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Dynamische Systeme

Organised by
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Abstract. This workshop continued the biannual series at Oberwolfach on Dynamical Systems that started as the "Moser-Zehnder meeting" in 1981. The main themes of the workshop are the new results and developments in the area of dynamical systems, in particular in Hamiltonian systems and symplectic geometry related to Hamiltonian dynamics. Highlights were new results on Arnold diffusion and a new approach to the study of Hamiltonian systems based on pseudoholomorphic curve methods.

Mathematics Subject Classification (2000): 37xx, 53D.

Introduction by the Organisers

The workshop was organized by H. Eliasson (Paris), H. Hofer (Princeton) and J.-C. Yoccoz (Paris). It was attended by more than 50 participants from 11 countries and covered a large area of dynamical systems with an emphasis on classical Hamiltonian dynamics: KAM theory, Arnold diffusion, celestial mechanics, geodesic flows, Reeb flows and Floer homology. Other subjects treated where dynamics of PDE's, motions in random potentials and random scatterers, actions of the mapping class group and higher rank abelian groups.

The topic of Arnold diffusion was treated in several talks. K. Zhang discussed diffusion along simple resonances and J. Mather discussed motion through double resonances in $2\frac{1}{2}$ degrees of freedom. Other aspects of diffusion were treated in the talks of A. Bounemoura and V. Kaloshin. J. Pöschel and D. Sauzin presented new results in classical KAM-theory. A surprising connection between the Horn problem for eigenvalues of sums of symmetric matrices and relative equilibria in celestial mechanics was revealed in the talk of A. Chenciner. New results on closed geodesics on Riemann and Finsler manifolds were presented by V. Bangert and Y. Long. B. Bramham reported on important progress on some old questions of Katok about the dynamics of symplectic maps in dimension two using pseudoholomorphic curves. Symplectic methods and periodic solutions were discussed in the talks of D. Hein, S. Hochloch, U. Hryniewicz, A.Momin and C. Wendl. Two talks were given on PDE's: W. Craig discussed the relevance of resonances for the Birkhoff normal form and P. Rabinowitz presented results on heteroclinic solutions in the Allen-Cahn equation. A. Knauf and T. Yarmola presented new results on motions in random environments and J. Franks and A. Katok reported on recent developments on the action of the mapping class group on surfaces and on rigidity of higher ranks abelian actions.

The meeting was held in an informal and stimulating atmosphere. The weather was in general very nice, except Wednesday which was slightly rainy. This however didn't prevent many of the participants to join Paul Rabinowitz for the traditional walk to St. Roman.

Workshop: Dynamische Systeme

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Abstracts

Rereb dynamics and obstructions to symplectic cobordisms

Chris Wendl
(joint work with Janko Latschev and occasionally Michael Hutchings)

In Hamiltonian dynamics, one often considers the question of whether a given hypersurface M^{2n-1} in a symplectic manifold (W^{2n}, ω) admits a periodic orbit for any Hamiltonian $H : W \to \mathbb{R}$ that has M as a regular energy surface. As is well known, the answer does not depend on the choice of H. The question is especially interesting when M is assumed to be of *contact type*, meaning it is transverse to a symplectically dilating vector field—in this case the dynamics near M are "stable" in the sense that for any 1-parameter family of hypersurfaces containing M, all hypersurfaces have the same orbits up to parametrization. The Weinstein conjecture asserts that any closed contact type hypersurface in a symplectic manifold admits a closed orbit.

The contact type condition also induces an intrinsic structure on the hypersurface M: we call a hyperplane field $\xi \subset TM$ a *contact structure* if it is the kernel of some 1-form α such that $\alpha \wedge (d\alpha)^{n-1} > 0$. Such a 1-form also determines the *Reeb vector field* X_α by the conditions

$$d\alpha(X_\alpha, \cdot) \equiv 0, \qquad \alpha(X_\alpha) \equiv 1,$$

and the Weinstein conjecture is then equivalent to the claim that all Reeb vector fields on closed contact manifolds admit periodic orbits.

Contact topology also studies a number of questions that, on the surface, have nothing to do with dynamics. The most fundamental is the classification of contact structures: given two contact structures ξ and ξ' on M, is there a diffeomorphism $\varphi : M \to M$ such that $\varphi_* \xi = \xi'$? Eliashberg showed [Eli89] that the classification question partitions closed contact 3-manifolds into two fundamentally different classes,

$$\{\text{contact 3-manifolds}\} = \{\text{tight}\} \sqcup \{\text{overtwisted}\},$$

of which the overtwisted ones are in some sense "easy" to classify and the tight ones are not. The overtwisted contact manifolds also have the property that they never appear as contact type boundaries of compact symplectic manifolds, i.e. they are not symplectically *fillable*. This is part of the inspiration for the following important conjecture in contact topology:

Conjecture 1. *If (M, ξ) is a tight contact manifold and (M', ξ') is obtained from (M, ξ) by contact surgery, then (M', ξ') is also tight.*

Contact surgery is a type of Dehn surgery that can be performed along any knot in M tangent to ξ, and it produces a new contact manifold (M', ξ') along with an exact symplectic cobordism from (M, ξ) to (M', ξ'). Because of this cobordism, (M', ξ') obviously admits a symplectic filling of (M, ξ) does, but the conjecture does not follow from this since tightness and fillability are not quite the same thing. One of the most interesting questions in this field is to understand which tight contact manifolds are *not* fillable.

On this subject, it turns out that dynamics has something interesting to tell us about contact topology. The following is, using modern technology, an easy exercise based on a result that Hofer proved using J-holomorphic disks:

Theorem 1. *(*[Hof93]*) Suppose (M, ξ) has a contact form with no contractible Reeb orbit. Then after contact surgery, (M', ξ') is always tight.*

This result is interesting for us because there are plenty of contact manifolds without a contractible orbit that are known to admit no symplectic fillings—the result thus provides a hint as to how one might attack the above conjecture using dynamical knowledge.

In a recent paper with Janko Latschev [LW10], we find that the above result of Hofer extends to an infinite hierarchy of nested subclasses of contact manifolds that are closed under contact surgery.

Main theorem. *(*[LW10]*) There exists a contact invariant $AT(M, \xi) \in \mathbb{N} \cup \{0, \infty\}$ with the following properties:*

(1) *If there is an exact symplectic cobordism from (M_-, ξ_-) to (M_+, ξ_+), then $AT(M_-, \xi_-) \leq AT(M_+, \xi_+)$.*
(2) *$AT(M, \xi) = 0$ if and only if (M, ξ) is* algebraically overtwisted*, i.e. it has trivial contact homology, cf.* [BN10].
(3) *If (M, ξ) is symplectically fillable then $AT(M, \xi) = \infty$.*
(4) *If $AT(M, \xi) = k < \infty$ then for every contact form defining ξ, there exists a surface Σ with nonempty boundary and*

$$\text{genus}(\Sigma) + \#\pi_0(\partial\Sigma) \leq k + 1,$$

and a continuous map $f : \Sigma \to M$ such that $f|_{\partial\Sigma}$ parametrizes a collection of closed Reeb orbits.
(5) *In dimension three, for every $k \geq 0$ there exist examples (M_k, ξ_k) with $AT(M_k, \xi_k) = k$.*

A corollary is that if contact surgery is performed on any (M, ξ) with $\text{AT}(M, \xi) \geq k$, then the new manifold (M', ξ') also has this property. It is known (cf. [Yau06]) that overtwistedness implies algebraic overtwistedness, and the converse is not known but is a reasonable conjecture. If it is true, then the above theorem proves the conjecture on surgery. More importantly, it establishes that within the class of tight contact manifolds, there are varying "degrees of tightness" that can be measured by the numerical invariant $\text{AT}(M, \xi)$; the "tightest" are the fillable contact manifolds, but there is also an infinite hierarchy of manifolds that are non-fillable but tight to varying degrees.

The construction of the invariant $\text{AT}(M, \xi)$ is based on Symplectic Field Theory, a very general algebraic formalism originally introduced by Eliashberg, Givental and Hofer [EGH00]. In the version we consider, one chooses a contact form α for (M, ξ) and associates to every closed Reeb orbit γ formal variables q_γ and p_γ, which are given the $\mathbb{Z}_2$-grading

$$|q_\gamma| = |p_\gamma| = \mathrm{CZ}(\gamma) + n - 1 \in \mathbb{Z}_2.$$

We then define the formal power series

$$H = \sum_{g,\Gamma^+,\Gamma^-} n_g(\Gamma^+,\Gamma^-)\ \hbar^{g-1} q^{\Gamma^-} p^{\Gamma^+},$$

where the sum is over all integers $g \ge 0$ and finite collections of Reeb orbits $\Gamma^\pm$, which play the role of multi-indices in the abbreviated expressions q^{Γ^-} and p^{Γ^+}. The numerical factor $n_g(\Gamma^+,\Gamma^-)$ is a suitably weighted algebraic count of rigid pseudoholomorphic curves of genus g in the symplectization $(\mathbb{R}\times M, d(e^t\alpha))$, with positive and negative cylindrical ends approaching the collections of orbits Γ^+ and Γ^- respectively, and it is understood that we set the count to zero whenever this space has the wrong dimension to be counted. Now if $\mathcal{A}$ denotes the free graded commutative $\mathbb{R}$-algebra with unit generated by all the variables q_γ, we can make the substitution

$$p_\gamma \rightsquigarrow \kappa_\gamma \hbar \frac{\partial}{\partial q_\gamma},$$

for a suitable combinatorial constant κ_γ, and use this to turn the power series H into a linear differential operator

$$D_H : \mathcal{A}[[\hbar]] \to \mathcal{A}[[\hbar]].$$

It then follows from the compactness and gluing theory of pseudoholomorphic curves that $D_H^2 = 0$, and the resulting homology

$$H_*^{\mathrm{SFT}}(M,\xi) = H_*\left(\mathcal{A}[[\hbar]], D_H\right)$$

is an invariant of the contact structure. Since D_H is not a derivation (it includes differential operators of all orders, not just 1) but is $\hbar$-linear, the homology $H_*^{\mathrm{SFT}}(M,\xi)$ does not inherit the algebra structure from $\mathcal{A}[[\hbar]]$, but it is at least an $\mathbb{R}[[\hbar]]$-module. Moreover, every power $\hbar^k$ for integers $k \ge 0$ satisfies $D_H \hbar^k = 0$ and thus defines a canonical element in $H_*^{\mathrm{SFT}}(M,\xi)$. A useful property is then that whenever there exists an exact symplectic cobordism from (M_-,ξ_-) to (M_+,ξ_+), this induces an $\mathbb{R}[[\hbar]]$-module morphism

$$H_*^{\mathrm{SFT}}(M_+,\xi_+) \to H_*^{\mathrm{SFT}}(M_-,\xi_-)$$

which maps $[\hbar^k] \mapsto [\hbar^k]$ for all $k \ge 0$. Our numerical invariant, called the *order of algebraic torsion*, is now defined by

$$\mathrm{AT}(M,\xi) = \sup\{k \ge 0 \mid [\hbar^{k-1}] \ne 0 \in H_*^{\mathrm{SFT}}(M,\xi)\},$$

and its monotonicity property with respect to cobordisms follows immediately from the above discussion.

In reality, we have cheated a bit with this discussion, because while SFT is not an especially new theory, its foundations present formidable analytical difficulties whose solution is still work in progress (cf. [Hof06]). For our three-dimensional examples however, Michael Hutchings has shown in the appendix to our paper

[LW10] that methods from the distinctly 3-dimensional theory of Embedded Contact Homology can be used to circumvent the analytical difficulties and prove some interesting corollaries more directly. The argument is, at this level, a direct generalization of Hofer's work in [Hof93], but using more general types of holomorphic curves with multiple positive ends for which compactness can fail in a wider variety of ways, thus producing more complicated ensembles of Reeb orbits that are not always contractible.

We conclude by mentioning the most obvious open question to arise from this work:

Question. *Are there examples in all dimensions of contact manifolds with all possible orders of algebraic torsion?*

Such examples would automatically satisfy the Weinstein conjecture, which remains open in dimension greater than three. Some candidates in all dimensions arise from a joint paper in progress with Patrick Massot and Klaus Niederkrüger [MNW], but the technical details are considerably harder than in dimension three.

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Ergodic properties of some canonical systems driven by thermostats

Tatiana Yarmola

Rigorous derivations of macroscopic heat conduction laws from microscopic dynamics of mechanical models coupled to heat reservoirs require good mixing properties of the invariant measures. For many such systems in non-equilibrium, i.e. with two or more unequal heat reservoirs, pure existence of invariant measures is a nontrivial question due to the non-compactness of the phase space. We present a simple mechanical system driven by thermostats for which the stationary measure exists, is unique, absolutely continuous and mixes with exponential rates.

The example we consider is motivated as follows. Consider a system of N non-interacting particles at various velocities bouncing elastically off the walls of a bounded domain. We can assume for visualization purposes that N is very large and the system is at temperature T_0 in the following sense: kinetic energies of the particles are distributed with a discrete approximation of the Gibbs distribution with parameter $\beta_0 = \frac{1}{T_0}$, i.e. the probability that a given particle has kinetic energy near E is approximately $ce^{-\beta_0 E}dE$.

Let us introduce a thermostat into the system set at a different temperature $T_1 \neq T_0$ such that when a particle collides with the thermostat, an energy exchange occurs in which the thermostat absorbs part of the particle's energy and the particle acquires an energy E from the thermostat drawn form Gibbs distribution with parameter β_1, where $\beta_1 = \frac{1}{T_1}$. Over time, such a system is expected settle at temperature T_1, i.e. the initial Gibbs distribution with parameter β_0 is expected to converge to the Gibbs distribution with parameter β_1. The questions of interest are whether the Gibbs distribution with parameter β_1 is indeed the unique invariant measure for the system to which all (or almost all) initial distributions converge, and if so, at which rate.

Now let us add another thermostat at a yet different temperature $T_2 \neq T_1$. Does invariant measure exist for such a system? Is it unique and if so, do reasonable initial distributions converge to is and at which rates? For the system we consider we answer all these questions affirmatively both for equilibrium and non-equilibrium invariant measures for the corresponding discrete dynamics on the Poincare section of the flow and conclude existence, uniqueness, and absolute continuity of the invariant distribution for the continuous dynamics.

Our settings are as follows. Let $\Gamma \subset \mathbb{T}^2$ be a bounded horizon billiard table with circular scatterers $D_1, \cdots, D_p$. We set all scatterers to act as thermostats at possibly different temperatures $T_1, \cdots, T_p$ such that upon a collision of a particle with a thermostat at temperature T_i, certain energy exchange occurs. More precisely, if $v = (v_\perp, v_t)$ is a decomposition of the particle's velocity at collision into a normal and tangential components with respect to the boundary of a thermostat, then after the collision, the $v_\perp$ component changes sign, i.e. $v'_\perp = -v_\perp$, v_t component gets absorbed by the thermostat, and a new tangential component $v'_\perp$ gets randomly drawn from the distribution with density $\sqrt{\frac{\beta_i}{\pi}}e^{-\beta_i v_t^2}dv_t$, where $\beta_i = \frac{1}{T_i}$. We assume that particles do not interact with each other. This type of energy exchange was used in [3].

Since there are no particle interactions, the system with many particles is simply the product of one particle systems and we can focus on studying the system with only one particle. The phase space of this system is

$$\tilde{\Omega} = \{(x, v) : x \in \Gamma, v \in \mathbb{R}^2\}/\sim,$$

where $\sim$ is a properly chosen identification of velocities at the collision manifold. The dynamics is described by a Markov Process Φ_τ, which is deterministic apart from collisions with thermostats and upon a collision of the particle with a thermostat, a random perturbation occurs.

The degeneracy of the Markov Process Φ_τ allows to restrict the study to the discrete time Markov Chain on the collision manifold. Choose the variables: $r \in$

$\partial\Gamma$, the position of the particle parameterized by arc-length, $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, the angle of reflection, and $v_\perp$, the absolute value of the normal component of the velocity. When the random perturbation occurs, i.e. v_t' gets randomly drawn, the only variable that gets affected is φ. This allows eliminate the φ variable and to reduce the dynamics to the Markov Chain Φ acting on a two-dimensional phase space $\Omega = \partial\Gamma \times [0, \infty]$. The dynamics of Φ is as follows: starting from a point $(r, v_\perp)$, we first draw v_t from the distribution with density $\sqrt{\frac{\beta_i}{\pi}} e^{-\beta_i v_t^2} dv_t$, from which we determine φ; then let the particle originate from $(r, \varphi, v_\perp)$ and flow until its next collision with the thermostat at $(r', \varphi', v_\perp')$; and finally forget the φ' component. Let $\mathcal{P}((r, v_\perp), \cdot)$ be the transition probability kernel of Φ, i.e. $\mathcal{P}((r, v_\perp), A) = P(\Phi_n \in A | \Phi_{n-1} = (r, v_\perp))$. Note that the transition probabilities are degenerate: $\mathcal{P}((r, v_\perp), \cdot) = \mathcal{P}_* \delta_{(r, v_\perp)}$ is supported on a family of one dimensional curves in the two dimensional phase space.

For the Markov Chain Φ both in equilibrium, $\beta_1 = \cdots = \beta_p$, and non-equilibrium setting, $\beta_i \neq \beta_j$ for some i, j, we show:

Theorem 1. *There exists an invariant measure μ for the Markov Chain Φ. Moreover, μ is unique (ergodic), absolutely continuous (w.r.t. Leb.), and mixing with exponential rates.*

By mixing we mean

$$\lim_{n \to \infty} \left| \int_{(r, v_\perp) \in B} \mathcal{P}^n((r, v_\perp), A) d\mu(r, v_\perp) - \mu(A)\mu(B) \right| = 0.$$

We also conclude that reasonable initial distributions converge to μ exponentially fast with control on the rates. It follows that for the Markov Process Φ_τ there exists and invariant measure and this measure is unique (ergodic) and absolutely continuous. The uniqueness requires additional argument which is very similar to the argument we use in the proof on the theorem since ergodicity of one particle systems does not directly imply the ergodicity of many particle system. Mixing and convergence of initial distributions to the invariant measure for Φ_τ do not follow from Theorem since under some scenarios particles might move extremely slowly and eventually freeze. We leave investigation of the mixing properties of the Markov Process Φ_τ for the future work.

The proof of the Theorem relies heavily on the general state Markov chain machinery, in particular, on the following Geometric Ergodicity Theorem also knows as Harris' Ergodic Theorem [1, 2]

Theorem 2. *Assume*

Potential Condition: .

There exists a function $V : \Omega \to [0, \infty)$, $K > 0$ and $\gamma \in (0, 1)$ such that $\forall (r, v_\perp) \in \Omega$.

$$\mathcal{P}V(r, v_\perp) \leq \gamma V(r, v_\perp) + K$$

and

Minorization Condition: .
There exists a probability measure ν*, N and* $\eta_N \in (0,1)$ *such that*

$$\inf_{(r,v_\perp)\in\mathcal{C}} \mathcal{P}^N((r,v_\perp),\cdot) \geq \eta_N \nu(\cdot),$$

where $\mathcal{C} = \{(r,v_\perp) \in \Omega : V(r,v_\perp) \leq S\}$ *for some* $S > 2K/(1-\gamma)$ *where* K *and* γ *are the constants from the Potential Condition.*

Then Φ *admits a unique invariant measure* μ*. Furthermore, there exist constants* $C > 0$ *and* $\tilde{\gamma} \in (0,1)$ *such that*

$$\sup_{A\in\Omega} |\mathcal{P}^n((r,v_\perp),A) - \mu(A)| \leq C\tilde{\gamma}^n(1 + V(r,v_\perp)).$$

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Arnold diffusion via normally hyperbolic cylinders

Ke Zhang

(joint work with Patrick Bernard and Vadim Kaloshin)

The question of Arnold diffusion is concerned with instabilities for nearly integrable systems. To describe the problem, we consider a Hamiltonian function of the form

$$H_\epsilon(\theta,p,t) = H_0(p) + \epsilon H_1(\theta,p,t), \quad p \in \mathbb{R}^n, \theta \in \mathbb{T}^n, t \in \mathbb{T},$$

where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. There does not seem to be a universally accepted definition for Arnold diffusion. In the context of this talk, we say that the Hamiltonian system exhibits Arnold diffusion if there exists $c > 0$ such that the following hold: for arbitrarily small ϵ, there exists an orbit $(\theta_\epsilon(t), p_\epsilon(t))$ of the system H_ϵ, and $T_\epsilon > 0$ with

$$|p_\epsilon(T_\epsilon) - p_\epsilon(0)| > c > 0.$$

The first example of Arnold diffusion was constructed by Arnold in [2], hence the name "Arnold diffusion". Arnold conjectured in [1] that this phenomenon should happen for general systems. In [10], J. Mather announced a result of a *stronger* form of Arnold diffusion for $n = 2$: in our definition, Arnold diffusion is defined as a property of the family H_ϵ, hence the "diffusion distance" c may depend on H_1; while in the case of [10], the diffusion distance c is independent of H_1.

In this talk we present a result on the weaker form of diffusion as described above.

Theorem 1. *Assume that the Hamiltonian $H_\epsilon(\theta, p, t)$ is C^r with $r \geq 4$. For a "typical" ϵH_1 (with $\|H_1\|_{C^r} = 1$), there exists $l(H_1) > 0$, an orbit $(\theta_\epsilon, p_\epsilon)(t)$ of the the Hamiltonian system $H_\epsilon = H_0 + \epsilon H_1$ and $T_\epsilon > 0$ such that*

$$|p_\epsilon(T_\epsilon) - p_\epsilon(0)| > l(H_1).$$

The "typical" condition is the cusp residue condition introduced by Mather in [10]. Due to restriction of the length we will not give details of this condition.

The orbit exhibiting diffusion is constructed near *resonances* (precise definitions will be given later). The idea of diffusing near resonances is well known among experts (in particular, some of the ideas we used has first appeared in [11]); the main novelty of our approach is the use of Hamiltonian averaging and normally hyperbolic cylinders to connect the diffusion problem to the widely studied *a priori unstable* systems.

We will assume that H_ϵ is a *Tonelli Hamiltonian*, which means $\partial_{pp} H_\epsilon$ is uniformly strictly convex and $\lim_{|p|\to\infty} H_\epsilon/|p| = \infty$. Denote by $\omega(p) = \partial_p H_0(p)$ the *frequency map*. A frequency vector ω is called resonant if there exists an integer vector $k \in \mathbb{Z}^n$ and $l \in \mathbb{Z}$ such that $k \cdot \omega + l = 0$. A frequency vector ω is called $m-$resonant if there exists $k_1, \cdots, k_m \in \mathbb{Z}^n$, $l_1, \cdots, l_m \in \mathbb{Z}$, $\{(k_i, l_i)\}$ linearly independent, and $k_i \cdot \omega + l_i = 0$ for $i = 1, \cdots, m$. The diffusion orbit we will construct will be close to $(n-1)-$resonant vectors under the frequency map. To be more specific, denote

$$\Gamma_{\mathbf{k}} = \{p \in \mathbb{R}^n : k_i \cdot \omega(p) + l_i = 0, i = 1, \cdots, n-1\}$$

for $\mathbf{k} = \{(k_i, l_i)\}$. A linear change of coordinates brings the resonance into the form

$$\Gamma = \{p \in \mathbb{R}^n : \partial_{p_i} H_0(p) = 0, i = 1, \cdots, n-1\}.$$

Denote $p^s = (p_1, \cdot, p_{n-1})$ and $p^f = p_n$. Γ can be naturally parametrized as $\Gamma = \{p^s = p^s_*(p^f)\}$. Denote $\theta^s = (\theta_1, \cdot, \theta_{n-1})$ and $\theta^f = \theta_n$, we write

$$Z(\theta^s, p) = \iint H_1^s(\theta^s, p^s, \theta^f, p^f, t)\, d\theta^f\, dt.$$

The Hamiltonian H_ϵ may be rewritten as

$$H_\epsilon = H_0(p) + \epsilon Z(\theta^s, p) + \epsilon H_2(\theta, p, t),$$

with $Z(\theta^s, p)$ being the *resonant* term, and $H_2 = H_1 - Z$ the *nonresonant* term. We have the following normal form theorem:

Theorem 2. *For any $\delta > 0$, there exists $\epsilon_0 > 0$ and $a_- < a_+$ depending only on δ, such that on the $\epsilon^{\frac{1}{6}}$ neighbourhood of $\{p^s = p^s_*(p^f), p^f \in (a_-, a_+)\} \subset \Gamma$, there is a change of coordinates Φ such that*

$$N_\epsilon = H_\epsilon \circ \Phi = H_0 + \epsilon Z + \epsilon R$$

with $\|R\|_{C^2} \leq \delta$.

If the function $Z(\theta^s, p)$ satisfies a set of genericity condition, we show that there exist normally hyperbolic cylinders for the normal form system N_ϵ. In the simplistic situation, this condition means that $Z(\theta^s, p)$ as a function of θ^s has a unique nondegenerate maximum at $\theta^s_*(p^f)$ for $p^f \in (a_-, a_+)$ and $p^s = p^s_*(p^f)$. The general picture is more complicated and we shall not discuss it here.

Proposition 1. *Assume that $Z(\theta^s, p)$ achieves its maximum at $\theta^s_*(p^f)$ for each $p^f \in (a_-, a_+)$ and $p^s = p^s_*(p^f)$. Then there exists $\delta > 0$ depending only on Z such that if $N_\epsilon = H_0 + \epsilon Z + \epsilon R$ satisfies $\|R\|_{C^2} \leq \delta$, the Hamiltonian flow of N_ϵ admits a normally hyperbolic invariant cylinder*

$$X = \{(\theta^s, p^s) = (\Theta^s, P^s)(\theta^f, p^f, t), \theta^f, t \in \mathbb{T}, p^f \in (a_-, a_+)\}.$$

For the a priori unstable systems, the existence of normally hyperbolic invariant cylinders can be used to construct diffusion orbits (see [3], [4], [5], [6], [7], [8], [12], [13]). While our system is not a priori unstable, we show that the variational methods of Bernard (see [3]) and Cheng-Yan (see [4], [5]) applies. In this simplified situation, it is possible to prove existence of a diffusion orbit near the set X, if we allow an additional generic perturbation. This proves our main theorem in the most simplified situation.

The diffusion orbit will have its p^f variable drift from a_- to a_+. We stress that the size of this interval depends on Z and hence on H_1. The reason that we cannot diffuse further is due to the existence of *additional resonances*. We say that $p \in \Gamma$ admits an additional resonance if there exist $k_n, l \in \mathbb{Z}$ such that $k_n \partial_{p_n} H_0(p) + l = 0$. The normal form theorem and the existence of normally hyperbolic cylinder fails near an additional resonance with small k_n and l.

In the case $n = 2$, the additional resonances on Γ are *double resonances*. Due to the low degree of freedom, it is still possible to characterize the system. In particular, invariant cylinders still exist, but they are no longer normally hyperbolic. They are, in general, nonuniformly partially hyperbolic. However, it is still possible to use them for Arnold diffusion. We intend to explore this in further research.

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Transition between stability and instability for Hamiltonian systems close to integrable

Abed Bounemoura

Let $n \geq 2$ be an integer, $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and $B = B_R$ be an open ball in $\mathbb{R}^n$ of radius $R > 1$ with respect to the supremum norm. Consider a Hamiltonian system close to integrable, of the form

$$\begin{cases} H(\theta, I) = h(I) + f(\theta, I) \\ |f| \leq \varepsilon \ll 1 \end{cases}$$

where $(\theta, I) \in \mathbb{T}^n \times B$ are angle-action coordinates for the integrable part h and f is a small perturbation in some suitable topology defined by a norm $|\,.\,|$. For simplicity we shall restrict ourself to the analytic case (but extensions to non-analytic systems are easily obtained), so we assume that h and f are bounded and real-analytic on $D = \mathbb{T}^n \times B$. Then they have holomorphic extensions to some σ-neighbourhood $V_\sigma(D)$ of D in the complex phase space, for some $\sigma > 0$, and we define $|\,.\,| = |\,.\,|_\sigma$ as the C^0 norm on $V_\sigma(D)$.

In the absence of perturbation, that is when ε is zero, for all solutions $(\theta(t), I(t))$ the action variables $I(t)$ are integrals of motion, but after perturbation the only (trivial) stability property that remains true for all solutions is that

$$\lim_{\varepsilon \to 0} \left(\sup_{0 \leq |t| \leq \varepsilon^{-c}} |I(t) - I_0| \right) = 0.$$

for any $0 \leq c < 1$.

Without hypotheses on h, one cannot take $c \geq 1$ in the equality above. Indeed, it follows from the work of Nekhoroshev and Niederman that if the restriction of h to some affine hyperplane, whose direction is generated by integer vectors, has a non-isolated critical point, then there exist $\delta > 0$ and an arbitrarily small perturbation of size ε such that

$$\sup_{0 \leq t \leq \varepsilon^{-1}} |I(t) - I_0| = |I(\varepsilon^{-1}) - I_0| \geq \delta.$$

This prompts us to introduce the following two definitions.

Definition 1. *An integrable Hamiltonian $h : B \to \mathbb{R}$ is rationally steep if its restriction to any affine hyperplane of the form $I_0 + \Lambda$, with $I_0 \in B$ and Λ a linear subspace of $\mathbb{R}^n$ generated by integer vectors, has only isolated critical points.*

Definition 2. *An integrable Hamiltonian $h : B \to \mathbb{R}$ is effectively stable if for any $f : \mathbb{T}^n \times B \to \mathbb{R}$ with $|f| \le \varepsilon$, all solutions $(\theta(t), I(t))$ of the Hamiltonian system $H = h + f$ starting at (θ_0, I_0) satisfy*

$$\lim_{\varepsilon \to 0} \left(\sup_{0 \le |t| \le \varepsilon^{-1}} |I(t) - I_0| \right) = 0.$$

Then we have the following result.

Theorem 1. *Effectively stable Hamiltonians are exactly rationally steep Hamiltonians.*

Hence for a rationally steep Hamiltonian, the time of stability $T(\varepsilon)$, that is the maximal time during which the variation $V(\varepsilon)$ of the action of all solutions of the perturbed system satisfy $\lim_{\varepsilon \to 0} V(\varepsilon) = 0$, is at least $1/\varepsilon$. As a stability result, this is very weak but if one quantifies correctly this notion of rational steepness, then a more precise estimate on $T(\varepsilon)$ can be obtained. Now a possible game one can play is to try to improve as much as possible this stability time $T(\varepsilon)$ to reach an "optimal" value, and by means of examples to show that instability occurs after this time-scale (this would somehow justify the word "optimal").

In the case where the integrable Hamiltonian is linear, that is $h(I) = \omega.I$ for some $\omega \in \mathbb{R}^n \setminus \{0\}$, the game described above is over as a consequence of the following two results. Note that the assumption that h is rationally steep translates into a non-resonant condition on ω. Up to a rescaling, we may assume $\omega = (1, \alpha)$ with $\alpha \in \mathbb{R}^{n-1}$. Then the function $\Psi = \Psi_\omega$ given by

$$\Psi(K) = \max \left\{ |d(k.\alpha, \mathbb{Z})|^{-1} \mid k \in \mathbb{Z}^{n-1}, \, 0 < |k| \le K \right\}, \quad K \in \mathbb{N}^*$$

is well-defined. It is obviously strictly increasing on $\mathbb{N}^*$, hence we can extend it (keeping the same notation) as a strictly increasing continuous function defined on $[1, +\infty)$, and then we can also define two additional functions

$$\Lambda(x) = x\Psi(x), \quad \Delta(x) = \Lambda^{-1}(x), \quad x \ge 1,$$

which are also strictly increasing and continuous.

Theorem 2. *For any non-resonant vector $\omega \in \mathbb{R}^n$, and any sufficiently small ε-perturbation f, all solutions $(\theta(t), I(t))$ of $H = h + f$ with $I_0 \in B_{R/2}$ satisfy the estimates*

$$|I(t) - I_0| \le \delta, \quad |t| \le \delta\varepsilon^{-1} \exp\left(c_1 \Delta(c_2 \varepsilon^{-1})\right).$$

for any $c_1 \left(\Delta\left(c_2 \varepsilon^{-1}\right)\right)^{-1} \le \delta < R/2$.

Theorem 3. *For any non-resonant vector $\omega \in \mathbb{R}^n$, there exists a sequence $(f_j)_{j \in \mathbb{N}^*}$ of ε_j-perturbation with $\varepsilon_j \to 0$ when $j \to +\infty$, such that the system $H_j = h + f_j$ has orbits which satisfy the equalities*

$$|I(t) - I_0| = \delta, \quad |t| = \delta\varepsilon_j^{-1} exp\left(c_3 \Delta(c_4 \varepsilon_j^{-1})\right).$$

for any $0 < \delta < R/2$, *if* $I_0 \in B_{R/2}$.

The constants c_i depend only on h (n, R, ω) and σ, but not on ε. In Theorem 2, we recover as a particular case the known results for a Diophantine frequency.

Now let conclude with some remarks concerning non-linear Hamiltonians. We plan to give an "optimal" result of stability under the rational steepness condition (in the same spirit as in Theorem 2), which would contain as particular cases the known results for convex, steep or Diophantine steep Hamiltonians. Also we are trying to construct examples of instability (as in Theorem 3) to justify the optimality, but this exercise is much harder for convex or steep Hamiltonians.

For convex systems, there are precise results of stability and instability which almost match, so that the game here is almost (but not quite) over.

For steep systems, results of stability are also known, but we believe that they can be improved, while there are no examples of instability for steep non-convex integrable Hamiltonians.

Let us finally point out that some non-steep systems can be almost as stable as steep (or convex) systems, and that showing stability and instability in this case can be more simple.

On an Allen-Cahn phase transition model

Paul H. Rabinowitz
(joint work with Jaeyoung Byeon)

Several authors have studied Allen-Cahn models in which the spatial phase transition manifests itself as a heteroclinic or homoclinic solution of the corresponding Allen-Cahn equation. These solutions are "unidirectional" in the sense that they are heteroclinic or homoclinic in one direction, e.g. in the x_1 direction. See e.g. [1]- [4], [6]-[9], [12]-[14] as well as the related works [5], [10], and [15]. The goal of this talk is to present a class of Allen-Cahn models for which the phase transitions are multidirectional. The only results of this nature that we know of are contained in the recent paper [11] which treats a different but related model.

To describe our results more precisely, let $G(u) = \frac{1}{2}u^2(1-u)^2$, a typical double well potential. Let $A \in C^1(\mathbb{R}^n)$ be a nonnegative function that is 1-periodic in the components of $x = (x_1, \cdots, x_n)$, i.e. $A(x) = A(x+i)$ for any $x \in \mathbb{R}^n$ and $i \in \mathbb{Z}^n$. Set $\Omega \equiv \{x \in (0,1)^n \mid A(x) > 0\}$. Assume $2\delta^* = |\partial\Omega - \partial[0,1]^n| > 0$ and $\partial\Omega$ is a smooth manifold. Set $\Omega_d \equiv \{x \in \Omega \mid |x - \partial\Omega| > d\}$ so for sufficiently small $d \in (0, \delta^*), \partial\Omega_d$ is diffeomorphic to $\partial\Omega$. Fix such a small d. Let $\varepsilon > 0$ and $A_\varepsilon = 1 + \frac{1}{\varepsilon}A$.

Our model PDE is

$$\text{(PDE)} \qquad -\Delta u + A_\varepsilon G'(u) = 0, \quad x \in \mathbb{R}^n$$

Let

$$T \subset \mathbb{Z}^n, \;\; A^T = \cup_{i \in T}(i + \Omega), \;\; B^T = \cup_{i \in \mathbb{Z}^n \setminus T}(i + \Omega).$$

We are interested in solutions of (PDE) satisfying $0 < U < 1$ and that are near 1 on A^T and near 0 on B^T. To obtain such solutions, take $d \in (0, \delta)$ and Ω_d as above. Define

$$A_T \equiv \cup\{i + \Omega_d \mid i \in T\}$$

and

$$B_T \equiv \cup\{i + \Omega_d \mid i \in \mathbb{Z}^N \setminus T\}.$$

Set

$$L_\varepsilon(u) = \frac{1}{2}|\nabla u|^2 + A_\varepsilon G(u) \quad \text{and} \quad J_\varepsilon(u) = \int_{\mathbb{R}^n} L_\varepsilon(u)\, dx.$$

Choose $0 < b < \frac{1}{2} < a < 1$ and define

$$\Gamma(T) = \{u \in C^2(\mathbb{R}^n, [0,1]) \mid u \geq a > 1/2 \text{ on } A_T \text{ and } u \leq b < 1/2 \text{ on } B_T\}.$$

Whenever a solution, $u \in \Gamma(T)$, of (PDE) satisfies $J_\varepsilon(u+\varphi) - J_\varepsilon(u) \geq 0$ for all φ with compact support such that $u + \varphi \in \Gamma(T)$, we say u is *minimal* in $\Gamma(T)$. Our main result is:

Theorem 1. *Under the above hypotheses on A and G, there is an $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any $T \subset \mathbb{Z}^n$, there is a solution $U = U_{\varepsilon,T} \in \Gamma(T)$ of (PDE), which is minimal in $\Gamma(T)$ and satisfies $0 < U_{\varepsilon,T} < 1$. Moreover as $\varepsilon \to 0$, $U_{\varepsilon,T}$ converges uniformly to 1 on A^T and to 0 on B^T.*

More can be said: as $\varepsilon \to 0, U_{\varepsilon,T} \to U_{0,T}$, a solution of a limit problem associated with (PDE).

The idea of the proof is first to use a constrained minimization argument to get the result for finite T and then pass to a limit to get the result for arbitrary $T \subset \mathbb{Z}^n$.

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The angular momentum of a relative equilibrium

Alain Chenciner

Analyzing the frequency structure of the angular momentum of a relative equilibrium solution of the N-body problem in a euclidean space of arbitrary dimension, leads to a conjecture which relates this question to a version of Horn's problem.

1. Angular momentum

Given an euclidean space (E,ϵ) of dimension d and N positive *masses* $m_1,\cdots m_N$, an element $x=(\vec{r}_1,\cdots,\vec{r}_N)\in E^N$ such that $\sum_{k=1}^N m_k\vec{r}_k=0$ will be called an *N-body configuration.* Calling $y=\dot{x}=(\vec{v}_1,\cdots,\vec{v}_N)$ a configuration of velocities, the *angular momentum* of (x,y) is the bivector $\mathcal{C}=\sum_{k=1}^N m_k\vec{r}_k\wedge\vec{v}_k\in\wedge^2E$. Given an orthonormal basis of E it can be identified with the antisymmetric matrix

$$C=-XM^tY+YM^tX,$$

where X (resp. Y) is the $d\times N$ matrix whose columns are the coordinates of the $\vec{r}_k$ (resp. $\vec{v}_k$), $M=diag(m_1,\cdots,m_N)$ and ${}^tX$ is the transpose of X. The coefficients c_{ij} of C are $c_{ij}=\sum_k m_k(-r_{ik}v_{jk}+r_{jk}v_{ik})$.

2. Relative equilibria

A relative equilibrium solution of the N-body problem is a an equilibrium of the "reduced" equations, obtained by going to the quotient by translations (this was implicitely done by choosing a galilean frame where the center of mass is fixed at the origin) and linear isometries. It is proved in[AC] that these are exactly the *rigid motions*, where the mutual distances $||\vec{r}_i-\vec{r}_j||_\epsilon$ stay constant, that is where the N-body configuration behaves as a rigid body. Moreover, the motion is necessarily of the form $X(t)=e^{t\Omega}X_0$ where Ω is an ϵ-antisymmetric operator on E and if we call E the actual space of motion (forgetting the non visited dimensions), Ω is non degenerate. Choosing an orthonormal basis which diagonalizes Ω amounts to saying that there exists a hermitian structure on the space E and an orthogonal decomposition $E \equiv \mathbb{C}^p = \mathbb{C}^{k_1} \times \cdots \times \mathbb{C}^{k_r}$ such that

$$x(t) = (x_1(t), \cdots, x_r(t)) = (e^{i\omega_1 t}x_1, \cdots, e^{i\omega_r t}x_r),$$

where x_m is the orthogonal projection on $\mathbb{C}^{k_m}$ of the N-body configuration x and the action of $e^{i\omega_m t}$ on x_m is the diagonal action on each body of the projected configuration. Such quasiperiodic motions exist only for very special configurations, called *balanced configurations* in [AC]; the classical case of central configurations, the only one to occur if the dimension of E is 3 or less, corresponds to the totally degenerate case where $\Omega = \omega J$, with J a hermitian structure on E, that is to

$$x(t) = (\vec{r}_1(t), \cdots, \vec{r}_N(t)) = e^{i\omega t}x_0 = (e^{i\omega t}\vec{r}_1, \cdots, e^{i\omega t}\vec{r}_N(t))$$

in the hermitian space $E \equiv \mathbb{C}^{2p}$. In particular, it is periodic.

3. The frequency mapping

The dynamics of a solid body is determined by its inertia tensor S_0 which in the case of an N-body configuration x_0 whose center of mass is at the origin is identified with the symmetric matrix $S_0 = X_0 M {}^t\!X_0$ with coefficients $s_{ij} = \sum_{k=1}^N m_k r_{ik} r_{jk}$. In particular, the angular momentum of a relative equilibrium is represented by the antisymmetric matrix $C = S_0\Omega + \Omega S_0$. Restricting to the case of central configurations, we associate in this way to any $2p \times 2p$ real symmetric matrix S_0 a mapping

$$J \mapsto \omega^{-1}C = S_0 J + J S_0$$

from the space of hermitian structures on E to the set of $2p \times 2p$ antisymmetric real matrices. We shall only be interested in the spectra of the matrices $\omega^{-1}C$, hence choosing an orientation for J is harmless and we shall consider only those of the form $J = R^{-1}J_0 R$, where $J_0 = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$ and $R \in SO(2p)$. The matrix C is actually J-skew-hermitian, with spectrum $i\omega\nu_1, \cdots, i\omega\nu_p$ if considered as a complex matrix. Replacing it by $J_0^{-1}RCR = \omega(J_0^{-1}SJ_0 + S)$, where $S = RS_0R^{-1}$, makes it J_0-hermitian with spectrum $\omega\nu_1, \cdots, \omega\nu_p$ that we can suppose to be ordered. We define the *frequency mapping*

$$\begin{cases} \mathcal{F} : U(p)\backslash SO(2p) \to W_p^+ = \{(\nu_1, \cdots \nu_p) \in \mathbb{R}^p, \nu_1 \geq \cdots \geq \nu_p\} \quad \text{by} \\ R \mapsto (\nu_1, \cdots, \nu_p) = \text{ ordered spectrum of } \Sigma = J_0^{-1}(RS_0R^{-1})J_0 + RS_0R^{-1}. \end{cases}$$

An obvious remark is that $\nu_1 + \nu_2 + \cdots + \nu_p = \text{trace } S_0$, which is the moment of inertia of the configuration with respect to its center of mass.

Question: *S_0 being given, describe the image of $\mathcal{F}$.*

4. Adapted hermitian structures and the Horn problem

We define a class of hermitian structures sharing some symmetries with the inertia ellipsoid defined by S_0.

From now on, we suppose that the othonormal basis of E was chosen so that

$$S_0 = diag(\sigma_1, \cdots, \sigma_{2p}).$$

Definition 1. *A hermitian structure is called "adapted " if it is of the form*

$$J_{\rho,P} = P^{-1}\begin{pmatrix} 0 & -\rho^{-1} \\ \rho & 0 \end{pmatrix} P = R^{-1}J_0R, \quad R = R_{\rho,P} = \begin{pmatrix} \rho & 0 \\ 0 & Id \end{pmatrix} P,$$

where $\rho \in SO(p)$ and $P \in SO(2p)$ is a signed permutation. When ρ can be chosen equal to Id, we speak of a "basic hermitian structure".

The permutation P being given, the adapted hermitian structures of the form $J_{\rho,P}$ are precisely the ones which send the real p-dimensional subspace generated by the basis vectors $\vec{e}_{\pi(1)}, \cdots, \vec{e}_{\pi(p)}$ onto the orthogonal subspace generated by $\vec{e}_{\pi(p+1)}, \cdots, \vec{e}_{\pi(2p)}$, where $P^{-1}(\vec{e}_i) = \epsilon_i \vec{e}_{\pi(i)}$. The basic hermitian structures are those for which the 2-planes generated by $\vec{e}_{\pi(i)}, \vec{e}_{\pi(p+i)}$ are complex lines. What makes the adapted structures remarkable is that the frequency map $\mathcal{F}$ associates to $R_{\rho,P}$ (that is to $J_{\rho,P}$) the ordered spectrum of the hermitian (in fact real symmetric) $p \times p$ matrix

$$\Sigma_{\rho,P} = \rho\sigma_-^\pi\rho^{-1} + \sigma_+^\pi,$$

where σ_-^π and σ_+^π are such that $PS_0P^{-1} = diag(\sigma_-^\pi, \sigma_+^\pi)$, that is

$$\sigma_-^\pi = diag(\sigma_{\pi(1)}, \sigma_{\pi(2)}, \cdots, \sigma_{\pi(p)}), \;\; \sigma_+^\pi = diag(\sigma_{\pi(p+1)}, \sigma_{\pi(p+2)}, \cdots, \sigma_{\pi(2p)}).$$

Finding the set $\mathcal{A}_P$ of ordered spectra of the matrices $\Sigma_{\rho,P}$ when P, that is the diagonal matrices σ_-^π and σ_+^π, is given is the real form of the *Horn problem.* Being the intersection with the positive Weyl chamber W_P^+ of a moment map[K], $\mathcal{A}_P$ is a convex polytope whose faces are given by the so called Horn's inequalities[F, KT]. Moreover, the following theorem asserts that, if $\sigma_1 \geq \cdots \geq \sigma_{2p}$, all these polytopes $\mathcal{A}_P$ are contained in the one, $\mathcal{A}_{P_0}$, corresponding to the permutation P_0 such that $\pi(i) = 2i-1, \; \pi(p+i) = 2i$.

Theorem 1 ([FFLP] Proposition 2.2)**.** *Let A and B be $p \times p$ Hermitian matrices. Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{2p}$ be the eigenvalues of A and B arranged in descending order. Then there exist Hermitian matrices $\tilde{A}$ and $\tilde{B}$ with eigenvalues $\sigma_1 \geq \sigma_3 \geq \cdots \geq \sigma_{2p-1}$ and $\sigma_2 \geq \sigma_4 \geq \cdots \geq \sigma_{2p}$ respectively, such that $\tilde{A} + \tilde{B} = A + B$.*

5. A Conjecture

We conjecture that the non-adapted hermitian structures do not contribute to the image of the frequency map, that is: $Im\mathcal{F} = \mathcal{A}_{P_0}$.

Theorem 2. *The conjecture is true when $p = 2$.*

The proof is a direct computation using spherical coordinates on the 2-sphere $U(2)\backslash SO(4)$. In this case, $Im\mathcal{F}$ is contained in an interval and it is shown that it coincides with $\cup_P \mathcal{A}_P = \mathcal{A}_{P_0}$ because the determinant mapping $J \mapsto \nu_1\nu_2$ has only the obvious critical points forced by the symmetries, that is the 6 basic hermitian structures.

Numerical testing made by Hugo Jimenez Perez for the case $p = 3$ has not infirmed the conjecture.

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Dynamical aspects of homoclinic Floer homology

Sonja Hohloch

Let (M, ω) be a symplectic manifold and φ a symplectomorphism with a hyperbolic fixed point x. Then the stable and unstable manifolds $W^s := W^s(\varphi, x)$ and $W^u := W^u(\varphi, x)$ are Lagrangian submanifolds. Thus the set of homoclinic points $W^s \cap W^u$ is the intersection set of a Lagrangian intersection problem.

In the 1960s, Arnold conjectured that, on a closed symplectic manifold, the number of fixed points of a (nondegenerate) Hamiltonian diffeomorphism ψ is greater or equal to the sum over the Betti numbers. Floer theory (cf. Floer [1]) was originally devised to detect the (minimal) number of intersection points of graph(ψ) with the diagonal in $(M \times M, \omega \oplus (-\omega))$ which is the associated Lagrangian intersection problem. For more general (usually compact) Lagrangian intersection problems, Floer theory has been studied by Fukaya & Oh & Ohta & Ono [2].

Since the homoclinic points of a symplectomorphism are associated to a Lagrangian intersection problem, one may ask if one can construct a Floer homology for this situation. In classical Floer theory, the Lagrangian submanifolds are usually compact (or at least sufficiently 'nice') whereas (un)stable manifolds are usually only injectively immersed and oscillate and accumulate wildly. This turns the analysis of J-holomorphic curves — an essentiel ingredient in Floer theory — into a quite hopeless task. On top of that, there are 'too many' intersection points.

This talk is based on two articles by the author (Hohloch [4], [5]) in which a Floer theory for homoclinic points is devised and where dynamical aspects are studied. The above mentioned analysis problem can be circumvented by considering only two-dimensional symplectic manifolds: then the analysis can be replaced by combinatorics. The problems caused by the (too) huge number of homoclinic points are untouched by this restriction. Fortunately, there are 'good' subsets of $W^s \cap W^u$, for instance the so-called 'primary points', which can serve as generator sets for the construction of Floer homology.

Depending on the chosen generator set and on the definition of the boundary operator, the properties of the resulting homoclinic Floer homologies vary drastically. For instance, some homoclinic Floer homologies transform

$$\operatorname{rk} H_*(\varphi^n, x) = n \operatorname{rk} H_*(\varphi, x)$$

whereas some stay invariant, i.e. $H_*(\varphi^n, x) = H_*(\varphi, x)$. In the latter case, one can observe growth by filtering the homology by the symplectic action

$$\operatorname{rk} H_*^{]b-\varepsilon, b+\varepsilon]}(\varphi^n, x) = n \operatorname{rk} H_*^{]b-\varepsilon, b+\varepsilon]}(\varphi, x).$$

Further growth observations are linked to the (absolute) flux studied in MacKay & Meiss & Percival [6]. In $(\mathbb{R}^2, \omega)$, define the flux of a symplectomorphism φ through a simply closed curve c by

$$\mathcal{F}lux_\varphi(c) := \operatorname{vol}_\omega(\varphi(\operatorname{Int}(c)) \cap \operatorname{Ext}(c)).$$

The flux also can be defined on a cylinder or annulus. To a homoclinic point p, associate a curve c_p which starts at x, runs through W^u to p and through W^s back to x. If c_p does not have self-intersections define

$$\mathcal{F}lux_\varphi(p) := \mathcal{F}lux_\varphi(c_p)$$

The flux through a homoclinic point coincides under certain conditions with its relative symplectic action, thus linking the flux to the action spectrum which is an important invariant in Floer homology. Moreover, the flux also interacts in a certain way with the boundary operator and the bifurcation behaviour of homoclinic points.

In our situation, the action and Maslov index are φ-invariant and thus both are no help for the observation of changes when comparing φ and φ^n. But the flux transforms $\mathcal{F}lux_{\varphi^n}(p) = n\mathcal{F}lux_\varphi(p)$. This is the same growth behaviour as the action and mean index display in classical Floer theory.

Growth phenomena are important features. Growth behaviour of symplectomorphisms has been studied for example by Polterovich [7] who used it to establish a Hamiltonian Zimmer program in [8]. Ginzburg & Gürel [3] used the growth rate of the symplectic action and the mean index in classical Floer homology for the proof of the Conley conjecture.

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Near a Double Resonance

John N. Mather

In [1], I announced results concerning Arnold diffusion. In [2], I corrected some errors in [1]. Part of the proof of these results is a detailed study of the location of Aubry sets. This talk briefly outlined these results about Aubry sets.

The results announced in [1] and [2] concern the Euler-Lagrange equation associated to a Lagrangian L in two and one half degrees of freedom that is a small perturbation of an integrable system. This means that L has the form $L(\theta,\dot\theta,t),=\ell_0(\dot\theta)+\epsilon P(\theta,\dot\theta,t)$, where $\dot\theta=(\dot\theta_1,\dot\theta_2)$ ranges over a closed ball B in $\mathbb{R}^2$, $\theta=(\theta_1,\theta_2)$ ranges over the 2-torus $\mathbb{T}^2:=\mathbb{R}^2/\mathbb{Z}^2$, and $t\in\mathbb{R}$. We assume that ℓ_0 is a C^4 function on B, P is a C^3 function on $\mathbb{T}^2\times P\times\mathbb{R}$, periodic of period 1 in t, such that $\|P\|_{C^3}=1$, and ϵ is a small positive number. In addition, we assume that $d^2P(\dot\theta)>0$, i.e. the Hessian matrix of second partial derivatives of ℓ_0 is positive definite at every point $\dot\theta\in B$.

We let $B^*\subset int\ B$ be a closed ball with the same center as B and suppose that $\Gamma\subset B^*$. We let $\widetilde{L}:T\mathbb{T}^2\times\mathbb{T}\to\mathbb{R}$ be a Tonelli Lagrangian of the form $\widetilde{L}=\widetilde{\ell}_0+\epsilon\widetilde{P}$ that extends L, and let $\beta=\beta_{\widetilde{L}}$ be the minimal average action function associated to $\widetilde{L}$. In other words if $h\in H_1(\mathbb{T}^2,\mathbb{R})$ then $\beta_{\widetilde{L}}(h)=min\{A(\mu)\}$, where μ ranges over probability measures on $T\mathbb{T}^2\times\mathbb{T}$, invariant under the Euler-Lagrange flow associated to $\widetilde{L}$, whose rotation vectors satisfy $\rho(\mu)=h$. We let $\alpha_{\widetilde{L}}:H^1\left(\mathbb{T}^2;\mathbb{R}\right)\to\mathbb{R}$ denote the Legendre-Fenchel dual of $\beta_{\widetilde{L}}$. We identify $H_1(\mathbb{T}^2;\mathbb{R})$ and $H^1(\mathbb{T}^2;\mathbb{R})$ with $\mathbb{R}^2$ in the standard way. Thus, we regard B^* as a subset of $H_1(\mathbb{T}^2;\mathbb{R})$ and $d\ell_0(B^*)$ as a subset of $H^1(\mathbb{T}^2;\mathbb{R})$, where $d\ell_0:B\to\mathbb{R}^2$ is the derivative of ℓ_0.

If ϵ is sufficiently small then $\beta|B^*$ and $\alpha|d\ell_0(B^*)$ are independent of the choice $\widetilde{L}$ of extension of L. We suppose that ϵ is so chosen. Consequently, the restriction to B^* of the Legendre-Fenchel transform $\mathcal{LF}$ associated to $\beta_{\widetilde{L}}$ is independent of the choice of extension $\widetilde{L}$. We recall that this associates to $h\in B^*$ a non-empty convex, compact subset $\mathcal{LF}(h)$ of $H^1(\mathbb{T}^2;\mathbb{R})$. We set $\mathcal{LF}(\Gamma):=\cup\{\mathcal{LF}(h):h\in\Gamma\}$.

Under suitable genericity hypotheses, described in [2], $\mathcal{LF}(\Gamma)$ has non-empty interior, and it is possible to prove results about the location of $Au(c)$ for $c\in int\ \mathcal{LF}(\Gamma)$. If $\omega\in\Gamma$ is not within a $\sqrt{\epsilon}-$neighborhood of an ω_0 admitting a strong second resonance, and c is in the relative interior of $\mathcal{LF}(\omega)$ (a compact interval) then $Au(c)$ (a subset of $\mathbb{T}^2\times\mathbb{T}$) is in a small neighborhood of a 2-torus that can be described explicitly in terms of the averaged potential

$P_{\Gamma,\omega}(\varphi) := \langle P(\theta,\omega,t)\rangle_{(\theta,t)\in\varphi}$, where $\varphi \in \mathbb{T}^1_\Gamma := (\mathbb{T}^2 \times \mathbb{T})/\mathbb{T}^2_\Gamma$ and

$$\mathbb{T}^2_\Gamma := \{(\theta_1,\theta_2,t) \in \mathbb{T}^2 \times \mathbb{T} : k_0 t + k_1\theta_1 + k_1\theta_2 = 0 \ (mod\ \mathbb{Z}^2 \times \mathbb{Z})\}.$$

The Aubry set Au_c is in a small neighborhood of $\pi^{-1}(Min(P_{\Gamma,\omega}))$, where $\pi : \mathbb{T}^2 \times \mathbb{T} \to \mathbb{T}^1_\Gamma$ is the projection and $Min\ P_{\Gamma,\omega}$ is the set of $\varphi \in \mathbb{T}^1_\Gamma$ where the average potential $P_{\Gamma,\omega}$ takes its minimum. (This set has at most two points under our genercity hypothesis.)

Near an ω_0 that admits a strong second resonance, matters are more complicated. In contrast to the discussion concerning ω away from strong second resonances, one needs to consider an averaged Lagrangian L_{ω_0} where one averages over only one fast variable. See [2, §3] for the definition of L_{ω_0}.) This averaged Lagrangian has the form $L_{\omega_0} = K_{\omega_0} + P_{\omega_0}$, where P_{ω_0} is a function defined on a 2-torus $\mathbb{T}^2_{\omega_0} = \mathbb{T}^2 \times \mathbb{T}/\mathbb{T}^1_{\omega_0}$ and $K_{\omega_0} := \frac{1}{2}g_{\omega_0}$, where g_{ω_0} is a Riemannian metric on $\mathbb{T}^2_{\omega_0}$.

We set $E_0 := -min\ P_{\omega_0}$. The Jacobi metric g_E associated to L_{ω_0} and the energy level E is a Riemannian metric when $E > E_0$, but vanishes at one point when $E = E_0$ (under our genericity hypothesis).

We let h_0 be the integral homology class associated to $\mathbb{T}^2_\Gamma/\mathbb{T}^1_{\omega_0}$ in $\mathbb{T}^2_{\omega_0} = (\mathbb{T}^2 \times \mathbb{T})/\mathbb{T}^1_{\omega_0}$. There are two cases depending on whether a $g_{E_0}-$shortest curve in h_0 is simple or not. (We overlooked the possibility that such a curve might not be simple when we wrote [1]. This led to the errors that we corrected in [2].)

In the case that all such shortest curves associated to $\omega_0 \in \Gamma$ that admit strong second resonances are simple, there is a connected component of $\mathcal{LF}(\Gamma)$ that contains both $\mathcal{LF}(\omega_0)$ and $\mathcal{LF}(\omega_1)$, where ω_0 and ω_1 are the endpoints of Γ; otherwise this is not true. The proof that I envisioned when I wrote [1] goes through in the first case; otherwise, it requires considerable modification.

The difference between the two cases appears in the discussion of $\mathcal{LF}(\mathbb{R}\cdot h_0)$, where now $\mathcal{LF}$ denotes the Legendre-Fenchel transform associated to $\beta_{L_{\omega_0}}$. Thus, $\mathcal{LF}(h)$ is a non-empty, compact, convex subset of $H^1(\mathbb{T}^2_{\omega_0};\mathbb{R})$ in the case that $h \in H_1(\mathbb{T}^2_{\omega_0};\mathbb{R})$. In the case that $g_{E_0}-$shortest curve in h_0 is simple, there is a uniform lower bound on the width of $\mathcal{LF}(\mathbb{R}\cdot h_0)$; otherwise $\mathcal{LF}(\lambda h_0)$ pinches to a point as $\lambda > 0$ converges to 0 and pinches to a second point as $\lambda < 0$ converges to 0.

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Global surfaces of section for Reeb flows on the tight 3-sphere

Umberto L. Hryniewicz

Methods from symplectic geometry have been successfully used many times to understand global questions in Hamiltonian dynamics, leading to the recent introduction of the term Symplectic Dynamics by Bramham and Hofer [1]. An important example of such methods is pseudo-holomorphic curve theory, which was introduced in the context of symplectizations by Hofer [2] to study the three-dimensional Weinstein conjecture.

A celebrated achievement of this set of ideas and techniques was obtained by Hofer, Wysocki and Zehnder [3] who proved that Hamiltonian dynamics on a strictly convex regular energy level $S \subset (\mathbb{R}^4, \omega_0)$ admits a disk-like global surface of section: this is an embedded disk $D \subset S$ such that ∂D is a periodic orbit and every trajectory (distinct of ∂D) hits $D \setminus \partial D$ transversely and ∞-many times in the future and in the past.

Such a Hamiltonian flow can be described as the Reeb flow associated to a contact form on the tight 3-sphere, and this result immediately prompts the question of which closed Reeb orbits bound a disk-like global section. This was first answered in [4] when the contact form is non-degenerate and arises from a strictly convex energy level. The more complicated situation of a general non-degenerate contact form on the tight S^3 is covered by the following statement proved in collaboration with Pedro A. S. Salomão [6].

Theorem 1. *If a contact form on the tight 3-sphere is non-degenerate then a prime closed Reeb orbit P bounds a disk-like global surface of section if, and only if, it is unknotted, has self-linking number -1, $\mu_{CZ}(P) \geq 3$ and all closed Reeb orbits P' satisfying $\mu_{CZ}(P') = 2$ are linked to P.*

Above μ_{CZ} denotes the Conley-Zehnder index. The main step of the proof is the analysis of a Bishop family of pseudo-holomorphic disks with boundary on a suitable disk spanning the orbit P. The lack of compactness of the Bishop family and our assumptions on P produce a finite-energy plane asymptotic to P. It turns out that this plane has a "fast asymptotic convergence" which plays a crucial role in foliating $S^3 \setminus P$ by planes asymptotic to P. Each plane is only a page of an open book decomposition of S^3 with binding P, and every page is a disk-like global section.

A contact form on the tight S^3 is dynamically convex if all closed Reeb orbits have Conley-Zehnder index ≥ 3. Examples are given by contact forms induced by strictly convex energy levels and, in fact, the results of [3] are proved for dynamically convex contact forms. In [5] we prove

Theorem 2. *Let D be the disk-like global section for the Reeb flow of a dynamically convex contact form on the tight S^3 obtained from* [3]. *Then any closed Reeb orbit simply linked to ∂D also bounds a disk-like global surface of section for the Reeb flow.*

Brouwer's translation theorem implies that the first return map to D has a fixed point. Consequently Theorem 2 gives new global sections.

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A compact Riemannian manifold with convex boundary that contains a complete geodesic, but no closed geodesic

Victor Bangert

(joint work with Nena Röttgen)

We construct a Riemannian metric g on the closed ball B of dimension $n \geq 4$ with the following properties:

(i) In a neighbourhood of ∂B the metric coincides with the Euclidean metric. In particular, ∂B is strictly convex.

(ii) There exists a non-constant geodesic $c : \mathbb{R} \to B$.

(iii) There does not exist a closed geodesic in B.

It is a result due to G. D. Birkhoff [2, VI.10] that such an example cannot exist in two dimensions.

Here is a brief outline of the construction of g:

First, we deform the standard metric g_0 on the ball $B \subset \mathbb{R}^n$ of radius 2 so that all the spheres $S(\rho) \subset B$ of radius $\rho \in]0, 2[$ remain strictly convex, except for $S(1)$ whose second fundamental form vanishes precisely on the vectors tangent to an irrational geodesic foliation $\mathcal{F}$ of the Clifford torus $\mathbb{T}^2 \subset S(1) \cap (\mathbb{R}^4 \times \{0\})$. This implies that there are no closed geodesics in B with respect to this metric. Moreover, we achieve that also the second fundamental form of the Clifford torus $\mathbb{T}^2$ vanishes in the direction of $\mathcal{F}$. Then the leaves of $\mathcal{F}$ are complete geodesics not only in $\mathbb{T}^2$ but also with respect to the metric on B.

Our example provides a negative answer to two natural questions. The first one was asked by W. Craig at an Oberwolfach meeting on dynamical systems. This question is related to the article [3] about microlocal analysis for Schrödinger equations. In [3] the non-existence of a trapped bicharacteristic (i.e. a bounded geodesic) is a standing hypothesis:

Suppose g is a Riemannian metric on $\mathbb{R}^n$ that is asymptotic to the Euclidean one at infinity. Assume there exists a non-constant geodesic $c : \mathbb{R} \to (\mathbb{R}^n, g)$ that *stays in a bounded subset of* $\mathbb{R}^n$. *Does this imply that there is a closed geodesic in* $(\mathbb{R}^n, g)$*?*

By property (i), our metric on B can be extended to $\mathbb{R}^n$ by the Euclidean metric and hence shows that the answer to W. Craig's question is no for $n \geq 4$.

The second question was asked by B. White [4, Remark 2.8] in connection with isoperimetric inequalities for submanifolds in Riemannian manifolds:

Let N *be a compact Riemannian manifold with* k*-convex boundary that contains a non-zero stationary* k*-varifold,* $1 \leq k \leq \dim N - 1$. *Does this imply that* N *contains a non-zero* integral *stationary* k*-varifold?*

B. White proved in [4] that this is true in the codimension one case, i.e. if $k = \dim N - 1$. In our example, the complete geodesic is part of a geodesic foliation of a 2-torus in B. This gives rise to a non-zero stationary 1-varifold that is supported on the tangent vectors to this geodesic foliation. The properties of our example can be used to prove that B does not contain a non-zero integral stationary 1-varifold.

It is not difficult to generalize our construction so as to obtain compact n-dimensional manifolds N with k-convex boundary showing that the answer to B. White's question is no for every $n \geq 4$ and every $1 \leq k \leq n - 3$, cf. [1].

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Periodic Reeb orbits in the complement of a Hopf link

Al Momin

(joint work with Umberto Hryniewicz, Pedro A. S. Salomão)

We consider Reeb vector fields on the (tight) three sphere which are tangent to a certain Hopf link and describe a condition on the infinitessimal rotation around these orbits which implies the existence of other periodic orbits. The assumptions - particularly the one that the vector field is tangent to this particular link - may seem quite restrictive, but are not actually as restrictive as they appear at first glance. For instance, it can be shown that for certain flows arising from what are known as "dynamically convex" contact forms (defined first in [HWZ98]) there is always such a link up to contact isotopy, and thus after an isotopy the above assumption is satisfied. In this convex case, the theorem we describe below can be demonstrated using global surfaces of section constructed by U. Hryniewicz and Pedro Salomão [HS10] (refining work of Hofer-Wysocki-Zehnder - see e.g. [HWZ98]) and a version of the Poincaré-Birkhoff Theorem due to J. Franks [Fr88]. However, in the general case it is not at all clear that these methods can be applied: the argument we sketch in this talk is rather variational in nature, instead of the more geometric methods that work in the dynamically convex case. Finally, we describe an application to closed geodesics on the two sphere.

Let us describe this in somewhat more detail. On $\mathbb{C} \times \mathbb{C}$, let us denote the complex coordinates by (z_1, z_2) and polar coordinates on each $\mathbb{C}$ factor ($z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$). On the unit 3-sphere $S^3 \subset \mathbb{C} \times \mathbb{C}$, there is a contact form obtained by restricting the following one-form to S^3:

$$\lambda_0(r_1, \theta_1, r_2, \theta_2) = \frac{r_1^2}{2} d\theta_1 + \frac{r_2^2}{2} d\theta_2.$$

We consider contact forms $\lambda = f \cdot \lambda_0$ where $f : S^3 \to (0, \infty)$ is a smooth positive function. We consider the following Hopf link in S^3

$$L_1 = S^3 \cap \{z_2 = 0\}, \qquad L_2 = S^3 \cap \{z_1 = 0\}$$

and now restrict consideration to forms $f\lambda_0$ such that the associated Reeb vector field X, defined by

$$\lambda(X) = 1, \qquad d\lambda(X, \cdot) = 0$$

is tangent to $L_1 \cup L_2$ along $L_1 \cup L_2$. This condition can be formulated in terms of the function f: it is true if and only if f satisfies

$$\forall z \in L_1 \cup L_2, \forall v \in \xi_z, \, df(z)v = 0$$

Remark 1. *In fact, we only need to assume the following. Suppose λ is a tight contact form on S^3, and that L is a transverse Hopf link of self-linking number* 0 *(this is a number associated with any transverse link) which is tangent to the Reeb vector field. By Gray's stability theorem we may find a diffeomorphism of S^3 so that $\phi^*\lambda = f\lambda_0$ for some function f. Then a theorem of Etnyre and Van Horn-Morris* [EHM] *implies that there is a contact isotopy which takes L to the standard example $L_1 \cup L_2$ above. This extends to an ambient isotopy (see e.g.* [Gei]*) ψ, and $\psi^*(f\lambda_0) = f' \cdot \lambda_0$, which is of the form described above and its Reeb vector field has the same dynamics as the Reeb vector field for the form λ we started out with.*

We may associate to the orbits L_1, L_2 rotation numbers as follows. First, we note that the contact structure ξ admits a global trivialization $\Phi : \xi \to \mathbb{C}$. Denote by ϕ_t the flow of the Reeb vector field X. Fixing a point $x \in L_i$ and $v \in \mathbb{C}$, we have a path $P(t)v$:

$$v \mapsto P(t)v = \Phi \circ d\phi_t(x)\Phi^{-1}(v) \in \mathbb{C}$$

We define the rotation number (let T_i denote the minimal period of the orbit L_i):

$$\rho(L_i) = \lim_{t \to \infty} \frac{\Delta \arg P(t) \cdot v}{t/T_i}$$

which exists and is independent of all choices made.

The set of homotopy classes of $S^3 \backslash (L_1 \cup L_2)$ are classified by the linking numbers with the components L_1, L_2. Given a loop a, let $p = \ell(a, L_2)$ and $q = \ell(a, L_1)$. The numbers (p, q) completely characterize the homotopy class of a i.e. another loop b is freely homotopic to a if and only if $p = \ell(b, L_2)$ and $q = \ell(b, L_1)$. Thus, we will denote homotopy classes of $S^3 \backslash (L_1 \cup L_2)$ by these integers (p, q).

To state our main theorem, we use the following notation. Suppose v, w are non-zero vectors in the portion D of $\mathbb{R}^2$ defined by $D = \{(x, y) | x > 0 \text{ or } y > 0\}$. Say $v > w$ if the argument of v is greater than the argument of w, where the argument function is defined by cutting along a ray in the fourth quadrant (say, along the ray pointing in the direction of the vector $(-1, -1)$).

Theorem 1. *Suppose $f\lambda_0$ is as above i.e.*

$$\forall z \in L_1 \cup L_2, \forall v \in \xi_z,\ df(z)v = 0$$

If $(1, \rho(L_1) - 1) < (p, q) < (\rho(L_2) - 1, 1)$ (in the notation of the previous paragraph, viewing each ordered pair as a vector in D), there is a periodic Reeb orbit in the homotopy class (p, q). If $(1, \rho(L_1) - 1) > (p, q) > (\rho(L_2) - 1, 1)$, there is a periodic Reeb orbit in the homotopy class (p, q).

We also give an application to closed geodesics on the two-sphere. Suppose g is a Riemannian or reversible Finsler metric on the two-sphere S^2, and that γ is a simple, closed geodesic. We mention the following corollary, which is a weaker version of a theorem of Angenent [A05], but which can be applied in more general situations (such as Finsler metrics, though one must change the statement appropriately).

Corollary 1. *Let $\rho(\gamma)$ denote Poincaré's inverse rotation number for the geodesic γ. Then for each (p, q) in the range $(1, 2\rho(\gamma) - 1) < (p, q) < (2\rho(\gamma) - 1, 1)$ (again using the notation described above), there is a closed geodesic. These geodesics are distinguished homotopically (up to the possibility that a geodesic is counted twice - once forwards and once backwards). The same statement holds if instead $(1, 2\rho(\gamma) - 1) > (p, q) > (2\rho(\gamma) - 1, 1)$.*

Finally, let us give a word about the proofs. The proof uses "cylindrical contact homology" - due to Eliashberg-Givental-Hofer [EGH00] - or rather a version defined on the complement of Reeb orbits described in [M10]. In fact, the theorem was already established in the case that both L_1, L_2 are elliptic non-degenerate in [M10]. The proof in the more general case uses the structure of the Hopf link to refine the arguments of [M10] about compactness of holomorphic cylinders in $\mathbb{R} \times (S^3 \backslash L)$. In the end, the orbits are produced by a "stretching the neck" argument applied to holomorphic curves that arise in the definition of certain chain maps between chain complexes of cylindrical contact homology.

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There is only one KAM curve

David Sauzin
(joint work with Carlo Carminati, Stefano Marmi)

We address the oldest open problem in KAM theory: in 1954, at the end of his ICM conference [Kol54], Kolmogorov asked whether the regularity of the solutions of small divisor problems with respect to the frequency could be investigated using appropriate analytical tools, suggesting a connection with the theory of "monogenic functions" in the sense of Émile Borel [Bo17]. In [CMS11], we provide evidence that Kolmogorov's intuition was correct by establishing a monogenic regularity result upon a complexified frequency for the KAM curves of a family of analytic twist maps of the annulus; as a consequence, these curves enjoy a property of "$\mathscr{H}^1$-quasianalyticity" with respect to the frequency.

Borel's monogenic functions may be considered as a substitute to holomorphic functions when the natural domain of definition is not open but can be written as an increasing union of closed subsets K_j, $j \in \mathbb{N}$, of the complex plane; monogenicity then means $\mathscr{C}^1$-holomorphy on each K_j, which is simply Whitney differentiability in the complex sense (*i.e.* usual Whitney differentiability with Cauchy-Rieman equations) on each K_j. As pointed out by Herman [He85], Borel's motivation was probably to ensure quasianalytic properties (unique monogenic continuation) by an appropriate choice of the sequence $(K_j)_{j\in\mathbb{N}}$, which turns out to be difficult in a general framework.

Kolmogorov's question has already been considered in small divisor problems other than KAM theory, particularly for circle maps. In his work on the local linearization problem of analytic diffeomorphisms of the circle, Arnold [Ar61] defined a complexified rotation number, with respect to which he showed the monogenicity of the solution of the linearized problem, but his method did not allow him to prove that the solution of the nonlinear conjugacy problem was monogenic. This point was dealt with by Herman [He85], who used quite a different method and also reformulated Borel's ideas using the modern terminology.

We consider the Lagrangian formulation of KAM theory for symplectic twist maps of the annulus [SZ89], [LM01], specifically for the standard family defined by

$$T_\varepsilon\colon (x,y) \mapsto (x_1,y_1), \qquad \begin{cases} x_1 = x + y + \varepsilon f(x) \\ y_1 = y + \varepsilon f(x) \end{cases}$$

in the phase space $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$, where f is a 1-periodic real analytic function with zero mean value and ε is a small real parameter. Let us fix $\tau > 0$: associated with a τ-Diophantine frequency, *i.e.* an element of

$$A_M^{\mathbb{R}} = \left\{ \omega \in \mathbb{R} \mid \forall (n,m) \in \mathbb{Z} \times \mathbb{N}^*,\ |\omega - \frac{n}{m}| \geq \frac{1}{Mm^{2+\tau}} \right\},$$

there is a KAM curve, which can be parametrized as $\theta \in \mathbb{R}/\mathbb{Z} \mapsto \gamma(\theta) = \big(\theta + u(\theta), \omega + v(\theta)\big)$ so that $T_\varepsilon\big(\gamma(\theta)\big) = \gamma(\theta+\omega)$, where u and v are small 1-periodic real analytic functions depending analytically on ε. One can impose that the mean value of u be zero and this function is then determined as the unique solution of the equation

$$u(\theta+\omega) - 2u(\theta) + u(\theta-\omega) = \varepsilon f\big(\theta + u(\theta)\big),$$

while $v(\theta) = u(\theta) - u(\theta-\omega)$. Given ω Diophantine, one can find $\rho > 0$ such that, as a function of (θ,ε), the solution u extends holomorphically to $S_R \times \mathbb{D}_\rho$, where

$$S_R = \{\theta \in \mathbb{C}/\mathbb{Z} \mid |\Im m\, \theta| < R\}, \quad \mathbb{D}_\rho = \{\varepsilon \in \mathbb{C} \mid |\varepsilon| < \rho\}.$$

We thus have a map $\omega \in A_M^{\mathbb{R}} \mapsto u \in \mathbb{B}_{R,\rho}$, where $\mathbb{B}_{R,\rho}$ is the complex Banach space of all bounded holomorphic functions on $S_R \times \mathbb{D}_\rho$. It turns out that our parametrization of the KAM curve depends periodically on the frequency, thus we can set $q = E(\omega) := \mathrm{e}^{2\pi \mathrm{i}\omega}$ and view the above as a function $q \in E(A_M^{\mathbb{R}}) \mapsto u \in \mathbb{B}_{R,\rho}$. Our answer to Kolmogorov's question in [CMS11] consists in

Theorem 1. *Suppose that f extends holomorphically to a neighbourhood of $\overline{S}_{R_0}$, $0 < R < R_0$ and $M > 2\zeta(1+\tau)$. Then there exists $\rho > 0$ such that the above u extends to a $\mathbb{B}_{R,\rho}$-valued $\mathscr{C}^1$-holomorphic function $\tilde u_M$ on the compact set K_M of the Riemann sphere $\widehat{\mathbb{C}}$ obtained as $K_M := E(A_M^{\mathbb{C}}) \cup \{0,\infty\}$ with*

$$A_M^{\mathbb{C}} = \big\{ \omega \in \mathbb{C} \mid \ \exists \omega_* \in A_M^{\mathbb{R}} \ \text{ such that } \ |\Im m\, \omega| \geq |\omega_* - \Re e\, \omega| \big\}.$$

We prove this by defining an appropriate norm on the space of all $\mathbb{B}_{R,\rho}$-valued $\mathscr{C}^1$-holomorphic functions which makes it a Banach algebra $\mathscr{C}^1_{\mathrm{hol}}(K_M, \mathbb{B}_{R,\rho})$ and by complexifying Levi-Moser's modified Newton algorithm [LM01].

Observe that the interior of K_M has two connected components, one inside the unit disk, which contains 0, and one outside, which contains ∞, and that our extension $\tilde u_M$ is holomorphic in these open sets. However, we also prove that *there is no point of the unit circle (*i.e. *corresponding to a real frequency) at which it has a holomorphic extension* (due to the density of the resonances).

Now, for any complex Banach space B, it is proved in [MS11] that $\mathscr{C}^1_{\mathrm{hol}}(K_M, B)$ is "$\mathscr{H}^1$-quasianalytic": any $C \subset K_M$ of positive linear Hausdorff measure is a uniqueness set for the functions of $\mathscr{C}^1_{\mathrm{hol}}(K_M, B)$ (if such a function vanishes on C then it must vanish on the whole of K_M). In particular, *the function $\tilde u_M$ which*

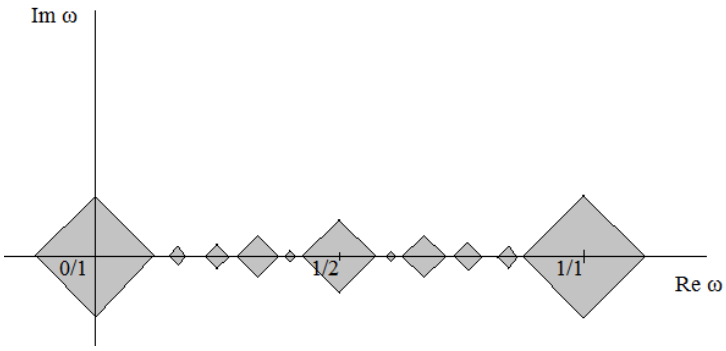

FIGURE 1. The perfect subset $A_M^{\mathbb{C}} \subset \mathbb{C}$

encodes the complexified KAM curves is determined by its restriction to any set of frequencies of positive linear measure.

Notice that Whitney smooth dependence on *real* Diophantine frequencies has been established long ago by Lazutkin and Pöschel for KAM curves and tori, but what is at stake in our work is the *complex* extension, its regularity and the uniqueness property this regularity implies. From the point of view of classical analytic continuation, the real axis in frequency space appears as a natural boundary, but our quasianalyticity result is sufficient to prove that some sort of "generalized analytic continuation" [RS02] through it is possible: the knowledge of the parametrizations on a set of positive linear measure of frequencies (real or complex) is sufficient to determine all the parametrized KAM curves: in this sense there is only one KAM curve, parametrized by one monogenic function of the frequency.

An interesting open problem would be to find a natural space of functions which contains the parametrization of the KAM curves and which is quasianalytic in the classical Hadamard sense (*i.e.* flatness at a point is sufficient to imply that the function is zero everywhere.)

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Homological periodicity and multiple closed geodesics on compact simply connected manifolds

Yiming Long

The problem of closed geodesics is a traditional and active topic in dynamical systems and differential geometry. The existence of at least one closed geodesic on every Riemannian sphere was proved by G. D. Birkhoff in 1917-1927. Then it was further proved by L. Lyusternik and A. Fet for every compact Riemannian manifold. A famous conjecture claims that there exist always infinitely many closed geodesics on every compact Riemannian manifold.

This conjecture has been extensively studied. There are many important achievements on this problem. Besides many partial results, here we recall the results of D. Gromoll and W. Meyer, V. Bangert and J. Franks.

In 1969, D. Gromoll and W. Meyer proved that if the Betti number sequence $\{b_j(\Lambda M)\}_{j\in\mathbf{N}}$ of the free loop space ΛM of a compact Riemannian manifold M is unbounded, then there exist infinitely many geometrically distinct closed geodesics on M. Then in 1976, M. Vigué-Poirrier and D. Sullivan proved that the free loop space of a compact simply connected Riemannian manifold M has no unbounded sequence of Betti numbers if and only if the rational cohomology algebra of M possess only one generator which means that

$$H^*(M;\mathbf{Q}) \cong T_{d,h+1}(x) = \mathbf{Q}[x]/(x^{h+1}=0)$$

with a generator x of degree $d \geq 2$ and hight $h+1 \geq 2$.

Around 1990, V. Bangert and J. Franks proved that on every Riemannian S^2, there always exist infinitely many geometrically distinct closed geodesics, and solved the conjecture for 2 dimensional manifolds.

For the Finsler manifolds, in 1973 A. Katok constructed a family of Finsler metrics on sphere S^d which possesses precisely $2[(d+1)/2]$ distinct closed geodesics. In 2003 H. Hofer, K. Wysocki and E. Zehnder proved that there exist either two or infinitely many distinct prime closed geodesics on a Finsler (S^2, F) provided that all the iterates of all closed geodesics are non-degenerate and the stable and unstable manifolds of all hyperbolic closed geodesics intersect transversally. In 2004, V. Bangert and Y. Long proved that on every irreversible Finsler S^2 there exist always at least two distinct prime closed geodesics (published in Math. Ann. 2010). Based on these result, it is natural to conjecture that for each positive integer n there exist positive integers $1 \leq p_n \leq q_n$ with $p_n \to +\infty$ as $n \to +\infty$

such that the number of distinct closed geodesics on a compact Finsler manifold (M, F) with $\dim M = n$ is either contained in $[p_n, q_n]$ or equals to $+\infty$.

Recently, H. Duan and Y. Long proved that on every compact simply connected Riemannian or Finsler manifold of dimension 3 or 4, there exist always at least two distinct closed geodesics (cf, Advances in Math. 2009, J. Funct. Anal., 2010).

The following theorems are the most recent results

Theorem 1. (H. Duan-Y. Long) *For every irreversible Finsler metric F on any compact simply connected manifold of dimension at least 2, there exist always at least two distinct prime closed geodesics.*

Theorem 2. (H. Duan-Y. Long) *For every reversible Finsler metric F on any compact simply connected manifold of dimension at least 2, there exist always at least two geometrically distinct closed geodesics. In particular, it holds for every such Riemannian manifold.*

The proof of these theorems starts from the $T^*(M, \mathbf{Q})$ condition mentioned before when the Betti number sequence of the free loop space of M is bounded, using corresponding information on the Betti numbers of the free loop space of M module the S^1 action, a new identity is established when there exists only one prime closed geodesic c on M. Then using estimates on the Morse indices of iterates of c, a contradiction is deduced to prove the the existence of at least one additional prime closed geodesic on M.

In this proof, prime closed geodesics are classified into rational and irrational two classes. For rational closed geodesic, a new homological periodicity theorem on the rational homologies of related level set pairs is proved. For irrational closed geodesic, a new homological quasi-periodicity theorem on such homologies is proved. Such theorems yield the mentioned new identity for the prime closed geodesic.

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Generalised Hopf bifurcations

Marc Chaperon

This talk, inspired by the author's article in memory of V.I. Arnol'd [2], was about the birth of dynamics out of statics (or the nonlinear coupling of oscillators). In generic smooth one-parameter families of vector fields, simple examples are

- the Hopf bifurcation, in which an attracting equilibrium point becomes unstable while giving rise to an attracting periodic orbit

- the "Hopf" (Sacker-Naimark) bifurcation, in which an attracting periodic orbit becomes unstable while giving rise to an attracting invariant 2-torus.

These $\mathbb{T}^0 \to \mathbb{T}^1$ and $\mathbb{T}^1 \to \mathbb{T}^2$ bifurcations are *not* paralleled by a $\mathbb{T}^2 \to \mathbb{T}^3$ bifurcation which, far from being generic, requires infinitely many conditions [3]: in generic one-parameter families, the invariant 2-torus will break down when it loses attractivity and chaos ("turbulence" [6]) will develop.

Thus, to study the birth of n-tori with $n > 2$, it is best to follow René Thom's advice: "Look for the organising centre or phenomena" and consider families depending on more parameters. The result is that $\mathbb{T}^0 \to \mathbb{T}^n$ and $\mathbb{T}^1 \to \mathbb{T}^{n+1}$ bifurcations (among others) occur smoothly in generic families depending on at least n parameters: attracting[1] invariant n-tori (and more suprising invariant submanifolds) are born smoothly at partially elliptic stationary points in generic families of vector fields (resp., transformations) depending on at least n parameters.

Here, "partially elliptic" means that the eigenvalues of the linearised dynamics which lie on the imaginary axis (resp., unit circle) are simple and consist of n pairs of conjugate complex numbers[2]. The corresponding values u_0 of the parameter u form a submanifold of codimension n, but we shall see soon that the set of those u for which the attracting invariant submanifold exists (and depends differentiably on u) contains an open subset with nonempty *open* tangent cone at u_0, implying that the phenomenon is not negligible—for $n > 1$, turbulence as above can be observed when the parameter crosses the boundary of this open subset.

Under a mild nonresonance condition, taking a suitable chart and restricting the dynamics to a central manifold, one may assume that, near u_0 and the partially elliptic stationary point considered in phase space, the dynamics under study form a local family Z_u (resp., h_u) of vector fields on (resp., transformations of) $\mathbb{C}^n$ having third order contact at $0 \in \mathbb{C}^n$ with a normal form

$$N_u(z) = \Big(z_j \Big(\lambda_j(u) + i\mu_j(u) - \sum_{\ell=1}^{n} \big(a_{j\ell}(u) + i b_{j\ell}(u) \big) |z_\ell|^2 \Big) \Big)_{1 \le j \le n},$$

(resp., the time one of its flow), where $\lambda_j, \mu_j, a_{j\ell}, b_{j\ell}$ are differentiable real functions with $\lambda_j(u_0) = 0$ (ellipticity). Here is the main result of [2]:

Theorem 1. (birth lemma) *Under those hypotheses*[3]*, assume that, for some tangent vector* v_0 *at* u_0*, the vector field*

$$\widetilde{\xi}_{u_0,v_0}(\zeta) = \Big(\zeta_j \Big(D(\lambda_j + i\mu_j)(u_0) v_0 - \sum_{\ell=1}^{n} \big(a_{j\ell}(u_0) + i b_{j\ell}(u_0) \big) |\zeta_\ell|^2 \Big) \Big)_{1 \le j \le n}$$

on $\mathbb{C}^n$ *admits a normally hyperbolic compact invariant manifold* $\widetilde{\Sigma} \subset \mathbb{C}^n$*. Then, there is an open subset* $\mathcal{U}_{u_0,v_0}$ *of parameter space* $\mathcal{U}$ *with the following properties:*

i) *Its closure contains* u_0*.*

ii) *Its tangent cone at* u_0 *is an open cone with vertex* 0 *containing* $\mathbb{R}_+^* v_0$*.*

[1] More generally, normally hyperbolic.

[2] For maps, [2] treats similarly the case where there are $n-1$ pairs, plus -1 but not $+1$.

[3] Even when there are less than n parameters or the family is not generic.

iii) *Every* Z_u *(resp.* h_u*) with* $u \in \mathcal{U}_{u_0,v_0}$ *has a compact normally hyperbolic invariant manifold* S_u *diffeomorphic to* $\widetilde{\Sigma}$*, whose index*[4] *is that of* $\widetilde{\Sigma}$ *for* $\widetilde{\xi}_{u_0,v_0}$*, depending nicely on* u *and tending to* $\{0\}$ *when* $u \to u_0$*.*

iv) *Precisely, there is an open cone* $V \ni v_0$ *of* $T_{u_0}\mathcal{U}$ *with vertex* 0 *such that each* $\widetilde{\xi}_{u_0,v}$ *with* $v \in V$ *has a unique normally hyperbolic compact invariant manifold* $\widetilde{\Sigma}_v$ *diffeomorphic to* $\widetilde{\Sigma}$ *and* C^1*-close to it up to homothety*[5]*; every smooth* $\gamma : (\mathbb{R}_+, 0) \to (\mathcal{U}, u_0)$ *with* $\dot\gamma(0) = v \in V$ *satisfies* $\gamma(\varepsilon) \in \mathcal{U}_{v_0,v_0}$ *for* $\varepsilon > 0$ *small enough, and* $\lim_{\varepsilon\to 0} \varepsilon^{-\frac{1}{2}} S_{\gamma(\varepsilon)} = \widetilde{\Sigma}_v$ *in the at least* C^1 *sense.*

The last statement shows that the invariant manifold S_u arises rather suddenly, as in classical Hopf bifurcations.

The vector field $\widetilde{\xi}_{u_0,v_0}$ being $\mathbb{U}(1)^n$-invariant, so is $\widetilde{\Sigma}$ by local uniqueness; passing to the quotient, we see that the $\mathbb{O}(1)^n$-invariant submanifold $\Sigma = \widetilde{\Sigma} \cap \mathbb{R}^n$ of $\mathbb{R}^n$ is a normally hyperbolic invariant manifold of the $\mathbb{O}(1)^n$-invariant vector field

$$\xi_{u_0,v_0}(r) = \sum_j r_j \Big(D\lambda_j(u_0)v_0 - \sum_\ell a_{j\ell}(u_0)\, r_\ell^2 \Big) \frac{\partial}{\partial r_j}$$

on $\mathbb{R}^n$. Here is a weak converse: if ξ_{u_0,v_0} has a normally hyperbolic $\mathbb{O}(1)^n$-invariant submanifold Σ *on which it vanishes identically*, then the hypothesis of the birth lemma is satisfied by $\widetilde{\Sigma} = \{z \in \mathbb{C}^n : (|z_1|, \dots, |z_n|) \in \Sigma\}$. Two cases of interest:

- *Tori.* When $\Sigma = \{r : \forall j\, D\lambda_j(u_0)v_0 = \sum_\ell a_{j\ell}(u_0)\, r_\ell^2\}$ with $\big(a_{j\ell}(u_0)\big)$ invertible, it consists of equilibrium points; if they are hyperbolic, we get the n-tori mentioned before.
- *Moment-angle manifolds.* If $\Sigma = \{r : F(r) = b\}$ with $F(r) = \sum_j \Lambda_j r_j^2$, $b, \Lambda_1, \dots, \Lambda_n \in \mathbb{R}^c$, $\operatorname{conv}(\Lambda_1, \dots, \Lambda_n) \not\ni 0$ and b a regular value of F, then $\widetilde{\Sigma}$ is called a *moment-angle manifold* and can have various topologies [1, 5]. When $\xi_{u_0,v_0}(r) = \sum_j r_j \Lambda_j \cdot \big(b - F(r)\big) \frac{\partial}{\partial r_j}$ (the dot stands for the scalar product), it equals $-\frac{1}{2}\nabla|F(r) - b|^2$ and therefore admits Σ as a pointwise invariant normally hyperbolic attractor; hence, the birth lemma applies.

In this example, $\widetilde{\Sigma}$ is a $(2n-1)$-sphere if $c = 1$, an n-torus if $c = n$. Of course, this ξ_{u_0,v_0} is too particular to arise in *generic* n-parameter families[6] but, as normal hyperbolicity is open, the birth of normally hyperbolic attractors diffeomorphic to $\widetilde{\Sigma}$ will be observed in generic n-parameter families nearby. This is the spirit of the following result, where M and $\mathcal{U}$ are separable manifolds:

Corollary 1. *Assume that the vector field* $\xi(r) = \sum_j r_j \Big(\nu_j v - \sum_\ell \gamma_{j\ell}\, r_\ell^2 \Big) \frac{\partial}{\partial r_j}$ *on* $\mathbb{R}^n$*,* $\nu_j, \gamma_{j\ell} \in \mathbb{R}$*, has an* $\mathbb{O}(1)^n$*-invariant normally hyperbolic invariant manifold* Σ

[4] Dimension of the leaves of the stable foliation.

[5] Because normal hyperbolicity is open and $\eta^{\frac{1}{2}}\widetilde{\Sigma}_v$ is invariant by the flow of $\widetilde{\xi}_{u_0,\eta v}$ for all positive η when $\widetilde{\Sigma}_v$ is invariant by the flow of $\widetilde{\xi}_{u_0,v}$

[6] Normally hyperbolic invariant manifolds Σ of large dimension are not always easy to find.

whose intersection with $\mathbb{R}^n_+$ *is connected (normal hyperbolicity being absolute when* Σ *meets some coordinate hyperplane* $\{r_j = 0\}$ *in which it is not contained). Then, for* $\dim M \geq 2n$ *and* $\dim \mathcal{U} \geq n$*, there exists a* nonempty, C^3*-open set of smooth families* $X : \mathcal{U} \times M \to TM$ *of vector fields (resp.,* $f : \mathcal{U} \times M \to M$*) for which the birth lemma ensures at some point* (u_0, x_0) *the birth of normally hyperbolic invariant submanifolds of* X_u *(resp.,* f_u*) diffeomorphic to* $\widetilde{\Sigma}$*. The same holds true if "normally hyperbolic" is replaced by "normally hyperbolic and attracting".*

For example, if $n = 3$, Σ can be a periodic orbit, yielding a 4-torus $\widetilde{\Sigma}$ in $\mathbb{C}^3$. Of special interest is the case of the birth lemma where $\widetilde{\Sigma}$ is an attracting embedded sphere of codimension 1 around the origin, a *bona fide* generalisation of the Hopf bifurcation in which every nonzero forward orbit of Z_u (resp., h_u) in a fixed neighbourhood of the origin in $\mathbb{C}^n$ tends to S_u for $u \in \mathcal{U}_{u_0,v_0}$ close enough to u_0. The approach *via* the case $c = 1$ of moment-angle manifolds [4] is interesting because the dynamics on S_u varies a lot, ξ_{u_0,v_0} having no dynamics on Σ, but this provides quite a narrow set in parameter space. A very wide set is furnished by the rough birth lemma [2] stating that, for positive $D\lambda_j(u_0)v_0$ and $a_{j\ell}(u_0)$, a family of attracting Čech homology $(2n-1)$-spheres bifurcates as in the bir! th lemma. Conditions for these "spheres" to be normally hyperbolic differentiable hypersurfaces S_u will be studied in a forthcoming paper with Santiago López de Medrano, together with the bifurcations that can occur inside S_u—the birth lemma can indeed apply for the same (u_0, v_0) (with different $\mathcal{U}_{u_0,v_0}$) to many manifolds $\widetilde{\Sigma}$, among which a big embedded $(2n-1)$-sphere containing all the others.

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KAM à la R

Jürgen Pöschel

In [4] Rüssmann proposed – quoting from his abstract – a new variant of the KAM-theory, containing an artificial parameter q, $0 < q < 1$, which makes the steps of the KAM-iteration infinitely small in the limit $q \downarrow 1$. ... The new technique of estimation differs completely from all what has appeared about KAM-theory in the literature up to date. Only Kolmogorov's idea of local linearization and Moser's modifying terms are left. The basic idea is to use the polynomial structure in order to transfer, at least partially, the whole KAM-procedure outside of the original domain of definition of the given dynamical system.

It is the purpose of this talk to make this scheme accessible in an even simpler setting, namely for analytic perturbations of constant vector fields on a torus.

Let N denote a constant vector field on the n-torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ describing uniform rotational motions with frequencies $\omega = (\omega_1, \dots, \omega_n)$. Putting N into normal form, we have $N = \omega$. A small perturbation $X = N + P$ usually destroys this simple flow, due to frequency drifts and the effect of resonances. If, however, the frequencies ω are strongly nonresonant, the perturbation P is sufficiently smooth and small, and if we are allowed to add a small correctional n-vector to adjust frequencies, then X is conjugate to ω. This is the content of the classical KAM theorem with modifying terms for this model problem, as introduced by Moser [2].

The precise setting is the following. We consider N as a vector field in normal form depending on the frequencies ω as parameters. These vary in some neighbourhood of a fixed compact set $\Omega \subset \mathbb{R}^n$ consisting of strongly nonresonant frequencies. That is, each $\omega \in \Omega$ satisfies

$$|\langle k, \omega \rangle| \geq \frac{\alpha}{\Delta(|k|)}, \qquad 0 \neq k \in \mathbb{Z}^n,$$

with some $\alpha > 0$ and some *Rüssmann approximation function* Δ. These are continuous, increasing functions $\Delta \colon [1, \infty) \to [1, \infty)$ such that $\Delta(1) = 1$ and

$$\int_1^\infty \frac{\log \Delta(t)}{t^2} \mathrm{d}t < \infty.$$

The perturbation P is assumed to be analytic in the angles $\theta \in \mathbb{T}^n$ and may depend analytically on the parameters ω as well. The complex domains are

$$D_s = \{\theta : |\mathrm{Im}\,\theta| < s\}, \qquad \Omega_h = \{z : |z - \Omega| < h\},$$

where $|\cdot|$ denotes the *max*-norm for complex vectors, while it denotes the *sum*-norm for integer vectors. To simplify matters considerably, we employ the *weighted norms*

$$|P|_{s,h} = |P|_{s,\Omega_h} = \sup_{\omega \in \Omega_h} \sum_{k \in \mathbb{Z}^n} |p_k(\omega)| \mathrm{e}^{|k|s}, \qquad P = \sum_{k \in \mathbb{Z}^n} p_k(\omega) \mathrm{e}^{\mathrm{i}\langle k, \theta \rangle}.$$

Finally, with any approximation function Δ we associate another such function Λ by setting $\Lambda(t) = t\Delta(t)$.

KAM Theorem. *Suppose $X = N + P$ is real analytic on $D_s \times \Omega_h$ with*

$$|P|_{s,h} = \epsilon < \frac{h}{16} \leq \frac{\alpha}{32\Lambda(\tau)},$$

where τ is so large that

$$r := 8 \int_\tau^\infty \frac{\log \Lambda(t)}{t^2} \mathrm{d}t < \frac{s}{2}.$$

Then there exists a real map $\varphi\colon \Omega \to \Omega_h$, *and for each* $\omega \in \Omega$ *a real analytic diffeomorphism* Φ_ω *of the* n*-torus, such that*

$$\Phi_\omega^*(\varphi(\omega) + P) = \omega.$$

Moreover, $|\varphi - \mathrm{id}|_\Omega \le \epsilon$ *and* $|\Phi - \mathrm{id}|_{s-2r,\Omega} \le \Lambda(\tau)\alpha^{-1}\epsilon$.

The above smallness condition does not depend explicitly on the dimension n of the problem. However, this dimension enters implicitly through the small divisor conditions and Dirichlet's lemma which states that for nonresonant vectors ω,

$$\min_{0<|k|\le K} |\langle k, \omega\rangle| \le \frac{|\omega|}{K^{n-1}}.$$

Hence the approximation function Δ has to grow at a rate depending on n in order to obtain admissible frequencies. A typical example is $\Delta(t) = t^\nu$ with $\nu > n - 1$.

We prove the theorem by an iterative process of successive coordinate transformations proposed by Kolmogorov [1]. However, at variance with the crustimoney proseedcake [5, Chapter IV], we use a scheme of estimates proposed by Rüssmann, which does not rely on superlinear convergence speeds, but aims to decrease the size of the perturbation just a tiny bit at each step.

To this end, we split P into an infrared part $\tilde{P}$ and an ultraviolet part $\hat{P}$. However – and this is a new twist – $\hat{P}$ also contains fractions of the Fourier coefficients of *low order.* As a result, $\tilde{P}$ will be bounded on a *larger domain*, with even a *better bound* than P itself.

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Approximating pseudo-rotations by integrable systems using holomorphic curves

Barney Bramham

Anatole Katok asks the following fairly informal question.

Question 1 ([13]) In low dimensions (2 for maps, 3 for flows) are all conservative dynamical systems with zero topological entropy a limit of integrable systems?

In a step towards answering this affirmatively, we prove the following in [6]. All maps we discuss here are orientation preserving, and area preservation is always with reference to the Euclidean volume form $dx \wedge dy$.

Theorem 1 ([6])**.** *Let φ be a smooth, area preserving, diffeomorphism of the closed 2-disk. Assume that φ has precisely one periodic point which without loss of generality is the origin $0 \in D$. (In particular φ has zero topological entropy.) Then there exists a sequence $\{\varphi_j\}_{j\in\mathbb{N}}$ of smooth diffeomorphisms converging in $\mathrm{Diff}^\infty(D)$ to φ, such that for each $j \in \mathbb{N}$ there exists $g_j \in \mathrm{Diff}^\infty(D)$ with $g_j(0) = 0$, so that*

$$g_j^{-1} \circ \varphi_j \circ g_j = R_{p_j/q_j}$$

for some $p_j, q_j \in \mathbb{Z}$, $q_j \geq 1$, and $R_{p_j/q_j} : D \to D$ denotes the periodic rotation map $z \mapsto e^{2\pi i p_j/q_j} z$.

The class of conservative dynamical systems that this result addresses has an interesting history, and are referred to as smooth pseudo-rotations:

Definition 1. *An area preserving homeomorphism of the closed disk having precisely one periodic point (which must automatically be a fixed point) is called a* pseudo-rotation*.*

Note that Franks [9] shows that in fact any area preserving disk homeomorphism with a finite number of periodic points is a pseudo-rotation. Obvious examples of pseudo-rotations are maps that are conjugate to a rotation through an irrational angle, hence the terminology.

However these are not the only examples. In 1970 Anosov and Katok [1] discovered "exotic" pseudo-rotations which cannot be conjugated to a rotation. More precisely, they developed a technique, known as the "approximation by conjugation method", which allowed them to construct pseudo-rotations exhibiting surprising dynamical properties, including ergodicity. This of course implies the existence of dense orbits. It is not obvious how one approximates a system with a dense orbit by integrable ones. Indeed, if we take the definition of integrable to mean the time-1 map of an *autonomous* Hamiltonian $H : D \to \mathbb{R}$ on the disk, then each orbit of an integrable system is confined to a closed set of measure zero. From this point of view the two behaviors are entirely different. Nevertheless, the Anosov-Katok ergodic examples were, by construction, limits of integrable systems. More precisely, limits of maps conjugate to rational rotations, providing some evidence for question 1. Theorem 1 above shows that infact all pseudo-rotations are a limit of maps conjugate to rational rotations.

We point out that it is currently unclear whether the approximation maps provided by theorem 1 can be made to preserve the standard area form. This seems most likely possible, but is still work in progress.

The proof of theorem 1 uses pseudo-holomorphic curve techniques from symplectic geometry. The relevant setup is described in [7], where the focus is on other applications. Very roughly the construction of the approximating maps goes as follows. The discrete dynamical system under study given by the area preserving disk map φ, is viewed as the time-1 map of a smooth time dependent Hamiltonian on the disk. The trajectories of this continuous system can be identified with a class of pseudo-holomorphic planes in a four dimensional almost complex manifold $(\mathbb{R}^2 \times D, J)$. These are characterized as minimizing a certain energy amongst all pseudo-holomorphic planes. The planes combine to foliate the whole four dimensional space. Denote this foliation $\mathcal{F}_\varphi$. (All the dynamical information is contained in the almost complex structure J which depends explicitly on the Hamiltonian.)

It turns out that to some degree the process just described is reversible. That is, any suitable foliation of $(\mathbb{R}^2 \times D, J)$ by pseudo-holomorphic curves gives rise to a diffeomorphism of the disk. Foliations by pseudo-holomorphic curves were first constructed by Hofer, Wysocki, and Zehnder in their seminal papers on Reeb flows on energy surfaces diffeomorphic to the three-sphere [11, 12]. In [3] these techniques were adapted for the setup described here, and much further developed in [5]. By constructing a sequence of foliations $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \dots$ whose energy decays to zero, a well developed compactness theory of pseudo-holomorphic curves [2] can be applied and we conclude that $\mathcal{F}_j \to \mathcal{F}_\varphi$ in a C^∞_{loc}-sense, as $j \to \infty$. From this we obtain maps φ_j converging to φ, now in a C^∞-sense on the whole disk. It is possible to prove that each foliation $\mathcal{F}_j$ has a certain amount of symmetry which translates into the property that each φ_j is a root of unity. That is, for each $j \in \mathbb{N}$ there exists $q_j \in \mathbb{N}$ such that $\varphi_j^{(q_j)} = \mathrm{id}_D$. Such maps can be smoothly conjugated to a rational rotation R_{p_j/q_j} for some $p_j \in \mathbb{Z}$, completing the argument.

The fact that φ is a pseudo-rotation enters the proof when showing that each approximating foliation $\mathcal{F}_j$ has the desired symmetry. One would of course like to know whether this framework can used to give a complete answer to Katok's question. This will be the subject of future work.

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Arithmeticity for some higher rank abelian actions

Federico Rodriguez Hertz
(joint work with Anatole Katok)

Given a matrix $A \in SL(d,\mathbb{Z})$, we say that A is totally irreducible if the characteristic polynomial of all powers of A are irreducible. Dirchlet's Unit Theorem gives that the centralizer in $SL(d,\mathbb{Z})$ of a totally irreducible matrix $A \in SL(d,\mathbb{Z})$, $Z(A) \subset SL(d,\mathbb{Z})$ is a finite extension of $\mathbb{Z}^{r+c-1}$ where r is the number of real eigenvalues and c is the number of pairs of complex (non-real) eigenvalues of A (hence $r + 2c = d$). This shows that the maximal possible rank of a totally irreducible linear action of $\mathbb{Z}^k \in SL(d,\mathbb{Z})$ is $d - 1$ and is precisely when all the eigenvalues are real.

In this way we get interesting non trivial actions of $\mathbb{Z}^{d-1}$ into a d dimensional manifold ($\mathbb{T}^d$ in the present case). Also one can define affine actions simply by adding a cocycle to the linear action. It turns out that any such cocycle trivializes on a finite index subgroup. The natural question arises, what are other possible non trivial actions.

A first natural construction is to blow up a fixed point of the action and then fill the blown up point with something. Or blow up two fixed points (take a finite index subgroup to get two fixed points) and add a handle. Finally, in this maximal rank case the centralizer of A is $\mathbb{Z}^{d-1} \times \{\pm Id\}$ we can just take the quotient $\mathbb{T}^d/\{\pm Id\}$ and blow up the fixed points of $-Id$ to get again a manifold. This type of construction was first done in [8], (see also [9]).

Finally one can try to blow up a non periodic orbit. It is worth noting that from Berend's Theorem [1] we have that the only closed invariant sets are either finite or $\mathbb{T}^d$. So a non periodic orbit will be a dense orbit, and hence the blow up would be like the building of a Denjoy counterexample for circle rotations. It is yet not clear to us if such a construction can be done, and in case it can be done, what is the degree of smoothness allowed.

Our next step is to try to understand general non trivial $\mathbb{Z}^{d-1}$ actions on a d-dimensional manifold M. The non triviality condition will come in a condition on Lyapunov exponents and entropy w.r.t. some invariant measure. Our first result in this direction is a joint work of B. Kalinin and A. Katok with the author in [5] where we get that such a measure has to be absolutely continuous w.r.t. Lebesgue measure.

Theorem 1. *If α is a $\mathbb{Z}^{d-1}$ action on a d-dimensional manifold, $d \geq 3$, preserving an ergodic measure μ and the kernels of its Lyapunov exponents are in general position then μ is absolutely continuous w.r.t. Lebesgue measure.*

After the prove of this theorem, with the examples at hand and in light of the results in [4] and [9] it was natural to ask the following (see Oberwolfach report [6] and [5]):

What are the possible values of the entropy for different elements of the action, or what is equivalent by Pesin's entropy formula, what are the possible values of Lyapunov exponents. Are they logarithms of algebraic numbers and in this case is the degree the dimension of M?.

Also it is asked if it is possible to build such an action on any manifold as in the rank 1 case, see [2, 3], or if on the contrary there is some restriction for instance in the topology of the manifold.

Also, another natural question is about finiteness property of such measure, since the measure is absolutely continuous w.r.t. Lebesgue then there are at most countably many ergodic components, are they finite?

And finally what about the measurable classification of such systems? Are they always measurably conjugated to algebraic actions? It is worth remarking that in [7] it is proven that knowing the entropy values is not enough to solve the conjugacy problem unlike the rank one Bernoulli systems.

In the following theorem we solve all this questions:

Theorem 2. *Let $\alpha : \mathbb{Z}^d \to Diff^r(M^{d+1})$, $d \geq 2$ and $r > 1$ be a maximal rank abelian action preserving an ergodic measure μ and such that the kernels of its Lyapunov exponents are in general position. Then there is a finite index subgroup $\Gamma \subset \mathbb{Z}^d$, a measure ν on M^{d+1} invariant by the restriction of α to Γ, $\alpha|\Gamma$, and finitely many elements $n_1, \ldots, n_k \in \mathbb{Z}^d$, $k = card(\mathbb{Z}^d/\Gamma)$ such that*

$$\mu = \frac{\alpha(n_1)_*\nu + \cdots + \alpha(n_k)_*\nu}{k}.$$

There is an affine action α_0 on an infratorus T and a bi-measurable bijection $H : (T, \lambda) \to (M, \nu)$ where λ is Lebesgue measure on T (projected Haar measure) conjugating the restriction of α to Γ with α_0.

Moreover there is an open $\alpha|\Gamma$-invariant set $U \subset M^{d+1}$ of full ν measure and an open α_0 invariant set $V \subset T$, the complement of a finite α_0-invariant set such that the inverse of $H : V \to U$ coincides (mod 0) with a continuous onto map, $h : U \to V$ conjugating $\alpha|\Gamma$ with α_0.

Finally, for every ϵ there is a set Λ_ϵ of ν-measure bigger than $1 - \epsilon$ and a diffeomorphism $h_\epsilon : U \to V$ that coincides with h on Λ_ϵ.

In particular whenever one has such an action, then the fundamental group of M^{d+1} contains a copy of $\mathbb{Z}^{d+1}$ and hence this gives a highly nontrivial restriction on the topology of M^{d+1}. Also, the support of the measure has non trivial topology hence there are at most finitely many such ergodic measures.

Finally, we get as a Corollary, using the information on the topology of the manifold and the global rigidity result in [10] that if in addition the actions has an Anosov element then the action is smoothly conjugated to the affine action.

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Birkhoff normal form and nonlinear scattering for PDEs

Walter Craig

The theory of Hamiltonian partial differential equations (PDEs) takes the point of view that a PDE of evolution is a dynamical system posed on an appropriate space of functions for its phase space, studying the details of orbits and the principal structures that are invariant under the flow. Examples of this include the constructions of KAM tori [5][2], the analysis of invarant measures, descriptions of stable and unstable varieties to fixed points (and to other geometrical objects), Nekhorashov stability theorems, and the construction of orbits which exhibit action cascades. Normal forms have long played a rôle in Hamiltonian dynamical systems, being useful to reduce a Hamitonian system to exhibit its essential nonlinearities. They are increasingly being used in the analysis of Hamiltonian PDEs, see [5] for example. However there is an important difference in the character of a normal form and the importance of classical resonances between problems whose linearization has continuous spectrum, as opposed to the case of discrete spectrum [6]. In this talk this difference is made explicit in a Birkhoff normal form for the nonlinear Schrödinger equation posed on the space $x \in \mathbb{R}^d$, for which all nonresonant and all *resonant* terms of the fourth order term of the Hamiltonian can be removed by an appropriate canonical transformation of a given Hilbert space. The normal form and the Birkhoff normal forms transformation have implications for the nonlinear scattering of solutions. We pursue the analysis as an instructive example in the particular case of the nonlinear Schrödinger equation, however we believe that the phenomenon is more general, and relevant to the analysis of the evolution of other PDEs whose linearization is dominated by continuous spectrum. An averaging theory approach to nonlinear PDEs which is formally from a quite different point of view, but which has surprising connections to the present work, appears in [3]. The results in this note represent work in progress, which is joint with A. Selvitella and Y. Wang of McMaster University.

Nonlinear Schrödinger equation. The cubic nonlinear Schrödinger equation on $\mathbb{R}^d$is

$$i\partial_t u = \tfrac{1}{2}\Delta_x u - \sigma|u|^2 u \ , \tag{1}$$

where $\sigma = +1$ is the defocusing case and $\sigma = -1$ is the focusing case. The Hamiltonian, or energy functional is given by the expression

$$\begin{aligned} H(u) &= \int \tfrac{1}{2}|\nabla u|^2 \, dx + \int \tfrac{\sigma}{2}|u|^4 \, dx \\ &= H^{(2)} + H^{(4)} \ . \end{aligned} \tag{2}$$

With regard to the Hamiltonian (2), the equations (1) can be rewritten in complex symplectic coordinates as the system

$$\partial_t u = i\mathrm{grad}_{\bar{u}} H := X^H(u) \ ,$$

whose flow, or solution map, we will denote by $\varphi_t(u)$. The linearization of equations (1) about the equilibrium solution $u = 0$ is just the free Schrödinger equation, with Hamlitonian $H^{(2)}$, whose frequencies are given by the dispersion relation

$$\omega(k) = \tfrac{1}{2}|k|^2 \ .$$

We denote the linear flow by $\Phi_t(u)$. From the character of the flow about the equilibrium $u = 0$, it is clear that it is an elliptic stationary point whose linear eigenvalues are given by $\pm i\omega(k)$.

Normal forms. A normal forms transformation is intended to simplify a Hamiltonian system, retaining only essential nonlinearities. Considering the Hamiltonian (2) about the elliptic equilibrium $u = 0$, a normal form entails a near identity canonical transformation $v = \tau(u)$ such that $\tilde{H}(v) := H(u)$ is of the form

$$\tilde{H}(v) = H^{(2)} + \big(Z^{(3)} + \cdots + Z^{(N)}\big) + \tilde{R}^{(N+1)} \ .$$

In finite dimensional Hamiltonian systems the vector field $X^{H^{(2)}}$ has finitely many frequencies $\omega(k)$, $k = 1, \dots m$, and the *resonant terms* $Z^{(n)}$ are associated with resonance conditions

$$\sum_k \big(\omega(k)(p_k - q_k)\big) = \langle \omega, P - Q \rangle \ , \qquad P,\, Q \in \mathbb{Z}^m \ , \quad |P| + |Q| = n \ .$$

All other terms of the Hamiltonian up to order N can be eliminated by the appropriate choice of $\tau(u)$. The situation is similar for Hamiltonian PDE posed over compact domains, such as $x \in \mathbb{T}^d$ as in [5].

This situation contrasts with the case (1) above when posed over $x \in \mathbb{R}^d$. Firstly, the Fourier transform represents a canonical transformation,

$$\hat{u}(k) = \mathcal{F}(u)(k) = \frac{1}{\sqrt{2\pi}^d} \int e^{-ik\cdot x} u(x)\, dx \ .$$

From the Plancherel identity, the Hamiltonian is given by

$$(3) \quad H^{(2)} + H^{(4)} = \int \omega(k) |\hat{u}(k)|^2 \, dk + \iiint_{\{k_1+k_2=k_3+k_4\}} \tfrac{\sigma}{2} \hat{u}(k_1)\hat{u}(k_2)\hat{\bar{u}}(k_3)\hat{\bar{u}}(k_4)\, dk_1 dk_2 dk_3 \ .$$

Following Birkhoff's receipe, a fourth order normal form is obtained through a transformation given by the flow of a Hamiltonian vector field $X^{G^{(4)}}$, where $G^{(4)}$ is a solution of the homological equation

$$(4) \qquad \{H^{(2)}, G^{(4)}\} = H^{(4)} \ .$$

In the case of the nonlinear Schrödinger equation this has as solution

$$G^{(4)} = -i \iiint_{\{k_1+k_2=k_3+k_4\}} \tfrac{\sigma}{2} \frac{\hat{u}(k_1)\hat{u}(k_2)\hat{\bar{u}}(k_3)\hat{\bar{u}}(k_4)}{\omega(k_1)+\omega(k_2)-\omega(k_3)-\omega(k_4)}\, dk_1 dk_2 dk_3 \ .$$

The denominator is rewritten to exhibit the form of a convolution kernel

$$\omega(k_1)+\omega(k_2)-\omega(k_3)-\omega(k_4) = \tfrac{1}{2}(|k_1|^2+|k_2|^2-|k_3|^2-|k_4|^2) = -(k_1-k_3)\cdot(k_2-k_3) \ ,$$

where we have used the constraint that $k_1 + k_2 - k_3 - k_4 = 0$, which is imposed by the conservation of momentum. The fourth order Birkhoff normal forms transformation is given by the time-1 flow of the Hamiltonian vector field

$$(5) \quad X^{G^{(4)}}(u)(x) = i\mathrm{grad}_{\bar{u}(x)} G^{(4)} = -\iint_{\mathbb{R}^{2d}} u(x_1)u(x_2)\bar{u}(x-x_1-x_2)\ \mathrm{sgn}((x_1-x)\cdot(x_2-x))\, dx_1 dx_2 \ .$$

The basic question presented by this situation is whether the flow $\psi_s(u)$ of the vector field $X^{G^{(4)}}(u)$ exists, and on which Banach space of functions $u(x)$.
Define the Hilbert space $H^{r,r}(\mathbb{R}^d) := \{u \in L^2 \, : \, \partial_x^r u \ , \ x^r u \in L^2\}$.

Theorem 1. *Set $r > d/2$, then the vector field $X^{G^{(4)}}(u)$ is holomorphic in the variables $(u, \bar{u}) \in (H^{r,r})^2$. It follows that the flow ψ_s exists, and for sufficiently small R it gives rise to a holomorphic canonical transformation $v = \tau(u) := \psi_s\big|_s(u)$ on a ball $B_R(0) \subseteq H^{r,r}$.*

Proof: The vector field (5) satisfies a Lipschitz estimate, for $u_1, u_2 \in H^{r,r}$

$$\|X^{G^{(4)}}(u_1) - X^{G^{(4)}}(u_2)\|_{r,r} \le CR^2 \|u_1 - u_2\|_{r,r} \ ,$$

from which the theorem follows. □

We remark that this transformation removes the nonresonant terms as well as the *resonant* terms of the quadratic Hamiltonian $H^{(4)}$, a phenomenon which is quite different from the finite dimensional analogs of Hamiltonian dynamical systems, as well as for Hamiltonian PDEs defined over compact spatial domains.

Scattering. Restrict our attention to the nonlinear Schrödinger equation in the defocusing case, there are other methods with which to approach the problem of normal forms. These are through the scattering map.

Theorem 2 (J. Ginibre & G. Velo [4]). *For $d > 1$ the following limit exists in $H^1(\mathbb{R}^d)$*

$$\lim_{t\to\pm\infty} \Phi_{-t}\varphi_t(u) := u_\pm = \Omega_\pm(u)$$

Related results have been discussed in many other papers on the subject, and a survey is beyond the scope of this talk. However we know that the maps $\Omega_\pm(u)$ are holomorphic in $(u, \bar{u})$ in the space $H^{1,1}$ [1]. Furthermore the scattering variables linearize the flow;

$$\Omega_\pm(\varphi_t(u)) = \Phi_t(\Omega_\pm(u)) \ .$$

This is a stronger statement than Theorem 1 above, as there is no restriction to the ball $B_R(0)$ and there are no remaining error terms. However the scattering map is given by a limiting process, in contrast to being given by an explicit singular integral kernel as in $G^{(4)}$. We therefore have three normal forms, namely

$$u_+ = \Omega_+(u) \ , \quad u_- = \Omega_-(u) = \overline{\Omega_+(\bar{u})} \ , \quad v = \tau(u) \ .$$

It turns out that there is a relation between these, and in particular the Birkhoff normal forms transformation $v = \tau(u)$ describes the asymptotics of the time decay of the quantities

$$\Phi_{-t}\varphi_t(u) - \Omega_\pm(u) \ .$$

This is work in progress, and is beyond the scope of the present short note.

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Diffusion along mean motion resonance for the restricted planar three body problem

Vadim Kaloshin

(joint work with J. Féjoz, M. Guardia, P. Roldan)

We study dynamics of the restricted planar three body problem near a mean motion resonance, i.e. resonance between periods of Jupiter and Asteroid. This problem often used to model the Sun–Jupiter–Asteroid system. We pick a realistic mass ratio $\mu = 10^{-3}$ and small Jupiter eccentricity e_0. The main result is a construction of a variety of diffusion orbits with varying eccentricity. In the proof we verify certain non-degeneracy conditions numerically.

Based on work of Treschev and Piftankin it is natural to conjecture that speed of diffusion for this problem is at least $\sim \frac{-\ln(\mu e_0)}{\mu e_0} t$. We expect our mechanism to apply to small values of μ and e_0 and give heuristic arguments in its favor. If so, applicability of Nekhoroshev theory to the three body problem as well as long time stability becomes problematic.

It is well known that in the Asteroid belt distribution of Asteroids has so-called *Kirkwood gaps* exactly at low order mean motion resonances and our mechanism could be one of possible explanations. To relate existence of Kirkwood gaps with Arnold diffusion we also state a conjecture on its existence for a typical ε-perturbation of the product of the pendulum and the rotator. Namely, we predict that a positive conditional measure of initial conditions concentrated in the main resonance exhibits Arnold diffusion on time scales $-\ln\varepsilon/\varepsilon^2$.

Localization of filtered Floer homology and implications for the Conley conjecture

Doris Hein

In 1984, C. Conley conjectured that on any symplectic torus, any Hamiltonian must have infinitely many periodic orbits, which are geometrically distinct. More concretely, the conjecture states that there are simple periodic orbits of arbitrarily large period, if the number of one-periodic orbits is finite. This conjecture has now been established for Hamiltonians on more general manifolds: If the first Chern class of a closed symplectic manifold vanishes over the second fundamental group, then the Conley conjecture is true. The Conley conjecture also holds for Hamiltonians on the cotangent bundle of a closed manifold, provided that the Hamiltonian is quadratic at infinity.

Let M be a symplectic manifold and $H\colon S^1 \times M \to \mathbb{R}$ a Hamiltonian. Denote the time-one-map of a Hamiltonian H by φ_H.

Theorem 1. *(*[He1, He2]*) Assume that φ_H has only finitely many fixed points and M and H satisfy one of the conditions:*

(1) *the manifold M is closed and $c_1(M)|_{\pi_2(M)}$ or*

(2) *the manifold $M = T^*B$ is the cotangent bundle of a closed, oriented base manifold B and the Hamiltonian H is quadratic at infinity.*

Then there exist simple periodic orbits of arbitrarily large period.

The crucial ingredient of the proof is the localization of filtered Floer homology. Using this localization, we can reduce the proof to the case of a closed, symplectically aspherical manifold M in [Gi], since the proof there is essentially a local argument near a special one-periodic orbit.

The Floer homology is, roughly speaking, the Morse homology for the action functional on the space of capped loops in M. The action decreases along trajectories connecting the critical points of the action, which are capped periodic orbits of the underlying Hamiltonian system. Denote the filtered Floer homology considering only orbits with action in the interval (a, b) by $HF_*^{(a,\, b)}(H)$. For a small action interval, the filtered Floer homology can be localized as follows:

Theorem 2. *Let $U \subset V$ be open sets in M such that they are homotopy equivalent and that H does not have periodic solutions on $\bar{V} \setminus U$ and is autonomous on this shell. For sufficiently small action interval (a, b), the filtered Floer homology has a direct sum decomposition*

$$HF_*^{(a,\, b)}(K) = HF_*^{(a,\, b)}(K, U) \oplus HF_*^{(a,\, b)}(K; M, U),$$

where the first summand contains only homology classes of periodic orbits in U with a capping contained in U.

This theorem is proved using energy bounds for the Floer trajectories passing through the difference $\bar{V} \setminus U$. In particular, the first summand depends only on the restriction of the Hamiltonian and the symplectic form to V. The bound for the length of the action interval depends only on the open sets U and V and the behavior on the Hamiltonian on the difference. For a sufficiently small open set V (e.g., V can be contained in a Darboux chart), this summand is therefore independent of the surrounding manifold. Restricting to this summand, we can reduce the proof of the above theorems to the proof in the case of a closed, symplectically aspherical manifold.

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KAM and rigidity in partially hyperbolic and parabolic dynamics

Anatole Katok

We consider a broad class of partially hyperbolic algebraic actions of higher rank abelian groups. Those actions appear as restriction of full Cartan actions on homogeneous spaces of Lie groups and their factors by compact subgroups of their centralizer. A common property of their actions is that hyperbolic directions and their brackets generate the whole tangent space. For those actions we prove differentiable rigidity for perturbations of sufficiently high regularity. The method of proof is a KAM type iteration scheme. The principal difference with previous work that used a similar approach is the very general nature of our proofs: the only tool from analysis on groups is exponential decay of matrix coefficients and no more specific information about unitary representations is required.

We also consider the rigidity problem for a model parabolic action: that is the unipotent subgroup on $Sl(2,\mathbb{R}) \times Sl(2,\mathbb{R})/\Gamma$ where Γ is an irreducible lattice. In this case there is a conditional rigidity for 2-parametric families of perturbations satisfying a natural transversality assumption. The method is also based on KAM but the iterative step is more specific and uses a description of irreducible unitary representation of $Sl(2,\mathbb{R})$.

Classical Motion in Random Potentials

Andreas Knauf

(joint work with Christoph Schumacher)

We assume the random potential to be based on short range *single site* potentials $W_j \in C^\eta(\mathbb{R}^d, \mathbb{R}),\ j \in J$, for $|J| < \infty$, $\eta \geq 2$ and

$$|\partial^\alpha W_j(q)| \leq \frac{C_\alpha}{\langle q \rangle^{d+\varepsilon}} \qquad (q \in \mathbb{R}^d,\ \alpha \in \mathbb{N}_0^d,\ |\alpha| \leq \eta)$$

with $\langle q \rangle := \sqrt{1+|q|^2}$, for constants $C_\alpha > 0$.

- In the *lattice case* the W_j are placed on a regular lattice $\mathcal{L} \subseteq \mathbb{R}^d$ according to $\omega \in \Omega := J^{\mathcal{L}}$ to form the random potential on *extended configuration space* $\Omega \times \mathbb{R}^d$:

$$V : \Omega \times \mathbb{R}^d \to \mathbb{R} \quad , \quad V(\omega, q) := \sum_{\ell \in \mathcal{L}} W_{\omega(\ell)}(q - \ell).$$

  An $\mathcal{L}$-ergodic probability measure β on Ω is chosen.
- In the *marked Poisson case* intensities $\rho_j > 0 \quad (j \in J)$ are chosen,

$$\tilde{\Omega} := \big\{ \omega \bigm| \omega \text{ measure on } (\mathbb{R}^d \times J,\ \mathcal{B}(\mathbb{R}^d \times J)) \text{ with } \\ \omega(K \times \{j\}) \in \mathbb{N}_0 \text{ if } K \subseteq \mathbb{R}^d \text{ is compact} \big\}.$$

Then (with Lebesgue measure λ^d) the probability measure on $\tilde{\Omega}$ is given by

$$\beta\big(\{\omega \in \tilde{\Omega} \mid \omega(K \times \{j\}) = m\}\big) = \frac{(\rho_j \lambda^d(K))^m}{m! \exp(\rho_j \lambda^d(K))},$$

$(m \in \mathbb{N}_0,\ j \in J)$. This is mixing with respect to translations by $\mathbb{R}^d$, and induces the random potential

$$V : \tilde{\Omega} \times \mathbb{R}^d \to \mathbb{R} \quad , \quad (\omega, q) \mapsto \int_{\mathbb{R}^d \times J} W_j(q - x)\, d\omega(x, j).$$

After restricting to a subset $\Omega \subseteq \tilde{\Omega}$ of full measure in the Poisson case, on *extended phase space* $P := \Omega \times \mathbb{R}^d \times \mathbb{R}^d$ the hamiltonian function

$$H \colon P \to \mathbb{R} \quad , \quad (\omega, p, q) \mapsto \tfrac{1}{2}|p|^2 + V(\omega, q)$$

induces in both cases a continuous hamiltonian flow

$$\Phi \colon \mathbb{R} \times P \to P.$$

A natural $\mathcal{L}$ action on P leaves H as well as the measure $\mu := \beta \otimes \lambda^{2d}$ invariant.

Besides others, we derive the following statements:

- *asymptotic* velocity

$$\overline{v}^{\pm} \colon P \to \mathbb{R}^d \quad , \quad \overline{v}^{\pm}(\omega, x_0) := \lim_{T \to \pm\infty} \frac{q_\omega(T, x_0)}{T}.$$

  exists almost surely. It leads to β–a.e. deterministic distributions ν_ω on the space $\mathbb{R} \times \mathbb{R}^d$ of energies and asymptotic velocities.
- For $d = 1$ the support of ν is explicitly known.
- If the motion on the energy surface $H_\omega^{-1}(E)$ is ergodic, $\overline{v} = 0$, but the motion is unbounded (β–a.s.)
- For $d = 1$ and in the Poisson case ergodicity does not occur.
- Generally in the lattice case the motion on the energy surfaces is not of Anosov type and thus lacks uniform hyperbolicity.
- For coulombic random potentials on $\mathcal{L}$ (with single site potentials like $W_j(q) = -\exp(-\mu_j |q|)/|q|$) for large energies E one has ergodicity on a compactified space, and asymptotic velocity $\overline{v} = 0$.
- In that case the motion is topologically transitive on $H_\omega^{-1}(E)$, and the closed orbits are dense. The motion is related to a geodesic flow on a *visibility manifold*.

Triviality of Some Actions of the Mapping Class Group

John Franks
(joint work with Michael Handel)

If S is a surface of genus g with a (perhaps empty) finite set of punctures and boundary components we will denote by $\mathrm{MCG}(S)$ the group of isotopy classes of homeomorphisms of S which pointwise fix the boundary and punctures of S. In this talk we show the triviality of representations of $\mathrm{MCG}(S)$ in $GL(n, \mathbb{C}), \mathrm{Diff}(S^2)$ and $\mathrm{Homeo}(\mathbb{T}^2)$ under various additional hypotheses.

For a closed surface S of genus g there is a natural representation of $\mathrm{MCG}(S)$ into the group of symplectic matrices of size $n = 2g$ obtained by taking the induced action on $H_1(S, \mathbb{R})$. It is natural to ask if there are linear representations of lower dimension. This is one of the questions we address.

Theorem 1 (F, Handel [2])**.** *Suppose that S is a genus $g \geq 1$ surface of finite type (perhaps with boundary and punctures), that $n < 2g$ and that $\phi : \mathrm{MCG}(S) \to GL(n, xC)$ is a homomorphism. Then if $g = 2$ the image of ϕ is finite cyclic and if $g \geq 3$ then ϕ trivial.*

Idea of Proof: The proof is by induction on g (when g decreases by one n decreases by 2). It suffices to show that one Dehn twist about a non-separating simple closed curve has trivial ϕ image. Consider S_g as the connected sum of a torus and a surface S' of genus $g - 1$. Let $L = \phi(T_\alpha)$ where T_α is a Dehn twist about α, an essential curve in the torus.

We analyse the canonical form of L and the fact that $\mathrm{MCG}(S')$ lies in the centralizer of L. □

This theorem and a classical result of Thurston are used to reduce the problem of showing triviality to finding a global fixed point.

Theorem 2 (Global Fixed Point ⇒ Triviality)**.** *Suppose that M^n is a connected manifold of dimension n, that $n < 2g$ and that $\phi : \mathrm{MCG}(S_g) \to \mathrm{Diff}^1(M^n)$ is a homomorphism. If the action ϕ has a global fixed point then it is trivial.*

As an application we have the following two results about triviality of actions of the mapping class group.

Theorem 3 (F, Handel [2])**.** *Suppose that S is a closed surface with genus $g > 6$. Then every homomorphism $\phi : \mathrm{MCG}(S) \to Diff(S^2)$ is trivial.*

Theorem 4 (F, Handel [2])**.** *Suppose that S is a closed oriented surface with genus $g > 2$. Then if $n = 1, 2$, every homomorphism $\phi : \mathrm{MCG}(S) \to \mathrm{Homeo}(\mathbb{T}^n)$ is trivial.*

The proofs of these two results show the existence of a global fixed point for the action and then apply the "global fixed point theorem."

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