Mathematisches Forschungsinstitut Oberwolfach

Report No. 22/2011

DOI: 10.4171/OWR/2011/22

Mini-Workshop: Mathematical Models, Analysis, and Numerical Methods for Dynamic Fracture

Organised by
Gianni Dal Maso, Trieste
Christopher J. Larsen, Worcester
Christoph Ortner, Oxford

April 24th – April 30th, 2011

Abstract. The mathematical foundation of fracture mechanics has seen considerable advances in the last fifteen years. While this progress has been substantial, it has been largely limited to quasi-static evolutions based on global energy minimization, which is known to produce non-physical results. What is missing is a generally accepted mathematical theory of dynamic crack growth, which accounts for material inertia. Such a theory would not only be able to describe the most physically realistic setting, but it would also provide a trusted starting point to resolve pressing questions about quasi-static evolutions, e.g., a rigorous justification of the quasi-static setting as an asymptotic limit of inertial dynamics. This workshop brought together researchers in mathematical analysis, mechanics, applied mathematics, and numerical analysis and laid the groundwork for progress on these questions.

Mathematics Subject Classification (2000): 74R10, 74R15, 74H20.

Introduction by the Organisers

The mini-workshop *Mathematical Models, Analysis, and Numerical Methods*, organised by Gianni Dal Maso (Trieste), Christopher J. Larsen (Worcester), and Christoph Ortner (Oxford) was held April 24th–April 30th, 2011. It was attended by 15 participants representing a broad range of expertise.

The mathematical foundations of fracture mechanics have seen considerable advances in the last fifteen years. While mathematical modelling of fracture has been a serious scientific discipline at least since the pioneering work of Griffith, there had not been a mathematically well-posed model for the prediction of crack paths until new formulations were proposed, using the framework of the calculus

of variations in the function spaces BV and SBV. While there had been much previous progress, and success, in engineering models of fracture, it is the extra assumptions (e.g., regularity) that often prevent rigorous mathematical analysis. Aside from establishing a rigorous mathematical theory, the main achievement of this recent research effort was the creation of models of fracture that surpass all previous models in their flexibility of predicting crack paths.

While this progress has been substantial, it has been largely limited to quasi-static evolutions based on global energy minimization, which is known to produce non-physical results. What has been missing is a generally accepted mathematical theory of dynamic crack growth, which accounts for inertia. Such a theory would not only be able to describe the physically most realistic setting, but it would also provide a trusted starting point to resolve pressing questions about quasi-static evolutions, in particular, to rigorously justify the quasi-static setting as an asymptotic limit of inertial dynamics.

This mini-workshop brought together mathematical analysts, numerical analysts, mechanicians and applied mathematicians with expertise in fracture mechanics. Key issues in modelling, analysis, and simulation of dynamic fracture were identified. Specifically:

- Fundamental experimental results in both quasi-static and dynamic settings, which will serve as benchmarks for models and their simulation
- Comparison of energy minimization and the principle of local symmetry
- Recent progress and fundamental open problems in the mathematical analysis of quasi-static fracture
- Modeling of the free fracture surface
- Existence and uniqueness results for the wave equation off a growing straight crack, and formulas for exact solutions, in both quasi-statics and dynamics
- Issues in applying X-FEM to dynamic fracture
- Comparison of simplified 1-D dynamic models with quasi-static models
- Survey of energy release rates and their relation to Griffith fracture

Mini-Workshop: Mathematical Models, Analysis, and Numerical Methods for Dynamic Fracture

Table of Contents

Abstracts

Dynamic Fracture in Nominally Brittle Materials

K. Ravi-Chandar

Griffith's theory of fracture [5] is based on the idea of minimization of the total energy in the body, usually partitioned into the elastic (strain) energy in the volume of body and the surface energy associated with the crack geometry; this represents a necessary condition, but is clearly not sufficient. Consider a simple example of a stretched plate without a crack; the total energy that can be stored in such a plate may be brought to zero (minimized) if the plate is broken into at least two pieces, each with no strain energy; however, this does not occur spontaneously and needs a pathway such as an initial crack. A perfectly reversible system as implied by the Griffith theory is indeed found in cleavage of pure mica sheets in vacuum, as demonstrated by the experiments of Obreimoff [9]. However, even the slightest contamination on the newly created fracture surface (such as oxidation of the fracture surface in ambient environment) is enough to destroy the reversibility of the fracture process; more generally, there is a region just below the fracture surface (whose physical dimension depends on the type of material) that is typically irreversibly deformed, that fracture is not really a reversible process. Perfect cleavage fracture consumes very little energy typically, the surface energy is around 2-10J/m^2 for most materials; however, due to the inelastic processes that occur over a small volume just beneath or near the fracture surface, most materials consume significantly more energy — three to six orders of magnitude more — in the creation of the fracture surface. Nevertheless, stress analysis based on elastic or elastic-plastic constitutive models is useful, as long as the dimension over which the fracture processes occurs is small and the energy for fracture includes the surface energy plus the energy dissipated within the fracture process zone. Orowan [10] suggested incorporating the plastic dissipation into the fracture criterion, yielding $(G - \gamma_P)\dot{a} \geq 0.$ as the fracture criterion. Note that G, called the *energy release rate*, is the change in potential energy of the system with crack extension, and depends on the applied loading, boundary conditions, the crack length, and the presumed direction of crack extension. In the engineering literature, following the ideas of Irwin [6] and Williams [13], the fracture criterion is posed in terms of the singular elastic stress field parameters; the amplitude of the singular stress field is denoted as the *stress intensity factor*, and the fracture criterion becomes $K_I = K_{IC}$, where K_I is the opening mode (or Mode I) stress intensity factor (note: $G = K_I^2/E$, where E is the modulus of elasticity), and K_{IC} is the *fracture toughness* for an opening mode of fracture with the loading condition being symmetric with respect to the crack geometry. This theory works quite well under quasi-static loading conditions, and management of fracture critical structures in civil infrastructure, transportation systems, aerospace vehicles, power generation equipment, microelectronic packaging and many other engineering endeavors.

For loading conditions in *in-plane antisymmetry* with respect to the crack line, Goldstein and Salganik [4] generalized ideas of Erdogan and Sih [1] and presented the fracture criterion based on the *principle of local symmetry*: the crack will find a path such that at the tip of the growing crack the Mode-I stress intensity factor on the growing crack is equal to the fracture toughness: $K_I = K_{IC}$, and the antisymmetric mode (Mode-II) stress intensity factor for the growing crack is zero: $K_{II} = 0$. This criterion also works quite well; there are numerous examples of its successful application, most famously in the oscillating thermal quench crack experiments of Yuse and Sano [16]; see also Yang and Ravi-Chandar [15].

For loading conditions that introduce *anti-plane symmetry* with respect to the crack plane, Goldstein and Salganik [4] suggested a criterion of the form $K_I = K_{IC}$, with $f(K_{II}, K_{III}) = 0$, where K_{III} is the Mode-III or antiplane stress intensity factor on the growing crack. However, experiments indicate that such a criterion could not be imposed continuously along the crack front; typically, the crack front fragments into multiple cracks. Lin et al. [8] suggested that the principle of local symmetry would also be appropriate to this case, with $K_I = K_{IC}, K_{II} = 0$, and $K_{III} = 0$ along the growing crack. There are a number of open issues related to this problem, particularly ones associated with estimating the spacing of the fragmented cracks. Lazarus [7] has provided a nice review of the mathematical issues related to calculations of the energy release rates for mixed mode, three-dimensional problems.

For dynamic fracture problems, the generalization of the ideas of Griffith is straightforward: incorporate the kinetic energy variation into the energy balance equation. This is typically done in a rate form: $P - (\dot{U} + \dot{K}) = \dot{D}$, over any closed contour surrounding the crack tip, where P is the power input through the boundaries of the contour, $\dot{U}$ and $\dot{K}$ are the rate of change of strain and kinetic energy, respectively, and $\dot{D} = \gamma_P v$ is the rate of dissipation at the tip of a crack moving at a speed v. The sum of terms on the left hand side is typically called the *energy flux* to the crack tip region and denoted by F; a *dynamic energy release rate* (per unit crack extension) can be defined as $G = F/v$. The power-balance equation then yields the crack tip equation of motion $(G(v) - \gamma_P)v = 0$, or, in terms of the *dynamic stress intensity factor*: $A(v)K_I^2(t, v)/E = \gamma_P$ (see [3] for details). This equation suggests that the Rayleigh wave speed, C_R, sets the upper limit for a Mode-I crack. There are numerous experiments that address the dynamic fracture problem (see [11] for a summary of these investigations). The key experimental results are summarized here: for crack speeds below about $0.2C_R$ the crack tip equation of motion above works quite well. However, with increasing crack speed, a dependence of γ_P on crack speed, attributed to evolution of the fracture process even in nominally brittle materials, is observed; numerous investigations have documented $\gamma_P(v)$ for different materials (again, see [11] for specific details). In particular, a large increase in γ_P at $\approx 0.5C_R$ results in a practical speed limit for cracks at this value. The fracture surface becomes quite rough, and cracks branch into multiple cracks beyond this point. Yoffe instability [14] arising from a redistribution of the dynamic stress field, crack front fragmentation and

roughening [12], microbranching instability [2], and other mechanisms have been proposed to explain the limiting speed, crack branching and other dynamic effects; however, a quantitative theory is not yet available.

References

[1] F. Erdogan and G.C. Sih. On the crack extension in plates under plane loading and transverse shear. *Journal of Basic Engineering* 85: 519-523, 1963.

[2] J. Fineberg and M. Marder. Instability in dynamic fracture. *Physics Reports* 313:1-108, 1999.

[3] L. B. Freund. *Dynamic Fracture Mechanics*. Cambridge University Press, 1990.

[4] R.V Goldstein and R.L Salganik. Brittle fracture of solids with arbitrary cracks. *International Journal of Fracture* 10:507-523, 1974.

[5] A.A. Griffith. The phenomena of rupture and flow in solids. *Philosophical Transactions of the Royal Society of London* A221:163-198, 1920.

[6] G.R. Irwin. Analysis of stresses and strains near the end of a crack traversing a plate. *Journal of Applied Mechanics* 24:361-364, 1957.

[7] V. Lazarus. Perturbation approaches of a planar crack in linear elastic fracture mechanics: A review. *Journal of the Mechanics and Physics of Solids* 59:121-144, 2011.

[8] B. Lin, M.E. Mear and K. Ravi-Chandar. Criterion for initiation of cracks under mixed-mode I + III loading. *International Journal of Fracture* 165:175188, 2010.

[9] J. W. Obreimoff. The splitting strength of mica. *Proceedings of the Royal Society of London* A127:290-297, 1930.

[10] E. Orowan. Fracture and strength of solids. *Report on Progress in Physics* 12:185-232, 1948.

[11] K. Ravi-Chandar. *Dynamic Fracture*. Elsevier, 2004.

[12] K. Ravi-Chandar and W.G. Knauss. An experimental investigation into dynamic fracture - III. On steady state crack propagation and branching. *International Journal of Fracture* 26:141-154, 1984.

[13] M.L. Williams. On the stress distribution at the base of a stationary crack. *Journal of Applied Mechanics* 24:109-113, 1957.

[14] E. Yoffe. The moving Griffith crack. *Philosophical Magazine* 42:739-750, 1951.

[15] B. Yang and K. Ravi-Chandar. Crack path instabilities in a quenched glass plate. *Journal of the Mechanics and Physics of Solids* 49:91-130, 2011.

[16] A. Yuse and M. Sano. Transition between crack patterns in quenched glass plates. *Nature* 362:329-331.

Quasistatic crack growth

Gianni Dal Maso

In recent years there has been a remarkable progress in the mathematical analysis of several variational models of quasistatic crack growth [12, 8, 9, 4, 11, 6, 5, 1]. The deformed cracked body is described by a time dependent pair $(u(t), \Gamma(t))$, where $u(t)$ is the deforamation at time t (or the dispalcement, in the linearized models) and $\Gamma(t)$ is the crack at time t in the reference configuration. It is assumed that $\Gamma(t)$ has dimension $n-1$ and that the discontinuity set of $u(t)$ is contained in $\Gamma(t)$ for every t.

The relevant energetic terms are the stored elastic energy $\mathcal{E}(u)$, depending only on the deformation u, the work done by the loads $\mathcal{L}(u)$, and the energy dissipated by the crack $\mathcal{K}(\Gamma)$, depending only on Γ in the case of brittle cracks.

The mathematical formulation of these models is based on a stability condition and on an energy balance. The strongest form of stability is the global minimality condition:

$$\mathcal{E}(u(t)) + \mathcal{K}(\Gamma(t)) - \mathcal{L}(u(t)) \leq \mathcal{E}(u) + \mathcal{K}(\Gamma) - \mathcal{L}(u) \tag{1}$$

for every crack $\Gamma \supset \Gamma(t)$ and for every deformation (resp. displacement) u satisfying the boundary conditions and whose discontinuity set is contained in Γ. The energy balance requires that

$$\mathcal{E}(u(t)) + \mathcal{K}(\Gamma(t)) = \mathcal{E}(u(0)) + \mathcal{K}(\Gamma(0)) + \text{work of the loads in } [0,t]\,.$$

An approximate solution is usually constructed by solving incremental minimum problems; passing to the limit as the time step tends to zero one obtains a solution of the evolution problem.

The main results have been proved for models where the stability is interpreted as the global minimality condition considered in (1): this simplifies the mathematics of the problem. Some of these models allow to predict the crack path and to deal with large deformations in the framework of finite elasticity, taking also into account the noninterpenetration condition [7].

So far there are only two ways to replace global minimality by a local stability condition. One is based on the minimization on a set of pairs that are reachable from the present pair $(u(t), \Gamma(t))$ by means of an ε-slide [15]: this is a continuous path along which the increase of energy is at most ε. Another one is based on a sort of viscosity approximation, but so far has been applied only to the case of a prescribed crack path [2, 13, 18, 19, 14, 17], or assuming a priori bounds on the curvature of the cracks [16]. It would be useful to extend these techniques to the case of a free crack path.

These models deal with the case of brittle cracks. Similar results for cohesive zone models, where the crack energy depend also on the opening of the crack, have been obtained only in the case of a prescribed crack path under special hypotheses on the cohesive forces [10, 3].

References

[1] B. Bourdin, G.A. Francfort, J.-J. Marigo, *The variational approach to fracture* J. Elasticity **91** (2008), 5–148.

[2] F. Cagnetti, *A vanishing viscosity approach to fracture growth in a cohesive zone model with prescribed crack path* Math. Models Methods Appl. Sci. **18** (2008), 1027–1071.

[3] F. Cagnetti, R. Toader, *Quasistatic crack evolution for a cohesive zone model with different response to loading and unloading: a Young measure approach*, ESAIM Control Optim. Calc. Var., **17** (2011), 1–27..

[4] A. Chambolle, *A density result in two-dimensional linearized elasticity, and applications*, Arch. Ration. Mech. Anal. **167** (2003), 211–233.

[5] A. Chambolle, A. Giacomini, M. Ponsiglione, *Crack initiation in brittle materials*, Arch. Ration. Mech. Anal. **188** (2008), 309–349.

[6] G. Dal Maso, G.A. Francfort, R. Toader, *Quasistatic crack growth in nonlinear elasticity*, Arch. Ration. Mech. Anal. **176** (2005), 165–225.

[7] G. Dal Maso, G. Lazzaroni, *Quasistatic crack growth in finite elasticity with non-interpenetration*, Ann. Inst. H. Poincaré Anal. Non Linéaire **27** (2010), 257–290.

[8] G. Dal Maso, R. Toader, *A model for the quasi-static growth of brittle fractures: existence and approximation results*, Arch. Ration. Mech. Anal. **162** (2002), 101–135.
[9] G. Dal Maso, R. Toader, *A model for the quasi-static growth of brittle fractures based on local minimization*, Math. Models Methods Appl. Sci. **12** (2002), 1773–1799.
[10] G. Dal Maso, C. Zanini: *Quasistatic crack growth for a cohesive zone model with prescribed crack*, Proc. Roy. Soc. Edinburgh Sect. A **137** (2007), 253–279.
[11] G.A. Francfort, C.J. Larsen, *Existence and convergence for quasi-static evolution in brittle fracture*, Comm. Pure Appl. Math. **56** (2003), 1465–1500.
[12] G.A. Francfort, J.-J. Marigo, *Revisiting brittle fracture as an energy minimization problem.* J. Mech. Phys. Solids **46** (1998), 1319–1342.
[13] D. Knees, A. Mielke, C. Zanini, *On the inviscid limit of a model for crack propagation*, Math. Models Methods Appl. Sci. **18** (2008), 1529–1569.
[14] D. Knees, C. Zanini, A. Mielke, *Crack growth in polyconvex materials*, Physica D **239** (2010), 1470–1484.
[15] C.J. Larsen, *Epsilon-stable quasi-static brittle fracture evolution*, Comm. Pure Appl. Math. **63** (2010), 630–654.
[16] G. Lazzaroni, R. Toader, *A model for crack propagation based on viscous approximation* Math. Models Methods Appl. Sci., to appear.
[17] M. Negri, *From rate-dependent to rate-independent brittle crack propagation*, J. Elasticity **98** (2010), 159–178.
[18] M. Negri, C. Ortner, *Quasi-static crack propagation by Griffiths criterion*, Math. Models Methods Appl. Sci. **18** (2008), 1895–1925.
[19] R. Toader, C. Zanini, *An artificial viscosity approach to quasistatic crack growth*, Boll. Unione Mat. Ital. (9) **2** (2009), 1–35.

Variational Models of Dynamic Fracture

Michael Ortiz

Crack fronts play a fundamental role in engineering models of fracture. They are the location of both the crack growth and the energy dissipation due to growth. However, there has not been a rigorous mathematical definition of crack front, nor rigorous mathematical analysis predicting fracture paths using these fronts as the location of growth and dissipation. Here, we give a natural weak definition of crack front and front speed, and consider models of crack growth in which the energy dissipation is a function of the front speed, that is, the dissipation rate potential at the time t is of the form

$$\int_{F(t)} \psi(v(x,t)) d\mathcal{H}^{n-2}(x),$$

where $F(t)$ is the front at time t, and v is the front speed. We show how this dissipation can be used within existing models of fracture, including dynamics and inertia.

A short survey on energy release rates

Dorothee Knees

The energy release rate plays a central role in the modeling of crack propagation in elastic materials. It quantifies the amount of stored energy that is released at an infinitesimal crack extension. For a single crack of length s in two dimensions it is defined as

$$ERR(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(u_{\min}(s), s),$$

where $\mathcal{E}(v,s) = \int_{\Omega_s} W_{\mathrm{el}}(\nabla v)\,\mathrm{d}x - \langle \ell, v\rangle$ describes the stored energy for a displacement field v on a body $\Omega_s = \Omega\backslash C_s$ with a crack C_s of length s, and $u_{\min} \in \operatorname{argmin}\{\,\mathcal{E}(v,s)\,;\, v \in V_{\mathrm{adm}}(\Omega_s)\,\}$ is the minimizing displacement or deformation field. Formulas for the energy release rate are well known since long time. Here, we describe shortly, which of these formulas are justified rigorously from a mathematical point of view. Thereby we discuss linear elasticity, models of power-law type and finite strain elasticity. We focus on the simple case of a straight crack and assume that the crack tip is located in $(0,0)$ and $C_s = (-\infty, 0] \times \{0\} \cap \Omega$. While the extension to smooth curved cracks is straightforward, only few results are known for cracks with kinks or even with several branches.

1. Linear elasticity

In the case of linear elastic materials the energy release rate is well defined and there are basically three equivalent formulas expressing this quantity. The elastic energy density is of the form $W_{\mathrm{el}}(F) = \frac{1}{2}\mathbb{C}F_{\mathrm{sym}} : F_{\mathrm{sym}}$, where $F_{\mathrm{sym}} \in \mathbb{R}^{2\times 2}$ stands for the linearized strain tensor and $\mathbb{C}$ denotes the positive definite, symmetric fourth order elasticity tensor, which is assumed to be constant in Ω. In the following we assume that the volume forces are zero in a neighborhood of the crack tip. Otherwise, additional terms enter into the formulas presented here below.

Based on a smooth one-parameter family of spatial transformations, by which domains $\Omega_{s+\rho}$ with a crack extended by ρ are mapped to the domain Ω_s, it was shown in [5] that the energy release rate is well defined and satisfies

$$ERR(s) = \int_{\Omega_s} \mathbb{E}(\nabla u_{\min}(s)) : e_1 \otimes \nabla\theta\,\mathrm{d}x. \tag{1}$$

Here, $\theta \in C_0^\infty(\mathbb{R}^2)$ is an arbitrary cut-off function with $\operatorname{supp}\theta \subset \Omega$ and $\theta = 1$ in a neighborhood of the crack tip and $e_1 = (1,0)^\top$ is tangential to the crack. Moreover, $\mathbb{E}(\nabla u) = \nabla u^\top DW_{\mathrm{el}}(\nabla u) - W_{\mathrm{el}}(\nabla u)\mathbb{I}$ denotes the Eshelby tensor associated with the energy density W_{el}. We use the notation $A : B = \sum_{i,j=1}^2 a_{ij}b_{ij}$ for the inner product of matrices A, B. In [10] the investigations were extended to cracks with contact conditions on the crack surface and in [14] continuity properties with respect to s were added. In the derivation of (1) no higher regularity properties of the minimizers are needed. Hence, this approach opens the possibility to study the

existence of the energy release rate also for more general energy densities, where no additional information about the regularity of minimizers is available.

In the case of linear elasticity, very detailed information on the regularity of the minimizer $u_{\min}$ is known: Let $R > 0$ such that the ball $B_R(0)$ is contained in Ω. Then, if the datum ℓ is smooth enough, $u_{\min}(s)$ belongs to the Besov space $B^{\frac{3}{2}}_{2,\infty}(B_R(0)\backslash C_s)$ and $u_{\min}(s) \in H^2(B_R(0)\backslash(B_{R/2}(0)\cup C_s))$. Based on these regularity properties the representation formula (1) can be integrated by parts. Using the fact that minimizers satisfy the Noether equations (derived via inner variations), one finally obtains the representation of the energy release rate via the J-integral, also called Cherepanov-Rice-integral:

$$ERR(s) = \int_\Gamma \mathbb{E}(\nabla u_{\min}(s))\mathbf{n} \cdot e_1 \mathrm{d}\Gamma. \tag{2}$$

Here, Γ is an arbitrary path enclosing the crack tip and $\mathbf{n}$ denotes the inner normal vector.

Finally, the following asymptotic expansion is valid in a neighborhood of the crack tip

$$u_{\min}(x) = \eta(x)\Big(\sum_{i\in\{1,2\}} K_i r^{\frac{1}{2}} v_i(\phi)\Big) + u_{\mathrm{reg}}(x),$$

where (r, ϕ) are polar coordinates and η a cut off function centered at the crack tip, K_i are the stress intensity factors depending on the applied loading and on the domain Ω_s, and v_i are smooth functions and depend on the coefficient tensor $\mathbb{C}$. These expansions were first described in [20, 21, 9] and mathematically rigorously justified in [16, 7, 17, 4], to name a few. Inserting this expansion into the J-integral results in a formula for the energy release rate that is solely based on the stress intensity factors. It is of the general form

$$ERR(s) = \mathbb{M}\left(\begin{smallmatrix} K_1 \\ K_2 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} K_1 \\ K_2 \end{smallmatrix}\right) \tag{3}$$

with a 2×2-matrix $\mathbb{M}$ that depends on the coefficient tensor $\mathbb{C}$. In the case of isotropic materials and assuming a certain normalization for the functions v_i it holds $ERR = \frac{1-\nu^2}{E}(K_1^2 + K_2^2)$, where ν is the Poisson ratio and E the Young modulus, [8].

The above described relations can easily be extended to smooth curved cracks and to energy densities, where $\mathbb{C}$ depends smoothly on x.

In [1, 18, 19] the question of a kinking crack was investigated. It turns out that again a formula of the type (3) is valid, where the matrix $\mathbb{M}$ depends also on the kink angle. More general crack variations were studied in [3] using Γ-convergence methods. To the author's knowledge it is a completely open problem to characterize the energy release rate for the very general crack geometries that occur in the Francfort/Marigo model [6].

2. Power-law models

Within the deformation theory of plasticity, power-law models are frequently used to characterize the stress strain relation. A typical class of such models is

given by the Ramberg/Osgood or Norton/Hoff models. Here, the stress strain relation is of the structure

$$\mathbf{e}(u) = \mathbb{A}\sigma + \kappa\,|\sigma_D|^{p-2}\,\sigma_D, \tag{4}$$

where $\mathbf{e}(u) = (\nabla u)_{\mathrm{sym}}$ is the linearized strain tensor, σ the stress tensor, $\sigma_D = \sigma - \frac{1}{d}\operatorname{tr}\sigma\mathbb{I}$ the deviatoric part of σ, $\mathbb{A}$ is the inverse of the elasticity tensor $\mathbb{C}$, and $p \in (1,2)$ and $\kappa > 0$ are further material dependent constants. Given some load ℓ and suitable boundary conditions, the corresponding displacement and stress fields are determined from (4) and the force balance equation $\operatorname{div}\sigma + \ell = 0$ in Ω. Based on the regularity results for the solutions proved in [2, 11], the formulas (1) and (2) were justified from a mathematical point of view in [12].

3. Finite strain elasticity

In the case of finite strain elasticity the situation changes completely. Here, only very little is known about the regularity of minimizers, the energy in general is not Gateaux differentiable and there might exist several minimizers. Nevertheless, using the method of variations of the domain it can be shown that the energy has left and right derivatives with respect to the crack length. Due to the non-uniqueness of the minimizers these derivatives might differ, and hence it is not clear, which of these generalized energy release rates should occur in a crack evolution model. A first answer to this question was given in [15].

Let us briefly give some more details. In the finite strain case it assumed that the energy density $W_{\mathrm{el}} : \mathbb{R}^{2\times 2} \to [0,\infty]$ is frame-indifferent, polyconvex and coercive. Moreover, a multiplicative stress-control assumption should be satisfied i.e. for all $F \in \mathbb{R}^{2\times 2}$ with $\det F > 0$ it holds $\left|F^\top DW_{\mathrm{el}}(F)\right| \le c(1 + W_{\mathrm{el}}(F))$. Let $\mathcal{I}(s) := \min\{\, \mathcal{E}(\varphi, s) = \int_{\Omega_s} W_{\mathrm{el}}(\nabla\varphi)\,\mathrm{d}x - \langle \ell, \varphi\rangle\,;\, \varphi \in V_{\mathrm{adm}}(\Omega_s)\,\}$. Then $\mathcal{I}$ depends locally Lipschitz continuously on the crack length s, the left and right derivatives with respect to s exist and are given by [13, 15]

$$\partial_s^- \mathcal{I}(s) = \max\{\, -G(\varphi, s)\,;\, \varphi \text{ minimizes } \mathcal{E}(\cdot, s)\,\},$$
$$\partial_s^+ \mathcal{I}(s) = \min\{\, -G(\varphi, s)\,;\, \varphi \text{ minimizes } \mathcal{E}(\cdot, s)\,\}.$$

Here, $G(\varphi, s) = \int_{\Omega_s} \mathbb{E}(\nabla\varphi) : e_1 \otimes \nabla\theta\,\mathrm{d}x$ is the same expression as in (1). Moreover, $\partial_s^+\mathcal{I}$ is lower semicontinuous and right continuous and $\partial_s^-\mathcal{I}(\cdot)$ is upper semicontinuous and left continuous. The proof relies on a variation of domains in connection with a weak continuity property of the Eshelby tensor along sequences of weakly converging deformation fields $(\varphi_n)_n$ with converging energies $\mathcal{E}(\varphi_n, s)$.

References

[1] I. I. Argatov and S. A. Nazarov, *Energy release rate caused by the kinking of a crack in a plane anisotropic solid*, J. Appl. Math. Mech. **66** (2002), no. 3, 491–503.

[2] A. Bensoussan and J. Frehse, *Asymptotic behaviour of Norton-Hoff's law in plasticity theory and H^1 regularity*, Boundary value problems for partial differential equations and applications (J.-L. Lions and C. Baiocchi, eds.), RMA, vol. 29, Masson, Paris, 1993, pp. 3–25.

[3] A. Chambolle, G.A. Francfort, and J.-J. Marigo, *Revisiting energy release rates in brittle fracture*, J. Nonlinear Sci. **20** (2010), no. 4, 395–424 (English).

[4] M. Costabel and M. Dauge, *Crack singularities for general elliptic systems*, Math. Nachrichten **235** (2002), 29–49.
[5] P. Destuynder and M. Djaoua, *Sur une interpretation mathématique de l'intégrale de Rice en théorie de la rupture fragile*, Math. Methods Appl. Sci. **3** (1981), 70–87.
[6] G.A. Francfort and J.-J. Marigo, *Revisiting brittle fracture as an energy minimization problem.*, J. Mech. Phys. Solids **46** (1998), no. 8, 1319–1342 (English).
[7] P. Grisvard, *Singularité en élasticité*, Arch. Ration. Mech. Anal. **107** (1989), 157–180.
[8] D. Gross, *Bruchmechanik*, Springer Verlag, Berlin, Heidelberg, 1996.
[9] G. R. Irwin, *Analysis of stresses and strains near the end of a crack traversing a plate*, J. Appl. Mech. **24** (1957), 361–364.
[10] A. M. Khludnev and J. Sokolowski, *Griffith formulae for elasticity systems with unilateral conditions in domains with cracks*, Eur. J. Mech., A/Solids **19** (2000), 105–119.
[11] D. Knees, *Global regularity of the elastic fields of a power-law model on Lipschitz domains*, Math. Methods Appl. Sci. **29** (2006), 1363–1391.
[12] ———, *Griffith-formula and J-integral for a crack in a power-law hardening material*, Math. Models Methods Appl. Sci. **16** (2006), no. 11, 1723–1749.
[13] D. Knees and A. Mielke, *Energy release rate for cracks in finite-strain elasticity*, Math. Methods Appl. Sci. **31** (2008), no. 5, 501–528.
[14] D. Knees, A. Mielke, and C. Zanini, *On the inviscid limit of a model for crack propagation*, Math. Models Methods Appl. Sci. **18** (2008), no. 9, 1529–1569.
[15] D. Knees, C. Zanini, and A. Mielke, *Crack growth in polyconvex materials*, Physica D **239** (2010), no. 15, 1470 – 1484.
[16] V. A. Kondrat'ev, *Boundary value problems for elliptic equations in domains with conical or angular points*, Trans. Moscow Math. Soc. **10** (1967), 227–313.
[17] V.G. Maz'ya, S. A. Nazarov, and B. A. Plamenevsky, *Asymptotische Theorie elliptischer Randwertaufgaben in singulär gestörten Gebieten I,II*, Akademie-Verlag, Berlin, 1991.
[18] M. Negri, *Energy release rate along a kinked path*, Math. Methods Appl. Sci. **34** (2011), no. 4, 384–396 (English).
[19] M. Steigemann, *Generalized eigenfunctions and local integral characteristics for quasi-static propagation of cracks in anisotropic materials*, Ph.D. thesis, Berichte aus der Mathematik. Aachen: Shaker; Univ. Kassel, Inst. für Analysis und Angew. Mathematik (Diss. 2008).
[20] H. M. Westergaard, *Bearing pressures and cracks*, J. Appl. Mech. **6** (1939), A49–A53.
[21] M.L. Williams, *On the stress distribution at the base of a stationary crack*, J. Appl. Mech. **24** (1957), 109–114.

A Continuum Model of Dynamic Steady-State Fracture Incorporating Interfacial Mechanics

Jay R. Walton
(joint work with Tsvetanka Sendova)

Classical brittle fracture modeling is set in the context of the linearized theory of elasticity and a crack is modeled as a surface (in 3-dimensions) or a line (in 2-dimensions) across which the displacement can exhibit a jump discontinuity when the body is placed under an external thermal or mechanical load. Moreover, classically a crack surface is viewed as a free-surface on which tractions can be freely imposed as a boundary condition. In this setting, solutions to the corresponding crack boundary value problem for both the static and dynamic Navier linear system

of equations for the displacement field in the body exhibit the well-known square-root stress/strain singularity at a crack-tip (in 2-dimensions) or crack-edge (in 3-dimensions).

In this talk, an alternative fracture modeling paradigm is described in which crack surfaces are viewed as "dividing surfaces" endowed with "excess" physical properties (mass, internal energy, stress, entropy, temperature). In similar fashion, crack edges (the boundaries of crack surfaces) are also modeled as possessing excess physical properties. One can derive such models by a homogenization procedure whereby the excess properties on material interfaces (such as fracture surfaces) arise as an asymptotic limit of an infinitesimally thin interfacial region adjoining distinct material phases. For a fracture surface, the distinct, adjoined phases are the intact bulk solid phase and the vacuum (or gas or fluid) phase occupying the opened crack. In this setting, crack surfaces are no longer free surfaces on which boundary conditions can be freely imposed. Rather, the appropriate boundary conditions are prescribed through the jump momentum balance. One must then impose a constitutive model for stress-deformation behavior within the interface.

In [1], the problem of a single, finite length, static crack in an infinite, linear elastic body (the classical Griffith crack problem) is analyzed in this new fracture setting. It is shown that if stresses in the fracture surface are modeled as Eulerian, i.e. consists only of surface tension, and if surface tension is assumed to have a linear dependence upon crack-surface mean curvature (in the deformed configuration), then solutions to the corresponding boundary value problem for the static Navier equations exhibit bounded stresses/strains at a crack tip or edge. That paper also contains a discussion of thermomechanics for the corresponding dynamic, transient crack growth problem, and derives a new necessary condition for crack growth based upon the second-law of thermomechanics. This new fracture condition replaces the classical energy release rate based Griffith criterion which no longer applies since crack-tips (edges) are not singular energy sinks.

The jump momentum balance boundary condition is most simply formulated in the deformed configuration (opened crack). But elastic boundary value problems are more simply studied when formulated in the reference configuration (closed, unloaded crack). However, in the reference configuration, the jump momentum balance boundary condition with curvature dependent surface tension takes a very complicated nonlinear form. When linearized under the assumption of infinitesimal strains and utilizing the static, linear-elastic Dirichlet-to-Neumann map on the fracture-surface, the fracture boundary condition becomes a system of singular, integro-differential equations involving higher-order tangential derivatives of the crack-surface displacement components. For the finite length, static Griffith crack problem studied in [1], it is shown that given a suitably strong dependence of surface-tension upon (linearized) mean-curvature, this boundary integro-differential system possesses smooth solutions resulting in bounded crack-tip stresses/strains and a cusp-shaped crack-tip opening profile.

As expected, incorporating dynamics into the fracture problem complicates the analysis considerably. A useful intermediate problem to study is dynamic, steady-state fracture in which the (constant) crack-speed is not small relative to the speed of Rayleigh waves requiring retention of the inertial terms in the balance of linear momentum, but for which the motion of the body appears static relative to a coordinate system moving with the crack-tip. The governing equations in the deformed configuration are the (bulk) balance of linear momentum

$$\rho\ddot{\mathbf{u}} = \operatorname{div}\mathbf{T} + \mathbf{b} \quad \text{in} \quad \mathbf{B}_t$$

subject to the jump momentum boundary condition on the crack surfaces

$$\rho^{(\sigma)}\ddot{\mathbf{u}} = \operatorname{div}^{(\sigma)}\mathbf{T}^{(\sigma)} + [[\mathbf{T}]]\mathbf{n} \quad \text{on} \quad \partial\mathbf{B}_t, \tag{1}$$

and a uniform tensile traction σ^{∞} applied at infinity, where $\mathbf{u}$ denotes the displacement, $\mathbf{T}$ and $\mathbf{T}^{(\sigma)}$ denote the bulk and surface Cauchy stress tensors, div and $\operatorname{div}^{(\sigma)}$ denote the bulk and surface divergence operators, $\mathbf{b}$ denotes the bulk body force density, and $[[\mathbf{T}]]\mathbf{n}$ denotes the bulk traction jump across the crack surfaces ($\mathbf{n}$ being the unit normal to the crack surface pointing out of the bulk). Modeling the surface stress-deformation behavior as purely Eulerian, that is

$$\mathbf{T}^{(\sigma)} := \tilde{\gamma}\mathbf{P} \tag{2}$$

where $\tilde{\gamma}$ denotes the surface tension and $\mathbf{P} := \mathbf{I} - \mathbf{n}\otimes\mathbf{n}$ is the projection operator onto the crack-surface tangent plane, the surface divergence operator takes the form

$$\operatorname{div}^{(\sigma)}\mathbf{T}^{(\sigma)} = \operatorname{grad}^{(\sigma)}\tilde{\gamma} - 2\tilde{\gamma}H\mathbf{n} \tag{3}$$

with H denoting the crack-surface mean-curvature in the deformed (open crack) configuration and $\operatorname{grad}^{(\sigma)}$ is the surface gradient operator.

Surface tension must be specified constitutively. While surface tension is often taken to be constant, it was shown in [1] that when $\tilde{\gamma}$ is constant, the solution to the static Griffith crack boundary value problem exhibits a logarithmic crack-tip stress singularity and a crack-tip opening angle θ satisfying $0 < \theta < \pi$ with $\theta = 0$ corresponding to a cusp-shaped crack-surface profile at the tip and $\theta = \pi$ is the blunt crack-tip profile associated to the classical square-root stress/strain crack-tip singularity. However, it was further demonstrated in [1] that assuming surface tension has a dependence upon mean-curvature of the form

$$\tilde{\gamma} = \gamma_0 + \gamma_1 H, \tag{4}$$

then the solution to the crack boundary value problem has a cusp-shaped crack-surface profile and bounded stresses/strains at a crack-tip provided the ratio γ_1/γ_0 is sufficiently large. In (4), γ_0 should be viewed as the (constant) surface tension on an infinite, flat interface between the bulk solid phase and the vacuum (or gas) phase occupying the opened crack.

Consider now the dynamic, plane-strain, steady-state problem of a semi-infinite crack propagating at constant speed V in an infinite, linearly elastic body under the action of an applied crack-face traction of finite impulse. Relative to a coordinate

system attached to the crack-tip, the crack lies along the negative, horizontal coordinate axis and the dynamic, Navier equations take the (static) form

$$(V/c_T)^2\mathbf{w}_{,11} = \triangle\mathbf{w} + (2+\lambda/\mu)\nabla\mathrm{div}\,\mathbf{w} \tag{5}$$

where $\mathbf{w}$ denotes the displacement (in the moving frame), $c_T := \sqrt{\mu/\rho}$ is the speed of transverse waves, and μ and λ are the Lamè parameters. The jump momentum balance boundary condition on the crack becomes

$$V^2\rho^{(\sigma)}\mathbf{w}_{,11} = \mathrm{div}^{(\sigma)}\mathbf{T}^{(\sigma)} + [[\mathbf{T}]]\mathbf{n} \tag{6}$$

with $\mathbf{T}^{(\sigma)}$ given by (2), (4). After linearization (under the assumption of infinitesimal strains), the crack boundary condition (6) takes the form (for $z_1 < 0$)

$$\begin{aligned}\tau_{12}(z_1,0+) &= -\gamma_1 w_{2,111}(z_1,0+) + V^2\rho^{(\sigma)}w_{1,11}(z_1,0+)\\ \tau_{22}(z_1,0+) &= -P\phi(-z_1/l) + (-\gamma_0 + V^2\rho^{(\sigma)})w_{2,11}(z_1,0+)\end{aligned}$$

where $P\phi(z/l)$ is the applied crack-face loading (l being a length-scale associated with the loading) satisfying the finite impulse condition

$$\int_0^\infty |\phi(z)|dz < \infty.$$

After nondimensionalization and appealing to the Dirichlet-to-Neumann map for the dynamic, steady-state Navier system (5), the boundary value problem can be recast as a boundary, singular integrodifferential equation of the form

$$\begin{aligned}-p\frac{\xi\omega^2}{\beta_2(\omega)}\phi'(x) - p\mathcal{H}_-^{-1}\{\phi\} &= \frac{A(\omega)}{\beta_2(\omega)}v'''(x) + \frac{R(\omega)}{\omega^2\gamma_L(\omega)}v'(x)\\ &\quad + \xi\omega^2\frac{\gamma_T(\omega)}{\omega_L(\omega)}\mathcal{H}_-\{v'\} - (\gamma_0+\xi\omega^2)\mathcal{H}_-^{-1}\{v''\}\end{aligned} \tag{7}$$

where $\mathcal{H}\{\cdot\}$ denotes the (restricted) Hilbert transform

$$\mathcal{H}_-\{f\}(x) := \frac{1}{\pi}\int_{-\infty}^0 f(r)\frac{dr}{r-x},$$

$\mathcal{H}_-^{-1}\{g\}(x)$ is the (left) inverse

$$\mathcal{H}_-^{-1}\{g\}(x) := -\frac{\sqrt{|x|}}{\pi}\int_{-\infty}^0 \frac{g(r)}{\sqrt{|r|}}\frac{dr}{r-x},$$

and various (dimensionless) parameters and functions defined through: $\omega := V/c_T$, $p := P/\mu$, $x := z_1/L$, $v := w_2/L$, $\tau := \rho^{(\sigma)}/\rho$, $\xi := \tau/L$, $\tilde{\gamma}_0 := \gamma_0/(\mu L)$, $\tilde{\gamma}_1 := \gamma_1/(\mu L^2)$, $g := (\mu\gamma_1)/\gamma_0^2$, $k := (\tau\mu)/\gamma_0$, $\gamma_T(\omega) := \sqrt{1-\omega^2}$, $\gamma_L(\omega) := \sqrt{1-(\eta\omega)^2}$, $\eta^2 := (c_T/c_L)^2 = (1/2-\nu)/(2-3\nu)$, $\beta_1(\omega) := \omega^2\gamma_T(\omega)/(1-\gamma_L(\omega)\gamma_T(\omega))$, $\beta_2(\omega) := \omega^2\gamma_L(\omega)/(1-\gamma_L(\omega)\gamma_T(\omega))$, $\alpha(\omega) := (2\gamma_L(\omega)\gamma_T(\omega) - (2-\omega^2))/(1-\gamma_L(\omega)\gamma_T(\omega))$, $A(\omega) := g\alpha(\omega) + k\omega^2(1-k\omega^2)$, and $R(\omega) := 4\gamma_L(\omega)\gamma_T(\omega) - (2-\omega^2)^2$ is the Rayleigh function with respect to which ω_R is the (dimensionless) is the speed of Rayleigh surface waves given as the unique positive number satisfying $R(\omega_R) = 0$. One should note that $\tau = \rho^{(\sigma)}/\rho$ is a length-scale corresponding to the thickness

of the interfacial layer. Also, L is any suitably chosen length-scale for the purpose of nondimensionalization and ν is Poisson's ratio for the bulk material.

One now concludes from the analysis in [1] that a necessary condition for (7) to have a continuous solution $v'''(x)$ (and hence for the boundary value problem to predict bounded crack-tip stresses and strains) is that the lead coefficient $A(\omega)$ not vanish. Depending upon material parameters this condition places an upper-bound upon allowable crack-speeds that is strictly below the speed of Rayleigh waves, the upper bound predicted from the classical fracture problem with singular crack-tip stresses. This prediction is in agreement with experimental observations that suggest an upper bound on crack-speeds in most materials of around 40% – 60% of the speed of Rayleigh waves.

There remains the key question of defining a criterion for predicting the dynamic, steady-state crack-speed. Two candidate criteria are suggested in [1], one based upon the thermodynamic analysis presented there together with the crack-edge jump momentum balance, and the other based upon the notion of the maximum crack-tip cleavage stress which is well-defined for the present model due to its prediction of a finite crack-tip stress. Future work will explore these and possibly other strategies for devising a physically motivated crack-tip speed selection criterion appropriate to the modeling paradigm described above, and investigate the challenge of developing direct numerical procedures, such as the finite element method, for performing fracture simulations based upon this crack model setting.

References

[1] T. Sendova and Jay R. Walton, A new approach to the modeling and analysis of fracture through extension of continuum mechanics to the nanoscale, *Math. Mech. Solids*, **15**(3), (2010), 368-413.

X-FEM level sets methods: application to dynamic crack propagation

Alain Combescure

(joint work with Antony Gravouil, Nicolas Moes, T Elguedj)

The computation of dynamic crack propagation is a difficult topic. The main issue is due to the fact that crack propagation mechanisms generally use a small part on the available energy. As a consequence a small error in the global energy conservation leads to a large error in the crack propagation prediction. Many attempts have been done in the past to produce good simulations.
One may cite the following methods:

- The element erosion method associated with finite elements: when one finite element has reached a critical state (excessive strain, critical damage, too large dissipated energy,...) it is removed from the mesh.
- The EFG method: when one "node" has reached a critical state defined as in the erosion method, the connections between this node and its neighbours is cut and the "node" is free ([1],[2],[3],[4],[5],[6]).

- The cohesive zone method: all element sides are initially or progressively connected by cohesive elements whose fracture energy is the material energy release rate G_C: the connection starts to fracture when the stress in the cohesive zone exceeds some critical value σ_C.

The simulations which have encountered the most success until the end of last century all need a very fine mesh and a local vision of fracture. The cohesive zone approach first introduced by Needleman for quasistatic prediction and then extended for dynamic cases (There are many interesting papers on this topic [7] [10] [9], [8]). These methods allow to predict in the same formalism crack propagation, branching and well as fragmentation: the brittle cracks are governed by only two parameters the critical stress and the fracture energy release rate. Some precautions have to be employed for the success of the predictions. They will be given later. The second approach is based of EFG or eroding elements in standard finite elements: these methods also request extremely fine meshes and ad hoc material models, for instance one has to avoid artificial localization if one uses damage modeling of fracture. If one does not use damage theory associated with a critical damage of 1, one removes from the computation the elastic energy contained in the reference volume around the "broken" "node" each time it occurs. The X-FEM level set method has developed recently (between 1999 and 2011 [11]) and are based on two concepts:

- The mesh of the interface is independent from that of the structure
- When one element is cut by the interface an extended finite element is introduced which contains not only the standard continuous degrees of freedom but also additional degrees of freedom which take into account the presence of the discontinuity within the element. For instance xhen this discontinuity is a crack and that this crack cuts the element in two parts 3 jumps degree of freedom are added to the 3 usual translations.

This contribution is devoted to the X-FEM Level set technology (one may refer to [13] to get many details of the method) . It will be focused on the essential aspects and to dynamic crack propagation. The main interest of the method is first the concept of independant meshing of discontinuity surface and structure. This allows easy meshing of interfaces like cracks agregates or foams. The level set concept is still more interesting because it allows to represent the interface by signed distance to the usual geometrical nodes. This has a very important practical consequence : when one wants to perform parametric studies on interface shapes (e. g. crack shapes and or position) one is not required to change the structural mesh but only the numerical values of the level sets at the nodes of the mesh: hence there is huge economy in manpower for mesh generation. The second main interest of the method is that it allows very easy crack propagation simulation with a controlled precision. If one moves a crack or an interface one only has to change the level sets values. The discussion on the precision shall be given later.

1. The discontinuity representation.

Two manners can be chosen to represent the interface.

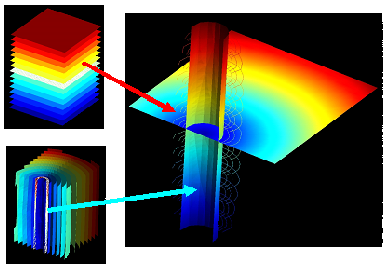

FIGURE 1. Semi elliptic crack Level Set representation

- A simple "mesh" (line (resp. surface)for 2D (resp. 3D))
- A level set which is the zero isovalue of a signed distance scalar field. (the image of geographic maps where the sea level is zero, the deep blue represents large negative height whereas dark brown represent high positive alitudes)

A crack is represented by two interfaces: one represents the crack surface and the second intersects the first one on the crack front. Figure 1 represents the two level sets which trace a semi elliptic 3D plane crack The values of the level sets are given on the nodes of the basic mesh. This mesh has to be rather fine to ensure a "good" and easy computation of level set values changes when crack propagates. The level set values are propagated using an Hamilton Jacobi equation which is the simple transport equation of the field Φ with the known velocity field $\vec{V}$ on the surface.

$$\frac{\partial \Phi}{\partial t} + \vec{V} \nabla (\Phi) = 0. \tag{1}$$

The stationary solution of this equation is sought at each time increment: this is the new Level Set field. Robust simple solutions are only available on structured meshes. For this reason some authors have chosen to represent the level sets on a 3D auxiliary Cube attached to the structure [14].

2. The X-FEM concept.

One the position of the crack is known one now has to determine how this crack intersects the basic structural mesh. This is a simple geometric problem when the structure is meshed with finite elements. This problem is rather simple in 2D and more complex in 3D cases. Figure 2 represents a simple case. One can observe different three classes of nodes: the nodes belonging to elements whose intersection with the crack is null, the nodes belonging to elements completely cut by the crack (blue nodes) and the nodes belonging to elements where the crack tip is (red nodes). This partition allows to define 3 types of elements. The first type are elements which are completely cut by the crack on which a jump displacement field $[\underline{U}]$ shall be added to the usual continuous field $\underline{U}$. The elements in which the crack tip is for which an additional singular field is added: this field

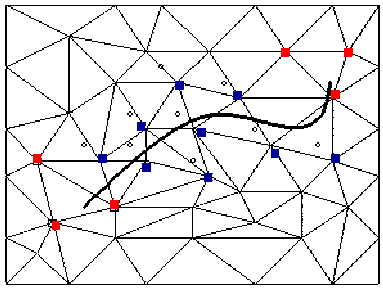

FIGURE 2. Different types of nodes and elements in case of X-FEM mesh: blue Jump added DOF, red Singular added DOF

has 4 unknowns and is such that their combination exactly represents the known singular displacement elastic field around crack tip. The last class of elements are denoted "blended" elements. These elements contain a mixture of standard usual nodes and of extended nodes. The displacement field is discretized using the partition of unity concept and the displacement field within an element writes in all cases:
(2)

$$\mathbf{u}(\mathbf{x},t) = \sum_{i\in N_{nodes}} N_i(\mathbf{x})\,\mathbf{u}_i(t) + \sum_{j\in N_{cut}} N_j(\mathbf{x})\,H_j(\mathbf{x})\,a_j(t) + \sum_{k\in N_{tip}} N_k(\mathbf{x}) \sum_{l=1}^{l=4} \psi_l(\mathbf{x})\,\mathbf{b}_{kl}(t).$$

In (2) $H_j(\mathbf{x})$ are the Heaviside jump functions whereas $\psi_l(\mathbf{x})$ are the usual four singular functions at the crack tip (see for example Black and Belytschko [12]). Moreover $\mathbf{u}_i(t)$ is the nodal continuous displacement at node i, $\mathbf{a}_j(t)$ is the discontinuous displacement amplitude at node j and $\mathbf{b}_{kl}(t)$ the amplitude of the l^{th} singular degree of freedom at node k, $N_i(\mathbf{x})$ is the usual shape function of node i. The four singular functions ψ_l are:
$\sqrt{r}sin\left(\frac{\theta}{2}\right), \sqrt{r}cos\left(\frac{\theta}{2}\right), \sqrt{r}sin\left(\frac{\theta}{2}\right)sin(\theta), \sqrt{r}cos\left(\frac{\theta}{2}\right)sin(\theta)$.
The numerical integration scheme has to be adapted for the non standard elements. The first technique was to subdivide these elements in simple sub elements (triangles in 2D) and to adopt a simple standard integration rule in each of them. This technique is simple but leads to field transfer problems in case of crack propapagation simulation when the material is history dependent or for transient cases for which the velocity state also depends on history. Elguedj [15] introduced a simple integration law based on a fixed sampling of Gauss integration points within the element. This method avoids any remeshing when the crack propagates and consequently is of good quality in case of plasticity damage of transient computations.

3. Dynamic crack propagation examples.

This section is devoted to the application of X-FEM to the simulation of dynamic crack propagation. The numerical integrator must be chosen with care not

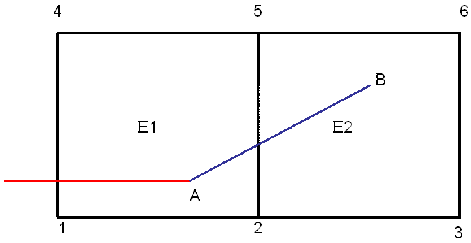

FIGURE 3. X-FEM Mesh of 2 elements for crack advance between points A and B

to dissipate energy. The authors have chosen either the implicit mean acceleration scheme or the central differnce one which is explicit. The se two schemes belong to the family of the Newmark integrators. A special attention must be paid to explicit scheme to get efficient algorithms because if no care is taken the stability time step is close to zero when the crack passes very close to a node. Menouillard [18] [19] has proposed a simple modification of the mass matrix to solve this issue: the resulting CFL time step is only one half of the usual standard time step of an uncracked element. One first explains why X-FEM is a good tool for that purpose and then develops two examples. X-FEM is especially interesting in case of history dependent problem because no remeshing is necessary when the crack propagates.

- Energy conservation. Let us suppose that the crack tip is in element 1 at one time step (position A in Figure 3) and in its neighbor (element 2, tip B) at the next one.

  One will add singular degree of freedom corresponding at crack tip position B at nodes 2 3 5 6 and jump degree of freedom at nodes 1 and 4. Note that theses nodes (e. g. 1 and 4) will have at the same time the singular DOF corresponding to crack tip position A (which is no longer active) and jump degree of freedom corresponding to crack tip position B. If one initializes all the newly added degrees of freedom (singular and jump, displacements and velocities) at zero the energy is not changed and the new crack segment is closed (opening displacement is zero). The work of the corresponding forces which are released when the new segment (blue one) is opened is exactly the energy release rate because the 4 singular functions exactly represent the singular field at crak tip ([16], [17]). The same argumentation holds for the kinetic energy change when one adds extended velocities DOF. This point is crucial for the quality of the dynamic crack propagation simulations. One can also use this argumentation for adaptative cohesive modeling of crack propagation but one must take care of dividing by two the mass of splitted nodes.

- Propagation criteria
  The following questions are to be answered:

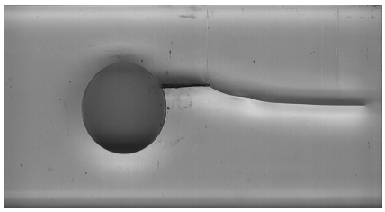

FIGURE 4. Experimental broken specimen

1) can the crack propagate?
There are two criteria to answer to this question: the first one is based on the energy release rate concept, the second on a stress state near the crack tip.
-For the energy release rate one considers the available elastic energy in the structure evaluated for instance with the interaction integral [21] which gives the two stress intensity factors in 2D (K_I *and* K_{II}). Once these two numbers are known one computes the available energy in the structure G ($G = \frac{1-\nu^2}{E}\left(K_I^2 + K_{II}^2\right)$ in plane strain). If this available energy is higher than the energy release rate G_{IC} (which is a material parameter) the crack propagates.
-For the local stress state one computes a "mean" stress tensor in a small half disk ahead the crack tip. Once a measure of this tensor is large enough the crack propagates. One of these measures is the maximum possible hoop mean stress. If this stress is higher than a critical stress (which has to be fitted from material properties as well as mesh size) the crack propagates.
2) In which direction does the crack propagate?
For brittle fracture this direction is given by the maximum hoop stress direction which can be computed from the stress invariants of is the corresponding direction in case of local crack tip stress field estimation.
3) What is the crack speed?
One uses most often the Kanninen [20] empirical formula which gives the crack tip velocity ($\dot{a} = c_R(1 - \frac{K_{IC}}{K_I})$). Similar expression exists for the local evaluation of crack propagation. Let us quote that this equation imposes a crack propagation velocity $\dot{a}$ which is less than Rayleigh wave speed c_R. This assumption is not so sure: experimentalists have observed faster cracks.

- Example 1 : stop and restart of a crack: G_{IC} prediction.
  This example is fully described in [22]. A PMMA specimen with an initial crack is impacted in an Hopkinson bar system. The crack tip is loaded dynamically in mixed mode. It turns stops and restarts: the final broken plate is shown in Figure 3. The prediction of its path and its velocity history was an interesting challenge. Figure 3 compares the experimental

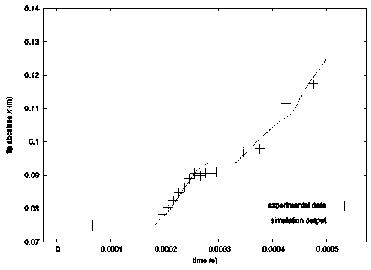

FIGURE 5. Crack tip velocity: experimental-X-FEM simulation comparison.

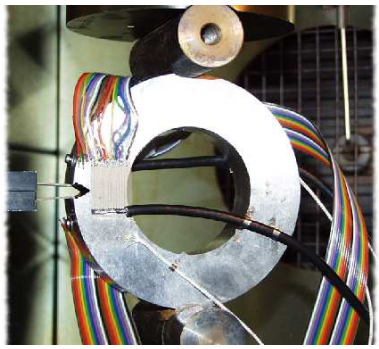

FIGURE 6. Test ring specimen

crack velocities and the predicted one with X-FEM using two different critical toughnesses K_{IC} one for the initial crack and a second one for the propagation phase.

- Example 2: Local stress criterion
  This example is fully described in [23]. One uses in this example a local stress propagation criterion which whose parameters are identified using a CT specimen equiped with a system to measure crack tip velocity. The ring (Figure 3) is loaded by a quasi static compression and is initially cracked. Once the loading is large enough the crack starts with a velocity of about $800m/s$ and stops. The trajectory was difficult to guess. The experimental broken ring is displayed in Figure 3.

  The comparison of measured and predicted crack trajectories are shown in Figure 3.

REFERENCES

[1] Belytschko T. et al., Meshless methods: An overview and recent developments, *Computer Methods in Applied Mechanical Engineering* 1996; **32**:3-47.

[2] Nguyen V.P., Rabczuk T., Bordas S., Duflot M., Meshless methods: A review and computer implementation aspects, *Mathematics and Computers in Simulation* 2008; **32**:763-813.

FIGURE 7. Broken ring specimen

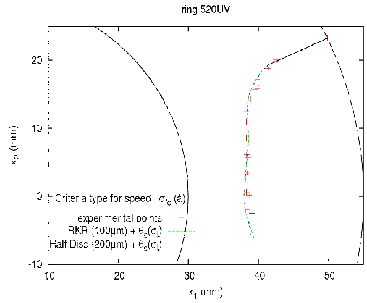

FIGURE 8. Crack path: experimental-X-FEM simulation comparison.

[3] Belytschko T., Lu Y.Y. , Gu L., Element-free Galerkin methods, *International Journal of Numerical Methods in Engineering* 1994; **37**:229-256.
[4] Belytschko T., Gu L., Lu Y.Y. Fracture and crack growth by element-free Galerkin methods, *Modelling and Simulation in Materials Science and Engineering* 1994; **2**:519-534.
[5] Belytschko T., Tabbara T., Dynamic fracture using element-free Galerkin methods, *International Journal of Numerical Methods in Engineering* 1996; **39**:923-938.
[6] Bordas S., Rabczuk T., Goangseup Z., Three-dimendional crack initiation, propagation, branching and junction in non-linear materials by an extended meshfree method without asymptotic enrichment, *Engineering Fracture Mechanics* 2008; **75**:943-960.
[7] de Borst R, Remmers J, Needleman A. Mesh-independent discrete numerical representations of cohesive-zone models. *Engineering fracture mechanics* 2006; **73**:160–177.
[8] Camacho G T, Ortiz M. Computational modelling of impact damage in brittle materials. *International journal of solids and structures* 1996; **33(20)**:2899–2938.
[9] Comi C., Mariani S., Perego U. An extended finite element strategy for transition from continuum damage to mode I cohesive crack propagation. *International journal for numerical and analytical methods in geomechanics* 2007; **31(2)**:213–238.
[10] Remmers J, de Borst R, Needleman A. A cohesive segments method for the simulation of crack growth. *Computational Mechanics* 2003; **31**:69–77.
[11] Moes N., Dolbow J., Belytschko T., A finte element method for crack growth without remeshing, *Internation Journal of Numerical Methods in Engineering* 1999; **32**:133-150.

[12] Black T, BelytschkoT . Elastic crack growth in finite elements with minimal remeshing. *International Journal for Numerical Methods in Engineering* 1999; **45**:601–620.
[13] S. Pommier, A. Gravouil, N. Moes, A. Combescure, *Extended finite elements methods for crack propagations*, **ISTE Wiley** (2011), ISBN: 978-1-84821-209-1.
[14] Prabel B, Gravouil A, Combescure A, Marie S. Level set X-FEM non matching meshes : Application to dynamic crack propagation in elastic-plastic media. *International Journal for Numerical Methods in Engineering* 2007; **69**:1553–1569.
[15] T Elguedj, A Gravouil, A Combescure, A mixed augmented lagrangian-extended finite element method for modelling elastic-plastic fatigue crack growth with unilateral contact, International Journal of Numerical Methods in Engineering 2007, Vol 71, 1569–1597
[16] Réthoré J, Gravouil A, Combescure A. An energy-conserving scheme for dynamic crack growth using the extended finite element method. *International Journal for Numerical Methods in Engineering* 2005; **63**:631–659.
[17] Combescure A, Gravouil A, Grégoire D, Réthoré J. X-FEM a good candidate for energy conser- vation in simulation of brittle dynamic crack propagation. *Computer Methods in Applied Mechanics and Engineering* 2008; **197**(5):309–318.
[18] Menouillard T, Réthoré J, Combescure A, Bung H. Efficient explicit time stepping for the extended finite element method. *International Journal for Numerical Methods in Engineering* 2008; **68**:11–38.
[19] Menouillard T, Réthoré J, Moës, Combescure A, Bung H. Mass lumping strategies for X-FEM explicit dynamics : Application to crack propagation. *International Journal for Numerical Methods in Engineering* 2008; **74**(3):447–474.
[20] Kanninen M, Popelar C. Advanced fracture mechanics. *Oxford University Press* , 1985.
[21] Freund L B. Dynamic Fracture Mechanics. *Cambridge University Press*, Cambridge, UK, 1990.
[22] Grégoire D, Maigre H, Combescure A. Dynamic crack propagation under mixed-mode loading – Comparison between experiments and X-FEM simulations. *International Journal of Solids and Structures* 2007; **44**:6517–6534.
[23] Prabel B, Combescure A, Marie S. Using the X-FEM method to model the dynamic propagation and arrest of cleavage cracks in ferritic steel. *Engineering Fracture Mechanics* 2008; **75(10)**:2984–3009.

A toy model for dynamic debonding

Giuliano Lazzaroni

(joint work with Renaud Bargellini, Pierre-Emmanuel Dumouchel, Jean-Jacques Marigo)

We present a model of peeling test [1, 2] for a one-dimensional inextensible thin film, subject to a monotonic loading with vanishing speed, under the hypothesis that the toughness is piecewise constant and assumes only two possible values $\gamma_1 < \gamma_2$. This simplified context allows us to highlight the contribution of the kinetic energy to crack propagation in a heterogeneous material (with variable toughness). In particular, we compare quasistatic and dynamic models as the speed of loading tends to zero.

Indeed, even under quasistatic loading (i.e., assuming that the speed of loading is smaller than the speed of the internal vibrations), the material's answer is a priori dynamic, because Griffith's theory does not provide an absolutely continuous solution if the total quasistatic energy is not a convex function of the crack length. In this case, an unstable phase of propagation takes place, where Griffith's

quasistatic law fails; in the quasistatic framework, this fast propagation is seen as a jump in time.

In our one-dimensional example, the film is a semi-infinite line $(0, +\infty)$. The debonded (or cracked) region is assumed to be a segment $(0, \ell_\varepsilon(T))$ and the debonding front is a point $\ell_\varepsilon(T)$, which has to be determined in dependence of the loading level $T = \varepsilon\, t$, where ε is a small speed and t the time. Griffith's law, formulated in terms of the dynamical energy release rate [3], allows to predict the evolution $T \mapsto \ell_\varepsilon(T)$ in the regions where the material's toughness is constant; when the debonding front meets a discontinuity of the toughness (from γ_1 to γ_2 or vice versa), further criteria should be provided. The case of a single discontinuity is analyzed in [4], while in [5] we study the interaction of more discontinuities.

If the toughness has only one discontinuity, the dynamic evolutions $T \mapsto \ell_\varepsilon(T)$ converge, as the loading speed tends to zero, towards an evolution $T \mapsto \ell(T)$ of the quasistatic type. If at the discontinuity ℓ_1 the toughness is increasing (i.e., it passes from γ_1 to γ_2), $\ell(T)$ is absolutely continuous and presents a phase of arrest after ℓ_1. On the contrary, in the decreasing case (from γ_2 to γ_1), the energy release rate at ℓ_1 turns to be greater than the toughness, so $\ell(T)$ presents a jump, limit of fast propagations for the dynamic evolutions $\ell_\varepsilon(T)$. The total quasistatic energy is conserved after the jump; at each continuity point, $\ell(T)$ satisfies Griffith's quasistatic law, so $\ell(T)$ can be determined using only quasistatic quantities under the assumption of energy conservation after the jump.

The behaviour is very different when the toughness has two discontinuities ℓ_1, ℓ_2 and equals γ_2 in $(0, \ell_1)$, γ_1 in (ℓ_1, ℓ_2), and γ_2 again in $(\ell_2, +\infty)$ (this corresponds to a defect in the material). If the size of the defect does not exceed a threshold depending only on the two possible values of the toughness, we observe for $\varepsilon > 0$ the interaction of two shock waves, originating at the discontinuities and propagating in the debonded part of the film: at each reflection of the shock waves with the debonding front, the system switches from a phase of arrest to a phase of motion or vice versa. As a consequence, the limit evolution $\ell(T)$ as $\varepsilon \to 0$ presents a jump between ℓ_1 and ℓ_2, while after ℓ_2 the evolution is continuous but Griffith's quasistatic criterion does not hold. Hence, the dynamic evolutions $\ell_\varepsilon(T)$ do *not* converge to a quasistatic one; it is possible to see also that the quasistatic evolution designed under the principle of energy conservation overestimates the jump's length.

This phenomenon is due to the presence of kinetic energy, which cannot be neglected in this case even under slow loading. It turns out that for $\varepsilon > 0$ the kinetic energy increases during the fast propagation between ℓ_1 and ℓ_2, then decreases with oscillations and is transformed in potential energy; finally it reaches the order of ε, so the system becomes close to the quasistatic regime. The upper and lower bounds of the oscillations can be characterized in terms of Γ-convergence as $\varepsilon \to 0$, which allows us to estimate the amount of kinetic energy not yet transformed in potential.

We study also the case of many defects with a periodic distribution, with toughness γ_1 in $(id, (i+\theta)d)$ and γ_2 in $((i+\theta)d, (i+1)d)$ for every $i \geq 0$, where $\theta \in (0, 1)$

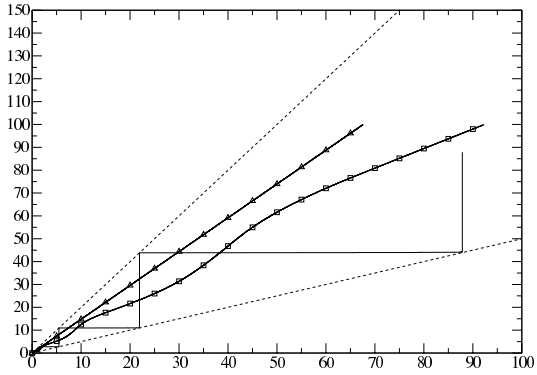

FIGURE 1. Dynamic solutions in the case of many defects. Here, $\gamma_1 = 0.5$, $\gamma_2 = 2$, $d = 0.001$, $\varepsilon = 0.001$. *Solid line*: $\theta = 0.99999$. *Squares*: $\theta = 0.97$. *Triangles*: $\theta = 0.7$. *Dashed lines*: quasistatic evolution for homogeneous materials of toughness γ_1 and γ_2, respectively.

and $d > 0$. In the limit as $d \to 0$ we find a homogenized material; here the density θ of material with the lowest toughness plays a major role. We consider the quasistatic model under the assumption of energy conservation: the solutions show phases of arrest and phases of jump and are then staircase functions in the space/time plane. The points where the behaviour switches from arrest to jump or vice versa, lay on two lines whose slope depends on θ. In particular, fixed a level of load, the length of debonding can be higher than the one corresponding to a material with a constant low toughness.

Due to the difficulty of controlling the several wave interactions that perturb the debonding process, in the case of many defects the dynamic problem is treated using an exact numerical solution of the wave equation. The simulations show that for $d \ll 1$ and $\varepsilon \ll 1$ the qualitative aspect of the dynamic solutions depends strongly on θ, while it is not affected by the choice of the defects' distribution (e.g., the behaviour does not change if one considers a random distribution of defects with mean density θ). If the density of the lowest toughness is close to one, the dynamic solutions seem to converge to a quasistatic staircase evolution: indeed, during the fast propagations the debonding passes through many defects. As this density decreases, the curves in the space/time plane seem more smooth, showing a wave trend; if the density is close to 1/2, their shape approaches a line. In all these situations the behaviour shown as $d \to 0$ is different from the case of a homogeneous material with the mean toughness $\bar{\gamma} := \theta\gamma_1 + (1 - \theta\gamma_2)$.

This behaviour reflects the influence of the kinetic energy on the dynamics. In particular, the mean toughness $\bar{\gamma}$ of the material should be substituted by an effective toughness in order to retrieve the limit of dynamic evolutions by quasistatic quantities. The effective toughness can be easily determined in the case of a single defect; on the contrary, the rigorous study of the homogenized material is still an open problem.

References

[1] B. Lawn, *Fracture of Brittle Solids*, 2nd edn., Cambridge Solid State Science Series, Cambridge University Press, London, 1993.

[2] J. W. Obreimoff, *The splitting strength of mica*, Proc. R. Soc. Lond. Ser. A **127** (1930), 290–297.

[3] L. B. Freund, *Dynamic Fracture Mechanics*, Cambridge Monographs on Mechanics and Applied Mathematics, Cambridge University Press, London, 1998.

[4] P.-E. Dumouchel, J.-J. Marigo, M. Charlotte, *Dynamic fracture: an example of convergence towards a discontinuous quasistatic solution*, Contin. Mech. Thermodyn. **20** (1990), 1–19.

[5] R. Bargellini, P.-E. Dumouchel, G. Lazzaroni, J.-J. Marigo, *On the role of the kinetic energy during unstable crack propagation under quasistatic loading*, in preparation.

The Eshelby-Kostrov property in the Theory of Dynamic Fracture

Juan J. L. Velázquez

The asymptotic behaviour of an elastic field near the tip of a moving rectilinear crack $\Gamma_t = \{x = (x_1, 0) : -\infty < x_1 < \lambda(t)\}$ can be described by means of a combination of asymptotic formulas with the form:

$$u \sim K(t)\, r^{\frac{1}{2}} B\left(\theta, \dot{\lambda}(t)\right) \tag{1}$$

where r is the distance of a given point to the tip of the crack and θ the angle with respect to a fixed direction of a line connecting the tip of the crack with the point under consideration.

A remarkable property of the elasticity equations which was first described in [1] and [3] and whose meaning was greatly clarified in [2] states that for a given set of the initial data of the elastic field, the function $K(t)$ in (1) does not depend on the whole history followed by the crack Γ_t but only in the values of $\lambda(t)$ and $\dot{\lambda}(t)$. Due to this property the study of description of the growth of a crack reduces to a first order ODE if the energy required to increase the length of a crack moving at a given speed is known.

A new proof of the Eshelby-Kostrov property in the case of antiplane loading has been obtained in the paper [5] where some representation formulas for the elastic field in the whole space has been also obtained for arbitrary initial data. The proof shows that in the particular case of rectilinear cracks the solutions of the wave equation have a very peculiar structure. On the other hand, it has been shown in [4] that this property is not satisfied in the case of nonrectilinear cracks. This has been proved by means of the study of a kinked crack in which its tip is followed in two different ways. The particular geometry considered in [4] makes

possible to compute the dynamic elastic field in explicit form using separation of variables and integral representation formulas.

However, the geometry and the type of motion considered in [4] is in some sense too singular, because kinked cracks are considered and part of the motion takes place at constant speed. It could be relevant to obtain counterexamples of the Eshelby-Kostrov property for cracks moving at subsonic speeds along smooth curves, perhaps using perturbative arguments. It could be relevant also to extend the representation formulas in [5] to general solutions of the elastic equations.

References

[1] J. D. Eshelby. The elastic field of a crack extending nonuniformly under general antiplane loading. J. Mech. Phys. Solids 17, 177–199, (1969).

[2] L. B. Freund. Dynamic fracture mechanics. Cambridge University Press, (1998).

[3] B. V. Kostrov. On the crack propagation with variable velocity. Int. J. Fract. 11, 47–56, (1975).

[4] M. A. Herrero, G. E. Oleaga and J. J. L. Velázquez, Planar cracks running along piecewise linear paths. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 460, 2042, 581-601, (2004).

[5] M. A. Herrero, G. Oleaga and J. J. L. Velázquez, On the Eshelby-Kostrov property for the wave equation in the plane. Trans. AMS, 358, 3673-3695, (2006).

Crack Propagation by PLS

Matteo Negri

We consider as a reference uncracked configuration an open, bounded Lipschitz set Ω contained in $\mathbb{R}^2$. We assume that the initial crack Γ_0 is a closed line segment with one end point on the boundary $\partial\Omega$. For convenience we will fix a system of coordinates with the origin in the other end point (the crack tip) with $\hat{e}_1$ aligned with Γ_0 (see Figure 1). We will also assume that $\Omega \setminus \Gamma_0$ is connected and that $\Omega \setminus \Gamma_0$ can be represented by the union of a finite number of Lipschitz subsets, so that Korn's inequality still holds true. Considering the system of coordinates introduced above, the crack path will be represented by the graph of a Lipschitz function y belonging to

$$\mathcal{Y} = \{y \in C^{0,1}([0,S]) : y(0) = 0\,, \|y'\|_\infty \le C\}\,, \tag{1}$$

where C will be chosen later (sufficiently large). For notational convenience, let us also introduce the curve $\gamma : [0,S] \to \Omega \setminus \Gamma_0$ given by $\gamma(s) = (s, y(s))$. In this way, denoting $\Gamma_s = \Gamma_0 \cup \{(x, y(x)) : x \in [0,s]\}$, the set $\Omega \setminus \Gamma_s$ is still connected and can be represented again as a finite union of Lipschitz sets.

For the elastic displacement u we set a Dirichlet boundary conditions on a subset $\partial_D\Omega$ of $\partial\Omega \setminus \Gamma_0$ with $\mathcal{H}^1(\partial_D\Omega) > 0$. For sake of simplicity this boundary condition will be of proportional type, i.e. $u = cg$ for $c \in W^{1.1}(0,T)$ with $c(0) = 0$ and $g \in H^{1/2}(\partial_D\Omega, \mathbb{R}^2)$. On the rest of the boundary an homogeneous Neumann boundary condition is set. With proportional boundary conditions it is enough for many pourposes to consider a reference problem with space of admissible displacements given by $\mathcal{U}(\Omega \setminus \Gamma_s) = \{u \in H^1(\Omega \setminus \Gamma_s, \mathbb{R}^2) : u = g \ \partial_D\Omega\}$. We will employ linearized elasticity, with density $W^e(u) = \mu|\boldsymbol{\epsilon}(u)|^2 + (\lambda/2)|\mathrm{tr}\,\boldsymbol{\epsilon}(u)|^2$. Then, for $u \in \mathcal{U}(\Omega \setminus \Gamma_s)$

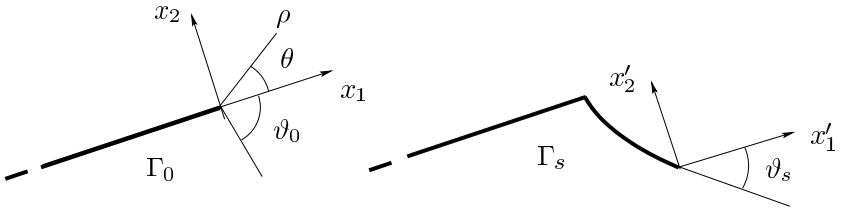

FIGURE 1. Absolute (left) and local (right) systems of Cartesian and polar coordinates employed in this work. The local system translates with the crack.

the energy is $E_s(u) = \int_{\Omega\setminus\Gamma_s} W^e(u)$. As the evolution is quasi-static, given Γ_s it is sufficient to consider the unique equilibrium configuration for the displacement, that is $\{u_s\} = \operatorname{argmin}\{E_s(u) : u \in \mathcal{U}(\Omega \setminus \Gamma_s)\}$. Clearly, when the boundary condition is $u = cg$ by linearity it is sufficient to consider the displacement field cu_s and the energy $c^2E_s(u_s)$.

In general, for a $C^{0,1}$ path it is not known whether the stress intensity factors exist, hence we will employ a regularization with a volume integral representation. Given $r > 0$ let $c_r = (2\pi)^{-1}r^{-5/2}$. For $i = 1, 2$ let

$$\begin{aligned} k_1(\theta) &= c_r\big(a_1\cos(\theta/2) + a_3\cos(3\theta/2), a_2\sin(\theta/2) + a_4\sin(3\theta/2)\big) \\ k_2(\theta) &= c_r\big(b_1\sin(\theta/2) + b_3\sin(3\theta/2), b_2\cos(\theta/2) + b_4\cos(3\theta/2)\big)\,, \end{aligned} \tag{2}$$

for a suitable choice of the coefficients a_i and b_i. Let $\hat\theta_s$ denote the argument of $x-\gamma(s)$ in the local system of polar coordinates (see Figure 1). Then, for $i = 1, 2$, the non-local stress intensity factors are defined by

$$\widetilde K_i(\Gamma_s) = \int_{B_r\setminus\Gamma_s} (u_s - \mathring u_s)\cdot k_i(\hat\theta_s - \vartheta_s)\,dx\,, \tag{3}$$

where B_r denotes the ball $B_r(\gamma(s))$ and $\mathring u_s$ is the average of u in a ball $B_{r'}(\gamma(s))$ for $0 < r' \le r^2$. Within this approximation the right SIF are then given by

$$\widetilde K_i^*(\Gamma_s, \vartheta) = \int_{B_r\setminus\Gamma_\tau} (u_0 - \mathring u_0)\cdot k_i(\theta - \vartheta)\,dx\ .$$

For a suitable choice of the coefficients a_i and b_i the values $\widetilde K_i(\Gamma_s, \vartheta)$ provide a good approximation of the stress intensity factors and of their right limits, indeed, for every $\varepsilon > 0$ there exists C_ε such that (uniformly with respect to ϑ)

$$|\tilde K(\Gamma_0, \vartheta) - \widetilde{\boldsymbol T}(\vartheta)K(\Gamma_0)| \le C_\varepsilon r^{1/2-\varepsilon} \tag{4}$$

where $\widetilde{\boldsymbol T}(\vartheta)$ is an approximation of the transfer matrix suggested in [4] and given by

$$\widetilde{\boldsymbol T}(\vartheta) = \frac14\begin{pmatrix} 3\cos(\vartheta/2) + \cos(3\vartheta/2) & -3\sin(\vartheta/2) - 3\sin(3\vartheta/2) \\ \sin(\vartheta/2) + \sin(3\vartheta/2) & \cos(\vartheta/2) + 3\cos(3\vartheta/2) \end{pmatrix}. \tag{5}$$

Given $y \in \mathcal{Y}$ and $s \in [0, S]$ let $V(\Gamma_s) = \operatorname{tg} \vartheta_s$ where θ_s solves $\widetilde{K}_{\mathrm{II}}(\Gamma_s, \vartheta_s) = 0$. The crack path is found by solving the first order functional differential equation

$$\begin{cases} y'(s) = V(\Gamma_s) & \text{for a.e. } s \in (0, S) \\ y(0) = 0\,. \end{cases}$$

Indeed, if $y'(s) = V(\Gamma_s) = \operatorname{tg} \vartheta_s$ then $\widetilde{K}_{\mathrm{II}}(\Gamma_s, \vartheta_s) = \widetilde{K}_{\mathrm{II}}(\Gamma_s) = 0$. In this way, the path will satisfy the Principle of Local Symmetry for a.e. $s \in [0, S]$.

First, it is necessary to prove that V is well defined and that it depends continuously on s.

Lemma 1. *Let S be sufficiently small. For every $y \in \mathcal{Y}$ and $s \in [0, S]$ there exists a unique $\vartheta_s \in (-\arccos(1/3), 0)$ such that $\widetilde{K}_{\mathrm{II}}(\Gamma_s, \vartheta_s) = 0$. Moreover ϑ_s is continuous in $[0, S]$.*

Now, we can state our existence result for the crack path.

Theorem 2. *For $S > 0$ sufficiently small and C sufficiently large there exists a fixed point for the functional*

$$[\mathcal{F}(y)](s) = \int_0^s V(\Gamma_z)\, dz\,.$$

Moreover, the fixed point is of class $C^1([0, S])$. In other terms, $\widetilde{K}_{\mathrm{II}}(\Gamma_s) = 0$ for every $s \in (0, S]$ and $\widetilde{K}_{\mathrm{II}}(\Gamma_0, \vartheta_0) = 0$.

Once the path Γ_s is given (by Theorem 2) the Principle of Local Symmetry is true for every parametrization $s(t)$. We are therefore free to choose it in such a way that the approximated Griffith's criterion is satisfied. Since the parametrization is non-decreasing (to model irreversibility) we will define the initiation time $t_{init} = \sup\{t : s(t) = 0\}$. In this way, for $t > t_{init}$ we have $\Gamma_t \supsetneq \Gamma_0$. Then we define

$$\widetilde{K}^*_{\mathrm{I}}(t, \Gamma_s) = \begin{cases} \widetilde{K}_{\mathrm{I}}(t, \Gamma_0, \vartheta_0) & \text{for } \Gamma_s = \Gamma_0 \\ \widetilde{K}_{\mathrm{I}}(t, \Gamma_s) & \text{otherwise.} \end{cases}$$

The precise statement, which characterizes the quasi-static evolution [3], is contained in the next Theorem.

Theorem 3. *Given a fixed point $y \in \mathcal{Y} \cap C^1$ there exists a non-decreasing, left-continuous parametrization $s(t)$ such that for $\Gamma_t = \Gamma_{s(t)}$ the following Kuhn-Tucker conditions are satisfied:*

$$\widetilde{K}^*_{\mathrm{I}}(t, \Gamma_t) \le K^c_{\mathrm{I}} \quad \textit{for } t \in [0, T]$$

$$\left(\widetilde{K}^*_{\mathrm{I}}(t, \Gamma_t) - K^c_{\mathrm{I}}\right) ds(t) = 0 \qquad \textit{(in the sense of measures) in } [0, T].$$

Moreover for $t \in J(s)$ we have

$$\widetilde{K}^*_{\mathrm{I}}(t, l) \ge K^c_{\mathrm{I}} \qquad \textit{for } l \in [s^-(t), s^+(t)]\,,$$

so that discontinuities represents the unstable regimes of the evolution. Finally, if $c(t)$ is strictly increasing the parametrization s is unique.

References

[1] R.V. Goldstein and R.L. Salganik. Brittle fracture of solids with arbitrary cracks. *Internat. J. Fracture*, (10):507–523, 1974.
[2] P. Grisvard. Singularités en elasticité. *Arch. Rational Mech. Anal.*, 107(2):157–180, 1989.
[3] M. Negri. A comparative analysis on variational models for quasi-static brittle crack propagation. *Adv. Calc. Var.*, 3:149–212, 2010.
[4] M.L. Williams. On the stress distribution at the base of a stationary crack. *J. Appl. Mech.*, 24:109–114, 1957.

About Numerics

Blaise Bourdin

I will first present a panorama of numerical methods for quasi-static fracture, and illustrate it with numerical experiments under mechanical and thermal loads. I will then describe an extension to dynamic fracture and present more experiments illustrating the properties of the model.

Reporter: Christoph Ortner

Participants

Prof. Dr. Blaise Bourdin
Department of Mathematics
Louisiana State University
Baton Rouge LA 70803-4918
USA

Prof. Dr. Alain Combescure
Laboratoire de Mecanique des Contacts
et des Structures, LaMCoS
UMR CNRS 5259, INSA
18-20, rue des Sciences
F-69621 Villeurbanne

Prof. Dr. Gianni Dal Maso
S.I.S.S.A.
Via Bonomea 265
I-34136 Trieste

Dr. David Doyen
Institut de Mathematiques de Toulouse
Universite Paul Sabatier
118, route de Narbonne
F-31062 Toulouse Cedex 9

Dr. Dorothee Knees
Weierstrass-Institute for Applied
Analysis and Stochastics
Mohrenstr. 39
10117 Berlin

Prof. Dr. Christopher J. Larsen
Department of Mathematics
Worcester Polytechnic Institute
100 Institute Road
Worcester , MA 01609
USA

Dr. Giuliano Lazzaroni
Institut für Angewandte Mathematik
Universität Bonn
Endenicher Allee 60
53115 Bonn

Dr. Matteo Negri
Dipartimento di Matematica
Universita di Pavia
Via Ferrata, 1
I-27100 Pavia

Prof. Dr. Michael Ortiz
Division of Engineering and
Applied Sciences; MS 104-44
California Institute of Technology
Pasadena , CA 91125
USA

Dr. Christoph Ortner
Mathematical Institute
Oxford University
24-29 St. Giles
GB-Oxford OX1 3LB

Prof. Dr. Krishnaswamy Ravi-Chandar
Center for Mechanics of Solids,
Structure and Materials
University of Texas at Austin
210 E. 24th Street, WRW 117B
Austin , TX 78712-0235
USA

Dr. Valeriy Slastikov
Department of Mathematics
University of Bristol
University Walk
GB-Bristol BS8 1TW

Prof. Dr. Natarajan Sukumar
PULSE Institute, Photon Science
SLAC National Accelerator Lab.
2575 Sand Hill Road
Menlo Park , CA 94025
USA

Prof. Dr. Juan J. L. Velazquez
Hausdorff Center for Mathematics
Institute for Applied Mathematics
Endenicher Allee 60
53115 Bonn

Prof. Dr. Jay R. Walton
Department of Mathematics
Texas A & M University
College Station , TX 77843-3368
USA