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Discrete Geometry

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Abstract. A number of remarkable recent developments in many branches of discrete geometry have been presented at the workshop, some of them demonstrating strong interactions with other fields of mathematics (such as algebraic geometry, harmonic analysis, and topology). The field is very active with lots of open questions and many solutions. There was a large number of young participants who are eager to work on these problems. The future of discrete geometry looks more than promising.

Mathematics Subject Classification (2000): 52Cxx.

Introduction by the Organisers

Discrete Geometry deals with the structure and complexity of discrete geometric objects ranging from finite point sets in the plane to more complex structures like arrangements of n-dimensional convex bodies. Classical problems such as Kepler's conjecture and Hilbert's third problem on decomposing polyhedra, as well as classical works by mathematicians such as Minkowski, Steinitz, Hadwiger and Erdős are part of the heritage of this area. In the last couple of years several outstanding open problems have been solved. Here we list a few of them: (1) Erdős distinct distances problem by Guth and Katz, using algebraic geometry (based on ideas of Elekes and Sharir), (2) tight lower bounds for geometric ε-nets by Pach and Tardos, and a weaker but still superlinear lower bound by Alon, (3) a superlinear lower bound on the size of weak ε-nets by Bukh, Matoušek, and Nivasch, (4) disproof by Santos of the famous Hirsch conjecture from 1957, (5) topological extension of the first selection lemma by Gromov (which also improves the constant) and which has lead to further use of algebraic topology in discrete geometry.

By its nature, this area is interdisciplinary and has relations to many other vital mathematical fields. The breakthrough results above use methods of algebraic geometry, topology, combinatorics, computational geometry, convexity, discrepancy theory, and probability. At the same time it is on the cutting edge of applications such as geographic information systems, mathematical programming, coding theory, solid modelling, computational structural biology and crystallography.

The workshop was attended by 47 participants. There was a series of 10 survey talks giving an overview of developments in Discrete Geometry and related fields:

- Micha Sharir: *From Joints to Distinct Distances and Beyond: The Dawn of an Algebraic Era in Combinatorial Geometry*
- Roman Karasev: *A simpler proof of the Boros–Füredi–Bárány–Pach–Gromov theorem*
- János Pach: *Piercing convex sets*
- Luis Montejano: *When is a disk trapped with four lines?*
- Uli Wagner: *Isoperimetry, Crossing Numbers, and Multiplicities of (Equivariant) Maps*
- József Solymosi: *Point-pseudoline incidences in higher dimensions*
- Boris Bukh: *Space crossing numbers*
- Nati Linial: *What are high-dimensional permutations? How many are there?*
- Igor Pak: *Finite tilings*
- Günter M. Ziegler: *Polytopes with low-dimensional realization spaces*

In addition, there were 26 shorter talks and an open problem session chaired by János Pach on Tuesday evening—a collection of open problems resulting from this session can be found in this report. The program left ample time for research and discussions in the stimulating atmosphere of the Oberwolfach Institute. In particular, there were several special informal sessions, attanded by smaller groups of the participants, on specific topics of common interest.

On Wednesday we had a very pleasant excursion leading to the MiMa (Museum for Minerals and Mathematics in Oberwolfach Kirche) where many participants of the workshop (with the help of the museum's staff) worked on the construction of the "Exploded stellated Dodecahedron" using Zometool building blocks.

Workshop: Discrete Geometry

Table of Contents

Abstracts

From Joints to Distinct Distances and Beyond: The Dawn of an Algebraic Era in Combinatorial Geometry

Micha Sharir

Summary. In the past three years the landscape of combinatorial geometry has considerably changed, due to two groundbreaking papers by Guth and Katz ([6] in 2008 and [7] in 2010). They have introduced reasonably simple techniques from algebraic geometry that enabled them to tackle successfully several major problems in combinatorial geometry. Their first paper obtained a complete solution to the *joints problem*, a problem involving incidences between points and lines in three dimensions which has been open since it was first posed (by myself and others) in 1992. The second Guth–Katz paper was even more dramatic. They obtained a nearly complete solution to the classical problem of Erdős [4] on distinct distances in the plane, which was open since 1946. Both problems have been extensively studied over the years, using more traditional, and progressively more complex methods of combinatorial geometry, but with only partial and incomplete results.

Together with my colleagues and students, we have been working intensively during the past three years, exploiting, simplifying, and extending the new paradigm to tackle a variety of related problems in combinatorial geometry. As a matter of fact, the second Guth–Katz paper was a follow-up of a recent study of myself (with the late Gy. Elekes), where we have laid out a program for tackling Erdős's distinct distances problem by reducing it to an incidence problem in three dimensions, similar to those arising in the joints problem.

A review of the recent developments. In its simplest, original form, the *joints problem*, posed in [1] in 1992, is to obtain a sharp upper bound on the number of points that can be incident to at least three non-coplanar lines, in any set of n lines in three dimensions; these points are called *joints*. Simple constructions show that the number of joints can be $\Omega(n^{3/2})$ and the goal was to obtain a matching upper bound. After 15 years of frustrating research, the best upper bound that could be obtained, with the "traditional" machinery, was $O(n^{1.623})$ [5].

Then, in December 2008, Guth and Katz [6] established the upper bound $O(n^{3/2})$, thus solving the problem completely. They used in the proof several reasonably simple tools from algebraic geometry, and we mention here two of them: **(i)** Given a set P of m points in $\mathbb{R}^3$, one can find a trivariate polynomial f of degree $D = O(m^{1/3})$ that vanishes at all the points of P. (An appropriate generalization holds in any dimension $d \geq 1$, except that the degree of the resulting polynomial is $O(m^{1/d})$.) **(ii)** Given two trivariate polynomials f and g with no common factor, and with corresponding zero sets $Z(f)$, $Z(g)$, the number of lines that are fully contained in $Z(f) \cap Z(g)$ is at most $\deg(f) \cdot \deg(g)$.

Here is a brief, rough, and informal description of the analysis of Guth and Katz. Given a set L of n lines in $\mathbb{R}^3$, they "force", in a preliminary pruning and sampling step, most of the joints of L to lie on the zero set $Z(f)$ of a polynomial f of degree $D \leq cn^{1/2}$, with a sufficiently small constant c, and then only consider lines of L that are also fully contained in $Z(f)$ (the other lines do not generate too many joints). Now a joint incident to three non-coplanar lines, all contained in $Z(f)$, must be a singular point of f, and lines that contain more than D such joints must consist exclusively of singular points. Lines that contain fewer than D joints contribute at most $nD = O(n^{3/2})$ joints, so they can be ignored. Now, assuming f to be irreducible, and applying the preceding result (ii) to f and one of its partial derivatives, say f_x, we conclude that the number of such "critical" lines is at most $D^2 \leq c^2 n$, which we can make smaller than, say, $n/2$. An inductive argument on n then completes the proof.

The actual proof in [6] is more involved and technical. It has been greatly simplified in two subsequent papers by Kaplan et al. [10] and by Quilodrán [14], and has also been extended to any dimension $d \geq 3$.

Incidences. Although the joints problem might appear, on the face of it, only a minor curious problem, the recent developments, as being reviewed here, show that it is in fact a significant pillar in the study of *incidences* between points and lines, curves, hyperplanes or surfaces. In the simplest form of the problem, the celebrated theorem of Szemerédi and Trotter [18] from 1983 asserts that, for a set P of m points and a set L of n lines in the plane, the the number $I(P, L)$ of *incidences* between the points of P and the lines of L, is $O(m^{2/3}n^{2/3} + m + n)$, and this bound is tight in the worst case. Many extensions of the problem have been considered, and can be found in the recent comprehensive survey [13].

If one considers incidences between m points and n lines in higher dimensions, say in $d = 3$ dimensions, the problem, on first sight, seems totally uninteresting. Indeed, one can project the points and lines onto some generic plane, observe that incidences are preserved in the projection, and apply the Szemerédi–Trotter bound. Since the bound is worst-case tight in the plane, it continues to be so in any higher dimension. The joints problem, in retrospect, was an attempt to remove the triviality from this extension, by forcing the input, in a sense, to be "truly three-dimensional". As follows from the results of Guth and Katz (and even from the weaker previous results), one does indeed get improved bounds in truly three-dimensional scenes, in which the amount of coplanarity of the input points and lines is kept in control. A subsequent paper of Elekes et al. [2] has extended the study of [6] to consider not just the number of joints but also the number of incidences between the joints of L and the lines of L, and to more general scenarios involving incidences between points and lines in $\mathbb{R}^3$.

Distinct distances. The next development took place in an attempt to apply the new machinery to the planar *distinct distances* problem of Erdős [4]. In this problem the goal is to establish a sharp lower bound on the minimum possible number of distinct distances between the elements of a set S of s points in the plane. Erdős noticed that the $\sqrt{s} \times \sqrt{s}$ integer grid generates $O(s/\sqrt{\log s})$ distinct distances, and conjectured this to be also the lower bound, namely, that any set of s points in the plane determines at least $\Omega(s/\sqrt{\log s})$ distinct distances. Again, traditional techniques, becoming progressively more sophisticated during the 65 years since the original problem statement, have been unable to settle the conjecture, and the best lower bound that was achieved, by Katz and Tardos [11], was $\Omega(n^{0.8641})$.

Nevertheless, about 10 years ago, Elekes had come up, in an unpublished manuscript, with an ingenious program to reduce the planar distinct distances problem to an incidence problem between points and curves in three dimensions. To tackle the latter problem, though, he needed a couple of fairly deep conjectures, which neither he nor anybody else knew how to solve at that time. If these conjectures could be established, they would have led to the almost tight lower bound $\Omega(s/\log s)$ on the number of distinct distances. In a joint paper with Elekes [3], written after the passing away of Elekes in 2008, I have presented Elekes's program, and applied the new algebraic machinery to it, but I was still unable to settle Elekes's conjectures, as the algebraic machinery, available from the Guth–Katz paper of 2008 and the follow-up ones, was still too weak.

This was taken care of in the second dramatic breakthrough of Guth and Katz [7], in November 2010, where they introduced new algebraic machinery, based on the *polynomial ham sandwich theorem* of Stone and Tukey [17], which allowed them to establish Elekes's conjectures and thereby obtain the aforementioned lower bound $\Omega(s/\log s)$ for distinct distances. Specifically, their main result, an extension of the main conjecture of Elekes, is: Given n lines in three dimensions, the number of points that are incident to at least $k \geq 3$ of these lines is $O(n^{3/2}/k^2)$, provided that no plane contains more than $n^{1/2}$ lines. The case $k = 2$ is also treated in [7]. There one needs to assume that no plane or regulus contains more than $n^{1/2}$ lines, and the analysis is based on algebaic properties of *ruled surfaces*, established by Salmon and Cayley in the 19th century [15].

The application of the polynomial ham sandwich theorem in [7] results in a so-called *polynomial partitioning* scheme, a new tool that appears to be very powerful in combinatorial and computational geometry, nicely complementing the 20-years-old arsenal of geometric partitions based on *cuttings* [12] and on *simplicial partitions* [12]. Roughly, it states that, given a set P of m points in $\mathbb{R}^d$, and a parameter $t < m$, one can find a d-variate polynomial f, of degree $D = O(t^{1/d})$, such that each connected component ("cell") of $\mathbb{R}^d \setminus Z(f)$ contains at most m/t points of P; the number of cells is $O(D^d) = O(t)$. This partitioning of P is not exhaustive, as some (perhaps many, or all) points of P may lie on $Z(f)$, and they require a special treatment, depending on the specific problem at hand.

Among the results that exploit the new polynomial partitioning technique are: (i) an expository paper by Kaplan, Matoušek and Sharir [8] where the new technique is exposed and applied to derive new and simple proofs of several classical results in combinatorial geometry, including the Szemerédi–Trotter incidence bound for points and lines in the plane, and the existence of spanning trees with small stabbing number in any dimension; (ii) two independent derivations, by Kaplan et al. [9] and by Zahl [19], of the improved bound $O(n^{3/2})$ on the number of repeated distances in a set of n points in $\mathbb{R}^3$; and (iii) a paper by Solymosi and Tao [16] on incidences between points and low-dimensional varieties that behave like pseudo-lines.

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A simpler proof of the Boros–Füredi–Bárány–Pach–Gromov theorem

Roman Karasev

1. The problem

The main topic of this talk is:

Problem. *Let $d+1$ random points $x_0, \dots, x_d$ be distributed independently in $\mathbb{R}^d$. Show that one point $c \in \mathbb{R}^d$ is covered by the simplex* $\operatorname{conv}\{x_0, \dots, x_d\}$ *with probability p_d with largest possible value of p_d.*

2. The history

Endre Boros and Zoltán Füredi [3] established the best constant $p_2 = 2/9$ when the points are distributed with the same discrete distribution.

Imre Bárány [1] considered arbitrary dimension and random points distributed by the same discrete distribution. The constant was roughly $p_d = (d+1)^{-d}$. This result was obtained by partitioning the N distribution points into $\sim \frac{N}{d+1}$ groups of $d+1$ each by the Tverberg theorem and then applying the colorful Carathéodory theorem to every $(d+1)$-tuple of $(d+1)$-tuples.

János Pach [9] considered arbitrary dimension and points distributed with different discrete distributions. The constant p_d was approximately $\frac{1}{(5d)^{d^2}(d+1)}$.

In case of the same discrete distribution for all points Uli Wagner [11] has improved the bound to $p_d = \frac{d^2+1}{(d+1)^{d+1}}$.

Recently Mikhail Gromov [6] has developed a topological approach to estimating multiplicity of maps, in particular, giving a better bound $p_d = \frac{1}{(d+1)!}$ for the probability of covering by the convex hull, which improves to $p_d \geq \frac{2d}{(d+1)!(d+1)}$ when some two points have the same distribution.

Gromov actually proved a much stronger result: Instead of several finite point sets in $\mathbb{R}^d$ one can consider a continuous map of the join of $d+1$ finite sets to $\mathbb{R}^d$ (or the d-skeleton of large enough simplex) and study covering by the images of faces of maximal dimension under this map. But we will not consider such generalizations here.

The proof of Gromov is not easy to understand. It used an abstract notion of the space of cocycles. Moreover, the space of cocycles was defined as a simplicial set, which is even harder to imagine.

3. The short proof

Now we are going to decipher Gromov's proof. It turns out that the proof becomes almost elementary and the only needed topological notion is the degree of a piecewise-smooth map.

Let us make more precise definitions. Consider a set of $d+1$ absolutely continuous probability measures $\mu_0, \mu_1, \dots, \mu_d$ on $\mathbb{R}^d$.

Define a *random simplex* of dimension k as a simplex spanned by $k+1$ points $x_{d-k}, \dots, x_d \in \mathbb{R}^d$, where the point x_i is distributed according to the measure μ_i. Note the indices that we choose for the case $k < d$.

Let us state the theorem:

Theorem 1. *Under the above assumptions there exists a point $c \in \mathbb{R}^d$ such that the probability for a random d-simplex to contain c is at least*

$$p_d = \frac{1}{(d+1)!}.$$

We assume that $\mathbb{R}^d$ is contained in its one-point compactification $S^d = \mathbb{R}^d \cup \{\infty\}$.

Assume the contrary. Take some small $\varepsilon > 0$. Take fine enough finite triangulation Y of S^d with one vertex at ∞ so that for any $0 < k \leq d$ and any k-face σ of Y the probability of a random $(d-k)$-simplex $x_k x_{k+1} \dots x_d$ to intersect σ is $< \varepsilon$. Here and below we always assume that μ_i is the distribution of x_i.

To make such a triangulation it is sufficient to take a large enough ball B so that at least $1-\varepsilon$ of every measure is inside B. Then we take the simplices of Y that intersect B small enough, other simplices may be arbitrary. From the absolute continuity of measures it follows that for small enough simplices the probabilities become arbitrarily small, and for simplices in $\mathbb{R}^d \setminus B$ they are $< \varepsilon$ by the choice of B.

Consider a $(d+1)$-dimensional simplicial complex $Y * 0$ (the cone over Y with apex 0). Now we are going to build a (piece-wise smooth) map $f : (Y * 0)^{(d)} \to S^d$ (from the d-skeleton) which is "economical" with respect to the measures μ_i (this phrase will be clarified below), and coincides with the identification $Y = S^d$ on $Y \subset (Y * 0)^{(d)}$.

Now proceed by induction:

Map 0 to $\infty \in S^d$;

For any vertex $v \in Y$ map $[v0]$ to an open ray starting from v (and ending at $\infty \in S^d$) so that the probability for a random $(d-1)$-simplex $x_1 \dots x_d$ to meet $f([v0])$ is $< p_d$. This is possible because a simplex $x_0 x_1 \dots x_d$ contains v if and only of the $(d-1)$-simplex $x_1 \dots x_d$ intersects the ray from v opposite to $x_0 - v$. Since the probability for a random d-simplex to contain v is $< p_d$, for some of such rays the corresponding probability is also $< p_d$.

Step to the k-skeleton of $Y * 0$ as follows. Let $\sigma = v_1 \dots v_k 0$ be a k-simplex of $Y * 0$. The map f is already defined for $\partial\sigma$. We know that the probability for a random $(d-k+1)$-simplex $x_{k-1} \dots x_d$ to meet some $f(v_1 \dots \hat{v}_i \dots v_k 0)$ $(i = 1, \dots, k)$ is $< (k-1)! p_d$, and the probability to meet $f(v_1 \dots v_k)$ is $< \varepsilon$. If ε is chosen small enough we see that a random $(d-k+1)$-simplex $x_{k-1} \dots x_d$ intersects $f(\partial\sigma)$ with probability $< k! p_d$.

There exists a point x_{k-1} not in $f(\partial\sigma)$ such that the probability for $x_{k-1} x_k \dots x_d$ (with random last $d-k+1$ points) to meet $f(\partial\sigma)$ is $< k! p_d$; here the independence of the distributions of vertices is essential.

Let us define the map f on the simplex σ treated as a join $\partial\sigma * c$ so that c is mapped to $\infty \in S^d$, and every segment $[vc]$ ($v \in \partial\sigma$) is mapped to the infinite ray from $f(v)$ in the direction opposite to $x_{k-1} - v$.

Finally, for any d-simplex σ of Y, the boundary of the cone $\sigma * 0$ is mapped by f so that

$$\mu_d(f(\partial(\sigma * 0))) < (d+1)!\, p_d = 1,$$

if we selected small enough ε at the beginning.

Therefore $f(\partial(\sigma * 0)) \neq S^d$ and the restriction $f|_{\partial(\sigma*0)}$ has zero degree. By summing up the degrees (the d-faces of $(\partial\sigma) * 0$ go pairwise and cancel, because Y is a triangulation) we see that the map $f|Y$ has even degree but it is the identity map, which is a contradiction.

4. The case of equal measures

This theorem can be sharpened (following [6]) if two of the measures coincide.

Theorem 2. *If some two measures coincide then the bound in Theorem 1 can be improved to*

$$p_d = \frac{2d}{(d+1)!(d+1)}.$$

The proof is essentially the same, but it uses more careful definition of f on the last step (for d-dimensional faces).

5. Further questions

- What is the best value of p_2 for different distributions of three point in $\mathbb{R}^2$? Is it 1/6?
- For large d, it is not known whether the inverse factorial in p_d is the right order of magnitude.
- Existing examples (by Boris Bukh, Jiří Matoušek, and Gabriel Nivasch [4]) only show that p_d cannot be better than $\sqrt{2\pi d}e^{-d}$.
- In the paper [8] of Jiří Matoušek and Uli Wagner further improvements of the constant (using Gromov's proof and some new ideas) are made.

6. Other applications

A similar technique allows to prove the following result, conjectured by Jiří Matoušek and Aleš Přívětivý [7]:

Theorem 3. *Suppose that a d-dimensional cube Q^d is partitioned into n^d small cubes in an obvious way. Let $0 \le m < d$. If the set of small cubes of Q^d is colored into $m+1$ colors then there exists a connected monochromatic component of size at least*

$$f(d,m)n^{d-m}.$$

Here $f(d,m)$ is a function depending on d and m and not depending *on n.*

The function $f(d,m)$ quickly decreases both in d and m. For $m = 1$ and $m = d-1$ this theorem can be proved with much better constant $f(d,m)$, see [7].

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When is the number of hyperplanes determined by n points in d-space at least the number of $(d-2)$-dimensional flats?

George Purdy

1. Erdős's Question

Circa 1978, stimulated by work we were doing together, Erdős asked the question: When is the number of planes determined by n points in 3-space at least the number of lines? I conjectured what we now call

Conjecture $\mathbf{P}_3$. *The number p of planes determined by n points in 3-space satisfies*

$$p \geq l, \qquad (1)$$

where l is the number of lines, provided

- *n is sufficiently large,*
- *the n points do not all lie on two skew lines, and*
- *no plane contains $n-1$ of the points.*

Remark: Putting $n/2$ points on each of two skew lines L and L' results in $\Omega(n^2)$ lines and $O(n)$ planes, falsifying (1).

In 1983, with the help of the Szemerédi–Trotter theorem, I was able to prove the weaker

Theorem $\mathbf{P}_3^*$.

(2) $$p > cl,$$

for a constant $c > 0$, under the same hypothesis.

We have since made the conjecture

Conjecture $\mathbf{P}_d^*$. *If W_k is the number of flats of dimension k determined by n points in d-space, then*

(3) $$W_{d-1} > c_d W_{d-2},$$

provided the n points cannot be covered by a collection of flats whose dimensions add up to less than d.

For example, when $d = 3$, the n points cannot be covered by two skew lines, since the dimensions of the lines would add up to 2, which is less than $d = 3$.

Comments: In 1951, Motzkin proved that $W_{d-1} \geq W_0 = n$ for n points spanning d-space, and this has been shown to be true, for example, for simple matroids.

But, P_d^* is false for finite projective d-spaces for sufficiently large finite fields, so P_3^* seems to capture some properties of real 3-space. But we can also prove P_3^* in complex 3-space using the fact, recently proved by Józef Solymosi and Terence Tao that the Szemerédi–Trotter theorem is (within epsilon) true in $\mathbb{C}^2$, together with a 1986 inequality by F. Hirzebruch:

Given n points in $\mathbb{C}^2$, if t_i is the number of lines containing exactly i points, and if $t_n = t_{n-1} = 0$, then

$$t_2 + t_3 \geq n + t_5 + 2t_6 + 3t_7 + \cdots$$

Remark: Our proof of P_3^* actually proves a stronger result,

Theorem $\mathbf{Q}_3^*$. *Under the hypothesis of theorem P_3^*, one of the n points is incident with cl of the planes.*

We thus have a result reminiscent of Erdős's weak Dirac conjecture, now a theorem:

Given n points in the plane, not all collinear, one of the points is incident with cn lines determined by the points, where $c > 0$.

We naturally make,

Conjecture $\mathbf{Q}_d^*$. *Under the same hypothesis as P_d^*, one of the original n points is incident with $c_d W_{d-2}$ hyperplanes.*

Recent Work: My students Ben Lund, Justin Smith and I have launched a major assault on P_d^* and have been developing new tools analogous to the 1983 results of Szemerédi, Trotter, and Beck for this purpose. So far, we have proved P_4^* and the following:

Theorem. *Given n points, of which k are colored red, there are only*

$$O_d(m^{2/3}k^{2/3}n^{(d-2)/3} + kn^{d-2} + m)$$

incidences between the k red points and m hyperplanes spanned by the n points, provided that $m = \Omega(nd^2)$.

The monochromatic case $k = n$ was proved by Agarwal and Aronov. We use this incidence bound to prove that n points, no more than $n - k$ of which lie on any plane or two lines, span $\Omega(nk^2)$ planes, which is a special case of a general conjecture of ours that implies P_d^*. We are hopeful that this conjecture, and another, even stronger one, will admit proofs by induction.

A pseudoline counterexample to the Strong Dirac conjecture

Ben Lund

The Strong Dirac conjecture, open in some form since 1951 [5], is that every set of n points in $\mathbb{R}^2$ includes a member incident to at least $n/2 - c$ lines spanned by the set, for some universal constant c. The less frequently stated dual of this conjecture is that every arrangement of n lines includes a line incident to at least $n/2 - c$ vertices of the arrangement. It is known that an arrangement of n pseudolines includes a line incident to at least $c_d n$ vertices of the arrangement, for some universal constant c_d [2, 8]. Every known infinite family of arrangements includes a line incident to at least $n/2 - 3/2$ vertices of the arrangement, and such a family was found by Felsner [4, p. 313]. I presented an infinite family of arrangements of n pseudolines that does not include any pseudoline incident to more than $4n/9 - 10/9$ vertices.

Felsner found an infinite family of arrangements with $n = 12k + 7$ lines, each line incident to at most $n/2 - 3/2$ vertices. The first member of this family is shown in Figure 1.

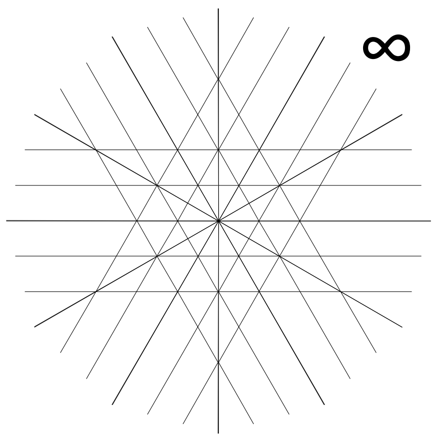

Figure 1. Felsner's arrangement with $n = 19$

Since this visualization of the arrangement has D_6 symmetry, a wedge with central angle of $\pi/6$ contains sufficient information to examine the entire arrangement. Figure 2 shows wedges for the first two members of Felsner's family. This method of depiction was introduced by Eppstein [6].

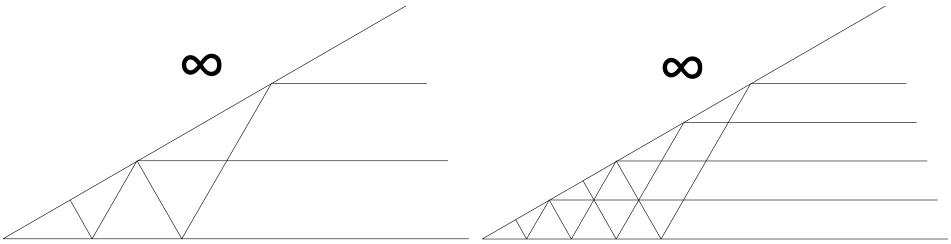

FIGURE 2. Felsner's arrangement with $n = 19$ and $n = 31$

The first member of a family of simple arrangements, each having $n = 4k + 9$ lines, no line incident to more than $n/2 - 1/2$ vertices, is shown in Figure 3.

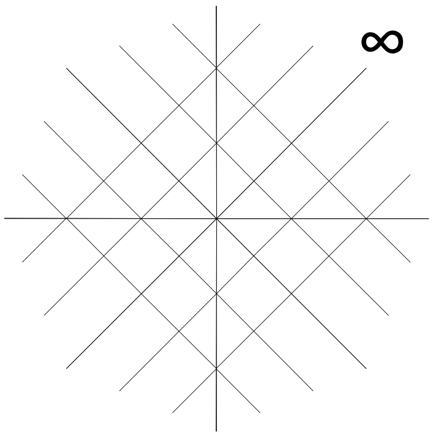

FIGURE 3. Simple arrangement with 13 lines

Akiyama et. al. [1] published a catalog of known infinite families of point sets representing extreme examples for various large n that includes dual versions of both of the arrangements shown here. Grünbaum cataloged several finite line arrangements such that no line is incident more than $n/2-c$ vertices, with $c > 3/2$ [7].

Wedges for the two smallest members of a family of pseudoline arrangements with no line incident to more than $4n/9 - 10/9$ vertices are shown in Figures 5 and 6; Figure 4 shows the complete arrangement with 25 pseudolines. Members of this family drawn in the style used here have k-fold dihedral symmetry for $k = 6j + 2,\ j = [0, \infty)$. Such an arrangement with k-fold symmetry contains

$n = 3k+1$ pseudolines. This arrangement was previously investigated by Berman [3, Fig. 11], in the context of simplicial pseudoline arrangements.

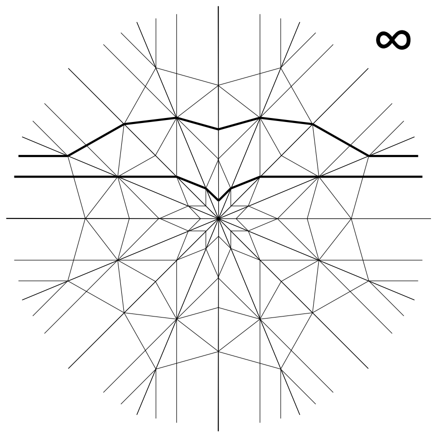

FIGURE 4. Pseudoline arrangement with $n = 25$

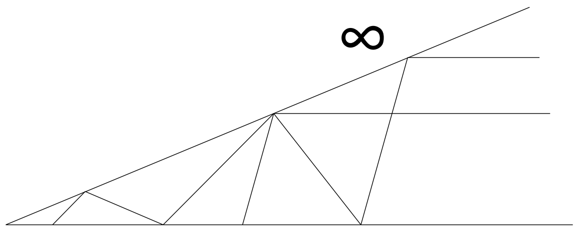

FIGURE 5. Wedge for pseudoline arrangement with $n = 25$

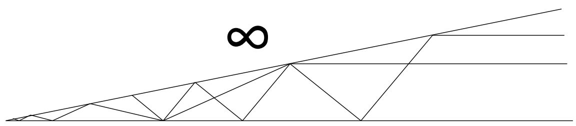

FIGURE 6. Wedge for pseudoline arrangement with $n = 49$

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Helly numbers of acyclic families

Xavier Goaoc
(joint work with Éric Colin de Verdière and Grégory Ginot)

The *nerve* $N(F)$ of a family F of sets is the family of its subsets with nonempty intersection, that is:

$$N(F) = \left\{ G \subseteq F \mid \bigcap_{a \in G} a \neq \emptyset \right\}.$$

The nerve is an *abstract simplicial complex*, that is a family of sets closed under taking subsets (and sometimes also called a *monotone hypergraph*). Just like a graph can be embedded, in the plane for planar graphs or in $\mathbb{R}^3$ in general, we can associate to an abstract simplicial complex a *geometric realization*, making these discrete objects amenable to topological methods. A typical example is Borsuk's *Nerve theorem*, which asserts that for good covers the geometric realization of the nerve has the same homotopy type as the union. (Recall that a *good cover* is a family of subsets of a topological space such that the intersection of every subfamily is empty or contractible.) The Nerve theorem implies, for instance, that if F is an open good cover in $\mathbb{R}^d$ such that $\bigcap_{a\in F} a = \emptyset$ and $\bigcap_{a\in G} a \neq \emptyset$ for *all* proper subfamilies $G \subsetneq F$ then F has cardinality at most $d+1$. Indeed, the nerve of such a family F is the boundary of a $(|F|-1)$-simplex, which has nontrivial homology in dimension $|F|-2$, whereas any open[1] subset of $\mathbb{R}^d$ has vanishing homology in dimension d. In other words, the Helly number of any open good cover is at most $d+1$. (Recall that the *Helly number* of a family is defined as the cardinality of its largest inclusion-minimal subfamily with empty intersection.) The Nerve theorem can be easily seen to fail when the sets are non-connected, or have non-connected intersections. We propose a refinement of the notion of nerve that enjoys an analogue of the Nerve theorem not only for good cover, but for any acyclic family; here a family F of subspaces of a topological space is *acyclic* if

[1]Interestingly, this assertion becomes false if "open" is replaced by "compact" without further care, as for instance the union of all circles with centers $(0, \frac{1}{n})$ and radii $\frac{1}{n}$, the so-called "Hawaiian earrings", has non-trivial homotopy groups in arbitrary high dimensions.

the connected components of the intersection of any subfamily of F are homology cells. This allows us to generalize some Helly numbers.

Let F denote a family of subsets of a topological space. We associate to every subset $G \subseteq F$ with non-empty intersection not just a simplex, but as many simplices as G has connected components. Specifically, we put:

$$M(F) = \left\{ (G, C) \mid G \subset F \text{ and } C \text{ is a connected component of } \bigcap_{a \in G} a \right\},$$

with the convention that $(\emptyset, \bigcup_{a \in F} a) \in M(F)$. The incidences between simplices, given by the inclusion relation in the case of the nerve, are now recorded by the partial order

$$(G, C) \preceq (G', C') \Leftrightarrow G \subseteq G' \quad \text{and} \quad C \supseteq C'$$

which turns $M(F)$ into a simplicial partially ordered set (or *simplicial poset*): it has a unique minimum element and for any element $\sigma \in M(F)$ there exists a bijection between the lower interval $\{\tau \mid \tau \preceq \sigma\}$ and the face lattice of a simplex. Like graphs and abstract simplicial complexes, simplicial posets have geometric realizations and can be studied via topological methods. In fact simplicial posets are particular cases of *simplicial sets*, a combinatorial description of spaces used in topology.

Given a topological space Γ, let d_Γ denote the smallest integer such that the i^{th} homology of any open subset of Γ vanishes for all $i \geq d_\Gamma$. (In particular, $d_{\mathbb{R}^d} = d$ and, more generally, $d_\Gamma = d$ for any non-compact or non-orientable d-manifold and increases to $d_\Gamma = d + 1$ for compact and orientable d-manifolds.) We prove:

Theorem 1. *Let F be a finite acyclic family of open subsets of a locally arc-wise connected topological space Γ. If any sub-family of F intersects in at most r connected components, then the Helly number of F is at most $r(d_\Gamma + 1)$.*

The case $r = 1$ is essentially Helly's topological theorem (in an arbitrary manifold) and the case where F is a r-family[2] over an open good cover of $\mathbb{R}^d$ was previously established by Kalai and Meshulam [3] (on whose approach we modelled our proof). The acyclicity condition can be weakened to accomodate some non-trivial homology in low dimension for the intersection of few sets, allowing applications in geometric transversal theory. See the preprint [1] for full details.

There are at least two reasons to believe the story does not end here. First, Matoušek [4] proved a similar statement in homotopy (with sensibly larger constants) and his proof allows non-trivial homotopy in high dimension; as our proof permits some amount of non-trivial homology in low dimension, it is not clear how much acyclicity suffices to ensure bounded Helly number. Second, there is a combinatorial generalization of Kalai and Meshulam's theorem, due to Eckhoff and Nischke [2], that we cannot, at the moment, bring under the same umbrella as Theorem 1.

[2]A family F is a r-family over G if the intersection of any subfamily of F is a disjoint union of at most r members of G.

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Generalizations of the Kakeya problem

Otfried Cheong

(joint work with Antoine Vigneron)

Given a compact convex figure P in the plane, we call a compact convex figure C a *keyhole* for P if for any orientation P' of P, the set P' can be translated to lie in C. Intuitively, we are looking for a set in which the "key" P can be placed and rotated by a full turn.

This problem was perhaps first considered in the famous Kakeya needle problem, which asks for the smallest-area keyhole for the unit segment. This problem was solved by Pàl in 1921, who showed the solution to be an equilateral triangle of height one [4]. Bezdek and Connelly show that the unique smallest-perimeter convex figure in which an equilateral triangle can be fully rotated is its circumcircle [2]. For centrally symmetric figures P, it follows immediately from the Cauchy–Crofton formula [3] that rotating P around its center produces a smallest-perimeter keyhole for P. Finding smallest-area keyholes seems hard, for instance it is not even known for the equilateral triangle, the square, or the Reuleaux triangle [1, 2].

We prove that for any compact convex figure P the smallest enclosing disk of P is a smallest-perimeter keyhole for P.

We observe first that a keyhole for a segment of length a must necessarily have width a in every direction. Since the perimeter is the integral over the width [3], this implies that any convex figure of constant width a has diameter at least πa, and in particular the circle with diameter a is a smallest-perimeter keyhole.

Consider next an acute triangle T. Choose a coordinate system with origin at the center of the circumcircle of T, and such that the circumcircle has radius one. We wish to prove that any keyhole for T must have perimeter at least 2π, implying that the circumcircle is optimal.

We borrow an idea of Bezdek and Connelly [2]. Let u_1, u_2, u_3 be the three corners of T. By our assumptions, the origin lies in the interior of their convex hull, and the three vectors have length one. The origin can be expressed as a convex combination $0 = \sum_{i=1}^{3} \alpha_i u_i$ with $\alpha_i \geq 0$ and $\sum_{i=1}^{3} \alpha_i = 1$. Let δ_i, for $i = 1, 2, 3$, be the angle formed by u_1 and u_i (so $\delta_i = 0$).

Let C be a any keyhole for T and let $w(\theta)$ be the width of C in direction θ (that is, the length of the projection of C on a line with slope θ).

Let p be the length of the perimeter of C. By the Cauchy–Crofton formula, we have

$$p = \int_0^{\pi} w(\theta) d\theta.$$

Since the width is a periodic function with period π, we have

$$p = \int_0^{\pi} w(\theta) d\theta = \int_{\delta_i}^{\pi+\delta_i} w(\theta) d\theta = \int_0^{\pi} w(\theta + \delta_i) d\theta.$$

It follows that

$$p = \sum_{i=1}^{3} \alpha_i p = \sum_{i=1}^{3} \alpha_i \int_0^{\pi} w(\theta + \delta_i) d\theta = \int_0^{\pi} \big(\sum_{i=1}^{3} \alpha_i w(\theta + \delta_i) \big) d\theta.$$

Consider now a fixed orientation θ. The keyhole C must contain a copy of T rotated such that u_1 has direction $\theta = \theta + \delta_1$. This implies that u_2 has direction $\theta + \delta_2$, and u_3 has direction $\theta + \delta_3$. For simplicity, we let T denote this rotated copy, with its circumcenter still at the origin. Since C is a keyhole, it must also contain a translated copy of $-T$, let's say $t - T$.

Since u_i and $t - u_i$ lie in C, the width of C in the direction of u_i can be lower bounded by $\langle u_i, u_i \rangle - \langle t - u_i, u_i \rangle = 2\langle u_i, u_i \rangle - \langle t, u_i \rangle = 2 - \langle t, u_i \rangle$. We therefore have

$$\sum_{i=1}^{3} \alpha_i w(\theta + \delta_i) \geq \sum_{i=1}^{3} \alpha_i (2 - \langle t, u_i \rangle) = 2 - \langle t, \sum_{i=1}^{3} \alpha_i u_i \rangle = 2.$$

It now follows from the Cauchy–Crofton formula that $p \geq 2\pi$.

Consider finally a compact convex figure P, and let D be the smallest enclosing disk of P. Either D touches P in two points that form a diameter of D, or D touches P in three points that form an acute triangle. In both cases, our previous results imply that D is a shortest-perimeter keyhole for either the segment or the triangle, and therefore for P.

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Good covers are algorithmically unrecognizable

Martin Tancer
(joint work with Dmitry Tonkonog)

Many results in discrete geometry are devoted to studying intersection patterns of convex sets. Pioneering result in this respect is the Helly theorem [1] which states that whenever $C_1, \dots, C_n$ are convex sets in $\mathbb{R}^d$ with $n \geq d+1$ such that the intersection of any $d+1$ among these sets is nonempty, then the intersection of all sets is nonempty. Many results of similar flavor are known and the interested reader is referred to the survey paper [3] for more details.

d-representable complexes. Intersection patterns of collections of convex sets can be studied systematically via their nerves. Given a collection of sets $\mathcal{F} = \{F_1, \dots, F_n\}$ the *nerve* of this collection is the simplicial complex with vertex set $\mathcal{F}$ and whose faces are the subcollections of $\mathcal{F}$ with a nonempty intersection. A simplicial complex is *d-representable* if it is isomorphic to the nerve of a finite collection of convex sets in $\mathbb{R}^d$. Using this notion, classifying intersection patterns of convex sets in $\mathbb{R}^d$ is equivalent to classifying d-representable complexes.

Topological d-representability. Many results on intersection patterns of convex sets were generalized to the situation where the sets are not necessarily convex, however, their intersections are not too complicated (we will discuss the precise definition in the following paragraph). In such case we talk about 'topological versions'. For instance there is a topological version of the Helly theorem obtained again by Helly [2].

A *good cover* is a finite collection of open sets in $\mathbb{R}^d$ such that the intersection of any nonempty subcollection is either empty or contractible (in particular, the sets of the collection are contractible).

A simplicial complex K is *topologically d-representable* if it is isomorphic to the nerve of a good cover. A *topological d-representation* of K is a good cover whose nerve is isomorphic to K.

Computational complexity. In this contribution we focus on the computational complexity of recognition of topologically d-representable complexes. We show that this problem is algorithmically undecidable. (More precisely, we fix a positive integer d; the input of the problem is a simplicial complex given abstractly as a collection of faces; and the question is whether this complex is topologically d-representable.)

Theorem 1. *For $d \geq 5$, it is algorithmically undecidable whether a given simplicial complex is topologically d-representable.*

In convex setting, d-representable complexes are algorithmically recognizable. This result dates back at least to Wegner [5]; and there is actually a PSPACE algorithm. It is interesting to see this contrast, since many Helly-type theorems are valid in almost the same settings for d-representable and topologically d-representable complexes. Our result is thus in opposite spirit.

The proof method; a brief sketch. Our main ingredient is the theorem of Novikov [4] on algorithmic unrecognizability of d-sphere for $d \geq 5$. Using this theorem we are able to construct a sequence of simplicial complexes $\{C_i\}_{i=1}^{\infty}$ such that the elements of this sequence are either "nice" d-balls (piecewise linearly embeddable into $\mathbb{R}^d$), or they have a nontrivial fundamental group (and some additional properties, that we do not discuss here in detail); and there is no algorithm deciding which of the two cases holds.

In case of "nice" d-balls we can first construct a piecewise linear embedding of these balls and then it is possible to deduce a topological d-representation from this embedding. (This part is a bit technical; however, quite straightforward.)

In the second case, the nontriviality of the fundamental group together with "additional properties" implies the nonexistence of a topological d-representation. This case is in our opinion not so obvious, however, still manageable.

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Equalities on empty polygons

Pavel Valtr

Let P be a set of n points in general position in the plane. Consider the complex, C, of empty convex polygons in P. C is clearly a simplicial complex. Let $f_k(P)$ be its f-vector ($k = 1, 2, \dots$), that is, $f_k(P)$ is the number of empty convex k-gons in P. Clearly $f_1(P) = n$, and $f_2(P) = \binom{n}{2}$. It is proved by Edelman and Rainer [2] that C is contractible. Then it satisfies the Euler equation:

$$f_1(P) - f_2(P) + f_3(P) - f_4(P) \cdots = 1.$$

There is another linear relation satisfied by the f-vector: it is shown by Ahrens et al. [1] that

$$f_1(P) - 2f_2(P) + 3f_3(P) - 4f_4(P) \cdots = |P \cap \text{ intconv} P|.$$

Also other proofs of these two equalities have been published. So far, the simplest proof was using the so-called "continuous motion technique" and was given in a paper by R. Pinchasi, R. Radoičić and M. Sharir [4]. In my Oberwolfach talk, I presented a very elementary proof technique which allows to prove the two equalities and it can be also used to prove other equalities, in particular two equalities of García [3] on the number of specific empty triangles and quadrilaterals in a finite planar point set in general position.

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Piercing quasi-rectangles — On a problem of Danzer and Rogers

János Pach
(joint work with Gábor Tardos)

An old problem of Danzer and Rogers [8, 2, 5, 3] is the following: What is the area of the largest convex region not containing in its interior any one of n given points in a unit square? By drawing parallel lines through the points, the square falls into $n+1$ rectangles. At least one of these rectangles has area at least $\frac{1}{n+1}$, so this is clearly a lower bound. Can the order of magnitude of this bound be improved for all point sets, as n tends to infinity? We do not know. In 1982, Moser [8] reported only a fairly weak upper bound, $O\left(\frac{\sqrt{\log n}}{n^{3/4}}\right)$, due to Fan Chung. Since then, the problem has been analyzed a little better. To explain the new developments, we need some preparation.

It is more convenient to rephrase the question as follows. Given $\varepsilon > 0$, what is the size of the smallest set of points with the property that every compact convex set of area ε within the unit square contains at least one of them. Denoting this minimum by $f(\varepsilon)$, we clearly have $f(\varepsilon) = \Omega(1/\varepsilon)$. The question is whether $f(\varepsilon) = O(1/\varepsilon)$ holds.

This problem can be regarded as a continuous version of the ε-net problem in an infinite *range space* $(X, \mathcal{R})$, where the ground set X is the unit square, the *ranges* $R \in \mathcal{R}$ are compact convex subsets of X, and we want to "hit" every range R with $|R \cap X| = |R| \geq \varepsilon|X|$, where $|.|$ stands for the Lebesgue measure (area). A subset $N \subset X$ that intersects every such range is said to be an *ε-net* for the range space $(X, \mathcal{R})$.

A subset $A \subseteq X$ of the ground set is called *shattered* if for every subset $B \subseteq A$, one can find a range $R_B \in \mathcal{R}$ with $R_B \cap A = B$. The size of the largest shattered subset of points, $A \subseteq X$, is said to be the *Vapnik–Chervonenkis dimension* (or *VC-dimension*) of the range space $(X, \mathcal{R})$ (see [13, 9, 4]). It follows from the celebrated results of Haussler and Welzl [6] that every range space of VC-dimension at most Δ admits an ε-net of size $O_\Delta\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$.

We apply these ideas to our original problem. The area of the largest rectangle contained in a plane convex set R is at least half of the area of R [12]. Thus, in order to hit (pierce) all plane convex sets of area ε in the unit square, it is sufficient to find an $\varepsilon/2$-net for all rectangles. The family of rectangles has bounded VC-dimension $\Delta < 10$. Therefore, the theorem of Haussler and Welzl implies that $f(\varepsilon) = O\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$.

It has been known for a long time that, in the "abstract" combinatorial setting, the logarithmic factor in the Haussler–Welzl theorem cannot be removed [11, 7]. More recently, following the work of Alon [1], the present authors constructed a variety of geometric range spaces with the same property [10].

Nevertheless, it is perfectly possible that $f(\varepsilon) = O\left(\frac{1}{\varepsilon}\right)$, that is, all rectangles of area at least $\varepsilon > 0$ in the unit square can be pierced by $O\left(\frac{1}{\varepsilon}\right)$ points.

We show that, if we slightly enlarge the family of rectangles, by including "quasi-rectangles," then $O\left(\frac{1}{\varepsilon}\right)$ points do not suffice.

A rectangle is a region swept out by a line segment s moving orthogonally to itself. If we continuously translate s almost orthogonally to itself, without rotating it, so that the angle between s and the trajectory of its center always remains between $90-\delta$ and $90+\delta$ degrees for a fixed small $\delta > 0$, then we call the resulting region a *quasi-rectangle*. To be concrete, set $\delta = 1°$. The motion of the segment s is supposed to be monotone in the direction orthogonal to it, so that the segment is not allowed to turn back. Therefore, the area of a quasi-rectangle is equal to the length of s multiplied by the distance it traveled in the direction orthogonal to s.

A quasi-rectangle is not necessarily convex, but it is "almost" convex. Although the VC-dimension of the family of quasi-rectangles is unbounded, it is not hard to see that all quasi-rectangles of area ε inside the unit square can be stabbed by $O\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$ points. Our main theorem shows that this bound is tight up to a constant factor.

Theorem. *For any $\varepsilon > 0$, let $F(\varepsilon)$ denote the smallest number of points with the property that every quasi-rectangle of area ε inside the unit square contains at least one of them. We have $F(\varepsilon) = \Theta\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$.*

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When is a disk trapped with four lines?

Luis Montejano
(joint work with Tudor Zamfirescu)

Let Ω be a closed subset of euclidean 3-space $\mathbb{R}^3$ and let $D \subset \mathbb{R}^3 - \Omega$ be a planar disk. We say that D is trapped by Ω if D can continuously moved to infinity without intersecting Ω. For example, the 1-skeleton of a tetrahedra of side $\frac{\sqrt{3}}{2}$ always traps a disk of diameter slightly smaller than one, but the union of three lines in $\mathbb{R}^3$ does not trap a disk. We will characterize when a disk is trapped by 4 lines.

Theorem 1. *A disk of diameter slightly smaller than h_0 is trapped by four lines if and only if there is a direction v_0 such that:*

(1) *orthogonal to every direction sufficiently close to v_0, there is a disk of diameter h_0, intersecting the four lines,*
(2) *there is not a disk of diameter smaller than h_0, orthogonal to v_0, intersecting the four lines,*
(3) *there is a disk of diameter smaller than h_0, orthogonal to v_0, intersecting every three of the lines.*

Example. Let a_1, a_2, a_3 and a_4 be the four vertices of a regular tetrahedron of sides $\frac{\sqrt{3}h_0}{2}$ in $\mathbb{R}^3$. Let L_i be the line through the vertices a_1 and a_i, $i = 2, 3, 4$, and let L_1 be the line through the points a_2 and $\frac{a_2+a_3+a_4}{3}$. Then, it is intuitively clear that there is a disk of diameter slightly smaller than h_0 trapped by our four lines. Indeed, this is so because if $v_0 = \frac{3a_1-(a_2+a_3+a_4)}{\|3a_1-(a_2+a_3+a_4)\|}$, then orthogonal to every direction sufficiently close to v_0, there is a disk of diameter h_0, intersecting our four lines but there is not a disk of diameter smaller than h_0, orthogonal to v_0, intersecting our four lines, moreover there is a disk of diameter smaller than h_0, orthogonal to v_0, intersecting every three of our lines.

Let $L \subset \mathbb{R}^3$ be a line and let $D(L)$ be the collection of disks of diameter h intersecting L. Clearly $D(L)$ can be thought as a subset of $(L \times D_0) \times \mathbb{RP}^2$, where D_0 is a disk of diameter h orthogonal to L and centered at L. This is so, because a disk of diameter h that intersects L is uniquely determined by its center in $(L \times D_0)$ and a direction in $\mathbb{R}^3$. Similarly the collection of disks of diameter h in $\mathbb{R}^3$ is homeomorphic to $\mathbb{R}^3 \times \mathbb{RP}^2$.

It is convenient now to consider the oriented case. Let Ω be a closed subset of euclidean 3-space $\mathbb{R}^3$ and let $\mathfrak{D} \subset \mathbb{R}^3 - \Omega$ be an oriented planar disk. We say that $\mathfrak{D}$ is trapped by Ω if $\mathfrak{D}$ can continuously moved to infinity without intersecting Ω. Of course, the oriented disk $\mathfrak{D}$ is trapped by Ω if and only if the corresponding non oriented disk D is trapped by Ω. So we will work by convenience in the oriented case.

We shall consider $\mathbb{R}^3 \times \mathbb{S}^2 = \mathbb{R}^2 \times (\mathbb{R} \times \mathbb{S}^2)$ as $\mathbb{R}^2 \times (\mathbb{R}^3 - \{0\})$. If we denote by Δ the collection of oriented disks of diameter h in $\mathbb{R}^3$, we may think that Δ is a subset of $\mathbb{R}^5$. Let $\pi : \Delta \to \mathbb{S}^2$ be the map that sends its normal vector to every oriented disk of diameter h. Let now $\Omega \subset \mathbb{R}^3$ be a closed subset of $\mathbb{R}^3$ and let $\mathfrak{D}(\Omega) \subset \Delta \subset \mathbb{R}^5$ be the collection of oriented disks of diameter h intersecting Ω. We are interested in the bounded components of $\mathbb{R}^5 - \mathfrak{D}(\Omega)$, because It is clear that there is an oriented disk of diameter h trapped by Ω if and only if $\mathbb{R}^5 - \mathfrak{D}(\Omega)$ has a bounded component.

At this point the strategy of the proof of our theorem is to use the fact that for a compact set $X \subset \mathbb{R}^n$, the complement of X in $\mathbb{R}^n$ has a bounded component if and only if $H_{n-1}(X) \neq 0$, where in this paper we always use Cech homology groups with Z_2 coefficients.

Let L_1, L_2, L_3 and L_4 be four lines in $\mathbb{R}^3$ and from now on let $\Omega = L_1 \cup L_2 \cup L_3 \cup L_4$. We have to deal with the problem that $\mathfrak{D}(\Omega)$ is not compact. Here we will assume allways that the four directions of the lines are independent. The proof of the theorem in its generality follows from the ideas exposed here.

Let $B \subset \mathbb{R}^3$ be a closed ball sufficiently big so that any oriented disk of diameter h, not intersecting Ω, with center outside B, is not trapped by Ω. This is so because the four directions of the lines are independent and B was chosen in such a way that outside B the lines are very far away one of each other. Let $\mathfrak{D}_i \subset \Delta \subset \mathbb{R}^5$ be the collection of oriented disks of diameter h intersecting L_i, with center at B. So, the bounded components of $\mathbb{R}^5 - (\mathfrak{D}_1 \cup \mathfrak{D}_2 \cup \mathfrak{D}_3 \cup \mathfrak{D}_4)$ are exactly the bounded components of $\mathbb{R}^5 - \mathfrak{D}(\Omega)$. Since every $\mathfrak{D}_i$ is compact, this implies that there is an oriented disk of diameter h trapped by $\Omega = L_1 \cup L_2 \cup L_3 \cup L_4$ if and only if

$$H_4(\mathfrak{D}_1 \cup \mathfrak{D}_2 \cup \mathfrak{D}_3 \cup \mathfrak{D}_4) \neq 0.$$

We next consider the commutative diagram where the first row correspond to the Mayer Vietories sequence of the pair $(\mathfrak{D}_1 \cap [\mathfrak{D}_3 \cup \mathfrak{D}_4]; \mathfrak{D}_2 \cap [\mathfrak{D}_3 \cup \mathfrak{D}_4])$, the second row correspond to the Mayer Vietories sequence of the pair $(\mathfrak{D}_1 \cap \mathfrak{D}_3 \cap \mathfrak{D}_4; \mathfrak{D}_2 \cap \mathfrak{D}_3 \cap \mathfrak{D}_4)$, the third row correspond to the Mayer Vietories sequence of the pair $(\mathfrak{D}_1 \cap \mathfrak{D}_3; \mathfrak{D}_2 \cap \mathfrak{D}_3)$ and the forth row correspond to the Mayer Vietories sequence of the pair $(\mathfrak{D}_1 \cap \mathfrak{D}_3; \mathfrak{D}_2 \cap \mathfrak{D}_4)$

Similarly, the first column correspond to the Mayer Vietories sequence of the pair $([\mathfrak{D}_1 \cup \mathfrak{D}_2] \cap \mathfrak{D}_3; [\mathfrak{D}_1 \cup \mathfrak{D}_2] \cap \mathfrak{D}_4)$, the second row correspond to the Mayer Vietories sequence of the pair $(\mathfrak{D}_1 \cap \mathfrak{D}_2 \cap \mathfrak{D}_3; \mathfrak{D}_1 \cap \mathfrak{D}_2 \cap \mathfrak{D}_4)$, and the third row correspond to the Mayer Vietories sequences of the pairs $(\mathfrak{D}_1 \cap \mathfrak{D}_3; \mathfrak{D}_1 \cap \mathfrak{D}_4)$ and $(\mathfrak{D}_2 \cap \mathfrak{D}_3; \mathfrak{D}_2 \cap \mathfrak{D}_4)$. Using this commutative diagram we prove the following lemma.

Lemma 2. *There is a disk of diameter h trapped by the lines L_1, L_2, L_3 and L_4 if and only if, in the above commutative diagram, $H_3([\mathfrak{D}_1 \cup \mathfrak{D}_2] \cap [\mathfrak{D}_3 \cup \mathfrak{D}_3]) \neq 0$ if and only if there is $0 \neq \alpha \in H_1(\mathfrak{D}_1 \cap \mathfrak{D}_2 \cap \mathfrak{D}_3 \cap \mathfrak{D}_4)$, such that $i_*(\alpha) = 0 = j_*(\alpha)$.*

Proof of Theorem. An oriented disk is trapped by the lines L_1, L_2, L_3 and L_4 if and only if the corresponding non oriented disk is trapped by the lines L_1, L_2, L_3 and L_4. So we will work by convenience in the oriented case. For every pair of non intersecting, lines L_i and L_j, $1 \leq i < j \leq 4$, let $\pm\mathfrak{v}_{ij} \in \mathbb{S}^2$ be such that $\pm\mathfrak{v}_{ij}$ is orthogonal to L_i and L_j and let Υ be the set of all these unit vectors. Let $E : \mathbb{S}^2 - \Upsilon \to \mathbb{R}$ be the map defined as follows: for every direction $v \in \mathbb{S}^2 - \Upsilon$, $E(v)$ is the smallest diameter of an oriented disk orthogonal to v, intersecting the four lines. Since for every $t \in \mathbb{R}$, $E^{-1}((-\infty, t))$ is open in $\mathbb{S}^2$ and $E^{-1}([t, \infty))$ is closed in $\mathbb{S}^2 - \Upsilon$, we have that E is a continuous map. Furthermore, E is clearly analytic in the open subset of $\mathbb{S}^2$ consisting of those vectors non orthogonal to any of our four lines. Finally, the restriction of E to all vectors of $\mathbb{S}^2 - \Upsilon$, orthogonal to L_i, $1 \leq i \leq 4$, is also analytic. The above implies that the set of local maximus of E are isolated points.

Suppose there is an oriented disk of diameter h trapped by the lines L_1, L_2, L_3 and L_4. Then $H_4(\mathfrak{D}_1 \cup \mathfrak{D}_2 \cup \mathfrak{D}_3 \cup \mathfrak{D}_4) \neq 0$ and by e) of Lemma 1, $H_3([\mathfrak{D}_1 \cup \mathfrak{D}_2] \cap [\mathfrak{D}_3 \cup \mathfrak{D}_3]) \neq 0$. This implies, by Lemma 2, that there is $\alpha \in H_1(\mathfrak{D}_1 \cap \mathfrak{D}_2 \cap \mathfrak{D}_3 \cap \mathfrak{D}_4)$, $\alpha \neq 0$, such that $i_*(\alpha) = 0 = j_*(\alpha)$ (see commutative diagram 1). Let us now consider the map $E : \mathbb{S}^2 - \Upsilon \to \mathbb{R}$. First note that $(\mathfrak{D}_1 \cap \mathfrak{D}_2 \cap \mathfrak{D}_3 \cap \mathfrak{D}_4) \subset E^{-1}((-\infty, h])$. So without loss of generality, we may assume there is a component Γ of the boundary of $E^{-1}((-\infty, h])$ contained in $E^{-1}(h)$ such that $\Gamma \subset (\mathfrak{D}_2 \cap \mathfrak{D}_3 \cap \mathfrak{D}_4) \cup (\mathfrak{D}_1 \cap \mathfrak{D}_3 \cap \mathfrak{D}_4) \cup (\mathfrak{D}_1 \cap \mathfrak{D}_2 \cap \mathfrak{D}_4) \cup (\mathfrak{D}_1 \cap \mathfrak{D}_2 \cap \mathfrak{D}_4)$ and there is $0 \neq \alpha \in H_1(\mathfrak{d})$, such that $i_*(\alpha) = 0 = j_*(\alpha)$, where $i : \Gamma \to (\mathfrak{D}_2 \cap \mathfrak{D}_3 \cap \mathfrak{D}_4) \cup (\mathfrak{D}_1 \cap \mathfrak{D}_3 \cap \mathfrak{D}_4)$ and $j : \Gamma \to (\mathfrak{D}_1 \cap \mathfrak{D}_2 \cap \mathfrak{D}_4) \cup (\mathfrak{D}_1 \cap \mathfrak{D}_2 \cap \mathfrak{D}_3)$ are the inclusions. Consider the component A of $E^{-1}([h, \infty))$ whose boundary is Γ. The fact that $i_*(\alpha) = 0 = j_*(\alpha)$ implies that $A \subset (\mathfrak{D}_2 \cap \mathfrak{D}_3 \cap \mathfrak{D}_4) \cup (\mathfrak{D}_1 \cap \mathfrak{D}_3 \cap \mathfrak{D}_4) \cup (\mathfrak{D}_1 \cap \mathfrak{D}_2 \cap \mathfrak{D}_4) \cup (\mathfrak{D}_1 \cap \mathfrak{D}_2 \cap \mathfrak{D}_4)$. This implies that A is compact and therefore that $E \mid A$ has a maximum value $h_0 > h$ obtained on a single point $v_0 \in \mathbb{S}^2 - \Upsilon$. Since h_0 is a local maximum value of E obtained at h_0, we have that orthogonal to every direction sufficiently close to v_0, there is an oriented disk of diameter h_0, intersecting the four lines but there is not an oriented disk of diameter smaller than h_0, orthogonal to v_0, intersecting the four lines. Furthermore, since $v_0 \in (\mathfrak{D}_2 \cap \mathfrak{D}_3 \cap \mathfrak{D}_4) \cup (\mathfrak{D}_1 \cap \mathfrak{D}_3 \cap \mathfrak{D}_4) \cup (\mathfrak{D}_1 \cap \mathfrak{D}_2 \cap \mathfrak{D}_4) \cup (\mathfrak{D}_1 \cap \mathfrak{D}_2 \cap \mathfrak{D}_4)$, and $h_0 > h$, then there is an oriented disk of diameter smaller than h_0, orthogonal to v_0, intersecting every three of the lines.

Conversely, suppose that orthogonal to every direction sufficiently close to v_0, there is an oriented disk of diameter h_0, intersecting the four lines, but there is not an oriented disk of diameter smaller than h_0, orthogonal to v_0, intersecting the four lines. This immediately implies that h_0 is a local maximum value of E obtained at v_0. Since local maximum values of E are isolated, let U be a small neighborhood of v_0 and h small but sufficiently close to h_0 such that there is a compact component Γ of $E^{-1}([h,\infty])$ that contains v_0 and it is contained in U. The boundary $\partial\Gamma$ of Γ is contained in $E^{-1}(h)$ and is such that $H_1(\partial\Gamma)$ is not zero but $k_* : H_1(\partial\Gamma) \to H_1(\Gamma)$ is zero, where $k : \partial\Gamma \to \Gamma$ denotes the inclusion map. Furthermore, since there is an oriented disk of diameter smaller than h_0, orthogonal to v_0, intersecting every three of the lines, then without loss of generality, there is an oriented disk of diameter h, orthogonal to every $v \in U$, intersecting every three of the lines. By Lemma 2 and Lemma 1 a), there is an oriented disk of diameter h trapped by our four lines. $\square$

Integer partitions from a geometric viewpoint

Matthias Beck
(joint work with Benjamin Braun, Ira Gessel, Nguyen Le, Sunyoung Lee, Carla Savage)

This talk was based on the two recent papers [5] and [6].

In a series of papers starting with [1], George Andrews and various coauthors successfully revitalized seemingly forgotten, powerful machinery based on MacMahon's Ω operator [9] to systematically compute generating functions related to various families of integer partitions. Andrews et al's papers concern generating functions of the form

$$f_P(z_1,\dots,z_n) := \sum_{\lambda\in P} z_1^{\lambda_1}\cdots z_n^{\lambda_n} \quad \text{and} \quad f_P(q) := f_P(q,\dots,q) = \sum_{\lambda\in P} q^{\lambda_1+\cdots+\lambda_n},$$

for some set P of *partitions* $\lambda = (\lambda_1,\dots,\lambda_n)$; i.e., we think of the integers $\lambda_n \ge \cdots \ge \lambda_1 \ge 0$ as the *parts* when some integer k is written as $k = \lambda_1+\cdots+\lambda_n$. If we do not force an order onto the λ_j's, we call λ a *composition* of k. Here is one sample theorem:

Theorem 1 (Andrews–Paule–Riese [4]). *Let $n \ge 3$ and*

$$\tau := \{(\lambda_1,\dots,\lambda_n) \in \mathbb{Z}^n : \lambda_n \ge \cdots \ge \lambda_1 \ge 1 \quad \text{and} \quad \lambda_1+\cdots+\lambda_{n-1} > \lambda_n\},$$

the set of all "n-gon partitions." Then

$$f_\tau(q) = \frac{q^n}{(1-q)(1-q^2)\cdots(1-q^n)} - \frac{q^{2n-2}}{(1-q)(1-q^2)(1-q^4)(1-q^6)\cdots(1-q^{2n-2})}.$$

More generally,

$$f_\tau(z_1,\dots,z_n) = \frac{Z_1}{(1-Z_1)(1-Z_2)\cdots(1-Z_n)} - \frac{Z_1 Z_n^{n-2}}{(1-Z_n)(1-Z_{n-1})(1-Z_{n-2}Z_n)(1-Z_{n-3}Z_n^2)\cdots(1-Z_1 Z_n^{n-2})},$$

where $Z_j := z_j z_{j+1}\cdots z_n$ for $1 \le j \le n$.

Our main goal is to prove these theorems geometrically, and more, by realizing a given family of partitions as the set of integer lattice points in a certain polyhedron. This approach is not new: Pak illustrated in [10, 11] how one can obtain bijective proofs by realizing when both sides of a partition identity are generating functions of lattice points in unimodular cones (which we will define below); this included most of the identities appearing in [2]. Corteel, Savage, and Wilf [8] implicitly used the extreme-ray description of a cone to derive product formulas for partition generating functions, including those appearing in [2]. However, we feel that each of these papers only scratched the surface of a polyhedral approach to partition identities, and we see the current paper as a further step towards a systematic study of this approach.

While the Ω-operator approach to partition identities is elegant and powerful (not to mention useful in the search for such identities), we see several reasons for pursuing a geometric interpretation of these results. As discussed in [7], partition analysis and the Ω operator are useful tools for studying partitions and compositions defined by linear constraints, which is equivalent to studying integer points in polyhedra. An explicit geometric approach to these problems often reveals interesting connections to geometric combinatorics, e.g., conjectures on volumes of certain polytopes.

Our main new results are of two kinds: we prove multivariate versions of partition-generating-function theorems, and we prove the following theorem (using triangulations of cones) to extend several theorems in [3] on "symmetrically constrained compositions."

Theorem 2. *Given integers $a_1 \le a_2 \le \cdots \le a_n$ satisfying $\sum_{i=1}^n a_i = 1$, the generating function for those $\lambda \in \mathbb{Z}_{\ge 0}^n$ satisfying*

$$\sum_{j=1}^{n} a_j \lambda_{\pi(j)} \ge 0 \qquad \text{for all } \pi \in S_n$$

is

$$F(z_1, z_2, \ldots, z_n) = \sum_{\pi \in S_n} \frac{\prod_{j \in D_\pi} \left(z_{\pi(1)}^{b_{1,j}} z_{\pi(2)}^{b_{2,j}} \cdots z_{\pi(n)}^{b_{n,j}} \right)}{\prod_{j=1}^{n} \left(1 - z_{\pi(1)}^{b_{1,j}} z_{\pi(2)}^{b_{2,j}} \cdots z_{\pi(n)}^{b_{n,j}} \right)}$$

where $D_\pi = \{ j : \pi(j) > \pi(j+1) \}$ and

$$b_{i,j} = \begin{cases} 1 & \text{if } j = n, \\ -(a_1 + \cdots + a_j) & \text{if } n \ge i > j \ge 1, \\ 1 - (a_1 + \cdots + a_j) & \text{if } 1 \le i \le j < n. \end{cases}$$

In particular, setting $z_1 = \cdots = z_n = q$ yields

$$F(q) = \frac{\sum_{\pi \in S_n} \prod_{j \in D_\pi} q^{j - n \sum_{i=1}^{j} a_i}}{(1 - q^n) \prod_{j=1}^{n-1} \left(1 - q^{j - n \sum_{i=1}^{j} a_i} \right)}.$$

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On stability of polyhedra

András Bezdek

We all have seen different versions of the popular children's toy called 'stand up kid'. These figures are easy to make as they are loaded figures which have only one stable equilibrium. Such bodies are called *monostatic*. The problem gets more interesting if one wants to make convex 'stand up kids' using homogeneous material. Three dimensional bodies can be defined by a distance function $R(\varphi, \theta)$ in a spherical coordinate system around their centers of gravity. It is easy to see that local minima and maxima of $R(\varphi, \theta)$ correspond to stable and unstable equilibria, but the bodies can have additional equilibria positions at saddle points of $R(\varphi, \theta)$. A recent construction of G. Domokos and P. Várkonyi (2006) amazed people and thus generated lot of media attention. Mathematically speaking they answered a question of V. Arnold by constructing a homogeneous, convex body (called Gömböc) which has exactly one stable equilibrium, exactly one unstable equilibrium and does not have any saddle type equilibrium.

Arnold's question is closely related, but is different from a question of J. Conway, which he asked about forty years ago. One says that a polyhedron in three dimensions is *stable on a facet* if and only if the perpendicular to that face through the center of gravity meets the facet itself. The center of gravity is that point, which would be the physical center of gravity when the body is composed of material with uniform density. Conway wanted to find an example of a homogeneous convex polyhedron which will rest in a stable position when lying on *only one* of its faces.

From the very beginning the problem of finding a monostatic (stable only on one facet) polyhedron with the smallest number of faces seemed to be of special interest. J. Conway and R. Guy (1969) constructed a monostatic polyhedron with 19 faces. It was long believed that 19 is the smallest such face number. In the same paper a constructive argument of M. Goldberg was given proving that no tetrahedra can be monostatic (stable only on one facet). Although the statement was correct, the proof seemed to be incomplete, as it was not using any information on the position of the mass center of the tetrahedron.

1. We started the talk with describing a polyhedron which showed that Goldberg's [CG69] direct approach cannot be made complete. The first correct proof ruling out the existence of a monostatic tetrahedron is attributed to J. Conway and was mentioned in a paper of R. Dawson [D85]. In subsequent papers R. Dawson and W. Finbow proved that monostatic simplices do not exist in dimensions smaller than nine.

2. We proved that in contrast to the common belief there are monostatic polyhedra which are bounded with less than 19 faces. In fact we described a monostatic skewed pyramid which has only 18 faces. The base of our pyramid was a special 17 sided symmetrical polygon, which allowed us to verify the needed stability properties.

3. We also considered skeletal versions of Conway's stability problem. In such cases the center of mass is determined by a uniform distribution on the n-skeleton ($n = 0, 1, 2$) of the body. Note that in case of tetrahedra the mass center of the 0-skeleton is the same as the center of the gravity of the tetrahedron composed of material with uniform density.

4. Among others we proved that if tetrahedron is constructed so that it has uniform (but possible different) mass distribution on its three skeletons then it has at least two stable faces. The proof utilizes an idea of Conway, which was mentioned in a paper of Dawson [D85].

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A cell complex in number theory

Anders Björner

Let Δ_n be the simplicial complex of squarefree positive integers less than or equal to n ordered by divisibility. It is known that the asymptotic rate of growth of its Euler characteristic (the Mertens function) is closely related to deep properties of the prime number system.

We study the asymptotic behavior of the individual Betti numbers $\beta_k(\Delta_n)$ and of their sum. We show that Δ_n has the homotopy type of a wedge of spheres. The following estimates are established: As $n \to \infty$

(1) $\sum_{k\geq 0} \beta_k(\Delta_n) = \frac{2n}{\pi^2} + O(n^\theta)$, for all $\theta > \frac{17}{54}$
(2) $\sum_{k \text{ even}} \beta_k(\Delta_n) \sim \frac{n}{\pi^2}$
(3) $\sum_{k \text{ odd}} \beta_k(\Delta_n) \sim \frac{n}{\pi^2}$
(4) For fixed k: $\beta_k(\Delta_n) \sim \frac{n}{2\log n} \frac{(\log\log n)^k}{k!}$

As a number-theoretic byproduct we obtain inequalities

$$\partial_k \left(\sigma_{k+1}^{odd}(n)\right) \leq \sigma_k^{odd}(n/2),$$

where $\sigma_k^{odd}(n)$ denotes the number of odd squarefree integers $\leq n$ with k prime factors, and ∂_k is the Kruskal–Katona shadow function.

We also study a CW complex $\widetilde{\Delta_n}$ that extends the previous simplicial complex. In $\widetilde{\Delta_n}$ *all* numbers $\leq n$ correspond to cells and its Euler characteristic is the summatory Liouville function. This cell complex $\widetilde{\Delta_n}$ is shown to be homotopy equivalent to a wedge of spheres, and as $n \to \infty$

$$\sum \beta_k(\widetilde{\Delta_n}) = \frac{n}{3} + O(n^\theta), \quad \text{for all } \theta > \frac{22}{27}.$$

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Uniform hypergraphs containing no grids, a problem concerning superimposed codes

Zoltán Füredi
(joint work with Miklós Ruszinkó)

A hypergraph is called an $r \times r$ *grid*, $\mathbb{G}_{r\times r}$, if it is isomorphic to a pattern of r horizontal and r vertical lines. Three sets form a *triangle* if they pairwise intersect in three distinct singletons. A hypergraph is *linear* if every pair of edges meet in at most one vertex.

Our aim is to construct a large linear r-hypergraphs which contain no grids. Moreover, a similar construction gives large linear r-hypergraphs which contain neither grids nor triangles. For $r \geq 4$ our constructions are almost optimal. These investigations are also motivated by coding theory: we get new bounds for optimal superimposed codes and designs.

Our main tool is a natural algebraic construction and some properties of pseudoline arrangements.

Union-free and sparse designs

Investigating the Rényi's search model Dyachkov and Rykov [6] obtained several sufficient conditions for the existence of regular binary superimposed codes. In [13] we answered their question asymptotically which lead to union-free designs. A Steiner system $S(v, r, 2)$ is a collection of r-subsets (blocks) of a v-set which has the property that every pair of distinct elements occurs in one block. Two families of r-sets $\mathcal{A}$ and $\mathcal{B}$ form a **grid**, $\mathbb{G}_{r\times r}$, if $|\mathcal{A}| = |\mathcal{B}| = r$, $\cup\mathcal{A} = \cup\mathcal{B}$ and $|\cup\mathcal{A}| = r^2$, i.e., both $\mathcal{A}$ and $\mathcal{B}$ consists of disjoint sets and every $A \in \mathcal{A}$ meets every $B \in \mathcal{B}$ in exactly one element. The *Turán number* of the r-uniform hypergraph $\mathcal{H}$, denoted by $\mathrm{ex}(n, \mathcal{H})$, is the size of the largest $\mathcal{H}$-free r-graph on n vertices.

Problem 1. *Given r, construct infinitely many grid-free Steiner systems, $S(v, r, 2)$.*

Theorem 2 (Füredi and Ruszinkó [13])**.** *There exist a linear, grid-free, r-uniform hypergraph on n vertices almost as big as a Steiner system,*

$$\frac{n(n-1)}{r(r-1)} - c_r n^{8/5} < \mathrm{ex}_r(n, \{\mathbb{I}_{\geq 2}, \mathbb{G}_{r\times r}\}) \leq \frac{n(n-1)}{r(r-1)}$$

holds for every $n, r \geq 4$.

In the case of $r = 3$ with probabilistic method we only have

$$\Omega(n^{1.8}) \leq \mathrm{ex}_3(n, \{\mathbb{I}_{\geq 2}, \mathbb{G}_{3\times 3}\}) \leq \frac{1}{6}n(n-1).$$

The real question is to determine the unavoidable substructures in designs. Only a few results are know, all in the case $r = 3$. A Steiner triple system is said to be *s-sparse* if it contains no i blocks on $i+2$ elements for any i, $4 \leq i \leq s$. The question whether s-sparse $\mathrm{STS}(v)$ exists was proposed by Erdős [7]. A 4-sparse $\mathrm{STS}(v)$ is known to exist for each admissible $v \neq 7, 13$ (Brouwer [3], Grannell, Griggs, and Whitehead [14]). Recently there has been substantial progress towards the goal of establishing that a 5-sparse $\mathrm{STS}(v)$ exists for each sufficiently large admissible v (Wolfe [19]). An infinite class of 6-sparse $\mathrm{STS}(v)$ is described by Forbes, Grannell, and Griggs [10], but no 7-sparse $\mathrm{STS}(v)$ is known for any v.

More on union-free and cover-free codes

A family $\mathcal{F} \subseteq 2^{[n]}$ is *g-cover-free* if for arbitrary distinct members $A_0, A_1, \ldots, A_g$

$$A_0 \not\subseteq \bigcup_{i=1}^{g} A_i.$$

Let $\mathbf{CF}_g(n)$ ($\mathbf{CF}_g(n, r)$) be the maximum size of a g-cover-free n vertex code (r-uniform hypergraph, resp.).

A family $\mathcal{F} \subseteq 2^{[n]}$ is *t-union-free* if for distinct t-multisets $\{A_1, \ldots, A_t\}$ and $\{B_1, \ldots, B_t\}$ $A_i, B_j \in \mathcal{F}$ we have
$$A_1 \cup A_2 \cup \cdots \cup A_t \neq B_1 \cup \cdots \cup B_t.$$
Let $\mathbf{UF}_t(n)$ ($\mathbf{UF}_t(n, r)$) be the maximum size of a t-union-free n vertex code (r-uniform hypergraph, resp.).

If $\mathcal{F}$ is t-**CF** then it is t-**UF**, and if $\mathcal{F}$ is t-**UF** then it is $(t-1)$-**CF**. Hence
$$\mathbf{CF}(n, t) \leq \mathbf{UF}(n, t) \leq \mathbf{CF}(n, t-1) \leq \mathbf{UF}(t-1, n) \leq \ldots$$
$$\mathbf{CF}_r(n, t) \leq \mathbf{UF}_r(n, t) \leq \mathbf{CF}_r(n, t-1) \leq \mathbf{UF}_r(t-1, n) \leq \cdots \leq \mathbf{UF}_2(n, r).$$

Union free and cover free families were introduced by Kautz and Singleton [17]. They studied binary codes with the property that the disjunctions of distinct at most g-tuples of codewords are all different. In information theory usually these codes are called **superimposed** and they have been investigated in several papers on multiple access communication (see, e.g., Nguyen Quang A and Zeisel [1], D'yachkov and Rykov [5], Johnson [16]). The same problem has been posed by Erdős, Frankl and Füredi [8, 9] in combinatorics, by Sós [18] in combinatorial number theory, and by Hwang and Sós [15] in group testing. For recent generalizations see, e.g., Alon and Asodi [2], and De Bonis and Vaccaro [4]. D'yachkov and Rykov [5] proved that there are positive constants α_1 and α_2 such that
$$\alpha_1 \frac{1}{g^2} < \frac{\log \mathbf{CF}_g(n)}{n} < \alpha_2 \frac{\log g}{g^2}$$
holds for every g and $n > n_0(g)$.

Frankl and the present author [12] determined asymptotically the maximum size of an r-uniform g-cover-free family showing that there exists a positive constant $\gamma := \gamma(r, g)$ such that $\mathbf{CF}_g(n, r) = (\gamma + o(1))n^{\lceil r/g \rceil}$.

Problem 3. *Given $r \geq t \geq 1$ find an asymptotic for $\mathbf{UF}_t(n, r)$.*

The order of magnitude of $\mathbf{UF}_r(n, 2)$ was determined by Frankl et al. [11].

Theorem 4 (Füredi and Ruszinkó [13])**.** *There exists a $\beta = \beta(r) > 0$ such that for all $n \geq r \geq 4$*
$$n^2 e^{-\beta_r \sqrt{\log n}} < \mathrm{ex}(n, \{\mathbb{I}_{\geq 2}, \mathbb{T}_3, \mathbb{G}_{r \times r}\}) \leq \mathbf{UF}_r(n, r) \leq \frac{n(n-1)}{r(r-1)}.$$

We have only weaker estimates for $r = 3$
$$\Omega(n^{5/3}) \leq \mathbf{UF}_3(n, 3) \leq \frac{1}{6} n(n-1).$$

The determination of the size of maximal ***t*-union-free** families is one of the important and likely solvable Turán type problems.

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Various aspects of Frobenius numbers

Martin Henk

(joint work with Iskander Aliev, Lenny Fukshansky)

Let $a \in \mathbb{Z}^n_{>0}$ be a positive integral n-dimensional primitive vector, i.e., $a = (a_1, \dots, a_n)^\intercal$ with $\gcd(a) := \gcd(a_1, \dots, a_n) = 1$, so that $0 < a_1 < a_2 < \cdots < a_n$. For a positive integer s the *s-Frobenius number* $\mathrm{F}_s(a)$, is the largest number that cannot be represented in at least s different ways as a non-negative integral combination of the a_i's, i.e.,

$$\mathrm{F}_s(a) = \max\{b \in \mathbb{Z} : \#\{z \in \mathbb{Z}^n_{\geq 0} : \langle a, z\rangle = b\} < s\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$.

This generalized Frobenius number has been introduced and studied by Beck and Robins [7] (see also [6]), who showed, among other results, that for $n = 2$

$$\mathrm{F}_s(a) = s\, a_1\, a_2 - (a_1 + a_2). \tag{1}$$

In particular, this identity generalizes the well-known result in the setting of the (classical) Frobenius number which corresponds to $s = 1$. The origin of this classical result is unclear, it was most likely known already to Sylvester, see, e.g., [19]. The literature on the Frobenius number $\mathrm{F}_1(a)$ is vast; for a comprehensive and extensive survey we refer the reader to the book of Ramírez Alfonsín [15]. Despite the exact formula in the case $n = 2$, for general n only bounds on the Frobenius number $\mathrm{F}_1(a)$ are available. For instance, for $n \geq 3$

$$((n-1)!\, a_1 \cdot \ldots \cdot a_n)^{\frac{1}{n-1}} - (a_1 + \cdots + a_n) < \mathrm{F}_1(a) \leq 2\, a_n \left\lfloor \frac{a_1}{n} \right\rfloor - a_1. \tag{2}$$

Here the lower bound follows from a sharp lower bound due to Aliev and Gruber [1], and the upper bound is due to Erdős and Graham [9]. Hence, in the worst case scenario we have an upper bound of the order $|a|_\infty^2$ on the Frobenius number with respect to the maximum norm of the input vector a. It is worth a mention that an upper bound on $\mathrm{F}_1(a)$, which is symmetric in all of the a_i's has recently been produced by Fukshansky and Robins [10]. The quadratic order of the upper bound is known to be optimal (see, e.g., [9]) and in view of the lower bound which is at most of size $|a|_\infty^{\frac{n}{n-1}}$ it is quite natural to study the average behavior of $\mathrm{F}_1(a)$. This research was initiated and strongly influenced by Arnold [4]–[5], and due to recent results of Bourgain and Sinai [8], Aliev and Henk [2], Aliev, Henk and Hinrichs [3], Marklof [14], Li [13], Shur, Sinai and Ustinov [16], Strömbergsson [18] and Ustinov [20] we have a pretty clear picture of "the average Frobenius number". In order to describe some of these results, which are going to extend to the s-Frobenius number $\mathrm{F}_s(a)$, we need a bit more notation. Let

$$\mathrm{G}(T) = \{a \in \mathbb{Z}^n_{>0} : \gcd(a) = 1,\ |a|_\infty \leq T\},$$

be the set of all possible input vectors of the Frobenius problem of size (in maximum norm) at most T. Aliev, Henk and Hinrichs [3] showed that

$$\sup_T \frac{\sum_{a \in \mathrm{G}(T)} \mathrm{F}_1(a) / \, (a_1\, a_2 \cdot \ldots \cdot a_n)^{\frac{1}{n-1}}}{\#\mathrm{G}(T)} \ll\gg_n 1, \tag{3}$$

i.e., the expected size of $\mathrm{F}_1(a)$ is "close" to the size of its lower bound in (2); here and below $\ll_n$ and $\gg_n$ denote the Vinogradov symbols with the constant depending on n only. Recently, Li [13] gave the bound

$$\mathrm{Prob}\Big(\mathrm{F}_1(a) / \, (a_1\, a_2 \cdot \ldots \cdot a_n)^{\frac{1}{n-1}} \geq D\Big) \ll_n D^{-(n-1)}, \tag{4}$$

where $\mathrm{Prob}(\cdot)$ is meant with respect to the uniform distribution among all points in the set $\mathrm{G}(T)$. The bound (4) is best possible due to an unpublished result of Marklof, and clearly implies (3).

Here we extend the results stated above, i.e., (2), (3) and (4), to the generalized Frobenius number $\mathrm{F}_s(a)$ in the following way:

Theorem 1. *Let $n \geq 2$, $s \geq 1$. Then*

$$\mathrm{F}_s(a) \geq s^{\frac{1}{n-1}} \left((n-1)!\, a_1 \cdot \ldots \cdot a_n\right)^{\frac{1}{n-1}} - (a_1 + \cdots + a_n)\,,$$
$$\mathrm{F}_s(a) \leq \mathrm{F}_1(a) + (s-1)^{\frac{1}{n-1}} \left((n-1)!\, a_1 \cdot \ldots \cdot a_n\right)^{\frac{1}{n-1}}\,.$$

Bounds with almost the same dependencies on s were recently obtained by Fukshansky and Schürmann [11]. Their lower bound, however, is only valid for sufficiently large s. Aliev and Gruber [1] applied the results of Schinzel [17] to obtain a sharp lower bound for the Frobenius number in terms of the covering radius of a simplex. The same approach can be used to obtain a sharp lower bound for the s-Frobenius number as well. We postpone a detailed discussion of these matters to a future paper.

As an almost immediate consequence of Theorem 1 we obtain:

Corollary 2. *Let $n \geq 3$, $s \geq 1$. Then*

$$\text{i)} \qquad \mathrm{Prob}\left(\mathrm{F}_s(a)/\left(s \cdot a_1\, a_2 \cdot \ldots \cdot a_n\right)^{\frac{1}{n-1}} \geq D\right) \ll_n D^{-(n-1)},$$

$$\text{ii)} \qquad \sup_T \frac{\sum_{a \in \mathrm{G}(T)} \mathrm{F}_s(a)/\left(s \cdot a_1\, a_2 \cdot \ldots \cdot a_n\right)^{\frac{1}{n-1}}}{\#\mathrm{G}(T)} \ll\gg_n 1.$$

Hence in this generalized setting the average s-Frobenius number is of the size $(s \cdot a_1\, a_2 \cdot \ldots \cdot a_n)^{\frac{1}{n-1}}$, which again is the size of its lower bound as stated in Theorem 1.

The proof of Theorem 1 is based on a generalization of a beautiful result of Kannan [12] which relates the classical Frobenius number to the covering radius of a certain simplex with respect to a certain lattice. In our setting we need a kind of generalized (so called) s-covering radius, which allows us to extend Kannan's result to the s-Frobenius number.

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To hold a convex body by a circle

Hiroshi Maehara

A circle Γ is said to hold a convex body K if

(1) $\Gamma \cap int(K) = \emptyset$, $conv(\Gamma) \cap int(K) \neq \emptyset$, and
(2) it is impossible to move Γior Kj with keeping $\Gamma \cap int(K) = \emptyset$ until K goes far away from Γ.

A convex body is called **circle-free** if no circle can hold the convex body.

For example, every ball is circle-free. It is also not difficult to show that very circular cylinder is circle-free, and every circular cone is circle-free.

T. Zamfirescu prove the following in 1995.

Theorem ([5]). *The set of circle-free convex bodies in $\mathbb{R}^3$ is a nowhere dense subset of the set of all convex bodies in $\mathbb{R}^3$ with Haussdorff metric.*

Thus, circle free convex bodies are rather rare.

Theorem ([3]). *For every planar compact convex set X and a ball B in $\mathbb{R}^3$, the Minkowski sum $X + B$ is circle-free.*

Problem ([3]). *Is it true that for every circle-free convex body K, the set $K + B$ is also circle-free?*

For some regular pyramids, it is possible to determine whether a given one is circle-free or not. Define the **slope** ρ of a regular pyramid by

$$\rho = \frac{\text{height}}{\text{circum-radius of the base}}.$$

Theorem ([3]). *Every regular pyramid with slope $\rho \geq 1$ can be held by a circle. Moreover, for every $0 < \varepsilon < 1$, there is a circle-free regular pyramid with slope $\rho = 1 - \varepsilon$.*

Y. Tanoue [4] proved the if part of the next theorem, but he did not consider the only if part.

Theorem ([4, 3]). *A regular pyramid with equilateral triangular base can be held by a circle if and only if*

$$\rho > \sqrt{\frac{3\sqrt{17} - 5}{32}} \approx 0.4799.$$

Theorem ([3]). *A regular pyramid with square base can be held by a circle if and only if $\rho > \sqrt{\frac{\sqrt{33}-3}{4}} \approx 0.828$.*

The great pyramid of Giza has base-edge $230m$ and height $140m$. Since $140\sqrt{2}/230 \approx 0.860 > 0.828$, it can be held by a circle.

Concerning the diameter of a circle that can hold a regular tetrahedron of unit edge, Itoh, Tanoue, and Zamfirescu proved the following.

Theorem ([1]). *A circle of diameter d can hold a regular tetrahedron of unit edge if and only if $\frac{1}{\sqrt{2}} \leq d < \phi_t \approx 0.8956$, where ϕ_t is the minimum value of $\frac{2(x^2-x+1)}{\sqrt{3x^2-4x+4}}$.*

For a cube and a regular octahedron, I could prove the followings.

Theorem ([2]). *A circle of diameter d can hold a unit cube if and only if $\sqrt{2} \leq d < \phi_c \approx 1.53477$, where ϕ_c is the minimum value of $\frac{\sqrt{2}(x^2+2)}{\sqrt{x^2+2x+3}}$.*

Theorem ([2]). *A circle of diameter d can hold a regular octahedron of unit edge if and only if $1 \leq d < \phi_o \approx 1.1066$, where ϕ_o is the minimum value of $\frac{2(x^2+1)}{\sqrt{3x^2+2x+3}}$.*

Y. Tanoue also obtained the octahedral case independently.

Problem. *Find similar results for the regular dodecahedron and the regular icosahedron.*

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Isoperimetry, Crossing Numbers, and Multiplicities of (Equivariant) Maps

Uli Wagner

Consider a set S of n points in $\mathbf{R}^d$. A result of Boros and Füredi [5] (for $d = 2$) and of Bárány Bárány [3] (for general d) asserts that there exists a point $p \in \mathbf{R}^d$ (not necessarily a member of S) that is contained in a fraction of at least c_d of the d-dimensional simplices spanned by S, where $c_d > 0$ is a constant that depends only on d.

The result can be restated as saying that for any *affine* map f from the $(n-1)$-simplex Δ^{n-1} to $\mathbf{R}^d$, there exists a point $p \in \mathbf{R}^d$ that is covered by the f-images of a fraction of at least c_d of the d-faces of Δ^{n-1}.

For $d = 2$, the optimal value of the constant is known to be $c_2 = 2/9$. For higher dimensions, the largest possible value of c_d has been the subject of ongoing research, see [12, 7, 6, 4]. In particular, it is known that $c_d \leq \frac{(d+1)!}{(d+1)^{d+1}}$ [6].

Recently, Gromov [8] introduced a new, topological proof method which, firstly, yields an improved lower bound of $c_d \geq \frac{2d}{(d+1)!(d+1)}$ (for $d = 2$, this coincides with the optimal $2/9$) and which, secondly, applies to arbitrary *continuous* maps $f\colon \Delta^{n-1} \to \mathbf{R}$.

Via combinatorial Poincaré duality in $\partial\Delta^{n-1} \cong \mathbf{S}^{n-2}$, this result can be seen as a discrete analogue of the following result, due to Almgren [1] and to Gromov [10] (see Memarian [11] for a detailed proof).

The Waist of the Sphere Theorem (Almgren, Gromov). *Let $\mathbf{S}^n$ be the unit sphere in Euclidean space. For every map $f\colon \mathbf{S}^n \to \mathbf{R}^d$, there exists a point $p \in \mathbf{R}^d$ such that the fiber $f^{-1}(p)$ has $(n-d)$-dimensional* volume[1] *at least as large as $\mathbf{S}^{n-d}$.*

In [9, Appendix 1.F], Gromov described a general approach to such results based on isoperimetric inequalities in $\mathbf{S}^n$. This apporach does not yield optimal constants, but applies more generally when $\mathbf{S}^n$ is replaced by a different source manifold M^n. Gromov's topological lower bound for the constant c_d can be seen as a discrete version of this isoperimetric approach (I am indebted to Larry Guth for helpful explanations concerning the Waist Theorem and this connection).

The same approach can be used to prove the following *equivariant* version of the Waist Theorem, with a sharp constant (we note that the topological part of the argument is much simpler in this case):

Theorem 1. *Let $f\colon \mathbf{S}^n \to \mathbf{R}^d$ be an* equivariant *map (i.e., $f(-x) = -f(x)$). Then the preimage $f^{-1}(o)$ of the origin $o \in \mathbf{R}^d$ has $(n-d)$-dimensional volume at least as large as $\mathbf{S}^{n-d}$.*

Again, one can consider discrete analogues of this result. A very simple one is the following: Let $\lozenge^n$ be the n-dimensional cross-polytope, i.e., the convex hull

[1] One very general way of making this precise is to say that, for every $\varepsilon > 0$, the ε-neighborhood of $f^{-1}(p)$ in $\mathbf{S}^d$ has d-dimensional volume at least as large as the ε-neighborhood of $\mathbf{S}^{n-d}$.

of the vectors $\pm e_i$, $1 \leq i \leq n$ (the standard basis vectors and their negatives). Then, for any equivariant map $f\colon \partial\Diamond^n \to \mathbf{R}^d$, the origin $o \in \mathbf{R}^d$ is covered by the f-images of at least $2\binom{n}{d+1}$ d-faces of $\partial\Diamond^n$. This particular fact is not very exciting, since it is also a straightforward consequence of the *Topological Radon Lemma* [2], which says that for every equivariant map $f\colon \partial\Diamond^{d+1} \to \mathbf{R}^d$, the origin is covered by the images of at least two d-faces.

Things become more interesting if we do not treat all d-faces of $\partial\Diamond^n$ in the same way. For instance, consider the case $d = 3$ and suppose that we only count those 3-faces $\mathrm{conv}\{+e_i, +e_j, -e_k, -e_\ell\}$ that are spanned by two positive and two negative basis vectors. In this setting, we can prove the following:

Theorem 2. *Let $u_1, \ldots, u_n \in \mathbf{S}^2$ be in general position, and suppose that we connect any two points u_i, u_j by the shorter geodesic arc between them. Then the number of crossings in the resulting* spherical geodesic drawing *of the complete graph K_n is at least $\frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$.*

We remark that the lower bound is conjectured to hold for arbitrary drawings. We hope that the approach sketched here will also be useful to attack the general conjecture.

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Point-pseudoline incidences in higher dimensions

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(joint work with Terence Tao)

Our goal is to prove almost tight bounds on incidences between points and k-dimensional varieties of bounded degree in $\mathbb{R}^d$. Our main tools are the Polynomial Ham Sandwich Theorem and induction on both the dimension and the number of points.

Given a collection P of points in some space, and a collection L of sets in that same space, let $I(P,L) := \{(p,\ell) \in P \times L : p \in \ell\}$ be the set of incidences. One of the objectives in combinatorial incidence geometry is to obtain good bounds on the cardinality $|I(P,L)|$ on the number of incidences between finite collections P, L, subject to various hypotheses on P and L. For instance, we have the classical result of Szemerédi and Trotter [1]:

Szemerédi–Trotter Theorem. *Let P be a finite set of points in $\mathbb{R}^d$ for some $d \geq 2$, and let L be a finite set of lines in $\mathbb{R}^d$. Then*

$$|I(P,L)| \leq C(|P|^{2/3}|L|^{2/3} + |P| + |L|) \tag{1}$$

for some absolute constant C.

We establish near-sharp Szemerédi–Trotter type bounds on incidences between points and k-dimensional algebraic varieties in $\mathbb{R}^d$ for various values of k and d, under some "pseudoline" hypotheses on the algebraic varieties. Our argument is based on the polynomial method as used by Guth and Katz [2], combined with an induction on the size of the point set P. The inductive nature of our arguments causes us to lose epsilon terms in the exponents, but the bounds are otherwise sharp.

Let $k, d \geq 0$ be integers such that $d \geq 2k$, and let $\epsilon > 0$ and $C_0 \geq 1$ be real numbers. Let P be be a finite collection of distinct points in $\mathbb{R}^d$, let L be a finite collection of real algebraic varieties in $\mathbb{R}^d$, and let $I \subset I(P,L)$ be a set of incidences between P and L. Assume the following "pseudoline-type" axioms:

(i) For each $\ell \in L$, ℓ is a k-dimensional real algebraic variety of degree at most C_0 (and is thus the restriction to $\mathbb{R}^d$ of a complex algebraic variety ℓ_C of the same dimension and degree).
(ii) If $\ell, \ell' \in L$ are distinct, then there are at most C_0 points p in P such that $(p,\ell), (p,\ell') \in I$.
(iii) If $p, p' \in P$ are distinct, then there are at most C_0 varieties ℓ in L such that $(p,\ell), (p',\ell) \in I$. (Note that for $C_0 = 2$, this is equivalent to (ii).)
(iv) If $(p,\ell) \in I$, then p is a smooth (real) point of ℓ. In particular, for each $(p,\ell) \in I$, there is a unique tangent space $T_p\ell$ of ℓ at p.
(v) If $\ell, \ell' \in L$ are distinct, and $p \in P$ are such that $(p,\ell), (p,\ell') \in I$, then the tangent spaces $T_p\ell$ and $T_p\ell'$ are transverse, in the sense that they only intersect at p.

Then one has

$$|I| \leq A|P|^{\frac{2}{3}+\epsilon}|L|^{\frac{2}{3}} + \frac{3}{2}|P| + \frac{3}{2}|L| \tag{2}$$

for some constant $A = A_{k,\epsilon,C_0}$ that depends only on the quantities k, ϵ, C_0.

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A center transversal theorem for hyperplanes

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(joint work with Vida Dujmović)

Motivated by an open problem from graph drawing, we study several partitioning problems for line and hyperplane arrangements. We prove a ham-sandwich cut theorem: given two sets of n lines in $\mathbb{R}^2$, there is a line ℓ such that in both line sets, for both halfplanes delimited by ℓ, there are $\sqrt{n}$ lines which pairwise intersect in that halfplane, and this bound is tight; a centerpoint theorem: for any set of n lines there is a point such that for any halfplane containing that point there are $\sqrt{n/3}$ of the lines which pairwise intersect in that halfplane. We generalize those results in higher dimension and obtain a center transversal theorem, a same-type lemma, and a positive portion Erdős–Szekeres theorem for hyperplane arrangements. This is done by formulating a generalization of the center transversal theorem which applies to set functions that are much more general than measures.

Tight bounds on the maximum size of a set of permutations of bounded VC-dimension

Josef Cibulka

(joint work with Jan Kynčl)

Motivated by the so-called acyclic linear orders problem, Raz [14] defined the *Vapnik–Chervonenkis dimension (VC-dimension)* of a set $\mathcal{P}$ of permutations: Let S_n be the set of all n-*permutations*, that is, permutations of $[n]$. The *restriction* of $\pi \in S_n$ to the k-tuple $(a_1, a_2, \dots, a_k)$ of positions (where $1 \leq a_1 < a_2 < \dots < a_k \leq n$) is the k-permutation π' satisfying $\forall i, j : \pi'(i) < \pi'(j) \Leftrightarrow \pi(a_i) < \pi(a_j)$. The k-tuple of positions $(a_1, \dots, a_k)$ is *shattered by* $\mathcal{P}$ if each k-permutation appears as a restriction of some $\pi \in \mathcal{P}$ to $(a_1, \dots, a_k)$. The *VC-dimension of* $\mathcal{P}$ is the size of the largest set of positions shattered by $\mathcal{P}$.

Let $r_k(n)$ be the size of the largest set of n-permutations with VC-dimension k.

Raz [14] proved that $r_2(n) \leq C^n$ for some constant C and asked whether an exponential upper bound on $r_k(n)$ can also be found for every $k \geq 3$.

An n-permutation π *avoids* a k-permutation ρ if none of the restrictions of π to a k-tuple of positions is ρ. Clearly, the set of permutations avoiding $\rho \in S_k$ has VC-dimension smaller than k. Thus, Raz's question generalizes the Stanley–Wilf conjecture which states that the number of n-permutations that avoid an arbitrary fixed permutation ρ grows exponentially in n. The conjecture was settled by Marcus and Tardos [9] using a result of Klazar [7].

Let $\alpha(n)$ be the inverse of the Ackermann function.

We show that the size of a set of n-permutations with VC-dimension k cannot be much larger than exponential in n.

Theorem 1.

$$\begin{aligned} r_3(n) &\leq \alpha(n)^{(4+o(1))n}, \\ r_4(n) &\leq 2^{n\cdot(2\alpha(n)+3\log_2(\alpha(n))+O(1))}, \\ r_{2t+2}(n) &\leq 2^{n\cdot((2/t!)\alpha(n)^t+O(\alpha(n)^{t-1}))} \qquad \textit{for every } t \geq 2 \textit{ and} \\ r_{2t+3}(n) &\leq 2^{n\cdot((2/t!)\alpha(n)^t\log_2(\alpha(n))+O(\alpha(n)^t)} \qquad \textit{for every } t \geq 1. \end{aligned}$$

The result has an application in the enumeration of simple complete topological graphs [8].

On the other hand, we give a negative answer to the Raz's question:

Theorem 2.

$$r_3(n) \geq (\alpha(n)/2 - O(1))^n.$$

Let $k \geq 4$ be a fixed integer and let $t := \lfloor (k-2)/2 \rfloor$. We have

$$r_k(n) \geq 2^{n((1/t!)\alpha(n)^t - O(\alpha(n)^{t-1}))}.$$

An *n-permutation matrix* is an $n \times n$ $(0,1)$-matrix with exactly one 1-entry in every row and every column. Permutations and permutation matrices are in a one-to-one correspondence that assigns to a permutation π a permutation matrix A_π with $A_\pi(i,j) = 1 \Leftrightarrow \pi(j) = i$.

An $m \times n$ $(0,1)$-matrix B *contains* a $k \times l$ $(0,1)$-matrix S if B has a $k \times l$ submatrix T that can be obtained from S by changing some (possibly none) 0-entries to 1-entries. Otherwise B *avoids* S. Thus, a permutation π avoids ρ if and only if A_π avoids A_ρ. Füredi and Hajnal [6] initiated the study of the following problems from the extremal theory of $(0,1)$-matrices. Given a matrix S (the *forbidden matrix*), what is the maximum number $\mathrm{ex}_S(n)$ of 1-entries in an $n \times n$ matrix that avoids S? This area is closely related to the Turán problems on graphs and to the Davenport–Schinzel sequences. Functions ex_S or their asymptotics have been determined for some matrices S [6, 13, 16] and these results have found applications mostly in discrete geometry [1, 4, 5, 11] and also in the analysis of algorithms [12]. The Füredi–Hajnal conjecture states that $\mathrm{ex}_P(n)$ is linear in n whenever P is a permutation matrix. Marcus and Tardos proved this conjecture by a surprisingly simple argument [9]. This implied the relatively long standing Stanley–Wilf conjecture by Klazar's reduction [7]. An improved reduction yielding the upper bound $2^{O(k \log k)n}$ on the size of a set of n-permutations with a forbidden k-permutation was found by the first author [2].

We modify the question of Füredi and Hajnal and study the maximal number $p_k(n)$ of 1-entries in an $n \times n$ matrix such that no $(k+1)$-tuple of columns contains all $(k+1)$-permutation matrices. It can be easily shown that $p_2(n) \le 4n - 4$.

Theorem 3.

$$2n\alpha(n) - O(n) \le p_3(n) \le O(n\alpha(n)),$$

$$p_{2t+2}(n) = n2^{(1/t!)\alpha(n)^t \pm O(\alpha(n)^{t-1})} \quad \textit{for every } t \ge 1 \textit{ and}$$

$$p_{2t+3}(n) \ge n2^{(1/t!)\alpha(n)^t - O(\alpha(n)^{t-1})}$$

$$p_{2t+3}(n) \le n2^{(1/t!)\alpha(n)^t \log_2(\alpha(n)) + O(\alpha(n)^t)} \quad \textit{for every } t \ge 1.$$

Let S and T be sequences. We say that S *contains* a pattern T if S contains a subsequence T' isomorphic to T, that is, T can be obtained from T' by a one-to-one renaming of the symbols. A sequence S over an alphabet Γ is a *Davenport–Schinzel sequence of order* s (a DS(s)-sequence for short) if no symbol appears on two consecutive positions and S does not contain the pattern $abab\dots ba$ of length $s+2$. These sequences were introduced by Davenport and Schinzel [3] and found numerous applications in computational and combinatorial geometry. More can be found in the book of Sharir and Agarwal [15]. The currently best known bounds on the maximum length of a Davenport–Schinzel sequence over n symbols are summarized in a paper of Nivasch [10].

The maximum length of a DS(s)-sequence over n symbols almost exactly corresponds to the maximum number of 1-entries in an $n \times n$ matrix avoiding a specific $(s+1) \times 2$ matrix [6, 13]. Our proofs use these correspondences between matrices and DS-sequences and also sequences with other forbidden patterns.

An *s-partition of the rows* of a matrix M is a partition of the interval of integers $\{1, \dots, m\}$ into s intervals $\{1 = m_1, \dots, m_2 - 1\}$, $\{m_2, \dots, m_3 - 1\}$, $\dots$, $\{m_s, \dots, m = m_{s+1} - 1\}$. A matrix M contains a *B-fat* (r, s)*-formation* if there exists an s-partition of the rows and an r-tuple of columns each of which has B 1-entries in each interval of rows.

The proof of Theorem 1 uses the following lemma, analogously to the use of Raz's Technical Lemma [14]. In the lemma, $\beta_s(m)$ are extremely slowly growing functions defined in terms of the inverse Ackermann function.

Lemma 4. *For all positive integers m, n, s and B, an $m \times n$ matrix M with at least $2(\beta_s(m) + 2)Bn$ 1-entries contains a B-fat $(\lfloor (n-1)B/(mc'_s) \rfloor, s)$-formation, where c'_s is a constant depending only on s.*

The proof of the lemma is based on a proof of the upper bound on the number of symbols in the so-called formation-free sequences by Nivasch [10].

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Space crossing numbers

Boris Bukh
(joint work with Alfredo Hubard)

The *crossing number* of a graph $G = (V, E)$ is the minimum number of crossings between edges of G among all the ways to draw G in the plane. It is denoted $\operatorname{cr}(G)$. The edges in a drawing of G need not be line segments, they are allowed to be arbitrary continuous curves. If one restricts to the straight-line drawings, then one obtains the *rectilinear crossing number* $\operatorname{lin-cr}(G)$. It is clear that $\operatorname{cr}(G) \leq \operatorname{lin-cr}(G)$, and there are examples where $\operatorname{cr}(G) = 4$, but $\operatorname{lin-cr}(G)$ is unbounded [2]. The principal result about crossing numbers is the crossing lemma of Ajtai–Chvátal–Newborn–Szemerédi and Leighton [1, 6] which states that

$$\operatorname{cr}(G) \geq c\frac{|E|^3}{|V|^2} \qquad \text{whenever } |E| \geq C|V|. \tag{1}$$

The inequality is sharp apart from the values of c and C (see [7] for the best known estimate on c). The most famous applications of the crossing lemma are elegant proofs by Székely [8] of Szemerédi–Trotter theorem on point-line incidences and of Spencer–Szemerédi–Trotter theorem on the unit distances. Another remarkable application is the bound on the number of halving lines by Dey [4]. We extend the crossing number to $\mathbb{R}^3$, in such a way that the corresponding "space crossing lemma" (Theorem 2 below) implies (1) (up to a logarithmic factor).

A *spatial drawing* of a graph G is representation of vertices of G by points in $\mathbb{R}^3$, and edges of G by continuous curves. A *space crossing* consists of a quadruple of vertex-disjoint edges $(e_1, \ldots, e_4)$ and a line l that meets these four edges. The *space crossing number* of G, denoted $\mathrm{cr}_4(G)$ is the least number of crossings in any spatial drawing of G. As in the planar case, the *spatial rectilinear crossing number* $\text{lin-cr}_4(G)$ is obtained by restricting to straight-line spatial drawings.

For a graph G pick a drawing of G in the plane with the fewest crossings. By perturbing the drawing slightly, we may assume that there are no points where three vertex-disjoint edges meet. The drawing can be lifted to a drawing G on a large sphere without changing any of the crossings. Since no line meets the sphere in more than two points, every space crossings in the resulting spatial drawing comes from a pair of crossings in the planar drawing. Thus,

$$\mathrm{cr}_4(G) \leq \binom{\mathrm{cr}(G)}{2}. \tag{2}$$

Let us note that the space crossing number is not the usual crossing number in disguise, for the inequality in the reverse direction does not hold:

Proposition 1. *For every natural number n there is a graph G with $\mathrm{cr}_4(G) = 0$ and $\mathrm{cr}(G) \geq n$.*

The principal result that justifies the introduction of the space crossing number is the following generalization of the crossing lemma.

Theorem 2. *Let $G = (V, E)$ be an arbitrary graph, then*

$$\mathrm{cr}_4(G) \geq \frac{|E|^6}{4^{179}|V|^4 \log_2^2|V|}$$

whenever $|E| \geq 4^{41}|V|$.

Since (1) is sharp, in the light of the argument that led to (2) there are graphs on the sphere for which the bound in Theorem 2 is tight up to the logarithmic factor. In the drawings of the these graphs, the edges are of course not straight. It turns out that there are also straight-line spatial drawings for which Theorem 2 is tight.

Theorem 3. *For all positive integers m and n satisfying $m \leq \binom{n}{2}$ there is a graph G with n vertices and m edges, and the rectilinear space crossing number at most $6720m^6/n^4$.*

Our original construction in the proof of Theorem 3 used the idea of stair-convexity introduced in [3]. However, at the workshop Geza Tóth greatly simplified the construction. His graph is a union of disjoint cliques of appropriate sized. The vertices of each clique are placed in small clusters, which are themselves in general position, so that no line comes close through three or more clusters.

Our final result is the lower bound on the space crossing number of (possibly sparse) pseudo-random graphs.

Theorem 4. *There is an absolute constant $\varepsilon > 0$ such that the following holds. Let $G = (V, E)$ be a graph such that whenever V_1, V_2 are any two subsets of V of size $\varepsilon|V|$, the number of edges between V_1 and V_2 is at least N. Then* $\text{lin-cr}_4(G) \geq N^4$.

The condition of the theorem holds for several models of random graphs, as well as for (n, d, λ)-graph (see for example [5, Theorem 2.11]).

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What are high-dimensional permutations? How many are there?

Nati Linial

(joint work with Zur Luria)

The permanent of an $n \times n$ matrix $A = (a_{ij})$ is defined by

$$\text{Per}(A) = \sum_{\sigma \in \mathbb{S}_n} \prod_{i=1}^{n} a_{i,\sigma_i}$$

Permanents have attracted a lot of attention [8]. They play an important role in combinatorics. Thus if A is a $0-1$ matrix, then $\text{Per}(A)$ counts perfect matchings in the bipartite graph whose adjacency matrix is A. They are also of great interest from the computational perspective. It is $\#P$-hard to calculate the permanent of a given 0-1 matrix [10], and following a long line of research, an approximation scheme was found [5] for the permanents of nonnegative matrices. Bounds on permanents have also been studied at great depth. Van der Waerden conjectured that $\text{Per}(A) \geq \frac{n!}{n^n}$ for every $n \times n$ doubly stochastic matrix A, and this was established more than fifty years later by Falikman and Egorychev [3, 2]. More recently, Gurvits [4] discovered a new conceptual proof for this conjecture (see [7] for a very readable presentation). What is more relevant for us here are upper bounds on permanents. These are the subject of Minc's conjecture which was proved by Brègman [1].

Theorem 1. *If A is an $n \times n$ 0-1 matrix with r_i ones in the i-th row, then*

$$\mathrm{Per}(A) \leq \prod_{i=1}^{n} (r_i!)^{1/r_i}.$$

Radhakrishnan's proof uses the entropy method. Our approach can be seen as an adaptation of this proof strategy to study a d-dimensional analogue of the permanent.

Definition 2. (1) *Let A be an $[n]^d$ array. A line of A is vector of the form*

$$(A(i_1, ..., i_{j-1}, t, i_{j+1}, ..., i_d))_{t=1}^{n},$$

where $1 \leq j \leq d$ and $i_1, ..., i_{j-1}, i_{j+1}, ..., i_d \in [n]$.

(2) *A d-dimensional permutation of order n is an $[n]^{d+1}$ array P of zeros and ones such that every line of P contains a single one and $n-1$ zeros. Denote the set of all d-dimensional permutations of order n by $S_{d,n}$.*

For example, a two dimensional array is a matrix. It has two kinds of lines, usually called rows and columns. Thus a 1-permutation is an $n \times n$ 0-1 matrix with a single one in each row and a single one in each column, namely a permutation matrix. A 2-permutation is identical to a Latin square and $S_{2,n}$ is the same as the set $\mathcal{L}_n$, of order-n Latin squares. We now explain the correspondence between the two sets. If X is a 2-permutation of order n, then we associate with it a Latin square L, where $L(i,j)$ as the (unique) index of a 1 entry in the line $A(i,j,*)$. For more on the subject of Latin squares, see [11]. The same definition yields a one-to-one correspondence between 3-dimensional permutations and Latin cubes. In general, d-dimensional permutations are synonymous with d-dimensional Latin hypercubes. For more on d-dimensional Latin hypercubes, see [12]. To summarize, the following is an equivalent definition of a d-dimensional permutation. It is an $[n]^d$ array with entries from $[n]$ in which every line contains each $i \in [n]$ exactly once. We interchange freely between these two definitions according to context.

Our main concern here is to estimate $|S_{d,n}|$, the number of d-dimensional permutations of order n. By Stirling's formula

$$|S_{1,n}| = n! = \left((1+o(1))\frac{n}{e}\right)^{n}.$$

As we saw, $|S_{2,n}|$ is the number of order n Latin squares. The best known estimate [11] is

$$|S_{2,n}| = |\mathcal{L}_n| = \left((1+o(1))\frac{n}{e^2}\right)^{n^2}.$$

This relation is proved using bounds on permanents. Brégman's theorem for the upper bound, and the Falikman–Egorychev theorem for the lower bound.

This suggests

Conjecture 3.

$$|S_{d,n}| = \left((1+o(1))\frac{n}{e^d}\right)^{n^d}.$$

We prove the upper bound

Theorem 4.

$$|S_{d,n}| \leq \left((1+o(1))\frac{n}{e^d}\right)^{n^d}.$$

As mentioned, our method of proof is an adaptation of [9]. We first need

Definition 5. (1) *An $[n]^{d+1}$ 0-1 array M_1 is said to include an array M_2 if*

$$M_2(i_1,...,i_{d+1}) = 1 \Rightarrow M_1(i_1,...,i_{d+1}) = 1.$$

(2) *The d-permanent of a $[n]^{d+1}$ 0-1 array A is*

$$\mathrm{Per}_d(A) = \text{The number of } d\text{-dimensional permutations included in } A.$$

Note that in the one-dimensional case, this is indeed the usual definition of $\mathrm{Per}(A)$. It is not hard to see that for $d = 1$ following theorem coincides with Brègman's theorem.

Theorem 6. *Define the function $f : \mathbb{N}_{\geq 0} \times \mathbb{N} \longrightarrow \mathbb{R}$ recursively by:*

- $f(0, r) = \log(r)$, *where the logarithm is in base e.*
- $f(d, r) = \frac{1}{r}\sum_{k=1}^{r} f(d-1, k)$.

Let A be an $[n]^{d+1}$ 0-1 array with $r_{i_1,...i_d}$ ones in the line $A(i_1,...,i_d,*)$. Then

$$\mathrm{Per}_d(A) \leq \prod_{i_1,\ldots,i_d} e^{f(d,r_{i_1,\ldots i_d})}.$$

Somel analysis yields fairly tight bounds on the function f from theorem 6. Theorem 4 follows by applying these estimates to the all-ones array.

What about proving a matching lower bound on $S_{d,n}$ (and thus proving conjecture 3)? In order to follow the footsteps of [11], we would need a lower bound on $\mathrm{Per}_d A$, namely, a higher-dimensional analog of the van der Waerden conjecture. The entries of a *multi-stochastic* array are nonnegative reals and the sum of entries along every line is 1. This is the higher-dimensional counterpart of a doubly-stochastic matrix. It should be clear how to extend the notion of $\mathrm{Per}_d(A)$ to real-valued arrays. In this approach we would need a lower bound on $\mathrm{Per}_d(A)$ that holds for every multi-stochastic array A. However, this attempt (or at least its most simplistic version) is bound to fail. An easy consequence of Hall's theorem says that a 0-1 matrix in which every line or column contains the same (positive) number of one-entries, has a *positive* permanent. However, the higher dimensional analog of this is simply incorrect. There exist multi-stochastic arrays whose d-permanent vanishes, as can easily be deduced e.g., from [6].

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On a problem of Grünbaum and Motzkin, and Erdős and Purdy

Rom Pinchasi

Let P be a set of n blue points in the plane, not all collinear. Let G be a set of m red points such that $G \cap P = \emptyset$ and every line determined by P contains a point from G. Grünbaum and Motzkin (1975) and independently Erdős and Purdy (1978) conjectured that m must be large in terms of n. The fact that $m \geq cn$ for some constant c follows from the weak Dirac's theorem by Beck and Széméredi and trotter (1983), where c is very small. No example is known where $m < n - 6$. This suggests that $m \geq cn$ for some reasonably large constant c.

We show that $m \geq \frac{n-1}{3}$.

New results on decomposability of geometric coverings

Dömötör Pálvölgyi
(joint work with János Pach, Géza Tóth)

The study of multiple coverings was initiated by Davenport and L. Fejes Tóth 50 years ago. In 1986 J. Pach published the first papers about *decomposability* of multiple coverings. It was discovered recently that besides its theoretical interest, this area has important practical applications. Now there is a great activity in this field with several breakthrough results, although, the most important questions are still unsolved.

Let $\mathcal{P} = \{ P_i \mid i \in I \}$ be a collection of planar sets. We say that $\mathcal{P}$ is an *m-fold covering* if every point in the plane is contained in at least m members of $\mathcal{P}$. The biggest such k is called the thickness of the covering. A 1-fold covering is simply called a *covering*.

Definition. A planar set P is said to be *cover-decomposable* if there exists a (minimal) constant $m = m(P)$ such that every m-fold covering of the plane with translates of P can be decomposed into two coverings.

Pach [6] proposed the problem of determining all cover-decomposable sets in 1980 and made the following conjecture.

Conjecture (Pach)**.** *All planar convex sets are cover-decomposable.*

For a cover-decomposable set P, one can ask for the exact value of $m(P)$. In most of the cases, the best known upper and lower bounds are very far from each other. The only case where the gap is relatively small is for open triangles where we know that $4 \leq m(P) \leq 12$ [4].

Decomposition of open polygons. This conjecture has been verified for open polygons through a series of papers.

Theorem A. (i) [7] *Every centrally symmetric open convex polygon is cover-decomposable.*
(ii) [13] *Every open triangle is cover-decomposable.*
(iii) [12] *Every open convex polygon is cover-decomposable.*

There are several recent negative results as well.

Theorem B. (i) [8] *Concave quadrilaterals are not cover-decomposable.*
(ii) [11] *"Typical" concave polygons are not cover-decomposable.*

In fact we have a complete characterization if instead of the plane, we require that there exists a (minimal) constant $m = m(P)$ such that every m-fold covering of ANY planar set with translates of P can be decomposed into two coverings, which property is called *totally-cover-decomposability.*

Theorem A–B ([12] and [11])**.** *An open polygon, P, is totally-cover-decomposable if and only if among its "wedges" there is no two that belong to a "special" type.*

Two halflines, both of endpoint O, divide the plane into two parts which we call *wedges.* A polygon with n vertices has n wedges, each corresponding to the lengthening of two adjacent sides. Two wedges belong to the special type if the union of the wedges is in an open halfplane whose boundary contains O, but none of them contain the other.

However, unfortunately we still do not have such a nice characterization for plane-cover-decomposability. Some special cases, when such extensions are possible, were studied in [11].

Decomposition to multiple parts.

Definition. Let P be a planar set and $k \geq 2$ integer. If it exists, let $m_k(P)$ denote the smallest number m with the property that every m-fold covering of the plane with translates of P can be decomposed into k coverings.

We conjecture that $m_k(P)$ exists for all cover-decomposable P. This was established for polygons in the above mentioned papers with some weak bounds, like $m_k(P) \leq O(m(P))^k$. However, all these results were improved to the optimal linear bound in a series of papers.

Theorem C. (i) [9] *For any centrally symmetric open convex polygon P, $m_k(P) = O(k^2)$.*
(ii) [1] *For any centrally symmetric open convex polygon P, $m_k(P) = O(k)$.*
(iii) [3] *For any open convex polygon P, $m_k(P) = O(k)$.*
(iv) [14] *For any open, cover-decomposable polygon[1] P, $m_k(P) = O(k)$.*

Space and homothetic copies. We can similarly say that a spatial set P is *cover-decomposable* if there exists a (minimal) constant $m = m(P)$ such that every m-fold covering of the space with translates of P can be decomposed into two coverings. Here less sets seem to have this property.

Theorem D. (i) [5] *The unit ball is not cover-decomposable*
(ii) [11] *Polytopes are not cover-decomposable in the space and in higher dimensions.*
(iii) [2] *Orthants in 4 and higher dimensions are not cover-decomposable.*
(iv) [14] *"Four sided spatial wedges" are not cover-decomposable in the space and in higher dimensions.*

In all four proofs some indecomposable planar configurations are simulated by intersecting a plane by translates. Recently a positive theorem was also established.

Theorem E. [4] *Open octants are cover-decomposable.*

This theorem has the following very interesting corollary, which is unique in the sense that all the other proofs only work for translates of planar sets.

Corollary. [4] *Any 12-fold covering of any subset of the plane with a finite number of homothetic copies of any given triangle can be decomposed into two coverings.*

Closed polygons and making the covering finite. Notice that all our positive Theorems (A, C and E) are about open polygons. This is because using a compactness argument we can suppose that the considered coverings are locally finite and reduce them to the finite case. This cannot be done for closed polygons. In fact, for non-locally finite coverings with closed polygons, no positive results were known until recently. (The indecomposable constructions are finite, thus, of course, also work for closed polygons.) We only knew that if the underlying

[1] I.e. polygon satisfying the requirements of Theorem A-B.

closed set is convex then we can suppose that the covering only contains countably many copies (see [11]. Also, very recently Tóth [14] showed that closed, symmetric polygons are cover-decomposable.

Open questions

Most importantly, are all planar convex sets cover-decomposable?

Is there a polygon that is cover-decomposable but not totally-cover-decomposable?

Are closed, convex polygons cover-decomposable?

Is it true that for any cover-decomposable polygon/set P, $m_k(P) = O(k)$? Is it true that at all that for any set P, if $m_2(P)$ exists then $m_3(P)$ also exists?

Is there a nice set in three dimensions that is cover-decomposable?

For the shift-chain conjecture and other related problems, also see [10].

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On the tolerated Tverberg Theorem

Pablo Soberón
(joint work with Ricardo Strausz)

In 1921 Johann Radon proved the following: *given $d+2$ point in $\mathbb{R}^d$ they can be partitioned in two disjoint set A and B such that their convex hulls intersect* [1]; this is now known as Radon's Theorem. This Theorem has been the basis for a wide number of generalisations. One of these generalisations is due to Helge Tverberg. In 1966 he showed that *given $(k-1)(d+1)+1$ points in $\mathbb{R}^d$ there is a partition of them in k disjoint sets $A_1, \dots, A_k$ such that their convex hulls intersect* [2]. Both these theorems have optimal numbers.

The versions with tolerance of these numbers started to appear in 1972 when David Larman showed that *given $2d+3$ points in $\mathbb{R}^d$ there is a partition of them in two disjoint sets A and B such that for all x, the convex hulls of $A \backslash \{x\}$ and $B \backslash \{x\}$ intersect* [3]. This would be a Radon Theorem with tolerance 1. Later in 2007, this result was generalised by Natalia García-Colín in her PhD thesis, with David Larman as supervisor. The result was that *given $(r+1)(d+1)+1$ points in $\mathbb{R}^d$, they can be partitioned in two sets A and B such that for any $x_1, \dots, x_r$ the convex hulls of $A \backslash \{x_1, \dots, x_r\}$ and $B \backslash \{x_1, \dots, x_r\}$ intersect* [4]. This is what we call the Radon Theorem with tolerance.

Larman's result has been proven to be optimal for $d \leq 4$. The best lower bound in general is $\lceil \frac{5d}{3} \rceil + 3$, proven by Jorge Ramírez-Alfonsín [5]. Other theorems with tolerance have begun to appear recently. In the talk I present a positive answer to a conjecture in [4], namely

Theorem 1 (Tverberg with tolerance [6])**.** *Let r, k, d be nonnegative integers such that $d \geq 1$. Then, given $(r+1)(k-1)(d+1)+1$ points in $\mathbb{R}^d$, there is a partition of them in k disjoint sets $A_1, \dots, A_k$ such that for any $x_1, x_2, \dots, x_r$ the convex huls of the sets $A_i \backslash \{x_1, x_2, \dots, x_r\}$ are intersecting.*

We have not proven that this result is optimal, however we conjecture that for any r, k, d there must be a set of $(r+1)(k-1)(d+1)$ points in $\mathbb{R}^d$ with no k-Tverberg partition with tolerance r.

It is interesting to not that some theorems do not admit nontrivial versions with tolerance. This would be true for the Bárány–Lovász generalisation of Carathéodory's Theorem. Their version says that *given $d+1$ color clases that capture the origin in $\mathbb{R}^d$ there is a colorful choice that captures the origin.* The version with tolerance says that *given $(r+1)(d+1)$ color classes that capture the origin in $\mathbb{R}^d$ there is a colorful choice that captures the origin with tolerance r.* The interesting thing about this result is that it is trivial with $(r+1)(d+1)$ classes but false with less.

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On line transversals

Edgardo Roldán-Pensado
(joint work with Jesús Jerónimo-Castro)

Let $\mathcal{F}$ be a finite family of convex bodies in $\mathbb{R}^d$. It is said that $\mathcal{F}$ has property T if there exists a line that intersects all of its members. Furthermore, if $k \in \mathbb{N}$ then $\mathcal{F}$ has property $T(k)$ if every subfamily of $\mathcal{F}$ with at most k members has property T.

In 1935, P. Vincensini [8] posed the problem of finding conditions on $\mathcal{F}$ so that property $T(k)$ would imply property T. The first result of this type was due to Santaló [7] who showed the following: *If $\mathcal{F}$ is a family of parallelotopes in $\mathbb{R}^d$ with edges parallel to the coordinate axes and $\mathcal{F}$ has property $T(2^{d-1}(2d-1))$, then $\mathcal{F}$ has property T.*

After this, several variations of this problem emerged. We are interested in one posed by B. Grünbaum [2] in 1964:

Let K be a convex body in $\mathbb{R}^d$ and $\mathcal{F} = \{x_1 + K, \dots, x_n + K\}$ be a family of translates of K with property $T(k)$. Determine the smallest $\lambda = \lambda(K, k) > 0$ such that the family $\lambda\mathcal{F} = \{x_1 + \lambda K, \dots, x_n + \lambda K\}$ has property T.

There have been several results on this such as the following (see [1, 3, 4]):

Theorem 1 (Eckhoff)**.** *Let D be a disk in $\mathbb{R}^2$, then $\lambda(D, 3) \leq 2$.*

Theorem 2 (Heppes)**.** *Let $\mathcal{F}$ be a family of disjoint translates of a disc in $\mathbb{R}^2$ with the property $T(3)$, then $1.65\mathcal{F}$ has property T.*

Theorem 3 (Jerónimo)**.** *Let D be a disk in $\mathbb{R}^2$, then $\lambda(D, 4) = \frac{1+\sqrt{5}}{2}$.*

It was known that $\lambda(D, 3) \geq \frac{1+\sqrt{5}}{2}$, as can be seen by placing five disks in $\mathbb{R}^2$ with centres on the vertices of a regular pentagon of appropriate size. Eckhoff conjectured that this was actually the correct value for $\lambda(D, 3)$. However, the best upper bound known was $\lambda(D, 3) < 2$.

Definition. Let K be a convex body in $\mathbb{R}^2$, the number $\mu(K)$ is defined as the smallest $\mu > 0$ such that the following holds: If x_1, x_2, x_3 and x_4 form the vertices of a parallelogram and the family $\mathcal{F} = \{x_1 + K, \dots, x_4 + K\}$ has property $T(3)$, then the family $\mu\mathcal{F}$ has property T.

For simple convex bodies $\mu(K)$ is easy to compute, for example, $\mu(D) = \sqrt{2}$ if D is a disk and $\mu(S) = 2$ if S is a square. In terms of this number, we managed to improve the bounds on $\lambda(K, 3)$ and in particular we improved the bounds on Eckhoff's conjecture (see [5]).

Theorem 4. *There exists $\varepsilon > 0$ such that for every convex body $K \subset \mathbb{R}^2$,*

$$\frac{4}{3} + \varepsilon \leq \lambda(K, 3) \leq \max\left\{\frac{2 + \sqrt{1 + 4\mu(K)}}{2}, \rho\right\},$$

where $\rho \approx 1.76$ is the real root of $x^3 - 2x^2 - x + 1$.

Corollary 5. *If D is a disk, then*

$$\frac{1 + \sqrt{5}}{2} \leq \lambda(D, 3) \leq \frac{1 + \sqrt{1 + 4\sqrt{2}}}{2} \approx 1.79.$$

Using some of the same techniques, we generalised Jerónimo's theorem above to all convex bodies.

Theorem 6. *Let $K \subset \mathbb{R}^2$ be any convex body, then*

$$\lambda(K, 4) \leq \frac{1 + \sqrt{5}}{2}.$$

Another variant of Grünbaum's problem closely related to this is the following: Let K be a convex body and $\mathcal{F} = \{x_1 + t_1K, \ldots, x_n + t_nK\}$ be a family of homothets of K with property $T(k)$. Determine the smallest $\lambda_h = \lambda_h(K, k) > 0$ such that $\lambda_h\mathcal{F}$ has property T.

As far as we know, there is nothing in the literature about this problem. We obtained some results for the case when K is a disk (see [6]).

Theorem 7. *Let B be a ball in $\mathbb{R}^d$, then*

$$\lambda_h(B, d + 1) \leq 4.$$

This theorem is far from being optimal. The main idea used here is to fix the smallest ball in $\mathcal{F}$ and shrink it to a point P while expanding the other so that the $T(d+1)$ property is not lost. Then we represent a subset of the lines through P as $\mathbb{R}^{d-1}$. Finally we project each ball to this space and use Helly's theorem to find a line transversal. This theorem can be improved considerably in $\mathbb{R}^2$.

Theorem 8. *If $D \subset \mathbb{R}^2$ is a disk, then*

$$\sqrt{3} \leq \lambda_h(D, 3) \leq \rho \approx 2.875,$$

where ρ is the real root of $x^3 - x^2 - 4x - 4$.

This theorem uses some of the same ideas but is done more carefully, we conjecture that $\lambda_h(D, 3) = \sqrt{3}$. If instead of $T(3)$ we use the stronger $T(4)$ property we obtain the following.

Theorem 9. *If $D \subset \mathbb{R}^2$ is a disk, then*

$$\sqrt{\frac{2+\sqrt{2}}{2}} \leq \lambda_h(D,4) \leq 2\sqrt{2}.$$

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Counting the number of ranking patterns

Norihide Tokushige
(joint work with Hidehiko Kamiya, Akimichi Takemura)

Let m objects $x_1, \ldots, x_m \in \mathbf{R}$ be given. A judge $y \in \mathbf{R}$ sorts these objects according to the Euclidean distance from y. Namely, for $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbf{R}^m$ and $y \in \mathbf{R}$, we say that a judge y gives a ranking $f(\mathbf{x}, y) := (i_1, \ldots, i_m)$ iff $|y - x_{i_1}| < |y - x_{i_2}| < \cdots < |y - x_{i_m}|$. Let $F(\mathbf{x}) = \{f(\mathbf{x}, y) : y \in \mathbf{R}\}$ be the ranking pattern of $\mathbf{x} \in \mathbf{R}^m$, and let $\mathcal{F}_m = \{F(\mathbf{x}) : \mathbf{x} \in \mathbf{R}^m, x_1 < x_2 < \cdots < x_m\}$ be the family of ranking patterns. For example,

$$\mathcal{F}_3 = \{\{123, 213, 231, 321\}\}$$

and

$$\begin{aligned} \mathcal{F}_4 \quad = \quad \{ & \{1234, 2134, 2314, 3214, 3241, 3421, 4321\}, \\ & \{1234, 2134, 2314, 2341, 3241, 3421, 4321\} \}. \end{aligned}$$

Finally, let $r(m) = |\mathcal{F}_m|$ be the number of ranking patterns arising from m objects. The following values are known: $r(3) = 1$, $r(4) = 2$, $r(5) = 12$, $r(6) = 168$, $r(7) = 4680$, $r(8) = 229386$, $r(9) = 18330206$, see [1].

I presented that

$$c_1 < \frac{r(m)^{1/m}}{m^2} < c_2$$

holds for some positive constants c_1, c_2. The lower bound comes from a simple combinatorial construction, while the upper bound is obtained by counting the cells in a corresponding hyperplane arrangement. I sketch the proof below.

Let $x_{ij} = \frac{x_i+x_j}{2}$ be the midpoint of x_i and x_j. Then the set of midpoints $\{x_{ij} : 1 \le i < j \le m\}$ is a poset with partial order $\prec$, where $x_{ij} \prec x_{i'j'}$ iff $i \le i'$ and $j \le j'$. There are exactly $r(m)$ ways of extending this poset to a totally ordered set. For example, if $m = 4$, then there are two possible orderings: $x_{12} < x_{13} < x_{23} < x_{14} < x_{24} < x_{34}$ and $x_{12} < x_{13} < x_{14} < x_{23} < x_{24} < x_{34}$. This motivates the mid-hyperplane arrangement $\mathcal{M}_m$, which is a collection of hyperplanes in $\mathbb{R}^m$ of type I $\{x_i = x_j : 1 \le i < j \le m\}$ and type II $\{x_i + x_j = x_k + x_\ell : i, j, k, l \text{ all distinct}\}$. In [1] it is observed that the number of chambers of $\mathcal{M}_m$ equals $m!\, r(m)$. In general the number of chambers created by n hyperplanes in $\mathbb{R}^d$ is at most $(en/d)^d$. In our case $d = m$ and $n = |\mathcal{M}_m| = \binom{m}{2} + 3\binom{m}{4}$, which give $r(m)^{1/m} < c_2 m^2$ with $c_2 = e^2/8 \approx 0.92$ for m large.

For the lower bound, we first fix $r(m)$ orderings of $\{x_{ij} : 1 \le i < j \le m\}$, and we then add x_{m+1} to see how many orderings of $\{x_{ij} : 1 \le i < j \le m+1\}$ appear on these fixed $r(m)$ orderings. A simple averaging yields roughly $r(m+1) > \frac{3}{4} m^2 r(m)$, which gives $c_1 m^2 < r(m)^{1/m}$ with $c_1 = 3/(4e^2) \approx 0.1$ for m large.

The first 5 terms of $r(m)$ coincide with the first 5 terms of the sequence A059522 of the OEIS [3]. This is related to the number of acyclic-function digraphs (A058128). Does A059522 give a lower bound for $r(m)$? (This is true for $3 \le m \le 11$.) If so, then this gives $r(m)^{1/m} > m^2/3 \approx 0.37m^2$ for m large.

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Construction of locally plane graphs

Gábor Tardos

A graph drawn in the plane with straight-line edges is called a geometric graph. If no path of length k or shorter is self-intersecting in a geometric graph we call it k-locally plane. The main result of this paper is a construction of k-locally plane graphs with a super-linear number of edges. For the proof we develop randomized thinning procedures for edge-colored bipartite (abstract) graphs that can be applied to other problems as well.

A *geometric graph* G is a straight-line drawing of a simple, finite (abstract) graph (V, E), i.e., we identify the vertices $x \in V$ with distinct points in the Euclidean plane, and we identify any edge $\{x, y\} \in E$ with the straight line segments xy in the plane. We assume that the edge xy does not pass through any vertex of G besides x and y. We call (V, E) the abstract graph *underlying* G. We say that the edges $e_1, e_2 \in E$ *cross* if the corresponding line segments cross each other, i.e., if they have a common interior point. We say that a subgraph of G is *self-intersecting* if it contains a pair of crossing edges.

Geometric graphs without crossing edges are plane drawings of planar graphs: they have at most $3n-6$ edges if n is the number of vertices.

Avital and Hanani [3], Erdős, and Perles initiated in the mid 1960s the systematic study of similar questions for more complicated *forbidden configurations*: Let H be set of forbidden configurations (geometric subgraphs). What is the maximal number of edges of an n vertex geometric graph not containing any configuration belonging to H? This problem can be regarded as a geometric version of the fundamental problem of extremal graph theory: What is the maximum number of edges that an abstract graph on n vertices can have without containing subgraphs of a certain kind.

Many questions of the above type on geometric graphs have been addressed in recent years. In a number of papers linear upper bounds have been established for the number of edges of a graph, under various forbidden configurations. They include the configurations of three pairwise crossing edges [2], four pairwise crossing edges [1], the configurations of an edge crossed by many edges [9], or even two large stars with all edges of one of them crossing all edges of the other [13].

For a constant number of 5 or more pairwise crossing edges the Pavel Valtr has the best result [11]: a geometric graph on n vertices avoiding this configuration has $O(n \log n)$ edges. Adam Marcus and the present author [4] building on an earlier result of Pinchasi and Radoičić [10] prove an $O(n^{3/2} \log n)$ bound on the number of edges of an n vertex geometric graph not containing self-intersecting cycles of length four. No construction is known beating the $\Omega(n^{3/2})$ edges of an abstract graph having no cycles of length four.

For surveys on geometric graph theory, consult [5], [6] and [8].

In this talk we consider forbidding self-intersecting path. For $k \geq 3$ we call a geometric graph *k-locally plane* if it has no self-intersecting subgraph (whose underlying abstract graph is) isomorphic to a path of length at most k.

Pach et al. [7] consider 3-locally plane graphs, i.e., the case of geometric graphs with no self-intersecting paths of length three. They prove matching lower and upper bounds of $\Theta(n \log n)$ on the maximal number of edges of a 3-locally plane graph on n vertices.

We extend the the lower bound result of [7] by forbidding all self-intersecting drawings of longer paths. Technically k-locally plane graphs are defined by forbidding self intersecting paths of length k *or shorter*, but forbidding only self-intersecting paths of length *exactly* k would lead to almost the same extremal function. Indeed, one can delete at most nk edges from any graph on n vertices, such that all the non-zero degrees in the remaining graph are larger than k. This ensures that all paths can be extended to a path of length k. It is possible, but not likely, that if one only forbids paths of length k with *the first and last edges crossing* significantly higher number of edges is achievable.

For even k a geometric graph is k-locally plane if and only if the $k/2$-neighborhood of any vertex x is intersection free. Note that this requirement is much stronger than the similar condition on abstract graphs, namely that the $k/2$-neighborhood of any point is planar. One can construct graphs with girth larger than k

and $\Omega(n^{\frac{k}{k-1}})$ edges. In such a graph the $k/2$-neighborhood of any vertex is a tree, still by [7] the graph does not even have 3-locally plane drawing.

Extending the lower bound result in [7] we prove that for arbitrary fixed $k \geq 3$ there exist k-locally plane graphs on n vertices with $\Omega(n \log^{(\lfloor k/2 \rfloor)} n)$ edges. Here $\log^{(t)}$ denotes t times iterated logarithm and the hidden constant in Ω depends on k.

As a simple corollary we can characterize the abstract graphs H such that any geometric graph having no self-intersecting subgraph isomorphic to H has a linear number of edges. These graphs H are the forests with at least two nontrivial components. To see the linear bound for the number of edges of a geometric graph avoiding a self-intersecting copy of such a forest H first delete a linear number of edges from an arbitrary geometric graph G until all non-zero degrees of the remaining geometric graph G' are at least $|V(H)|$. If G' is crossing free the linear bound of the number of edges follows. If you find a pair of crossing edges in G' they can be extended to a subgraph of G' isomorphic to H. On the other hand, if H contains a cycle, then even an abstract graph avoiding it can have a super-linear number of edges. If H is a tree of diameter k, then a k-locally plane geometric graph has no self-intersecting copy of H. Notice that the extremal number of edges in this case (assuming $k > 2$) is $O(n \log n)$ by [7], thus much smaller than the $\Omega(n^\alpha)$ edges ($\alpha > 1$) for forbidden cycles.

The main tool used in the proof of the above result is a randomized thinning procedure that takes a d edge colored bipartite graph of average degree $\Theta(d)$ and returns a subgraph on the same vertex set with average degree $\Theta(\log d)$ that does not have a special type of colored path (walk) of length four. The procedure can be applied recursively to obtain a subgraph avoiding longer paths of certain types. We believe this thinning procedure to be of independent interest. In particular it can be used to obtain optimal 0-1 matrix constructions for certain avoided submatrix problems, see the the details in [12].

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Finite tilings

Igor Pak
(joint work with Jed Yang)

Beauquier et al (1995) showed that tiling of general regions with two rectangles is NP-complete, except for few trivial special cases. In a different direction, Rémila showed that for simply connected regions and two rectangles, the tileability can be solved in quadratic time (in the area). We prove that there is a finite set of at most 10^6 rectangles for which the tileability problem of simply connected regions is NP-complete, closing the gap between positive and negative results in the field. We also prove that counting such rectangular tilings is #P-complete, a first result of this kind.

In the talk, I will survey known results and then give a general outline of the proof.

Polytopes with low-dimensional realization spaces

Günter M. Ziegler
(joint work with Karim Adiprasito)

We discuss, dimension-by-dimension, the two questions

Question 1. *Is it true that, up to projective transformations, there are only finitely many combinatorial types of d-dimensional convex polytopes?*

(This was conjectured to be true by Shephard and McMullen in the sixties; compare [4].)

Question 2. *How does, for d-dimensional polytopes, the dimension of the realization space grow with the size of the polytopes?*

(Suitable definitions of “realization space” and “size” are given below.)

1. Definitions

Let P be a convex d-polytope of *size* $\mathrm{size}(P) := f_0(P) + f_{d-1}(P)$. We define its *(centered) realization space* as

$$\mathcal{R}_0(P) := \big\{(V,C) \in \mathbb{R}^{d\times(f_0+f_{d-1})} : \mathrm{conv}(V) = \{x \in \mathbb{R}^d : C^t x \leq \mathbf{1}\}\mathrm{realizes} P\big\},$$

that is, as the set of combined vertex and facet descriptions of realizations of P that have the origin in the interior. Note that $(C,V) \in \mathbb{R}^{d\times(f_0+f_{d-1})}$ lies in this set iff $c_i^t v_j = 1$ whenever v_j lies on the facet $F_i = \{x \in \mathbb{R}^d : c_i^t x = 1\}$ and $c_i^t v_j < 1$ otherwise. Thus this realization space is a primary semialgebraic set in $\mathbb{R}^{d\,\mathrm{size}(P)}$; it particular, its *dimension* is well-defined.

The polytope P is called *projectively unique* if the group $\mathrm{PGL}(\mathbb{R}^d)$ of projective transformations on $\mathbb{R}^d$ acts transitively on $\mathcal{R}_0(P)$. In particular, for projectively unique polytopes we have $\dim \mathcal{R}_0(P) \leq \dim \mathrm{PGL}(\mathbb{R}^d) = d(d+2)$, with equality in all non-trivial cases (if the vertex set of the polytope contains a projective basis, that is, it is not a join).

The *naive guess* for the dimension of the (centered) realization space is

$$\mathrm{NG}(P) := d(f_0 + f_{d-1}) - f_{0,d-1},$$

where $f_{0,d-1}$ is the number of vertex-facet incidences. This quantity is "the number of variables minus the number of equations" in the above system. (Compare [1].)

2. Examples

For $d = 2$, the only projectively unique 2-polytopes are triangle Δ_2 and square $\square$.

For $d = 3$, the projectively unique 3-polytopes are the four types of 3-polytopes with at most 9 edges, that is: tetrahedron Δ_3, square pyramid $\square * \Delta_0$, prism over triangle $\Delta_2 \times \Delta_1$, bipyramid over triangle $\Delta_2 \oplus \Delta_1$ according to Grünbaum [2, Exercise 4.8.30]. Indeed, this is compatible with the result of Steinitz [9] (see also Richter-Gebert [6, Sect. 13.3]) that the dimension of the realization space $\mathcal{R}_0(P)$ of a 3-polytope is $\mathrm{NG}(P) = f_1 + 6 = f_1 - 9 + \dim \mathrm{PGL}(\mathbb{R}^d)$.

This answers Questions 1 and 2 for $d \leq 3$: For Question 1 the answer is yes, and *the dimension of the realization space grows linearly with the size.*

3. Universality

The naive guess $\mathrm{NG}(P)$ yields the dimension of the (centered) realization space correctly for all polytopes of dimension $d \leq 3$, as well as for all simple or simplicial polytopes. However, Robertson's [7] claim that $\mathcal{R}_0(P)$ is always a differentiable manifold of dimension $\mathrm{NG}(P)$ is far from being true — indeed, according to Mnëv's universality theorem [5] [6], $\mathcal{R}_0(P)$ is "stably equivalent" to an arbitrary primary semialgebraic set defined over $\mathbb{Z}$; here "stable equivalence" means that we loose the control over the dimension, but keep singularities as well as the (non)existence of rational points. Thus, we cannot assume that $\mathrm{NG}(P)$ yields the dimension of the realization space in general — the guess could be way too high, or too low. Indeed, according to Richter-Gebert [6] this already happens for $d = 4$.

4. Shephard's list

Construction methods for projectively unique d-polytopes were developed by Peter McMullen in his doctoral thesis (Birmingham 1968) directed by G. C. Shephard; see [4], where McMullen writes: "Shephard (private communication) has independently made a list, believed to be complete, of the projectively unique 4-polytopes. All of these polytopes can be constructed by the methods described here."

If the conjecture is correct, then the following list of eleven projectively unique 4-polytopes (all of them generated by McMullen's techniques, duplicates removed) should be complete:

Construction		$(f_0, f_1, f_2, f_3; f_{03})$	NG	facets
Δ_4	selfdual	(5,10,10,5;20)	20	5 tetrahedra
$\square * \Delta_1$	selfdual	(6,11,11,6;26)	22	4 tetrah., 2 square pyramids
$(\Delta_2 \oplus \Delta_1) * \Delta_0$		(6,14,15,7;29)	23	6 tetrah., 1 bipyramid
$(\Delta_2 \times \Delta_1) * \Delta_0$		(7,15,14,6;29)	23	2 tetrah., 3 sq. pyr., 1 prism
$\Delta_3 \oplus \Delta_1$	simplicial	(6,14,16,8;32)	24	8 tetrah.
$\Delta_3 \times \Delta_1$	simple	(8,16,14,6;32)	24	2 tetrah., 4 prisms
$\Delta_2 \oplus \Delta_2$	simplicial	(6,15,18,9;36)	24	9 tetrah.
$\Delta_2 \times \Delta_2$	simple	(9,18,15,6;36)	24	6 prisms
$(\square, v) \oplus (\square, v)$		(7,17,18,8;36)	24	4 square pyramids, 4 tetrah.
... its dual		(8,18,17,7;36)	24	2 prisms, 4 sq. pyr., 1 tetrah.
v.split$(\Delta_2 \times \Delta_1)$	selfdual	(7,17,17,7;32)	24	3 tetrah., 2 sq. pyr., 2 bipyr.

5. Neighborly cubical polytopes

The *neighborly-cubical* 4*-polytopes* $\mathrm{NCP}_4(n)$ constructed by Joswig & Ziegler [3] and further analyzed by Sanyal & Ziegler [8] are cubical 4-polytopes with the graph of the n-cube. From these data, one can derive that the extended f-vector is

$$(f_0, f_1, f_2, f_3; f_{03}) = \tfrac{1}{4}2^n(4, 2n, 3(n-2), n-2, 8n-16).$$

In particular, we get that the naive guess is naive: $\mathrm{NG}(\mathrm{NCP}_4(n)) = 2^n(6-n)$ is negative for high n.

Nevertheless the neighborly-cubical polytopes are not projectively unique, but the dimension of the realization space is very small compared to the size:

Theorem. *The dimension of the realization space of* $\mathrm{NCP}_4(n)$ *grows quadratically in* n*, and thus only logarithmically in* $\mathrm{size}(\mathrm{NCP}_4(n))$.

The quadratic lower bound follows from the construction as a generic projection of a simple n-polytope with $2n$ facets. The upper bound uses the "cubical Gale evenness criterion" combinatorial description of $\mathrm{NCP}_4(n)$ from [3] and [8], and then a suitable ordering of the *vertices and facets*, such that after a quadratic number variables has been fixed, all further vertices and facets are determined when they occur in the ordering. We note that this proof technique can establish that a realization space is low-dimensional only if the naive guess is low as well.

6. Complexity

Candidate 4-polytopes that *could* have a low-dimensional realization space (or even be projectively unique) should thus have a low (or even negative) naive guess. We note that this is closely related to the "complexity" parameter for polytopes

$$C(P) := \frac{f_{0,3} - 20}{f_0 + f_3 - 10}$$

introduced in [10] being large; specifically, it should be at least 4. The only examples we seem to know for this in the moment are the neighborly cubical polytopes, and more generally the "projected deformed products of polygons" of [11] [8]. However, all these are not projectively unique, by construction.

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Anchored rectangle packing

Csaba D. Tóth
(joint work with Adrian Dumitrescu)

We consider a rectangle packing problem popularized by Peter Winkler [4, 5, 6], which has been open for decades. It is a one-round game between Alice and Bob. First, Alice chooses a finite point set S in the unit square $U = [0, 1]^2$ in the plane, including the origin, that is, $(0, 0) \in S$ (Fig. 1(a)). Then Bob chooses an axis-parallel rectangle $r(s) \subseteq U$ for each point $s \in S$ such that s is the lower left corner of $r(s)$, and the interior of $r(s)$ is disjoint form all other rectangles (Fig. 1(b)). The rectangle $r(s)$ is said to be *anchored* at s, but $r(s)$ contains no point from S in its interior. It is conjectured that for any finite set $S \subset U$, $(0,0) \in S$, Bob can choose such rectangles that jointly cover at least half of U [1, 4, 5, 6]. However, it has not even been known whether Bob can always cover at least a positive constant area. It is clear that Bob cannot always cover $\frac{1}{2} + \varepsilon$ area for any fixed $\varepsilon > 0$. If Alice chooses S to be a set of n equally spaced points along the diagonal $[(0,0),(1,1)]$, as in Fig. 1(c), then the total area of Bob's rectangles is at most $\frac{1}{2} + \frac{1}{2n}$.

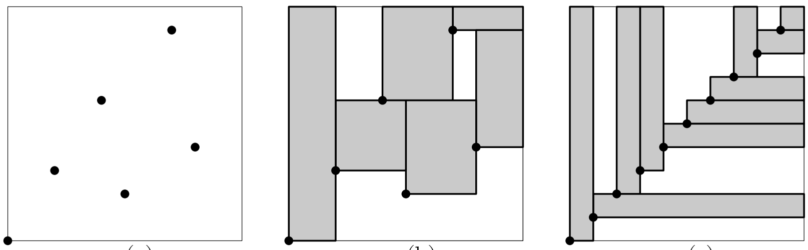

FIGURE 1. (a) A set S of 6 points in a unit square $[0,1]^2$, including the origin $(0,0)$. (b) A rectangle packing where the lower left corner of each rectangle is a point in S. (c) Ten equally spaced points along the diagonal $[(0,0),(1,1)]$, and a corresponding rectangle packing that covers roughly 1/2 area.

Very little is known about anchored rectangle packing. Recently, Christ *et al.* [2] proved that *if* Alice can force Bob's share to be less than $\frac{1}{r}$, then $n \geq 2^{2^{\Omega(r)}}$. Our result indicates that this condition does not materialize for large r, since Bob can always cover at least a constant fraction of the area for any $n \in \mathbb{N}$.

We present two simple strategies for Bob that cover at least 0.09121 area. These are the GreedyPacking and the TilePacking algorithms described below. Both algorithms process the points in the same specific order, namely the decreasing order of the sum of the two coordinates, with ties broken arbitrarily (hence $(0,0)$ is the last point processed).

The GreedyPacking algorithm chooses a rectangle of maximal area for each point in S sequentially, in the above order. The TilePacking algorithm constructs a packing of U with staircase-shaped tiles, and then chooses a rectangle of maximal area within each tile independently. We next describe how the tiling is obtained. Each tile is a staircase-shaped polygon, with a vertical left side, a horizontal bottom side, and a descending staircase connecting them. The lower left corner of each tile is a point in S. We say that the tile is *anchored* at that point. The algorithm maintains the invariant that the set of unprocessed points are in the interior of a staircase shaped polygon (super-tile), and in addition the *anchor* and possibly other points are on its left and lower sides. Processing a point amounts to shooting a horizontal ray to the right and a vertical ray upwards which together isolate a new tile anchored at that point, and the new staircase shaped polygon containing the remaining points is updated. Since $(0,0) \in S$, TilePacking does indeed compute a tiling of the unit square.

Theorem. *For every finite point set $S \subset U$, $(0,0) \in S$, each of* GreedyPacking *and* TilePacking *chooses a set of rectangles of total area $\rho \geq 0.09121$.*

We show that the GreedyPacking algorithm covers at least as much area as TilePacking. Hence it suffices to analyze the performance of the latter. The bulk of the work is in the analysis of this simple TilePacking algorithm, which involves geometric considerations and a charging scheme. A complete proof is available in [3].

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Upper bounds for densest packings with congruent copies of a convex body

Frank Vallentin
(joint work with Fernando Mario de Oliveira Filho)

Finding the maximum density of a packing of congruent copies of a compact convex body $\mathcal{K}$ in Euclidean space $\mathbb{R}^n$ is an extremely difficult problem in discrete geometry. A solution is known only for very few convex bodies. For instance, if $\mathcal{K}$ is the unit ball then this is the sphere packing problem. Recently, the case of regular tetrahedra got quite some attention, see Ziegler [7]. The last part of Hilbert's 18th problem states:

> Ich weise auf die hiermit in Zusammenhang stehende, für die Zahlentheorie wichtige und vielleicht auch der Physik und Chemie einmal Nutzen bringende Frage hin, wie man unendlich viele Körper von der gleichen vorgeschriebenen Gestalt, etwa Kugeln mit gegebenem Radius oder reguläre Tetraeter mit gegebener Kante (bez. in vorgeschriebener Stellung) im Raume am dichtesten einbetten, d.h. so lagern kann, daß das Verhältnis des erfüllten Raumes zum nichterfüllten Raume möglichst groß ausfällt.

Here we present a theorem that can be used to find upper bounds for the maximum density of a packing of any given compact convex body $\mathcal{K}$. The theorem relies on the harmonic analysis of the Euclidean motion group which is a non-compact and non-commutative group. It generalizes a theorem of Cohn and Elkies [1] that results the best known upper bounds for the densities of sphere packings in dimensions 4 to 36.

By $M(n)$ we denote the Euclidean motion group in dimension n which is generated by translations and rotations. It is the semidirect product of the non-compact translation group $\mathbb{R}^n$ and the non-commutative rotation group $\mathrm{SO}(n)$. We can model the body packing problem as the problem of determining the largest size of an independent set in the infinite graph $G = (M(n), E)$ where two vertices $(x, A), (y, B) \in \mathbb{R}^n \times \mathrm{SO(n)}$ are connected whenever the intersection of the $x + A\mathcal{K}^o$ and $y + B\mathcal{K}^o$ is not empty. Here $\mathcal{K}^o$ denotes the topological interior of $\mathcal{K}$.

In general, finding the independence number of a finite graph is a computational difficult problem. However, there are techniques using spectral properties of the graph and using convex optimization, going back to Hoffman [3], Delsarte [2], Lovász [4], which give upper bounds for it. These techniques can be transferred to infinite graphs:

Main Theorem. *Let $\mathcal{P}$ be a packing of congruent copies of a compact, convex body $\mathcal{K} \subseteq \mathbb{R}^n$. Let $f \in L^1(M(n))$ be a continuous function such that the conditions*

(1) *f is of positive type: for all $N \in \mathbb{N}$ and for all $(x_1, A_1), \ldots, (x_N, A_N) \in M(n)$ the matrix*

$$\left(f((x_i, A_i)(x_j, A_j)^{-1})\right)_{1 \le i,j \le N}$$

is positive semidefinite.

(2) *$f(x, A) \le 0$ whenever $\mathcal{K}^o \cap (x + A\mathcal{K}^o) = \emptyset$,*

(3) *$\lambda = \int_{M(n)} f(x, A) d(x, A) > 0$*

are fulfilled. Then, the density of $\mathcal{P}$ is bounded above by

$$\frac{f(0, I)}{\lambda} \mathrm{vol}\, \mathcal{K}.$$

In order to find a function f which proves a "good" upper bound, we make use of convex optimization (semidefinite programming) and harmonic analysis. For instance, if $n = 2$ we can parametrize the functions using their Fourier expansion, which is due to Vilenkin [5], as follows:

$$\begin{aligned} & f\left(\begin{pmatrix} a\cos\phi \\ a\sin\phi \end{pmatrix}, \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right) \\ &= \int_0^\infty \sum_{r,s=-\infty}^{\infty} \widehat{f}_{rs}(p) i^{s-r} e^{-i(s\theta + (r-s)\phi)} J_{s-r}(pa) p dp, \end{aligned}$$

where J_α is the Bessel function of the first kind of order α.

The fact that f is of positive type is reflected in the condition that for every $p \ge 0$ the infinite matrix $\left(\widehat{f}_{rs}(p)\right)_{r,s=-\infty}^{\infty}$ is positive semidefinite. Conditions (2) and (3) from the main theorem are linear conditions in these matrix coefficients.

Hence, finding an optimal function can formulated as an infinite-dimensional semidefinite program. This can be approximated using a sequence of finite-dimensional semidefinite programs.

In the case when the compact convex body $\mathcal{K}$ is the unit ball, condition (2) of the main theorem only depends on the length a of the vector x so one can assume that the function f also only depends on a. So instead of using the infinite semidefinite matrix $\left(\widehat{f}_{rs}(p)\right)_{r,s=-\infty}^{\infty}$ it suffices to use only the non-negative central coefficient $\widehat{f}_{00}(p)$. Doing this — the same reasoning works for $n > 2$ — we arrive at the result of Cohn and Elkies.

For explicit numerical computations, which turn out to be different form those of Cohn and Elkies, we may use tools from polynomial optimization (sums of squares): For this we consider functions of the form

$$\widehat{f}_{00}(p) = \sum_{k=0}^{d} f_{2k} p^{2k} e^{-p^2}.$$

Then, by a classical formula of Hankel, cf. Watson [6, p. 393], we have

$$f(a) = \int_0^\infty \widehat{f}_{00}(p) J_0(pa) p dp = \sum_{k=0}^{d} f_{2k} \frac{k!}{2} L_k(a^2/4) e^{-a^2},$$

where L_k is the Laguerre polynomial of degree k. Now we are trying to find real coefficients $f_0, f_2, \dots, f_{2d}$ such that

(1) $\sum_{k=0}^{d} f_{2k} p^{2k} \geq 0$ for all $p \geq 0$.

(2) $\sum_{k=0}^{d} f_{2k} \frac{k!}{2} L_k(a^2/4) \leq 0$ if $a \geq 2$,

(3) $f_0 = 1$,

and such that $\sum_{k=0}^{d} f_{2k} \frac{k!}{2}$ is minimized.

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On transversals of quasialgebraic families of sets

Vladimir Dolnikov
(joint work with Grigory Chelnokov)

We consider the Helly–Gallai numbers for families of sets that are similar to families of algebraic sets.

Definition 1. *A set X, $|X| \leq t$, is called a t-*transversal *for a family of sets F if $A \cap X \neq \emptyset$ for every $A \in F$.*

Definition 2. *The* Helly–Gallai number *$HG(t, F)$ of a family of sets F is called a minimal natural number k, such that if every subfamily $P \subseteq F$ with $|P| \leq k$ has a t-transversal, then the family F has a t-transversal. We put $HG(t, F) = \infty$ if such natural number k does not exists.*

Definition 3. *For every family F, the existence of a 1-transversal is equivalent to the condition that the intersection of all sets of F is nonempty. Therefore a number $HG(1, F)$ is called the* Helly number *for a family F.*

Remark. If F is a family of a convex compact sets in $\mathbb{R}^d$, $d \geq 2$, and $t \geq 2$ then $HG(t, F) = \infty$. If F is a family of intervals on the line then $HG(t, F) = t + 1$.

The Helly numbers $H(F)$ for a family of algebraic varieties were found by T. S. Motzkin.

Definition 4. *Let A^d_m be a family of sets of common zeroes in $\mathbb{R}^d$ for a finite collection of polynomials of d variables and degree not grater than m.*

Theorem 5 (Motzkin [5], 1955)**.**

$$H(A^d_m) = \binom{m+d}{d}.$$

The Helly–Gallai numbers for algebraic varieties A^d_n were determined by M. Deza and P. Frankl ([1], 1987), and Dol'nikov ([2], 1989). They are given by the formula:

$$HG(t, A^d_m) = \binom{\binom{m+d}{d} + t - 1}{t}.$$

In the papers [2, 3] the Helly–Gallai numbers for families of sets of more general kind were considered. More precisely, they were the zero sets of linear finite dimensional subspaces of functions on a ground set V with coefficients in a field F.

In particular, the Helly–Gallai numbers

$$HG(t, S_{d-1}) = \binom{d+t+1}{t}$$

for families of spheres S_{d-1} in $\mathbb{R}^d$ were found. Independently they were found by Maehara ([4], 1989).

Now we give some estimates for the Helly–Gallai numbers of quasialgebraic families of sets.

Definition 6. *Let F be a family of sets. Denote inductively*

$$F^1 = \{A_i \cap A_j : \textit{where}\, A_i, A_j \in F \textit{ and}\, A_i \neq A_j\}$$

and

$$F^{k+1} = \{A_i \cap A_j : \textit{where}\, A_i, A_j \in F^k \textit{ and}\, A_i \neq A_j\}.$$

Definition 7. *A family of sets F is called a (d, m)-*quasialgebraic family *or a* quasialgebraic family *of dimension m and degree d if $|B| \leq d$ for every $B \in F^m$.*

Denote the class of such families by QA^d_m.

A family of lines in $\mathbb{F}^d$, where $\mathbb{F}$ is a field, or a family of lines of a finite projective plain, or a family of edges of a graph G are quasialgebraic families of a dimension 1 and a degree 1. We also call such families linear families.

Elements of a linear family have many properties of usual lines. For example, the Sylvester–Gallai theorem is not true for linear families, but there we have a following result:

Theorem 8 (Bourbaki)**.** *Let F be a linear family. Suppose $\{x_1, x_2, \ldots x_n\}$ is a set such that for each two distinct points $\{x_i, x_j\}$ there exists a "line" $L_{i,j} \in F$ passing through $\{x_i, x_j\}$. Then the number of distinct such lines is at least n.*

A family of circles is a quasialgebraic family of a dimension 1 and a degree 2. More generally, the family A^d_m is a quasialgebraic family (*note that the numbers d and m are not the same*).

Let us give a lower bound for the Helly–Gallai number:

Theorem 9. *There exist quasialgebraic families $F \in QA^d_m$ such that*

$$HG(t, F) \geq \binom{d + m + t}{t}.$$

Sketch of the proof: Let F be a family of all $(d + m)$-element subsets of some $(d + m + t)$-element set. Obviously, the family F has (d, m)-property and has no t-transversal.

And the upper bound:

Theorem 10. *For any $F \in QA^d_m$:*

$$HG(t, F) \leq \frac{t^{d+m+1} - 1}{t - 1}.$$

The proof of the upper bound is less trivial. It based on an accurate investigation of properties of some critical configurations that are families F such that F doesn't have a t-transversal, but each subfamily of F has a t-transversal.

The lower estimate for $HG(t, F)$ is more natural than the upper estimate, which seems far from being tight.

Consider the case $F \in QA^1_1$ (a linear family). In this case we have more precise upper bound:

Theorem 11.
$$\binom{t+2}{2} \leq HG(t,F) \leq t^2+1.$$
If $1 \leq t \leq 4$ *then it can be improved to:*
$$HG(t,F) = \binom{t+2}{2}.$$

In this case we cannot improve the upper estimate even for lines of a finite projective plane.

Here we give another our result about sets behaving like circles:

Theorem 12.
$$HG(2, QA_1^d) = \binom{d+3}{2}.$$
if $1 \leq d \leq 5$.

Theorem 13.
$$HG(2, QA_1^d) \leq 2d^2+3.$$

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Kadets type theorems for partitions of a convex body

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(joint work with Roman Karasev)

Alfred Tarski [9] proved that for any covering of the unit disc by planks (the sets $a \leq \lambda(x) \leq b$ for a linear function λ and two reals $a < b$) the sum of plank widths is at least 2. Thøger Bang in [3] generalized this result for covering of a convex body B in $\mathbb{R}^d$ by planks showing that the sum of the widths is at least the width of B. He also posed the following question: Can the plank widths in the Euclidean metric be replaced by the widths relative to B (as in Definition 1 below)?

Keith Ball proved the conjecture of Bang in [2] for centrally symmetric bodies B or, in other words, for arbitrary normed spaces and coverings of the unit ball. For possibly non-symmetric B, it is known (see [1]) that the Bang conjecture is equivalent to the Davenport conjecture: If a convex body B is sliced by $n-1$ hyperplane cuts then there exists a piece that contains a translate of $\frac{1}{n}B$.

In [4, 5] András Bezdek and Károly Bezdek proved an analogue of the Davenport conjecture for binary partitions by hyperplanes. The difference is that they do not cut everything with every hyperplane; instead they divide one part into two parts and then proceed recursively.

The strongest possible result about coverings of a unit ball for the Hilbert (and finite dimensional Euclidean) space was proved by Vladimir Kadets in [8] (see also [6] for the proof in the two-dimensional case using the idea from [9]): For any convex covering $C_1, \dots, C_k$ of the unit ball the sum of inscribed ball radii $\sum_{i=1}^k r(C_i)$ is at least 1.

The reader is referred to [7] for a detailed historical survey on the Tarski plank problem.

In this work we prove analogues of the Kadets theorem for inscribing homothetic copies of a (not necessarily symmetric) convex body, replacing arbitrary coverings by certain convex partitions. By a *partition* of a convex set B we mean a covering of B by a family of closed convex sets with disjoint interiors. In the two-dimensional case the analogue of the Kadets theorem for possibly non-symmetric bodies (Theorem 2) holds for any partition, while in higher dimensions we need additional restrictions on the partition. In other words, we are solving positively certain particular cases of [7, Problem 7.2] about extending the Kadets theorem to Banach spaces.

Definition 1. *Let $B \subset \mathbb{R}^d$ be a convex body. For a convex set $C \subseteq \mathbb{R}^d$ define the analogue of the inscribed ball radius as follows:*

$$r_B(C) = \sup\{h \geq 0 : \exists t \in \mathbb{R}^d \text{ such that } hB + t \subseteq C\},$$

and put $r_B(C) = -\infty$ for empty C.

Theorem 2. *Let $B \subset \mathbb{R}^2$ be a convex body and let $C_1 \cup \dots \cup C_k = B$ be its convex partition. Then*

$$\sum_{i=1}^k r_B(C_i) \geq 1.$$

The proof is based on the following lemma

Lemma 3. *Any convex partition $C_1 \cup \dots \cup C_k = B \subset \mathbb{R}^2$ can be extended to a partition $V_1 \cup \dots \cup V_k = \mathbb{R}^2$.*

Using it we can prove the theorem.

Proof of Theorem 2. We extend the partition $C_1 \cup \dots \cup C_k = B$ to $V_1 \cup \dots \cup V_k = \mathbb{R}^2$ by Lemma 3. Then the function

$$r(y) = \sum_{i=1}^k r_B(B \cap (V_i + y))$$

is concave (since it sum of concave functions, which is minimum of linear functions), so by varying y we can make one of $B \cap (V_i + y)$ have empty interior without increasing $r(y)$. Then we omit V_i, obtain a partition of B into fewer parts, and use the inductive assumption. □

Since Lemma 3 does not hold in higher dimension we need certain restriction on the partitions.

Definition 4. *Call a convex partition* $V_1 \cup \cdots \cup V_k$ *of* $\mathbb{R}^d$ inductive *if for any* $1 \le i \le k$ *there exists an inductive partition* $W_1 \cup \cdots \cup W_{i-1} \cup W_{i+1} \cup \cdots \cup W_k$ *such that* $W_j \supseteq V_j$ *for any* $j \neq i$. *A partition into one part* $V_1 = \mathbb{R}^d$ *is assumed to be inductive.*

Definition 5. *Call a convex covering (by closed sets)* $V_1 \cup \cdots \cup V_k$ *of* $\mathbb{R}^d$ inductive *if for any* $1 \le i \le k$ *there exists an inductive covering* $W_1 \cup \cdots \cup W_{i-1} \cup W_{i+1} \cup \cdots \cup W_k$ *such that* $W_j \subseteq V_j \cup V_i$ *for any* $j \neq i$. *A covering by one set* $V_1 = \mathbb{R}^d$ *is assumed to be inductive.*

Theorem 6. *Let* $B \subset \mathbb{R}^d$ *be a convex body and let* $C_1 \cup \cdots \cup C_k = B$ *be induced from an inductive partition (or covering)* $V_1 \cup \cdots \cup V_k = \mathbb{R}^d$*; that is* $C_i = V_i \cap B$ *for any* i. *Then*

$$\sum_{i=1}^{k} r_B(C_i) \geq 1.$$

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Remark on Coresets for Minimum Enclosing Ellipsoids

Kenneth L. Clarkson

Let S be a set of n points in d dimensions. The Minimum Enclosing Ellipsoid $\mathbf{MEE}(S)$ is the unique ellipsoid of smallest volume that contains S. $\mathbf{MEE}(S)$ is determined by some $C \subset S$ of at most $d(d+3)/2$ points, called the *contact points* or *support points*.

A *coreset* of S for **MEE** is $S' \subset S$ such that $\mathbf{MEE}(S')$ is roughly the same as $\mathbf{MEE}(S)$. There are (at least) two varieties of such approximation, with corresponding coresets.

In one version, for given $\epsilon > 0$, an ϵ-coreset S' is such that $S \subset (1+\epsilon)\,\mathbf{MEE}(S')$. (Here the multiplication by $1+\epsilon$ denotes $c + (1+\epsilon)(\mathbf{MEE}(S') - c)$, where c is the center of $\mathbf{MEE}(S')$.) There are ϵ-coresets, in this sense, of size $O(d(\log d + 1/\epsilon))$ [2, 3]. Such coresets amount to a combinatorial specification of an approximate solution to the **MEE** problem; they give an *approximately minimum* ellipsoid.

Another kind of approximation, and ϵ-coreset, is ellipsoid $\mathcal{E}$ such that $(1-\epsilon)\mathcal{E} \subset \mathbf{MEE}(S) \subset (1+\epsilon)\mathcal{E}$. Call such an ellipsoid *approximate to minimum*. Rudelson showed that there is such an ellipsoid $\mathcal{E}$, with $O(d(\log d)/\epsilon^2)$ contact points, each of which is a small perturbation of a point of S [5]. Recently Naor has observed that the number of contact points can be reduced to $O(d/\epsilon^2)$, using results of Batson et al. [4, 1].

Approximate-to-minimum ellipsoids require more contact points, which is not surprising. Also, their contact points are not a subset of S, and the above constructions by make use of $\mathbf{MEE}(S)$. In light of approximately minimum coresets, it is natural to ask if it is possible to specify approximate-to-minimum ellipsoids using a small subset S'' of S, and to have the associated ellipsoid be $\mathbf{MEE}(S'')$. I can answer the first question affirmatively: there is a subset of size $O(d/\epsilon^2)$, such that an appropropriate linear algebraic construction using that subset yields an approximate-to-minimum ellipsoid. (I don't know the answer to the second question.)

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Dense favourite-distance digraphs

Konrad J. Swanepoel

The following is an extended abstract of [8] and [9]. Let S be a set of n points in the d-dimensional Euclidean space $\mathbb{R}^d$. Let $r\colon S \to (0,\infty)$ be a choice of a positive number for each point in S. Define the *favourite distance digraph on S determined by r* to be the directed graph $\vec{G}_r(S) = (S, \vec{E}_r(S))$ on the set S where

$$\vec{E}_r(S) := \{(\boldsymbol{x},\boldsymbol{y}) : \boldsymbol{x},\boldsymbol{y} \in S \text{ and } |\boldsymbol{xy}| = r(\boldsymbol{x})\}.$$

Here $|\boldsymbol{xy}|$ denotes the Euclidean distance between $\boldsymbol{x}$ and $\boldsymbol{y}$. Write $e_r(S) := |\vec{E}_r(S)|$. Define

$$f_d(n) := \max\left\{e_r(S) : S \subset \mathbb{R}^d, |S| = n \text{ and } r\colon S \to (0,\infty)\right\}.$$

The problem of determining $f_d(n)$ was originally introduced by Avis, Erdős and Pach [1]. They determined $f_3(n)$ asymptotically:

Theorem 1 (Avis–Erdős–Pach 1990). *$f_3(n) = \frac{n^2}{4} + O(n^{2-\varepsilon})$ for some $\varepsilon > 0$.*

We improve the asymptotics as follows:

Theorem A. *For sufficiently large n, $\frac{n^2}{4} + \frac{5n}{2} - 6 \leq f_3(n) \leq \frac{n^2}{4} + \frac{5n}{2} + 6$.*

This theorem follows from the following structural result and an improved construction. A finite set of points in $\mathbb{R}^3$ is called a *suspension* if it is contained in some circle and its axis of symmetry.

Theorem B. *For any $S \subset \mathbb{R}^3$ with $|S| = n$ sufficiently large and $r\colon S \to (0,\infty)$, if $|\vec{E}(S,r)| = f_3(n)$, then for some $T \subseteq S$ with $|T| \leq 2$, $S \setminus T$ is a suspension and, writing C for the circle of the suspension and ℓ for its axis of symmetry, $r(\boldsymbol{x})$ equals the distance from $\boldsymbol{x}$ to C for each $\boldsymbol{x} \in S \cap \ell$.*

We conjecture that the exceptional set T is empty.

Avis, Erdős and Pach also determined $f_d(n)$ asymptotically for even $d \geq 4$. Erdős and Pach [6] finished off the case of odd $d \geq 5$.

Theorem 2 (Avis–Erdős–Pach [1], Erdős–Pach [6]). *For any $d \geq 4$,*

$$f_d(n) = \left(1 - \frac{1}{\lfloor d/2 \rfloor} + o(1)\right) n^2.$$

For any set $S \subset \mathbb{R}^d$ of n points, let

$$u(S) := |\{\{\boldsymbol{x},\boldsymbol{y}\} \,:\, \boldsymbol{x},\boldsymbol{y} \in S \text{ and } |\boldsymbol{x}\boldsymbol{y}| = 1\}|$$

and set

$$u_d(n) := \max\left\{u(S) \,:\, S \subset \mathbb{R}^d \text{ and } |S| = n\right\}.$$

Clearly $f_d(n) \geq 2u_d(n)$. We give a simple derivation of an asymptotic upper bound for $f_d(n)$ (Theorem C below) using only the analogous upper bounds for $u_d(n)$ stated in the following theorem.

Theorem 3 (Erdős [5], Erdős–Pach [6]). *There exist constants $c_1, c_2 > 0$ such that for each $d \geq 4$ and all $n \in \mathbb{N}$,*

$$u_d(n) \leq \frac{1}{2}\left(1 - \frac{1}{\lfloor d/2 \rfloor}\right) n^2 + \begin{cases} c_1 n & \text{if } d \text{ is even,} \\ c_2 \left(\dfrac{n}{d}\right)^{4/3} & \text{if } d \text{ is odd.} \end{cases}$$

The above bounds are tight up to the values of c_1 and c_2 [6]. Since $f_d(n) \geq 2u_d(n)$, the bounds in the next theorem are also tight up to the values of the constants.

Theorem C. *With the same constants $c_1, c_2 > 0$ as in Theorem 3, for each $d \geq 4$ and all $n \in \mathbb{N}$,*

$$f_d(n) \leq \left(1 - \frac{1}{\lfloor d/2 \rfloor}\right) n^2 + \begin{cases} 2c_1 n & \text{if } d \text{ is even,} \\ 2c_2 \left(\dfrac{n}{d}\right)^{4/3} & \text{if } d \text{ is odd.} \end{cases}$$

By a *Lenz configuration for distance* $\lambda > 0$ we mean a finite set of the following type [4, 2]. For any $d \geq 4$, let $p = \lfloor d/2 \rfloor$ and consider any orthogonal decomposition $\mathbb{R}^d = V_1 \oplus \cdots \oplus V_p$ with all V_i 2-dimensional, except that V_1 is 3-dimensional when d is odd. In each V_i, let C_i be the circle (Σ_1 the sphere if $i = 1$ and d is odd) with centre the origin $\boldsymbol{o}$ and radius r_i, such that $r_i^2 + r_j^2 = \lambda^2$ for all distinct i and j. When $d \geq 6$ this implies that each $r_i = \lambda/\sqrt{2}$. Define a *Lenz configuration for the distance* λ to be any finite subset S of some translate $\boldsymbol{v} + \bigcup_{i=1}^p C_i$ when d is even, or of $\boldsymbol{v} + \Sigma_1 \cup \bigcup_{i=2}^p C_i$ when d is odd. The *partition associated to the Lenz configuration* S is the partition induced by the circles, i.e. the p subsets $S_1, \ldots, S_p$ where $S_i = S \cap (\boldsymbol{v} + C_i)$ ($S_1 = S \cap (\boldsymbol{v} + \Sigma_1)$ if $i = 1$ and d is odd).

The following theorem states that the extremal sets for unit distances are Lenz configurations, at least for a sufficiently large number of points.

Theorem 4 ([2, 7])**.** *Let $d \geq 4$ and let $S \subseteq \mathbb{R}^d$ be given with $|S| = n$ sufficiently large, depending on d, and such that $u(S) = u_d(n)$. Then S is a Lenz configuration for the distance* 1.

We show that when $d \geq 4$, the extremal favourite distance digraphs are exactly the same as the sets for which $u_d(n)$ is maximised, for all sufficiently large n, depending on d, except when $d = 4$, where there is an exceptional construction for all sufficiently large $n \equiv 1 \pmod 8$.

Theorem D. *Let $d \geq 4$ and let $S \subset \mathbb{R}^d$ and a function $r\colon S \to (0, \infty)$ be given such that $|S| = n$ is sufficiently large depending on d, and $e_r(S) = f_d(n)$. Then $r \equiv c$ for some $c > 0$, and S is a Lenz configuration for the distance c, except when $d = 4$ and $n - 1$ is divisible by* 8*, where the following situation is also possible: for some $a \in S$ and $c > 0$, $S \setminus \{a\}$ is a Lenz configuration for the distance c on two circles C_1 and C_2 of equal radius $c/\sqrt{2}$, a is the common centre of the two circles, $C_i \cap S$ consists of the vertices of $(n-1)/8$ squares inscribed in C_i $(i = 1, 2)$, and $r|_{S\setminus\{a\}} \equiv c$, $r(a) = c/\sqrt{2}$. In particular, $f_d(n) = 2u_d(n)$ for all $d \geq 4$ and $n \geq n_0(d)$.*

The proof of the above theorem needs a stability result. The following theorem is a known stability result for unit distances.

Theorem 5 ([7])**.** *For any $d \geq 4$, set $p = \lfloor d/2 \rfloor$. Then for any set $S \subset \mathbb{R}^d$ with $|S| = n$ sufficiently large that satisfies*

$$u(S) > \frac{1}{2}\left(1 - \frac{1}{\lfloor d/2 \rfloor} - o(1)\right) n^2,$$

there exists a subset $T \subseteq S$ such that $|T| = o(n)$ and $S \setminus T$ is a Lenz configuration. Furthermore, the partition $S_1, \ldots, S_p$ of $S \setminus T$ associated to the Lenz configuration satisfies $|S_i| = \frac{n}{p} + o(n)$ for all $i \in [p]$.

The next theorem is an analogue of the above theorem for favourite distance digraphs.

Theorem E. *For any $d \geq 4$, set $p = \lfloor d/2 \rfloor$. Then for any $S \subset \mathbb{R}^d$ with $|S| = n$ sufficiently large and any $r \colon S \to (0, \infty)$ that satisfy*

$$e_r(S) > \left(1 - \frac{1}{\lfloor d/2 \rfloor} - o(1)\right) n^2,$$

there exist $T \subseteq S$ and $c > 0$ such that $|T| = o(n)$, $S \setminus T$ is a Lenz configuration with distance c, and $r|_{S \setminus T} \equiv c$. Furthermore, the partition $S_1, \ldots, S_p$ of $S \setminus T$ associated to the Lenz configuration satisfies $|S_i| = \frac{n}{p} + o(n)$ for all $i \in [p]$.

By applying Theorem 5, the above theorem is relatively easy to prove for $d \geq 6$, but surprisingly, takes some work in the cases $d \in \{4, 5\}$. This is not so much because the Lenz construction is slightly more complicated in dimensions 4 and 5, but rather due to certain complications in the extremal theory of digraphs not shared by the extremal theory of ordinary graphs [3].

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Open Problems in Discrete Geometry

Collected by Dömötör Pálvölgyi

Problem 1 (Hiroshi Maehara)**.** A lattice L in a Euclidean space is called an integral lattice if the inner product $< x, y >$ is an integer for every $x, y \in L$.

Is every integral lattice L congruent to a sub-lattice of $\mathbb{Z}^n$ for some n?

Remark. 1. If the dimension of L is at most 3, then this is true. (I could prove that every 3-dimensional integral lattice is congruent to a sub-lattice of $\mathbb{Z}^{24}$.)

2. It is known that if L is an n-dimensional integral lattice, then L is congruent to a subset of Q^{n+3}, and hence L is similar to a sub-lattice of $\mathbb{Z}^{n+3}$.

Problem 2 (Günter M. Ziegler)**.** *The space of convex n-partitions of 3-space.*

What is the dimension of the space of all partitions of $\mathbb{R}^3$ into $n > 3$ convex pieces?

There are only finitely many combinatorial types of such partitions, so the question is for the maximal dimension of the realization space of such a combinatorial type.

For dimension $d = 2$, one can work out that the space of partitions of the plane into n convex polygons has dimension $4n-1$: This is achieved for a simple partition (all vertices of degree 3) with only 3 unbounded rays. Note that the space of power diagrams (also known as generalized Voronoi diagrams) has dimension only $3n-1$, as one has two coordinates for each site and one weight for each site, while the sum of weights can be normalized to be 0.

In dimension 3, all simple partitions are power diagrams, according to Whiteley (see Rybnikov [1]). The dimension of the space of convex n-partitions is at least the dimension of the space of power diagrams, which is $4n-1$, but it is not clear whether it is larger.

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Problem 3 (Günter Rote)**.** Is there a constant c such that, for every $n \geq 3$, there is a partition of the plane into convex regions whose boundaries form a connected graph with n vertices and 3 unbounded rays, such that every monotone path in the graph consists of at most c edges? Perhaps even with every region bounded by only 3 edges? A path is monotone if there is some linear function $\mathbb{R}^2 \to \mathbb{R}$ which is strictly monotone along the path. Without the bound on the number of unbounded rays, the statement is true with $c = 9$, even with triangular faces.

Note. Csaba D. Tóth and Adrian Dumitrescu have shown that for every convex subdivision of the plane into n convex cell, 3 of which are unbounded, there is a monotone path with $\Omega(\log n/\log\log n)$ vertices and this bound is best possible.

Problem 4 (Boris Bukh)**.** Is it true that for every finite set P of points in general position in $\mathbb{R}^d$ there are convex sets $C_1, \dots, C_k$ such that $|C \cap P| >= |P|/\mathrm{poly}(d)$ and every hyperplane H in $\mathbb{R}^d$ misses at least one of the $C_1, \dots, C_k$.

A theorem of Yao–Yao asserts that one can partition the space into 2^d convex cones, each containing approximately the same number of points. It is not known how to replace 2^d even by $2^d - 1$. No superlinear lower bound is known.

Problem 5 (Luis Montejano)**.** Let F be a set of points in $\mathbb{R}^d$ painted with colors $C = \{1, ...k\}$ (one point may be painted with several colors) and if β is contained in C, let $F\beta$ be the set of points painted with colors in β. I can prove that there is a function $f(.,.)$ such that if for every subset of colors β of size n, there is a subset of $F\beta$ in general position of size greater than $f(n,d)$, then there is a rainbow subset of F in general position of size k.

If $d = 1$, then $f(n, 1) = n$ by the Hall's Theorem for bipartite graph.

PROBLEM: give good bounds for $f(n, d)$.

Problem 6 (Zoltán Füredi)**.** A version of coin-weighting problems proposed by ApSimon [1], discussed also in [2] can be reformulated as follows.

Consider a set V of n non-zero vectors on the plane with non-negative integer coordinates. Let $S(V)$ be the set of the $2^n - 1$ non-empty subset sums. We are looking for the smallest $N = N(n)$ such that V is a subset of $0, 1, 2, 3, ..., N \times 0, 1, 2, ..., N$ and the *slopes* of the members of S are all distinct. Because they formulate the problem differently, they only show a lower bound $\Omega(n^c)$ and upper bound $O(n!)$ and ask if $N(n) = O(2^n)$. From our reformulation it is clear that

$$N(n) > 2^{n/2},$$

and a greedy algorithm gives $N(n) = O(2^{3n/2})$, and a random construction gives

$$N(n) < 2^n\sqrt{n}.$$

There are many question. Tighten these bounds. Find the limit $N(n)^{1/n}$, first prove it exists. Find higher dimensional generalizations.

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Problem 7 (József Solymosi)**.** Conjecture: There is a constant $c > 0$ and a threshold n_0 such that if the number of incidences between $n > n_0$ lines and n points in the plane is at least $n^{\frac{4}{3}-c}$ then there is triangle in the arrangement.

All I can prove is that if there is no triangle in the arrangement then the number of incidences is $o(n^{\frac{4}{3}})$ [1].

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Problem 8 (Volodya Dol'nikov)**.** Definition 1. A set X, $|X| \leq t$, is called a *t-transversal* of a family of sets F if $A \cap X \neq \emptyset$ for every $A \in F$.

1. *"Colored" Grünbaum's problem.*

Suppose F_1, F_2, F_3 are families of translates of a convex compact set A in plane, and $A_i \cap A_j \neq \emptyset$ for each $A_i \in F_i$, $A_j \in F_j$, $i \neq j$.

Is it true that some family F_i has 3-transveral?

Definition 2. A family of sets has a (p, q)-property if among every $A_1, A_2, \ldots, A_p \in F$ there exist q sets with a common point.

2. *Problem of Hadwiger–Debrunner type*

Let F be a family of hyperplanes in $\mathbb{R}^d$ with $(p, q)-$property, $d + 1 \leq q \leq p$.

Is it true that the family F has $(p-q+1)$-transversal?

2'. Problem 2' is a corollary of Problem 2.

Let F be a family of $r-$element sets with (p,q)-property, $r+1 \leq q \leq p$.

Is it true that the family F has $(p-q+1)$-transversal?

Problem 9 (Imre Bárány)**.** Let X be a finite point set in the plane with no three points collinear. The *degree* of a pair $x, y \in X$ is, by definition, the number of $z \in X$ such that the triangle with vertices x, y, z contains no further point from X. Let $d(X)$ denote the maximal degree of a pair in X. Show that $d(X)$ goes to infinity as $|X| \to \infty$.

It is known that the average degree of a pair is at least six, and it is shown in [1] that $d(X) \geq 10$ for large enough $|X|$, but that is far from infinity.

This question appeared in a paper by Paul Erdős [2], and in [1] as well.

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Problem 10 (Dömötör Pálvölgyi)**.** For $A \subset [n]$ denote by a_i the i^{th} smallest element of A.

For two k-element sets, $A, B \subset [n]$, we say that $A \leq B$ if $a_i \leq b_i$ for every i.

A k-uniform hypergraph $\mathcal{H} \subset [n]$ is called a *shift-chain* if for any hyperedges, $A, B \in \mathcal{H}$, we have $A \leq B$ or $B \leq A$. (So a shift-chain has at most $k(n-k)+1$ hyperedges.)

We say that a hypergraph $\mathcal{H}$ is 2-colorable if we can color its vertices with two colors such that no hyperedge is monochromatic.

Is it true that shift-chains are 2-colorable if k is large enough?

Remarks. For $k = 2$ there is a trivial counterexample: (12),(13),(23).

A very magical counterexample was found for $k = 3$ by Radoslav Fulek with a computer program:

(123),(124),(125),(135),(145),(245),(345),(346),(347),(357),
(367),(467),(567),(568),(569),(579),(589),(689),(789).

If we allow the hypergraph to be the union of two shift-chains (with the same order), then there is a counterexample for any k.

Problem 11 (Gábor Fejes Tóth)**.** Is it true that there exist a constant c such that for all coverings of the plane by closed unit circular discs any two points situated at distance d from one another and covered by at least two discs can be connected by a path of length $\sqrt{2}d + c$ traveling within at least doubly covered part of the plane?

The arrangement of discs in a square grid shows that the factor $\sqrt{2}$ cannot be improved. E. Roldán-Pensado showed that two points at distance d apart lying in at least doubly covered part of the plane can be connected by a path remaining in the at least doubly covered part of the plane whose length is at most $(\pi/3 + \sqrt{(3)})d + c$ with some constant $c < 17$.

There are two special types of path in the region covered at least twice by discs: one that uses only boundary arcs of the discs, and another type that travels along the sides of the Dirichlet cells. This gives rise to the following stronger conjectures.

There is a constat c_1 such that if closed unit circular discs cover the plane and a and b are two points at distance d apart, both lying on the boundary of some of the discs, then a and b can be connected by a path whose length is at most $\pi d/2 + c_1$ and which uses only boundary arcs of the discs.

There is a constat c_2 such that if closed unit circular discs cover the plane and a and b are two points at distance d apart, both lying on the boundary of the Dirichlet cell of some of the discs, then a and b can be connected by a path whose length is at most $\sqrt{2}d + c_1$ and which uses only boundary arcs of the Dirichlet cells.

In [1] the second conjecture is confirmed for lattice-coverings.

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Problem 12 (Géza Tóth)**.** The *monotone crossing number* of G is defined as the smallest number of crossing points in a drawing of G in the plane, where every edge is represented by an x-monotone curve, that is, by a connected continuous arc with the property that every vertical line intersects it in at most one point. This parameter can be strictly larger than the classical crossing number $\text{CR}(G)$, namely, there are graphs G_k with $\text{CR}(G_k) = 6k+6$ and $\text{MON-CR}(G) = 7k+6$. On the other hand, $\text{MON-CR}(G) < 2\text{CR}^2(G)$ holds for every graph G.

Is there a constant c such that $\text{MON-CR}(G) < c\text{CR}(G)$ holds for every G?

Problem 13 (Csaba D. Tóth)**.** Let $f(n)$ be the maximum integer such that every 3-connected cubic planar graph with n vertices contains a simple cycle of length at least $f(n)$. Find the order of magnitude of $f(n)$.

It is known that $f(n) \in \Omega(n^{\log_3 2})$. In fact, Chen and Yu (2002) proved that every 3-connected planar graph with n vertices (and no degree constraints) contains a simple cycle of length $\Omega(n^{\log_3 2})$. It is also known that $f(n) \in O(n^{\log_{38} 37})$, which follows from a recursive construction based on a counterexample to Tait's conjecture.

Problem 14 (Peter Braß)**.** Given a curve γ in the plane and a small positive number r, let $f_t(r,\gamma)$ denote the maximum density of a translative packing of the Minkowski-sum of γ and a disc of radius r, and $f_c(r,\gamma)$ the same for congruent copies. This corresponds to arranging copies of γ as densely as possible, such that any two are separated by a distance at least $2r$. I want to study the asymptotic behavior for r small. Clearly $\Omega(r) \leq f(r,\gamma) \leq 1$, and all these cases can occur: if γ_0 is the three-quarter square, γ_1 the semicircle, and γ_2 the V shape, then

$f_t(r, \gamma_0) = \Theta(r)$, $f_t(r, \gamma_1) = \Theta(\sqrt{r})$ and $f_t(r, \gamma_2) = \Theta(1)$. Also $f_c(r, \gamma_0) = \Theta(r)$ and $f_c(r, \gamma_2) = \Theta(1)$. Is it true that $f_c(r, \gamma_1) = \Theta(\sqrt{r})$?

Problem 15 (Jan Kynčl)**.** Is there a constant c such that for every simplicial 4-polytope P there is a coloring of its vertex set $V(P)$ with c colors such that no 2-face of P is monochromatic?

Remark 1. the simplex and the cyclic polytopes need only 3 colors. No example of a polytope which requires more than 3 colors is known.

Remark 2. there exist triangulations of $\mathbb{R}^3$ where arbitrarily many colors are needed.

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Problem 16 (Günter Rote)**.** Is there, for every c, a locally finite (perhaps even triply periodic) partition of 3-space into convex cells, such that every cell is incident to at least c vertices, and every vertex is incident to at least c cells?

A weaker requirement would be that these incidence counts hold on average: the number of incidences between vertices and cells is at least c times the combined number of vertices and cells. (This problem goes back to Günter M. Ziegler.)

Problem 17 (Ken Clarkson)**.** For a subset S of $\mathbb{R}^d$, let $\mathbf{MEE}(S)$ denote the ellipsoid of minimum volume that contains S and let D_{BM} denote the Banach–Mazur distance. For $\epsilon > 0$, let $f(d, \epsilon)$ denote the supremum, over all S, of the minimum cardinality $X \subset S$ such that $D_{BM}(\mathbf{MEE}(X), \mathbf{MEE}(S)) \leq \epsilon$. Is $f(d, \epsilon) = O(d/\epsilon^2)$?

Background: Since $f(d, 0) = \Theta(d^2)$, the interest is in trading a factor of d for some function of ϵ. Mark Rudelson showed that there is an $X \subset S$ of size $O(d(\log d)/\epsilon^2)$ such that there is a small perturbation $\widetilde{X}$ of X such that $D_{BM}(\mathbf{MEE}(\widetilde{X}), \mathbf{MEE}(S)) \leq \epsilon$, Asaf Naor discussed the application of the machinery of Batson, Spielman, and Srivasta to remove the factor of $\log(d)$. It is not hard to show, based on that prior work, that there is an X of size $O(d/\epsilon^2)$ such that an ellipsoid $E(X)$ determined by X so that $D_{BM}(E(X), \mathbf{MEE}(S)) < \epsilon$, but $E(X)$ is not $\mathbf{MEE}(X)$.

This problem is spiritually related to coresets for $\mathbf{MEE}$, where an ϵ-coreset X has S a subset of $(1 + \epsilon) \cdot \mathbf{MEE}(X)$. There are ϵ-coresets of size $O(d/\epsilon)$, as shown by Kumar and Yildirim, adapting an algorithm of Khachian that is an adaptation of the Frank–Wolfe technique.

Problem 18 (Gábor Tardos)**.**

Question 1. For any finite set S of points in the plane one can find another set H of cardinality at most $|S|/2$ such that any axis parallel rectangle contains a point of H or contains at most 1000 points of S.

Question 2. Q1 fails badly for almost all S. More concretely, for a uniform random set S of n points in the unit square with high probability the following holds. For any set H of at most $n/2$ points there exists an axis parallel rectangle containing no points from H and $\Omega(\log \log n)$ points of S.

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