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Mini-Workshop: Dynamics of Stochastic Systems and their Approximation

Organised by
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August 21st – August 27th, 2011

Abstract. The aim of this workshop was to bring together specialists in the area of stochastic dynamical systems and stochastic numerical analysis to exchange their ideas about the state of the art of approximations of stochastic dynamics. Here approximations are considered in the analytical sense in terms of deriving reduced dynamical systems, which are less complex, as well as in the numerical sense via appropriate simulation methods. The main theme is concerned with the efficient treatment of stochastic dynamical systems via both approaches assuming that ideas and methods from one ansatz may prove beneficial for the other. A particular goal was to systematically identify open problems and challenges in this area.

Mathematics Subject Classification (2000): 37H10, 37M99, 60H10, 60H15, 60H35, 65C30, 65P99.

Introduction by the Organisers

The workshop *Dynamics of Stochastic Systems and their Approximation*, organised by Evelyn Buckwar (Edinburgh), Barbara Gentz (Bielefeld) and Erika Hausenblas (Leoben) was held August 21st – August 27th, 2011. This meeting was well attended with 17 participants from 6 countries, with one participant having been awarded a US Junior Oberwolfach Fellowship to attend the workshop.

Stochastic modelling has become a standard tool in many areas of science and engineering, in fields as diverse as finance, molecular dynamics, neuroscience, laser physics, hydrogeology or climate research. Often, the mathematical description of real world phenomena is in terms of stochastic ordinary differential equations, stochastic delay differential equations or stochastic partial differential equations.

Although the aims of stochastic modelling are as manifold as the classes of equations and application areas, in several application areas the interest in stochastic modelling is in particular focussed on understanding the dynamical behaviour of the stochastic systems. Much of the research in, e.g., neuroscience or climate modelling is devoted to noise-induced or noise-related phenomena in the dynamical behaviour of the system.

Several approaches and techniques to describe and analyse the dynamics of stochastic systems exist. The mini-workshop aimed at bringing together people from different areas of expertise and the given talks have been grouped into the following themes:

- Stochastic dynamics in the Sciences,
- Numerics for stochastic dynamical systems,
- Reduced dynamics,
- Stochastic partial differential equations.

Participants had been asked to prepare a review talk on a specific subtopic fitting one of these themes, thus providing a basis for the subsequent productive discussions. In the first theme the following participants gave talks: Nils Berglund, Evelyn Buckwar, Barbara Gentz, Peter Imkeller, Rachel Kuske, Tony Lelièvre and Kevin Lin. The second theme was covered by Erika Hausenblas, Peter Kloeden, Andreas Neuenkirch and Tony Shardlow. Martin Riedler and Richard Sowers spoke about the third theme, and the final theme was addressed by Dirk Blömker, Anne de Bouard, Zdzislaw Brzezniak and Sandra Cerrai.

In the course of the mini-workshop, a number of challenging open questions were identified, examples are given by:

- Different concepts of numerical treatment of long-time dynamics of stochastic systems,
- Asymptotic dynamics of delay equations and reaction-diffusion equations with noise,
- Rigorous limit procedures between descriptions of a multi-scale system by continuous or discrete state spaces, depending on the relevant scale. This kind of problem arises, eg., in molecular dynamics and neuroscience.

The organisers thank the Mathematisches Forschungsinstitut Oberwolfach for providing an inspiring setting for this mini-workshop.

Mini-Workshop: Dynamics of Stochastic Systems and their Approximation

Table of Contents

Abstracts

Dynamical models of climate time series and the rate of convergence of power variations

Jan Gairing and Peter Imkeller
(joint work with Claudia Hein, Ilya Pavlyukevich)

Simple models of the earth's energy balance interpret some qualitative aspects of the dynamics of paleo-climatic data. They may be modeled by dynamical systems perturbed by noise, typically

$$X^\epsilon(t) = x - \int_0^t U'(X^\epsilon(s-))\,ds + \epsilon L(t), \quad t \geq 0, \epsilon > 0, \tag{1}$$

where L is a Lévy process. Its originally assumed Gaussian nature was questioned in the climate physics literature by Ditlevsen [2, 1] who analyzed global temperature proxies in a Greenland ice-core time series from the last glacial period and suggested jump diffusion models with *α-stable noise* instead.

In statistical terms, Ditlevsen's conjecture leads to a typical model selection problem. In the parametric version involved, one needs an efficient testing method for instance for the parameter α corresponding to the best fitting α-stable noise component. Continuing earlier work (see [3]) we develop a statistical testing method based on the p-variation of the solution trajectories of SDE with Lévy noise. If for some stochastic process Z we let

$$V_t^{p,n}(Z) = \sum_{i=1}^{[nt]} |Z(\frac{i}{n}) - Z(\frac{i-1}{n})|^p, \quad V_t^p(Z) = \lim_{n\to\infty} V_t^{p,n}(Z), \quad t \geq 0,$$

we take $V^p(X^\epsilon)$ as a test statistic for α. Let $X^\epsilon = Y^\epsilon + L^\epsilon$ denote the solution of (1), where Y^ϵ is the absolutely continuous part related to the potential gradient, and $L^\epsilon = \epsilon L$. In earlier work (see [3]) we proved the following key result on the asymptotic behavior of $V^{p,n}(L)$ which in the most interesting cases is identical to the asymptotic behavior of $V^{p,n}(X)$.

Theorem 1 ([7]). *Let $(L_t)_{t\geq 0}$ be an α-stable Lévy process. If $p > \alpha/2$ then*

$$\big(V_p^n(L)_t - nt\,B_n(\alpha,p)\big)_{t\geq 0} \xrightarrow{\mathcal{D}} (L'_t)_{t\geq 0} \quad \text{as } n \to \infty,$$

where L' is an independent $\frac{\alpha}{p}$-stable Lévy process, and $\xrightarrow{\mathcal{D}}$ denotes convergence in the Skorokhod topology. The normalizing sequence $(B_n(\alpha,p))_{n\geq 1}$ is deterministic and given by

$$B_n(\alpha,p) = \begin{cases} n^{-p/\alpha}\mathbb{E}|L_1|^p, & p \in (\alpha/2, \alpha), \\ \mathbb{E}\sin\left(n^{-1}|L_1|^\alpha\right), & p = \alpha, \\ 0, & p > \alpha. \end{cases}$$

Our first tests for α were based on the observation that if the underlying process L is α-stable, then the variations of power 2α, according to Theorem 1 will converge to a $\frac{1}{2}$-stable limit. Hence the *distance* of $V^{p,n}(L)_1$ and the distribution function of the $\frac{1}{2}$-stable law as a function of p will exhibit a minimum at $p = 2\alpha$, providing a method to test for α. However, a genuine statistical test needs quality criteria such as confidence intervals or bands. For the purpose of designing them, the speed of convergence of power variations in Theorem 1 to the $\frac{\alpha}{p}$-stable limits has to be described. In our approach of this problem, distances between random variables are measured by the Kolmogorov-Smirnov metric for their distribution functions or by the Wasserstein metric.

Definition 1. *Let F resp. G be the distribution functions of the real valued random variables X resp. Y, μ resp. ν their laws. The* Kolmogorov-Smirnov distance *between F and G (resp. X and Y) is given by*

$$D(F,G) = D(X,Y) = \sup_{x\in\mathbb{R}} |F(x) - G(x)|.$$

For $q > 0$, the Wasserstein distance of order q *between μ and ν (resp. X and Y) is given by*

$$\mathcal{W}_q(\mu,\nu) = \mathcal{W}_q(X,Y) = \inf_{\pi\in\mathcal{M}(\mu,\nu)} [\int_{\mathbb{R}^2} |x-y|^q d\pi(x,y)]^{1\wedge\frac{1}{q}},$$

where $\mathcal{M}(\mu,\nu)$ is the set of all probability measures on $\mathbb{R}^2$ having marginals μ and ν.

For Kolmogorov-Smirnov distances, we obtain the following results for an α-stable Lévy process L (see [4]).

Theorem 2. *For $t > 0, \alpha < p$ we have*

$$D(V^{p,n}(L)_t, V_t^p) = \begin{cases} \mathcal{O}(\frac{1}{n}), & \text{if } 2\alpha < p, \\ \mathcal{O}(\log(n)\frac{1}{n}), & \text{if } 2\alpha = p, \\ \mathcal{O}(n^{1-\frac{p}{\alpha}}), & \text{if } \alpha < p < 2\alpha. \end{cases}$$

For $t > 0, \frac{\alpha}{2} < p \leq \alpha$ we have

$$D(V^{p,n}(L)_t - ntB_n(\alpha,p)), V_t^p) = \begin{cases} \mathcal{O}(\log(n)^2\frac{1}{n}), & \text{if } \alpha = p, \\ \mathcal{O}(n^{1-2\frac{p}{\alpha}}), & \text{if } \frac{\alpha}{2} < p < 2\alpha. \end{cases}$$

For the rates of convergence of the Wasserstein distances, we obtain the following statements (see [4]).

Theorem 3. *For $t > 0, p > \alpha, \frac{\alpha}{p} < q < \frac{1}{\frac{p}{\alpha}-1}$, we have*

$$\mathcal{W}_q(V^{p,n}(L)_t, V_t^p) = \mathcal{O}(n^{1-\frac{p}{\alpha}(1\wedge q)}.$$

For $t > 0, p > \alpha, \frac{\alpha}{p} < q_0 \leq \frac{1}{\frac{p}{\alpha}-1}$, then we have for any $0 < q \leq q_0$

$$\mathcal{W}_q(V^{p,n}(L)_t, V_t^p) = \mathcal{O}(n^{-(1\wedge q)(\frac{p}{\alpha}-\frac{1}{1\wedge q_0})}).$$

The second statement of Theorem 3 allows to control the convergence of the laws as elements of Wasserstein spaces of order $q < \frac{\alpha}{p}$ since they both posses finite qth moments.

To prove these main results, we make use of the fact that unlike their laws which – except in the cases $\alpha = \frac{1}{2}, 1, 2$ – are not explicitly known, the characteristic functions of Lévy processes are easily available through the Lévy-Khinchin formula. So in order to obtain estimates of Berry-Esséen type as in Theorem 2 we make use of the following smoothness inequality relating distribution functions to their characteristic functions.

Lemma 1. *Let F and G be probability distribution functions with $\sup_{x\in\mathbb{R}} |G'(x)| \leq m < \infty$ and φ and γ their characteristic functions. Then we have for any $T > 0$:*

$$\sup_{x\in\mathbb{R}} |F(x) - G(x)| \leq \frac{1}{\pi} \int_{-T}^{T} \left| \frac{\varphi(\lambda) - \gamma(\lambda)}{\lambda} \right| d\lambda + \frac{24m}{\pi T} \ .$$

The differences of characteristic functions are further estimated via asymptotic expansions. For the estimation of Wasserstein distances, we adapt results of [9] that make use of a generalized central limit theorem and polynomial tails.

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Stochastic dynamical systems in neuroscience

Nils Berglund
(joint work with Barbara Gentz, Christian Kuehn, Damien Landon)

The success of Statistical Physics is largely due to the huge separation between microscopic and macroscopic scales, which enables us to apply limit theorems from probability theory, such as the law of large numbers and the central limit theorem. The difficulty of biological systems, such as those arising in neuroscience, is that instead of only two scales, they involve a whole hierarchy of them:

- The dynamics of the brain as a whole can be modeled in the continuum limit by *neural field equations*, which are partial differential or integro-differential equations for the synaptic electric potential [1].
- Localised populations of excitatory and inhibitory neurons are described by coupled integro-differential or delay equations, as in the *Wilson–Cowan model* for decision making [2].
- The dynamics of the action potential along the axons of individual neurons is described by partial or ordinary differential equations such as the *Hodgkin–Huxley equations* [3].
- The number of open and closed ion channels, which regulate action potentials across the neuron's membrane, can be described by a Markovian jump process [4]. Similarly, the interactions between genes, proteins and enzymes within cells involve complicated networks that can be modeled by large systems of coupled maps or differential equations [5].

Even though the scale separation is not as radical as in Statistical Physics, the different levels of the hierarchy are often still separated by one or two orders of magnitude.

Randomness arises in these models from simplifying the dynamics of higher or lower levels in the hierarchy, using reduction procedures such as stochastic averaging or continuum approximations (see Richard Sowers' and Martin Riedler's contributions to these Reports). For instance, models for action potential generation involve two variables: the voltage x across the membrane, and the vector y of proportions of open ion channels of different types. External noise, arising from fluctuations in synaptic currents coming from other neurons, originates in the next higher level of the hierarchy. Internal noise, stemming from the random dynamics of ion channels, comes from the level below. In the simplest case, one is thus led to a system of stochastic differential equations (SDEs) of the form

$$\begin{aligned} \mathrm{d}x_t &= \frac{1}{\varepsilon} f(x_t, y_t)\,\mathrm{d}t + \frac{\sigma_1}{\sqrt{\varepsilon}}\,\mathrm{d}W_t^{(1)}\,, \\ \mathrm{d}y_t &= g(x_t, y_t)\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_t^{(2)}\,, \end{aligned} \tag{1}$$

where ε describes the time scale separation, and $W_t^{(1)}$ and $W_t^{(2)}$ are independent Wiener processes, respectively modelling external and internal noise.

Choosing the appropriate model for noise is a difficult problem, influenced by parameters such as the existence of space and time correlations, and the discrete or

continuous nature of the dynamics. The simplest model for noise is Gaussian white noise as used in (1), but depending on the situation it may be more appropriate to use correlated noise such as Ornstein–Uhlenbeck processes, more general Lévy processes including jumps, or discrete-time noise such as Poisson or more general renewal processes.

Another important difference between Statistical Physics and biological problems is that while in the former, the emphasis lies on asymptotic behaviour, such as existence of stationary states and convergence to them, in biology transients play a fundamental rôle. There are several reasons for this: time-dependent forcing may prevent the existence of equilibrium states; the system may spend very long time spans in metastable states; non-equilibrium phenomena such as excitability and stochastic resonance often are of central importance to the system's behaviour.

As a paradigm illustrating transient behaviour, consider the SDE

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\,, \qquad x_t \in \mathbb{R}^n \tag{2}$$

for a dissipative vector field f. For small noise intensity σ, solutions of (2) tend to spend long time spans in localised regions of space, separated by quick transitions between these regions. The *stochastic exit problem* consists in quantifying this behaviour: Given a domain $D \subset \mathbb{R}^n$, determine the distribution of the *first-exit time* $\tau = \inf\{t > 0 \colon x_t \notin D\}$, and the law of the *first-exit location* $x_\tau \in \partial D$, known as the *harmonic measure*. If τ is likely to be large, the dynamics in D is called *metastable*, and can be described by a *quasistationary distribution* (QSD). The transitions between metastable domains can typically be reduced to a Markovian jump process, thereby providing an effective scale separation of the dynamics. A number of different techniques are available in order to achieve this program:

- The *theory of large deviations* for sample paths of the SDE (2) has been developed by Freidlin and Wentzell [6]. The probability of sample paths tracking a given deterministic path $\{\varphi_t\}$ behaves like $\mathrm{e}^{-I(\varphi)/2\sigma^2}$, for an explicitly known *rate function* I. This allows in particular to determine, by solving a variational problem, a constant V (the *quasipotential*) such that the expected first-exit time behaves like e^{V/σ^2} in the limit $\sigma \to 0$. Furthermore, this approach provides a way to characterise metastable regions and the transitions between them in this limit.
- A number of *analytic techniques* provide more detailed information on the exit problem. In particular, the expected first-exit time and location are linked, via Dynkin's formula, to the solutions of Dirichlet–Poisson boundary value problems involving the diffusion's infinitesimal generator [7]. These equations can be explicitly solved only in dimension 1 and in certain linear cases, but are accessible to WKB perturbation theory in more general cases.
- For fast–slow SDEs of the form (1), methods from *stochastic analysis* and *singular perturbation theory* provide a sharp control of the behaviour of sample paths in metastable regions and near bifurcation points [8] (see Barbara Gentz's contribution to these Reports).

- The theory of *random dynamical systems* [9] provides information on the asymptotic behaviour of the system, when it is meaningful to describe the system's reaction to the same random forcing, for different initial conditions. This is the case, e.g., for the phenomenon of *reliability* (see Kevin Lin's contribution to these Reports).

As an illustration, consider the FitzHugh–Nagumo equations, given by (1) with

$$f(x,y) = x - x^3 + y \ , \qquad \text{and} \qquad g(x,y) = a - x \ . \tag{3}$$

The deterministic equations admit a stationary point at $P = (a, a^3 - a)$, which is stable if $\delta = (3a^2 - 1)/2$ is positive. However, if δ is small, the system is *excitable*, meaning that small random perturbations may cause large excursions in phase space, corresponding to the neuron emitting a spike. For applications it is important to determine the interspike interval statistics. The invariant measure of the system is of little use here, as it only gives the probability of the neuron being in the spiking or quiescent state at any given time.

Let D be a neighbourhood of P, and let F be a curve joining P to the boundary ∂D, parametrised by a variable $r \in [0,1]$. The successive intersections of (x_t, y_t) with F up to its first exit from D define a substochastic Markov chain $(R_0, R_1, \ldots, R_{N-1})$ with kernel $K(r,A) = \mathbb{P}\{R_1 \in A | R_0 = r\}$.

Theorem 1 ([10])**.** *Assume $\sigma_1, \sigma_2 > 0$. Then K admits a QSD π_0, solution to $\pi_0 K = \lambda_0 \pi_0$, where the* principal eigenvalue λ_0 *is strictly less than* 1*. The survival time N of the Markov chain is almost surely finite and asymptotically geometric in the sense that*

$$\lim_{n\to\infty} \mathbb{P}\{N = n+1 | N > n\} = 1 - \lambda_0 \ . \tag{4}$$

The random variable N determines the length of the quiescent phase between spikes, and (4) shows that this length follows an approximately exponential law. More quantitative information is provided by the following result:

Theorem 2 ([10])**.** *For δ and ε small enough, and $\sigma_1^2 + \sigma_2^2 \leqslant (\varepsilon^{1/4}\delta)^2 / \log(\varepsilon/\sqrt{\delta})$, there exists $\kappa > 0$ such that*

$$1 - \lambda_0 \leqslant \exp\left\{ -\kappa \frac{(\varepsilon^{1/4}\delta)^2}{\sigma_1^2 + \sigma_2^2} \right\} \ . \tag{5}$$

Furthermore, the expectation of N satisfies

$$\mathbb{E}(N) \geqslant C(\mu_0) \exp\left\{ \kappa \frac{(\varepsilon^{1/4}\delta)^2}{\sigma_1^2 + \sigma_2^2} \right\} \ , \tag{6}$$

where $C(\mu_0)$ depends on the initial distribution μ_0 on F.

This result describes the weak-noise regime, and confirms the result of an approximate computation in [11]. An open problem which is currently under investigation is to obtain similar results in other parameter regimes, as well as for other models such as the Morris–Lecar equations.

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Reliable and Unreliable Behavior in Oscillator Networks

Kevin K. Lin

(joint work with Eric Shea-Brown and Lai-Sang Young)

This talk concerns the work in [5]. A related numerical study, summarized in [6], may also be of interest.

A dynamical system driven by an external stimulus is said to be *reliable* if, upon repeated presentations of the same stimulus, its response is essentially the same each time. Reliability is of interest in a number of biological and physical fields, in particular computational neuroscience, where the degree of reliability of a neuronal network constrains its ability to encode information via temporal patterns of spikes. Single neurons are well known to be typically reliable; much less is understood about networks. The present work is an attempt to uncover general network conditions conducive to reliability, and at the same time to study mechanisms for unreliability in some specific systems. We use a mixture of qualitative theory and numerical simulations; our findings range from fully rigorous results to conclusions based on a combination of numerical and theoretical evidence.

The class of models we consider are networks of coupled phase oscillators known as "Theta neurons" in mathematical neuroscience. In these models, oscillators are pulse-coupled to each other, and some – but not all – oscillators in the network receive fluctuating external stimuli. In our work, these stimuli are idealized as

white noise, so the network equations take on the form of SDEs. The question of reliability can then be stated as: *Given a realization of the driving process, is the network's trajectory essentially the same regardless of its initial condition (modulo transients)?* We show that this question can be given a precise formulation within the framework of stochastic flows and random dynamical systems theory. In particular, using known results on *sample measures* (also known as "statistical equilibrium" or "random invariant measures") [1, 3, 2], we argue that reliability is characterized by the condition $\lambda_{max} < 0$ (and unreliability by $\lambda_{max} > 0$), where λ_{max} denotes largest Lyapunov exponent. In other words, the reliability of a system receiving a stochastic stimulus is linked to its asymptotic linear stability.

Our main result on reliable network architectures is this: We proved that if a network is *acyclic* (i.e., its connectivity is described by a directed acyclic graph), then it must have $\lambda_{max} \leq 0$.[1] Hence, acyclic networks can never be unreliable; recurrent connections within the network are necessary for unreliable behavior. Our result can be further extended to systems composed of *modules* connected to each other via an acyclic graph (a module is a sub-network with clearly identified input and output nodes). This extension provides the basis for a "divide-and-conquer" approach to reliability: Given a network, we can first decompose it into an acyclic network of modules, then study the reliability of modules.

We also studied in detail the mechanism for unreliability in a system with two coupled oscillators. Our main finding here is a parameter regime in which the system is especially susceptible to unreliability; this parameter regime corresponds to the onset of phase-locking between the two oscillators in the absence of an external signal. Based on a combination of theoretical and numerical evidence, an explanation for this phenomenon is proposed in terms of "shear-induced chaos." This is a mechanism for the production of chaotic behavior (i.e., $\lambda_{max} > 0$) that has been rigorously studied in a different context, namely oscillators driven by periodic pulsatile forcing (see [8, 9], and also the review [4]).

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Stochastic dynamics and numerical methods in molecular dynamics

Tony Lelièvre

Molecular dynamics consists in generating trajectories of a molecular system, namely particles interacting through a given force field. As a typical example, one may consider the so-called overdamped Langevin dynamics:

$$dX_t = -\nabla V(X_t)\,dt + \sqrt{2\beta^{-1}}dW_t, \tag{1}$$

where $X_t \in \mathbb{R}^{3N}$ is the position of the N particles, $V : \mathbb{R}^{3N} \to \mathbb{R}$ is the potential, β^{-1} is proportional to the temperature and W_t is a standard $3N$-dimensional Brownian motion. A common feature of many molecular systems is that the generated trajectories are metastable: the positions X_t remain for a very long time in some region, before hopping to another region. Thus, it is in general very difficult to generate trajectories which are sufficiently long to observe such rare events. Depending on the output of interest, many techniques have been proposed in the literature to deal with this problem. There are roughly two situations.

In some cases, one is only interested in trajectorial averages of the form:

$$\frac{1}{T}\int_0^T \varphi(X_t)\,dt$$

which converge, in the limit $T \to \infty$, to averages with respect to the statistical ensemble associated to the dynamics. For (1) for example, we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \varphi(X_t)\,dt = Z^{-1}\int_{\mathbb{R}^{3N}} \varphi \exp(-\beta V)$$

where $Z = \int_{\mathbb{R}^{3N}} \exp(-\beta V) < \infty$. In this case, many methods have been proposed to accelerate the convergence to equilibrium and bypass the metastability of the original dynamics. They are typically based on two principles: (i) biasing (importance sampling techniques) and (ii) conditioning (constrained simulations). We refer for example to [3] for a general introduction to such techniques.

In other cases, one really wants to generate trajectories $(X_t)_{t\geq 0}$, for example to compute transport coefficients, or to identify the typical paths to go from one metastable state to another. For such computations in metastable situations, there are much less numerical methods available. Let us briefly describe two recent contributions in that direction.

In [1], we propose an adaptive multilevel splitting algorithm to efficiently generate reactive trajectories, namely a trajectory which leaves a metastable state (say $A \subset \mathbb{R}^{3N}$), and reach another one (say $B \subset \mathbb{R}^{3N}$) without going back to

A. An adaptation of the algorithm to compute reaction rates is also discussed. The principle of the algorithm is to consider a one-dimensional order parameter $\xi : \mathbb{R}^{3N} \to \mathbb{R}$ which indexes the transition from A to B in the sense that

$$A \subset \{x,\, \xi(x) < z_{\min}\} \text{ and } B \subset \{x,\, \xi(x) > z_{\max}\}.$$

We also assume that $\Sigma(z_{\min}) = \{x,\, \xi(x) = z_{\min}\}$ is close to A, in the sense that starting from $\Sigma(z_{\min})$, trajectories will go quickly to A (and very rarely to B). A similar assumption holds for $\Sigma(z_{\max})$ relatively to B. Then the algorithm goes as follows: starting from an ensemble of trajectories which start from A, reach $\Sigma(z_{\min})$ and go back to A, one deletes the trajectory which goes the less far in the ξ direction, and duplicates another one (chosen uniformly among the remaining trajectories) up to the level set of ξ which has been reached by the deleted trajectory. This new trajectory is then completed using a new driving random forcing (a new Brownian motion if one thinks of (1)) up to the time it reaches A or B. As the iterations proceed, an ensemble of trajectories which leave A, end in A or B, and conditioned to the fact that they reach a certain level set of ξ, is obtained. This algorithm is very simple to implement in existing molecular dynamics codes. It enables to generate trajectories starting from $\Sigma(z_{\min})$ and reaching B without going back to A, even if they are very rare (of probability as small as 10^{-18} in some examples in [1]). The mathematical analysis of this algorithm is quite limited so far, and it is an interesting subject to try to analyse the bias and variance of this algorithm, as the number of trajectories I grows. We expect, as always in such interacting walker algorithms, a bias which scales like $1/I$, and a statistical error which scales like $1/\sqrt{I}$.

In the manuscript [2], we consider the Parallel Replica Dynamics introduced by A. Voter in [4]. The aim of the method is to very efficiently generate a coarse-grained dynamics which contains the essential information from a macroscopic viewpoint. In mathematical terms, a mapping

$$\mathcal{S} : \mathbb{R}^{3N} \to \mathbb{N}$$

is introduced, which to a configuration $x \in \mathbb{R}^{3N}$ associates the number $\mathcal{S}(x)$ of the state in which the configuration x is. One could typically think of a numbering of the basins of attraction of the local minima of V, but any discrete partition of the space could be in principal considered. The question is then how to efficiently generate a discrete state-to-state dynamics $(S_t)_{t\geq 0}$ which has (almost) the same law as the reference one $(\mathcal{S}(X_t))_{t\geq 0}$. The algorithm goes in three step:

(1) During the *decorrelation step*, a reference walker is followed as it enters a new state. If it remains for a given time τ_{corr} in this new state, then the decorrelation step is considered as successful and one proceeds to the dephasing step, otherwise a new decorrelation step starts.
(2) In the *dephasing step*, the aim is to generate in parallel many initial conditions in the state visited by the reference walker. This is done by considering on (say) I processors in parallel, independent trajectories which start from the reference walker position and to let them go up to a time

$\tau_{dephase}$. If a trajectory happens to leave the state during this time interval, a new dephasing step is restarted for this walker, otherwise the final point is retained as one of the initial conditions for the parallel step.

(3) Once an initial condition in the state has been obtained on each processor, the *parallel step* starts. It simply consists in running in parallel and independently, the trajectories starting from the various initial conditions generated so far, and to track the first escaping event from the state, among all the walkers. When such an escaping event is detected, the reference walker is set to this walker which left the first the state, and the simulation clock is advanced by I times the escaping time for the first escaping event.

The idea behind the parallel step is the following: if, for one walker, the first exit time was exponentially distributed and independent of the first exit point, the parallel step would be exact, both in terms of time (this is related to the fact that $I \min(T_1, \ldots, T_I)$ where T_i are i.i.d. exponentially distributed random variables has the same law as T_1) and in terms of the next visited state (in other words, the law of the next visited state does not depend on the number of walkers I considered). Then, to understand the algorithm, a crucial tool is the so-called Quasi-Stationary Distribution (QSD) attached to the considered state. The QSD may be seen as the stationary probability measure for the dynamics starting in the state, conditioned to the fact that it does not leave the state. It has two very interesting properties: starting from the QSD in a given state, the first exit time is exponentially distributed and independent of the first exit point. Using, this tool, it is then natural to see the dephasing step as a way to generate I i.i.d. initial conditions distributed according to the QSD (this is very much related to the so-called Fleming-Viot process). And the decorrelation step may be seen as a way to probe if it is sensible to assume that the reference walker, after a time interval of length τ_{corr}, is indeed distributed according to the QSD. A value for τ_{corr} can then be given as a function of the first two eigenvalues of the Fokker-Planck operator (associated with the dynamics) with absorbing boundary conditions on the boundary of the state. With this point of view, one can give some estimates of what is the error associated to each iteration of the algorithm. More generally, this analysis shows the interest of the QSD as an intermediate between continuous state space dynamics (standard molecular dynamics) and discrete state space dynamics (typically kinetic Monte Carlo dynamics). This does not seem to have been explored so far in the literature.

We would like to mention that this last work benefited a lot from discussions with Pablo Ferrari during a previous meeting in Oberwolfach, in November 2010.

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Noise induced phenomena in dynamical systems

Rachel Kuske

Transient or unstable behavior is often ignored in considering long time dynamics in the deterministic world. However, stochastic effects can change the picture dramatically, so that the transients can dominate the long range behavior. This talk will compare a few seemingly unrelated canonical models with noise-driven regular behaviors, such as coherent oscillations, synchronization, mixed mode oscillations and even quiescence, that are transient in the absence of noise. Perspectives combining ideas from coherence resonance, meta-stability, and multiples scales show that the order in these models can be attributed to transients "stabilized" by stochastic effects. These perspectives suggest analysis on reduced models to better understand and predict these phenomena. Different mechanisms for the noise-driven order are considered, both where it is more complex than the deterministic order and where it is less complex.

One direction of study is in the area of coherence resonance, where an optimal noise input drives nearly regular oscillations in a system that is quiescent without this input. This idea has been developed in simple models such as the FitzHugh-Nagumo model, with parameter values near a critical value which marks the change in behavior from attraction to a stable fixed point to attraction to relaxation oscillations. We illustrate how this phenomenon transfers to networks of oscillators, where a range of optimal noise levels drives synchronized bursting in a network with excitatory coupling. A key element is the behavior on multiple scales, which separates the underlying dynamics into slow subthreshold oscillations that are sensitive to noise and fast bursting behavior and a refractory period that are insensitive to noise.

Another area where there are many open problems is in systems with delayed feedback. Dynamical systems with delay often exhibit an array of complex behaviours. These phenomena have been studied in the context of applications such as mechanical systems and neural dynamics. Stochastic effects can often change the picture dramatically, particularly if multiple time scales are present. In the context of machine tool dynamics, and more generally oscillators with delay, additive noise can sustain large oscillations that decay without noise, while randomness in the delay time can disrupt oscillations, causing attraction to fixed points that are unstable without noise.

More recently the mathematical techniques for models with delay have been transferred to the study of balance in the context of human postural sway and robotics. Similarities related underlying bifurcation structures can facilitate this transfer. However, additional complexity, such as discontinuous control of balance

require new ideas that allow an analysis of the interplay of nonlinearities, delays, randomness, and piecewise smooth dynamics. Many open problems that include this full range of complexity remain in both mathematical theory and applications in biology, mechanics, and neuroscience.

The fact that noise can sustain transient dynamics dominated by unstable deterministic behavior leads to questions of model calibration, comparing to observed dynamics or data. With noisy input driving nearly regular behavior, one can usually reproduce the same behavior with different models and different parameter combinations. This suggests that methods for model calibration must expose the interaction of stochastic effects with the underlying bifurcation structure or multiple scale dynamics. Characterizing the stochastic sensitivity of the observed dynamics can narrow down model type and parameter ranges, in order to facilitate model calibration with complex dynamics.

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Modified equations and long time approximation for SDEs

Tony Shardlow

We reviewed backward error analysis and its applications to ordinary differential equations, where it is of great value in understanding the approximation of Hamiltonian systems. The corresponding idea for SDEs was introduced and modified equations for the Euler-Maruyama and Milstein methods in the weak sense developed. For the Ornstein-Uhlenbeck process, a weak modified equation can be developed to arbitrarily high weak order. However, if the diffusion is nonlinear, no second order modified equation [1] currently exists for the Euler-Maruyama method. [2] has shown that a second order modified equation does exist for the Milstein method. Langevin systems are the natural way of adding noise to a Hamiltonian systems and we showed that in general modified equations do not have the general form of a Langevin equation. Special methods can be developed where the modified equation is a Langevin system and the invariant measure is understood [2].

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Mean-square random attractors

Peter E. Kloeden
(joint work with Thomas Lorenz)

Random dynamical systems as in Arnold [1] involve pathwise defined skew-product, but are not applicable to SDE that depend nonlocally on the different sample paths such as on the expectation, see e.g., [3, 6]. Here we discuss mean-square random dynamical system (MS-RDS) which were introduced by Kloeden & Lorenz [4].

Let $\mathbb{R}^2_{\geq} := \{(t,t_0) \in \mathbb{R}^2 \ : \ t \geq t_0\}$ and define $\mathfrak{X} := L^2\left((\Omega,\mathcal{F},\mathbb{P}),X\right)$ and $\mathfrak{X}_t := L^2\left((\Omega,\mathcal{F}_t,\mathbb{P}),X\right)$ for each $t \in \mathbb{R}$, where X is a Banach space.

Definition 1. *A mean-square random dynamical system (MS-RDS) φ on an underlying space X with a probability set-up $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in\mathbb{R}},\mathbb{P})$ is a family of mappings $\varphi(t,t_0,\cdot) : \mathfrak{X}_{t_0} \to \mathfrak{X}_t$, $(t,t_0) \in \mathbb{R}^2_{\geq}$, which satisfies*
1) initial value property*:* $\varphi(t_0,t_0,\phi_0) = \phi_0$ *for every $\phi_0 \in \mathfrak{X}_{t_0}$ and any $t_0 \in \mathbb{R}$;*
2) two-parameter semigroup property*: for each $\phi_0 \in \mathfrak{X}_{t_0}$ and all (t_2,t_1), $(t_1,t_0) \in \mathbb{R}^2_{\geq}$*

$$\varphi(t_2,t_0,\phi_0) = \varphi\left(t_2,t_1,\varphi(t_1,t_0,\phi_0)\right);$$

3) continuity property*:*

$$(t,t_0,\phi_0) \mapsto \varphi(t,t_0,\phi_0) \quad \textit{is continuous in the space } \mathbb{R}^2_{\geq} \times \mathfrak{X}.$$

These are in fact deterministic with the stochasticity built into or hidden in the time-dependent state spaces. They provide a mean-square analysis of the dynamics.

Definition 2. *A family $\mathcal{B} = \{B_t\}_{t\in\mathbb{R}}$ of nonempty subsets of $\mathfrak{X}$ with $B_t \subset \mathfrak{X}_t$ is said to be* φ-invariant *if $\varphi\left(t,t_0,B_{t_0}\right) = B_t$ for all $(t,t_0) \in \mathbb{R}^2_{\geq}$ and* φ-positively invariant *if $\varphi\left(t,t_0,B_{t_0}\right) \subset B_t$ for all $(t,t_0) \in \mathbb{R}^2_{\geq}$.*

For simplicity, we will say that $\mathcal{B}$ is a family of subsets of $\{\mathfrak{X}_t\}_{t\in\mathbb{R}}$ and that $\mathcal{B}$ is uniformly bounded if there is an $R < \infty$ such that $\mathbb{E}\|X_t\|^2 \leq R$ for all points $X_t \in B_t$ for every $t \in \mathbb{R}$.

The following definition are taken from [5] for deterministic nonautonomous dynamical systems.

Definition 3. *A φ-invariant family $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}}$ of nonempty compact subsets of $\{\mathfrak{X}_t\}_{t\in\mathbb{R}}$ is called a* mean-square forward attractor *if it forward attracts all families $\mathcal{B} = \{B_t\}_{t\in\mathbb{R}}$ of uniformly bounded subsets of $\{\mathfrak{X}_t\}_{t\in\mathbb{R}}$, i.e.,*

$$\operatorname{dist}\left(\varphi(t,t_0,B_{t_0}),A_t\right) \to 0 \quad \textit{as } t \to \infty \quad (t_0 \textit{ fixed}) \tag{1}$$

and a mean-square pullback attractor *if it pullback attracts all families $\mathcal{B} = \{B_t\}_{t\in\mathbb{R}}$ of uniformly bounded subsets of $\{\mathfrak{X}_t\}_{t\in\mathbb{R}}$, i.e., the limit (1) with $t_0 \to -\infty$ and t fixed.*

Various results involving pullback absorbing sets and pullback asymptotic compactness ensuring the existence of pullback attractors for processes are given in

[5]. These involve compact subsets of $L^2((\Omega, \mathcal{F}, \mathbb{P}), X)$ or asymptotic compactness conditions on the mappings φ in this space. Other methods depending on the contractive nature of the coefficients were used in [2].

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Pathwise Approximation of Stochastic Differential Equations

Andreas Neuenkirch

(joint work with Arnulf Jentzen and Peter E. Kloeden)

Approximation schemes for Itô stochastic differential equations of the form

$$dX(t) = a(X(t))\,dt + \sum_{j=1}^{m} b^j(X(t))\,dW^j(t), \qquad t \in [0,T], \tag{1}$$
$$X(0) = X_0 \in \mathbb{R}^d,$$

where $a, b^j : \mathbb{R}^d \to \mathbb{R}^d$, $j = 1, \dots, m$, and $W^j(t)$, $t \in [0,T]$, $j = 1, \dots, m$, are m independent Brownian motions on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, have been intensively studied in the recent years. For an overview, see, e.g. [4] or [5]. The vast majority of results, however, are concerned with error criteria that measure the error of the approximation on average. For instance, in the case of the so called "weak approximation" the error of an approximation $\overline{X}$ to X is measured by the quantity

$$|\mathbb{E}\phi(X(T)) - \mathbb{E}\phi(\overline{X}(T))|$$

for (smooth) functions $\phi : \mathbb{R}^d \to \mathbb{R}$, while for the "strong approximation" problem the p-th mean of the difference between X and $\overline{X}$ is considered, i.e.

$$\left(\mathbb{E} \sup_{i=0,\dots,n} \left|X(t_i) - \overline{X}(t_i)\right|^p\right)^{1/p}$$

for $p \geq 1$, where $|\cdot|$ denotes the Euclidean norm and $0 = t_0 \leq t_1 \leq \ldots \leq t_n = T$ are the time nodes of the discretization. In the latter case, usually the mean-square error, i.e. $p = 2$ is analyzed.

In general, the numerical calculation of the approximation $\overline{X}$ is actually carried out path by path, i.e. the real numbers $\overline{X}(t_1,\omega),\dots,\overline{X}(t_n,\omega)$ are calculated for a fixed $\omega\in\Omega$. In spite of this fact, only a few articles deal with the pathwise error

$$\sup_{i=0,\dots,n}|X(t_i)-\overline{X}(t_i)|,$$

which is a random quantity, but gives more information about the error of the calculated approximations $\overline{X}(t_1,\omega),\dots,\overline{X}(t_n,\omega)$ of $X(t_1,\omega),\dots,X(t_n,\omega)$ for a fixed $\omega\in\Omega$.

In [6] an upper bound for the pathwise error of the Milstein method is determined using the Doss-Sussmann approach to transform the stochastic differential equation and the Milstein scheme to a random ordinary differential equation and a corresponding approximation scheme, respectively. Gyöngy [1] shows that the explicit Euler-Maruyama scheme with equidistant step size $1/n$ converges pathwise with order $1/2-\varepsilon$ for arbitrary $\varepsilon>0$ without a global Lipschitz assumption on the coefficients. Hence the pathwise and the mean-square rate of convergence of the Euler method almost coincide. Using an idea in the proof of [1], one can show that this is not an exceptional case, but is, in fact, the rule for Itô-Taylor schemes:

Theorem 1 (see [2]). *Let $\gamma=0.5,1.0,1.5,\dots$. Assume that*

$$a\in C^{2\gamma+1}(\mathbb{R}^d;\mathbb{R}^d),\qquad b\in C^{2\gamma+1}(\mathbb{R}^d;\mathbb{R}^{d,m})$$

and that SDE (1) has a unique strong solution on finite time intervals. Moreover let $\overline{X}^{\gamma}_n$ be the Itô-Taylor method of strong order γ. Then for all $\varepsilon>0$ there exists a finite and non-negative random variable $\eta^{a,b}_{\gamma,\varepsilon}$ such that

$$\sup_{i=0,\dots,n}\left|X(iT/n,\omega)-\overline{X}^{\gamma}_n(iT/n,\omega)\right|\le\eta^{a,b}_{\gamma,\varepsilon}(\omega)\cdot n^{-\gamma+\varepsilon}$$

for almost all $\omega\in\Omega$ and all $n\in\mathbb{N}$.

Thus, we recover the pathwise convergence rate $\gamma-\varepsilon$ of the standard Itô-Taylor method of order γ, which has been derived under classical assumptions in [3]. This illustrates that the pathwise convergence rates of the standard Itô-Taylor schemes are very "robust" in the sense that they can be retained under almost minimal assumptions: No global Lipschitz assumption for the coefficients is required and a differentiability of order $2\gamma-1$ is in fact needed for the Itô-Taylor γ-scheme to be well defined. Moreover the above result holds also true for SDEs, which take values in a domain $D\subset\mathbb{R}^d$, if appropriately modified Itô-Taylor schemes are used, see [2].

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Approximation of SPDEs driven by Lévy noise

Erika Hausenblas
(joint work with Dunst, Thomas and Prohl, Andreas)

In recent years, stochastic partial differential equations driven by a space-time Lévy noise have received increasing interest; see e.g., Applebaum and Wu [1], Hausenblas [4], Knoche [8], Mueller [10], Saint Loubert Bié [12].

One motivation for this work is the model of river pollution proposed by Kwakernaak [7] and studied by Curtain [3]. Let $\mathcal{O} := (0,1) \times (0,3) \subset \mathbb{R}^2$, and $A = D\Delta - \gamma\frac{\partial}{\partial \xi_2} - a$; $D > 0$ is the dispersion rate, $\gamma > 0$ scales convection in the ξ_2-direction, and $a \geq 0$ is the leakage rate. Then $u : (0,T) \times \mathcal{O} \to \mathbb{R}$ describes the concentration of a chemical substance which solves the following problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= Au \qquad \text{in } \mathcal{O}_T := (0,T) \times \mathcal{O}\,, \\ \frac{\partial u}{\partial n} &= 0 \qquad \text{on } (0,T) \times \Gamma_N\,, \qquad \text{and} \qquad u = 0 \qquad \text{on } (0,T) \times \Gamma_D\,. \end{aligned}$$

A chemical substance is now deposited in the region $G \subset \mathcal{O}$. The deposits leak, and substance is dripping out with a certain mean rate $\lambda > 0$. This dripping out is modeled by a space-time Lévy noise.
Taking the dripping into account, processes $\big\{u(t,\xi);\, t \in [0,T],\, \xi \in \mathcal{O}\big\}$ can be described by the following stochastic partial differential equation (SPDE)

$$\begin{aligned} \frac{\partial u(t,\xi)}{\partial t} &= Au(t,\xi) + \dot{\mathbf{L}}(t,\xi) \quad \forall\,(t,\xi) \in \mathcal{O}_T\,, \\ u(0,\xi) &= 0 \quad \forall\,\xi \in \mathcal{O}\,, \\ \frac{\partial}{\partial \xi_2} u(t,\xi) &= 0 \quad \forall\,(t,\xi) \in (0,T) \times \Gamma_N\,,\ u(t,\xi) = 0 \quad \forall\,(t,\xi) \in (0,T) \times \Gamma_D\,, \end{aligned} \tag{1}$$

where $\dot{\mathbf{L}}$ represents space-time Lévy noise. Problem (1) is an example of the more general setting of SPDE's driven by a space-time Lévy process $\mathbf{L}$, which we consider in this work. A first numerical analysis of the problem is [9]. It is, however, that increments of the driving Lévy process were exactly simulated in these works. It is only for a small number of Lévy processes that the exact distribution function of the increments is available, while in general it is only the Lévy measure of the driving noise that is given. Therefore, the goal of this paper is to propose different

strategies to approximate a given Lévy process, and an analysis of errors inherent to a corresponding time-discretization of (1) .

Several approaches exist to simulate SDEs driven by Lévy processes. Rubenthaler [11] studied the *approximate Euler scheme*, which combines (explicit) time-discretization, and simulatable approximations of a given Lévy process (with unknown distribution) by truncating jumps below a certain threshold $\epsilon > 0$: the numerical strategy then uses 'interlacing', i.e., simulates subsequent stopping times where jumps occur, corresponding jump heights, and an intermediate Brownian motion. We remark that this strategy is less efficient in high dimensions, e.g. for SPDE's (after discretization in space). Convergence in law with rates for the scheme is shown in [11]. For related numerical works to solve SDE's without approximating the driving Lévy noise, we refer to cited works in [11].

Such a truncation strategy ignores small jumps, which may cause a rough approximation of Lévy measures of infinite variation, and a reduced precision of iterates of the related approximate Euler scheme; Asmussen and Rosiński [2] have shown that small jumps may be represented by a Wiener process, leading to an improved approximation of the given Lévy process. In [6], weak rates of convergence for iterates of the approximate Euler scheme for SDEs that employs this strategy are shown; strong rates of convergence are obtained in [5], where the proof relies on a central limit theorem when using the quadratic Wasserstein distance. In our work, we give a different proof that is based on Fourier transforms and uses tail estimates to show a corresponding L^p estimate ($1 \leq p \leq 2$), which is the proper setting of (1) where mild solutions are L^p-valued.

In the talk a strategy how to simulate the jumps of a Lévy process was presented. The main steps can be summerized as follows:

- **Truncation of small jumps:** First all jumps which are smaller than a given threshold ϵ are omitted.
- **Approximating the Lévy measure by a discrete measure:** The Lévy measure is approximated by a discrete measure. Here different strategies can be applied.
- **Generating a random variable due to the discrete measure** In fact, since the intensity measure is discrete, this step is now quite easy and straightforward.
- **Approximating the small jumps by a Wiener process:** If the underlying Lévy process is a Lévy process with unbounded variation close to two, the rate of convregence will be improved simulating the small jumps by a Wiener process.

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Approximation of Stochastic Hybrid Systems

Martin Riedler

Stochastic Hybrid Systems are càdlàg strong Markov processes that exhibit continuous evolution interlaced with jumps at random times. The continuous evolution may be either piecewise deterministic (Piecewise Deterministic Process, see [4, 5]) or given by an SDE (General Stochastic Hybrid Systems, see [3]). In the talk we discussed limit theorems for spatially extended stochastic hybrid systems, more precisely, for Hilbert space valued Piecewise Deterministic Processes developed in [5, 6]. These form the theoretical basis for an approximation of stochastic hybrid systems by less complex continuous processes, either deterministic, e.g., PDEs or integro-differential equations, or stochastic, i.e., SPDEs or SDEs in Banach spaces. Spatio-temporal dynamics are of interest in many applications in, e.g., mathematical neuroscience and related fields (waves in models of axons, cardiac tissue and in calcium dynamics, as well as various phenomena in neural field models). A common feature in these models is that the observed macroscopic dynamics are subject to *internal noise*. Internal noise arises as the observed macroscopic phenomena are an emergent effect of the coupled behaviour of a large number of individual stochastic objects on a smaller scale (see also Nils Berglund's contribution in this Report).

The limit theorems presented are, on the one hand, of the *law of large numbers* type connecting the stochastic microscopic models to a deterministic evolution equation, and, on the other hand, a *martingale central limit theorem* for the

martingale part of the stochastic process[1]. The latter theorem characterises the internal stochasticity of the models to be in the limit of a diffusive nature. Finally, and most importantly, these limit theorems provide an argument for the approximation by a diffusion process, usually a nonlinear stochastic infinite-dimensional evolution equation, the so called *Langevin approximation*. In the talk we exemplified the general limit theorems on a simple continuous time Markov chain model for brain activity. This model has been introduced to derive the Wilson-Cowan equation, see Eq. (1), via the van Kampen systems size expansion [1].

Macroscopic model. We consider the Wilson-Cowan equation

$$\tau\,\dot\nu(t,x) = -\nu(t,x) + f\Big(\int_D w(x,y)\nu(t,y)\,\mathrm{d}y + I(t,x)\Big), \quad x\in D, t\geq 0, \tag{1}$$

on a bounded spatial domain $D\subset\mathbb{R}^d$, $d\geq 1$. Here $\tau>0$ denotes the (synaptic) time constant, $f\in C^1(\mathbb{R})$ such that $0\leq f\leq 1$ is a Lipschitz continuous gain function that relates the input into firing activity, $w: D^2\to\mathbb{R}_+$ are sufficiently smooth weights such that $w(x,y)$ gives the strength of the connection of neurons at y to neurons at x and I is a sufficiently smooth external input. For $T>0$ and $s>d/2$ a unique solution $\nu\in C([0,T],H^s(D))$ exists for every initial condition $\nu_0\in H^s(D)$.

Microscopic model. For $n\in\mathbb{N}$ we define microscopic models such that the nth model consists of $P=P(n)$ neuron populations and each of these containing $N=N(n)$ identical neurons. Any neuron can be in one of two possible states

$$\textit{quiescent} \quad \rightleftharpoons \quad \textit{active}\,.$$

Changes between states of individual neurons are instantaneous and at random times governed by instantaneous rates. Thus we define a piecewise constant stochastic process $(\Theta(t))_{t\geq 0}$ taking values in $\mathbb{N}^P$ counting the active neurons in each population, i.e., $\Theta_k(t)$ is the number of active neuron in population k at time t. The dynamics of this process are governed by the rates

$$\theta_k \xrightarrow{\alpha N f_k(\theta,t)\,\mathbb{I}_{[\theta_k<N]}} \theta_k+1, \qquad \theta_k \xrightarrow{\alpha\,\theta_k} \theta_k-1 \tag{2}$$

depending on the current state $\theta\in\mathbb{N}^P$ of the process.

Connection. We connect the microscopic and macroscopic model in the following way. There exists a sequence of convex partitions $\mathcal{P}_n$ of the domain D with $|\mathcal{P}_n|=P$ and each subdomain $D_{k,n}$, $0\leq k\leq P$, contains one neuron population. We then define the rates in (2) by

$$\alpha := \tau^{-1}, \qquad f_k^n(\theta,t) := f\Big(\frac{1}{N}\sum_{j=1}^P W_{kj}^n\theta_j + I_k^n(t)\Big),$$

[1]The martingale part is the martingale arising in the Doob-Meyer Decomposition of the process.

where $W^n_{kj} := \frac{1}{|D_{k,n}|}\int_{D_{k,n}}\int_{D_{j,n}} w(x,y)\,\mathrm{d}y\,\mathrm{d}x$ and $I^n_k(t) := \frac{1}{|D_{k,n}|}\int_{D_{k,n}} I(t,x)\,\mathrm{d}x$. Furthermore, we define the *stochastic microscopic neural activity* ν^n by

$$\nu^n(t) := \nu^n(\Theta^n(t)) := \frac{1}{N}\sum_{k=1}^{P}\Theta^n_k(t)\,\mathbb{I}_{D_{k,n}} \;\in L^2(D)\,.$$

It allows for a comparison of the microscopic model to the solution of (1) as both map $[0,T]$ into $L^2(D)$. We now state the limit theorems for this model which are applications of the general theorems in [5, 6]. The derivation of the conditions and a thorough discussion of more general models of this type can be found in forthcoming work by the present author and E. Buckwar [2].

Theorem 1 (Law of large numbers)**.** *Assume that* $\lim_{n\to\infty} N(n) = \infty$*, the stochastic and deterministic initial conditions* Θ^n_0 *and* ν_0 *satisfy for all* $\epsilon > 0$

$$\lim_{n\to\infty}\mathbb{P}\big[\|\nu^n(\Theta^n(0)) - \nu_0\|_{L^2(D)} > \epsilon\big] = 0$$

and the sequence of partitions satisfies

$$\lim_{n\to\infty}\max_k \operatorname{diam}(D_{k,n}) = 0\,.$$

Then the jump-processes $(\nu^n(t))_{t\geq 0}$ *converge in probability to the solution* ν *of the Wilson-Cowan equation* (1) *on finite intervals, i.e., for all* $T, \epsilon > 0$

$$\lim_{n\to\infty}\mathbb{P}\big[\sup_{t\in[0,T]}\|\nu^n(t) - \nu(t)\|_{L^2(D)} > \epsilon\big] = 0\,.$$

Moreover, the convergence also holds in the mean.

Theorem 2 (Martingale central limit theorem)**.** *We define*

$$v_-(n) := \min_{k=1,\dots,P(n)}|D_{k,n}|, \qquad v_+(n) := \max_{k=1,\dots,P(n)}|D_{k,n}|\,.$$

Assume that the conditions of the law of large numbers are satisfied and

$$\lim_{n\to\infty}\frac{v_-(n)}{v_+(n)} = 1\,.$$

Then it follows that

$$\Big(\sqrt{\tfrac{N(n)}{v_+(n)}}\,M^n(t)\Big)_{t\geq 0} \xrightarrow{d} (M(t))_{t\geq 0}\,.$$

Here $\xrightarrow{d}$ *denotes convergence in distribution in the space of* $H^{-2s}(D)$*–valued càdlàg processes*[2] *and* $(M_t)_{t\geq 0}$ *is the* $H^{-2s}(D)$*–valued diffusion process defined by the covariance operator* $C(t) = \int_0^t G(t)\mathrm{d}s$ *with*

$$\begin{aligned}\langle G(s)\phi,\psi\rangle_{H^{2s}} &:= \langle G(\nu(s),s)\phi,\psi\rangle_{H^{2s}} := \\ &\int_D \phi(x)\left(\alpha\nu(s,x) + \alpha\, f\Big(\int_D w(x,y)\nu(s,y)\,\mathrm{d}y + I(s,x)\Big)\right)\psi(x)\,\mathrm{d}x.\end{aligned} \tag{3}$$

[2]Here $H^{-2s}(D)$ denotes the dual to the Sobolev space $H^{2s}(D)$.

Langevin approximation. Heuristically inserting the limit established in the martingale central limit theorem for the martingale part of the process and the deterministic limit for the initial condition and the finite variation part, we obtain the *Langevin approximation*. Exploiting the fact that the diffusion process defined by the covariance C can be represented as a stochastic integral yields the *Langevin approximation*

$$V_t^n = \nu_0 + \int_0^t -\alpha V_s^n + \alpha\, f\Big(\int_D w(\cdot,y) V_s^n(y)\,\mathrm{d}y + I(s,\cdot)\Big)\,\mathrm{d}s + \epsilon_n \int_0^t \sqrt{G(V_s^n,s)}\,\mathrm{d}W(s)$$

with $\epsilon_n = \sqrt{v_+(n)/N(n)}$. Here $\sqrt{G(\cdot,\cdot)}$ is a Hilbert-Schmidt operator which denotes the unique square root of the trace class operator $G(\cdot,\cdot)$ defined in (3) and W is a cylindrical Wiener process.

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Stochastic Averaging

Richard B. Sowers

The field of stochastic averaging has been around since the 1960's. The classical theory of deterministic averaging had been well-developed, but the work of Khasminskii [9, 11, 10] extended the theory to stochastic differential equations; see also [12, 13].

The basic idea can be seen by considering a system on the cylinder. Consider a system with fast angular drift and slow axial diffusion. It is instructive to view this simplest system through the lens of modern stochastic processes. The essential tools are: using test functions (as suggested by the martingale problem) to weakly characterize the limit points and using corrector functions to carry out the averaging. In fact, the appropriate corrector function exists only if one correctly guesses the averaged coefficients; there is a *geometric condition*, and a careful understanding of the PDE corrector function is crucial. These tools can be used to tackle more general averaging problems. Geometrically, we collapsed the cylinder in our basic problem to a line via orbit equivalence of the fast motions. What happens if the fast orbits have more complicated topologies? Consider a ball rolling around in a single-well potential; without noise or friction, the ball

oscillates as kinetic and potential energy switch. Action-angle coordinates allow us to more or less map the fast motion into angular rotation around a cylinder, and we can map the total energy into an axial action coordinate. Adding noise gives us stochastic averaging as the motion in the action variable is slow and stochastic.

In fact, even here there is something non-trivial. The fast motion collapses at the bottom of the well. However, by carefully using test functions and corrector functions, one can indeed get an averaged equation in the action variable, and show that this singularity doesn't have any effect [15]. Topologically, the bottom of the well is a point; it is furthermore inaccessible.

More complicated cases can appear. Freidlin and Wentzell [5, 8] and Freidlin and Weber [6, 7] considered the motion of a ball in a double-well potential; here there is a homoclinic orbit and two fixed points. The fixed points behave like the bottom of a potential well as discussed above. One can't collapse the space by orbits; the homoclinic orbit gives the wrong answer. One wants to understand how to collapse the state space by a combination of deterministic motion and small (stochastic) shocks. The theory of *chain equivalence* [14] does exactly this. One gets a graph. Stochastically, one gets *glueing conditions*, which are similar to coin flips, which identifies the likelihood that the reduced process will make an excursion into the different regions of orbit space. A careful analysis of test functions and PDE's can again be used; see [17, 18].

Even stranger topologies exist. Suppose that there is a region where the fast motion disappears. Chain equivalence again correctly describes the reduced state space; it is a combination of parts of different dimensions. Stochastic averaging still holds, but one needs slightly more complicated glueing procedures (and a more careful PDE analysis for the corrector functions); see [16]. Another interesting topology comes from pseudoperiodic flows; here there can be ergodic components to which the reduced motion will "stick"; see [1, 4, 19].

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Amplitude Equations – natural slow-fast systems

Dirk Blömker

Focusing on the stochastic Swift-Hohenberg equation (SH) only, we review results on the rigorous error estimates for amplitude equations. We discuss the impact of various models of the noise, together with open problems.

Introduction. Complicated models near a *change of stability* generate slow-fast systems in a natural way. The dominant pattern (or modes) evolve on a *slow time-scale*, while stable pattern decay and disappear on a *fast time-scale*. The evolution is then given by simplified models for the amplitudes of dominant pattern, the so called *amplitude equations*. There are many examples of formal derivation for such equations. For a review see [7].

For PDEs on bounded domains the theory of invariant or center manifolds is available, where solutions are well approximated by an ODE on the manifold. Unfortunately, invariant manifolds for stochastic PDEs move in time. Moreover, there is a lack of center manifold theory. This is similar to PDE on unbounded domains, where a whole band of eigenvalues changes stability, and amplitude or modulation equations are successfully applied. See [8, 10, 11].

Using amplitude equations, our aim is to understand the impact of noise on the dominant pattern and how noise is transported by the nonlinearity. For simplicity we consider only (SH) on the real line. In pattern formation (SH) is a celebrated

toy model for the convective instability in the Rayleigh-Bénard model.

$$\text{(SH)} \qquad \partial_t u = \mathcal{L}u + \nu\varepsilon^2 u - u^3 + \xi_\varepsilon \,,$$

where $\mathcal{L} = -(1+\partial_x^2)^2$. The dominant modes are $\mathcal{N} = \text{span}\{\sin, \cos\}$. Moreover ξ_ε is some small noise process, which may change for different applications.

Slow-Fast System. (SH) naturally generates a system, given by a slow SDE and a fast SPDE. To illustrate this, consider $\xi_\varepsilon = \varepsilon^2\partial_t W$ in (SH). Split $u(t) = \varepsilon v_c(\varepsilon^2 t) + \varepsilon v_s(\varepsilon^2 t)$ with $v_c \in \mathcal{N}$ and $v_s \perp \mathcal{N}$. Then

$$\text{(SLOW)} \qquad \partial_T v_c = \qquad \nu v_c - P_c(v_c+v_s)^3 + \partial_T \tilde{W}_c \,,$$

$$\text{(FAST)} \qquad \partial_T v_s = \varepsilon^{-2}\mathcal{L}v_s + \nu v_s - P_s(v_c+v_s)^3 + \partial_T \tilde{W}_s \,,$$

where P_c projects onto $\mathcal{N}$, $P_s = I - P_c$, and $\tilde{W}(T) = \varepsilon W(T\varepsilon^{-2})$ is a rescaled version of the driving Wiener process W.

Full Noise. Consider (SH) subject to periodic boundary conditions on $[0, 2\pi]$ with noise $\xi_\varepsilon = \varepsilon^2\partial_t W$, where W is some suitable Q-Wiener process W.

Theorem 1 (Approximation, [1])**.** *Consider $u(0) = \varepsilon a(0) + \varepsilon^2\psi(0)$ with $a(0)$ and $\psi(0)$ both $\mathcal{O}(1)$. Let $a(T) \in \mathcal{N}$ solve*

$$\partial_T a = \nu a - P_c a^3 + \partial_T \tilde{W}_c \,,$$

and let $\psi(t) \perp \mathcal{N}$ be an OU-process solving $\partial_t\psi = \mathcal{L}\psi + \partial_t W_s$. Then $u(t) = \varepsilon a(\varepsilon^2 t) + \varepsilon^2\psi(t) + \mathcal{O}(\varepsilon^{3-})$ for $t \in [0, T_0\varepsilon^{-2}]$.

The Approximation remains true for invariant measures. Moreover, Attractivity verifies that any solution scales as needed for the Theorem after some time.

Degenerate noise. Additive noise may lead to stabilization (or a shift of bifurcation) of dominant modes (pattern disappears). See [9]. Consider the noise $\xi_\varepsilon = \sigma\varepsilon\partial_t\beta$ for some real-valued Brownian motion β.

$$\text{Ansatz:} \qquad u(t,x) = \varepsilon A(\varepsilon^2 t)\mathrm{e}^{ix} + c.c. + \varepsilon Z(t) + \mathcal{O}(\varepsilon^2)$$

with some fast OU-process $Z(t) = \int_0^t e^{-(t-\tau)}d\beta(\tau)$ and a complex-valued amplitude A. Using explicit averaging results with error bounds, we obtain the Amplitude equation [5]

$$\text{(A)} \qquad \partial_T A = (\nu - \tfrac{3}{2}\sigma^2)A - 3A|A|^2 \,.$$

Open Problems. For higher order corrections [5] or quadratic nonlinearities and degenerate noise [3], averaging results with explicit error estimates for integrals of the type $\int_0^T X(\tau)Z(\tau\varepsilon^{-2})^q d\beta(\tau)$ are necessary. These results [3] are based on Levy's representation theorem, which restricts the result to $\dim(\mathcal{N}) = 1$. It remains open, how to obtain error estimates for the limit, if $X(t) \in \mathbb{R}^n$, $n > 1$.

Interesting results arise for (SH) on large [2] (full noise) or unbounded domains [6] (degenerate noise). In both cases the complex amplitude A is slowly modulated in space and given by a stochastic Ginzburg-Landau PDE. Here many questions

are still open, due to the lack of regularity and unboundedness of solutions of the stochastic Ginzburg-Landau equation on $\mathbb{R}$ with space-time white noise.

Another interesting question [4] is for non-Gaussian noise without scale invariance. For the driving process L we need limits of $\varepsilon^\alpha L(\epsilon^{-2}t)$ with error estimates.

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Nonlinear Dispersive Equations, Solitary Waves and Noise

Anne de Bouard

(joint work with A. Debussche, R. Fukuizumi, E. Gautier)

Nonlinear dispersive waves in general, and solitons in particular are universal objects in physics. They may as well describe the propagation of certain hydrodynamic waves, as localized waves in plasma physics, signal transmission in fiber optics, or phenomena such as energy transfer in excitable molecular systems. In all those cases, the formation of stable, coherent spatial structures have been experimentally observed, and may be mathematically explained by the theory of nonlinear integrable (or soliton) equations. However, none of those systems is exactly described by soliton equations, and those equations may only be seen as asymptotic models for the description of the physical phenomena. Moreover, as soon as microscopic systems are under consideration, thermal fluctuations may not be negligible. They give rise in general to stochastic fluctuations in the corresponding model, and their interaction with the waves has to be studied. In some other situations, the underlying asymptotic model is not even an integrable equation, even though it is a nonlinear dispersive equation. Solitary waves may still exist in this latter situation, and even if the mathematical theory is then much less

developed than in the integrable case, the hamiltonian structure of the equation may still be used to obtain dynamical properties of the waves. The object of the talk is to describe some results obtained in this latter situation, concerning the dynamical behaviour of stochastic perturbations of nonlinear dispersive equations, which do not make use of the integrability of the underlying deterministic model.

A first example is given by the Korteweg-de Vries equation, which may be written as

$$\partial_t u + \partial_x u + \partial_x^3 u + \partial_x(u^2) = 0, \quad x \in \mathbb{R}, \quad t > 0 \tag{1}$$

and is known to be an asymptotic model for the propagation of unidirectional, weakly nonlinear long waves on the surface of shallow water. In that context, the solution u stands for the (rescaled) elevation of the water surface. The equation also occurs in the context of plasma physics (see [7]). It is well known that soliton solutions exist, which form a two parameter family of localized traveling waves, given by $\varphi_c(x-(c-1)t+x_0)$, with $c>1$, and $x_0 \in \mathbb{R}$, where the profile φ_c may be explicitly computed :

$$\varphi_c(x) = \frac{3c}{2}\mathrm{sech}^2(\sqrt{c}\frac{x}{2}).$$

Any soliton solution is a local minimum of a Hamiltonian constrained to constant "mass" (or charge), both functionals being conserved by the evolution equation (1). This leads to the stability of the family $\{\varphi_c(x+x_0),\ x_0 \in \mathbb{R}\}$, or in other words to orbital stability of the soliton. The use of the parameter x_0, together with the fact that the mass is conserved, allows to get rid of two "secular modes" in the equation, which are generated by φ_c and $\partial_x\varphi_c$.

The question of the influence of a noise on the solutions of (1) is a natural question, as such random perturbations may have different origins, like a random pressure fields on the surface of the water, a random bottom topography, or simply thermal effects in the context of plasma physics. The first attempt in this direction was probably the works by Wadati [6], in which equation (1) is simply perturbed by a time dependent Gaussian white noise. In this case, the solution of the stochastic equation is easily found in terms of the solution of (1), by a change of frame, and it was deduced in [6] that the solution u, starting at time 0 from a soliton, satisfies, asymptotically for large time :

$$\max_x \mathbb{E}(u(t,x)) \sim ct^{-3/2}.$$

This "superdiffusion" phenomenon, as it was called by Wadati, is due to the random behavior of the center of mass $x(t)$ of the solution of the perturbed equation, which is a Gaussian process with variance $t^3/3$.

A natural question that arises is : does this behavior of the perturbed soliton persists with a more complicated noise, depending also on the space variable ? A partial answer to this question was given in [1, 2] and [4], where the solution is proved to decompose into a main modulated soliton and a remaining term, in the small noise limit.

Consider indeed the Itô equations

$$du + (\partial_x u + \partial_x^3 u + \partial_x(u^2))dt = \begin{cases} \varepsilon dW \\ \varepsilon u dW, \end{cases} \tag{2}$$

where $W(t)$ is an infinite dimensional Wiener process, regular in space, which is assumed stationary in space in the multiplicative case, and ε is a small parameter. A first natural question is the persistence time of the soliton which will necessarily be destroyed by the noise. In order to answer this question, one should naturally take into account random modulations in the center of mass and velocity of the soliton. For that purpose, let us write the solution u of Eq. (2) which has for initial state $\varphi_{c_0}(x)$ the form

$$u^\varepsilon(t,x) = \varphi_{c^\varepsilon(t)}(x - x^\varepsilon(t)) + \varepsilon\eta^\varepsilon(t, x - x^\varepsilon(t)), \tag{3}$$

where $c^\varepsilon(t)$ and $x^\varepsilon(t)$ are random modulation parameters, which are chosen in order that $\varphi_{c^\varepsilon}(x - x^\varepsilon)$ stays orthogonal to the two secular modes φ_{c_0} and $\partial_x\varphi_{c_0}$, so that the remaining part $\varepsilon\eta^\varepsilon$ of the solution will be small for a longer time. Note that Hamiltonian and mass are no more conserved by the perturbed evolution equation (2), but one may compute their evolution, and this allows in particular to get a bound on the exit time τ_α^ε of an α-neighborhood of the orbit of the soliton. It is found that for any time $T > 0$,

$$\mathbb{P}(\tau_\alpha^\varepsilon \leq T) \leq e^{\frac{-C_\alpha}{\varepsilon^2 T}} \tag{4}$$

for some constant $C_\alpha > 0$ (see [1] and [4] for details). This means that the solution will stay close to the randomly modulated soliton with high probability, for any time smaller than ε^{-2}.

In the case the noise is multiplicative and homogeneous (stationary) in space, the convergence of η^ε (see (3)) as ε goes to zero to a centered Gaussian process of Ornstein-Uhlenbeck type was also obtained in [2]. Moreover, keeping only the order one in ε, the equations for the modulation parameters c^ε and x^ε are found to be of the form

$$\begin{cases} dx^\varepsilon = c_0 dt + \varepsilon B_1 dt + \varepsilon dB_2 \\ dc^\varepsilon = \varepsilon dB_1, \end{cases}$$

where B_1 and B_2 correspond to the projection of the noise $\varphi_{c_0}W$ on the two-dimensional space generated by the secular modes φ_{c_0} and $\partial_x\varphi_{c_0}$.

Coming back to our problem of soliton diffusion, it is now possible, still keeping only the order one in ε, to compute explicitly the mean value of the modulated soliton, since the vector $(x^\varepsilon, c^\varepsilon)(t)$ is a Gaussian vector, and the asymptotic rate of diffusion in time is found :

$$\max_x \mathbb{E}(\varphi_{c^\varepsilon(t)}(x - x^\varepsilon(t))) \sim C\varepsilon^{-1/2}t^{-5/4}, \quad \text{as} \quad t \to +\infty.$$

Of course these asymptotics are valid only if ε tends to zero while t tends to infinity, with t much less than ε^{-2}. The behaviour of the solution of such equations perturbed by space time noise, at time exactly ε^{-2} is up to now still an open problem, even if some results where obtainded in [5] on the behaviour of the

scattering data for some time dependent perturbations, using the integrability of the deterministic equation.

The same results as above have been obtained in [3] for a nonlinear Schrödinger equation with harmonic potential and stochastic perturbation, arising in Bose-Einstein condensation. The solitary waves are then standing waves of the form $e^{i\omega t}\phi_\omega(x)$ and asymptotics in frequencies near the bottom of the spectrum are also established for the first order of the remainder.

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Fast transport asymptotics for stochastic RDE's with boundary noise

Sandra Cerrai

(joint work with Mark Freidlin)

In the classical chemical kinetics, the evolution of concentrations of various components in a reaction is described by ordinary differential equations. Such a description turns out to be unsatisfactory in a number of applications, especially in biology.

There are several ways to construct a more adequate mathematical model. If the reaction is fast enough, one should take into account that the concentration is not constant in space in the volume where the reaction takes place. Then the change of concentration due to the spatial transport, as a role the diffusion, should be taken into consideration and the system of ODE's should be replaced by a system of PDE's of reaction-diffusion type. In some cases, one should also take into account random change in time of the rates of reaction. Then the ODE is replaced by a stochastic differential equation. If the rates change randomly not just in time but also in space, the evolution of concentrations can be described by a system of SPDE's.

On the other hand, the rates of chemical reactions in the system and the diffusion coefficients may have, and as a rule have, different orders. Some of them

are much smaller than others and this allows to apply various versions of the averaging principle and some other asymptotic methods and, eventually, to obtain a relatively simple description of the system.

We have studied the case when the diffusion rate is much larger than the rate of reaction and we have shown that in this case it is possible to replace SPDEs of reaction-diffusion type by suitable SDEs. Such an approximation in particular is valid if the reaction occurs just on the boundary of the domain (this means that the nonlinearity is included in the boundary conditions). This replacement is a result of *averaging* in the fast spatial transport. We would like to stress that our approach allowed us also to calculate the main terms of deviations of the solution of the original problem from the simplified model. Moreover, the case when the diffusion coefficients and some of the reaction rates are large compared with other rates can be considered in a similar way.

We dealt with the following class of equations

(1)
$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial t}(t,x) = \dfrac{1}{\epsilon}\mathcal{A}u_\epsilon(t,x) + f(t,x,u_\epsilon(t,x)) + g(t,x,u_\epsilon(t,x))\,\dfrac{\partial w^Q}{\partial t}(t,x), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \geq 0,\ x \in D, \\ \dfrac{1}{\epsilon}\dfrac{\partial u_\epsilon}{\partial \nu}(t,x) = \sigma(t,x)\dfrac{\partial w^B}{\partial t}(t,x),\ t \geq 0,\ x \in \partial D, \qquad u_\epsilon(0,x) = u_0(x),\ x \in D, \end{cases}$$

for some $0 < \epsilon \ll 1$. These are reaction-diffusion equations perturbed by a noise of multiplicative type, where the diffusion term $\mathcal{A}$ is multiplied by a large parameter ϵ^{-1} and a noisy perturbation is also acting on the boundary of the domain D.

Here D is a bounded open subset of $\mathbb{R}^d$, with $d \geq 1$, having a regular boundary and in the case $d = 1$ we take $D = [a, b]$. $\mathcal{A}$ is a uniformly elliptic second order operator and $\partial/\partial\nu$ is the corresponding co-normal derivative. This is why the same constant ϵ^{-1}, which is in front of the operator $\mathcal{A}$, is also present in front of the co-normal derivative $\partial/\partial\nu$. In what follows, we shall denote by A the realization in $L^2(D)$ of the differential operator $\mathcal{A}$, endowed with the conormal boundary condition.

The coefficients $f, g : [0, \infty) \times D \times \mathbb{R} \to \mathbb{R}$ are assumed to be measurable and satisfy a Lipschitz condition with respect to the third variable, uniformly with respect to the first two ones, and the mapping $\sigma : [0, \infty) \times \partial D \to \mathbb{R}$ is bounded with respect to the space variable.

The noisy perturbations are given by two independent cylindrical Wiener processes w^Q and w^B, defined on the same stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ which take values on $L^2(D)$ and $L^2(\partial D)$ and have covariance operators $Q \in \mathcal{L}^+(L^2(D))$ and $B \in \mathcal{L}^+(L^2(\partial D))$, respectively. In space dimension $d = 1$ we could take Q equal to the identity operator, so that we could deal with space-time white noise. Moreover, as $L^2(\{a, b\}) = \mathbb{R}^2$, in space dimension $d = 1$ we did not assume any condition on B.

Stochastic partial differential equations with a noisy term acting also on the boundary have been studied by several authors. To this purpose we refer for

example to the paper by da Prato and Zabczyk [2] and to the papers by Freidlin and Wentzell [3] and by Sowers [4]. The last two mentioned papers, are also dealing with some limiting results with respect to small parameters appearing in front of the noise. However, the limiting results which we have studied seem to be completely new and we are not aware of any existing previous result dealing with the same sort of multi-scaling problem, even in the simpler case of homogeneous boundary conditions (that is $\sigma = 0$).

As mentioned above, our interest was to study the limiting behavior of the solution u_ϵ of problem (1), as the parameter ϵ goes to zero, under the assumption that the diffusion X_t associated with the operator $\mathcal{A}$, endowed with the co-normal boundary condition (this corresponds to a diffusion X_t on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{\mathbb{P}})$ which reflects on the boundary of D), admits a unique invariant measure μ and a spectral gap occurs. That is for any $h \in L^2(D, \mu)$

$$\int_D \left| \hat{\mathbb{E}}^x h(X_t) - \int_D h(y)\, \mu(dy) \right|^2 \mu(dx) \leq c\, e^{-2\gamma t} \int_D |h(y)|^2\, \mu(dy),$$

for some constant $\gamma > 0$. This can be expressed in terms of the semigroup e^{tA} associated with the diffusion X_t, by saying that

$$\left| e^{tA} h - \int_D h(x)\, \mu(dx) \right|_{L^2(D,\mu)} \leq c\, e^{-\gamma t} |h|_{L^2(D,\mu)}. \tag{2}$$

Our aim was proving that equation (1) can be substituted by a suitable one-dimensional stochastic differential equation, whose coefficients are obtained by averaging the coefficients and the noises in (1) with respect to the invariant measure μ. More precisely, for any $h \in L^2(D, \mu)$ we defined

$$\hat{F}(t, h) = \int_D f(t, x, h(x))\, \mu(dx), \qquad t \geq 0,$$

and for any $h \in L^2(D, \mu)$, $z \in L^2(D)$ and $k \in L^2(\partial D)$ we defined

$$\hat{G}(t, h)z = \int_D g(t, x, h(x))z(x)\, \mu(dx), \qquad t \geq 0,$$

and

$$\hat{\Sigma}(t)k = \delta_0 \int_D N_{\delta_0} \left[\sigma(t, \cdot)k\right](x)\, \mu(dx), \qquad t \geq 0,$$

where N_{δ_0} is the Neumann map associated with $\mathcal{A}$ and δ_0 is a suitable constant. We proved that for any $t \geq 0$ the mappings $\hat{F}(t, \cdot) : L^2(D, \mu) \to \mathbb{R}$ and $\hat{G}(t, \cdot) : L^2(D, \mu) \to L^2(D)$ are both well defined and Lipschitz-continuous and $\hat{\Sigma}(t) \in L^2(\partial D)$, so that the stochastic ordinary differential equation

$$\begin{cases} dv(t) = \hat{F}(t, v(t))\, dt + \hat{G}(t, v(t)) dw^Q(t) + \hat{\Sigma}(t) dw^B(t) \\ v(0) = \displaystyle\int_D u_0(x)\, \mu(dx), \end{cases} \tag{3}$$

admits for any $T > 0$ and $p \geq 1$ a unique strong solution $u \in L^p(\Omega; C([0, T]))$ which is adapted to the filtration of the noises w^Q and w^B. Notice that (3) is a one-dimensional stochastic equation, in the sense that the space variables desappeared.

When we say that equation (1) can be replaced by equation (3) we mean that the solution u_ϵ of (1) can be approximated by the solution v of (3) in the following sense

$$\lim_{\epsilon\to 0}\mathbb{E}\sup_{t\in[\delta,T]}\left|\int_D |u_\epsilon(t,x)-v(t)|^2\ \mu(dx)\right|^p = 0, \tag{4}$$

for any fixed $0<\delta<T$ and $p\geq 1/2$.

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Random attractors for stochastic Navier-Stokes equations in some unbounded domains

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In this talk I will present new developments in the theory of infinite dimensional random dynamical systems. The starting point is my recent paper with Y. Li [2]. In that paper we constructed a RDS for the stochastic Navier-Stokes equations in some unbounded domains $\mathcal{O}\subset\mathbb{R}^2$. We proved that that RDS is asymptotically compact, what roughly means that if a sequence (x_n) of initial data is bounded in the energy Hilbert space H and the sequence of initial times $(-t_n)$ converges to $-\infty$, then the sequence $\big(u(0,-t_n,x_n)\big)$, where $u(t,s,x)$, $t\geq s$ is a solution of the SNSEs such that $u(s,s,x)=x$, is relatively compact in H. A RDS satisfying this condition is called an asymptotically compact one. We also proved that for any asymptotically compact RDS on a separable Banach space, the Ω-limit set of any bounded *deterministic* set B is non-empty, compact and forward invariant with respect to the RDS (and attracting the set B). However, we were not able to show the existence of a random attractor. Indeed, such a proof would require that there exists a family of closed and bounded random sets such that each element of it is absorbed but another element form the same family. In our case we only showed that each deterministic bounded set is absorbed by some *random* bounded set. The family of deterministic bounded sets is too small. Nevertheless, our proof of the existence of a non-empty compact invariant set implied, as a byproduct, the existence of an invariant measure for the SNSEs, a non-trivial question as the domain $\mathcal{O}$ is unbounded.

In a recent paper [1] we overcame she shortcomings of [2] and, motivated by a recent paper by [3], we found a family of random sets having the properties mentioned above. The fundamental observation is that in the proof of the absorption

property, see the proof of Theorem 8.8 in [2], it is not necessary to assume that the sequence (x_n) of initial data is bounded but instead it is enough to assume that it satisfies a certain random growth condition. This random growth condition can be expresses in terms of large time behaviour of an auxiliary Ornstein-Uhlenbeck process. This observation suggest a choice of a certain family $\mathfrak{D}$ of closed random sets such that it has a compact random absorbing set $\mathfrak{C}$. Then the existence (as well uniqueness) of an attractor is a consequence of a random version of Theorem 3.4 from [4]. The proof of this auxiliary fact is given in [1] and it is based on the proof of Theorem 8.8 in [2]. Moreover, one can naturally modify a notion of an asymptotically compact random dynamical system with respect to such a family.

Definition 1. *Suppose that* X *is a Polish space, i.e. a metrizable complete separable topological space, $\mathcal{B}$ is its Borel $\sigma-$field and $\mathfrak{T} = (\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is a metric DS. A map $\psi : \mathbb{R}^2_{\geq} \times \Omega \times \mathrm{X} \ni (t, s, \omega, x) \mapsto \psi(t, s, \omega)x \in \mathrm{X}$, where $\mathbb{R}^2_{\geq} = \{(t, s) \in \mathbb{R}^2 : t \geq s\}$, is called a two parameter continuous* ***random dynamical system*** *(RDS) (on* X *over $\mathfrak{T}$), iff*

(i) *ψ is $\big(\mathcal{B}(\mathbb{R}^2_{\geq}) \bigotimes \mathcal{F} \bigotimes \mathcal{B}, \mathcal{B}\big)$-measurable;*

(ii) ***(Cocycle property)*** *for all $\omega \in \Omega$ and all $(t, s), (s, r) \in \mathbb{R}^2_{\geq}$, $\psi(s, s, \omega) = id$ and*

$$\psi(t, r, \omega) = \psi(t, s, \omega) \circ \psi(s, r, \omega)$$

and

$$\psi(t, s, \vartheta_s\omega) = \psi(t + s, r + s, \omega),$$

(ii) *for all $(t, s, \omega) \in \mathbb{R}^2_{\geq} \times \Omega$, $\psi(t, s, \omega) : \mathrm{X} \to \mathrm{X}$ is continuous.*

Definition 2. *A compact random set A on X is said to be a* ***random $\mathfrak{D}$-attractor*** *of a RDS ψ iff (i) A is ψ-invariant and (ii) A is $\mathfrak{D}$-attracting.*

Theorem 1. *Assume that $\mathfrak{T} = (\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is a metric DS,* X *is a separable Banach space, $\mathfrak{D}$ is a nonempty class of closed random sets on X and ψ is a continuous, $\mathfrak{D}$-asymptotically compact RDS on* X *(over $\mathfrak{T}$). Assume that there exists a $\mathfrak{D}$-absorbing closed random set B on X. Then there exists a minimal random $\mathfrak{D}$-attractor. Moreover, this minimal random $\mathfrak{D}$-attractor is equal to the Ω-limit set Ω_B of B defined by*

$$\Omega(B, \omega) = \Omega_B(\omega) = \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} \psi(0, -t, \vartheta_{-t}\omega) B(\vartheta_{-t}\omega)}. \tag{1}$$

The above result can then be applied to the 2-D stochastic NSEs which are asymptotically compact in this generalized sense.

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Stochastic Functional Differential Equations

Evelyn Buckwar

We consider Stochastic Functional Differential Equations (SFDEs) of the form

$$X(t) = X(0) + \int_0^t F(s, X_s)\,\mathrm{d}s + \int_0^t G(s, X_s)\,\mathrm{d}W(s)\,.$$

Here X_t denotes a memory functional or segment process $X_t(u) = \{X(t+u) : u \in [-\tau, 0]\}$ for $X_t \in C([-\tau, 0]; \mathbb{R}^n)$, and the initial data is given as $X(t) = \psi(t)$ for $t \in [-\tau, 0]$. The Wiener process $W = \{W(t, \omega), t \in [0, T], \omega \in \Omega\}$ is an m-dimensional Wiener process on filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$. The memory functional often takes special forms, e.g., by setting F and/or G to

$$\begin{aligned} &H_2(s, X(s), X(s-\tau)), && \text{(constant/discrete delay)}\\ &H_3(s, X(s), X(s-\tau(s))), && \text{(variable delay)}\\ &H_4(s, X(s), \int_{-\tau}^{0} K(s, u, X(s+u))\,\mathrm{d}u), && \text{(distributed delay).}\end{aligned}$$

Early deterministic systems of Functional Differential Equations (FDEs) were introduced to model mechanical, controlled systems, where the time delay accounted for the time the control needed to measure the state of the system and to apply the control countermeasure, if required. Biological systems represent another large area of applications of FDEs. In population dynamics, the delays may represent maturation or gestation times, in epidemiology delays account for incubation periods, where in neuroscience the delay may denote the finite transmission time of electrical impulses between neurons or neuron clusters. In the biosciences SFDE models appear since the early '90s, e.g., as models of human pupil light reflex [15], human postural sway [4], population dynamics [7], genetic regulatory networks [3, 5, 20, 21], and in neuroscience [8, 10, 12, 22]. Further applications can be found in many more areas, such as economics & finance, climate models describing the El Niño phenomenon, or laser dynamics.

Analysis of existence and uniqueness of solutions began with Itô and Nisio [13], more comprehensive treatments are in [16, 17]. Distinguishing features of SFDEs, compared to stochastic ordinary differential systems, are that they are infinite

dimensional systems, they are, in general, non-Markovian, and their analytical treatment may require Malliavin calculus [6, 11, 18].
SFDEs exhibit a rich dynamical behaviour, some key properties being oscillations, stabilising features, and stochastic resonance, cf., e.g., [1, 9], as well as most of the literature in applications. However, there are only few articles developing mathematical techniques to study the dynamics of SFDEs, e.g., [2, 14, 19]. In particular the interplay between dynamics induced by the delays and by the driving stochastic process is of high interest and provides challenging mathematical problems.

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Hunting French ducks in a noisy environment

Barbara Gentz
(joint work with Nils Berglund, Christian Kuehn)

During the last decades substantial progress has been made in the analysis of deterministic slow–fast systems, starting with Fenichel's seminal geometric approach to singularly perturbed systems [7] which provides a description of the dynamics in terms of invariant manifolds. Since then, different techniques have been developed for the study of slow–fast systems near bifurcations, see for instance [8] for the geometric approach, [9] for an analysis of canard solutions, and [6] for so-called mixed-mode oscillations (MMOs), consisting of small-amplitude oscillations alternating with large-amplitude oscillations.

For stochastic multiscale systems, the development of a comparable theory is still in its infancy. An analogue of Fenichel's geometric theory has been established in [2, 3] based on the observation that for small random perturbations the dynamics can be compared to its deterministic counterpart up to a suitable timescale. When the dynamics approaches a bifurcation point, noise-induced phenomena can be observed. Examples include early transitions between asymptotically stable equilibrium branches, stochastic resonance, a change from microscopic to macroscopic hysteresis cycles, noise-induced canards, changed mixed-mode patterns.

MMOs are observed in many neuronal systems and models. While different mechanisms have been identified which may cause this type of behaviour, we will focus on MMOs in a slow–fast dynamical system with one fast and two slow variables, containing a folded-node singularity. The main question we address is whether and how noise may change the dynamics.

Consider the normal form of a folded-node singularity [1, 9]

$$\begin{aligned} \varepsilon \dot{x}_t &= y_t - x_t^2 \\ \dot{y}_t &= -(\mu+1)x_t - z_t \\ \dot{z}_t &= \frac{\mu}{2} \end{aligned}$$

with timescale separation $\varepsilon \ll 1$. The critical manifold, given by $\{y = x^2\}$, has an attracting as well as a repelling part. Solutions which follow the repelling part for a substantial time and distance are called canards.

Let us assume that $\mu \in (0,1)$ satisfies $\mu^{-1} \notin \mathbb{N}$, then [1, 9, 10, 5]

- there exist a *strong* and a *weak* (maximal) canard $\gamma_\varepsilon^{\mathrm{s,w}}$;
- $\gamma_\varepsilon^{\mathrm{s}}$ makes half an oscillation (or one *twist*) around $\gamma_\varepsilon^{\mathrm{w}}$;

- for $2k+1 < \mu^{-1} < 2k+3$, there exist k *secondary* canards γ_ε^j;
- γ_ε^j makes $(2j+1)/2$ oscillations around $\gamma_\varepsilon^{\mathrm{w}}$.

Considering z as the time variable, adding noise to the other variables, rescaling by the so-called blow-up transformation, we see that the system takes the form

$$\begin{aligned} \mathrm{d}x_t &= (y_t - x_t^2)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^{(1)} \\ \mathrm{d}y_t &= [-(\mu+1)x_t - z_t]\,\mathrm{d}t + \sigma'\,\mathrm{d}W_t^{(2)} \\ \mathrm{d}z_t &= \frac{\mu}{2}\,\mathrm{d}t \end{aligned}$$

and can be rewritten as a two-dimensional system which again is a slow–fast system, provided μ is small. Thus we can apply the general approach developed in [2] to quantify the effect of noise by

- constructing small sets $\mathcal{B}(h)$ around canard solutions in which the sample paths are typically concentrated, and
- giving precise bounds on the exponentially small probability to observe atypical behaviour.

The small sets $\mathcal{B}(h)$ are determined by the covariance of the Gaussian process obtained by linearizing the equations around a canard solution and their width scales with h. We obtain the following result on the first-exit time $\tau_{\mathcal{B}}(h)$ from $\mathcal{B}(h)$, see [4]

$$\mathbb{P}\{\tau_{\mathcal{B}(h)} < z\} \le C(z_0, z)\,\exp\{-\kappa h^2/2\sigma^2\} \qquad \forall z \in [z_0, \sqrt{\mu}]$$

Thus, for h of larger order than σ, the probability of *not* being close to the canard is small.

Note that near $z = 0$, the distance between subsequent canards γ_ε^k and γ_ε^{k+1} is of order $\exp\{-c_0(2k+1)^2\mu\}$, while the section of $\mathcal{B}(h)$ is close to circular with radius $\mu^{-1/4}h$, see [4]. This implies that noisy canards become indistinguishable when the typical radius $\mu^{-1/4}\sigma$ becomes comparable to the distance between canards. We can rephrase this statement by saying that canards with $\frac{2k+1}{2}$ oscillations become indistinguishable from noisy fluctuations for

$$\sigma > \sigma_k(\mu) = \mu^{1/4}\exp\{-(2k+1)^2\mu\}\ .$$

Finally, let us point out that noise can cause sample paths to jump away from the canard solutions with high probability before deterministic orbits do. Indeed, we can define a suitable neighbourhood $\mathcal{S}_0$ of the weak canard such that

$$\mathbb{P}\{\tau_{\mathcal{S}_0} > z\} \le C|\log\sigma|^{\kappa_2}\exp\{-\kappa(z^2-\mu)/(\mu|\log\sigma|)\}$$

holds for $\sigma|\log\sigma|^{\kappa_1} \le \mu^{3/4}$ and all $z > \sqrt{\mu}$. The right-hand side becomes small as soon as z is of larger order than $(\mu|\log\sigma|/\kappa)^{1/2}$.

This early-jump mechanism can drastically influence the local and global dynamics of the system by changing the mixed-mode patterns, depending on the global return mechanism.

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