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Finite-dimensional Approximations of Discrete Groups

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May 15th – May 21st, 2011

Abstract. The main objective of this workshop was to bring together experts from various fields, which are all interested in finite and finite-dimensional approximations of infinite algebraic and analytic objects, such as groups, algebras, dynamical systems, group actions, or even von Neumann algebras.

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Introduction by the Organisers

The workshop *Finite-dimensional Approximations of Discrete Groups*, organized by Goulnara Arzhantseva (Wien), Andreas Thom (Leipzig) and Alain Valette (Neuchâtel) was held May 15th – May 21st, 2011. This meeting was well attended with 25 participants. Many of the participants are young researchers which just established themselves. This gave the workshop a very active atmosphere. Open discussions during talks were vivid and productive.

Over the last decade, finite approximation of groups has become an important ingredient in the understanding of various longstanding conjectures. A group is called *sofic* if its group law can be approximated on finite sets by finite permutations up to any given ε, where errors in the symmetric group are measured in terms of the usual Hamming distance. A series of breakthrough applications of this notion was started with Misha Gromov's solution of Gottschalk's Surjunctivity Conjecture for sofic groups. Later, Gabor Elek and Endre Szabo proved the Kaplansky Finiteness Conjecture for sofic groups. More recently, it was observed that Kervaire's Conjecture holds for hyperlinear groups; a class of groups which admits a modeling of the group law by unitary matrices. The notion of soficity has triggered groundbreaking work by Lewis Bowen, which resulted in a definition of entropy for actions of sofic groups, extending the existing definitions for amenable groups. The class of hyperlinear groups is well-studied by the operator-algebraic community, since it naturally arises from the definition of free entropy dimension given by Dan Voiculescu, which is based on matricial microstates.

The aim of the workshop was to bring together experts from various fields which share interest in finite and finite-dimensional approximation of infinite algebraic and analytic objects, such as groups, algebras, dynamical systems, group actions, and von Neumann algebras.

Since the audience was mixed, we started the workshop with a series of survey talks. Andreas Thom, Miklos Abert, Lewis Bowen and Narutaka Ozawa gave lectures about various aspects of finite and finite-dimensional approximation. The survey talks were followed by research talks on the remaining days, some of which will be mentioned in this section. Łukasz Grabowski presented in his talk the recent solution to the Atiyah Problem in the torsion case. Building on work of Tim Austin, and working independently of the team Pichot-Schick-Zuk, he could show that *every* non-negative real number can be the kernel-dimension of an element in the integral group ring of a countable group. Adam Timar presented the first example of a sequence of Cayley graphs which is convergent in the sense of Benjamini-Schramm, which however admits no coloring, so that the resulting sequence of Cayley diagrams converges as a sequence of Cayley diagrams. Miklos Abert (joint work with Glasner-Virag) presented a variant of Kesten's Theorem for discrete measured groupoids, which among other things imlies that every sequence of Ramanujan graphs has a sub-linear number of k-cycles. Ken Dykema (joint work with Kerr-Pichot) presented the recent invention of the sofic-dimension of a group or more generally a discrete measured groupoid. Wolfgang Lück presented new results on torsion-growth and homology-growth in the unexplored territory of homology in finite characteristic. Simon Thomas lectured about structure results about the ultraproduct of alternating groups, which shed light on the universal sofic group. Mark Sapir gave an introduction to the Higman embedding theorem and proved a new variant of it, which implies that expander graphs can be coarsely embedded into fundamental groups of manifolds. Erik Guentner (joint with Arzhantseva-Spakula) reported about coarse embeddings and related topics. Outlines of these and various other interesting talks appear with extended abstracts in this report.

The workshop included various evening sessions with additional talks and an extensive problem session on Thursday afternoon. After positive feedback from all sides, we (the organizers) came to the conclusion that the workshop achieved its ambitious goal of establishing contact across various areas of research.

Workshop: Finite-dimensional Approximations of Discrete Groups

Table of Contents

Abstracts

Introduction to sofic and hyperlinear groups and their applications

Andreas Thom

In this talk I gave a general overview about the use of finite and finite-dimensional approximation techniques. I reviewed the proof of the Kervaire Conjecture for hyperlinear groups (following the ideas of Gerstenhaber-Rothaus) and the proof of the Kaplansky Finiteness Conjecture for sofic groups (following Elek-Szabo).

Groups, graphs and dynamics

Miklos Abért

I gave two talks at the workshop: the first was more of introductory nature. This is a joint summary for them.

For a countable group Γ let

$$\mathcal{SUB}_\Gamma = \{H \leq \Gamma\}$$

be the space of subgroups of Γ. It is easy to see that $\mathcal{SUB}_\Gamma$ is closed in the product topology on $\{0,1\}^\Gamma$ and so $\mathcal{SUB}_\Gamma$ endowed with this topology (called the Chabauty topology) is compact. The group Γ acts continuously on $\mathcal{SUB}_\Gamma$ by conjugation.

Assume now that Γ is generated by the finite symmetric set S. Let

$$\mathcal{SCH}_\Gamma(S) = \{\mathrm{Sch}(\Gamma/H, S) \mid H \leq \Gamma\}$$

be the set of all connected Schreier graphs of Γ with respect to S. We endow $\mathcal{SCH}_\Gamma(S)$ with the rooted graph distance and topology. The group Γ acts on $\mathcal{SCH}_\Gamma(S)$ by moving the root. One can check that the map

$$H \longmapsto \mathrm{Sch}(\Gamma/H, S)$$

is a homeomorphism between $\mathcal{SUB}_\Gamma$ and $\mathcal{SCH}_\Gamma(S)$ that commutes with the Γ-action. So, these spaces are isomorphic as Γ-spaces.

By an *invariant random subgroup* (IRS) of Γ, we mean a Γ-invariant Borel probability distribution on $\mathcal{SUB}_\Gamma$. Using the above correspondance, for finitely generated groups, this is the same as a unimodular random Schreier graph of Γ, that is, a Borel probability distribution on $\mathcal{SCH}_\Gamma(S)$ that is invariant under moving the root in any S-direction.

Trivial examples for IRS-es are Dirac measures on normal subgroups and random conjugates of a finite index subgroup (or more generally a subgroup with normalizer of finite index in Γ). A general way to obtain an invariant random subgroup is to take the stabilizer of a random point in a measure preserving action of Γ on a Borel probability space. Our first result (joint with Yair Glasner and Bálint Virág, [1]) says that this is basically the only way.

Proposition 1. *Let H be an invariant random subgroup of the finitely generated group Γ. Then there exists a measure preserving action of Γ on a Borel probability space (X, μ) such that H is the stabilizer of a μ-random point of X in Γ.*

In the talk we discussed various properties of IRS-es and how to get them. The general theme is that IRS-s in many sense behave like normal subgroups. This is in accordance to the fact that unimodular random graphs tend to behave like vertex transitive graphs in many senses.

A major result in [1] is that Kesten's theorem holds for IRS-es. Let $\rho(G)$ denote the norm of the Markov operator on the connected regular infinite graph G. This can also be expressed as the exponent of the probabilities of returns for the simple random walk on G. When G is finite, then let $\rho(G)$ denote the norm of the Markov operator after removing the trivial eigenvalues ± 1.

Theorem 2. *Let Γ be a group generated by a finite symmetric set S and let H be an invariant random subgroup of Γ such that H has infinite index in Γ a.s. Then the following are equivalent:*
1) $\rho(\mathrm{Sch}(\Gamma/H, S)) = \rho(\mathrm{Cay}(\Gamma, S))$ *a.s.*
2) H is amenable a.s.

This has been proved by Kesten for normal subgroups in [3] and [4]. When applying this result to the free group $\Gamma = F_k$, we get that $\rho(\mathrm{Sch}(\Gamma/H, S)) > \rho(\mathrm{Cay}(\Gamma, S))$ for any IRS H in Γ, as free groups do not admit any nontrivial amenable IRS-es. More is true, as one can forget about groups and just look at graphs. A graph G is Ramanujan if $\rho(G) \leq \rho(T_d)$ where T_d is the d-regular tree.

Theorem 3. *Let G be an infinite d-regular unimodular random graph that is Ramanujan a.s. Then $G = T_d$ a.s.*

This has the following consequence on finite Ramanujan graphs in [1]. A sequence (G_n) of graphs has *essentially large girth*, if for all L, we have

$$\lim_{n\to\infty} \frac{c_L(G_n)}{|G_n|} = 0$$

where $c_L(G_n)$ denotes the number of cycles of length L in G.

Theorem 4. *Let (G_n) be a Ramanujan sequence of finite d-regular graphs. Then (G_n) has essentially large girth.*

One can also analyze IRS-es of locally compact groups. Let G be a simple Lie group with $\mathbb{R}$-rank at least two and let $X = G/K$ be the associated symmetric space. An X-manifold is a complete Riemannian manifold locally isomorphic to X, i.e. a manifold of the form $M = \Gamma\backslash X$ where $\Gamma \leq G$ is a discrete torsion free subgroup.

A theorem of Stuck and Zimmer [5] quickly implies that the only ergodic IRS-es in G are random conjugates of lattices in G. Using the theory of graph limits modified to Riemannian manifolds, this leads to the following theorem, that is joint with Nicolas Bergeron, Ian Biringer, Tsachik Gelander, Nikolay Nikolov, Jean Raimbault, and Iddo Samet [2].

Theorem 5. *Let G be a simple Lie group with $\mathbb{R}$-rank at least two. Then for every sequence (M_n) of distinct, closed X-manifolds and $r > 0$ we have*

$$\lim_{n\to\infty} \frac{\mathrm{vol}((M_n)_{<r})}{\mathrm{vol}(M_n)} = 0$$

where $M_{<r}$ denotes the set of points in M where the local injectivity radius is less than r.

We denote by $b_k(M)$ the k^{th} Betti number of M and by $\beta_k(X)$ the k^{th} L^2-Betti number of X. The last theorem now leads to a uniform Lück approximation theorem as follows [2].

Theorem 6. *Let (M_n) be a sequence of closed X-manifolds with injectivity radius uniformly bounded away from 0, and $\mathrm{vol}(M_n) \to \infty$. Then:*

$$\lim_{n\to\infty} \frac{b_k(M_n)}{\mathrm{vol}(M_n)} = \beta_k(X)$$

for $0 \le k \le \dim(X)$.

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Finite-dimensional approximations of discrete groups, analytic point of view

Narutaka Ozawa

In this introductory lecture, I surveyed about various approximation properties of (countable) discrete groups. Let Γ be a countable discrete group. Since we are fond of linear structure, we consider the complex group ring $\mathbb{C}\Gamma$. Further, let $\mathbb{C}\Gamma$ acts on the Hilbert space $\ell_2\Gamma$ as left convolution operators: $\lambda(f)\xi = f * \xi$ for $f \in \mathbb{C}$ and $\xi \in \ell_\Gamma$, and upgrade it to an *operator algebra* by completing $\lambda(\mathbb{C}\Gamma)$ in the Banach space $\mathbb{B}(\ell_2\Gamma)$ of bounded linear operators on $\ell_2\Gamma$. Because of completion, we are able to apply analytic methods to the study of the group Γ. There are two standard topologies on $\mathbb{B}(\ell_2\Gamma)$ to consider: the norm topology (the topology of uniform convergence on bounded subsets) and the strong operator topology (the topology of pointwise convergence). Depending on which topology we use, we obtain the *reduced group* C^**-algebra* $\mathrm{C}^*_\lambda\Gamma$ or the *group von Neumann algebra* $L\Gamma$. When $\Gamma = \mathbb{Z}$, the Fourier transform identifies $\ell_2\mathbb{Z}$ with $L^2(\mathbb{T})$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the Pontrjagin dual of $\mathbb{Z}$. Under this identification, the convolution operator $\lambda(f)$ becomes a pointwise multiplication operator $\widehat{f}$ on

$L^2(\mathbb{T})$. Thus, the reduced group C*-algebra $C^*_\lambda\mathbb{Z}$ of $\mathbb{Z}$ is isomorphic to the C*-algebra $C(\mathbb{T})$ of all continuous functions on $\mathbb{T}$, and the group von Neumann algebra $L\mathbb{Z}$ is isomorphic to the L^∞-function algebra $L^\infty(\mathbb{T})$. (The same thing holds for general Abelian groups.) One of the most basic tool in the study of harmonic analysis is Fejér's theorem that for $f \sim \sum_{k=-\infty}^{\infty} c_k z^k$ on the torus $\mathbb{T}$, the Cesáro mean $\frac{1}{n}\sum_{m=0}^{n-1}\sum_{k=-m}^{m} c_k z_k$ converges to f. More precisely, for $\phi_n(k) = (1-\frac{|k|}{n})\vee 0$ on $\Gamma = \mathbb{Z}$, the *multipliers*

$$m_{\phi_n} : \mathbb{C}\Gamma \ni f \mapsto \phi_n f \in \mathbb{C}\Gamma$$

are positive and contractive and converge to the identity on $\lambda(\mathbb{C}\Gamma) \subset \mathbb{B}(\ell_2\Gamma)$. A group Γ is *amenable* if Fejér's theorem holds for Γ, namely if there is a sequence (ϕ_n) of finitely supported functions of positive type which converges to 1 pointwise on Γ. The class of amenable groups contains all finite and Abelian groups and is closed under taking subgroups, quotients, extensions, and inductive limits. Amenability has many important consequences; e.g., Higson and Kasparov proved the Baum–Connes conjecture for amenable group. In particular, amenable groups satisfy the Kaplansky's conjecture that if Γ is a torsion-free group, then the group ring $\mathbb{C}\Gamma$ does not have non-trivial idempotents. Actually, Higson and Kasparov proved the Baum–Connes conjecture for all Haagerup groups. A group Γ is said to have the *Haagerup property* if there is a sequence (ϕ_n) of c_0 functions of positive type which converges to 1 pointwise on Γ. Free groups are important examples of Haagerup groups which are not amenable.

There are non-equivariant versions of these approximation properties. A length function on Γ is a non-negative symmetric function ℓ such that $\ell(xy) \le \ell(x)+\ell(y)$. It is proper if the inverse image of any bounded subset is finite. Every countable group Γ has a proper length function ℓ and $d(x,y) = \ell(x^{-1}y)$ is a left invariant proper metric on Γ. All left invariant proper metrics are coarsely equivalent in the sense of Gromov, i.e., $d_1(x_n,y_n) \to \infty$ if and only if $d_2(x_n,y_n) \to \infty$. Thus, a subset $E \subset \Gamma\times\Gamma$ has finite width iff $\{x^{-1}y : (x,y) \in E\}$ is a finite subset. Now, a group Γ is *exact* if there is a sequence (θ_n) of functions on $\Gamma\times\Gamma$ which are supported on subsets of finite width, positive definite and converge to 1 uniformly on subsets of finite width. Thus exactness is a property of the coarse metric space Γ. The c_0 variant of exactness is equivalent to coarse embeddability into a Hilbert space. The class of exact groups is very large: it contains all amenable groups, hyperbolic groups and linear groups, and is closed under taking subgroups, extensions and free products with amalgamation. Exactness also has many important consequences, such as the strong Novikov conjecture.

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Entropy theory for actions of sofic groups

Lewis Bowen

Entropy theory was developed in 1958 by Kolmogorov to answer a question posed by von Neumann more than 30 years prior: is the 2-shift (over $\mathbb{Z}$) isomorphic to the 3-shift? To be precise, let us recall the definition of a Bernoulli shift. Let (K, κ) be a standard probability space and G a countable group. Then $(K, \kappa)^G$ is the direct product of G copies of (K, κ). G acts on this space by "shifting": $(gx)(f) = x(g^{-1}f)$ for $x \in K^G$ and $g, f \in G$. The dynamical system $G \curvearrowright (K, \kappa)^G$ is called the *Bernoulli shift* over G with base space (K, κ). In the special case in which K is an n-element set and κ is the uniform distribution, this is called the *full n-shift*.

Von Neumann knew that all Bernoulli shifts over $\mathbb{Z}$ are spectrally isomorphic. Since spectral invariants were the only known invariant in measurable dynamics at the time, it was natural to ask whether the 2-shift could be isomorphic to the 3-shift. Kolmogorov's entropy is a real number invariant that, as proven later by Ornstein [Or70], completely classifies Bernoulli shifts over $\mathbb{Z}$. In the seventies and eighties, researchers extended entropy theory to actions of amenable groups G. This activity culminated in the Ornstein-Weiss landmark paper [OW87] which proved, among other things, that Bernoulli shifts over G are completely classified by entropy whenever G is amenable.

This left open the non-amenable case. However, Ornstein-Weiss presented a strange example: if G is the rank 2 free group, then the full 2-shift over G factors onto the full 4-shift over G. This is unusual because it violates basic axioms of entropy theory: (1) entropy does not increase under factor maps, (2) the full n shift has entropy $\log(n)$. It was widely believed, because of this example, that entropy theory can not be extended to free groups.

The Ornstein-Weiss example is very curious and it seems natural to wonder if it extends to other non-amenable groups and other Bernoulli shifts. I conjecture that in fact: if G is any non-amenable group then every Bernoulli shift over G factors onto every Bernoulli shift over G. This has been proven when G contains a rank 2 free subgroup [Bo11a]. It is also known that if G is any non-amenable group then there is some $m > 0$ such that the m-shift factors onto every Bernoulli shift [Ba05].

The main new result is that entropy theory extends to sofic groups, a large class of groups which contains many non-amenable groups such as free groups. Roughly speaking, a group is sofic if it admits an asymptotically free sequence of "partial" actions on finite sets. Here is the precise definition.

Definition 1. *Let G be a countable group. For $m \geq 1$, let $Sym(m)$ denote the full symmetric group on $\{1, \ldots, m\}$. Let $\sigma : G \to Sym(m)$ be a map. σ is not assumed to be a homomorphism! For $F \subset G$, let $V_\sigma(F) \subset \{1, \ldots, m\}$ be the set of all elements v such that for all $f_1, f_2 \in F$,*

$$\sigma(f_1)\sigma(f_2)v = \sigma(f_1 f_2)v$$

and $\sigma(f_1)v \neq \sigma(f_2)v$ *if* $f_1 \neq f_2$. σ *is an* (F,ϵ)*-approximation to* G *if* $|V_\sigma(F)| \geq (1-\epsilon)m$.

Let $\Sigma = \{\sigma_i\}_{i=1}^\infty$ *be a sequence of maps* $\sigma_i : G \to Sym(m_i)$. *Then* Σ *is a* **sofic approximation** *to* G *if each* σ_i *is an* (F_i,ϵ_i)*-approximation to* G *for some* (F_i,ϵ_i) *where* $F_i \subset F_{i+1}$ *for all* i, $\cup_i F_i = G$ *and* $\epsilon_i \to 0$ *as* $i \to \infty$. G *is* **sofic** *if there exists a sofic approximation to* G.

Sofic groups were defined implicitly by Gromov in [Gro99] and explicitly by Weiss in [We00]. An almost immediate consequence of the definition is that all residually amenable groups are sofic. In particular, since finitely generated linear groups (i.e., subgroups of $GL_n(F)$ where F is a field) are residually finite (by [Ma40]) they are sofic. It is unknown whether every countable group is sofic.

0.1. **Σ-entropy.** In [Bo10b], I defined for each system (G,X,μ) and sofic approximation $\Sigma = \{\sigma_i\}$ to G a number $h(\Sigma,G,X,\mu)$, called the Σ-entropy of (G,X,μ). Like Kolmogorov's entropy, its definition involves the use of an arbitrary finite-entropy generating partition α. Roughly speaking, $h(\Sigma,G,X,\mu)$ is the exponential rate of growth of the number of the partitions β of $\{1,\dots,m_i\}$ such that the "partial action" σ_i as viewed through β approximates the action (G,X,μ) as viewed through α. Here is the precise definition.

Definition 2. *Fix a map* $\sigma : G \to Sym(m)$. *Fix a partition* $\alpha = (A_1,\dots,A_u)$ *of* X *and a partition* $\beta = (B_1,\dots,B_u)$ *of* $\{1,\dots,m\}$. *Given a finite set* $F \subset G$ *and a function* $\phi : F \to \{1,\dots,u\}$, *define the atoms* $A_\phi \in \bigvee_{f\in F} f\alpha$ *and* $B_\phi \in \bigvee_{f\in F}\sigma(f)\beta$ *by*

$$A_\phi = \bigcap_{f\in F} fA_{\phi(f)},\ B_\phi = \bigcap_{f\in F} fB_{\phi(f)}.$$

The F*-distance between* α *and* β *is:*

$$d_F(\alpha,\beta) := \sum_{\phi:F\to\{1,\dots,u\}} \big|\mu(A_\phi) - \zeta(B_\phi)\big|$$

where ζ *is the uniform probability measure on* $\{1,\dots,m\}$.

Let $\mathcal{AP}(\sigma,\alpha : F,\epsilon)$ *be the set of all partitions* $\beta = (B_1,\dots,B_u)$ *of* $\{1,\dots,m\}$ *such that* $d_F(\alpha,\beta) \leq \epsilon$. *Let*

$$H(\Sigma,\alpha : F) = \lim_{\epsilon\to 0}\limsup_{i\to\infty}\frac{1}{m_i}\log\big|\mathcal{AP}(\sigma_i,\alpha : F,\epsilon)\big|, \tag{1}$$

$$h(\Sigma,\alpha) = \inf_{F\subset G} H(\Sigma,\alpha : F). \tag{2}$$

In [Bo10b], I showed that if α and β are finite-entropy generating partitions for (G,X,μ) then $h(\Sigma,\alpha) = h(\Sigma,\beta)$. This common number is the Σ-entropy of the system, denoted $h(\Sigma,G,X,\mu)$. A calculation in [Bo10b] shows that the Σ-entropy of a Bernoulli shift equals its base measure entropy. This proves that Bernoulli shifts over a countable sofic Ornstein group G are classified by base measure entropy i.e., two Bernoulli shifts over G are isomorphic if and only if they have the same base measure entropy. For example, every countably infinite linear group is sofic and Ornstein. It can be proven that if G is amenable, then the Σ-entropy equals the classical definition of entropy regardless of the choice of Σ. But if G is nonamenable then there are explicit examples showing that $h(\Sigma, G, X, \mu)$ can depend on Σ. Explicit computations of entropy are possible for a large class of algebraic actions [Bo11b].

For free groups $G = \langle s_1, \ldots, s_r \rangle$ there is an alternative to sofic entropy defined as follows. If $G \curvearrowright (X, \mu)$ and α is a finite partition of X then let

$$F(\alpha) = -(2r-1)H(\alpha) + \sum_{i=1}^{r} H(\alpha \vee s_i\alpha), \quad f(\alpha) = \inf_n F(\alpha^{B(e,n)}).$$

If α, β are generating partitions then $f(\alpha) = f(\beta)$ (by [Bo10a]), which allows us to define $f(G \curvearrowright (X, \mu)) = f(\alpha)$ for any generating partition α. This alternative notion of entropy is related to sofic entropy as follows. In the definition of sofic entropy above, let $\sigma_n : G \to \mathrm{Sym}(n)$ be a homomorphism chosen uniformly at random. Then define $h_*(\alpha)$ analogously to $h(\Sigma, \alpha)$ be replacing $\left|\mathcal{AP}(\sigma_i, \alpha : F, \epsilon)\right|$ with the expected value of $\left|\mathcal{AP}(\sigma_i, \alpha : F, \epsilon)\right|$. It is a theorem that $h_*(\alpha) = f(\alpha)$ [Bo10c].

The f-invariant appears to have some advantages over general sofic entropy. Using Markov chains over a free group, I have shown that there is an analogue of the Abramov-Rohlin formula and Yuzvinskii's addition formula for algebraic actions [Bo10d]. In work in progress, I hope to extend the f-invariant to general sofic groups.

An alternative approach to sofic entropy theory is being developed by David Kerr and Hanfeng Li [KL1, KL2]. This approach is based on operator algebras. There are two main advantages: (1) it is not necessary to require the existence of a finite generating partition, (2) by changing the definition slightly one obtains a topological version of sofic entropy which satisfies the variational principle.

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Approximating Cayley graphs versus Cayley diagrams

Ádám Timár

By a *diagram* we mean a graph with edges oriented and labelled by some finite set of labels. Several people have asked whether approximability of a Cayley graph by a sequence of finite graphs (in the sense of *weak* or *Benjamini-Schramm convergence*) implies the approximability of a corresponding Cayley diagram (and hence soficity of the group). A natural strengthening of the above question (asked by Lyons, see [1]) cannot hold, as shown by our example: we construct a Cayley graph G that can be weakly approximated by a sequence G_n of finite graphs, such that the corresponding Cayley diagram (i.e., G together with orientations of the edges labelled with generators) cannot be approximated by any orientation and labelling of the sequence G_n, [6].

In the heart of the construction is a result by Bollobás [3], showing a sequence of finite graphs that converge to the regular tree, and such that their independence ratio (the density of the largest independent set) does not converge to 1/2. This is actually true for the sequence of random regular graphs almost surely. Related questions and conjectures are discussed. The limit of the independence ratio of random regular graphs is known to exist by recent work of Bayati, Gamarnik and Tetali [2], and it is conjectured that this limit is the same as the supremum of the densities of independence sets on the regular tree that arise as factors (measurable equivariant maps) of i.i.d. Bernoulli labels on the vertices. Questions of similar flavor (but apparently easier to access) are about the existence of perfect matchings as factors of Bernoulli i.i.d. labels. The existence of a perfect matching factor was shown for nonamenable bipartite graphs [4], and for $\mathbb{Z}^d$ $(d > 1)$ [5], but the general case is open.

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L^2approximation and homological growth

Wolfgang Lück

Let G be a group together with an inverse system $\{G_i \mid i \in I\}$ of normal subgroups of G directed by inclusion over the directed set I such that $[G : G_i]$ is finite for all $i \in I$ and $\bigcap_{i\in I} G_i = \{1\}$. Let K be a field. We denote by mg the minimal number of generators, by $\rho^{\mathbb{Z}}$ the integral torsion, by $b_n^{(2)}$ the p-th L^2-Betti number, and by $\rho^{(2)}$ the L^2-torsion. The symbol $\mathcal{N}(G)$ stands for the von Neumann algebra of the group G.

The following two conjectures are motivated by [1, Conjecture 1.3] and [2, Conjecture 11.3 on page 418 and Question 13.52 on page 478].

Conjecture (*Approximation Conjecture for L^2-torsion*)
Let X be a finite connected CW-complex and let $\overline{X} \to X$ be a G-covering.

(1) If the G-CW-structure on $\overline{X}$ and for each $i \in I$ the CW-structure on $G_i\backslash\overline{X}$ come from a given CW-structure on X, then

$$\rho^{(2)}(\overline{X};\mathcal{N}(G)) = \lim_{i\to\infty} \frac{\rho^{(2)}(G_i\backslash\overline{X};\mathcal{N}(\{1\}))}{[G:G_i]};$$

(2) If X is a closed Riemannian manifold and we equip $G_i\backslash\overline{X}$ and $\overline{X}$ with the induced Riemannian metrics, one can replace the torsion in the equality appearing in (1) by the analytic versions;

(3) If $b_n^{(2)}(\overline{X};\mathcal{N}(G))$ vanishes for all $n \geq 0$, then

$$\rho^{(2)}(\overline{X};\mathcal{N}(G)) = \lim_{i\to\infty} \frac{\rho^{\mathbb{Z}}(G_i\backslash\overline{X})}{[G:G_i]}.$$

Conjecture (*Homological growth and L^2-torsion for aspherical closed manifolds*) Let M be an aspherical closed manifold of dimension d and fundamental group $G = \pi_1(M)$. Then

(1) For any natural number n with $2n \neq d$ we have

$$b_n^{(2)}(M;\mathcal{N}(G)) = \lim_{i\to\infty} \frac{b_n(G_i\backslash\widetilde{M};\mathbb{Q})}{[G:G_i]} = 0.$$

If $d = 2n$ is even, we get

$$b_n^{(2)}(M;\mathcal{N}(G)) = \lim_{i\to\infty} \frac{b_n(G_i\backslash\widetilde{M};\mathbb{Q})}{[G:G_i]} = (-1)^n \cdot \chi(M) \geq 0;$$

(2) For any natural number n with $2n + 1 \neq d$ we have

$$\lim_{i\in I} \frac{\ln\left(\left|\mathrm{tors}\left(H_n(G_i\backslash M)\right)\right|\right)}{[G:G_i]} = 0.$$

If $d = 2n+1$, we have

$$\lim_{i \in I} \frac{\ln\left(\left|\operatorname{tors}\left(H_p(G_i\backslash M)\right)\right|\right)}{[G:G_i]} = (-1)^n \cdot \rho^{(2)}\left(M; \mathcal{N}(G)\right) \geq 0.$$

Some evidence for the two conjectures above comes from the following result:

Theorem *(Lück)* Let M be an aspherical closed manifold with fundamental group $G = \pi_1(X)$. Suppose that M carries a non-trivial S^1-action or suppose that G contains a non-trivial elementary amenable normal subgroup. Then we get for all $n \geq 0$

$$\begin{aligned}
\lim_{i\to\infty} \frac{b_n(G_i\backslash\widetilde{M};K)}{[G:G_i]} &= 0;\\
\lim_{i\in I} \frac{\operatorname{mg}\left(H_n(G_i\backslash M)\right)}{[G:G_i]} &= 0;\\
\lim_{i\in I} \frac{\ln\left(\left|\operatorname{tors}\left(H_n(G_i\backslash M)\right)\right|\right)}{[G:G_i]} &= 0;\\
\lim_{i\in I} \frac{\rho^{(2)}\left(G_i\backslash\overline{X};\mathcal{N}(\{1\})\right)}{[G:G_i]} &= 0;\\
\lim_{i\in I} \frac{\rho^{\mathbb{Z}}\left(G_i\backslash\overline{X}\right)}{[G:G_i]} &= 0;\\
b_n^{(2)}(\widetilde{M};\mathcal{N}(G)) &= 0;\\
\rho^{(2)}(\widetilde{M};\mathcal{N}(G)) &= 0.
\end{aligned}$$

In particular the two conjectures above are true.

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Turing machines and Atiyah problem

Łukasz Grabowski

1. Introduction

In the talk I presented mainly results from my doctoral thesis based on the preprint [Gra10b].

We study countable discrete groups through so called l^2**-Betti numbers**. These were originally introduced by M. Atiyah in [Ati76] to study free cocompact actions of discrete groups on manifolds. Subsequently, they were studied and used in many different context in geometry and group theory (e.g. [Dod77], [CG86], [Gab02]).

For our purposes (see e.g. [Eck00] for equivalence with the original definition of Atiyah) a real number r is said to be an l^2**-Betti number arising from** G if and only if there exists a matrix $T \in M_k(\mathbb{Q}G)$ such that $\dim_{\mathrm{vN}} \ker T = r$, where $\mathbb{Q}G$ is the rational group ring of a discrete group G. Sometimes r will be referred to as the l^2-Betti number of T, denoted by $\beta^2(T)$.

A particular question Atiyah asked in [Ati76] was whether l^2-Betti numbers can be irrational. Since then, various statements about restrictions on possible values of l^2-Betti numbers bear the name *the Atiyah conjecture* (e.g. [DLM$^+$03]). We depart somewhat from this tradition. Given a countable discrete group G, the following question will be referred to as the *Atiyah problem for G*.

Question 1. *What is the set of l^2-Betti numbers arising from G?*

Let us introduce the notation $\mathcal{C}(G)$ for the set in the above question.

Before the work of R. Grigorchuk and A. Żuk in [GŻ01], it had been conjectured that $\mathcal{C}(G) \subset \mathbb{Z}(\frac{1}{a_1}, \frac{1}{a_2}, \ldots)$, where a_i are orders of torsion subgroups in G. However, in [GŻ01] the authors showed that $\dim_{\mathrm{vN}} \ker T = \frac{1}{3}$ for a certain operator T from the group ring of the lamplighter group $\mathbb{Z}_{/2} \wr \mathbb{Z}$. Recall that the latter group is a semi-direct product of $\mathbb{Z}_{/2}{}^{\oplus\mathbb{Z}}$ with $\mathbb{Z}$ with respect to the shift action of $\mathbb{Z}$ on $\mathbb{Z}_{/2}{}^{\oplus\mathbb{Z}}$. In particular torsion subgroups of the lamplighter group have orders which are powers of 2.

Shortly afterwards W. Dicks and T. Schick described in [DS02] an operator T from the group ring of $(\mathbb{Z}_{/2} \wr \mathbb{Z})^2$ and an heuristic evidence on why $\dim_{\mathrm{vN}} \ker T$ is probably irrational. Nonetheless, the question of irrationality of that specific number has remained open.

A breakthrough came in 2009, when T. Austin showed the following theorem.

Theorem 1 ([Aus09])**.** *The set of l^2-Betti numbers arising from finitely generated groups is uncountable.*

In particular there exist irrational l^2-Betti numbers. In the talk I focused on the following strengthening of Austin's result.

Theorem 2. *The set of l^2-Betti numbers arising from finitely generated groups is equal to the set of non-negative real numbers.*

Theorem 2 has been independently proven by M. Pichot, T. Schick and A. Żuk in [PS Z10]. Let us also mention two later developments: (1) In [LW10], F. Lehner and S. Wagner show that $\mathcal{C}(\mathbb{Z}_{/p} \wr F_d)$ contains irrational algebraic numbers, where F_d is the free group on d generators, $d > 2$, $p \geq 2d - 1$; (2) In [Gra10a] the present author shows that $\mathcal{C}(\mathbb{Z}_{/p} \wr \Gamma)$ contains transcendental numbers, for all $p > 1$ and all groups Γ which contain an element of infinite order.

Using the ideas from the proof one can also deduce the following two results (work in progress).

First, consider the following computational problem. Let a group G be generated by finite symmetric set S which includes the trivial element. Words in the alphabet $S \sqcup \{+, -\}$ give rise to elements of the integral group ring $\mathbb{Z}G$. Therefore we can ask whether there exists a Turing machine which decides whether given element of $\mathbb{Z}G$ (or more precisely the element of $\mathbb{Z}G$ represented by a given word in the alphabet $S \sqcup \{+,-\}$) is a 0-divisor in $\mathbb{Z}G$. Let us call this computational problem the **0-divisor problem for** G.

Theorem 3. *There exists a finite group* H *such that the* 0*-divisor problem for the group* $(\mathbb{Z}_{/2} \wr \mathbb{Z})^6 \times H$ *is undecidable.*

Note that the word problem and the torsion problem (i.e. deciding whether a given word in generators of a group represents a torsion element) are decidable in the group $(\mathbb{Z}_{/2} \wr \mathbb{Z})^6 \times H$.

Second, recall from [Ele02] that if G is amenable and $T \in \mathbb{Z}G$, one can define analogues of the l^2-Betti number of T over finite field by taking the following limit over a Følner sequence F_n:

$$\beta^2(T, \mathbb{F}_p) = \lim_{n\to\infty} \frac{\dim_{\mathbb{F}_p} \ker T}{|F_n|}$$

Theorem 4. *There exists an element* T *in the group ring of a finite extension of the group* $\mathbb{Z}_{/2} \wr \mathbb{Z}$ *whose* l^2*-Betti number is irrational and such that for all pairs of different prime numbers* p *and* q *the numbers* $\beta^2(T, \mathbb{F}_p)$ *and* $\beta^2(T, \mathbb{F}_q)$ *are different rational numbers.*

Detailed proofs of the last two results will appear in a future version of [Gra10a].

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The Atiyah conjecture for groups with torsion

Thomas Schick

(joint work with Anselm Knebusch, Peter Linnell)

Let X be a finite CW-complex, $\tilde{X} \to X$ a normal covering with covering group Γ. We obtain the cellular $\mathbb{Z}\Gamma$-chain complex of $\tilde{X}$. A choice of one cell in each Γ-orbit of cells in $\tilde{X}$ identifies the k-th chain groups with $\mathbb{Z}\Gamma^{n_k}$, where n_k is the number of k-cells in X, and identifies the differential with left multiplication by a matrix d_k over $\mathbb{Z}\Gamma$.

The L^2-Betti numbers of $\tilde{X}$ are now defined as follows: let d_k (from above) act by convolution multiplication on the Hilbert space $l^2(\Gamma)^{n_k}$, this is a bounded operator. Let p_k be the projection onto the kernel of $d_k^* d_k + d_{k+1} d_{k+1}^*$, let $e_j \in l^2(\Gamma)^{n_k}$ be the standard basis element, then $b_k(\tilde{X};\Gamma) = \sum_{j=1}^{n_k} \langle p_k e_j, e_j \rangle_{l^2(\Gamma)^{n_k}}$.

The Atiyah conjecture predicts the possible values of L^2-Betti numbers; in particular it states that these are integers if the group is torsion free, known for many examples of groups (compare e.g. [1, 2, 4, 5]).

In this context, one studies a diagram of ring inclusions as follows:

$$\begin{array}{ccc} \mathbb{Z}\Gamma & \longrightarrow & L\Gamma \\ \downarrow & & \downarrow \\ D\Gamma & \longrightarrow & U\Gamma \end{array}$$

Here, $L\Gamma$ is the group von Neumann algebra, the closure of $\mathbb{C}\Gamma$ in the weak operator topology of bounded operators on $l^2(\Gamma)$, $U\Gamma$ is the algebra of affiliated unbounded operators, i.e.! unbounded operators on $l^2\Gamma$ all whose spectral projections belong to $L\Gamma$. $D\Gamma$ is the division closure of $\mathbb{Z}\Gamma$ in $U\Gamma$, i.e. the smallest intermediate ring which is closed under taking inverses (in $U\Gamma$).

A crucial theorem in this context is due to Linnell [4]: if Γ is torsion-free then it satisfies the Atiyah conjecture if and only if $D\Gamma$ is a skew field.

If Γ is a group with torsion, $D\Gamma$ contains non-trivial zero-divisors.

We propose now a precise conjecture about the structure of $D\Gamma$ in general, for groups with a bound on the order of the finite subgroups: in this case $D\Gamma$ is conjectured to be a semisimple Artinian ring, i.e. a direct sum of matrix rings over skew-fields. Moreover, the structure of the lattice of finite subgroups and the representations of these finite subgroups give a precise recipe for the number of summands in the direct sum decomposition and for the dimension (over the ground skew field) of the matrix ring summands.

On the other hand, we propose a precise version of the Atiyah conjecture for the center valued L^2-Betti numbers: exactly those values should occur which occur from matrices over $\mathbb{Z}\Gamma$ which are contained in the image of the corresponding ring over a finite subgroup, again, the lattice of finite subgroups and their representations allow for a precise description of the values. As a special case: if Γ contains no finite normal subgroup then the center-valued L^2-Betti numbers coincide with the ordinary L^2-Betti numbers, and if l is the least common multiple of orders of finite subgroups, then the L^2-Betti numbers should be contained in the additive group $\frac{1}{l}\mathbb{Z}$. In this context, the conjecture for $D\Gamma$ states that it is a single $l \times l$-matrix ring over a skew field.

The main result now is that the two conjectures for groups with bounded torsion are equivalent. Moreover, we show that the conjectures are true essentially for all groups for which the classical Atiyah conjecture is true, as well. Most of these statements are proved in [3]. Preliminary results, in particular for the case where Γ has no non-trivial finite normal subgroups are contained in [6].

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Connes's Embeddings and the Structural Lück Approximation Problem

Gábor Élek

Let $\mathbb{C}\,\Gamma$ be the complex group algebra of a countable group Γ. We consider the embedding $i : \mathbb{C}\,\Gamma \to \mathcal{U}\Gamma$, where the $\mathcal{U}\Gamma$ is the algebra of affiliated operators, that is the Ore Localization of the group von Neumann algebra $\mathcal{N}\Gamma$. The algebra $\mathcal{U}\Gamma$ is a (von Neumann) regular $\star$-algebra equipped with a natural rank function [5]. Linnell and Schick [4] defined the regular closure of $\mathbb{C}\,\Gamma$, the minimal $\star$-regular algebra $\mathbb{R}\Gamma$ in $\mathcal{U}\Gamma$ containing $\mathbb{C}\,\Gamma$. They proved e.g. that the regular closure is a skew field if the Strong Atiyah Conjecture holds for Γ.

In order to define a regular closure one only needs an embedding of $\mathbb{C}\,\Gamma$ into a proper, $\star$-regular ring, not necessarily into the algebra of affiliated operators. Let $\mu = \{d_1 < d_2 < \dots\}$ be an infinite sequence of positive integers. Then one can consider the ultraproduct of the matrix algebras $\{\mathrm{Mat}_{d_i\times d_i}(\mathbb{C})\}_{i=1}^{\infty}$ as tracial algebras the following way (see [3]).

Let ω be a nonprincipal ultrafilter on the natural numbers and let $\lim_\omega$ be the corresponding ultralimit. First, consider the algebra of bounded elements

$$\mathcal{B} = \{(a_1, a_2, \dots) \in \prod_{i=1}^{\infty} \mathrm{Mat}_{d_i\times d_i}(\mathbb{C}) \mid \sup \|a_i\| < \infty\}.$$

Now let $\mathcal{I} \lhd \mathcal{B}$ be the ideal of elements $\{a_i\}_{i=1}^{\infty}$ such that $\lim_\omega \frac{tr(a_n^* a_n)}{d_n} = 0$. Then $\mathcal{B}/\mathcal{I} = \mathcal{M}_\mu$ is a type II_1-von Neumann factor with trace defined by

$$Tr_\omega[\{a_i\}_{i=1}^{\infty}] = \lim_\omega \frac{tr(a_n)}{d_n}.$$

The Ore Localization of $\mathcal{M}_\mu$ is again a proper $\star$-regular ring $\mathcal{U}(\mathcal{M}_\mu)$.
There is a purely algebraic version of the Connes Embedding Problem first considered in [2]. Namely, we can consider the ultraproduct of the matrix rings $\{\mathrm{Mat}_{d_i\times d_i}(\mathbb{C})\}$ as *rank algebras.*

Definition 1. *Let $\mathcal{J} \lhd \prod_{i=1}^{\infty} Mat_{d_i\times d_i}(\mathbb{C})$ be the following ideal,*

$$\mathcal{J} = \{\{a_i\}_{i=1}^{\infty} \mid lim_\omega \frac{rank(a_i)}{d_i} = 0\}.$$

Then $\prod_{i=1}^{\infty} Mat_{d_i\times d_i}(\mathbb{C})/\mathcal{J} = \mathcal{M}_\mu^{alg}$ is the ultraproduct of the matrix rings $\{Mat_{d_i\times d_i}(\mathbb{C})\}$.

$\mathcal{M}_\mu^{alg}$ is a simple complete $\star$-regular rank ring (Proposition 3.3 [2]).
If the group Γ is *sofic*, then its sofic representation defines an embedding

$$\phi : \mathbb{C}\Gamma \to \mathcal{M}_\mu^{alg}$$

[1].

Question 1 (The Structural Lück Approximation Conjecture)**.** *For any sofic approximation of a sofic group Γ the regular closure in $\mathcal{M}_\mu^{alg}$ is isomorphic (as a rank regular $\star$-ring) to the Linnel-Schick regular closure.*

Note that the Structural Lück Approximation Conjecture immediately implies the classical (numerical) Lück Approximation Conjecture. [1] We proved that the answer for this question is yes, if Γ is amenable. Also, if Γ is a free group, the conjecture still holds for the regular closure of the rational group algebra.

Question 2. *Is it true, that the rank completion of the regular closure of a group is either*

- *A skewfield*
- *A matrix ring of a skew field*
- *The von Neumann simple complete ring of a skew field.*
- *Direct product of some of the above algebras.*

Let α be a measure-preserving action of a countable group Γ on a probability measure space. Let A^α be the rank completion of the regular closure of its crossed product ring. How much does A^α "see" from the action ? For amenable group actions this ring seems to be always the von Neumann simple complete ring over the complex numbers. Is it possible that the completion of the regular closure is always the same algebra for certain non-amenable groups as well ? This would explain, why certain parameters of an action such as the cost, at least for certain groups, depend only on the group not on the action itself.

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Ultrapowers of II_1 factors

Ilijas Farah

In this talk I described some recent applications of model theory for metric structures, as introduced in [1] and adjusted to applications in operator algebras in [4], to ultrapowers of tracial von Neumann algebras.

Recall that if M is a von Neumann algebra with continuous normalized trace τ then on M we can define the ℓ_2-norm by $\|a\|_2 = \sqrt{\tau(a^*a)}$. This turns M into a pre-Hilbert space corresponding to the GNS representation given by τ (see e.g., [2]).

Given a nonrpincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ (and all ultrafilters in this abstract are assumed to be such) one can define the ultrapower of M, $M^{\mathcal{U}}$, as follows. Let $c_{\mathcal{U}}(M) = \{(x_n) \in L^\infty(M) : \lim_{n\to\mathcal{U}} \|x_n\|_2 = 0\}$ and

$$M^{\mathcal{U}} = L^\infty(M)/c_{\mathcal{U}}.$$

It is a classical result that $M^{\mathcal{U}}$ is again a tracial von Neumann algebra. Ultraproducts can also be defined in other categories (cf. Thomas's and Élek's abstracts in this workshop). Note the following simple facts:

(1) All ultrapowers of of $M_2(\mathbb{C})$ (with respect to its normalized trace) are isomorphic to $M_2(\mathbb{C})$.

(2) All ultrapowers of the (separable) Hilbert space $\ell_2(\aleph_0)$ are isomorphic to $\ell_2(2^{\aleph_0})$

(3) All ultrapowers of the Lebesgue measure algebra are isomorphic. This is a consequence of Maharam's characterization of homogeneous measure algebras as algebras corresponding to the product measure on 2^κ for some κ (see e.g., [6]).

(4) All ultrapowers of Abelian tracial atomless von Neumann algebras are isomorphic. This is a reformulation of (3), since categories of Abelian tracial von Neumann algebras and measure algebras are equivalent.

The question of whether all ultrapowers of a given II_1 factor (e.g., the hyperfinite II_1 factor) are isomorphic has a more interesting answer. Assuming the Continuum Hypothesis (CH), all ultrapowers of a fixed II_1 factor are isomorphic. This is a variant of a classical result proved by Keisler in the setting of discrete structures (see [7], [4]).

The following was proved in [3] and [5] (recall that all ultrapowers in this abstract are taken with respect to nonprincipal ultrafilters on $\mathbb{N}$).

Theorem 1. *Assume Continuum Hypothesis fails. Then every separable II_1 factor M has $2^{2^{\aleph_0}}$ nonisomorphic ultrapowers. If M moreover has property Γ then it has $2^{2^{\aleph_0}}$ nonisomorphic relative commutants in its ultrapowers.*

A dichotomy theorem, [4, Theorem 5.5], gives a characterization of those structures that can have nonisomorphic ultrapowers. Those are exactly models whose theories have the so-called *Order Property* of Shelah (see [10]). Quantitative improvement of this theorem given in [5] implies that every separable structure with two nonisomorphic ultrapowers has $2^{2^{\aleph_0}}$ nonisomorphic ultrapowers.

Order property was also used to sketch a purely model-theoretic proof of a slight improvement of a recent result of Ozawa [9] and Nicoara–Popa–Sasyk [8]. In a joint work with B. Hart (unpublished) we have constructed a family of $2^{\aleph_0}$ separable II_1 factors $\{M_\xi\}$ such that (i) every separable II_1 factor has at most countably many of M_ξ as subfactors and (ii) each M_ξ has an ultrapower isomorphic to an ultrapower of a II_1 factor. (In (ii) one has either to assume CH or allow ultrafilters on $2^{\aleph_0}$.)

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Survey on weak amenability

Narutaka Ozawa

In this lecture, I surveyed about weak amenability for groups.

Theorem (Grothendieck, Haagerup, Bożejko–Fendler). *For a function* ϕ *on a discrete group* Γ *and* $C > 0$*, the following are equivalent.*

(1) *The multiplier*

$$m_\phi \colon \mathrm{C}^*_\lambda\Gamma \ni \lambda(f) \mapsto \lambda(\phi f) \in \mathrm{C}^*_\lambda\Gamma$$

is completely bounded and $\|m_\phi\|_{\mathrm{cb}} \leq C$.

(2) *The Schur multiplier*

$$M_\phi \colon \mathbb{B}(\ell_2\Gamma) \ni [A_{x,y}]_{x,y\in\Gamma} \mapsto [\phi(x^{-1}y)A_{x,y}]_{x,y\in\Gamma} \in \mathbb{B}(\ell_2\Gamma)$$

is bounded and $\|M_\phi\| \leq C$.

(3) *There are Hilbert space valued functions* $\xi, \eta \in \ell_\infty(\Gamma, \mathcal{H})$ *such that*

$$\|\xi\|_\infty \|\eta\|_\infty \leq C \quad \text{and} \quad \phi(y^{-1}x) = \langle \xi(x), \eta(y) \rangle.$$

We say ϕ is a *Herz–Schur multiplier* if the above condition holds and set $\|\phi\|_{\mathrm{cb}} = \|M_\phi\|$. Note that if ϕ is a positive type function, then $\|\phi\|_{\mathrm{cb}} = \phi(1)$. A group Γ is *weakly amenable* if there is a sequence ϕ_n of finitely supported functions on Γ such that $\phi_n \to 1$ and $C := \limsup \|\phi_n\|_{\mathrm{cb}} < +\infty$. We denote the best possible C by $\Lambda_{\mathrm{cb}}(\Gamma)$ and call it the Cowling–Haagerup constant of Γ. The definition for locally compact groups is similar, and a lattice in a locally compact group is weakly amenable iff the ambient group is weakly amenable and their Cowling–Haagerup constants coincide. Moreover, Λ_{cb} is an ME invariant. (One can induce Herz–Schur multipliers through measurable cocycles.) Weak amenability is handy in the study of noncommutative harmonic analysis and von Neumann algebras. For instance, it is an open problem whether every bounded left convolution operator $\lambda_p(f)$ on $L^p(G)$, defined by $\lambda_p(f)\xi = f * \xi$, is in the closure (in the strong operator topology) of the space $\lambda_p(\mathbb{C}\Gamma)$ of finitely supported left convolution operators. (This is a theorem of Murray and von Neumann in the case $p = 2$.) Since Herz–Schur multipliers are also multipliers on $\mathbb{B}(L^p(G))$ for all p, the problem has an affirmative solution if G is weakly amenable (or has the weaker property AP). The classical results of Haagerup and his coauthors are summarized as follows.

Theorem (De Cannière–Haagerup, Cowling, Co.–Ha., Ha. 80s). *For a simple connected Lie group* G*, one has*

$$\Lambda_{\mathrm{cb}}(G) = \begin{cases} 1 & \text{if } G = \mathrm{SO}(1,n) \text{ or } \mathrm{SU}(1,n), \\ 2n-1 & \text{if } G = \mathrm{Sp}(1,n) \text{ or } F_{4(-20)} \text{ with } n = 11, \\ +\infty & \text{if } \mathrm{rk}_{\mathbb{R}}(G) \geq 2, \text{ e.g., } G = \mathrm{PSL}(n \geq 3, \mathbb{R}). \end{cases}$$

It is curious that weak amenability separates rank one from rank ≥ 2, and $\Lambda_{\mathrm{cb}} = 1$ iff it has Haagerup's property. Recently, I was able to give Lie group theory free proofs to the above theorem, although they are not as qualitative.

Theorem (Oz. 2007, Oz.–Popa 2007, Oz. 2010)**.**

(1) *Every hyperbolic group is weakly amenable.*
(2) *If G is weakly amenable and N is an amenable closed normal subgroup of G, then there is an N-invariant mean on N which is invariant under the conjugation by G.*

It follows that the semi-direct product $\mathrm{SL}(2, K) \ltimes K^2$ is not weakly amenable for any local field K, nor the wreath product $A \wr \Gamma$ with Γ non-amenable. Note that $A \wr \Gamma$ has Haagerup's property whenever both A and Γ have it (Cornulier–Stalder–Valette). I talked the outline of the proof of the above theorem, and proved some simple cases.

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Weakly-sofic groups and the profinite topology on free groups

Luis Manuel Rivera
(joint work with Lev Glebsky)

Sofic groups were introduced by M. Gromov in relation with the Gottschalk's surjunctivity Conjecture (the name was coined by B. Weiss). In the last years, several results has showed the importance of knowing that a given group is sofic (see, for example, the surveys [3], [5]). It remains an open problem if there exists a non-sofic group. The aim of this talk is to present the definition of weakly sofic groups (w-sofic groups) [2]. This definition is a natural extension of the definition of sofic groups, where instead of the normalized Hamming metric on symmetric groups we use general bi-invariant metrics on finite groups. Using the concept of metric approximation of groups [6], a group is called weakly sofic if and only if it has the $\mathcal{F}$-approximation property, where $\mathcal{F}$ is the class of all finite groups with an invariant length function.

We show that the existence of non w-sofic groups is equivalent to a conjecture about the closure of products of conjugacy classes in profinite topology. Let us give a definition of sofic groups that is equivalent to the more widely used [5]. First, we need the following

Definition 1. *Let H be a finite group with a bi-invariant metric d. Let G be a group, $A \subseteq G$ be a finite subset, $\epsilon > 0$, and $\alpha > 0$. A map $\phi : A \to H$ is said to be a (Φ, ϵ, α)-homomorphism if:*

(1) *For any two elements* $a, b \in A$*, with* $ab \in A$*,* $d(\phi(a)\phi(b), \phi(ab)) < \epsilon$
(2) *If* $e_G \in \Phi$*, then* $\phi(e_G) = e_H$
(3) *For any* $a \neq e_G$*,* $d(\phi(a), e_H) > \alpha$

Definition 2. *The group* G *is sofic if there exists* $\alpha > 0$ *such that for any finite set* $A \subseteq G$*, for any* $\epsilon > 0$ *there exists a* (A, ϵ, α)*-homomorphism to a symmetric group* S_n *with the normalized Hamming metric.*

This previous definition appeals to the following possible generalization

Definition 3. *A group* G *is called weakly sofic if there exists* $\alpha > 0$ *such that for any finite set* $A \subset G$*, for any* $\epsilon > 0$ *there exists a finite group* H *with a bi-invariant metric* d *and a* (A, ϵ, α)*-homomorphism to* (H, d)*.*

Note that in previous definition we do not ask the metric to be normalized. Therefore, α may be any fixed positive number. It is easy to see, that any sofic group is weakly-sofic. We don not know examples of non-weakly-sofic groups, and we posted the following

Conjecture 1. *There exists a non w-sofic group.*

The aim of this talk is to present the proof that previous conjecture is equivalent to one conjecture about the closure of products of conjugacy classes in the profinite topology on free groups. Before the formulation of such a conjecture, we remember some basics topics about profinite topology. The profinite topology of a free group G is formed by taking as a basis of neighbourhoods of the identity, the collection of all normal subgroups of finite index in G. This topology was first defined for M. Hall who also proved that any finitely generated subgroup of a finitely generated free group $F(P)$ is closed in the profinite topology. Another remarkable result is that the product of a finite number of finitely generated subgroups of $F(P)$ is closed in the profinite topology due to L. Ribes and P. A. Zalesskii, [4]).

Now, we present a second conjecture: Let F be a free group and $X \subseteq F$. By $\overline{X}$ let us denote the closure of X in the profinite topology on F. Let $[g]^G$ denote the conjugacy class of g in G.

Conjecture 2. *For a finitely generated free group* F*, there exists a sequence* $g_1, g_2, ..., g_k \in F$ *such that*

$$\overline{[g_1]^F[g_2]^F...[g_k]^F} \not\subseteq N(g_1, ..., g_k),$$

where $N(g_1, ..., g_k)$ denotes the normal subgroup generated by $g_1, ..., g_k$. The main result in [2] is the following

Theorem 1. *Let* F *be a finitely generated free group and* $N \lhd F$*. Then* F/N *is w-sofic if and only if for any finite sequence* $g_1, g_2, ..., g_k$ *from* N *one has* $\overline{[g_1]^F[g_2]^F...[g_k]^F} \subseteq N$*.*

This theorem implies the following

Corollary 1. *Conjecture 1 and Conjecture 2 are equivalent.*

That shows the relation between w-sofic group and the closure of finite products of conjugacy classes in the profinite topology on free groups.

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Cellular automata, surjunctivity, and sofic groups

Tullio Ceccherini-Silberstein and Michel Coornaert

The G-shift on A^G. Let G be a group and let A be a set (called the *alphabet*). The set $A^G = \{x \colon G \to A\} = \prod_{g \in G} A$ is endowed with its *prodiscrete topology*, i.e., the product topology obtained by taking the discrete topology on each factor A of A^G. Thus, if $x \in A^G$, a base of open neighborhoods of x is provided by the sets $V(x,\Omega) := \{y \in A^G : x|_\Omega = y|_\Omega\}$, where Ω runs over all finite subsets of G and $x|_\Omega \in A^\Omega$ denotes the restriction of $x \in A^G$ to Ω. The space A^G, which is called the space of *configurations*, is Hausdorff and totally disconnected, and it is compact if and only if the alphabet A is finite. There is a natural continuous left action of G on A^G given by $gx(h) = x(g^{-1}h)$ for all $g, h \in G$ and $x \in A^G$. This action is called the *G-shift* on A^G.

Cellular automata. Let G be a group and A and B two alphabet sets. A map $\tau \colon A^G \to B^G$ is called a *cellular automaton* if there exist a finite subset $M \subset G$ and a map $\mu_M \colon A^M \to B$ such that $\tau(x)(g) = \mu_M((g^{-1}x)|_M)$ for all $x \in A^G$ and $g \in G$. Such a set M is called a *memory set* and the map $\mu_M \colon A^M \to B$ is called the associated *local defining map.*

We shall be mostly interested in the case where $A = B$.

Example (The majority action on $\mathbb{Z}$). Take $G = \mathbb{Z}$, $A = \{0,1\}$, $M = \{-1,0,1\}$ and $\mu_M : A^M \equiv A^3 \to A$ defined by $\mu_M(a_{-1}, a_0, a_1) = 1$ if $a_{-1} + a_0 + a_1 \geq 2$ and $\mu_M(a_{-1}, a_0, a_1) = 0$ otherwise. Note that the corresponding cellular automaton $\tau \colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is surjective but not injective.

Example (Hedlund's marker). Let $G = \mathbb{Z}$, $A = \{0,1\}$, $M = \{-1,0,1,2\}$ and $\mu_M \colon A^M \equiv A^4 \to A$ defined by $\mu_M(a_{-1},a_0,a_1,a_2) = 1 - a_0$ if $(a_{-1},a_1,a_2) = (0,1,0)$ and $\mu_M(a_{-1},a_0,a_1,a_2) = a_0$ otherwise. The corresponding cellular automaton $\tau \colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a nontrivial involution of A^G.

Example (Conway's Game of Life). Let $G = \mathbb{Z}^2$, $A = \{0,1\}$, $M = \{-1,0,1\}^2 \subset \mathbb{Z}^2$ and $\mu_M \colon A^M \to A$ given by

$$\mu_M(y) = \begin{cases} 1 & \text{if} \begin{cases} \displaystyle\sum_{m\in M} y(m) = 3 \\ \text{or} \displaystyle\sum_{m\in M} y(m) = 4 \text{ and } y((0,0)) = 1 \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

for all $y \in A^M$. The corresponding cellular automaton $\tau \colon A^G \to A^G$ describes the *Game of Life* due to J.H. Conway. Note that τ is not injective. It can be shown that τ is not surjective either.

The generalized Curtis-Hedlund theorem. It easily follows from the definition that every cellular automaton $\tau \colon A^G \to A^G$ is G-equivariant, i.e., $\tau(gx) = g\tau(x)$ for all $g \in G$ and $x \in A^G$, and continuous (w.r. to the prodiscrete topology on A^G). The converse is also true in the finite alphabet case (the *Curtis-Hedlund theorem* [9]; see also [3, Theorem 1.8.1]). When A is infinite and the group G is non-periodic, one can always construct a G-equivariant continuous self-mapping of A^G which is not a cellular automaton (see [4] and [3, Example 1.8.2]). More generally, if we equip A^G with its *prodiscrete uniform structure*, that is with the uniform structure admitting the sets $W(\Omega) := \{(x,y) \in A^G \times A^G : x|_\Omega = y|_\Omega\}$, where $\Omega \subset G$ runs over all finites subsets of G, as a base of entourages, we have the following extension of the Curtis-Hedlund theorem:

Theorem ([4]; see also Theorem 1.9.1 in [3])**.** *Let G be a group and let A be a set. Then a map $\tau \colon A^G \to A^G$ is a cellular automaton if and only if it is uniformly continuous and G-equivariant.*

Linear cellular automata. Let $\mathbb{K}$ be a field and A and B two vector spaces over $\mathbb{K}$. Then A^G and B^G have a natural vector space structure. A cellular automaton $\tau \colon A^G \to B^G$ is called *linear* provided that τ is a $\mathbb{K}$-linear map. If $M \subset G$ is a memory set and $\mu_M \colon A^M \to B$ is the corresponding local defining map, then τ is linear if and only if μ_M is $\mathbb{K}$-linear.

Example (The Discrete Laplacian). Let G be a group, $A = \mathbb{R}$, and $S \subset G$ a nonempty finite subset. The map $\Delta_S \colon \mathbb{R}^G \to \mathbb{R}^G$, defined by $(\Delta_S x)(g) = x(g) - \frac{1}{|S|}\sum_{s\in S} x(gs)$ for all $x \in \mathbb{R}^G$ and $g \in G$, is a linear cellular automaton (with memory set $M = S \cup \{1_G\}$). Note that Δ_S is never injective since all constant configurations belong to its kernel. It can be shown (see [2]) that Δ_S is surjective if and only if the subgroup generated by S is infinite.

Algebraic cellular automata. Let $\mathbb{K}$ be a field. A subset $A \subset \mathbb{K}^m$ is called an *algebraic subset* if there exists a subset $\Sigma \subset \mathbb{K}[t_1, \dots, t_m]$ such that A is the set of common zeroes of the polynomials in Σ, i.e., $A = \{a \in \mathbb{K}^m : P(a) = 0$ for all $P \in \Sigma\}$. Let $A \subset \mathbb{K}^m$ and $B \subset \mathbb{K}^n$ be algebraic subsets. A map $f\colon A \to B$ is called *regular* (or *polynomial*) if f is the restriction of some *polynomial* map $\mathbb{K}^m \to \mathbb{K}^n$, that is, if there exist polynomials $P_1, \dots, P_n \in K[t_1, \dots, t_m]$ such that $f(a) = (P_1(a), \dots, P_n(a))$ for all $a \in A$. The identity map on any algebraic subset is regular and the composite of two regular maps is regular. Thus, the algebraic subsets of $\mathbb{K}^m$, $m = 0, 1, \dots$, are the objects of a category, the category of *affine algebraic sets* over $\mathbb{K}$, whose morphisms are the regular maps. Note that this category admits finite direct products. Indeed, if $A \subset \mathbb{K}^m$ and $B \subset \mathbb{K}^n$ are algebraic subsets then $A \times B \subset \mathbb{K}^m \times \mathbb{K}^n = \mathbb{K}^{m+n}$ is also an algebraic subset.

Let G be a group and A and B two affine algebraic sets over $\mathbb{K}$. One says that a cellular automaton $\tau\colon A^G \to B^G$ is an *algebraic cellular automaton* if for some (or, equivalently, any (cf. [6])) memory set M, the associated local defining map $\mu_M\colon A^M \to B$ is regular.

Example. The map $\tau\colon \mathbb{K}^{\mathbb{Z}} \to \mathbb{K}^{\mathbb{Z}}$, defined by $\tau(x)(n) = x(n+1) - x(n)^2$ for all $x \in \mathbb{K}^{\mathbb{Z}}$ and $n \in \mathbb{Z}$, is an algebraic cellular automaton with memory set $M = \{0, 1\}$. Note that τ is not injective. It is surjective if $\mathbb{K} = \mathbb{C}$ but not surjective for $\mathbb{K} = \mathbb{R}$ (cf. [6]).

Example. Let A be an affine algebraic group (e.g. $A = \mathrm{SL}_n(\mathbb{K})$). Then the map $\tau\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, defined by $\tau(x)(n) = x(n)^{-1}x(n+1)$ for all $x \in A^{\mathbb{Z}}$ and $n \in \mathbb{Z}$, is an algebraic cellular automaton with memory set $M = \{0, 1\}$. Note that τ is always surjective and that, unless A is a trivial group, τ is never injective.

Surjunctive categories. Let $\mathcal{C}$ be a category whose objects are sets and whose morphisms are maps between them. We say that the category $\mathcal{C}$ is *surjunctive* provided that every injective endomorphism $\varphi\colon A \to A$ is surjective (here injectivity and surjectivity are intended in the set-theoretical sense). For example, the category of finite sets, the category of finite-dimensional vector spaces over a given field $\mathbb{K}$, and the category of affine algebraic sets over a given algebraically closed field $\mathbb{K}$ (the *Ax-Grothendieck theorem*) are surjunctive.

Observe that in the category of finite sets (resp. of finite-dimensional vector spaces over a field $\mathbb{K}$) every surjective endomorphism is injective. This is no more the case for the category of affine algebraic sets over an algebraically closed field. Indeed, if $\mathbb{K}$ is algebraically closed, the polynomial map $f\colon \mathbb{K} \to \mathbb{K}$ given by $f(t) = t^2$ is surjective but not injective. Also note that the polynomial map $f\colon \mathbb{Q} \to \mathbb{Q}$ given by $f(t) = t^3$ is injective but not surjective.

The category of G-shifts. Let G be a group and let $\mathcal{C}$ be a category of sets admitting finite direct products. We make the additional assumption that if A and B are two objects in $\mathcal{C}$, then the canonical map $A \times_{\mathcal{C}} B \to A \times B$ is bijective. We consider the new category $\mathcal{C}(G)$ whose objects are the sets A^G, where A runs over all objects in $\mathcal{C}$, and whose morphisms are the cellular automata $\tau\colon A^G \to$

B^G whose local defining maps are morphisms in $\mathcal{C}$. One then says that G is $\mathcal{C}$-*surjunctive* if the category $\mathcal{C}(G)$ is surjunctive.

For example, the trivial group is $\mathcal{C}$-surjunctive if and only if the category $\mathcal{C}$ is surjunctive. When $\mathcal{C}$ is the category of finite sets, then G is $\mathcal{C}$-surjunctive if and only if G is *surjunctive* in the sense of W. Gottschalk [10]. If $\mathcal{C}$ is the category of finite dimensional vector spaces over a given field $\mathbb{K}$, then G is $\mathcal{C}$-surjunctive, if and only if the group ring $\mathbb{K}[G]$ is *stably finite*, that is, for every $n \geq 1$ if two $n \times n$ matrices a and b with coefficients in $\mathbb{K}[G]$ satisfy $ab = 1$, then they also satisfy $ba = 1$ (see [1, 7]).

Let Γ be a group. The set of quotients of Γ may be identified with the set $\mathcal{N}(\Gamma)$ of normal subgroups of Γ. The set $\mathcal{N}(\Gamma)$ is called the set of Γ-*marked groups*. It is a closed (and hence compact) subset of the compact space $\mathcal{P}(\Gamma) = \{0, 1\}^\Gamma$ for the prodiscrete topology.

Theorem ([5, 8])**.** *Let Γ be a group and $\mathcal{C}$ be the category of finite sets (resp. the category of finite dimensional vector spaces over a field $\mathbb{K}$). Then the set of normal subgroups $N \subset \Gamma$ such that the quotient group Γ/N is $\mathcal{C}$-surjunctive is closed (and hence compact) in $\mathcal{N}(\Gamma)$.*

Sofic groups. Let S be a finite set and let $\mathcal{G} = (V, E)$ be an S-labeled graph. For $v \in V$ and $r > 0$ we denote by $B_r^{\mathcal{G}}(v)$ the *ball of radius r around v* in $\mathcal{G}$.

Let G be a group and $S \subset G$ a finite subset. The *Cayley graph* of G with respect to S is the S-labeled graph $\mathcal{C}(G, S)$ with vertex set $V = G$, the set E of edges consisting of all pairs (g, gs), where $g \in G$ and $s \in S$, and the labeling map $\lambda \colon E \to S$ defined by $\lambda(g, gs) = s$.

A group G is called *sofic* if the following holds: for any finite subset S of G, and all $\varepsilon > 0$ and $r \in \mathbb{N}$, there exists a finite S-labeled graph $\mathcal{G} = (V, E)$ such that $|V(r)| \geq (1-\varepsilon)|V|$, where $V(r) \subset V$ denotes the set consisting of all vertices $v \in V$ such that $B_r^{\mathcal{G}}(v)$ and $B_r^{\mathcal{C}(G,S)}(1_G)$ are isomorphic as rooted S-labeled graphs.

Theorem (Gromov-Weiss [8, 11])**.** *Every sofic group is surjunctive in the sense of Gottschalk.*

Theorem ([1, 8])**.** *Let $\mathcal{C}$ be the category of finite dimensional vector spaces over a given field $\mathbb{K}$. Then every sofic group is $\mathcal{C}$-surjunctive.*

Theorem ([6, 8])**.** *Let $\mathcal{C}$ be the category of affine algebraic sets over an uncountable algebraically closed field $\mathbb{K}$. Then every residually finite group G is $\mathcal{C}$-surjunctive.*

We are working on the extension of this last theorem to all sofic groups.

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Coarse non-amenability and coarse embeddings

Erik Guentner
(joint work with Goulnara Arzhantseva, Jan Spakula)

This talk concerned two properties from coarse geometry – coarse embeddability[1] and coarse amenability ($\equiv$ Property A). Coarse embeddability was introduced by Gromov [3]. A metric space X is *coarsely embeddable* (in Hilbert space $\mathcal{H}$) if there exists a *coarse embedding* $f : X \to \mathcal{H}$; the function f is a coarse embedding if there exists non-decreasing, proper functions $\rho_\pm : [0, \infty) \to [0, \infty)$ satisfying

$$\rho_-(d(x, y)) \leq \|f(x) - f(y)\| \leq \rho_+(d(x, y)),$$

for every x and $y \in X$. Gromov speculated that coarse embeddability may be relevant for the Novikov conjecture, and his intuition was confirmed when Yu proved the Novikov conjecture for groups that coarsely embed in Hilbert space [10, 7]. In his work Yu introduced Property A as a readily verified criterion sufficient to imply coarse embeddability. There are now a variety of equivalent formulations of Property A – see, for example [8, 4, 2] or the survey [9]. We adopt the following 'non-equivariant' form of the Reiter condition for amenability as our definition: a (discrete) metric space X is coarsely amenable ($\equiv$ has Property A) if for every $R > 0$ and $\varepsilon > 0$ there exists an $S > 0$ and a function $\xi : X \to \ell_1(X)$ satisfying, for every x and $y \in X$:

(1) ξ_x is a probability measure;
(2) $\xi_x(y) = 0$ when $d(x, y) > S$;
(3) $\|\xi_x - \xi_y\| < \varepsilon$ when $d(x, y) < R$.

[1] In early literature on the subject, coarse embeddings were called uniform embeddings.

(The norm is the ℓ_1-norm.)[2] Just as coarse amenability appears as a coarse analogue of amenability, coarse embeddability appears as a coarse analogue of a-T-menability – the existence of a proper affine isometric action on a Hilbert space; indeed, such an action is simply an equivariant coarse embedding.

As mentioned above, a coarsely amenable space is coarsely embeddable. The analogous statement at the level of equivariant properties is that an amenable group is a-T-menable. The free group on two generators $\mathbb{F}_2$ is an example of a group which is a-T-menable, but not amenable and it is natural to ask whether there exist discrete metric spaces that are coarsely embeddable, but are not coarsely amenable ($\equiv$ are coarsely non-amenable). Examples of such spaces were given by Nowak [5]. A special case of his result is that the *coarse union* of cubes of increasing dimension:

$$\{0,1\} \sqcup \{0,1\}^2 \sqcup \{0,1\}^3 \sqcup \dots \qquad (\diamondsuit)$$

is coarsely embeddable and coarsely non-amenable. Here, each cube is given the Hamming distance and the union is given any metric satisfying the following two conditions:

(1) the inclusion of each cube is an isometry;
(2) the distance between individual cubes tends to infinity.[3]

A shortcoming of the example ($\diamondsuit$) is that it does not have bounded geometry; recall that a metric space has *bounded geometry* if for every $r > 0$ there exists N such that a ball of radius r can contain at most N points. Further, the construction of [5] cannot produce examples with bounded geometry. However, in [1] we prove the following result.

Theorem. There exists a bounded geometry discrete metric space that is coarsely embeddable and coarsely non-amenable.

In fact, we explicitly construct the first such space as follows. Let

$$N_0 = \mathbb{F}_2, \quad N_1 = \mathbb{F}_2^{(2)} = N_0^{(2)}, \quad N_2 = N_1^{(2)}, \quad \dots$$

Here, for example, $\mathbb{F}_2^{(2)}$ is the subgroup of $\mathbb{F}_2$ generated by the squares of elements of $\mathbb{F}_2$. These are finite index, characteristic subgroups of $\mathbb{F}_2$. Denote the coarse union of the corresponding finite quotients by

$$\Box\mathbb{F}_2 = \mathbb{F}_2/N_0 \sqcup \mathbb{F}_2/N_1 \sqcup \mathbb{F}_2/N_2 \sqcup \dots \qquad (\heartsuit)$$

Here, each quotient is given the word metric for the generators inherited from $\mathbb{F}_2$.

Coarse non-amenability of the example in ($\heartsuit$) is not difficult to check; it follows from the non-amenability of $\mathbb{F}_2$ and the fact that $\cap N_i$ is trivial [6]. Coarse embeddability, however, is more difficult and relies on the description of the finite spaces comprising $\Box\mathbb{F}_2$ as the iterated $\mathbb{Z}/2$-homology covers of the 'figure 8'.

[2] Most formulations of Property A, including the one we have adopted here, are known to be equivalent to Yu's original formulation only for spaces of bounded geometry.

[3] In general, the *coarse union* of finite metric spaces is their disjoint union equipped with a metric satisfying these two conditions. An elementary observation is that any two such metrics are coarsely equivalent.

The question of whether there exists a countable discrete *group* which is coarsely embeddable and coarsely non-amenable remains open.

The results described here are based on [1], to which we refer for details. The speaker was supported during this project in part by a grant from the U.S. National Science Foundation.

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Universal Sofic Groups

Simon Thomas

Let $\mathcal{U}$ be a nonprincipal ultrafilter over ω and let $G_{\mathcal{U}} = \prod_{\mathcal{U}} \mathrm{Sym}(n)$ be the corresponding ultraproduct of the finite symmetric groups. Then Allsup-Kaye [1] and Élek-Szabó [2] have independently shown that $G_{\mathcal{U}}$ has a unique maximal proper normal subgroup; namely,

$$M_{\mathcal{U}} = \left\{ (\pi_n)_{\mathcal{U}} \in G_{\mathcal{U}} : \lim_{\mathcal{U}} \frac{|\operatorname{supp}(\pi_n)|}{n} = 0 \right\},$$

where $\operatorname{supp}(\pi_n) = \{\ell \in n : \pi_n(\ell) \neq \ell\}$. Let $S_{\mathcal{U}} = G_{\mathcal{U}}/M_{\mathcal{U}}$. Then by Élek-Szabó [2], if Γ is a finitely generated group, the following statements are equivalent:

- Γ is a sofic group.
- Γ embeds into $S_{\mathcal{U}}$ for some (equivalently every) nonprincipal ultrafilter $\mathcal{U}$.

For this reason, $S_{\mathcal{U}}$ is said to be a *universal sofic group.* Of course, if $\mathcal{U} \neq \mathcal{D}$ are distinct nonprincipal ultrafilters over ω, then there is no reason to expect that $S_{\mathcal{U}}$ and $S_{\mathcal{D}}$ will be isomorphic. In this talk, we will consider the problem of computing the number of universal sofic groups $S_{\mathcal{U}}$ up to isomorphism. Perhaps surprisingly, this problem turns out to be much easier to handle under the assumption that the Continuum Hypothesis CH fails.

Theorem 1. *If CH fails, then there exist $2^{2^{\aleph_0}}$ universal sofic groups $S_{\mathcal{U}}$ up to isomorphism.*

On the other hand, if CH holds, then each ultraproduct $G_{\mathcal{U}} = \prod_{\mathcal{U}} \mathrm{Sym}(n)$ is saturated and hence is determined up to isomorphism by its first order theory. Thus there are at most $2^{\aleph_0}$ such ultraproducts up to isomorphism and hence also at most $2^{\aleph_0}$ universal sofic groups up to isomorphism. It is easily shown that (as expected) there are $2^{\aleph_0}$ such ultraproducts up to elementary equivalence. However, it is currently not even known whether there exist two nonisomorphic universal sofic groups if CH holds.

Conjecture 2. If CH holds, then there exist $2^{\aleph_0}$ universal sofic groups $S_{\mathcal{U}}$ up to isomorphism.

Question 3. Are all universal sofic groups $S_{\mathcal{U}}$ elementarily equivalent?

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Aspherical groups and manifolds with extreme properties

Mark V. Sapir

The main result reported in my talk is the following theorem.

Theorem 1. *Every finitely generated group with combinatorially aspherical recursive presentation complex embeds into a group with finite combinatorially aspherical presentation complex.*

Using Davis' construction this allows one to create closed aspherical manifolds of dimension 4 and higher with some previously unknown "extreme" properties. For example, by Gromov, there exists a finitely generated group with recursive combinatorially aspherical presentation whose Cayley graph coarsely contains an expander. Hence Theorem 1 implies that there exist closed aspherical manifolds of dimension 4 and higher whose fundamental groups coarsely contain expanders. These groups and manifolds are not coarsely embeddable into a Hilbert space, do not satisfy G.Yu's property A, and are counterexamples to the Baum-Connes conjecture with coefficients (by Higson, Lafforgue, Skandalis). They also have infinite asymptotic dimension. That solves a problem, formulated first by G. Yu asking whether the fundamental group of a closed aspherical manifold can have infinite asymptotic dimension. A weaker problem of whether the asymptotic dimension of a closed aspherical manifold can exceed its (ordinary) dimension was mentioned by Gromov and was open till now also. Note that Dranishnikov's problem whether the asymptotic dimension of an aspherical n-manifold is always n or infinity is still open. It seems that Gromov's random groups and our Theorem 1 cannot give an example where the dimensions are different while both are finite in view of the recent paper by Willett.

As another corollary one can deduce that a torsion-free Tarski monster (that is a finitely generated non-cyclic group all of whose proper subgroups are infinite cyclic) embeds into the fundamental group of a closed aspherical manifold. Indeed the torsion-free Tarski monsters constructed by Ol'shanskiĭ have recursive combinatorially aspherical presentations, it remains to apply Theorem 1 and the Davis construction. More generally, every *lacunary hyperbolic* group given by a recursive *graded small cancelation* presentation embeds into the fundamental group of a closed aspherical manifold. In our construction, we are using S-machines (which can be viewed as multiple HNN-extensions of free groups) used for some versions of Higman embedding before. The finitely presented aspherical is built from two (different) S-machines, and several hyperbolic and close to hyperbolic groups that "glue" these S-machines together. One of the main tools of the proof is the congruence extension property of certain subgroups of hyperbolic groups first established by Ol'shanskiĭ.

Sofic Dimension

Ken Dykema
(joint work with David Kerr, Mikael Pichot)

We begin by drawing some parallels between ideas and results in the theory of finite von Neumann algebras on the one hand and group theory on the other hand.

(i) In finite von Neuamm algebras, we may ask whether arbitrary n–tupes of operators have matrix approximants, in the sense of approximation of mixed moments w.r.t. traces. This is equivalent to Connes' embedding problem [3]. When specialized to group von Neumann algebras, this property is called hyperlinearity of the group, and is equivalent to asking whether arbitrary finite sets of elements in the group can be well modeled (w.r.t. the Euclidean norm coming from normalized traces) by unitary matrices. See Rădulescu [9]. An analogous property, in group theory is that of being sofic, which entails a similar sort of approximation in permutation groups S_d with respect to normalized Hamming distance.

(ii) Probabalistic methods for constructing matrix approximants for free products of finite von Neumann algebras were pioneered by Voiculescu [10], and these are fundamental to his microstates free entropy dimension (see below). The techniques and resutls were extended to the case of amalgamated free products of finite von Neumann algebras over a hyperfinite von Neumann aglebra by Brown, Dykema and Jung in [1]. In particular, this entails that every free product of hyperlinear groups with amalgamation over a (finite or) amenable group is hyperlinear. In the realm of sofic groups, Élek and Szabó proved [6] that every free product of sofic groups is sofic. Probabilistic methods were used by Collins and Dykema [2] to prove that the class of sofic groups is closed under taking free products with amalgamation over monotileably amenable groups. The requirement of monotileably amenable rather than just amenable is a technical restriction, which is overcome later in [5]. Also, completely different techniques were used, independently, by Paunescu [8] and by Élek and Szabó [7] to prove that the class of sofic groups is closed under taking free products with amalgamation over arbitrary (finite or) anemable groups.

(iii) Voiculescu's microstates free entropy dimension is an invariant for finite sets of elements of finite von Neumann algebras, which is based on answering the question "how many matrix approximants are there?" (asymptotically as the matrix size grows). An analogous notion for sofic groups (and more general related objects) is introduced by Dykema, Kerr and Pichot in [5]. We call this invariant the *sofic dimension* $s(G)$. For groups it is based on answering the question "how many approximants in S_d are there?" (asymptotically).

We show (1) if a group G is infinite and sofic, then it's sofic dimension is $s(G)$ is at least 1; (2) if G is finite, then $s(G) = 1 - (1/|G|)$ and if G is infinite amenable, then $s(G) = 1$; (3) if $G = G_1 *_H G_2$ is a free product of groups G_1 and G_2 with amalgamation over a (finite or) amenable group H, then $s(G) \leq s(G_1) + s(G_2) - s(H)$, with equality in the case that both G_1 and G_2 are approximation regular, which then entails that also G is approximation regular. Since finite and amenable groups are approximation regular, it follows that all groups in the bootstrap class that is constructed starting from amenable and finite groups, and taking free products with amalgamation over amenable or finite groups,

Finally, the sofic dimension is also defined in [5] for measurable discrete groupoids, which includes also (a) probability–measure–preserving actions of groups and (b) probability– measure– preserving measurable equivalence relations. Analogous results are proved in these settings: see also the related paper [4].

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Problem session

This section contains a collection of open problems, some of which are well-known. They were contributed by the workshop participants during the problem session.

Question 1 (Arzhantseva). *Are the following groups sofic or hyperlinear ? Thompson's group F, one-relator groups, Burnside's group, Burger-Mozes groups, Gromov's monster (a finitely generated group coarsely containing an infinite expander).*

Question 2 (Arzhantseva). *Is any word hyperbolic group sofic or hyperlinear ?*

Conjecture (Arzhantseva) *Every hyperbolic group is residually finite if and only if every hyperbolic group is sofic (or hyperlinear).*

Question 3 (Arzhantseva). *Let G_1, G_2 be sofic (or hyperlinear) groups and K be a residually finite group which has an isomorphic copy inside G_1 and G_2. Is the free amalgamated product $G_1 *_K G_2$ sofic (or hyperlinear)?*

Remark. If K is amenable, this is known to be true.

Question 4 (Arzhantseva). *Is a random finitely presented group sofic (or hyperlinear)?*

Question 5 (Arzhantseva). *Let G_1, G_2 be surjunctive finitely generated groups. Is the direct product $G_1 \times G_2$ surjunctive? What about the free product $G_1 * G_2$?*

Question 6 (Bowen). *A sofic approximation to a group and an ultrafilter on the natural numbers naturally determines a measure-preserving action of the group on an ultraproduct of finite spaces. The measure on this ultraproduct is non-standard. So it is not clear whether there is an ergodic decomposition. However, it is more interesting to ask whether there exists an ergodic decomposition which "lifts" to the finite spaces so that we really have a decomposition of the sofic approximation. It can be shown that if G is infinite and amenable then no such decomposition is possible. However, if G has property (T) then it is open.*

Question 7 (Bowen). *The problem is to classify the sofic approximations to a given group. For example, Élek has shown that all sofic approximations to an amenable group are essentially equivalent and come from a Følner sequence. Another example is that every sofic approximation to a free group essentially comes from a sequence of finite quotients. Lastly, all sofic approximations to a surface group come from finite quotients and "branched covers" of such. Is there an analogous result for the direct product of two free groups?*

Question 8 (Bowen). *If $H < G$ then any action of H can be co-induced to obtain an action of G. On the other hand, any sofic approximation to G can be restricted to obtain a sofic approximation to H. This raises several questions. For example, if F is a free subgroup of $SL_3(\mathbb{Z})$, can every sofic approximation to F be obtained by restricting a sofic approximation to $SL_3(\mathbb{Z})$? We expect not. This is important because if it is impossible then we should be able to obtain an action of F (for example, on its profinite completion) and co-induce to get an action of $SL_3(\mathbb{Z})$ that is non-sofic.*

More generally, let us say that (G, H) is sofic if every action of H co-induced to G is sofic. It is a curious problem to investigate permanence properties of sofic pairs.

Question 9 (Ozawa). *According to Kirchberg's theorem, Connes's embedding problem has a positive answer if and only if $C^*F_r \otimes C^*F_r$ has a unique C^*-norm. Here F_r is the free group of rank r and r can be any number $2, 3, \ldots, \infty$. This means that every unitary representation of $F_r \times F_r$ is weakly contained in a direct sum of finite representations. Kirchberg's theorem also implies the following: Fix a surjective homomorphism $q\colon F_r \to \Gamma$. Then, the group Γ is hyperlinear if and only if the representation μ of $F_r \times F_r$ on $\ell_2\Gamma$, given by*

$$\mu(g, h)\delta_x = \delta_{q(g)xq(h)^{-1}},$$

is weakly contained in a direct sum of finite representations.

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