisting of a single node labeled N, with 3 adjoining arrows which we denote as x, y, z, together with $W = \mathrm{Tr}(x[y, z])$. Indeed, at $N = 1$, $W = 0$, and $\mathcal{M} = \mathrm{Spec}(\mathbb{C}[x, y, z]) \simeq \mathbb{C}^3$ as required.

When $\mathcal{M}$ is a toric variety (note that $\mathcal{M}$ is non-compact) extraordinary structures exist and the purpose of this talk is draw an intricate web of inter-connections amongst them. We emphasize that the toric case is not merely of academic interest since almost all theories, especially the infinite families, studied in the AdS/CFT context fall into this category. An immediate toric constraint is that to ensure the F-terms give a binomomial ideal, such that each generator is of the form "monomial - monomial", we must have each field appearing in W exactly twice with opposite signs. Moreover, it turns out that for all such theories, $n_G - n_E + n_W = 0$ where n_W is the number of terms in W.

We thus have a remarkable topological condition, dictating that - as suggested by the Euler character of a torus - our quivers should be drawn periodically on a plane. Thus drawn, the F-terms are automatically incorporated as oriented loops, of which, due to the plus or minus sign of the corresponding term in W, there are only two types: counter-clockwise and clockwise. Upon graph-dualization, associating, say, counter-clockwise and clockwise loops to black and white nodes respectively, we arrive at a *dimer model*, a bipartite graph, on T^2, or equivalently a periodic *brane tiling* of the plane (cf. reviews in [2]). Hence, the study of D3-branes probing toric Calabi-Yau threefolds, an old subject dating to the first systematic analysis in [1], is reduced to the examination of dimers on a torus.

The key to dimer models is the so-called *Kasteleyn matrix*, a weighted adjacency matrix $K(z,w)$, whose determinant gives the generating polynomial of perfect matchings. Interestingly, $P(z,w) = \det K(z,w)$ is precisely the Newton polynomial of the toric diagram of $\mathcal{M}$, which, by the Calabi-Yau condition, is here a planar grid of lattice points. One could take two projections [3] of $P(z,w)$, the *amoeba projection* $(z,w) \to (\log|z|, \log|w|)$ and the *alga* or *co-amoeba projection* $(z,w) \to (\arg|z|, \arg|w|)$. The former gives a point-set in $\mathbb{R}^2$ whose "spine" is a tropical curve called (p,q)*-webs* in the physics literature. This is the graph dual of the resolution of the toric diagram of $\mathcal{M}$; moreover, the direction vectors (p_i, q_i) give rise to the matrix $a_{ij} = \det \begin{vmatrix} p_i & q_i \\ p_j & q_j \end{vmatrix}$, which is the anti-symmetrized adjacency matrix of the original quiver diagram. The latter, gives a shaded regions in $[-\pi,\pi)^2 \in \mathbb{R}^2$, and in clean cases, contract to precisely the dimer model diagram.

The mirror picture to the above is also enlightening [3]. The local mirror manifold to $\mathcal{M}$ is given by $P(z,w) = uv$ so that special Lagrangian 3-cycles S_i therein intersect to give $a_{ij} = S_i \circ S_j$ and the T^2 on which the dimer lives is in the T^3-fibre upon which one performs thrice T-duality for mirror symmetry. Furthermore, the zigzag cycles on the dimer can be untwisted - in a Seifert sense - to render a covering of the Riemann surface given by $P(z,w)$.

A final connection suggests itself [4] and adds an elegant number-theoreical touch to our geometrical story. The dimers are naturally interpreted as Groethendieck's *dessins d'enfants* where we form a ramified holomorphic branched cover map from the torus to $\mathbb{P}^1$, with black (resp. white) nodes mapped to 0 (resp. 1) and faces on the dimer, to ∞. The valency, i.e., the number of adjacent edges, of each node is then the ramification index of the pre-image of 0 or 1 while the number of edges per face is twice that of ∞. Specifically, we can assign, to each term of W (corresponding to a node in the dimer), a permutation tuple by associating edges to elements in Σ_{n_E}, the symmetric group on n_E elements. By Riemann existence theorem, this assignment uniquely determines our branched covering. Moreover, Belyi's theorem states that when the ramifications are only over $0, 1, \infty$, as in our case, the T^2 can be realized as an algebraic curve in $\overline{\mathbb{Q}}$. Thus, the elliptic curve with algebraic numbers as coefficients, together with the rational function which prescribes the branched cover over $\mathbb{P}^1$, known as a Belyi pair, completely encaptures the physics of the original quiver gauge theory.

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Wall Crossings in Hurwitz Theory

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(joint work with P. Johnson and H. Markwig)

1. Introduction

Hurwitz theory studies holomorphic maps between Riemann surfaces with specified ramification. Double Hurwitz numbers count covers of $\mathbb{P}^1$ with assigned ramification profiles over 0 and ∞, and simple ramification over a fixed branch divisor.

A systematic study of double Hurwitz numbers in [1] shows double Hurwitz numbers are piecewise polynomial in the entries of the partitions defining the special ramification. In [2], this result was investigated further in genus 0; the regions of polynomiality are determined, and a recursive wall crossing formula for how the polynomials change is obtained. This paper gives a unified approach to these results that strengthens them in several ways - the most important being the extension of the results of [2] to positive genus.

This extended abstract is based on [3].

2. Statement of Results

The double Hurwitz number $H_g(\mathbf{x})$ (where $\mathbf{x} = (x_1, \ldots, x_n)$) counts the number of maps $\pi : C \to \mathbb{P}^1$, where C is a connected, genus g curve and π has profiles $\mathbf{x_0} := \{x_i | x_i > 0\}$ (resp. $\mathbf{x}_\infty := \{x_i | x_i < 0\}$) over 0 (resp. ∞), and simple ramification over $r = 2g-2+n$ fixed other points. The preimages of 0 and ∞ are marked. Furthermore, each cover is counted with weight $1/|\mathrm{Aut}(\pi)|$. Frequently the natural numerical invariant is r, the number of simple ramifications, rather than the genus g. Since these are equivalent by the Riemann-Hurwitz formula (in this case, $r = 2g-2+n$), we will also sometimes use $H^r(\mathbf{x})$ to denote $H_g(\mathbf{x})$ when it makes formulas more attractive.

A ramified cover is essentially equivalent information to a monodromy representation it induces; thus, an equivalent definition of Hurwitz number counts the number of homomorphisms φ from the fundamental group Π_1 of $\mathbb{P}^1 \setminus \{0, \infty, p_1, \dots, p_r\}$ to the symmetric group S_d such that:

- the image of a loop around 0 has cycle type $\mathbf{x_0}$;
- the image of a loop around ∞ has cycle type $\mathbf{x}_\infty$;
- the image of a loop around p_i is a transposition;
- the subgroup $\varphi(\Pi_1)$ acts transitively on the set $\{1, \dots, d\}$.

This number is divided by $|S_d|$, to account both for automorphisms and for different monodromy representations corresponding to the same cover. One can organize this count in terms of graphs as in [4, Lemma 4.1], a fact which is the starting point of our investigation.

Let $\mathcal{H}$ be the hyperplane $\mathcal{H} = \{\sum_i x_i = 0\} \subset \mathbb{R}^n$. We think of H_g (resp. H^r) as a map

$$H_g : \mathcal{H} \cap \mathbb{Z}^n \to \mathbb{Q} : \mathbf{x} \mapsto H_g(\mathbf{x}).$$

Our first result is a new proof of the following theorem in [1]:

Theorem 1. The function $H_g(\mathbf{x})$ is a piecewise polynomial function of degree $4g - 3 + n$.

Our techniques allow us to extend this result and answer a question implicit in the work of Goulden, Jackson and Vakil:

Theorem 2. $H_g(\mathbf{x})$ is either even or odd, depending on the parity of the leading degree $4g - 3 + n$.

Further techniques allow us to extend the results of [2] to all genera. First, we determine the regions on which $H_g(\mathbf{x})$ is polynomial:

Theorem 3. The chambers of polynomiality of $H_g(\mathbf{x})$ are bounded by **walls** corresponding to the **resonance** hyperplanes W_I, given by the equation $W_I = \left\{\mathbf{x}_I = \sum_{i \in I} x_i = 0\right\}$, for any $I \subset \{1, \dots, n\}$.

Finally, we extend the wall crossing formula of [2] to all genera. We will denote the chambers of the resonance arrangement as H-chambers;

Definition 1. Let C_1 and C_2 be two H-chambers adjacent along the wall W_I, with C_1 being the chamber with $x_I < 0$. The Hurwitz number $H^r(\mathbf{x})$ is given by polynomials, say $P_1(\mathbf{x})$ and $P_2(\mathbf{x})$, on these two regions. By a wall crossing formula, we mean a formula for the polynomial

$$WC_I^r(\mathbf{x}) = P_2(\mathbf{x}) - P_1(\mathbf{x}).$$

Note that with the notation $WC_I^r(\mathbf{x})$ there is no ambiguity about which direction we cross the wall. Since $\mathbf{x}$ lies on the hyperplane $\sum_{i=1}^n x_i = 0$, each wall has two possible labels: W_I and W_{I^c} both denote the same hyperplane. We always choose the name so that $\mathbf{x}_I$ is increasing.

We use $H^{r\bullet}(\mathbf{x})$ to denote Hurwitz numbers with potentially disconnected covers. Our main theorem is the following wall crossing formula:

Theorem 4.

$$WC^r_I(\mathbf{x}) = \sum_{s+t+u=r} \sum_{|\mathbf{y}|=|\mathbf{z}|=|\mathbf{x}_I|} (-1)^t \binom{r}{s,t,u} \frac{\prod \mathbf{y}_i}{\ell(\mathbf{y})!} \frac{\prod \mathbf{z}_j}{\ell(\mathbf{z})!} H^s(\mathbf{x}_I, \mathbf{y}) H^{t\bullet}(-\mathbf{y}, \mathbf{z}) H^u(\mathbf{x}_{I^c}, -\mathbf{z})$$

Here $\mathbf{y}$ is an ordered tuple of $\ell(\mathbf{y})$ positive integers with sum $|\mathbf{y}|$, and similarly with $\mathbf{z}$.

Observe that the walls W_I correspond to values of $\mathbf{x}$ where the cover could potentially be disconnected, or where when $x_i = 0$. Crossing this second type of wall corresponds to moving a ramification between 0 and ∞. In the traditional view of double Hurwitz numbers, these were viewed as separate problems: the number of ramification points over 0 and ∞ were fixed separately, rather than just the total number of ramification points. However, Theorem 4 suggests that it is natural to treat them as part of the same problem: the wall crossing formula for $x_i = 0$ is identical to the other wall crossing formulas.

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Tropical geometry and correspondence theorems via toric stacks

Ilya Tyomkin

Introduction

Recently, tropical varieties appeared in various fields of study, such as string theory, mirror symmetry, and enumerative geometry. Roughly speaking, tropical variety (locally) is an integral piece-wise linear polyhedral complex equipped with an integral affine structure. One can also think about tropical varieties as algebraic varieties over the $(\max, +)$ semi-ring. Till now several applications of tropical geometry to algebraic geometry have been found.

In 2005 Mikhalkin [2] discovered a "tropical" formula for enumeration of curves of genus g in a linear system $\mathcal{L}$ on a toric surface X passing through an appropriate number of points in general position. The main ingredient in the proof of Mikhalkin's formula was a "Correspondence Theorem" that provided a relation between enumeration of algebraic and parameterized tropical curves. Mikhalkin showed that any algebraic curve on a toric surface defines a parameterized tropical curve in $\mathbb{R}^2$, and that under certain conditions one can compute the number of algebraic curves defining a given parameterized tropical curve.

To assign a parameterized tropical curve to an algebraic curve, Mikhalkin analyzed the Hausdorff limits of logarithmic degenerations of algebraic curves in the logarithmic image $\mathbb{R}^2$ of the complex torus $(\mathbb{C}^*)^2$, and showed that these limits are piece-wise linear immersed graphs in $\mathbb{R}^2$ that can be equipped with weights turning them into parameterized tropical curves. Then he introduced the notion of a complex tropical curve, counted the number of such curves corresponding to a given parameterized tropical curve, and used analytic and symplectic techniques to prove that under certain assumptions there exists unique algebraic curve defining a given complex tropical curve. In his ICM paper [3] Mikhalkin presents the "Correspondence Theorem" (Theorem 2) as an application of a result about realization of regular parameterized tropical curves by complex algebraic curves (Theorem 1), which holds true in arbitrary dimension.

Since 2005 several other correspondence theorems for real and complex curves have been established by Mikhalkin, Shustin, Nishinou and Siebert, and others. The approaches of Shustin and of Nishinou and Siebert are different from Mikhalkin's original approach. Shustin's approach is based on Viro's patchworking method, and the approach of Nishinou and Siebert is based on log-geometry. However, in all these approaches the parameterized tropical curve Γ corresponding to an algebraic curve C was constructed in terms of the morphism from C to the toric variety. As a result, the underlying tropical curve (i.e. the metric graph) depended and on the immersion of the curve into the toric variety.

The goals of the lecture

This lecture gives a brief overview of the results and ideas of our paper [6], and we refer the reader to the paper for details, precise statements, and complete proofs.

Canonical tropicalization. The first goal of the lecture is to describe a canonical procedure associating a tropical curve Γ to an algebraic curve with marked points (C, D) over the separable closure $\overline{\mathbb{F}}$ of the field of fractions $\mathbb{F}$ of a complete discrete valuation ring R. We define the underlying graph of Γ to be *the dual graph of the stable reduction of* (C, D), and we define the metric on Γ in a natural way in terms of the singularities of the total space of the stable model. If, in addition, we are given a morphism f from $C \setminus D$ to the algebraic torus $(\overline{\mathbb{F}}^*)^n$, then we construct a natural parameterized tropical curve $h\colon \Gamma \to \mathbb{R}^n$. Our construction is canonical, and the parameterized tropical curves constructed in [2, 4, 5] are obtained from Γ above by contraction of maximal connected subgraphs contracted by h.

We shall mention that there is an alternative description of Γ. Namely, given a curve with marked points (C, D), one considers the corresponding Berkovich analytification (B, D). If (C, D) is stable then B contains a distinguished skeleton, which is a metric graph; and it is possible to show that this graph is naturally isometric to Γ.

An algebraic approach to correspondence theorems. The second goal of the lecture is to present an algebra-geometric approach to various correspondence theorems. It can be used to give proofs of known correspondence theorems, as well as to prove new correspondence theorems for rational curves satisfying cross-ratio constraints, and for elliptic curves having given j-invariant [6]. Our methods are purely algebraic and work in sufficiently big positive characteristics. The case of small characteristic is more subtle, but we hope to adjust our approach to this case soon. Let us describe the approach in more detail:

Assume that the residue field $\Bbbk$ of R is algebraically closed. Let (C, D, f) be a triple, where (C, D) is a smooth curve with marked points over $\mathbb{F}$ and f is a morphism from $C \setminus D$ to the algebraic torus $(\overline{\mathbb{F}}^*)^n$. Then there exists the minimal partial toric compactification $X_{\mathbb{F}}$ of $(\overline{\mathbb{F}}^*)^n$ such that f extends to a morphism $C \to X_{\mathbb{F}}$. Furthermore, for a sufficiently ramified separable extension $\mathbb{L}/\mathbb{F}$, there exists a canonical (minimal) commutative diagram

$$\begin{array}{ccccc} \mathbb{C}_{\Bbbk} & \hookrightarrow & \mathbb{C}_{R_{\mathbb{L}}} & \hookleftarrow & C_{\mathbb{L}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X}_{\Bbbk} & \hookrightarrow & \mathcal{X}_{R_{\mathbb{L}}} & \hookleftarrow & X_{\mathbb{L}} \end{array}$$

where $R_{\mathbb{L}}$ is the ring of integers of $\mathbb{L}$, $\mathcal{X}_{R_{\mathbb{L}}}$ is a stacky toric degeneration of $X_{\mathbb{L}} := X_{\mathbb{F}} \times_{Spec(\mathbb{F})} Spec(\mathbb{L})$, $(\mathbb{C}_{R_{\mathbb{L}}}, D_{R_{\mathbb{L}}})$ is a semistable model equipped with an appropriate stacky structure, and $\mathbb{C}_{\Bbbk} \to \mathcal{X}_{\Bbbk}$ is the reduction of $\mathbb{C}_{R_{\mathbb{L}}} \to \mathcal{X}_{R_{\mathbb{L}}}$. Note that the stacky toric degenerations are generalizations of toric stacks of Borisov, Chen, and Smith [1]. In [1], the authors introduce smooth Deligne-Mumford stacks as a global quotients of quasi-affine toric varieties by the actions of certain subtori. We present a modification of their construction that produces a reacher class of toric stacks, in particular those that appear naturally in the tropicalization procedure.

If the triple (C, D, f) satisfies tropicalizable constraints (e.g. passing through given points, having given cross-ratios of certain quadruples of marked points if $p_g(C) = 0$, having given j-invariant if $p_g(C) = 1$, etc.) then the constraints can be included naturally in the diagram. Keeping this in mind, one can think about correspondence theorems as statements about one-to-one correspondences between algebraic triples (C, D, f) satisfying an appropriate number of tropicalizable constraints and solid parts of the diagrams

$$\begin{array}{ccccc} \mathbb{C}_{\Bbbk} & \dashrightarrow & \mathbb{C}_{R_{\mathbb{L}}} & \dashleftarrow & C_{\mathbb{L}} \\ \downarrow & & \vdots & & \vdots \\ \mathcal{X}_{\Bbbk} & \hookrightarrow & \mathcal{X}_{R_{\mathbb{L}}} & \hookleftarrow & X_{\mathbb{L}} \end{array} \tag{1}$$

(plus constraints that we omit here for the sake of simplicity). Thus, in order to prove a correspondence theorem, it is sufficient (i) to describe all possible solid parts of the diagram that may correspond to an algebraic triple (C, D, f), and (ii) to show that the deformation problem described by the diagram is zero-dimensional and unobstructed. Note that if one considers the diagrams of coarse moduli-spaces (the naive diagrams without the stacky structure) then, usually, the corresponding deformation problem is zero-dimensional, but obstructed.

Finally we would like to mention that there exists a mysterious complex L_Γ of length two of free abelian groups that controls the algebraic-tropical correspondence. Namely, the group $H^1(L_\Gamma \otimes_{\mathbb{Z}} \mathbb{R})$ is the versal deformation space of the parameterized tropical curve Γ, and the versal deformation space of Γ has expected dimension if and only if $H^2(L_\Gamma \otimes_{\mathbb{Z}} \mathbb{R}) = 0$. The group $H^1(L_\Gamma \otimes_{\mathbb{Z}} \Bbbk)$ is the space of first order deformations of the solid part of diagram (1), and the deformation space is unobstructed if and only if $H^2(L_\Gamma \otimes_{\mathbb{Z}} \Bbbk) = 0$. The group $H^1(L_\Gamma \otimes_{\mathbb{Z}} \Bbbk^*)$ acts simply-transitively on the set of all possible solid parts of diagram (1) for a given parameterized tropical curve (if this set is not empty), and if $H^2(L_\Gamma \otimes_{\mathbb{Z}} \Bbbk^*) = 1$ then the set of stacky tropical limits is not empty.

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Informal discussion: Enumeration of real elliptic curves

Grigory Mikhalkin

1. Objective of the discussion

1.1. **Gromov-Witten theory viewpoint.** Recall the current state of the real enumerative geometry viewed from the Gromov-Witten theory approach. According to this approach the number of curves in an algebraic variety X passing through some constraints is defined through an intersection theory in the moduli space $\mathcal{M}^d_{g,k}(X)$ of stable curves of degree d (d can be considered as an element of $H_2(X;\mathbb{Z})$) of denus g with k marked points (here k is the number of constraints).

For example, if X is $\mathbb{C}P^2$ then the space $\overline{\mathcal{M}}^d_{g,k}(\mathbb{C}P^2)$ is a $3d-1+g+k$-dimensional complex variety (once we restrict only to stable curves $h: S \to \mathbb{C}P^2$ that are approximable by holomorphic immersions of Riemann surfaces and make the usual identification $H_2(\mathbb{C}P^2;\mathbb{Z}) = \mathbb{Z}$). The simplest example of a constraint is given by fixing a point p and requiring that the jth marked point $q_j \in S$ of the curve is mapped to p. In the Gromov-Witten paradigm such constraint can be rewritten as a pullback of $p \in \mathbb{C}P^2$ under the evaluation map $ev_j : \overline{\mathcal{M}}^d_{g,k}(\mathbb{C}P^2) \to \mathbb{C}P^2$ that takes a curve h to $h(q_j)$. Furthermore, as p is a zero-dimensional oriented manifold it defines (by the Poincaré duality) a cohomology class $\xi_p \in H^4(\mathbb{C}P^2)$ and thus we get $ev^*(\xi_p) \in H^4(\overline{\mathcal{M}}^d_{g,k}(\mathbb{C}P^2) \to \mathbb{C}P^2)$.

If we fix k points $p_1, \ldots, p_k \in \mathbb{C}P^2$ and twice this number is equal to $\dim \overline{\mathcal{M}}^d_{g,k}(\mathbb{C}P^2) = 3d - 1 + g + k$, i.e. $k = 3d - 1 + g$ then the cup-product of all ξ_{p_j} has top dimension and we may evaluate it against the fundamental class of $\overline{\mathcal{M}}^d_{g,k}(\mathbb{C}P^2)$. The result in this case is the so-called Gromov-Witten number corresponding to a given enumerative.

In general these numbers are quite different from the original problem. However in the case we consider the meaning is indeed enumerative. With the help of the Poincaré duality we see that (in the case when the points are chosen so that the corresponding number of curves is finite) that this is just a way to associate a multiplicity to each curve passing through our configuration of $3d - 1 + g$ points in $\mathbb{C}P^2$. Note that by the maximum principle the multiplicity prescribed to each curve in this way is positive. If the points are chosen in general position then each curve gets counter with multiplicity $+1$.

The situation changes radically once we pass from complex to real numbers in this problem as there is no natural orientation of real varieties and furthermore they are not even orientable. Already the target space $\mathbb{R}P^2$ of the evaluation map is nonorientable.

Nevertheless, a solution in the case $g = 0$ was suggested by Welschinger in 2003 [4]. He noted esentially that the evaluation map

$$\mathrm{ev} : \overline{\mathcal{M}}^d_{0,3d-1}(\mathbb{R}P^2) \to (\mathbb{R}P^2)^{3d-1}$$

preserves the Stiefel-Whitney class w_1, i.e.

$$\mathrm{ev}^*(w_1((\mathbb{R}P^2)^{3d-1})) = w_1(\overline{\mathcal{M}}^d_{0,3d-1}(\mathbb{R}P^2)).$$

Because of this one may define the number of real curves in this case staying within the Gromov-Witten paradigm once we use $\mathbb{Z}$-coefficients twisted by w_1. As in the complex case for a generic choice of $3d - 1$ points in $\mathbb{R}P^2$ we have a finite collection of real rational curves of degree d through these points and every curve will have multiplicity of absolute value 1. But as we no longer have positivity of intersections in the real case each real curve in this collection contributes ± 1 to the corresponding enumerative number.

The sum of these multiplicities is known as the Welschinger invariant. It was defined for $g = 0$ curves through collection of points in $\mathbb{R}P^2$ (as well as other real Dell Pezzo surfaces) in [4]. It was generalized to $g = 0$ curves through collection of points in $\mathbb{R}P^3$ (as well as other real Fano 3-folds) in [5], but no essential generalizations of these invariants were found since then. Furthermore, as it was noted by Welschinger himself the rule of signs that he defined is non-invariant already in $\mathbb{R}P^2$ once we pass to curves of higher genus.

1.2. **Tropical geometry viewpoint.** Tropical geometry provides other tools for solving enumerative questions in addition to the *compactification+localization* technique (introduced by Kontsevich almost 20 years ago and still in active use as a principal tool by many complex enumerative geometers today). It allows to answer an enumerative algebro-geometric problem direclty in a combinatorial way after *tropicalization* of the problem, see [2] for the case of $\mathbb{C}P^2$ and $\mathbb{R}P^2$.

Furthermore, a tropical approach provides not only the number of solutions but topology and geometry of all curves solving a given enumerative problem. The tropical technique is applicable to constraints in *tropically general position* which is an asymptotic condition.

One may note (see e.g. [1]) that the number of curves of genus g and degree d through $3d-1+g$ points in tropically general position in $\mathbb{R}P^2$ and counted according to the (non-invariant in the classical set-up) Welschinger rule is invariant, i.e. does not depend on the choice of configuration. Furthermore, one can apply the technique developed in [2] to compute this number (e.g. for $d=4$ and $g=1$ one gets 93, recall that the corresponding number of complex curves is 225).

Thus our experimental data in enumeration of algebraic curves of degree d and genus $g>0$ through $3d-1+g$ points in $\mathbb{R}P^2$ that can be summarized as the following mysterious-looking observation.

The Welschinger rule of signs is decidedly non-invariant for $g>0$ in the classical set-up, nevertheless it produces an invariant in the tropical set-up.

2. Explanation of the "mystery"

In the case of $g>0$ the map

$$\mathrm{ev}:\overline{\mathcal{M}}^{d}_{g,3d-1+g}(\mathbb{R}P^2)\to(\mathbb{R}P^2)^{3d-1+g}$$

does not preserve w_1. The distortion is given by the hypersurface

$$W=\{h:(C;q_1,\dots,q_{3d-1+g}\to\mathbb{R}P^2\ :\ |K_C-h^*(K_{\mathbb{R}P^2})-\sum_{j=1}^{3d-1+g}q_j|\neq\emptyset\}$$

The hypersurface $W\subset\overline{\mathcal{M}}^{d}_{g,3d-1+g}(\mathbb{R}P^2)$ is not homologous to zero mod 2 for $g=1$ already because it has odd intersection number with the universal curve. Nevertheless its image under ev is always homologous to zero mod 2 (and even over $\mathbb{Z}$ coefficients in the w_1-twisted coefficients) as this is the locus on non-invariance of the Welschinger rule of signs.

Note that the degree of the line bundle $(K_C-h^*(K_{\mathbb{R}P^2})-\sum\limits_{j=1}^{3d-1+g}q_j)$ is $g-1$. Thus the locus W is determined as the intersection with the Theta-divisor which can be easily approached tropically, see [3].

Thus for a generic (in the classical sense) configuration P composed of points $p_1,\dots,p_{3d-1+g}$ we can define the number $W_{g,d}$ as the sum of the Welschinger signs for all genus g degree d curves passing through P and the correction term obtained as twice the linking number of $\mathrm{ev}(W)$ with the 0-cycle in $(\mathbb{R}P^2)^{3d-1+g}$ obtained as a difference of P and Q, where Q is any configuration of $3d-1+g$ points in *tropically* general position.

By the very definition $W_{g,d}$ coincides with the tropical real enumerative invariant obtained by applying the Welschinger rule of signs in the tropical situation. This makes it interesting to investigate the hypersurface W (in particular its homology class) which can be studied by tropical means. This is subject of a future research.

In particular the image $\mathrm{ev}(W)$ has to degenerate in the tropical limit as tropically generic configurations become dense and this explains tropical invariance. One can add that this can be seen very explicitly in the case $g=1$.

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Informal discussion: Superpotentials

Johannes Walcher

Mathematicians that interact with physicists studying supersymmetric gauge theory and string theory sometimes find themselves confronted with seemingly inpenetrable jargon. Much of the physicists' terminology is rooted in concepts designed to describe fundamental particles and their interactions. Over the years, mathematicians have been able to translate some of the statements into more familiar language, and to attach their own meaning to the new words, see [1]. The liberty with which this happens is sometimes disconcerting for physicists who then find themselves unable to understand even the relevance of the mathematicians' reply to their original question.

The purpose of this session was to present one of those concepts, the so-called "superpotential" in its original physical context, and to discuss some of its applications for topics relevant to this workshop.

A starting point for the construction of models of Quantum Field Theory are action principles based on Lagrangians (local functionals of space-time fields), that implement physical principles of the mathematical description of reality. The simplest possible model is the free massive real scalar field, say in $3+1$ dimensions:

$$S[\phi] = \int d^4x \big(\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\big) \tag{1}$$

The arena is Minkowski space $\mathbb{R}^{3,1} \ni (x_\mu)_{\mu=0,\ldots 3}$ with its metric $\eta_{\mu\nu}$, $\phi : \mathbb{R}^{3,1} \to \mathbb{R}$ is the scalar field, of mass m. Of importance is the invariance under the Poincaré group $G = L \ltimes \mathbb{R}^{3,1}$, the semi-direct product of the Lorentz group $L = O(3,1)$ with translations, generated geometrically by the action of the vector fields $P_\mu = \frac{\partial}{\partial x^\mu}$.

This is not the place to discuss the purpose of writing actions like (1). Suffice it to mention some generalizations. Keeping with scalars, one may consider complex fields, $\phi : \mathbb{R}^{3,1} \to \mathbb{C}$, or have several fields. In certain cases, those may live in a curved target space M (a Riemannian manifold), which could also be endowed with a potential function $V : M \to \mathbb{R}$, with suitable conditions. One may also enrich the model with fields in other representations of L, internal symmetry groups, that could be gauged, etc., etc. For a sensible physical model, one usually requires the Lagrangian to be a real scalar with no explicit dependence on the coordinates x^μ, so as to preserve Poincaré invariance, locality, and unitarity.

For appreciating the possibility of a superpotential, one should be aware also of the option of an additional modification, known as dimensional reduction. The obtuse way of doing this is to declare some of the fields to be independent of one of the coordinates, say x^3. If $\partial_{x^3}\varphi = 0$, then we may consider actions

$$S[\phi, \varphi] = \int d^4x \, \mathcal{L}^{(4)}(\phi, \varphi) + \int d^3x \, \mathcal{L}^{(3)}(\varphi)$$

consisting of a 4-dimensional, and a 3-dimensional part. Note that such Lagrangians preserve 4-dimensional translational symmetry, but break the Lorentz-group to its 3-dimensional version, and cannot be written as $\int d^4x$ without introducing x^3-dependence. The smarter way of going about dimensional reduction is Kaluza-Klein compactification: One imagines that physical space is a product $\mathbb{R}^{2,1} \times S^1$, expands all fields in Fourier modes, and studies the physics at low energies.

The very smart modification, of course, is supersymmetry, which is an extension of the symmetry group by certain "odd square-roots" of the translations. Geometrically, this can be achieved by replacing $\mathbb{R}^{3,1}$ with superspace $\mathbb{R}^{3,1|4} \cong V \oplus \Pi S$, the odd component of which carries a copy of the spin representation, S, of $L = O(3,1)$ (V being the vector representation). One may use 4 real or 2 complex coordinates for this, collectively denoted θ. The key to the geometry of supersymmetry are the odd vector fields $Q = \partial_\theta + \theta \, \not{\partial}_x$, implementing supersymmetry according to $\{Q, Q\} = 2\not{P}$ (using some hopefully suggestive notation).

The simplest models of supersymmetric quantum field theory are now based justly on scalar superfields, $\Phi : \mathbb{R}^{3,1|4} \to \mathbb{C}$ (the necessity to work with complex fields will become clear momentarily), and an action $\sim \int d^4x d^4\theta \, \mathcal{K}(\Phi, \bar{\Phi})$. The supersymmetry is obvious if $\mathcal{K}$ does not depend explicitly on the x and θ. A feature of supersymmetry is that derivatives are already built into the supermultiplet structure, so an ordinary function $\mathcal{K}$ (with non-degenerate second derivative $\partial_\Phi \bar{\partial}_{\bar{\Phi}} \mathcal{K}$) will serve as kinetic term for Φ.

Now the **key to the superpotential** is to realize that the decomposition $S \times \mathbb{C} = S^+ \oplus S^-$ and $S^+ \wedge S^+ \cong \mathbb{C}$ allows splitting the invariant measure $d^4\theta = d^2\theta d^2\bar{\theta}$ (using complex coordinates) into two measures that are separately

Lorentz-invariant (albeit complex). The "obtuse" option of dimensional reduction mentioned above now becomes acute. If $\mathcal{W}$ is a superfield independent of the anti-holomorphic $\bar{\theta}$ (in an appropriate way, compatible with the torsion on $\mathbb{R}^{3,1|4}$), the action

$$S[\Phi] = \int d^4x d^4\theta\, \mathcal{K}(\Phi, \bar{\Phi}) + \int d^4x d^2\theta\, \mathcal{W}(\Phi) + \int d^4x d^2\bar{\theta}\, \bar{\mathcal{W}}(\bar{\Phi})$$

is also supersymmetric (i.e., invariant under Lorentz transformations, *and* supertranslations). Here we anticipated that $\mathcal{W}$ can only depend on Φ in a holomorphic way, Φ itself must be a *chiral* field (i.e., independent of $\bar{\theta}$), and we have added the complex conjugate of $\mathcal{W}$ to ensure that the action is real. Note that the superpotential term in the action cannot be written as an integral over full superspace. In general then, Φ takes values in a Kähler manifold, and $\mathcal{K}$ is the Kähler potential.

The superpotential is one of the central quantities in the study of supersymmetric theories with 4 real supercharges, such as minimal supersymmetry in 4 dimensions. The other options of modifying the theory of course remain open. For instance, we may reduce from 4 to 2 dimensions, thus obtaining the worldsheet theory of the superstring. The target M in this case is the physical space-time manifold, which must be ten-dimensional for consistency of the theory. Using constructions of string theory, such as compactification, D-branes, fluxes, etc., we may land back with an effective theory in 4-dimensions that has scalar fields and supersymmetry, so may again carry a superpotential. In this situation, it is rewarding to differentiate between the superpotential of the 2-dimensional theory (called *world-sheet superpotential*, often denoted by W), from the superpotential of the 4-dimensional theory (called *space-time superpotential*, often denoted by $\mathcal{W}$).

Before proceeding with examples, some remarks are in order:
(i) The relation between Lagrangians and physical theories is just that—a relation. One and the same physical theory may have different Lagrangian definitions, and some physical theories have no Lagrangian description at all. Sometimes, information encoded in the superpotential in one description is encoded in other data in a dual description.
(ii) Mathematically of course, the superpotential is just a holomorphic function (or, more often than not, a section of a line bundle), sometimes with additional properties such as a degree of homogeneity, or other symmetries. The critical structure of this function (in the sense of singularity theory) determines the vacuum structure of the physical theory.
(iii) Of interest to this workshop, interesting enumerative information is encoded in the *expansion* of the superpotential around certain limit, often singular, points in field space.

There is clearly a certain tension between (i) and (iii), from which much of the richness of the superpotential derives. A few examples of this include:
1. The sigma model with target space $\mathbb{CP}^1$, and Kähler potential $K(z, \bar{z}) = t \ln(1+|z|^2)$ (no superpotential being possible in this case) is, according to Hori-Vafa, mirror dual to the Landau-Ginzburg model on $\mathbb{C}^* \ni Y$ with superpotential $W =$

$Y + e^t/Y$. This superpotential can be interpreted [2] as encoding the holomorphic disks ending on a latitudinal circle on $\mathbb{CP}^1 \cong S^2$ (Y being the exponentiated area of the disk on the North cap, e^t/Y that of the South cap).

2. The sigma model with target space the resolved conifold $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ is mirror dual to a conic bundle over the plane $\mathbb{C}^* \times \mathbb{C}^* \ni (X, Y)$ defined by $F(X, Y) = Y + e^t X/Y - 1 - X = UV$, where $(U, V) \in \mathbb{C}^2$. D-branes wrapped on components of reducible fibers are mirror duals of Lagrangian submanifolds invariant under a torus action on the conifold. According [3], the spacetime superpotential associated with such a D-brane construction is $\mathcal{W} = \int^p \ln X \, d \ln Y$ given by the Abel-Jacobi map of the point $p \in \{F(X, Y) = 0\}$ on the spectral curve over which the reducible fiber resides. This superpotential can be written as $\mathcal{W} = \mathrm{Li}_2(Y) + \mathrm{Li}_2(e^t/Y)$, again counting the same disks as under 1., albeit with slightly different weight.

3. Finally, consider the space-time superpotential $\mathcal{W}(a; \psi) = \frac{a^3}{3} - \psi a$. According [4], this is the space-time superpotential encoding the vacuum structure on a D-brane wrapped on the real Fermat quintic threefold. The dynamical field is a, and ψ is a parameter (related to the Kähler class of the quintic). The only *invariant* information in this superpotential is the difference of critical values $\mathcal{W}_+ - \mathcal{W}_- \sim \psi^{3/2}$, which encodes nothing less than the numbers of real rational curves on the generic real quintic threefold.

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Informal discussion: Enumeration of nodal curves on surfaces in non-linear systems

Maxim Kazarian

Let $N_n(d)$ be the number of plane curves of degree d having n nodes and passing through an appropriate number (to be precise, $\binom{d+2}{2} - 1 - n$) of generic points on the plane $\mathbb{C}P^2$. Counting these numbers was one of the first non-trivial computations both in Gromov-Witten theory (Kontsevich and Manin [2], Caporaso and Harris [3]) and in tropical geometry (Mikhalkin [4]). Consider the projective space $B = |O(d)| = \mathbb{C}P^{\binom{d+2}{2}-1}$ of all degree d curves. Then $N_n(d)$ is the degree of the subvariety Δ_n parameterizing all curves with n double points. In other terms, the cohomology class represented by this subvariety is equal to

$$[\Delta_n] = N_n(d) \, h^n \in H^*(B) = \mathbb{Z}[h]$$

where $h \in H^2(B)$ is the class of the hyperplane.

The subvariety Δ_n is invariant under the natural action of the group PGL(3) of projective linear transformations of the plane. One of the problems that we address to in our discussion is the computation of the cohomology class represented by the subvariety $\Delta_n \subset B$ in the *equivariant* setting:

$$[\Delta_n]_{\mathrm{PGL}(3)} \in H^*_{\mathrm{PGL}(3)}(B).$$

The equivariant cohomology group $H^*_{\mathrm{PGL}(3)}(B)$ is much bigger then the usual one and the above cohomology class carries much more information than just the number $N_n(d)$.

One of the reformulations of the problem is as follows. Replace the plane $\mathbb{C}P^2$ by the total space of the projective bundle PE where E is a given rank 3 vector bundle over some nonsingular base. Replace B by the total space of the projective bundle $P\,\mathrm{Sym}^d E^\vee$. This space parameterizes degree d curves on the fibers of PE. We compute the cohomology class represented by the subvariety Δ_n in $P\,\mathrm{Sym}^d E^\vee$ consisting of n-nodal curves. This cohomology class $[\Delta_n] \in H^*(P\,\mathrm{Sym}^d E^\vee)$ can be written as a polynomial in the class $h = c_1(O(1))$ and the Chern classes of the bundle E. The coefficient of h^n in this polynomial is equal to $N_n(d)$, while the other coefficients are subject to our computation.

In a **more general setting**, we consider an arbitrary generic family of curves on surfaces, that is, a diagram of the form

$$H \xrightarrow{j} W \xrightarrow{\pi} B,$$

where π is a locally trivial fibration with compact smooth 2-dimensional fibers and j is an embedding of codimension 1. The genericity condition means that H is smooth and generally embedded. We regard B as the parameter space: to each parameter value we associate a surface (the fiber of π) and a curve on it (the one cut off by the hypersurface H).

The **Kleiman-Piene conjecture** [5] is an explicit formula for the cohomology class $[\Delta_n]$ represented by the cycle of n-nodal curves $\Delta_N \subset B$ in this general setting. This formula involves a sequence of universal polynomials $R_n(u, c_1, c_2)$ with rational coefficients, $n = 1, 2, \ldots$. The polynomial R_n is called the *residual* polynomial. It has the quasihomogeneous degree $n-1$ with respect to the grading with $\deg u = 1$, $\deg c_i = i$. In fact, the conjecture postulates the existence of these polynomials, while their explicit values are subject to further computations.

The variables u, c_1, c_2 have the meaning of certain cohomology classes on W related to the problem. Namely, $u = [H]$ is the class of the divisor H, and c_i are the Chern classes of the relative tangent bundle for the fibration π.

We propose [1] an **algorithm for the explicit computation of the residual polynomials**. The idea is to restrict the Kleiman-Piene formula to the universal unfoldings of quasihomogeneous plane curve singularities. One can list explicitly the cases when Δ_n is empty and thus $\deg \Delta_n = 0$. Every such case provides a linear equation on the (unknown) coefficients of R_n. Resolving these equations one gets explicitly the coefficients of R_n:

$$\begin{aligned}
R_1 &= 1,\\
2\,R_2 &= -u + 6\,(c_1 - u),\\
3\,R_3 &= u^2 + (60\,c_1 - 68\,u)\,(c_1 - u),\\
4\,R_4 &= -u^3 + (840\,c_1^2 - 156\,c_2 - 1582\,c_1 u + 803\,u^2)\,(c_1 - u),\\
&\dots
\end{aligned}$$

Substituting the found polynomials to the Kleiman-Piene formula one gets the answer in the particular enumerative problems, for example, in the one formulated at the beginning of this abstract.

The proposed algorithm has the following peculiar properties:

1) **The algorithm is simple, efficient, and reliable**. It is interesting that it gets an answer in a huge variety of geometric problems without detailed geometric study of any of them! The computation is absolutely formal. There is no place in it to miss strata or miscalculate multiplicities or signs: it uses just a simple combinatorics of the Newton polygons of the quasihomogeneous singularities. The actual code of the implemented (*Mathematica*) program contains about 10 lines. It reduces essentially to solving a linear system of equations of big size. The obtained system of equations is highly overdetermined, and the very its consistency is surprising enough. Some impression about these systems can be obtained from the following table.

n	2	3	4	5	6	7	8	9	10	⋯	19	20
# of unknown coefficients of R_n	1	2	4	6	9	12	16	20	25		90	100
# of equations on these coefficients	1	3	8	14	18	27	36	41	51		143	157

2) **The algorithm is not completely justified** since it is based on a conjecture which is not yet proved in its full generality (Kleiman and Piene announced its proof for $n \leq 8$ [5]). This fact does not contradict to the mentioned above reliability of the algorithm since numerous computational, geometric, and topological evidences leave no doubts in the validity of the conjecture.

3) **The algorithm leads to the answer for** ≤ 19 **only**. For $n \geq 20$ the obtained linear system has no full rank (for some unclear reason) leaving several coefficients of R_n undetermined.

Remarks. 1. For the case of a *linear* system of divisors on a *fixed* surface, the degree of the Δ_n-stratum is uniquely determined by the first four coefficients $a_{0,0}$, $a_{1,0}$, $a_{2,0}$, and $a_{0,1}$ in the polynomial

$$R_n = a_{0,0}u^{n-1} + a_{1,0}c_1u^{n-2} + a_{2,0}c_1^2u^{n-3} + a_{0,1}c_2u^{n-3} + \dots,$$

and the Kleiman-Piene conjecture reduces in this case to the **Göttsche conjecture** [6]. Thus, all known results on enumeration of nodal curves on fixed surfaces determine only 4 out of all coefficients in each residual polynomial R_n. It shows that the problem of counting of nodal curves in general families cannot be reduced, in general, to the special case of counting nodal curves in a linear system on a fixed surface. Combining the above mentioned algorithm with the known results on nodal curves in linear systems one can extend the computation of the polynomials R_n up to $n = 23$. Starting from $n = 24$ the knowledge of the coefficients $a_{0,0}, \dots, a_{0,1}$ is not sufficient to identify all coefficients of R_n.

2. The genericity condition is essential. It is an open condition but its verification is in general not so simple. For example, for counting plane nodal curves the universal formula predicts $N_n(d)$ as a polynomial in d for each fixed n. This polynomial provides a correct answer if d is big enough (presumably, for $d > n/2$). For small d the value of the polynomial does not equal to $N_n(d)$ and even can be negative. This remark shows that the formula of Kleiman and Piene does not pretend to cover the enumeration of nodal curves in all known cases.

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Informal discussion: Amoebas, coamoebas, and hypergeometric monodromy

Mikael Passare

We present some results, obtained with Lisa Nilsson and August Tsikh, on the representation of A-hypergeometric functions in terms of Γ-series and Mellin–Bernes integrals.

The A-hypergeometric system $H_A(\beta)$ is determined by an integer $(n\times N)$-matrix A and a homogeneity vector $\beta \in \mathbf{C}^n$. We assume the columns of A generate the lattice $\mathbf{Z}^n$, and the rows of A contain the vector $(1, 1, \dots, 1)$ in their span. The vector β is assumed to be generic. One fixes a Gale dual of A, that is, an integer $(N \times d)$-matrix B, where $d = N - n$, whose columns give a basis for the kernel of A. The rows of B are denoted $b_1, b_2, \dots, b_N$.

By choosing suitable vectors $\gamma \in \mathbf{C}^N$, with $A\gamma = \beta$, one can produce Γ-series solutions to $H_A(\beta)$ of the form

$$\Phi_\gamma(z) = \sum_{k\in\mathbf{Z}^d} \prod_{j=1}^{N} \frac{z_j^{\gamma_j+b_jk}}{\Gamma(1+\gamma_j+b_jk)}\,,$$

and since d of the γ_j are chosen to be integers, the summation actually only takes place in a convex subcone of the lattice $\mathbf{Z}^d$. For any choice of maximal minor A_I of A, there will be $\mu_I = |\det A_I|$ different such Γ-series, and we describe their convergence domains in terms of the connected components of the complement $\mathbf{R}^N \setminus \mathcal{A}_{E_A}$ of the amoeba of the full A-discriminant E_A, also known as the principal A-determinant.

More precisely, each connected component corresponds to a regular triangulation of the convex hull $Q(A)$ of the point configuration A, and a Γ-series corresponding to a simplex A_I converges in the union of those complement components for which the associated triangulations contain the simplex A_I.

An alternative representation of solutions to the system $H_A(\beta)$ is in terms of Mellin–Barnes integrals of the form

$$MB(z) = \int_{i\mathbf{R}^d} \prod_{j=1}^{N} \Gamma(-\gamma_j - b_j s)\, z_j^{\gamma_j + b_j s}\, ds_1 \wedge ds_2 \wedge \ldots \wedge ds_d\,.$$

Here we assume that all the γ_j have negative real part. This can be achieved by adding an integer vector to the homogeneity vector β if necessary, and such a translation does not change the monodromy properties of the system $H_A(\beta)$.

A key observation is that the above Mellin–Barnes integral has quite a large domain of convergence, determined by the arguments θ_j of the variables z_j. In fact, letting the row vectors b_j also denote the line segments connecting the origin to the point $b_j \in \mathbf{Z}^d$, we form the zonotope $Z_B = \pi b_1 + \ldots + \pi b_N$ as the Minkowski sum of the dilated segments. The convergence domain of the integral $MB(z)$ is then given by all argument vectors θ for which the product θB is contained in the interior of the zonotope Z_B.

There is a close relation, so far only proved for $d \leq 2$, between the zonotope and the coamoeba $\mathcal{A}'_{D_B}$ of the reduced discriminant D_B. Namely, the coamoeba $\mathcal{A}'_{D_B}$ and the zonotope Z_B, when considered as chains, together give a μ-fold cover of the torus $\mathbf{T}^d$. Here μ is the normalized volume of $Q(A)$, which is also known to be the dimension of the solution space to the system $H_A(\beta)$. This implies in particular that if the complement of the coamoeba is non-empty, then the Mellin–Barnes integral produces a full basis of solutions to $H_A(\beta)$.

In a recent paper Frits Beukers has made use of the existence of such Mellin–Barnes bases of solutions as a combinatorial approach to the study of the monodromy of A-hypergeometric functions. His idea is to let the Mellin–Barnes basis connect the various local monodromy groups, obtained by letting $\mathbf{Z}^N$ act by coordinatewise angular loops on the Γ-series Φ_γ. Since this action just amounts to multiplying each series with a suitable exponential, and since the transition from one Mellin–Barnes basis element to another is also effectuated by purely angular moves, it is possible in this way to get a grip on the transition matrices between the Γ-series bases and the Mellin–Barnes integral bases. Beukers conjectures that the group generated by such transitions is in fact the full monodromy group for the system $H_A(\beta)$.

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