Billiards, Flat Surfaces, and Dynamics on Moduli Spaces

Organised by
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ABSTRACT. This workshop brought together people working on the dynamics of various flows on moduli spaces, in particular the action of $SL_2(\mathbb{R})$ on flat surfaces. The new results presented covered properties of interval exchange transformations, Lyapunov spectrum of this flow and the geometry of Teichmüller space.

Introduction by the Organisers

Billiards in polygons provide much of the motivation and the many of the main examples of the subject of this workshop. More generally, flat surfaces were the objects of focus of all the participants of this conference. Dynamics on moduli space is a good description of an area that has become very well established in recent years.

Some of the most interesting recent results in this area have arisen from applying methods from quite different subjects. Consequently the workshop had participants with very different backgrounds. These included ergodic theory, topology, (Teichmüller) geometry, geometric group theory, and algebraic geometry. Nevertheless the level of expertise of participants in the common subject of the conference was extremely high. This allowed the organizers to schedule short research talks by almost all individuals or groups who had recent results, and these results were understood by essentially everybody whether the talks were ergodic theoretic, differential geometric, or algebraic geometric in nature.

The area of dynamics on moduli spaces is rapidly evolving. Basically the results presented were limited to new results obtained since the research summer trimester

in HIM in 2010. This suggests that it might be reasonable to preserve the tradition of have yearly conferences in this area of mathematics at institutions such as CIRM, MSRI, Oberwolfach, and HIM.

It has become commonplace that papers written in this area are the collaboration of two or three authors who often live on different continents. The evenings were very densely charged, since all these mostly overlapping small groups were working hard taking advantage of being unified in a nice and creative environment.

Workshop: Billiards, Flat Surfaces, and Dynamics on Moduli Spaces

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Abstracts

Gaps for Saddle Connection Directions

JAYADEV ATHREYA

(joint work with Jon Chaika)

1. INTRODUCTION

1.1. **Generalized diagonals for rational billiards.** Let P be a Euclidean polygon with angles in $\pi\mathbb{Q}$. We call such a polygon *rational*. A classical dynamical system is given by the idealized motion of a billiard ball on P: the (frictionless) motion of a point mass at unit speed with elastic collisions with the sides.

A *generalized diagonal* for the polygon P is a trajectory for the billiard flow that starts at one vertex of P and ends at another vertex. Since the group Δ_P generated by reflections in the sides of P is finite, the *angle* of a trajectory is well defined in $S^1 \cong S^1/\Delta_P$. A motivating question for our paper is the following: *how close in angle can two generalized diagonals of (less than) a given length be* (in terms of the length)?

Masur [4] showed that the number of generalized diagonals of length at most R grows quadratically in R. We show, for some families of billiards, that the smallest gap γ_R^P between two generalized diagonals on P of length at most R satisfies

$$(1.1) \qquad \lim_{R\to\infty} R^2 \gamma_R^P = 0,$$

and for other specific billiard tables that

$$(1.2) \qquad \liminf_{R\to\infty} R^2 \gamma_R^P > 0.$$

1.2. **Translation surfaces.** Let Σ_g be a compact surface of genus $g \geq 2$. Let Ω_g be the moduli space of holomorphic differentials on Σ_g. That is, a point $\omega \in \Omega_g$ is an equivalence class of pairs (M, ω), where M is a genus g Riemann surface, and ω is a holomorphic differential on M, i.e., a tensor with the form $f(z)dz$ in local coordinates, such that $\frac{i}{2}\int_{\Sigma_g} \omega \wedge \bar{\omega} = 1$.

$|\omega|$ determines a flat metric on M with conical singularities at the zeros of the differential ω. Geometrically, a zero of the form $z^k(dz)$ corresponds to a cone angle of order $(2k+2)\pi$. Zeroes of ω are *singular* points for the flat metric. We refer to non-singular points as *regular points*. The space Ω_g can be decomposed naturally into *strata* $\mathcal{H}$, each carrying a natural measure $\mu_\mathcal{H}$.

1.3. **Saddle connections and cylinders.** Fix $\omega \in \Omega_g$. A *saddle connection* on ω is a geodesic segment in the flat metric connecting two singular points (that is, zeros of ω) with no singularities in its interior. Given a regular point p, a *regular closed geodesic* through p is a closed geodesic not passing through any singular points. Regular closed geodesics appear in families of parallel geodesics of the same length, which fill a cylindrical subset of the surface.

1.3.1. *Holonomy vectors.* Let γ be an (oriented) saddle connection or regular closed geodesic. Define the associated holonomy vector

$$\text{(1.3)} \qquad \mathbf{v}_\gamma := \int_\gamma \omega.$$

Note that if γ is a closed geodesic, $\mathbf{v}_\gamma$ only depends on the cylinder it is contained in, since regular closed geodesics appearing in a fixed cylinder all have the same length and direction. View $\mathbf{v}_\gamma$ as an element of $\mathbb{R}^2$ by identifying $\mathbb{C}$ with $\mathbb{R}^2$.

Let

$$\text{(1.4)} \qquad \begin{aligned} \Lambda_\omega^{sc} &= \{\mathbf{v}_\gamma : \gamma \text{ a saddle connection on } \omega\} \\ \Lambda_\omega^{cyl} &= \{\mathbf{v}_\gamma : \gamma \text{ a cylinder on } \omega\} \end{aligned}$$

be the set of holonomy vectors of saddle connections and cylinders respectively. For $\Lambda_\omega = \Lambda_\omega^{sc}$ or Λ_ω^{cyl}, we have that Λ_ω is discrete in $\mathbb{R}^2$ (see, e.g., [5, Proposition 3.1]), but Masur [3] showed that associated set of directions

$$\Theta^\omega := \{\arg(\mathbf{v}) : \mathbf{v} \in \Lambda_\omega\}$$

is dense in $[0, 2\pi)$ for any $\omega \in \Omega_g$.

1.4. **Decay of gaps.** In this paper, we give a measure of the quantitative nature of this density by considering fine questions about the *distribution* of saddle connection directions. Given $R > 0$, let

$$\text{(1.5)} \qquad \Theta_R^\omega := \{\arg(\mathbf{v}) : \mathbf{v} \in \Lambda_\omega \cap B(0, R)\}$$

denote the set of directions of saddle connections (or cylinders) of length at most R. Let $\gamma^\omega(R)$ be the size of the smallest gap, that is $\gamma^\omega(R) = \min_{\theta_i \in \Theta_R} |\theta_i - \theta_{i+1}|$, where we view θ_{n+1} as θ_1, where

$$n = N(\omega, R) := |\Lambda_\omega \cap B(0, R)|$$

is the cardinality of Θ_R^ω. Masur [4] showed that the counting function $N(\omega, R)$ grows quadratically in R for any ω, thus, one would expect the $\gamma^\omega(R)$ to decay quadratically. Our main theorem addresses the asymptotic behavior of the rescaled quantity $R^2 \gamma^\omega(R)$. Let $\mathcal{H}$ be a stratum of Ω_g, and let $\mu = \mu_\mathcal{H}$.

Theorem 1.1. *For μ-almost every $\omega \in \mathcal{H}$,*

$$\text{(1.6)} \qquad \lim_{R \to \infty} R^2 \gamma^\omega(R) = 0.$$

Moreover, for any $\epsilon > 0$, the proportion of gaps less than ϵ/R^2 is positive. That is, writing $\Theta_R^\omega := \{0 \le \theta_1 \le \theta_2 \le \ldots \le \theta_n\}$, we have

$$\text{(1.7)} \qquad \lim_{R \to \infty} \frac{|\{1 \le i \le N(\omega, R) : (\theta_{i+1} - \theta_i) \le \epsilon/R^2\}|}{N(\omega, R)} > 0.$$

Theorem 1.1 cannot be extended to *all* $\omega \in \mathcal{H}$, since for any stratum $\mathcal{H}$ there are many examples $\omega \in \mathcal{H}$ for which

$$\text{(1.8)} \qquad \liminf_{R \to \infty} R^2 \gamma^\omega(R) > 0.$$

We say that ω has *no small gaps* (NSG) if (1.8) holds. An important motivating example of a surface with NSG is the case of the square torus $(\mathbb{C}/\mathbb{Z}^2, dz)$. Since there are no singular points, there are no saddle connections, but cylinders are given by integer vectors, and Θ^{ω_0} then corresponds to rational slopes. It can be shown that $3/\pi^2$ is a lower bound for $R^2\gamma^{\omega_0}(R)$ (see, for example [1]).

The torus is an example of a lattice surface. Recall that ω is said to be a *lattice surface* if the group of derivatives of affine diffeomorphisms of ω is a lattice in $SL(2,\mathbb{R})$ We have:

Theorem 1.2. *ω is a lattice surface if and only if it has no small gaps.*

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Ellipses in translation surfaces

CHRISTOPHER JUDGE

(joint work with S. Allen Broughton)

In this talk, I describe joint work with S. Allen Broughton of the Rose-Hulman Institute of Technology that will soon appear as [3].

A translation structure μ on a (connected) topological surface X is an equivalence class of atlases whose transition functions are translations. Translation surfaces are fundamental objects in Teichmüller theory, the study of polygonal billiards, and the study of interval exchange maps.

The cylinders that are isometrically embedded in a translation surface play a central role in the theory. In Teichmüller theory, they appear as solutions to moduli problems. In rational billiards and interval exchange maps, cylinders correspond to periodic orbits.

Indeed, each periodic geodesic γ on a translation surface belongs to a unique 'maximal' cylinder that is foliated by the geodesics that are both parallel and homotopic to γ. One method for producing such periodic geodesics implicitly uses ellipse interiors: If X admits an isometric immersion of an ellipse with area greater than that of X, then the image of the immersion contains a cylinder, and hence a periodic geodesic.

Ellipses interiors also serve to interpolate between maximal cylinders. The set, $\mathcal{E}(X,\mu)$, of ellipse interiors isometrically immersed in X has a natural geometry

coming from the space of quadratic forms. The set of maximal cylinders is a discrete set lying in the frontier of the path connected space $\mathcal{E}(X, \mu)$.

If the frontier of a translation surface X is finite, then each point in the frontier may be naturally regarded as a *cone point* with angle equal to an integral multiple of 2π. If the frontier of an immersed ellipse interior U contains a cone point c, then we will say that U *meets x*. If an ellipse interior meets a cone point, then the ellipse interior belongs to the frontier of $\mathcal{E}(X, \mu)$. The remainder of the frontier consists of cylinders.

The number of cone points met by an ellipse interior determines a natural stratification of $\mathcal{E}(X, \mu)$.[1] We show that $\mathcal{E}(X, \mu)$ is homotopy equivalent to the stratum consisting of ellipse interiors that meet at least three cone points. We prove that the completion of this stratum is naturally a (non-manifold) 2-dimensional cell complex whose 2-cells are convex polygons.

We show that the topology of this polygonal complex and the geometry of the immersed ellipses and cylinders that serve as its vertices together encode the geometry of (X, μ) up to homothety.

Theorem. *Suppose that there is a homeomorphism Φ that maps the polygonal complex associated to (X, μ) onto the polygonal complex associated to (X', μ'). If for each vertex U, the ellipses (or strips) U and $\Phi(U)$ differ by a homothety, then (X, μ) and (X', μ') are equivalent up to homothety.*

Affine mappings naturally act on planar ellipses, and hence the group of affine homeomorphisms of (X, μ) acts on $\mathcal{E}(X, \mu)$. Because μ is a translation structure, the differential of an orientation preserving affine homeomorphism is a well-defined 2×2 matrix of unit determinant. The set of all differentials is a discrete subgroup of $SL_2(\mathbb{R})$ that is sometimes called the *Veech group* and is denoted $\Gamma(X, \mu)$. Using Theorem , one can characterize $\Gamma(X, \mu)$.

Theorem. *The group $\Gamma(X, \mu)$ consists of the $g \in SL_2(\mathbb{R})$ for which there exists an orientation preserving self homeomorphism of the polygonal complex associated to (X, μ) such that for each vertex U there exist a homothety h_U such that U differs from $\Phi(U)$ by $h_U \circ g$.*

The group $\Gamma(X, \mu)$ is closely related to the subgroup of the mapping class group of X that stabilizes the Teichmüller disc associated to (X, μ). To be precise, each mapping class in the stabilizer has a unique representative that is affine with respect to μ. The Veech group is the set of differentials of these affine maps, and is isomorphic to the stabilizer modulo automorphisms. In particular, if there are no nontrivial automorphisms in the stabilizer, then the quotient of the hyperbolic plane by a lattice Veech group is isometric to a Teichmüller curve.

There is a natural map that sends each ellipse interior $U \subset \mathbb{R}^2$ to the coset of $SO(2) \backslash SL_2(\mathbb{R})$ consisting of g such that $g(U)$ is a disc. This map naturally determines a map from $\mathcal{E}(X, \mu)$ onto the Poincaré disc. The image of the 1-skeleton of the polygonal cell complex determines a tessellation of the upper half-plane that

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[1]To be precise one must lift to the universal cover before counting.

coincides with a tessellation defined by William Veech [6] [7]. Indeed, our work began with a reading of a 2004 preprint of [7]. Later, we discovered that Joshua Bowman had independently defined the tessellation [1] [2]. In a companion paper [4], we will discuss the connection between $\mathcal{E}_3(X, \mu)$ and the tessellation in more detail.

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Diffusion rate in the wind-tree model

Vincent Delecroix

(joint work with Pascal Hubert and Samuel Lelièvre)

We study periodic versions of the wind-tree model introduced by P. & T. Ehrenfest in 1912 [3]. A point moves in the plane $\mathbb{R}^2$ and bounces elastically off rectangular scatterers following the usual law of reflection. The scatterers are translates of the rectangle $[0, a] \times [0, b]$, $0 < a < 1$ and $0 < b < 1$, one centered at each point of $\mathbb{Z}^2$. We denote the complement of obstacles in the plane by $T(a, b)$ and refer to it as *the wind-tree model.* Our aim is to understand its dynamical properties following the general scheme.

- Does there exists a typical behavior for trajectories ? If so describe it ?
- Quantify the set of non-typical behavior.

Typical behavior can be thought in topological or measurable sense and with respect to different dynamical properties: recurrence/divergence, diffusion rate, ergodicity, ...

The first study of the periodic wind-tree model is due to J. Hardy and J. Weber [7]. They proved that the rate of diffusion is $\log(t) \log \log(t)$ for very specific directions (generalized diagonals). Their result was recently completed by J.P. Conze and E. Gutkin [2] who build the ergodic decomposition of the billiard flow for those specific directions. In another direction, P. Hubert, S. Lelièvre and S. Troubetzkoy [6] proved that for a dense set of parameters a, b, for almost every direction θ, the

flow in the direction θ is recurrent. In this paper, we compute the polynomial rate of diffusion.

Let $0 < a, b < 1$ be a fixed size for the scatterer. If we consider an initial condition with angle θ then, as the barriers are either horizontal or vertical, the ball will only takes direction θ, $-\theta$, $\pi - \theta$ and $\pi + \theta$. We call *flow in direction* θ and denote by ϕ_t^θ the billiard flow associated to the quadruple of directions $\{\theta, -\theta, \pi - \theta, \pi + \theta\}$. The phase space for the billiard flow is $T(a,b) \times (++, +-, -+, --)$ where $++, +-, -+, --$ refers to the four possible directions.

Theorem. *Let ϕ_t^θ be the billiard flow in direction θ in the table $T(a,b)$ and $d(.,.)$ be the euclidean distance on $\mathbb{R}^2$.*

(1) *If (a,b) are rational numbers, then for almost every θ and for every point x in $T(a,b)$ (with an infinite trajectory), we have*

$$\limsup_{t \to \infty} \frac{\log(d(x, \phi_t^\theta(x)))}{\log(t)} = \frac{2}{3}.$$

(2) *If $(a,b) \in \mathbb{Q}[\sqrt{D}]$ are quadratic numbers with the additional condition that: $1/(1-a) = x + z\sqrt{D}$ and $1/(1-b) = (1-x) + z\sqrt{D}$ then for almost every θ and for every point x in $T(a,b)$ (with an infinite trajectory), we have*

$$\limsup_{t \to \infty} \frac{\log(d(x, \phi_t^\theta(x)))}{\log(t)} = \frac{2}{3}.$$

(3) *For almost all $(a,b) \in (0,1)^2$, for almost every θ and for every point x in $T(a,b)$ (with an infinite trajectory), we have*

$$\limsup_{t \to \infty} \frac{\log(d(x, \phi_t^\theta(x)))}{\log(t)} = \frac{2}{3}.$$

The conclusion of the first and second statement holds for specific parameters while the third one is the answer in the generic case. We do not know if the latter result holds for *every* parameters $(a,b) \in (0,1)^2$.

By the $\mathbb{Z}^2$ periodicity of the billiard table $T(a,b)$, our problem reduces to estimations of a $\mathbb{Z}^2$ cocycle over the billiard in a fundamental domain. On the other hand, a standard construction consisting of unfolding the trajectories [12], the billiard flow can be replaced by a linear flow on a (non compact) translation surface that we denote $X_\infty(a,b)$. The surface $X_\infty(a,b)$ keeps the $\mathbb{Z}^2$-periodicity of the billiard table $T(a,b)$. We denote $X(a,b)$ the quotient of $X_\infty(a,b)$ under this $\mathbb{Z}^2$ action. As, the unfolding procedure of the billiard flow is equivariant with respect to the $\mathbb{Z}^2$ action $X(a,b)$ can be also be seen as the unfolding of the billiard in a fundamental domain of $T(a,b)/\mathbb{Z}^2$.

The position of the particle in $T(a,b)$ can be tracked from $X(a,b)$. The position of the particle starting from x in direction θ can be approximated by the intersection of a geodesic in $X(a,b)$ with a cocycle $f \in H^1(X(a,b); \mathbb{Z}^2)$ describing the infinite cover $X_\infty(a,b)/X(a,b)$. Theorem has an immediate translation in this language. The growth of such quantities has been studied since a long time by A. Zorich [13, 14] and G. Forni [5] (see also [8]) and are related to Lyapunov

exponents of the Teichmüller flow. In our case which does not fit into the preceding general theory, we prove that the exponents do control the growth of the intersection. That's the main part of the paper. From results by M. Bainbridge [1] and A. Eskin, M. Kontsevich and A. Zorich [4], we deduce that the value of the Lyapunov exponent under consideration is $2/3$ which explains the right term in Theorem .

The surface $X(a, b)$ is a covering of the genus 2 surface $L(a, b)$ which is a so called L-shaped surface. By C. McMullen's fundamental work [9, 10, 11], the only $SL_2(\mathbb{R})$ invariant submanifolds of the stratum $\mathcal{H}(2)$ are the Teichmüller curves (cases 1 and 2 in Theorem) and the stratum itself (case 3). The only $SL_2(\mathbb{R})$ invariant probability measures are the Lebesgue measures on these loci. To prove Theorem we use asymptotic theorems (namely Birkhoff and Oseledets ergodic theorem) with respect to those measure.

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On the neutral Oseledets bundle of Kontsevich-Zorich cocycle over certain cyclic covers

CARLOS MATHEUS

(joint work with Giovanni Forni and Anton Zorich)

The moduli space $\mathcal{H}_g^{(1)}$ of unit area Abelian differentials ω on a genus $g \geq 1$ Riemann surface M is naturally stratified by prescribing the list $(k_1, \ldots, k_\sigma)$ of orders of zeros of ω. Here $\sum_{n=1}^\sigma k_i = 2g - 2$ in view of classical index theorems (Poincaré-Hopf, Gauss-Bonnet, Riemann-Roch, etc.). Denoting by $\mathcal{H}(k_1, \ldots, k_\sigma)$ the corresponding stratum, it is possible to define a natural $\mathrm{SL}(2, \mathbb{R})$-action on each connected component[1] $\mathcal{C}$ of $\mathcal{H}(k_1, \ldots, k_\sigma)$. By the seminal works of Howard Masur [9] and William Veech [11], we know that the action of the diagonal subgroup $g_t = \mathrm{diag}(e^t, e^{-t})$ of $\mathrm{SL}(2, \mathbb{R})$ is ergodic (and actually mixing) with respect to a natural $\mathrm{SL}(2, \mathbb{R})$-invariant probability μ_{MV} on $\mathcal{C}$ (sometimes called *Masur-Veech measure* in the literature). The action of g_t is the so-called *Teichmüller geodesic flow*. This flow is known to act as a renormalization dynamics for interval exchange transformations, certain rational billiards and vertical flows on translation surfaces. In particular, the study of Lyapunov exponents of g_t is a relevant subject connected to the deviations of ergodic means of the systems quoted above.

Following M. Kontsevich and A. Zorich, the Lyapunov spectrum (i.e., the collection of Lyapunov exponents) of Teichmüller geodesic flow can be computed from the nowadays called *Kontsevich-Zorich (KZ) cocycle*. In few words, KZ cocycle G_t^{KZ} is obtained from the quotient by the mapping class group $\Gamma_g := \mathrm{Diff}^+(M)/\mathrm{Diff}_0^+(M)$ of the trivial cocycle $\widehat{G_t^{KZ}} : \mathcal{T}_g \times H^1(M, \mathbb{R}) \to \mathcal{T}_g \times H^1(M, \mathbb{R})$, $\widehat{G_t^{KZ}}(\omega, c) = (g_t(\omega), c)$. Here, $\mathrm{Diff}^+(M)$ is the set of orientation-preserving diffeomorphisms of M, $\mathrm{Diff}_0^+(M)$ is the connected component of the identity inside $\mathrm{Diff}^+(M)$, and $\mathcal{T}_g$ is the Teichmüller space of Abelian differentials on M. By definition, G_t^{KZ} is a symplectic cocycle on the $2g$-dimensional real vector space $H^1(M, \mathbb{R})$ (since it preserves the symplectic natural intersection form on $H^1(M, \mathbb{R})$, and thus the Lyapunov spectrum of G_t^{KZ} with respect to any g_t-invariant probability μ is symmetric under sign changes, i.e., it has the form

$$\lambda_1^\mu \geq \cdots \geq \lambda_g^\mu \geq 0 \geq \lambda_{g+1}^\mu = -\lambda_g^\mu \geq \cdots \geq \lambda_{2g}^\mu = -\lambda_1^\mu.$$

Also, it is possible to show that the Lyapunov exponents of g_t with respect to μ have the form $\pm 1 \pm \lambda_i^\mu$, so that the Lyapunov spectra of g_t are completely determined by the Lyapunov spectra of G_t^{KZ}.

For the Masur-Veech measure μ_{MV}, after several computer experiments, it was conjectured by Kontsevich and Zorich that the Lyapunov spectrum of KZ cocycle was *simple*, i.e., all exponents $\lambda_i^{\mu_{MV}}$ have multiplicity 1 and, in particular, 0 doesn't

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[1]After the results of Maxim Kontsevich and Anton Zorich [8], there is a complete classification of such connected components and, in particular, we know that there are at most 3 of them per stratum.

belong to it. Nowadays, after the works of Giovanni Forni [3], and Artur Avila and Marcelo Viana [1], we have that this conjecture is true.

Partly motivated by this, W. Veech asked whether such a conjecture would remain true for other $SL(2, \mathbb{R})$-invariant probabilities. In general, a simple geometrical argument reveals that $\lambda_1^\mu = 1$. Also, as it was shown by W. Veech in the case of μ_{MV} and by G. Forni in the general case of a g_t-invariant probability μ, one always has $1 = \lambda_1^\mu > \lambda_2^\mu$, i.e., λ_1^μ always has multiplicity 1, so that Veech's question concerns only the exponents $\lambda_2^\mu \geq \cdots \geq \lambda_g^\mu (\geq 0)$. In 2005, G. Forni [4] found an example of $SL(2, \mathbb{R})$-invariant probability μ_{EW} in the stratum $\mathcal{H}(1, 1, 1, 1)$ of genus 3 Riemann surfaces equipped with Abelian differential with 4 simple zeroes such that $\lambda_2^{\mu_{EW}} = \lambda_3^{\mu_{EW}} = 0$. Hence, this shows that Kontsevich-Zorich conjecture is far from being true for other measures than μ_{MV}. In 2008, Forni's example was rediscovered by Martin Möller, Frank Herrlich and Gabriela Schmithüsen [7] as an example of Teichmüller curve with plenty of unusual properties and they coined the term *Eierlegende Wollmilchsau* for the (unique) square-tiled surface in the support of μ_{EW}. In the same year, G. Forni and the present author announced the existence of another $SL(2, \mathbb{R})$-invariant probability μ_O in the (even spin connected component of the) stratum $\mathcal{H}(2, 2, 2)$ of genus 4 Riemann surfaces with 3 double zeroes such that $\lambda_2^{\mu_O} = \lambda_3^{\mu_O} = \lambda_4^{\mu_O} = 0$, i.e., the spectrum is *totally degenerate*. It was recently suggested to the author by Vincent Delecroix and Barak Weiss that, in analogy to the *Eierlegende Wollmilchsau*, the unique square-tiled surface in the support of μ_O should be called *Ornithorynque* (i.e., *Platypus* in French), *Ornitorinco* (i.e., *Platypus* in Italian) or even *Ornitorrinco* (i.e., *Platypus* in Portuguese), and that's why we denoted μ_O the corresponding measure.

In any case, even though it was shown recently by M. Möller [10] that, except possibly for certain strata in genus 5, there are no further totally degenerate examples among Teichmüller curves besides the previous examples, it is possible to "include" these examples in a larger class of Teichmüller curves called *square-tiled cyclic covers* obtained by cyclic covers of the Riemann sphere branched at four points. After the works of G. Forni, the present author and A. Zorich [5], and Alex Eskin, Maxim Kontsevich and Anton Zorich [2], the geometry, combinatorics and the *precise* value of *individual* Lyapunov exponents of square-tiled cyclic covers were studied in details, and, in particular, we know that this is a rich class of examples with *partially degenerate* spectrum (i.e., some of the Lyapunov exponents vanish). Also, in a work [6] still in preparation, G. Forni, A. Zorich and the present author found the geometric reason responsible for the presence of vanishing exponents in square-tiled cyclic covers: indeed, denoting by B the *second fundamental form* (also known as *Kodaira-Spencer map*) of the *Gauss-Manin connection* on the *Hodge bundle $H_g^1 := (\mathcal{T}_g \times H^1(M, \mathbb{R}))/\Gamma_g$*, the neutral Oseledets bundle E_0^μ (i.e., the Oseledets subspaces associated to vanishing Lyapunov exponents) coincides with the annihilator $\text{Ann}(B)$ of B. In particular, since B is a real-analytic function of the base point $\omega \in \mathcal{C}$, we see that the neutral Oseledets bundles of square-tiled cyclic covers depend *real-analytically* on the base point. Notice that

this is very far from being true for general cocycles (and the best one can say in general is that Oseledets subspaces depend measurably on the base point).

Of course, it is tempting to conjecture that this picture for square-tiled cyclic covers could be generalized for all $SL(2, \mathbb{R})$-invariant probabilities under KZ cocycle. One of the main results of the work [6] is the fact that the family of genus 10 curves $y^6 = \prod_{n=1}^{6} (x - x_n)$ equipped with the Abelian differentials $\omega = (x-x_1)dx/y^3 \in \mathcal{H}(8, 2, 2, 2, 2, 2)$ for a $SL(2, \mathbb{R})$-invariant locus supporting $SL(2, \mathbb{R})$-invariant probabilities such that the corresponding neutral Oseledets bundles doesn't coincide with the annihilator of B even though these subspaces have the same dimension!

Closing our discussion, we present in a nutshell the proof of this result. A direct inspection reveals that $\mathrm{Ann}(B)$ is $SO(2, \mathbb{R})$-invariant, so that, if $E_0^{\mu} = \mathrm{Ann}(B)$, we would conclude that $\mathrm{Ann}(B)$ is $SO(2, \mathbb{R})$ and g_t invariant at the same time. Hence, it would follow that $\mathrm{Ann}(B)$ is $SL(2, \mathbb{R})$-invariant. However, this last property can be easily contradicted if one can find an adequate pair of pseudo-Anosov (i.e., a pair of periodic g_t-orbits) associated to two Abelian differentials ω and ω' sitting on the *same* Riemann surface M and deduced one from the other by rotation (i.e., an element in $SO(2, \mathbb{R})$, or equivalently, $\omega' = e^{i\theta}\omega$ for some $\theta \in \mathbb{R}$). Indeed, in this context, $\mathrm{Ann}(B)$ would be a common subspace of the matrices associated to the actions on homology of these pseudo-Anosovs, and so we get a contradiction as soon as these matrices don't share common subspaces, a simple (linear algebra) property to check from explicit realizations of them.

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Geometry of Teichmüller curves

Dawei Chen

(joint work with Martin Möller)

My research centers around the geometry of moduli spaces. One of my projects related to this meeting is to study the *algebro-geometric* properties of $SL(2, \mathbb{R})$-submanifolds in the moduli space $\mathcal{H}$ of Abelian differentials.

Take a Riemann surface along with a holomorphic 1-form parameterized in $\mathcal{H}$. Its complex structure varies naturally with the 1-form via the $SL(2, \mathbb{R})$ action. An *$SL(2, \mathbb{R})$-submanifold* is an orbit closure in $\mathcal{H}$ under this action. To name a few examples, if an orbit itself forms a closed complex curve, we call it a *Teichmüller curve*. The Hurwitz space parameterizing branched covers of tori and the strata in $\mathcal{H}$ parameterizing 1-forms with prescribed type of zeros are also $SL(2, \mathbb{R})$-submanifolds.

In algebraic geometry it is often desirable to work with a compactified moduli space, like passing from the moduli space $\mathcal{M}_g$ of genus g Riemann surfaces to its Deligne-Mumford compactification $\overline{\mathcal{M}}_g$, i.e. we allow a slight degeneration of Riemann surfaces by pinching two points together. Here I would like to emphasize the significance of this viewpoint for the study of $SL(2, \mathbb{R})$-submanifolds.

Take Teichmüller curves as illustration of the idea. One can associate three numbers: *the sum of Lyapunov exponents L, Siegel-Veech constant c* and *slope s* to a Teichmüller curve. The first two come from dynamics. Roughly speaking, Lyapunov exponents characterize the rate of separation of infinitesimally closed trajectories under the Teichmüller geodesic flow. The Siegel-Veech constant represents the average number of weighted horizontal cylinders in the orbit that generates the Teichmüller curve, where the Abelian differential defines a flat structure on the Riemann surface such that it decomposes into cylinders along a fixed direction and the weight of a cylinder is given by its height/length. The third one, slope, comes from algebraic geometry, by taking the quotient of the intersection of a Teichmüller curve with the boundary of $\overline{\mathcal{M}}_g$ and the intersection with the first Chern class of the Hodge bundle on $\overline{\mathcal{M}}_g$.

Although these three numbers seem unrelated, after the work of Kontsevich [5], Bouw-Möller [1], Eskin-Kontsevich-Zorich [4] (in much more generality) and myself [2], we now know a simple relation among them:

$$s = \frac{12c}{L} = \frac{12c}{c + \kappa},$$

where κ is a constant determined by the type of zeros of a generating Abelian differential. Namely, knowing any one of the three immediately tells the other two!

As an application, joint with Möller [3] we show that for many strata of Abelian differentials in low genus the sum of Lyapunov exponents is *non-varying* for *all* Teichmüller curves in that stratum. Our idea is to prove that the slope is non-varying first, by exhibiting a geometrically defined *divisor* on $\overline{\mathcal{M}}_g$ that does not

intersect Teichmüller curves. Then we can translate back to the dynamical side by the above relation.

Currently I am interested in generalizing the results of Teichmüller curves to quadratic differentials as well as higher dimensional $SL(2, \mathbb{R})$-submanifolds.

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Ends of strata of the moduli space of quadratic differentials
Corentin Boissy

In this talk, we present the main result of the paper [2].

1. Introduction and statement of the result

We study compact surfaces endowed with a flat metric with isolated conical singularities and $\mathbb{Z}/2\mathbb{Z}$ linear holonomy. Such surface is naturally identified with a Riemann surface endowed with a meromorphic quadratic differential with at most simple poles. The moduli space of such surfaces with fixed combinatorial data is a noncompact complex-analytic orbifold $\mathcal{Q}$ and is called a stratum of the moduli space of quadratic differentials.

There is an obvious way to leave any compact subset of $\mathcal{Q}$ by rescaling the metric so that the area tends to infinity or to zero. Hence we usually consider normalized strata that corresponds to area one flat surfaces. A normalized strata is still noncompact, and a neighborhood of the boundary corresponds to flat surfaces with a short saddle connection.

Very few results are known about the topology of these strata. Kontsevich, Zorich and Lanneau have classified their connected components (see [5] and [6]). Here, we show the following theorem:

Theorem 1.1. *Let $\mathcal{C}$ be a connected component of a normalized stratum of the moduli space of quadratic differentials. Then, $\mathcal{C}$ has only one topological end.*

We will consider the subset $\mathcal{C}_\varepsilon \subset \mathcal{C}$ of area one flat surfaces that have a saddle connection of length less than ε. And show that it is connected.

2. Combinatorics of a surface near the boundary

The most natural approach to prove the theorem is to describe a typical flat surface in the neighborhood of the boundary. A saddle connection is a geodesic joining two singularities. A flat surface is near the boundary if it has a saddle connection of short length. One can look at the set of saddle connections that are of minimal length. In general, there can be several such saddle connections and we can show that they are parallel for a generic flat surface. Furthermore, they stay parallel and of the same length for any small perturbation of the surface. One can associate to such collection of saddle connection a "configuration" that describes how the collection splits the surface (see [4, 9], and also [1]). The number of different configurations tends to infinity when the genus tends to infinity. Also, there is no obvious way to relate the different configurations that occur on a connected component of a stratum, as illustrated by the following example.

2.1. **Example.** We consider the stratum of quadratic differentials $\mathcal{Q}(-1, 9)$. This stratum has two connected component: the *regular* one and the *irreducible* one. For each $k \in \{1, 2, 3, 4\}$, there exists $S_k \in \mathcal{Q}(-1, 9)$ with the following decomposition:

- on S_k, there are exactly two smallest closed saddle connections $\gamma_{1,k}$ and $\gamma_{2,k}$.
- $\gamma_{1,k}$ and $\gamma_{2,k}$ start and end at the singularity of order 9 and are the boundary of a metric cylinder embedded in the surface.
- the angle between γ_1 and γ_2 is $k\pi$.

Lanneau ([6]) has proven the following.

- If $k \in \{1, 2, 4\}$ then S_k belongs to the *regular* connected component of $\mathcal{Q}(-1, 9)$.
- If $k = 3$, then S belongs to the *irreducible* connected component of $\mathcal{Q}(-1, 9)$.

Therefore, in we can start from the surface S_1, we can continuously deform it so that we get S_2 or S_4, but it is impossible to continuously deform the surface so that we have S_3. Note that in the stratum $\mathcal{Q}(-1, 1+4n)$, for $n \geq 3$, one can find surfaces with analogous decomposition by a pair of saddle connections bounding a cylinder. But in this case all parameters k, can be reached by continuous deformations, since the underlying stratum is connected.

3. Sketch of the proof

In order to bypass these difficulties, we use a construction, that build a translation surface $S(\pi, \zeta)$ from to a (irreducible) permutation $\pi \in \Sigma_d$ and a continuous parameter $\zeta \in \mathbb{C}^d$. The continuous parameter is called a *suspension data* and must satisfy some linear inequalities (see [7]), and the connected component in which the constructed surface lies depends only of π. The original construction is due to Veech [10], but the equivalent point of view in terms of suspension data

is due to Marmi-Moussa-Yoccoz [7]. Such construction was generalized quadratic differentials by the author and Lanneau [3].

Given a permutation π that corresponds to a connected component $\mathcal{C}$, we can define the set $S(\mathcal{D}_{\pi,\varepsilon})$ of area one flat surfaces obtained with the Veech construction, and with a parameter ζ having at least one coordinate of length smaller than ε. This set is naturally a subset of $\mathcal{C}_\varepsilon$, and one can show that it is connected.

The set of permutations that can appear with the Veech construction in a connected component of a stratum is called the *extended Rauzy class*. The important fact is that for each pair π, π' in such class, we can join π to π' using a sequence of elementary operations called the (extended) Rauzy moves. This moves are related to the well known *Rauzy induction*. Using this moves, one can show the following lemma:

Lemma 3.1. *All the subsets $S(\mathcal{D}_{\pi,\varepsilon})$ are in the same connected component of $\mathcal{C}_\varepsilon$.*

A difficulty now is that a generic surface near the boundary, even if it is obtained by the Veech construction, it does not necessarily appear from the construction with a "short" parameter. But we can show the following:

Lemma 3.2. *For any flat surface S in $\mathcal{C}_\varepsilon$, there exists a permutation π and a path that stays in $\mathcal{C}_\varepsilon$ and joins S to $S(D_{\pi,\varepsilon})$.*

Theorem 1.1 is then obtained by a combination of the to previous lemma.

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Abelian Square-Tiled Surfaces
Alex Wright

The flat pillowcase metric on $\mathbb{CP}^1 \setminus \{z_1, z_2, z_3, z_4\}$ is given by the quadratic differential

$$q_0 = \frac{(dz)^2}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)}.$$

For some choice of z_1, z_2, z_3, z_4, the result is two squares glued together along corresponding edges.

A cyclic square-tiled surface is a normal cover of $\mathbb{CP}^1 \setminus \{z_1, z_2, z_3, z_4\}$ whose deck group is cyclic, endowed with a lift of q_0. Such covers can be given as algebraic curves by

$$w^N = (z - z_1)^{a_1}(z - z_2)^{a_2}(z - z_3)^{a_3}(z - z_4)^{a_4}.$$

Historically, interest in cyclic square-tiled surfaces arose from examples of Teichmüller curves with totally degenerate Lyapunov spectrum [5, 3, 4]. (All known examples of Teichmüller curves with totally degenerate Lyapunov spectrum are cyclic square-tiled surfaces.)

Cyclic square-tiled surfaces have been studied systematically in [2] and [4]. Notably, Eskin-Kontsevich-Zorich have computed all individual Lyapunov exponents of the Hodge bundle for cyclic square-tiled surfaces. The moral reason for their success is that the underlying Riemann surface is nicely described as an algebraic curve, allowing computations to be done explicitly.

We define an abelian square-tiled surface to be a normal cover of $\mathbb{CP}^1 \setminus \{z_1, z_2, z_3, z_4\}$ whose deck group is abelian, endowed with a lift of q_0. Despite the fact that there is no longer a single nice formula for the underlying algebraic curve, it is possible to write down the function field, and once again all Lyapunov exponents may be computed. This computation was in fact the original motivation for studying abelian square-tiled surfaces, since the holonomy double cover of a cyclic square tiled-surfaces is abelian, and the Lyapunov exponents of the Hodge bundle of the double cover give the Lyapunov exponents of the full tangent bundle to the cyclic square-tiled surface.

Perhaps the most surprising thing about abelian square-tiled surfaces is that they can be used to rephrase the construction of the Bouw-Möller Teichmüller curves. These are Teichmüller curves whose affine group is typically a (n, m, ∞) triangle group [1]. This construction includes the Veech and Ward curves as a special case.

Hooper has given an elementary construction of lattice surfaces whose affine group is typically a (n, m, ∞) triangle group [4].

Theorem (W). *Hooper's lattice surfaces generate the Bouw-Möller curves in all cases.*

Previously this was known in some cases. As a consequence, the Bouw-Möller curves are generated by the flat surfaces with semi-regular polygon decomposition, discovered independently by Hooper and Mukamel.

Bouw-Möller's construction is novel in that, instead of using the $SL_2(\mathbb{R})$ action, it uses a variant of the following result.

Theorem (Rephrasing of Möller's Criterion). *A curve in moduli space is a Teichmüller curve if and only if there is a rank two sub-VHS (i.e., rank two bundle that splits into a $(1,0)$ part and a $(0,1)$ part) of H^1 such that*

- *the rank two bundle has parabolic monodromy around cusps, and*
- *the associated period map has non-vanishing derivative.*

The monodromy of the Hodge bundle for cyclic square-tiled surfaces may essentially be computed using hypergeometric differential equations, and as a result rank two subbundles can be found with triangle group monodromy, whose period maps have non-vanishing derivative. However, these bundles have finite order monodromy around some cusps, so Möller's Criterion does not apply.

The solution is to use abelian square-tiled surfaces which are exceptionally symmetric, in that they admit a lift of the Klein four group of pillowcase symmetries. Upon taking a fiberwise quotient, some of the noded Riemann surfaces at the cusps of the arithmetic Teichmüller curve become smooth, and then, after forgetting marked points, Möller's criterion applies. This is only a rephrasing of the original construction of Bouw-Möller, but the square-tiled surface perspective directly gives the following surprising result.

Theorem (W). *The closures of the Bouw-Möller and Veech Teichmüller curves in the Deligne-Mumford compactification of $\mathcal{M}_g$ are images of the closures of arithmetic Teichmüller curves under a tautological forgetful map $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_g$.*

A quadratic differential with simple poles may be assigned to all but at most four fibers of a Veech or Bouw-Möller Teichmüller curve, giving the fibers the structure of a square-tiled surface in such a way that all the square-tiled surfaces thus obtained are related by the $SL_2(\mathbb{R})$ action on the square-tiled flat structures.

We suggest that Teichmüller curves in $\mathcal{M}_g$ which are images of arithmetic Teichmüller curves in some $\mathcal{M}_{g,n}$ under a tautological forgetful map be thought of as "pseudo-arithmetic." We do not yet know if all Teichmüller curves are pseudo-arithmetic.

The high-tech nature of the construction of the Veech-Ward-Bouw-Möller curves gives the Lyapunov exponents and monodromy almost for free, and allows us to observe that these curves are pseudo-arithmetic. Furthermore, the connection to abelian square-tiled surfaces as well as to the semi-regular polygon decomposition allows the Veech-Ward-Bouw-Möller curves to be studied by elementary means.

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Weak mixing of induced IETs

MICHAEL BOSHERNITZAN

For an ergodic IET (interval exchange transformation) $f : X \to X$, $X = [0,1]$ we prove that the IETs $f_t : X_t \to X_t$, $X_t := [0,t]$, $0 < t < 1$ (obtained from f by inducing f to X_t) are weakly mixing for Lebesgue almost all t.

Ergodic infinite extensions of locally Hamiltonian flows

CORINNA ULCIGRAI

(joint work with Krzysztof Fraczek)

Let S be a closed connected surface of genus $g \geq 1$ with a smooth area form ν. Consider a vector field X which preserves ν and denote by $(\phi_t)_{t \in \mathbb{R}}$ the associated area-preserving flow. Given a smooth real valued function $f : S \to \mathbb{R}$, the *extension* $(\Phi_t^f)_{t \in \mathbb{R}}$ of $(\phi_t)_{t \in \mathbb{R}}$ given by f is the flow on the trivial bundle $M = S \times \mathbb{R}$ given by the solutions of the differential equations

$$(0.1) \qquad \begin{cases} \frac{dx}{dt} &= X(x), \\ \frac{dy}{dt} &= f(x), \end{cases} \quad \Rightarrow \quad \Phi_t^f(x,y) = \left(\phi_t x, \ y + \int_0^t f(\phi_s x) \, ds \right),$$

where $(x,y) \in S \times \mathbb{R}$. Hence, the flow (0.1) projects in the first coordinate to the the surface flow $(\phi_t)_{t \in \mathbb{R}}$, while the motion in the $\mathbb{R}$-coordinate is determined by the ergodic integrals of f along the flow trajectories. Thus, the extension $(\Phi_t^f)_{t \in \mathbb{R}}$ provides a geometric way of visualizing the fluctuations of the ergodic integrals. The flow $(\Phi_t^f)_{t \in \mathbb{R}}$ preserves the *infinite* invariant measure $\mu = \nu \times Leb$, where Leb denotes the Lebesgue measure on the fiber $\mathbb{R}$. Thus one can investigate its ergodic properties. Let us recall that a flow $(\Phi_t)_{t \in \mathbb{R}}$ preserving a invariant measure μ (possibly infinite, as in our case) is *ergodic* if for any measurable set A which is invariant, i.e. such that $A = \Phi_t(A)$ for all $t \in \mathbb{R}$, either $\mu(A) = 0$ or $\mu(A^c) = 0$ where A^c denotes the complement.

For $g = 1$, extensions of linear flows on tori where studied by Herman and Krygin, who showed that the extension can be ergodic only when the rotation number of the linear flow is *Liouville* (see [7, 9]). On the other hand, Fayad and Lemańczyk [3] studied flows on tori with singularities (more precisely, they considered locally Hamiltonian flows, which are defined below) and proved that in this case for almost every rotation number one can construct ergodic extensions

over their minimal components. The ergodicity of extensions of minimal flows of the same type in higher genus $g \geq 2$ is left as an open problem in [3].

We prove the *existence* of ergodic extensions of area-preserving flows on surfaces any genus $g \geq 2$. More precisely, we consider locally Hamiltonian flows $(\phi_t)_{t \in \mathbb{R}}$ (also known as flows given by a multivalued Hamiltonian), which are a natural class of symplectic flows introduced and studied by S.P. Novikov and his school (see [1, 11, 17]). Given a closed 1-form on M, the locally Hamiltonian flow $(\phi_t)_{t \in \mathbb{R}}$ given by η is the the smooth flow on M associated to the vector field X determined by $\eta = i_X \omega = \omega(X, \cdot)$, where ω is the non-degenerate 2-form which gives ν. Let us assume that $(\phi_t)_{t \in \mathbb{R}}$ has *no saddle connections*, (this implies minimality) and that the fixed points of the flow are only *Morse saddles*. The ergodic properties of these locally Hamiltonian surface flows are now well understood (see [8, 4, 12, 15, 16]) and the deviations of their ergodic integrals were studied by Forni in [6]. We restrict our attention to locally Hamiltonian flows of *hyperbolic periodic type*, which are a natural generalization of linear flows on tori whose rotation number has periodic continued fraction (more precisely, we say that a flow is of *periodic type* if it induces, as Poincaré map on a cross section, an interval exchange transformation with periodic Rauzy-Veech expansion and it is of *hyperbolic* periodic type if $2g$ eigenvalues of the period matrix have modulus different than 1). This class, which has measure zero, exhibit nevertheless the same ergodic properties of the typical locally Hamiltonian flows with simple saddles (that is, they are minimal, uniquely ergodic and weakly mixing, but not mixing). Our main result is the following.

Theorem. *Let $(\phi_t)_{t \in \mathbb{R}}$ be a locally Hamiltonian flow of hyperbolic periodic type on a compact surface S of genus $g \geq 2$. There exists a closed $(\phi_t)_{t \in \mathbb{R}}$-invariant subspace $K \subset \mathcal{C}^{2+\epsilon}(S)$ with codimension g in $\mathcal{C}^{2+\epsilon}(S)$, where g is the genus of S, such that if $f \in K$ and there is a fixed point of $(\phi_t)_{t \in \mathbb{R}}$ on which f does not vanish, then the extension $(\Phi_t^f)_{t \in \mathbb{R}}$ is ergodic.*

We remark that the space K is infinite dimensional and it is an extension of the space of invariant distributions introduced by Forni [5]. In particular, the Theorem allows to construct examples of ergodic extensions of area-preserving flows on surfaces of any genus $g \geq 2$. Furthermore, for functions $f \in K$ we can prove a dynamical dichotomy. If $f \in K$ and, otherwise, f vanishes on all fixed points, then the extension $(\Phi_t^f)_{t \in \mathbb{R}}$ is *topologically reducible* (that is, it is isomorphic to the trivial extension $(\Phi_t^0)_{t \in \mathbb{R}}$ given by $\Phi_t^0(x, y) = (\phi_t x, y)$, via an isomorphism of the form $G(x, y) = (x, y + g(x))$, where $g : M \to \mathbb{R}$ is continuous). In this case, which is opposite to ergodicity, the phase space is foliated into invariant sets for $(\Phi_t^f)_{t \in \mathbb{R}}$ given by graphs of g. No other phenomenon (in particular, no *irregular* extensions, see [13]) arise for functions in K.

The proof reduces to proving ergodicity for a class of skew products over interval exchange transformations (IETS). We recall that IETs are Poincaré maps of $(\phi_t)_{t \in \mathbb{R}}$ on a transverse interval $I \subset S$ (in suitably chosen coordinates). The Poincaré map of the extension $(\Phi_t^f)_{t \in \mathbb{R}}$ on the transverse hypersurface $\Sigma = I \times \mathbb{R}$ has the form $T_{\varphi_f}(x, y) = (Tx, y + \varphi_f(x))$, where $(x, y) \in I \times \mathbb{R}$, T is the IET

obtained inducing $(\phi_t)_{t\in\mathbb{R}}$ on I and $\varphi_f : I \to \mathbb{R}$ is given by

$$(0.2) \qquad \varphi_f(x) = \int_0^{\tau(x)} f(\phi_s x)ds,$$

where $\tau(x)$ is the first return time of $x \in I$ to I under $(\phi_t)_{t\in\mathbb{R}}$. It turns out that φ_f has symmetric logarithmic singularities. In order to prove ergodicity of T_φ, we use the technique of *essential values*, developed by K. Schmidt and J.-P. Conze (see for example [13]). To control essential values, we investigate the behavior of Birkhoff sums $\varphi_f^{(n)}(x) = \sum_{k=0}^{n-1} \varphi_f(T^k x)$ of φ_f. Ergodicity follows if one can prove that the sequence $\varphi_f^{(n_j)}$ is partially tight along a subsequence $(n_j)_{j\in\mathbb{N}}$ of partial rigidity times for the IET and at the same time, exploiting the presence of the logarithmic singularities, $\varphi_f^{(n)}$ has enough oscillations. In order to achieve *tightness*, we need *correct* the function φ_f by a piecewise constant function χ (which is equivalent to requiring that f belongs to the space K, which is defined as kernel of the correction operator), following an idea introduced by Marmi, Moussa and Yoccoz in order to solve the cohomological equation for IETs in [10]. The correction operator that we use is closely related to the correction operator used by the Fraczek and Conze in [2]. The additional difficulty that we have to face to achieve tightness is the presence of logarithmic singularities. Here the assumption that the singularities are *symmetric* is crucial to exploit the cancellation mechanism introduced in [16] in order to show absence of mixing of the surface flow.

We conclude with some open questions. It is natural to ask what are the ergodic properties of extensions $(\Phi_t^f)_{t\in\mathbb{R}}$ of a *typical* locally Hamiltonian flow. We believe that if $f \in K$ our techniques could extend to prove ergodicity for almost every flow under a Roth-type condition (see [10]). On the other hand, our approach does not seem to be suitable for functions outside the space K, for which new tools are required.

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Renormalization of Polygon Exchange Maps arising from Corner Percolation

W. Patrick Hooper

Let X be a finite disjoint union of polygons in the plane. A *polygon exchange map* of X, $T : X \to X$, cuts X into finitely many pieces, then and applies a translation to each piece so that the image $T(X)$ has full area in X.

Polygon exchange maps are natural generalizations of interval exchange maps, and yet comparatively little is understood about polygon exchange maps. In particular, it is not understood how effective renormalization arguments will be for understanding the long-term behavior of iterating polygon exchange maps.

We will consider a family of rectangle exchange maps parameterized by a choice of a point (α, β) in the square $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$. We denote these maps by $\Psi_{\alpha,\beta} : X \to X$, where X is a union of four tori. (These maps are defined at the end of this abstract.) We show that for irrational choices of α and β, there are points whose orbits under $\Psi_{\alpha,\beta}$ are periodic, with arbitrarily large period. The space X comes equipped with Lebesgue measure, λ, which we normalize so that $\lambda(X) = 1$. We define $M(\alpha, \beta)$ to be the λ-measure of the collection of all periodic points under $\Psi_{\alpha,\beta}$. In a forthcoming paper, we prove the following three results about this quantity.

Theorem (Periodicity almost everywhere). *$M(\alpha, \beta) = 1$ for Lebesgue-almost every parameter $(\alpha, \beta) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}]$.*

As mentioned above, there are always periodic points. In fact, $M(\alpha, \beta) > 0$ for all (α, β). However,

Theorem (Existence of periodicity only on small sets). *For any $\epsilon > 0$, there are irrational parameters α and β so that $M(\alpha, \beta) < \epsilon$.*

Theorem (Topologically generic aperiodicity). *There is a dense set of irrational parameters (α, β) so that $M(\alpha, \beta) \neq 1$.*

Renormalization Dynamics

A *renormalization* of a polygon exchange map $T : X \to X$, is the choice of a finite union Y of sub-polygons of X with disjoint interiors such that the return map $T|_Y : Y \to Y$ is also a polygon exchange map. In the case of interval exchange maps, the return map to an interval is always another interval exchange map. For polygon exchange maps, however, not all such return maps yield polygon exchange maps.

Let G be the group of isometries of $\mathbb{R}$ generated by the maps $z \mapsto z + 1$ and $z \mapsto -z$. This group has $[0, \frac{1}{2}]$ as a fundamental domain, and for $x \in \mathbb{R}$ we write $x \pmod{G}$ to denote the unique element $y \in [0, \frac{1}{2}]$ so that there is a $g \in G$ with $g(x) = y$. We define the action f on the irrationals in $(0, \frac{1}{2})$ by

$$(0.1) \qquad\qquad f(x) = \frac{x}{1 - 2x} \quad \pmod{G}.$$

This map governs our renormalization.

For the maps $\Psi_{\alpha,\beta}$, we actually renormalize on a double cover. So, for each pair (α, β) we consider a lift $\widetilde{\Psi}_{\alpha,\beta} : \widetilde{X} \to \widetilde{X}$, where $\widetilde{X}$ is a particular double cover of X. For each pair of parameters, we show that there is a subset $Z = Z(\alpha, \beta) \subset \widetilde{X}$ so that the return map of $\widetilde{\Psi}_{\alpha,\beta}$ to Z is affinely conjugate to $\widetilde{\Psi}_{f(\alpha),f(\beta)}$.

So, we are implicitly interested in the dynamics of $f \times f$ on $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$. The following results concern the dynamics of this map.

Proposition. *The measure ν on $[0, \frac{1}{2}]$ which is absolutely continuous with respect to λ with Radon-Nikodym derivative given by $\frac{d\nu}{d\lambda}(x) = \frac{1}{x} + \frac{1}{1-x}$ is f invariant.*

It should be noted that $\nu([0, \frac{1}{2}]) = \infty$. Nonetheless:

Theorem (Poincaré recurrence). *Let $A \subset [0, \frac{1}{2}] \times [0, \frac{1}{2}]$ be Lebesgue measurable. Then, for $\nu \times \nu$-a.e. pair $(\alpha, \beta) \in A$ there is an $n \geq 1$ so that $(f \times f)^n(\alpha, \beta) \in A$.*

It remains to explain what the orbit of (α, β) under $f \times f$ says about $M(\alpha, \beta)$. In fact, there is a formula which gives the quantity $M(\alpha, \beta)$ in terms of a limit involving a finite dimensional cocycle over $f \times f$. (We omit a formal description for brevity.) By explicitly working with this formula, we are able to show prove Theorem 3, as well as the following.

Lemma. *For α and β irrational, there is an increasing sequence $\{m_k\}_{k \in \mathbb{N}}$ such that $\lim_{i \to \infty} m_i = M(\alpha, \beta)$. Moreover, for all integers $k > 0$,*

$$1 - m_k < (1 - m_{k-1})\left(1 - \frac{4}{3} f^{k-1}(\alpha) f^{k-1}(\beta)\right).$$

In particular, if the orbit of (α, β) has an accumulation point in $(0, \frac{1}{2}] \times (0, \frac{1}{2}]$, we have $M(\alpha, \beta) = 1$. In particular, Theorem implies $M(\alpha, \beta) = 1$ for Lebesgue-almost every (α, β). (This sketches the proof of Theorem 1.)

Definition of the polygon exchange maps $\Psi_{\alpha,\beta}$

We will now define the examples of interest to us. Consider the lattice $\Lambda = \mathbb{Z}^2 \cup \left[(\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2\right]$, and let Y be the torus $\mathbb{R}^2/\Lambda$. This torus may be cut into two squares, $A_1 = [0, \frac{1}{2}) \times [0, \frac{1}{2})$ and $A_{-1} = [0, \frac{1}{2}) \times [\frac{1}{2}, 1)$, whose union forms a fundamental domain for the action of Λ on $\mathbb{R}^2$ by translation. Let N be the finite set of four elements, $N = \{(1, 0), (-1, 0), (0, 1), (0, -1)\}$. Fix two parameters $\alpha, \beta \in [0, \frac{1}{2}]$. We will define a polygon exchange map $\Psi_{\alpha,\beta} : Y \times N \to Y \times N$. Assume $(x, y) \in A_s$ and $\mathbf{v} = (a, b) \in N$. Then we define

$$(0.2) \qquad \Psi_{\alpha,\beta}\big((x, y), \mathbf{v}\big) = \big((x + bs\alpha, y + as\beta) \pmod{\Lambda}, (bs, as)\big).$$

Figure 1 illustrates this map.

Connections to Corner Percolation

We will very briefly describe the corner percolation model introduced by Bálint Tóth, and studied in depth by Gábor Pete [1].

Consider four squares decorated by arcs joining a pair of midpoints of adjacent edges:
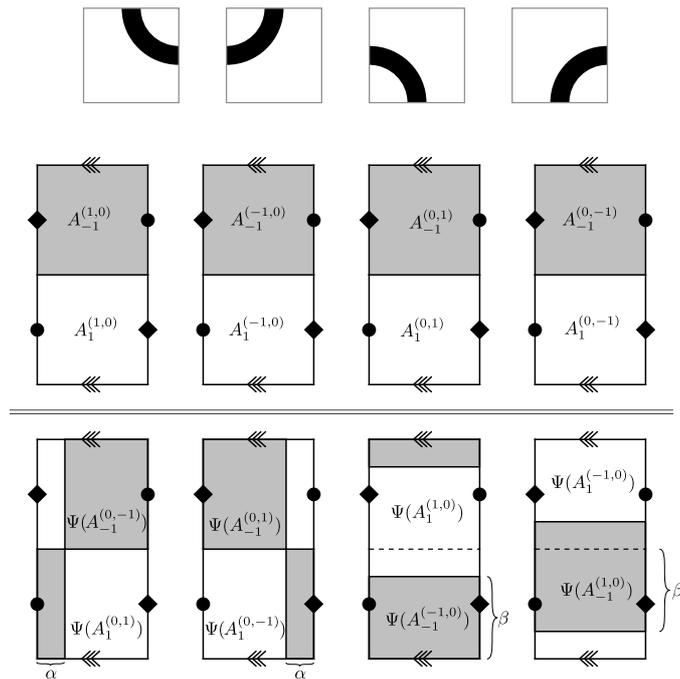
FIGURE 1. This illustrates the map $\Psi = \Psi_{\alpha,\beta}$ defined in equation 0.2. Above the line indicates the sets $A_s^{(a,b)} = A_s \times \{(a, b)\}$, and below illustrates their images under Ψ. In both cases, the tori are drawn $Y \times \{(1, 0)\}$, $Y \times \{(-1, 0)\}$, $Y \times \{(0, 1)\}$ and $Y \times \{(0, -1)\}$, from left to right.

A *corner percolation tiling* is formed by tiling the plane with these tiles, so that whenever two tiles are adjacent along an edge, the arcs of these two tiles either both have endpoints on the edge or neither have endpoints along the edge. Thus, the arcs of the tiles join together to form a family of disjoint simple curves, which are either closed or bi-infinite.

In [1], it was shown that in a "random" corner percolation tiling, all loops were closed. (Much stronger results were shown as well.) Corner percolation tilings can also be generated using symbolic dynamics applied to a pair of rotations by α and β. The quantity $M(\alpha, \beta)$ can then be interpreted as representing the probability that a curve of the tiling is closed, where the curve is chosen by fixing an edge in the tiling and looking a the curve through that edge. Thus, Theorems 1-3 are also theorems about tilings which are random with respect to a zero-entropy measure $\mu_{\alpha,\beta}$ on the space of corner percolation tilings.

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Translation surfaces satisfying Pérez Marco's condition
Yitwah Cheung
(joint work with Pascal Hubert and Howard Masur)

For the billiard in a rational polygon P, Kerckhoff-Masur-Smillie [5] showed that the flow in almost every direction is uniquely ergodic. Thus, the set $\mathrm{NE}(P)$ of non-ergodic directions has measure zero. In [6] Masur showed that every non-ergodic direction determines a Teichmüller geodesic that is divergent in the stratum. In other words, denoting by $\mathrm{DIV}(P)$ the set of such directions, we have the inclusion

$$\mathrm{NE}(P) \subset \mathrm{DIV}(P)$$

for any rational polygon P. Moreover, the Hausdorff dimension of $\mathrm{DIV}(P)$ is bounded above [6]:

$$\mathrm{H.\,Dim\,DIV}(P) \leq \frac{1}{2}.$$

Both sets can be defined more generally for the class of translation surfaces. In [7] Masur-Smillie showed that the Hausdorff dimension of $\mathrm{NE}(X)$ is positive for a generic translation surface. In the few examples where it has been determined, the value of the Hausdorff dimension (for either of these sets) is either zero or $\frac{1}{2}$ and it remains unknown if any value strictly between these can be achieved.

For Veech surfaces, both of these sets are countable, and hence, of Hausdorff dimension zero. Smillie-Weiss [8] showed that this holds more generally for branched covers of Veech surfaces that are branched over a single point, as well as for iterates of this construction. Parking garages exhibiting this phenomenon have been found in [4], suggesting that billiard examples may also exist. In these examples, the converse to Masur's theorem [6] holds, i.e. every uniquely ergodic direction

determines a Teichmüller geodesic that eventually returns to a compact set. However, it is well-known that the converse does not hold in general [3].

Consider the billiard in the rectangular table P_λ of width one and height two with a wall of length $\lambda \in (0,1)$ parallel to the shorter sides inserted at the midpoint of one of its longer sides. In joint work with Pascal Hubert and Howard Masur [2], we show that

(i) $\text{H.Dim} \, \text{DIV}(P_\lambda) = \frac{1}{2}$ if λ is irrational, and is otherwise zero;

(ii) $\text{H.Dim} \, \text{NE}(P_\lambda) = \frac{1}{2}$ if λ is an irrational satisfying Pérez Marco's condition

$$\sum_k \frac{\log \log q_{k+1}}{q_k} < \infty$$

on the denominators of the continued fraction of λ, and is otherwise zero.

The statement (ii) extends earlier results of Boshernitzan and Cheung [1].

The associated translation surface X_λ obtained by unfolding P_λ is a branched double cover of the square torus T, branched over two points that form the endpoints of a horizontal slit. We expect the main result to hold more generally if X_λ is replaced with a cyclic branched cover of T branched over an arbitrary two points. In this case, q_k is understood to be the denominators of the sequence of simultaneous best approximants of the holonomy vector (x_0, y_0) that joins the branch points.

We further expect this generalization to hold if T is replaced by a square-tiled surface. More specifically, the main result is expected to hold for the class of cyclic branched covers of arithmetic Veech surfaces, branched over two points, one of which is assumed to be a singularity.

Question: Does the dichotomy for the Hausdorff dimension of the set of nonergodic directions or the set of divergent directions still hold if we replace T with a non-arithmetic Veech surface ? And if so, what is the analog of Pérez Marco's condition ?

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Surfaces with Completely Degenerate Kontsevich-Zorich Spectrum
David Aulicino

1. Introduction

In [8], Kontsevich and Zorich introduced the Kontsevich-Zorich cocycle, denoted G_t^{KZ}, which is a continuous version of the Rauzy-Veech-Zorich cocycle. They showed that this cocycle has a spectrum of $2g$ non-trivial Lyapunov exponents with the property

$$1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g \geq 0 \geq -\lambda_g \geq \cdots \geq -\lambda_2 \geq -\lambda_1 = -1.$$

These exponents have strong implications about the dynamics of flows on Riemann surfaces, interval exchange transformations, rational billiards, and related systems. These exponents also describe how generic trajectories of an Abelian differential distribute over a surface [14]. Furthermore, Zorich [14] proved that they fully describe the non-trivial exponents of the Teichmüller geodesic flow, denoted G_t. Veech [13] proved $\lambda_2 < 1$, which implies that G_t is non-uniformly hyperbolic. Since then, the study of the Lyapunov spectrum of the Kontsevich-Zorich cocycle has become of widespread interest. Forni [4] proved the first part of the Kontsevich-Zorich conjecture [8]: $\lambda_g > 0$ for almost every $\mathrm{SL}_2(\mathbb{R})$-invariant measure in the moduli space of holomorphic quadratic differentials. His result implies G_t^{KZ} is also non-uniformly hyperbolic. Avila and Viana [1] then used independent techniques to show that the spectrum is simple for the canonical measures on the strata of Abelian differentials, i.e. $\lambda_k > \lambda_{k+1}$, for all k.

Veech asked how degenerate the spectrum could be. Forni [5] found an example of a surface in genus three whose Teichmüller disk has completely degenerate Lyapunov spectrum, i.e. $\lambda_1 = 1 > \lambda_2 = \lambda_3 = 0$. In the literature, Forni's genus three example is known as the Eierlegende Wollmilchsau for its numerous remarkable properties [7]. Forni and Matheus [6] then found an example in genus four with $\lambda_1 = 1 > \lambda_2 = \lambda_3 = \lambda_4 = 0$. Both examples are Veech surfaces and in particular cyclic covers of square tiled surfaces. Using techniques from algebraic geometry Möller [10] proved that these two examples are the only examples of Veech surfaces with completely degenerate Lyapunov spectrum except for possible examples in certain strata of Abelian differentials in genus five.

2. Complete Periodicity and the Rank One Locus

Let $\mathcal{D}_g(1)$ denote the subset of the moduli space of Abelian differentials such that the derivative of the period matrix has rank one. For this note it suffices to interpret $\mathcal{D}_g(1)$ as the subset of the moduli space containing all surfaces with completely degenerate Lyapunov spectrum. Though the Kontsevich-Zorich exponents are only defined almost everywhere, the following corollary allows us to state a problem that is defined everywhere.

Corollary 2.1. *Let μ be a $SL_2(\mathbb{R})$-invariant ergodic probability measure on the moduli space of Abelian differentials. The support $supp(\mu) \subset \mathcal{D}_g(1)$ if and only if*

$$\lambda_2 = \cdots = \lambda_g = 0.$$

Since we choose to focus on $SL_2(\mathbb{R})$-invariant measures, we consider $SL_2(\mathbb{R})$ orbits of surfaces carrying Abelian differentials, known as *Teichmüller disks*. It is natural to ask if we can classify all Teichmüller disks contained in $\mathcal{D}_g(1)$. We prove a strong restriction on such disks.

Definition 2.2. *A surface (X, ω) is* completely periodic *if for all $\theta \in \mathbb{R}$ such that $(X, e^{i\theta}\omega)$ has a periodic trajectory γ, every trajectory parallel to γ is either periodic or a union of saddle connections.*

Recall that a surface can be completely periodic with nearly all the properties of a Veech surface, without being a Veech surface [12].

Theorem 2.3. *If the Teichmüller disk of (X, ω) is contained in $\mathcal{D}_g(1)$, then (X, ω) is completely periodic.*

We prove this by looking at the Deligne-Mumford compactification of the moduli space. Using expansions of Abelian differentials in terms of pinching coordinates [9], we can estimate the derivative of the period matrix as in [4][Section 4]. Then certain phenomena that would prevent the surface from being completely periodic are excluded. A key tool in this proof is the extension of the $SL_2(\mathbb{R})$ action on holomorphic Abelian differentials to meromorphic Abelian differentials with simple poles and the study of the $SL_2(\mathbb{R})$ orbits of such differentials.

3. Progress Toward Classifying Teichmüller Disks in $\mathcal{D}_g(1)$

It is conjectured that the genus three and four examples above represent the only Teichmüller disks contained in $\mathcal{D}_g(1)$. In the recent work of [3], they prove that there are no $SL_2(\mathbb{R})$-invariant orbifolds contained in $\mathcal{D}_g(1)$, for $g \geq 7$. We provide a completely different approach to this problem than that of [3] with the hope that it may yield a stronger result, while relying on far less sophisticated techniques. We summarize our progress here. The moduli space of Abelian differentials is stratified by the orders of the zeros of the differentials. Let $\mathcal{H}(\kappa)$ denote such a stratum.

Proposition 3.1. *Let n and m be odd numbers such that $n + m = 2g - 2$. There are no Teichmüller disks contained in either $\mathcal{D}_g(1) \cap \mathcal{H}(2g-2)$ or $\mathcal{D}_g(1) \cap \mathcal{H}(n, m)$.*

Corollary 3.2. *There are no Teichmüller disks contained in $\mathcal{D}_2(1)$.*

We can extend $\mathcal{D}_g(1)$ to the boundary of the moduli space given by the Deligne-Mumford compactification and denote this extension by $\overline{\mathcal{D}_g(1)}$.

Theorem 3.3. *The Teichmüller disk D of a completely periodic surface (X, ω) is contained in $\mathcal{D}_g(1)$ only if there is a Veech surface $(X', \omega') \in \overline{\mathcal{M}_g}$ such that the Teichmüller disk D' of (X', ω') is contained in $\overline{\mathcal{D}_g(1)}$ and ω' is holomorphic.*

This theorem is proven by finding a sequence of completely periodic surfaces that converge to a surface satisfying topological dichotomy in the sense of [2], followed by constructing another sequence of surfaces satisfying topological dichotomy that converge to a surface which is uniformly completely periodic. By [11], uniform complete periodicity is equivalent to being a Veech surface.

Möller [10] showed that any Veech surface whose Teichmüller disk is contained in $\mathcal{D}_g(1)$ is a square-tiled surface. In the context of Theorem 3.3, the Teichmüller disk of a surface of high genus can only be contained in $\mathcal{D}_g(1)$ if it can degenerate to one of the square-tiled surfaces in $\mathcal{D}_g(1)$. Such a surface obviously has punctures because it is of lower genus. By analyzing where the punctures must lie, we hope to reach a contradiction to show that no such degeneration is possible, thereby eliminating the possibility that Teichmüller disks are contained in $\mathcal{D}_g(1)$ in high genus. So far this method has yielded the following theorem.

Theorem 3.4. *The Eierlegende Wollmilchsau generates the only Teichmüller disk contained in $\mathcal{D}_3(1)$.*

<div align="center">REFERENCES</div>

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Monodromy Representations of Origamis
André Kappes

This talk is based on the author's thesis [4].

Consider the affine group $\mathrm{Aff}(X,\omega)$ of a translation surface $(X,\omega) \in \Omega\mathcal{M}_g$. This group has an action ρ by pullback on the first cohomology $H^1(X,\mathbb{Z})$ of X. On the other hand, $\mathrm{Aff}(X,\omega)$ acts as a Fuchsian group on the upper half plane $\mathbb{H}$, seen as the Teichmüller disk for (X,ω^2). Here, an element $f \in \mathrm{Aff}(X,\omega)$ acts by the action of its derivative $D(f) \in \mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$.

If the surface (X,ω) is a Veech surface, then the quotient $\mathbb{H}/\mathrm{Aff}(X,\omega)$ is an algebraic curve with an immersion into the moduli space of curves, and is called a Teichmüller curve. Passing to a finite index subgroup $\Gamma \subset \mathrm{Aff}(X,\omega)$, we can achieve that $C = \mathbb{H}/\Gamma$ is smooth, that Γ is the fundamental group of C and that $C \to \mathcal{M}_g$ factors over a fine moduli space. Then the restriction of ρ to Γ is the monodromy representation of the family of curves $\phi : \mathcal{X} \to C$ that parametrizes all compact Riemann surfaces obtained from affine deformations of the initial translation surface (X,ω) by matrices in $\mathrm{SL}_2(\mathbb{R})$.

By a theorem of Deligne [2], the monodromy representation of an algebraic family is semi-simple. Furthermore, Möller [6] showed that in the case of a Teichmüller curve, there is a splitting

$$\rho \otimes \mathbb{Q} = \rho_T \oplus \rho_R$$

where $\rho_T \otimes \mathbb{R}$ consists of all Galois conjugates to the standard fuchsian representation $\mathrm{Aff}(X,\omega) \to \mathrm{SL}_2(\mathbb{R})$, coming from the subspace spanned by $\Re\omega$ and $\Im\omega$ in $H^1(X,\mathbb{R})$.

I address the question how to decompose the second factor ρ_R. A general principle how to obtain splittings is to use covering maps from (X,ω) to another Veech surface (Y,ν). I present two examples of Veech surfaces, where a complete splitting of the first cohomology into invariant rank-2 subspaces is found. The examples are both origamis, i.e. coverings of the once-punctured square torus E^*, and my computations relie on the concrete description of origamis as conjugacy classes of subgroups of $F_2 = \pi_1(E^*)$ initiated by Weitze-Schmithüsen [7]. One is an origami M in the stratum $\Omega\mathcal{M}_4(2,2,2)^{\mathrm{odd}}$ consisting of 9 squares, the other one is an origami N in the stratum $\Omega\mathcal{M}_{10}(2^9)^{\mathrm{even}}$, which is a 3-fold cover of M. Both are covered by the characteristic origami with 108 squares constructed by Herrlich [3].

The question of finding Γ-invariant subspaces in the cohomology of (X,ω) is strongly connected with the question how to compute the Lyapunov exponents of the Kontsevich-Zorich cocycle over the Teichmüller curve. These are $2g$ real numbers

$$1 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_g \geq 0 \geq -\lambda_g \geq \cdots \geq -\lambda_1 = -1$$

that govern the mean logarithmic growth behavior of the norms of vectors in $H^1(X,\mathbb{R})$ along a generic Teichmüller geodesic on the Teichmüller curve, measured in the Hodge norm. In general, not much is known about the individual exponents.

However, by [5], their sum is given by

$$\sum_{i=1}^{g} \lambda_i = \frac{2 \deg(\phi_* \Omega_{\mathcal{X}/C})}{-\chi(C)}.$$

A variation of this formula arises in the following way: Let $V \subset H^1(X, \mathbb{R})$ be a rank 2-subspace, which is Γ-invariant, and whose associated local system carries a polarized variation of Hodge structures (pVHS). Then as is shown in [1], there are two Lyapunov exponents associated with V, and the non-negative one is given by

$$\lambda_V = \frac{\deg(\mathcal{L})}{-\chi(C)},$$

where $\mathcal{L}$ is the $(1,0)$-part of the pVHS associated with V (on the completion $\overline{C}$ of C). Using an idea of M. Möller, I show in [4] that one can rewrite this equation to obtain

$$\lambda_V = \frac{\deg(p)\mathrm{vol}(\mathbb{H}/\rho_{|V}(\Gamma))}{\mathrm{vol}(\mathbb{H}/\Gamma)}$$

in the case when V is defined over $\mathbb{Z}$ and $\rho_{|V}(\Gamma)$ is a finite-index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Here, p is the period map $\mathbb{H}/\Gamma \to \mathbb{H}/\rho_{|V}(\Gamma)$ associated with the pVHS on V, or rather its extension to the closed curves. The volumes and even the degree of p can entirely be determined from the monodromy representation ρ if the latter is given in terms of a matrix for each generator of $\mathrm{Aff}(X, \omega)$ representing its action on cohomology. In this way, I am able to determine all the Lyapunov exponents of the two examples. For M,

$$1 > \tfrac{1}{3} \geq \tfrac{1}{3} \geq \tfrac{1}{3}$$

is the non-negative part of the Lyapunov spectrum, and for N,

$$1 > \underbrace{\tfrac{1}{3} \geq \cdots \geq \tfrac{1}{3}}_{6} \geq 0 \geq 0 \geq 0$$

is the non-negative part of the Lyapunov spectrum. In particular, the action of the affine group of N on a six-dimensional subspace of $H^1(N, \mathbb{Z})$ is by a finite group.

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Finite blocking property on translation surfaces
Thierry Monteil

The following problem was stated for the Leningrad's Olympiad of 1989 [2]:

> "Professor Smith stands in a square hall with mirrored walls. Professor Jones intends to arrange several students in the hall so that Smith can't see his own reflection. Can Jones reach her goal? (Professor Smith and the students are considered points; students can be arranged by the walls and in the corners)."

Note that there are infinitely many light (billiard) trajectories between Jones and Smith. The square billiard table can be unfolded into a flat torus $\mathbb{R}^2/\mathbb{Z}^2$. A translation surface T is said to have the *finite blocking property* (FBP) if, for every pair (S, J) of points in T, there exists a finite number of "blocking" points $B_1, \ldots, B_n$ (different from S and J) such that every geodesic from S to J meets one of the B_i's. Let us solve the Olympiad's problem by showing that $\mathbb{R}^2/\mathbb{Z}^2$ has the FBP.

Let us write the professors' positions in coordinates: $S = (x, y)$, $J = (x', y')$. Any trajectory between S and J can be unfolded in $\mathbb{R}^2$ into a line between S and $J' = (x' + k, y' + l)$ for some $(k, l) \in \mathbb{Z}^2$.
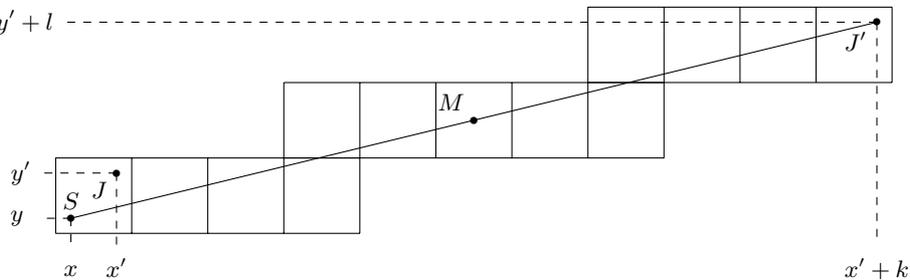
The middle of the trajectory is $M = ((x + x' + k)/2, (y + y' + l)/2)$. If we project M back to $\mathbb{R}^2/\mathbb{Z}^2$, we get a point $\tilde{M} = ((x + x')/2, (y + y')/2) + (k/2, l/2) \bmod \mathbb{Z}^2$. Since $(k/2, l/2) \bmod \mathbb{Z}^2$ can only take four values, the infinite set of trajectories between S and J in $\mathbb{R}^2/\mathbb{Z}^2$ is blocked by at most four points (in some particular cases, some of the four points could correspond to J or S and should be removed from the blocking configuration). $\qquad\square$

Since the FBP is stable under branched coverings and under the action of $SL(2, \mathbb{R})$, we just saw that **any torus branched covering has the FBP**. If we try to generalise the previous construction to another surface T, " $\bmod \mathbb{Z}^2$ " should be replaced by " $\bmod G$ ", where G is the group generated by the translations used

to identify the pairs of edges in some representation of T by a glued polygon. The previous construction of a finite set of points $\tilde{M}$ back in T works when G is discrete.

The easiest way to make G non-discrete is to have two adjacent parallel cylinders of uncommensurable perimeters. It turns out that in such a situation, the surface fails to have the FBP [6], hence we have a local criterion to start a classification. Any periodic orbit in a translation surface can be thickened into a cylinder. Unfortunately, the set of translation surfaces that contains two parallel cylinders with uncommensurable perimeters has zero measure, so this local criterion cannot be often directly used.

In [8], we proved that a translation surface with the finite blocking property is completely periodic. If we merge this result with the local criterion, we proved that **any translation surface with the FBP is purely periodic**, where a translation surface T is said to be *purely periodic* if, for any direction $\theta \in \mathbb{S}^1$, the existence of a (non-singular) periodic orbit in the direction θ implies that the directional flow ϕ_θ is periodic (*i.e.* there exists $t > 0$ such that $\phi_\theta^t = Id_S$ a.e.). Indeed the periodicity of the flow ϕ_θ is equivalent to the existence of a decomposition of T into cylinders of commensurable perimeters in the direction θ. The geodesic flow on a translation surface T is defined on its unit tangent bundle $T \times \mathbb{S}^1$, it admits two subflows depending on whether we fix the direction $\theta \in \mathbb{S}^1$ (directional flow) or the starting point $J \in T$ (exponential flow). Hence, the previous results establish a surprising relation between three notions on translation surfaces, the first involving the global geometry of the surface (being a torus branched covering), the second involving the exponential flow (the FBP) and the third involving the directional flow (the pure periodicity). It would be nice to have an equivalence between those three notions, hence we would like the pure periodicity to imply being a torus branched covering.

Torus branched coverings can be characterised using *translational holonomy*: any curve $\gamma : [0, 1] \to T$ on a translation surface T can be lifted as a planar curve $\tilde{\gamma}$ which is defined up to translation so that $hol(\gamma) = \tilde{\gamma}(1) - \tilde{\gamma}(0)$ is well defined. Restricted to the closed curves, the map hol induces a morphism from $H_1(T, \mathbb{Z})$ to $\mathbb{R}^2$. The "unfolding group" G previously introduced is actually $hol(H^1(T, \mathbb{Z}))$. A translation surface is a torus branched covering if, and only if, $hol(H^1(T, \mathbb{Z}))$ is a lattice.

For purely periodic translation surfaces, the J-invariant introduced in [5] can be computed through any pair of periodic directions, and the fact that it does not depend on such a pair implies that the holonomy of any three periodic orbits are rationally dependent. Hence, if $P(T)$ denotes the subgroup of $H^1(T, \mathbb{Z})$ generated by the periodic orbits (considered as closed curves), then $hol(P(T))$ is a lattice [7]. Hence, the previous three notions are equivalent when the periodic orbits of the translation surface generates its homology.

In a nutshell, the three notions are known to be equivalent:

- on a dense open subset of full measure in every stratum,

- for Veech surfaces, and more generally for surfaces whose Veech group contains two non-commuting parabolic elements,
- in genus 2 (using the classification of completely periodic surfaces [1]),
- for surfaces that admit a representation by a convex glued polygon, and more generally for surfaces which are named face-to-face surfaces.

A natural challenge is therefore to **describe the surfaces whose homology is not generated by the periodic orbits of their geodesic flow**.

The eierlegende Wollmilchsau [4] and the translation surface introduced in [3] constitute the first examples. Indeed, in both cases, the two horizontal cylinders are homologous. Moreover, the Veech group of those two surfaces is equal to $SL(2, \mathbb{Z})$, hence the vertical and horizontal cylinders generate all cylinders (by making successive twists along both directions), hence the periodic orbits generate only a subgroup of dimension 2 in $H^1(T, \mathbb{Z})$. Those two examples are torus branched coverings, we do not know any primitive example.

Note that

- the set of translation surfaces that do not admit a strictly convex pattern,
- the non face-to-face surfaces,
- the set of translation surfaces whose homology is not generated by periodic orbits (and some variations on the dimension of the space generated by the periodic orbits)

are closed $SL(2, \mathbb{R})$-invariant spaces (containing each other).

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Small eigenvalues of the Laplacian in moduli space

SÉBASTIEN GOUËZEL

(joint work with Artur Avila)

Let M be a compact hyperbolic surface, we will first mention several classical facts on such objects.

(1) Analysis. Consider the hyperbolic Laplacian on M coming from the riemannian metric. Since Δ is a self-adjoint elliptic differential operator, its spectrum is a sequence of eigenvalues $\lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$.

(2) Geometry. Let π_T be the number of closed geodesics of length at most T. Using Margulis' techniques, one can show that $\pi_T \sim e^T/T$. However, in this situation, much more is known thanks to Selberg's trace formula. A bold application of this formula leads to the conjecture $\pi_T = \sum_{j=0}^K e^{a_j T}/(a_j T) + O(e^{T/2})$, where a_j is a sequence of numbers in $[1/2, 1]$ in bijection with the $\lambda_j \in [0, 1/4]$. This formula should hold in full generality, but it is only known up to an error term $O(e^{3T/4})$ (Huber).

(3) Dynamics. The geodesic flow g_t on the unit cotangent bundle $T^1 M$ of M is mixing, i.e., for all smooth functions u, v, $\int u \cdot v \circ g_t$ tends to $\int u \cdot \int v$. In this situation, one can be more precise: one has $\int u \cdot v \circ g_t = \sum_{j=0}^K e^{-b_j t} + O(e^{-t})$, for some numbers b_j in $[0, 1]$ which are again in bijection with the $\lambda_j \in [1/4]$ (Moore, Ratner).

(4) Representation theory. The unitary action of $SL(2, \mathbb{R})$ on $L^2(T^1 M)$ can be decomposed as a direct integral of irreducible representations. The irreducible representations have been classified by Bargmann, and belong to three series (the principal series, the discrete series and the complementary series). It turns out that the irreducible representations arising in $L^2(T^1 M)$ are in bijection with the eigenvalues of the Laplacian, and the eigenvalues in $[0, 1/4]$ correspond to the representations in the complementary series, i.e., the most exotic ones in some sense.

This shows that the eigenvalues of the Laplacian in $[0, 1/4]$ really play a special role. In finite volume hyperbolic surfaces, the analytic approach above fails (while Δ still has a regularizing effect, the presence of the cusp prevents $(I + \Delta)^{-1}$ from being compact). Nevertheless, using a more algebraic approach, one may show that the spectrum of the Laplacian is made of the whole interval $[1/4, +\infty)$, and finitely many eigenvalues in the interval $[0, 1/4]$. Therefore, the above geometric, dynamical and representation theory facts can still be formulated in this context, and they all hold.

Our main object of interest is a stratum $\mathcal{H}$ in the moduli space of abelian (or quadratic) differentials, together with an $SL(2, \mathbb{R})$-invariant probability measure μ. The space $\mathcal{H}$ is not compact, and leaves of the foliation by $SL(2, \mathbb{R})$-orbits have large codimension. Therefore, the Laplacian is not elliptic (nor hypo-elliptic): it can have smoothing properties in all directions only thanks to recurrence properties

of the foliation, that are hard to quantify. Using a different approach (that avoids completely the use of the Laplacian), we prove the following theorem.

Theorem. *The measure μ satisfies the following properties:*

(1) *The (foliated) Laplacian acting on $L^2(\mu)$ has finite spectrum in $[0, 1/4 - \delta]$ for all $\delta > 0$.*

(2)

(3) *For all $\delta > 0$, for all C^∞ compactly supported functions in $\mathcal{H}$, one has $\int u \cdot v \circ g_t = \sum_{j=0}^{K} e^{-b_j t} + O(e^{-(1-\delta)t})$, for some numbers b_j in $[0, 1 - \delta]$.*

(4) *When one decomposes $L^2(\mu)$ into irreducible representations for the canonical unitary $SL(2, \mathbb{R})$-action, representations in the complementary series occur only discretely.*

It turns out that those different items are all equivalent. We prove a part of the third one, deduce the fourth one, and then also obtain the first and third one. Unfortunately, we have nothing to say on point (2), i.e., the geometric counting of closed geodesics in the support of the measure μ.

The main idea of the proof is to use a functional analytic approach. We introduce a suitable Banach space $\mathcal{B}$ such that the operator $\mathcal{L} : u \mapsto u \circ g_t$ (for some fixed t) acts continuously on $\mathcal{B}$, with good spectral properties: while there is no hope to get a compact operator since the space is not compact, one can ensure that $\mathcal{L}$ is quasi-compact, meaning that its spectrum is made of finitely many eigenvalues outside of a suitably small disk. The space $\mathcal{B}$ should be chosen to take advantage of the good dynamical properties of the flow g_t. One takes for $\mathcal{B}$ a space of distributions which are smooth in the stable direction and dual of smooth in the unstable direction, so that $u \circ g_t$ is better behaved than u in all respects. The main issue is to control infinity, where there is no hyperbolicity. Here, we use quantitative recurrence estimates due to Eskin-Masur and Athreya, that we incorporate into the definition of the space.

Decompositions and Genericity in $\mathcal{H}^{\mathrm{hyp}}(4)$

Duc-Manh Nguyen

In this talk we are concerned with the stratum $\mathcal{H}^{\mathrm{hyp}}(4)$ of translation surfaces of genus three. This stratum consists of pairs (X, ω), where X is a hyper-elliptic Riemann surface of genus 3, and ω is a holomorphic 1-form on X which has only one zero, the order of which must be 4. Note that the unique zero of ω must be a Weierstrass point of X. The holomorphic 1-form induces a flat metric structure on X with cone singularity at the zero of ω whose transition maps are translations of $\mathbb{R}^2$, such a surface is called a *translation surface*. The hyper-elliptic involution of X induces an isometry of the corresponding translation surface. Note that this involution acts like $-\mathrm{Id}$ on $H_1(X, \mathbb{Z})$, and fixes 8 points on X. We denote by $\mathcal{H}_1^{\mathrm{hyp}}(4)$ the subset of $\mathcal{H}^{\mathrm{hyp}}(4)$ consisting of surfaces with unit area.

On a translation surface, a *saddle connection* is a geodesic segment whose endpoints are singularities of the surface, which may coincide. For surfaces in $\mathcal{H}^{\mathrm{hyp}}(4)$, a saddle connection is then a geodesic loop joining the unique singularity to itself. We can associate to a saddle connection γ (together with a choice of orientation) a vector $V(\gamma) \in \mathbb{R}^2$, which is the integral of the holomorphic 1-form defining the flat metric along γ. In fact, the integral gives us a complex number, we view it as a vector in $\mathbb{R}^2$ by the standard identification $\mathbb{C} = \mathbb{R} \oplus \imath\mathbb{R}$.

Given a translation surface Σ, a *cylinder* in Σ is an open subset which is isometric to the quotient $\mathbb{R} \times]0; h[/\mathbb{Z}$, where $\mathbb{Z}$ is the cyclic group generated by $(x, y) \mapsto (x + \ell, y)$, and maximal with respect to this property. We will call h the *height*, and ℓ the *width* of C, the modulus of C is defined to be the ratio h/ℓ. By definition, we have a map from $\mathbb{R} \times]0; h[$ to Σ, which is locally isometric, with image C. This map can be extended by continuity to a map from $\mathbb{R} \times [0; h]$ to Σ. We call the images of $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{h\}$ under this map the *boundary components* of C. Each boundary component of C is a concatenation of saddle connections, and freely homotopic to the simple closed geodesics in C. Remark that the two boundary components of C are, in general, not disjoint subsets of Σ, they can even coincide. We call C a *simple cylinder* when each of its boundary components consists of only one saddle connection. First, we have

Theorem. *On every surface in $\mathcal{H}^{\mathrm{hyp}}(4)$, there always exist four pairs of homologous saddle connections $\delta_i^{\pm}, i = 1, \ldots, 4$, such that*

- *$\delta_1^{\pm}$ bound a simple cylinder,*
- *for $i = 1, 2, 3$, $\delta_i^+ \cup \delta_{i+1}^+ \cup \delta_i^- \cup \delta_{i+1}^-$ is the boundary of a topological disk isometric to a parallelogram in $\mathbb{R}^2$,*
- *$\delta_4^{\pm}$ bound a simple cylinder.*

There exists an action of $\mathrm{SL}(2, \mathbb{R})$ on the moduli space of translation surfaces which leaves invariant the Lebesgue measure, and preserves the area. It is now a classical fact, due to Masur and Veech, that the $\mathrm{SL}(2, \mathbb{R})$ action is ergodic in each connected component of the moduli space, a surface whose $\mathrm{SL}(2, \mathbb{R})$-orbit is dense in its component is called *generic*. The $\mathrm{SL}(2, \mathbb{R})$-orbit of almost all surfaces in each component is dense, however, the problem of determining whether the orbit of a particular surface is dense in its component is wide open. We only have a complete classification, due to McMullen and Calta, for the case of genus 2. Recall that the Veech group of a translation surface is the stabilizer subgroup for the action of $\mathrm{SL}(2, \mathbb{R})$. It is a well-known fact that the $\mathrm{SL}(2, \mathbb{R})$-orbit of a surface is a closed subset in its stratum if and only if its Veech group is a lattice of $\mathrm{SL}(2, \mathbb{R})$. It turns out from the work of McMullen that, for translation surfaces of genus two, if the Veech group contains a hyperbolic element, then the $\mathrm{SL}(2, \mathbb{R})$-orbit cannot be dense in the corresponding stratum. More recently, Hubert-Lanneau-Moeller give some results on generic surfaces in the hyper-elliptic locus $\mathcal{L}$ of $\mathcal{H}^{\mathrm{odd}}(2, 2)$, which is one of the two components of $\mathcal{H}(2, 2)$. They show that, in contrast with the case of genus 2, there are generic surfaces in $\mathcal{L}$, that is the $\mathrm{SL}(2, \mathbb{R})$-orbit is

dense in $\mathcal{L}$, whose Veech group contains hyperbolic elements.

Back to the case of $\mathcal{H}^{\mathrm{hyp}}(4)$, let Σ be a surface in $\mathcal{H}_1^{\mathrm{hyp}}(4)$, and $\delta_i^{\pm}, i = 1, \dots, 4$, be as in Theorem . Cutting Σ along $\delta_3^{\pm}$, we get two connected components whose boundary consists of two geodesic segments. Gluing those geodesic segments together, we then get a flat torus, which will be denoted by Σ', and a surface in $\mathcal{H}(2)$. We can identify Σ' with the quotient $\mathbb{R}^2/\Lambda$, where Λ is a lattice in $\mathbb{R}^2$, which is the image of the map $H_1(\Sigma', \mathbb{Z}) \to \mathbb{C} \simeq \mathbb{R}^2 : c \mapsto \int_c \omega$. A vector in $\mathbb{R}^2$ is said to be *generic* with respect to Λ if it is not collinear with any vector in Λ. We have

Theorem. *Suppose that $\delta_1^{\pm}$ and $\delta_3^{\pm}$ are parallel, that is $V(\delta_1^{\pm})$ and $V(\delta_3^{\pm})$ are collinear, and $V(\delta_3) = V(\delta_3^{\pm})$ is generic with respect to Λ, then $\mathrm{SL}(2, \mathbb{R}) \cdot \Sigma$ is dense in $\mathcal{H}_1^{\mathrm{hyp}}(4)$.*

A consequence of Theorem is the following

Corollary. *Let Σ be a surface in $\mathcal{H}_1^{\mathrm{hyp}}(4)$. Suppose that the horizontal direction is completely periodic for Σ, and that Σ is decomposed into three horizontal cylinders whose moduli are independent over $\mathbb{Q}$. Then $\mathrm{SL}(2, \mathbb{R}) \cdot \Sigma$ is dense in $\mathcal{H}_1^{\mathrm{hyp}}(4)$.*

Using this result, we can show that there exists generic surfaces in $\mathcal{H}^{\mathrm{hyp}}(4)$ with coordinates in a real quadratic field over $\mathbb{Q}$. We can also construct explicitly Thurston-Veech surfaces with cubic trace field over $\mathbb{Q}$ which are generic in $\mathcal{H}^{\mathrm{hyp}}(4)$.

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Deviation of ergodic averages for substitution dynamical systems with eigenvalues of modulus one

PASCAL HUBERT

(joint work with X. Bressaud and A. Bufetov)

Let σ be a primitive substitution over a finite alphabet $\mathcal{A}$, let M_σ be the matrix substitution, and let X_σ be the corresponding subshift. The aim of this report is to study the asymptotic behavior of ergodic sums for the minimal dynamical system

(X_σ, T) in the (non-hyperbolic) case when the matrix M_σ has an eigenvalue of modulus one. For a function $f : X_\sigma \to \mathbb{R}$, all $x \in X_\sigma$ and all $n \in \mathbb{N}$, set

$$S_n f(x) = \sum_{k=0}^{n-1} f(T^k x).$$

We shall only consider functions f depending on the first coordinate of the symbolic sequence. In what follows, we will identify such a function f with the corresponding vector in $\mathbb{C}^{\#\mathcal{A}}$. Deviation of ergodic sums is studied for substitution dynamical systems with a matrix that admits eigenvalues of modulus 1. The functions γ we consider are the corresponding eigenfunctions.

Deviations of ergodic sums for interval exchange transformations have been studied by Zorich [9] [10] and Forni [4]. Part of this analysis applies to interval exchange transformations arising from pseudo-Anosov diffeomorphisms. More precisely, an interval exchange transformation defining a periodic path in its Rauzy diagram provides an example of a substitutive subshift. Examples of pseudo-Anosov diffeomorphisms with eigenvalues of modulus one are known till the work of Veech in 1982 [8]. Now, infinitely such examples have been described (see for instance [5]).

We prove that the limit inferior of the ergodic sums

$$(n, \gamma(x_0) + \ldots + \gamma(x_{n-1}))_{n \in \mathbb{N}}$$

is bounded for *every* point x in the phase space. This result has corollaries concerning the theory of affine interval exchange transformations (see for instance [2] and [6]). Given a self similar interval exchange transformation T and γ as above, any affine interval exchange transformation with log slopes vector γ semi-conjugate to T is in fact conjugate to T.

We prove the existence of limit distributions along certain exponential subsequences of times for substitutions of constant length ($S_n f(x)$ is considered as a random variable). Under additional assumptions, we prove that ergodic integrals satisfy the Central Limit Theorem. The proof of this result is based on a Markov approximation of our system and Dubroshin's CLT on non homogeneous Markov chains (see [3] and [7]).

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Non-uniquely ergodic billiards and flows on flat surfaces
Jon Chaika

Dynamical systems seeks to understand the orbits of points. Motivated by this and the Birkhoff Ergodic Theorem we are interested in understanding the invariant measures of systems. Flows on flat surfaces are typically uniquely ergodic [7],[10], as are flows in rational billiards [6]. This talk considers the other situation, when flows are minimal (every orbit is dense) but not uniquely ergodic. There are two different constructions of this phenomena and they can be used to construct different behaviors.

1. Skew products of rotations

It was shown in [9] and [8] that irrational rotations have $\mathbb{Z}_2$ skew products that are minimal but not uniquely ergodic. This example can arise as the first return of a billiard by examining a table that is a rectangle with a barrier of carefully chosen length placed halfway up the side and parallel to two sides. In this example there is symmetry that comes from the fact that the barrier is placed halfway. (This is also visible by examining the symmetry coming from the $\mathbb{Z}_2$ group action on the first return map.)

2. Keane Construction

Another construction of minimal but not uniquely ergodic billiard flows arises from discrete dynamical systems. Induction procedures on interval exchange transformations (IET) can give rise to minimal but not uniquely ergodic dynamical systems [5] that can be viewed as first return maps of billiards. These are connected to billiards in L-shaped polygons. Under the unfolding [4] we obtain an L shaped table with opposite sides identified. The flow in these tables parametrize an open set of IETs with permutation (2413). After the first few steps of Rauzy-Veech induction one can obtain any IET with permutation in this Rauzy class. Many dynamical properties behave well under this induction procedure so we gain examples about flows in polygons with strange properties by constructing IETs with strange properties.

3. Strange metric behavior

Theorem 3.1. [1] *There exists a minimal flow in an L shaped polygon with two ergodic measures, λ_2 and λ_3 such that for any $\epsilon > 0$ we have $\liminf\limits_{n \to \infty} t^{1-\epsilon} d(F^t x, y) = 0$ for $\lambda_2 \times \lambda_3$ almost every (x, y) and $\liminf\limits_{n \to \infty} t^{\epsilon} d(F^t x, y) = \infty$ for $\lambda_3 \times \lambda_2$ almost every (x, y).*

Due to the symmetry in the construction this can not arise from Veech-Sataev constructions.

Theorem 3.2. [2] *There exists a minimal, non-uniquely ergodic flow in a rectangle with barriers and a constant $c > 0$ such that $\inf\limits_{n > 0} t \, |F^t x - x| > c$ for all x.*

4. Strange measure behavior

4.1. **Eigenfunctions and weak mixing.**

Theorem 4.1. *(Ferenczi, Zamboni [3]) There exists a minimal, non-uniquely ergodic 4-IET with two ergodic measures where it is weakly mixing with respect to one ergodic measure and not the other.*

This is another example of a 4-IET where the behavior of the two ergodic measures are different.

4.2. **Quasi-generic measures.**

Definition 4.2. *Let (X, T, μ) be a dynamical system. We say $x \in X$ is* generic *for μ if $\delta_x, \frac{1}{2}(\delta_x + \delta_{Tx}), \dots$ converges in the weak-* topology to μ. (δ_u denotes point mass at u.) We say it is* quasi-generic *for μ if μ is a weak-* limit point. We say a measure is* quasi-generic *if there is a point quasi-generic for it.*

One can think of quasi-generic measures are the measures that can be seen at some time by some points of the dynamical system. If T is continuous, μ is Borel and X is a compact metric space then quasi-generic measures are invariant. If T is minimal and continuous then the set of quasi-generic measures is connected and closed. This should be contrasted to Ratner's Theorem where the quasi-generic measures are ergodic measures, but the map is not minimal. Therefore when T is minimal and there are only 2 ergodic measures all invariant measures are quasi-generic.

Theorem 4.3. *(Chaika, Cheung, Masur) There exists a minimal non-uniquely ergodic flows with invariant measures that are not quasi-generic.*

A complementary result:

Theorem 4.4. *There are flows with arbitrarily many ergodic measures such that all the invariant measures are quasi-generic.*

4.3. **Hausdorff dimensions of ergodic measures.** Given a metric D let $diam(U) = \sup_{x,y \in U} D(x,y)$. Consider a set $S \subset [0,1)$. We say a collection of open sets $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ is a $\delta > 0$ cover of S if $S \subset \bigcup_{i=1}^{\infty} U_i$ and $diam(U_i) \leq \delta \ \forall i$. Let $H_\delta^s(S) = \inf\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\}$ is a δ cover of $S\}$. Let $H^s(S) = \lim_{\delta \to 0^+} H_\delta^s(S)$. Notice that the limit exists. Let $H_{dim}(S) = \inf\{s : H^s(S) = 0\}$. This is equivalent to defining $H_{dim}(S) = \sup\{s : H^s(S) = \infty\}$.

We can create two new metrics on $[0,1)$, $d_{\mu_i}(a,b) = \mu_i([a,b])$ and ask what is the Hausdorff dimension of μ_1 with respect to the metric d_{μ_2} and vice-versa. Let $H_{dim}(\mu, d_\nu)$ denote the Hausdorff dimension μ with respect to the metric d_ν.

Theorem 4.5. [1] *There exists a 4-IET with two ergodic measures μ and ν such that $H_{dim}(\mu, d_\nu) = 1$ and $H_{dim}(\nu, d_\mu) = 0$.*

This example shows two ergodic measures that interact with each other differently. It is reminiscent of Theorem 3.1. It can arise from a billiard in an L-shaped polygon.

We end with a question:

Question 1. *(Kornfeld) Can any residual set carry an ergodic measure of a minimal non-uniquely ergodic IET?*

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On the geometry of the handlebody group

Sebastian Hensel

(joint work with Ursula Hamenstädt)

A handlebody of genus g is a 3–manifold with boundary that is obtained from a 3–ball by attaching g one-handles. The boundary of such a handlebody V_g is a closed surface S_g of genus g. Every closed 3–manifold can be obtained by gluing two handlebodies of the same genus along their boundaries with a suitable homeomorphism (this is a so called *Heegaard splitting* of the 3–manifold). In this sense, handlebodies are basic building blocks for closed 3–manifolds.

In this work, we are interested in the mapping class group of the handlebody V_g, which is often called the *handlebody group* of V_g. Explicitly, the handlebody group is the group of all self-homeomorphisms of V_g up to homotopy.

By a theorem of Wajnryb [8], the handlebody group is finitely presented. Therefore, one can equip the handlebody group with a word metric. Our main goal is to study the large-scale geometric properties of the handlebody group induced by such a metric.

It is easy to see that the handlebody group of V_g embeds into the mapping class group of the boundary surface S_g. The coarse geometry of mapping class groups of closed surfaces has been thoroughly studied in recent years, and there are many powerful tools available to answer geometric questions about such mapping class groups (see for example [5] or [2] for explicit constructions of quasigeodesics).

Therefore, a natural approach to study the geometry of handlebody groups is to study the geometry of the inclusion map into the mapping class group of the boundary surface. If the inclusion were a quasi-isometric embedding, then the geometry of the handlebody group would be completely inherited from the ambient surface mapping class group.

Our main result states (see [3]) that this is not the case for handlebody groups of genus $g \geq 2$. More precisely, we show that the handlebody group is exponentially distorted as a subgroup of the mapping class group.

In addition to mapping class groups of surfaces, the handlebody group is connected to another important group. Namely, the action of homeomorphisms on the fundamental group induces a homomorphism from the handlebody group onto the outer automorphism group of a free group on g generators. The outer automorphism group of a free group is finitely generated, and hence also can be equipped with a word metric. However, the kernel of the projection from the handlebody group to the outer automorphism group of a free group is infinitely generated (see [6]) and therefore it is not easily possible to transfer geometric properties between the two groups.

To shed more light on the geometry of the handlebody group, we next consider geometric properties that might distinguish handlebody groups from surface mapping class groups on the one side, and outer automorphism groups of free groups on the other side. One such invariant is the growth rate of the Dehn function. The Dehn functions of surface mapping class groups have quadratic growth (this follows from the stronger statement that they are automatic groups, compare [7]) while the Dehn functions of outer automorphism groups of free groups have exponential growth type (compare [4] and [1]). In an upcoming preprint, we show that for any genus $g \geq 2$ the growth rate of the Dehn function of handlebody groups is at most exponential.

At least for genus 2, this estimate is probably not sharp. In fact, we conjecture that the Dehn function for genus 2 handlebody groups is of quadratic growth type. In the study of surface mapping class groups, the case of genus 2 is often exceptional, and it is not clear what growth rate to expect for Dehn functions of higher genus handlebody groups.

A different geometric property one might consider is the structure of curve stabilizers. The stabilizer of a simple closed curve in the mapping class group of a surface is an undistorted subgroup (i.e. the inclusion is a quasi-isometric embedding).

In analogy to this result, we show in an upcoming preprint that the stabilizer of an essential disk in a handlebody is undistorted in the handlebody group. Hence, although the handlebody group is not quasi-isometrically embedded in the surface mapping class group, the intrinsic geometries have certain features in common. The analogous statement for outer automorphism groups of free groups is also true (see [4]): the stabilizer of a free splitting of a free group (corresponding to a separating disk) or a corank 1 factor (corresponding to a nonseparating disk) is undistorted. It remains to be shown if the geometry of handlebody groups is more related to the geometry of mapping class groups of surfaces or the one of outer automorphism groups of free groups.

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Measures and hyperbolicity in Teichmüller space

MOON DUCHIN

(joint work with Spencer Dowdall and Howard Masur)

We seek to make precise the usual intuition that exceptions to hyperbolicity in Teichmüller space are "rare." To do this we consider various measures on Teichmüller space, and prove that, with respect to either Hausdorff measure or the push-down of Masur-Veech measure,

If two points are selected at random in the ball of radius r then the expected distance between them is additively close to $2r$.

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Asymptotics for pseudo-Anosov elements in Teichmüller lattices

JOSEPH MAHER

A *Teichmüller lattice* Γy is the orbit of a point y in Teichmüller space under the action of the mapping class group Γ. Athreya, Bufetov, Eskin and Mirzakhani [1] showed that the asymptotic growth rate of the number of lattice points in a ball of radius r is

$$|\Gamma y \cap B_r(x)| \sim \Lambda(x)\Lambda(y)he^{hr}.$$

Here $B_r(x)$ denotes the ball of radius r centered at x in the Teichmüller metric, $h = 6g - 6$ is the topological entropy of the Teichmüller geodesic flow, Λ is the Hubbard-Masur function, $|X|$ denotes the number of elements in a finite set X, and $f(r) \sim g(r)$ means $f(r)/g(r)$ tends to one as r tends to infinity. We use their work, together with some results from [6], to show that the number of lattice points corresponding to pseudo-Anosov elements in the ball of radius r is asymptotically the same as the total number of lattice points in the ball of radius r. More generally, let $R \subset \Gamma$ be a set of elements for which there is an upper bound on their translation length on the complex of curves, for example, the set of non-pseudo-Anosov elements. We shall write Ry for the orbit of the point y under the subset R. We show that the proportion of lattice points Γy in $B_r(x)$ which lie in Ry tends to zero as r tends to infinity. In fact, we show a version of this result for bisectors. Let Q be the space of unit area quadratic differentials on the surface

§, and given $x \in \mathcal{T}$, let $S(x)$ be the subset of Q consisting of unit area quadratic differentials on x. The space Q has a canonical measure, known as the Masur-Veech measure, which we shall denote μ, and we will write s_x for the conditional measure on $S(x)$ induced by μ. We may think of $S(x)$ as the (co-)tangent space at x. Given $x, y \in \mathcal{T}$, let $q_x(y)$ be the unit area quadratic differential on x corresponding to the geodesic ray through y. Given a lattice point γy, we will write $q(x, \gamma y)$ for the pair $(q_x(\gamma y), q_y(\gamma^{-1} x)) \in S(x) \times S(y)$. Given subsets $U \subset S(x)$ and $V \subset S(y)$, we may consider those lattice points γy which lie in the bisector determined by U and V, i.e. those γy for which $q(x, \gamma y) \in U \times V$. If X is a finite subset of Γ, we will write $|X, condition|$ to denote the number of elements $\gamma \in X$ which also satisfy *condition*. We say a surface of finite type is *sporadic* if it is a torus with at most one puncture, or a sphere with at most four punctures.

Theorem. *Let Γ be the mapping class group of a non-sporadic surface. Let $R \subset \Gamma$ be a set of elements of the for which there is an upper bound on their translation distance on the complex of curves. Let $x, y \in \mathcal{T}$, and let $U \subset S(x)$ and $V \subset S(y)$ be Borel sets whose boundaries have measure zero. Then*

$$(0.1) \qquad \frac{|Ry \cap B_r(x), \ q(x, \gamma y) \in U \times V|}{|\Gamma y \cap B_r(x), \ q(x, \gamma y) \in U \times V|} \to 0, \ as \ r \to \infty.$$

This shows that pseudo-Anosov elements are "generic" in the mapping class group , at least for one particular definition of generic, see Farb [4] for a discussion of similar questions. In the case in which R consists of the non-pseudo-Anosov elements of the mapping class group , this result should also follow from the methods of Eskin and Mirzakhani [3], which they use to show that the number of conjugacy classes of pseudo-Anosov elements of translation length at most r on Teichmüller space is asymptotic to e^{hr}/hr. In the sporadic cases, the mapping class group is either finite, or already well understood, as the mapping class group is $SL(2, \mathbb{Z})$, up to finite index.

We now give a more detailed outline of the argument. Let $R \subset \Gamma$ be a set of elements for which there is an upper bound on their translation length on the complex of curves, for example, the set of non-pseudo-Anosov elements in the mapping class group . We wish to consider the distribution of elements of Ry inside Teichmüller space $\mathcal{T}$. In some parts of $\mathcal{T}$ elements of Ry are close together, and in other parts elements of Ry are widely separated. We quantify this by by defining R_k to be the k-dense elements of Ry, namely those elements of Ry which are distance at most k in the Teichmüller metric from some other element of R. If two lattice points γy and $\gamma' y$ are a bounded Teichmüller distance apart, then γ and γ' are a bounded distance apart in the word metric on Γ. In [6] we showed that the limit set of the k-dense elements in the word metric has measure zero in the Gromov boundary of the relative space, and we use this to show that the k-dense elements in Teichmüller space have a limit set in the visual boundary $S(x)$ which has s_x-measure zero. A straightforward application of the results of [1] then shows that the proportion of lattice points $\Gamma y \cap B_r(x)$ which lie in $R_k y$ tends to zero as r tends to infinity.

It remains to consider $Ry \setminus R_k y$, which we shall denote $R'_k y$. We say a subset of $\mathcal{T}$ is k-separated, if any two elements of the set are Teichmüller distance at least k apart, so $R'_k y$ is a k-separated subset of $\mathcal{T}$. Naively, one might hope that the proportion of k-separated elements of Γy in $B_r(x)$ is at most $1/\left|\Gamma y \cap B_k(y)\right|$, as each lattice point $\gamma y \in R'_k y$ is contained in a ball of radius k in Teichmüller space containing $\left|\Gamma y \cap B_k(y)\right|$ other lattice points, none of which lie in $R'_k y$. Such a bound would imply the required result, as this would give a collection of upper bounds for

$$\lim_{r \to \infty} \frac{|Ry \cap B_r(x)|}{|\Gamma y \cap B_r(x)|}$$

which depend on k, and furthermore these upper bounds would decay exponentially in k, so this implies that the limit above is zero. However, this argument only works for those lattice points in the interior of $B_r(x)$ for which $B_k(\gamma y) \subset B_r(x)$. If a lattice point γy is within distance k of $\partial B_r(x)$, then many of the lattice points in $B_k(\gamma y)$ may lie outside $B_r(x)$, and a definite proportion of lattice points are close to the boundary, as the volume of $B_r(x)$ grows exponentially. We use the mixing property of the geodesic flow to show that that $\partial B_r(x)$ becomes equidistributed on compact sets of the quotient space $\mathcal{T}/\Gamma$, and this in turn shows that the intersections of $\partial B_r(x)$ with $B_k(\gamma y)$ are evenly distributed. This implies that we can find an upper bound for the average number of lattice points of Γy near some γy close to the boundary, which do in fact lie inside $B_r(x)$. In fact, we prove a result that works for bisectors, so we also need to show that the proportion of lattice points near the geodesics rays through ∂U tends to zero as r tends to infinity. These arguments using mixing originate in work of Margulis [7], and our treatment of conditional mixing is essentially due to Eskin and McMullen [2], see also Gorodnik and Oh [5], for the higher rank case.

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Dynamics of the horocycle flow on the eigenform loci in $\mathcal{H}(1,1)$

Barak Weiss

(joint work with Matt Bainbridge and John Smillie)

We discuss dynamics of the horocycle flow on the eigenform loci in the stratum $\mathcal{H}(1,1)$, discovered by Calta and McMullen. We reprove and improve the measure classification result of Calta and Wortman, and obtain a corresponding orbit-closure classification. The new phenomenon we display is orbit-closures which are linear manifolds *with boundary*. We also obtain some statements regarding limits of sequences of measures arising in some counting problems. In particular we show that every orbit is equidistributed in its closure.

In other strata, we describe new examples of closed horocycle invariant sets and measures. These do *not* arise via the previously known mechanisms: minimal sets, invariant horizontal saddle connections, $\mathrm{SL}(2,R)$-invariant sets and measures, and their pushforward via the real REL action. Instead their construction involves new invariant vector fields which commute with the horocycle actions, on infinite covers of the stratum, corresponding to invariant subspaces in the monodromy action of affine automorphism groups.

Particular compatible sequences of periodic orbits of the Koch snowflake fractal billiard

Robert G. Niemeyer

(joint work with Michel L. Lapidus)

Those familiar with the subject of mathematical billiards are well aware of the friendly nature of a billiard table with a smooth boundary. Even a billiard table with finitely many singularities (i.e., points for which reflection is not defined) still maintains a rather pleasant demeanor. An example of such a billiard is a *rational billiard* $\Omega(R)$. Such a billiard table has a polygonal boundary R where each interior angle is a rational multiple of π. The Koch snowflake is a fractal that is comprised of three self-similar Koch curves; see Figure 1. The Koch snowflake KS is an everywhere nondifferentiable curve. As one then expects, the billiard $\Omega(KS)$ with boundary KS is a particularly unfriendly (and unwieldy) billiard table. However, closely related to the Koch snowflake are its prefractal approximations. A prefractal approximation to the Koch snowflake billiard, denoted by $\Omega(KS_n)$ (with $n = 0$ being the equilateral triangle billiard), is a rational billiard. It is our intention to gain insight into the nature of the Koch snowflake billiard by examining the behavior of the billiard flow on successive prefractal approximations.

Let us first state a very lofty goal: to show that an analogue of the Veech dichotomy holds for a particular family of fractal billiard tables. Given that the subject of fractal billiards is still in its infancy, this very difficult problem will likely not be solved by the authors in the near future. However, such a goal serves to guide the development of the subject. In [1], we detailed a number of conjectures and open questions regarding the existence of a well-defined billiard $\Omega(KS)$. A
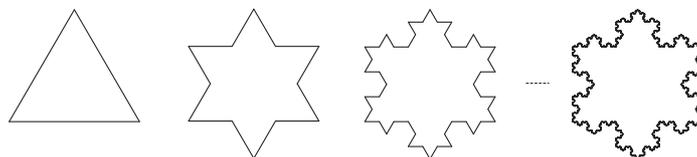
FIGURE 1. The construction of the Koch snowflake, via its pre-fractal approximations KS_n for $n = 0, 1, 2, ...$

number of simulations were detailed and experimental evidence motivated many of the definitions given therein. In [2], we began to investigate the behavior of orbits in a particular direction, namely, the direction of $\theta = \pi/3$.

In [4], the billiard flow associated with a rational billiard is carefully described. Consider the billiard map f_n that describes the billiard flow on the phase space $\Omega(KS_n) \times S^1/ \sim$. Second, fix a coordinate system by which every angle will be measured. Then $f_n(x_n^{k_n}, \theta_n^{k_n}) = (x_n^{(k+1)_n}, \theta_n^{(k+1)_n})$ is the image of a basepoint $x_n^{k_n}$ and angle $\theta_n^{k_n}$ under the billiard map f_n associated with $\Omega(KS_n)$, and both $\theta_n^{k_n}$ and $\theta_n^{(k+1)_n}$ are measured relative to the same coordinate system, rather than their respective sides in $\Omega(KS_n)$. Now, suppose one takes $(x_n^0, \pi/3)$ as an initial condition of the billiard flow (we are assuming $\pi/3$ is an inward pointing direction based at x_n^0). As described in great detail in [2], one may then show there is an intimate relationship between the orbit $\mathcal{O}_n(x_n^0, \pi/3)$ and an orbit $\mathcal{O}_0(x_0^0, \pi/3)$ of the equilateral triangle billiard $\Omega(KS_0)$ (assuming x_n^0 was not a corner of $\Omega(KS_n)$). We describe this intimate relationship as the two orbits being *compatible*.

Such language allows us to construct what we call *a compatible sequence of piecewise Fagnano orbits*, *an eventually constant compatible sequence of periodic orbits* and *a compatible sequence of generalized piecewise Fagnano orbits*. The notion of a compatible sequence of piecewise Fagnano orbits is motivated by the Fagnano orbit of the equilateral triangle billiard $\Omega(KS_0)$. Simply put, a piecewise Fagnano orbit is constructed by appending scaled copies of the Fagnano orbit to each basepoint; the result will then be an orbit that is compatible with the orbit to which the scaled copies were appended. An interesting fact to point out is that a piecewise Fagnano orbit can be determined by utilizing a particular iterated function system of non-expansive mappings. An orbit in an eventually constant compatible sequence of orbits is referred to as a Cantor orbit (or $\mathcal{C}$-orbit), because of the nature of the basepoints of such an orbit; they are points which coincide with elements of Cantor sets that do not have finite ternary representations. A generalized piecewise Fagnano orbit is named for the fact that it is very much a generalization of a piecewise Fagnano orbit, which is clearly seen in a symbolic setting. Specifically, the graphical representation of a generalized piecewise Fagnano orbit may be obtained by a suitably adjusted iterated function system consisting of finitely many non-expansive mappings. By this we mean that the contraction

ratio may not necessarily be $1/3^k$, for some k, nor can one simply append suitably scaled copies of a Fagnano orbit to produce the next orbit in the compatible sequence of generalized piecewise Fagnano orbits.

A more formal description of these orbits can be given in terms of the ternary representation of the initial basepoint x_0^0 of the initial orbit in the particular compatible sequence of orbits. In particular, if x_0^0 has a ternary representation consisting of infinitely many 1's and finitely many 0's and 2's, then the resulting compatible sequence to which $\mathcal{O}_0(x_0^0, \pi/3)$ belongs will be a compatible sequence of piecewise Fagnano orbits. If x_0^0 has a ternary representation consisting of finitely many 1's and infinitely many 0's and 2's, then the resulting compatible sequence is an eventually constant compatible sequence of orbits (eventually comprised of a single $\mathcal{C}$-orbit). Finally, if x_0^0 has a ternary representation consisting of either infinitely many 1's and 2's, or 1's and 0's or 1's, 2's and 0's, then the resulting compatible sequence to which the orbit $\mathcal{O}_0(x_0^0, \pi/3)$ belongs to is a compatible sequence of generalized piecewise Fagnano orbits.

In [2], we give a plausibility argument as to why it is that the inverse limit of a compatible sequence of piecewise Fagnano orbits constitutes a periodic orbit of the Koch snowflake billiard $\Omega(KS)$. In addition to this, we show that the trivial limit of an eventually constant compatible sequence of periodic orbits constitutes what we call a *stabilizing periodic orbit* of the Koch snowflake billiard $\Omega(KS)$. Because of the elusive nature of a compatible sequence of generalized piecewise Fagnano orbits, we do not give a plausibility argument as to why it is the suitably defined inverse limit of such a sequence may constitute a periodic orbit of the Koch snowflake. Instead, we provide a conjecture about the existence of such an orbit.

Recent work in [3] providing support for the conjecture that $\Omega(KS)$ constitutes a well-defined billiard involves analyzing what we are calling *hybrid periodic orbits*. Current computer simulations of these orbits in their respective prefractal approximations show 1) that one can construct a well-defined compatible sequence of hybrid periodic orbits and 2) such orbits may be generated by a particular iterated function system of non-expansive mappings (in much the same way one can generate a piecewise Fagnano orbit). Such simulations provide the most substantial evidence in support of the existence of a well-defined billiard $\Omega(KS)$. Additional results indicate that is may be possible to construct well-defined compatible sequences of periodic orbits in directions other than $\pi/3$.

It is our hope that once a suitable notion of reflection in the boundary KS of the fractal billiard $\Omega(KS)$ can be determined, the particular orbits that we have described above will have analogues in the limiting case that are also periodic orbits of the well-defined Koch snowflake billiard. Once (and if) this has been accomplished, we hope that out of this comes a natural notion of *non periodic* orbit. Then, with a notion of periodic and not periodic at hand, it should become much clearer how to formulate a suitable analogue of the Veech dichotomy.

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Coarse Differentiation and the rank of Teichmüller space

Kasra Rafi

We study the quasi-isometry group of Teichmüller space. As a first step we give a local description of floats in the Teichmüller space. The key ingredients are: Eskin's coarse differentiations and the construction of a combinatorial model for Teichmüller space.

Prym-Tyurin classes and tau-functions

Dmitry Korotkin

(joint work with Peter Zograf)

In this talk we study the moduli space $\mathcal{M}$ of holomorphic n-differentials with simple zeros on Riemann surfaces for $n \geq 2$, following our recent work [1] devoted to the moduli spaces of holomorphic Abelian differentials ($n = 1$). The space $\mathcal{M}$ consists of pairs (C, W), where C is a compact Riemann surface of genus g and W is a holomorphic n-differential on C with simple zeros (the number of the zeros then equals $n(2g - 2)$). The dimension of the space $\mathcal{M}$ equals $(n+1)(2g-2)$ (here and below we always assume that $n \geq 2$). To each pair (C, W) one can naturally associate the canonical n-sheeted covering $p : \hat{C} \to C$ where $\hat{C}$ is a Riemann surface of genus $n^2(g-1)$ and W becomes an nth power of an abelian differential w: $W = w^n$. The covering has ramification points of multiplicity $n-1$ at the zeros of W. Moreover, $\hat{C}$ possesses an automorphizm μ such that $\mu^n = id$ and $w(\mu x) = \epsilon w(x)$ where $x \in \hat{C}$ and ϵ is the primitive root of unity, $\epsilon := e^{2\pi i/n}$.

Consider the space $H^{(1,0)}(\hat{C}, \mathbb{C})$ of holomorphic differentials on $\hat{C}$; the dimension of this space equals $(n + 1)(2g - 2)$. The automorphizm μ defines a linear automorphizm Q^μ of order n in $H^{(1,0)}(\hat{C}, \mathbb{C})$. Denote the eigenspace of Q^μ corresponding to the eigenvalue ϵ^k by $H^{(k)}$ ($k = 0, 1, \ldots, n - 1$). The dimensions of these eigenspaces are given by: $\dim H^{(0)} = g$, $\dim H^{(k)} = (2n - 2k + 1)(g - 1)$ for $k = 1, \ldots, n-1$. In this way we get a set of n vector bundles over the moduli space

$\mathcal{M}$. The vector space $H^{(0)}$ coincides with the vector space of holomorphic differentials on C. Other $n-1$ vector bundles obtained in this way we call *Prym-Tyurin vector bundles*. The first Chern classes of the corresponding determinant line bundles we call *Prym-Tyurin classes* and denote them by λ_k^{PT} for $k = 1, \ldots, n-1$.

The boundary of the space $\mathcal{M}$ is formed by the following divisors: the divisors $D_0, D_1, \ldots, D_{[g/2]}$ corresponding to Deligne-Mumford compactification of the moduli space of Riemann surfaces of genus g, and an additional divisor D_{deg}, which corresponds to n-differentials W with double zeros.

Our main result is the expression for the classes λ_k^{PT} (as elements of the rational Picard group) in terms of the boundary classes of $\mathcal{M}$ and the tautological class ψ.

The main analytical tool we are using is the formalism of Bergman tau-functions, appropriately adjusted to the present situation. The definition of these tau-functions is inspired by the theory of isomonodormic deformations [2], the spectral theory of Laplace operator in flat singular metrics [3] and the theory of random matrices [4].

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Hilbert modular varieties do not lie in the Schottky locus

Matt Bainbridge

(joint work with Martin Möller)

Consider the moduli space of genus g Riemann surfaces $\mathcal{M}_g$ and the moduli space of g-dimensional principally polarized Abelian varieties $\mathcal{A}_g = \mathbb{H}_g/\mathrm{Sp}_{2g}\mathbb{Z}$, where $\mathbb{H}_g$ is the g-dimensional Siegel upper half space. The Torelli map $t\colon \mathcal{M}_g \to \mathcal{A}_g$ embeds $\mathcal{M}_g$ into $\mathcal{A}_g$, and the image $t(\mathcal{M}_g)$ is called the Schottky locus. The image of the Torelli map is dense in $\mathcal{A}_g$ if $g = 2$ or 3, and is otherwise a complicated subvariety of $\mathcal{A}_g$.

The moduli space $\mathcal{A}_g$ possesses a natural locally symmetric metric. There are many subvarieties of $\mathcal{A}_g$ which are totally geodesic with respect to this metric. Such a subvariety is called a *Shimura variety*. (To be completely precise, a Shimura variety is a totally geodesic subvariety which contains a CM point.) In this talk, we consider the question of which Shimura subvarieties of $\mathcal{A}_g$ are contained in the closure of the Schottky locus $\overline{t(\mathcal{M}_g)}$. It is conjectured that over all g there are only finitely many Shimura subvarieties of $\mathcal{A}_g$ which are contained in $\overline{t(\mathcal{M}_g)}$ and meet $t(\mathcal{M}_g)$. See the survey [MO] for more about these questions.

Examples of Shimura varieties are the Hilbert modular varieties $X_{\mathcal{O}} = \mathbb{H}^g / \operatorname{SL}_2 \mathcal{O}$. Here $\mathcal{O}$ is an order in a totally real number field F of degree g (for example the ring of integers). $X_{\mathcal{O}}$ can be regarded as the moduli space of principally polarized Abelian varieties with real multiplication by $\mathcal{O}$, and there is a canonical totally geodesic immersion $X_{\mathcal{O}} \to \mathcal{A}_g$.

A special case of the above problem is the question of which Hilbert modular varieties are contained in the closure of the Schottky locus. In [dJZ07], de Jong and Zhang showed that if $g > 5$, then no Hilbert modular variety is contained in $\overline{t(\mathcal{M}_g)}$. This question is trivial if $g = 2$ or 3 as the Schottky locus is dense, but the question remained open if $g = 4$.

A more delicate question is to study the dimension of intersection of Hilbert modular varieties with the Schottky locus. It is natural to conjecture for dimension reasons that for any sufficiently large g, all but finitely many Hilbert modular varieties are disjoint from the Schottky locus. This would likely imply that there are only finitely many algebraically primitive Teichmüller curves in $\mathcal{M}_g$ for any sufficiently large g.

In recent work with Martin Möller, we proved:

Theorem. *No Hilbert modular variety in $\mathcal{A}_4$ is contained in $\overline{t(\mathcal{M}_4)}$.*

Combined with [dJZ07], this shows that no Hilbert modular variety is contained in the Schottky locus for $g > 3$. After this result was proved, we learned from [MO] of a mistake in [dJZ07]. In fact, their proof works for the case of $g = 4$, although they claimed otherwise. So our theorem is not in fact new, though the methods are different and could potentially be applied to study for example the dimension of intersection with the Schottky locus.

The proof uses an explicit computation of the boundary of $t^{-1}(X_{\mathcal{O}})$ in the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ from [BM]. The boundary of $\overline{\mathcal{M}}_g$ can be divided into strata composed of stable curves of a fixed topological type. Given such a stratum $\mathcal{S} \subset \overline{\mathcal{M}}_g$, we defined in [BM] a morphism $t_{\mathcal{S}}$ from $\mathcal{S}$ to an algebraic torus $(\mathbb{C}^*)^N$. Each component of the boundary of $t^{-1}(X_{\mathcal{O}})$ is $t_{\mathcal{S}}^{-1}(T)$ for some subtorus $T \subset (\mathbb{C}^*)^N$ of appropriate dimension. The proof of the theorem then roughly amounts to showing that certain explicit subvarieties of algebraic tori do not contain subtori of a given dimension.

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Wild singularities of translation surfaces.

Ferrán Valdez

(joint work with Joshua P. Bowman)

Overview. In this talk we define *wild singularities* for a special class of translation surfaces and introduce an affine invariant topological space called *the space of linear approaches to a singularity*. This space provides local criteria to distinguish among several recently discovered classes of translation surfaces presenting wild singularities but having the same (infinite) topological type. We address topological aspects of this space such as functoriality and its decomposition into *rotational components*.

Wild singularities. A *translation surface* is a pair (X, ω) formed by a Riemann surface X and a non identically zero holomorphic 1-form ω on X. We denote by $Z(\omega) \subset X$ the set of zeroes of ω and by $\widehat{X}$ the metric completion of $X \setminus Z(\omega)$ with respect to its natural translation invariant flat metric. Henceforth, we deal with translation surfaces for which the *set of singularities* $Sing(\widehat{X}) := \widehat{X} \setminus X$ is a discrete subset of $Sing(\widehat{X})$. Points in $\widehat{X} \setminus X$ are classified into:

(1) Cone angle singularities. These are points $x \in \widehat{X}$ for which either the Riemann surface structure of X extends to $X \cup x$ or there exists a punctured neighborhood of x which is isometric to an infinite cyclic covering of the punctured disc $(0 < |z| < \varepsilon, dz) \subset \mathbb{C}$. The point x is called finite or infinite angle cone singularity respectively.

(2) The rest. We call such points *wild* singularities of the flat surface.

Examples. Cone angle singularities naturally appear in translation surfaces associated to polygonal billiards or infinite staircases [3]. On the other hand, wild singularities arise when studying the "Baker's map" [2], generalizing Thurston's construction to infinite bipartite graphs [4] or as geometric limits of a family of compact translation surfaces [1].

The space of linear approaches. In the following paragraphs we introduce the topological spaces $\mathcal{L}(X)$ and $\mathcal{L}(x)$. The latter provides the desired invariant for a wild singularity $x \in Sing(\widehat{X})$.

Consider for each $\varepsilon > 0$ the space

$$\mathcal{L}^\varepsilon(X) := \{\text{unit speed geodesic trajectories } \gamma : (0, \varepsilon) \to X'\}$$

endowed with the uniform metric. Two linear approaches $\gamma_1 \in \mathcal{L}^\varepsilon(X)$ and $\gamma_2 \in \mathcal{L}^{\varepsilon'}(X)$ are said to be equivalent if and only if $\gamma_1(t) = \gamma_2(t)$ for all $t \in (0, \min\{\varepsilon, \varepsilon'\})$. We denote by $\sim$ this equivalence relation and define:

$$(0.1) \qquad \mathcal{L}(X) := \bigsqcup_{\varepsilon > 0} \mathcal{L}^\varepsilon(X) / \sim$$

The equivalence class defined γ will be denoted by $[\gamma]$. Every class $[\gamma] \in \mathcal{L}(X)$ is called a *linear approach* to the point $\lim_{t \to 0} \gamma(t) \in \widehat{X}$. Remark that for each $\varepsilon' \le \varepsilon$ the restriction of each linear approach in $\mathcal{L}^\varepsilon(X)$ to the interval $(0, \varepsilon')$ defines

a continuous injection $\rho_\varepsilon^{\varepsilon'} : \mathcal{L}^\varepsilon(X) \to \mathcal{L}^{\varepsilon'}(X)$. Define $\varepsilon \trianglelefteq \varepsilon'$ if and only if $\varepsilon' \leq \varepsilon$, where $\leq$ is the standard order in $\mathbb{R}$. Then $< \mathcal{L}^\varepsilon(X), \rho_\varepsilon^{\varepsilon'} >$ is a direct system of topological spaces over $(\mathbb{R}^+, \trianglelefteq)$. We have the equality of sets

$$(0.2) \qquad\qquad \mathcal{L}(X) = \varinjlim \mathcal{L}^\varepsilon(X)$$

Henceforth we endow $\mathcal{L}(X)$ with the direct limit topology. This topology is generated by the family of sets $U^t = \{[\gamma] \in \mathcal{L}(X) \mid \gamma(t) \in U\}$, where U ranges over open subsets of X and t over $\mathbb{R}^+$. Using this subbasis, one can prove that the space $\mathcal{L}(X)$ is Hausdorff. Nevertheless, in general $\mathcal{L}(X)$ is not metrizable.

For every $[\gamma] \in \mathcal{L}(X)$ we define two maps $\mathrm{bp}([\gamma]) := \lim_{t \to 0} \gamma(t) \in \widehat{X}$ and $\mathrm{dir}([\gamma]) := \gamma'(t) \in \mathbf{S}^1$. These are called the *basepoint* and *direction* maps respectively. Their continuity follows from the universal property of the direct limit.

For every $x \in \widehat{X}$ we call $\mathcal{L}(x) := \mathrm{bp}^{-1}(x)$, endowed with the subspace topology, *the set of linear approaches to the point x*. The topological type of this space is invariant under affine orientation preserving diffeomorphisms of X. Remark that a point $x \in Sing(\widehat{X})$ is a cone angle singularity (of finite or infinite type) if and only if $\mathcal{L}(x) \subset \mathcal{L}^\varepsilon(X)$ for some $0 < \varepsilon$. This corresponds to the fact that *short saddle connections* do not accumulate on x.

Theorem. *The space $\mathcal{L}(x)$ is the closure of a union of immersed connected 1-manifolds (possibly with boundary), each of which carries a canonical (angular) metric.*

Each 1-manifold is called a *rotational component* and its obtained by rotating a fixed $[\gamma] \in \mathcal{L}(x)$ around the basepoint $\mathrm{bp}([\gamma])$. A rotational component is called a *spire* if it is unbounded with respect to its angular metric, a *double spire* if it is unbounded in both directions or an *arc* if it is unbounded and not homeomorphic to $\mathbf{S}^1$.

Examples revisited. Denote by X_b and X_{AY} the translation surface associated to the Baker's map [2] and the geometric limit of the Arnoux-Yoccoz family [1]. Both surfaces have the same topological type and we can now distinguish them using only the space of linear approaches. Remark that $\widehat{X_b} = X_b \cup x$, $\widehat{X_{AY}} = X_b \cup x'$ and that both x and x' are wild singularities. On the other hand, $\mathcal{L}(x)$ is formed by two double spires plus an infinite number of arcs, whereas $\mathcal{L}(x')$ is formed by two double spires and 4 arcs. Since the space of linear approaches is an affine invariant, there is no affine diffeomorphism from X_b to X_{AY}. A similar argumentation can be used to distinguish between other pairs of translation surfaces presenting wild singularities.

Functoriality. Every affine map $f : X \to Y$ between two translation surfaces whose differential lies in $\mathrm{GL}_+(2, \mathbb{R})$ induces a continuous "push-forward" map $f_* : \mathcal{L}(X) \to \mathcal{L}(Y)$. Moreover, if we denote by $\widehat{f}$ the continuous extension of f to $\widehat{X}$ and by Df the *normalized differential* of f, then the following diagrams

commute:

$$
\begin{array}{ccc}
\mathcal{L}(X) \xrightarrow{\ f_*\ } \mathcal{L}(Y) & \qquad & \mathcal{L}(X) \xrightarrow{\ f_*\ } \mathcal{L}(Y) \\
{\scriptstyle \mathrm{bp}}\downarrow \qquad \downarrow{\scriptstyle \mathrm{bp}} & & {\scriptstyle \mathrm{dir}}\downarrow \qquad \downarrow{\scriptstyle \mathrm{dir}} \\
\overline{X} \xrightarrow[\bar{f}]{} \overline{Y} & & \mathbf{S}^1 \xrightarrow[Df]{} \mathbf{S}^1
\end{array}
$$

In this sense, the space $\mathcal{L}(X)$ extends the unit tangent bundle of X to points in $Sing(X)$.

Questions for future work.

(1) Let $f(z)$ be holomorphic in the punctured unit disc $\mathbf{D}^* \subset \mathbb{C}$ and suppose that $z = 0$ is an essential singularity. Consider the holomorphic 1-form $\omega := f(z)dz$ defined in U^*. Is it possible to tell from the Laurent series $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ what kind of rotational components $\mathcal{L}(z = 0)$ presents?

(2) Can the topological spaces that arise as $\mathcal{L}(x)$ be characterized?

(3) What is the analog of theorem 1 to be obtained if we let $Sing(X)$ be non-discrete in $\widehat{X}$ (*e.g.*, what if $Sing(X)$ contains a Cantor set)?

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Bounded combinatorics and the Lipschitz metric on Teichmüller space

Anna Lenzhen

(joint work with Kasra Rafi and Jing Tao)

Let $\mathcal{T}(S)$ be the Teichmüller space of a surface S of finite type. Given a metric on $\mathcal{T}(S)$, it is natural to ask to what extent it is hyperbolic. One way hyperbolicity manifests itself is in strongly contracting closest point projection to geodesics.

We would like to consider the *Lipschitz distance* on $\mathcal{T}(S)$, that was introduced by Thurston in [10]. For $x, y \in \mathcal{T}(S)$, the distance $d_L(x, y)$ between x and y is defined in terms of the best Lipschitz map from x to y. The distance can be formulated in terms of ratios of hyperbolic lengths of curves [10]:

$$
(0.1) \qquad d_L(x, y) = \log \sup_{\alpha} \frac{\ell_y(\alpha)}{\ell_x(\alpha)},
$$

where $\ell_x(\alpha)$ is the hyperbolic length of α in the unique hyperbolic metric in the conformal class of x.

This metric is in some ways similar to the much more studied and better understood Teichmüller metric on $\mathcal{T}(S)$. The *Teichmüller distance* $d_T(x,y)$ is defined in terms of the best quasiconformal map between x and y, and can be formulated in terms of ratios of extremal lengths of curves [3]:

$$(0.2) \qquad d_T(x,y) = \frac{1}{2} \log \sup_\alpha \frac{\mathrm{Ext}_y(\alpha)}{\mathrm{Ext}_x(\alpha)}.$$

It is easy to see that the Lipschitz metric, unlike the Teichmüller metric, is not symmetric. On the other hand, we know from [1] that in the thick part of $\mathcal{T}(S)$, the distances $d_L(x,y)$, $d_L(y,x)$ and $d_T(x,y)$ are equal up to a universal additive error.

When x is in the thick part, the geometry of x can be coarsely encoded by its associated short marking μ_x, which is a finite collection of simple closed curves. There are many results relating the combinatorics of μ_x and μ_y to the behavior of the Teichmüller geodesic $\mathcal{G}_T$ between x and y. (See [7, 8, 2], or [9] for a review of some of these results in one paper.) Contrasting with the Teichmüller metric, there is no unique geodesic in the Lipschitz metric from x to y. But one hopes that qualitative information about Lipschitz geodesics can still be extracted from the end invariants μ_x and μ_y.

The first natural situation to consider is when x and y have bounded combinatorics. That is when, for every proper subsurface Y of S, the distance $d_Y(\mu_x, \mu_y)$ in the curve complex of Y between the projections of μ_x and μ_y to Y is uniformly bounded. For the Teichmüller metric, this is in fact equivalent to $\mathcal{G}_T$ being cobounded (See [7] and [9]. The fact that endpoints of a cobounded Teichmüller geodesic have bounded combinatorics follows also from the work of Minsky [4, 6].)

Our first result is that bounded combinatorics also guarantees cobounded for every Lipschitz geodesic $\mathcal{G}_L$ from x to y. In fact, $\mathcal{G}_L$ is well approximated by $\mathcal{G}_T$.

Theorem (Bounded combinatorics implies cobounded). *Assume, for $x, y \in \mathcal{T}(S)$ in the thick part of Teichmüller space, that*

$$d_Y(\mu_x, \mu_y) = O(1)$$

for every proper subsurface $Y \subset S$. Then any geodesic $\mathcal{G}_L$ in the Lipschitz metric connecting x to y fellow travels the Teichmüller geodesic $\mathcal{G}_T$ with endpoints x and y. Consequently, $\mathcal{G}_L$ is cobounded.

To restate Theorem more succinctly is to say that $\mathcal{G}_T$, viewed as a set in the Lipschitz metric, is quasi-convex. A standard argument for showing a set is quasi-convex is to show the closest-point projection to the set is strongly contracting. Indeed, this is how we prove Theorem .

Theorem (Lipschitz projection to Teichmüller geodesics). *Let $\mathcal{G}_T$ be a cobounded Teichmüller geodesic. Then the image of a Lipschitz ball disjoint from $\mathcal{G}_T$ under*

the closest-point projection to $\mathcal{G}_T$ (with respect to the Lipschitz metric) has uniformly bounded diameter. That is, the closest-point projection to $\mathcal{G}_T$ is strongly contracting.

This is analogous to Minsky's theorem ([5]) that the closest-point projection in the Teichmüller metric to a cobounded Teichmüller geodesic is strongly contracting. Combining Theorem and Theorem , we obtain:

Theorem (Strongly contracting for projections to Lipschitz geodesics)**.** *Let $\mathcal{G}_L$ be a Lipschitz geodesic whose endpoints have bounded combinatorics. Then the closest-point projection to $\mathcal{G}_L$ is strongly contracting.*

Theorem is a negative-curvature phenomenon. A natural consequence is stability of $\mathcal{G}_L$. In other words,

Corollary (Stability of Lipschitz geodesics)**.** *If $\mathcal{G}_L$ is a Lipschitz geodesic whose endpoints have bounded combinatorics, then any quasi-geodesic with the same endpoints as $\mathcal{G}_L$ fellow travels $\mathcal{G}_L$.*

For a Teichmüller geodesic to be cobounded, it is necessary for the endpoints to have bounded combinatorics. Naturally, one should ask whether this holds for the Lipschitz metric as well. We claim that the answer is no. That is, there are arbitrarily long Lipschitz geodesics which stay in the thick part of Teichmüller space, but whose endpoints *do not* have bounded combinatorics.

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Continued fractions and translation surfaces

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(joint work with Kariane Calta)

Arnoux and Yoccoz [2] gave examples in genus $g \geq 3$ of pseudo-Anosov maps with dilatation of degree g. These were the first examples of such maps that realized the lower bound on their degree. Their examples have led to various interesting studies, see especially [9] and [12].

When a pseudo-Anosov stabilizes an abelian differential, it acts as an affine diffeomorphism, whose linear part is a hyperbolic matrix. This matrix can be diagonalized to the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, with λ the dilatation. Following Long and Reid [11], we call the matrix and the pseudo-Anosov map *special* if $\lambda \in \mathbb{Q}(\lambda + \lambda^{-1})$. The Arnoux-Yoccoz examples are thus special in this sense.

Recall that the Sah-Arnoux-Fathi (SAF) invariant for interval exchange maps is zero whenever the map is periodic, and that the Kenyon-Smillie J-invariant [10] associated to a translation surface (defined by an abelian differential) "projects" so as to give the SAF-invariant of each direction on the surface. Calta [5] showed that if a surface has three directions with vanishing SAF-invariant, then the set of directions form the projective line over a field as soon as the surface is normalized to be in *standard form*: 0, 1, ∞ have vanishing SAF-invariant. Calta and Smillie show that furthermore, the presence of a pseudo-Anosov implies that this periodic field is exactly the *trace field* formed by adjoining onto $\mathbb{Q}$ the traces of the elements of the Veech group. Thus, a pseudo-Anosov map has linear part with trace in this periodic field, and is special exactly when λ itself lies in the field (in general, λ lies in a quadratic extension of the trace field).

Veech [15] gave examples of translation surfaces with large affine diffeomorphism groups. His examples, formed by identifying sides of pairs of regular n-gons, have so-called Veech groups (formed by the linear part of the affine diffeomorphisms) that are triangle Fuchsian groups of signature $(2, n, \infty)$, $(2m, \infty, \infty)$. In the 1930s, Hecke studied a particular family of such groups; in the 1950s, D. Rosen [13] introduced his continued fractions to address the "word problem" for the Hecke groups. A decade later, Rosen posed the problem of identifying the parabolic fixed points of each of the Hecke groups. A German school of Leutbecher, Borho, Rosenberger, Wolfart and others attacked the problem throughout the 1970s.

Arnoux and Schmidt [1] point out that results of Rosen and Towse [14] (obtained almost 50 years after Rosen had written his thesis) show that the double septagon has a special pseudo-Anosov map, and that work of Towse and others [8] imply that the double nonagon has at least four such maps. Arnoux and Schmidt further exhibited special hyperbolics for the lattice surface examples of Veech corresponding to (double) 14-, 18- 20-, and 24-gons. This complemented work of Leutbecher, Wolfart *et al.* to imply that: Every Veech example of $g > 2$ has non-parabolic elements in its periodic field.

Veech's student Ward [16] gave a second infinite family of lattice translation surfaces, having Veech group that are triangle Fuchsian groups of signature $(3, n, \infty)$. More recently, Bouw and Möller [3] have shown that virtually all signatures (m, n, ∞) are realized as Veech groups. We show the following.

Theorem. *Any Bouw-Möller surface of signature $(2m', 2n', \infty)$ has a special pseudo-Anosov map.*

We prove this by first exhibiting a representative group of signature (m, n, ∞) that lies in $\mathrm{PSL}_2(\mathcal{O}_K)$ where $\mathcal{O}_K$ is the ring of integers of the trace field of the group. Whereas the representative is in standard form (that is has parabolic fixed points including 0, 1 and ∞) when at least one of m, n is odd, when both m, n are even one finds rather that 1 is a hyperbolic fixed point. An easy argument shows that since the group must be conjugated to one in standard form, and $0, \infty$ are parabolic fixed points of the group, there is an element of K that is fixed by the conjugate group in standard form.

Furthermore, reduction modulo the prime ideal of $\mathcal{O}_K$ above $\langle 2 \rangle$ leads to the following.

Theorem. *Any Bouw-Möller surface of signature $(2^k, n, \infty)$ with n odd, $n \neq 2^f + 1$ has non-parabolic elements in periodic field.*

Finally, in [6] we create a continued fraction algorithm for the Ward examples, with the aim to detect non-parabolic elements of the periodic field. Our continued fractions have various desirable properties, including detecting transcendence; this property was only recently shown for the Rosen fractions [4].

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