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The Analytic Theory of Automorphic Forms

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Abstract. The workshop brought together leading experts in the theory of automorphic forms. Recently developed analytic, geometric and ergodic methods have enabled several important results in the theory for $GL(2)$, and also made it possible to investigate groups of higher rank. This includes in particular recent work on periods of automorphic forms, subconvexity bounds for automorphic L-functions, and equidistribution results on homogenous spaces.

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Introduction by the Organisers

Automorphic forms are a very interdisciplinary topic in modern mathematics at the interface of number theory, analysis, representation theory and algebraic geometry. Among these different view points, the workshop focused in particular on the analytic theory of automorphic forms and their associated L-functions. There has been much progress in the analytic theory of automorphic forms in the past years. In particular, new and powerful techniques have been and are being developed that not only substantially enrich the $GL(2)$ theory, but make it possible to consider more general Lie groups of higher rank, such as $GL(n)$ or Sp_4. Forty-four leading experts in the field came together to exchange ideas, present newly developed methods and start or continue their collaboration on projects related to the subject of the workshop. The programme included 24 talks all of which presented very substantial new results. We highlight a few general principles:

Period bounds for automorphic forms, and equidistribution: If $Y \subset \Gamma\backslash G$ is a subset equipped with a natural measure and f is a function on $\Gamma\backslash G$, one is interested in bounds for the periods of f along Y. Typically such bounds reflect certain equidistribution properties and, equally important, can be translated into deep statements of L-functions. Classical examples (on the homogeneous space $\Gamma\backslash\mathbb{H}$) include integrals of f over the vertical non-compact geodesic yielding standard L-functions; over closed geodesics yielding square-roots of base-change L-functions (by Waldspurger's theorem); and over closed horocycles yielding weighted Fourier coefficients. Coupled with standard methods from harmonic analysis and possibly ergodic theory, such period formulas belong currently to the most powerful tools in the theory.

Multiple Dirichlet series: It has recently been observed that the theory of multiple Dirichlet series in connection with the theory of several complex variables leads to very precise moment calculations of families of L-functions that the classical methods do not seem to be capable of. The key point here is a non-trivial group of functional equations that gives rise to a finer analysis and a number of unexpected results. Moreover, it is conjectured that Weyl group multiple Dirichlet series are in fact Whittaker coefficients of Eisenstein series on a certain metaplectic cover which establishes a direct connection to automorphic forms.

Ergodic methods: The introduction of ergodic theory into the theory of numbers goes back at least to Linnik; it became very attractive when Ratner's classication and equidistribution results came out, but has developed its full strength only recently. If it is possible to reformulate number theoretical questions in the language of dynamical systems, then one can hope to obtain results in a great degree of generality. One of the most famous applications in automorphic forms is "quantum unique ergodicity", motivated by theoretical physics. On the Riemann surface $\Gamma\backslash\mathbb{H}$, one can interpret the Laplacian as the quantization of the Hamiltonian that generates the geodesic flow. In this formulation, automorphic forms as eigenfunctions of the Laplacian play the role of eigenstates of a particle. It is one of the central questions if (or under what conditions) automorphic forms in the semiclassical limit $t_j \to \infty$ behave like random waves, or if they display some structure.

An unexpected, but very interesting application of the general theory of Fuchsian groups and their sub-semigroups included an almost complete solution of Zaremba's conjecture, a 40-year-old unsolved problem in elementary number theory that claims that for every integer q there is a coprime number p such that the continued fraction expansion p/q has uniformly bounded denominators.

As fruitful as the talks presented at this workshop were informal discussion after lunch, after dinner and at the traditional hike that initiated several new exciting projects. Thursday evening featured a lively problem session in which 7 open problems were posed and thoroughly discussed.

Workshop: The Analytic Theory of Automorphic Forms

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Abstracts

Growth and non-vanishing of Siegel modular forms arising as Saito-Kurokawa lifts

Matthew Young

Suppose that $F : \mathcal{H}_2 \to \mathbb{C}$ is a Siegel modular form of weight k for $Sp_4(\mathbb{Z})$, where Siegel's upper half space $\mathcal{H}_2$ is defined by

$$\mathcal{H}_2 = \{Z \in Mat_{2\times 2}(\mathbb{C}) : Z = Z^t \text{ and } \mathrm{Im}(Z) \text{ is positive definite}\}. \tag{1}$$

In coordinates, say $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$. Recall that $Sp_4(\mathbb{Z})$ acts on $\mathcal{H}_2$ via $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $\gamma(Z) = (AZ + B)(CZ + D)^{-1}$. Then F is a Siegel modular form of weight k if $F(\gamma(Z)) = \det(CZ + D)^k F(Z)$ for all $\gamma \in Sp_4(\mathbb{Z})$ and $Z \in \mathcal{H}_2$ and if F is holomorphic in τ, z, τ'.

Here we discuss the behavior of Siegel modular forms when restricted to $z = 0$. One can easily check that $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ embeds into $Sp_4(\mathbb{Z})$ as follows. Say $\alpha, \alpha' \in SL_2(\mathbb{Z})$ with $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\alpha' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, then

$$\gamma = \begin{pmatrix} a & & b & \\ & a' & & b' \\ c & & d & \\ & c' & & d' \end{pmatrix} \in Sp_4(\mathbb{Z}); \tag{2}$$

furthermore, for $Z = \begin{pmatrix} \tau & 0 \\ 0 & \tau' \end{pmatrix}$, we have $\gamma(Z) = \begin{pmatrix} \frac{a\tau+b}{c\tau+d} & 0 \\ 0 & \frac{a'\tau'+b'}{c'\tau'+d'} \end{pmatrix}$, and hence $F\begin{pmatrix} \tau & 0 \\ 0 & \tau' \end{pmatrix}$ is a modular form in τ and in τ'. It is an interesting question to understand how this restricted function $F|_{z=0}$ behaves. In particular, we wish to understand when $F|_{z=0}$ vanishes identically, and also to compare the Petersson L^2 norm of $F|_{z=0}$ on $SL_2(\mathbb{Z})\backslash\mathbb{H} \times SL_2(\mathbb{Z})\backslash\mathbb{H}$ to the Petersson norm of F on $Sp_4(\mathbb{Z})\backslash\mathcal{H}_2$. This measures in some sense how concentrated F is along $F|_{z=0}$.

By linear algebra, one can write $F\begin{pmatrix} \tau & 0 \\ 0 & \tau' \end{pmatrix} = \sum_{g_1,g_2} c_{g_1,g_2} g_1(\tau) g_2(\tau')$, where g_1 and g_2 range over an orthonormal basis of weight k modular forms and c_{g_1,g_2} is the projection of F onto $g_1 \times g_2$, with respect to the Petersson inner product. The size and nonvanishing properties of $F|_{z=0}$ are then encoded by the behavior of these coefficients. A beautiful result of Ichino [3] gives a formula for $|c_{g,g}|^2$ as a central value of a $GL_3 \times GL_2$ L-function in the case that F arises as a Saito-Kurokawa lift of a Hecke eigenform, and when g is a Hecke eigenform. Under these assumptions, it is not hard to show that $c_{g_1,g_2} = 0$ unless $g_1 = g_2$.

To state Ichino's formula we need to review some theory of Saito-Kurokawa lifts. In brief summary, there is a chain of isomorphisms between various spaces of modular forms which when combined allows one to construct a Siegel cusp form

F_f of even weight ℓ from a given weight $2\ell - 2$ cusp form f for $SL_2(\mathbb{Z})$. The Shimura correspondence as progressed by Kohnen [6] constructs from f a weight $\ell - \frac{1}{2}$ form h_f for $\Gamma_0(4)$ lying in the Kohnen + space. If $h_f(\tau) = \sum_{n\geq 3} c(n)e(n\tau)$, then the Fourier expansion of F_f can be given explicitly in terms of the $c(n)$'s.

Let k be an odd positive integer. Let $f \in S_{2k}(SL_2(\mathbb{Z})$ be a normalized (first Fourier coefficient equals 1) Hecke eigenform and $h \in S^+_{k+1/2}(\Gamma_0(4))$ a Hecke eigenform associated to f by the Shimura correspondence. Let $F_f \in S_{k+1}(Sp_4(\mathbb{Z}))$ be the Saito-Kurokawa lift of f.

Let B_{k+1} denote a Hecke basis of $S_{k+1}(SL_2(\mathbb{Z}))$, normalized so that the first Fourier coefficient equals 1. For each $g \in B_{k+1}$, the period integral $\langle F_f|_{z=0}, g \times g\rangle$ is given by

$$\langle F_f|_{z=0}, g\times g\rangle = \int_{SL_2(\mathbb{Z})\backslash\mathbb{H}}\int_{SL_2(\mathbb{Z})\backslash\mathbb{H}} F\left(\begin{pmatrix}\tau & \\ & \tau'\end{pmatrix}\right)\overline{g(\tau)g(\tau')}\mathrm{Im}(\tau)^{k+1}\mathrm{Im}(\tau')^{k+1}d\mu(\tau)d\mu(\tau'),$$

where $d\mu(z) = y^{-2}dxdy$ if $z = x + iy$. Define the constants

$$\begin{aligned} v_1 &:= \mathrm{vol}(SL_2(\mathbb{Z})\backslash\mathbb{H}) = 2\pi^{-1}\zeta(2) = \tfrac{\pi}{3}, \\ v_2 &:= \mathrm{vol}(Sp_4(\mathbb{Z})\backslash\mathcal{H}_2) = 2\pi^{-3}\zeta(2)\zeta(4) = \tfrac{\pi^3}{270}. \end{aligned} \tag{3}$$

Let $\lambda_f(n)$ (resp. $\lambda_g(n)$) denote the n-th Hecke eigenvalue of f (resp. g), scaled so Deligne's bound gives $|\lambda_f(n)| \leq d(n)$. The $GL_3 \times GL_2$ Rankin-Selberg L-function is entire and satisfies the functional equation

$$\Lambda(s, \mathrm{sym}^2 g\otimes f) = \Lambda(1-s, \mathrm{sym}^2 g\otimes f)$$

where

$$\Lambda(s, \mathrm{sym}^2 g\otimes f) = 2^3(2\pi)^{-(3s+3k-\frac{1}{2})}\Gamma\left(s+2k-\tfrac{1}{2}\right)\Gamma\left(s+k-\tfrac{1}{2}\right)\Gamma\left(s+\tfrac{1}{2}\right) \times L(s, \mathrm{sym}^2 g\otimes f).$$

Now we can state Ichino's result.

Theorem 1 ([3] Theorem 2.1)**.** *For each $g \in B_{k+1}$,*

$$\Lambda\left(\tfrac{1}{2}, \mathrm{sym}^2 g\otimes f\right) = 2^{k+1}\frac{\langle f,f\rangle}{\langle h,h\rangle}\frac{|\langle F|_{z=0}, g\times g\rangle|^2}{\langle g,g\rangle^2}. \tag{4}$$

Then we define the restricted L^2 norm of F_f by

$$N(F_f) = \frac{\frac{1}{v_1^2}\langle F_f|_{z=0}, F_f|_{z=0}\rangle}{\frac{1}{v_2}\langle F_f, F_f\rangle}, \tag{5}$$

where the inner product in the numerator is the product of Petersson inner products on $SL_2(\mathbb{Z})\backslash\mathbb{H} \times SL_2(\mathbb{Z})\backslash\mathbb{H}$. Using Ichino's formula as well as some nice simplifications we derive the following:

Theorem 2. *With notation as above, we have*

$$N(F_f) = \frac{v_1^{-2}}{v_2^{-1}}\frac{24\pi}{L(3/2, f)L(1, \mathrm{sym}^2 f)}\sum_{g\in B_{k+1}}\frac{1}{k}L\left(\tfrac{1}{2}, \mathrm{sym}^2 g\otimes f\right). \tag{6}$$

By convexity, and the well-known Hoffstein-Lockhart bound [2], we deduce

Corollary 3. *With the same notation as in this section, we have*

$$N(F_f) \ll k \log k.$$

The Lindelöf hypothesis implies $N(F_f) \ll k^{\varepsilon}$ but in fact we have:

Conjecture 4. *As $k \to \infty$, we have*

$$N(F_f) \sim 2.$$

This is a special case of the conjectures of [1], though it takes a little work to derive this particular answer from their conjecture. It is a challenging problem to unconditionally prove Conjecture 4; to help measure the diffculty, one can consider the mean-value problem with f replaced by an Eisenstein series leading to the problem of estimating the second moment of symmetric-square L-functions in the weight aspect, which is listed as an open problem in [5, Conjecture 1.2]. We can prove Conjecture 4 on average over f and the weight k (of course f depends on the weight so we cannot try to fix f and vary the weight):

Theorem 5. *Suppose that w is a function satisfying*

$$\begin{cases} w \text{ is smooth with compact support on } [K, 2K] \\ |w^{(j)}(x)| \leq C_j K^{-j} \text{ for some } C_j > 0, \quad j = 0, 1, 2, \ldots. \end{cases} \tag{7}$$

Then as $K \to \infty$,

$$\sum_{k \text{ odd}} w(k) \sum_{f \in B_{2k}} N(F_f) \sim \sum_{k \text{ odd}} w(k) \sum_{f \in B_{2k}} 2. \tag{8}$$

We also have some results on the nonvanishing of $N(F_f)$.

Theorem 6. *Of the* $\dim(S_{2\ell-2}) = \frac{\ell}{6} + O(1)$ *Hecke eigenforms lifting to Siegel modular forms of weight ℓ under the Saito-Kurokawa correspondence, no more than* $\dim(M_{\ell-10}) = \frac{\ell}{12} + O(1)$ *have vanishing restricted L^2 norm $N(F_f)$. Consequently, for such f we have that there exists a Hecke eigenform g of weight ℓ such that* $L(1/2, \mathrm{sym}^2 g \otimes f) \neq 0$.

The proof of this theorem is based on the theory of Jacobi forms and does not directly show nonvanishing of any $GL_3 \times GL_2$ L-functions.

With a different argument, we also have

Theorem 7. *Suppose that for ℓ even large enough and for given f a Hecke eigenform of weight $2\ell - 2$, there exists a fundamental discriminant $D < 0$, $4|D$, such that $|D| \ll \ell^{1-\varepsilon}$ and satisfying $L(\frac{1}{2}, f \otimes \chi_D) \geq \ell^{-100}$. Then $N(F_f) \neq 0$ for such f.*

The result of Iwaniec and Sarnak [4] shows that the hypothesis holds with $D = -4$, say, for at least $(\frac{1}{2} - \varepsilon) \cdot \dim(S_{2\ell-2}))$ of the forms (for ℓ large enough in terms of ε), on average over ℓ; note that Theorem 6 holds for individual ℓ.

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Local spectral equidistribution for Siegel modular forms

Abhishek Saha
(joint work with Emmanuel Kowalski, Jacob Tsimerman)

This joint project with Emmanuel Kowalski and Jacob Tsimerman is a first exploration for the group GSp_4 of the following philosophy: given a "family" Π of cusp forms and a finite place v, the local components π_v of the elements $\pi \in \Pi$ (which are represented as restricted tensor products $\pi = \otimes\pi_v$ over all places) should be well-behaved, and more specifically, under averaging over finite subsets of the family, (π_v) should become equidistributed with respect to a suitable measure μ_v. This principle shows up in various guises in the orthogonality relations for Dirichlet characters, the trace formula (or the Petersson formula) and the "vertical" Sato-Tate type Theorems.

For example, Serre (and independently Conrey–Duke–Farmer, in the pure weight aspect) in 1997 proved the following theorem.

Theorem 1 (Serre, 1997)**.** *Let p be a prime and let (k_λ, N_λ) be an arbitrary sequence of positive integers such that k_λ is even, $p \nmid N_\lambda$, and $k_\lambda + N_\lambda \to \infty$ as $\lambda \to \infty$. For each λ, let $\mathbf{x}_\lambda$ denote the set of eigenvalues of the operator $p^{(1-k_\lambda)/2}T_p$ acting on the space $S(N_\lambda, k_\lambda)$ of holomorphic cusp forms of level N_λ and weight k_λ. Then the sequence $\mathbf{x}_\lambda$ gets equidistributed in $[-2, 2]$ with respect to the local Plancherel measure.*

Our main result is a weighted version of the above Theorem for the case of Siegel modular forms. For even $k \geq 2$, let $\mathcal{S}_k^*$ be a Hecke basis of the space of Siegel cusp forms of weight k on $\mathrm{Sp}(4, \mathbb{Z})$. For $F \in \mathcal{S}_k^*$, let

$$F(Z) = \sum_{T>0} a(F,T)e(\mathrm{Tr}(TZ))$$

be its Fourier expansion, where T runs over symmetric positive-definite semi-integral matrices, and let

$$\omega_k^F = c_k \frac{|a(F,1)|^2}{\langle F, F\rangle}. \tag{1}$$

where c_k is an explicit constant depending only on k. For each prime p, let $(a_{F,p}, b_{F,p})$ be the Satake parameters of the local representation $\pi_{F,p}$ attached to F at p. Then we prove,

Theorem 2 (Kowalski–Saha–Tsimerman, 2011)**.** *As $k \to \infty$ the family of points $\{(a_{F,p}, b_{F,p}) : F \in \mathcal{S}_k^*\}$, with weighting ω_k^F, gets equidistributed in $S^1 \times S^1$ with respect to a certain measure.*

The measure which appears in the above theorem was defined by Furusawa and Shalika and they called it the Plancherel measure associated to the relevant Bessel model; it is completely explicit. Moreover, the theorem as stated above is only a special case of what is actually proved in our paper. We work with more general weights $\omega_{k,d,\Lambda}^F$ which involve averages of $a(F,T)$ over positive definite T with a fixed discriminant; we treat the case of a finite set of primes p; we give a quantitative statement about the rate of equidistribution.

We also provide several applications of Theorem 2. For instance, our computation of the density of low-lying zeros of the spinor L-functions (for restricted test functions) gives global evidence for a well-known conjecture of Böcherer concerning the arithmetic nature of Fourier coefficients of Siegel cusp forms. Another application is to averages of L-functions: we give the limit of the average of the spinor L-function with the above weighting as $k \to \infty$.

In future work, we would like to extend Theorem 2 in various directions. First, we would like to treat the case of the level going to infinity; this will probably involve only technical difficulties. Next, it would be interesting to try and improve the quantitative rates of equidistribution, this will have important applications to L-values inside the critical strip. We would also like to extend our equidistribution results to the case of automorphic forms on other groups. A particularly suitable case is that of the split special orthogonal groups.

L_2 restriction norm of a $GL(3)$ Maass form

Xiaoqing Li
(joint work with Matt Young)

In quantum chaos, a key issue is the behavior of the eigenfunctions as the eigenvalue becomes large. In particular, one would like to know if the eigenfunctions behave like random waves, or if they can concentrate on certain subdomains. The influentical QUE conjecture of Rudnick and Sarnak asserts that the quantum measures associates to the eigenstates tend (in the weak-* sense) to the volume measure provided that the manifold has negative curvature.

We are naturally led to studying the sizes of Laplace eigenfunctions which can be measured in various ways. For instance, one may consider the L^p norms for $p \geq 2$. Alternatively, one may consider L^p norms of the eigenfunction restricted to some subset of its domain. In the arithmetical setting one has a commuting family of Hecke operators in addition to the Laplacian, and so it is natural to consider the behavior of these Maass forms. There are a small handful of results in this direction for GL_2 automorphic forms. In particular, Sarnak and Watson have announced a proof of a sharp bound (up to λ^{ε}) on the L^4 norm of Maass forms in the spectral aspect, conditional on the Ramanujan conjecture.

Reznikov wrote an influential preprint studying L^2 restriction problems of automorphic forms restricted to certain curves. Since then, there have appeared a number of papers studying very general problems of bounding the L^p norm of the restriction of the eigenfunction of the Laplacian to a submanifold of a Riemannian manifold with some very general results which are sharp in their generality, and which in particular stresses the problem of finding lower bounds. However, in the context of automorphic forms these general results are not sharp and it is desirable to prove stronger results and to understand what the true order of magnitude should be, whether it can be proven or not. Sarnak nicely explains some of the issues in studying such restriction problems, especially the connection with the Lindelöf hypothesis on pages 5 and 6 in his letter to Reznikov.

In a slightly different direction, Michel and Venkatesh proved a "subconvex" geodesic restriction theorem (see their Section 1.4) for the geodesic Fourier coefficients of GL_2 automorphic forms.

In our paper, we study a novel restriction problem for a GL_3 Maass form restricted to a codimension 2 submanifold (essentially arising from embedding GL_2 in the upper diagonal). Such a restricted function has nice invariance properties; it is invariant by $SL_2(\mathbb{Z})$ on the left and by $O_2(\mathbb{Z})$ on the right, and it is natural to understand how it fits into the GL_2 picture. For instance, one can ask what is the inner product of this restricted function with a given $SL_2(\mathbb{Z})$ Maass form (or more generally, we ask for the spectral decomposition).

This is the first sharp codimension 2 restriction result, as well as the first such result in a higher rank (GL_3) context. By Parseval's formula, the problem becomes bounding averages of different families of GL(3)xGL(2) L-functions. Assuming the Lindelof hypothesis for these GL(3)xGL(2) L-functions as we usually do, one can achieve a sharp bound in terms of the analytic conductor of the varying GL(3) Maass form. However, we will give an unconditional proof of this sharp bound for selfdual GL(3) Maass forms. For nonselfdual GL(3) Maass forms, our bounds depend on the bounds of the first Fourier coefficients of the GL(3) Maass forms.

L-functions twisted by *n*-th order characters

Benoît Louvel
(joint work with Valentin Blomer, Leo Goldmakher)

Analytic properties of L-functions in the critical strip often reflect subtle arithmetic properties of the number theoretic object encoded in the coefficients. In particular the central value has received much attention. If one has a suitable family of L-functions, their central values can again be encoded into an L-function, and one obtains a multiple Dirichlet series. Typically such multiple Dirichlet series no longer have Euler products, but they may have other interesting structure such as a non-trivial group of functional equations. One example of such a double Dirichlet series was investigated by Friedberg, Hoffstein, and Lieman in [2]. Their construction is somewhat technical, but the idea is straightforward. Fix $n \geq 3$, let K be a totally imaginary number field of degree d over $\mathbb{Q}$ containing the n-th roots of unity. We fix a sufficiently large set S of finite places of K and a sufficiently small ideal $\mathfrak{c}$, depending only on K. There is a family of n-th order Hecke characters $\chi_{\mathfrak{a}}$ of K indexed by integral ideals $\mathfrak{a}$ coprime to S, which are a natural analogue of the family of quadratic Dirichlet characters $\chi_D = \left(\frac{D}{\cdot}\right)$. While the quadratic characters χ_D have been studied extensively, little is known about their higher order relatives.

One can now construct a double Dirichlet series $Z(s, w)$ which (in the region of absolute convergence) looks roughly like $\sum_{\mathfrak{a}} L(s, \chi_{\mathfrak{a}})(\mathcal{N}\mathfrak{a})^{-w}$ where $\mathcal{N}$ denotes the norm of an ideal. Let $H_{\mathfrak{c}}$ be the ray class group modulo $\mathfrak{c}$, $R_{\mathfrak{c}} := H_{\mathfrak{c}} \otimes \mathbb{Z}/n\mathbb{Z}$, and let ψ be a (unitary) idèle class character of conductor dividing $\mathfrak{c}$ and order dividing n, that is, a character of $R_{\mathfrak{c}}$. Let $S_{\mathfrak{c}}$ denote the support of the conductor of $\chi_{\mathfrak{a}}$. For $s \in \mathbb{C}$ with sufficiently large real part define

$$L^*(s, \psi, \mathfrak{a}) = L_{S \cup S_{\mathfrak{a}}}(s, \psi\chi_{\mathfrak{a}})\, \mathfrak{A}(s, \psi, \mathfrak{a}) \tag{1}$$

where the subscript $S \cup S_{\mathfrak{a}}$ indicates that the Euler factors at places in $S \cup S_{\mathfrak{a}}$ have been removed, and where $\mathfrak{A}(s, \psi, \mathfrak{a})$ is a suitable correction factor at the ramified primes. For two characters ψ, ψ' of $R_{\mathfrak{c}}$ and s, w with sufficiently large real part we define

$$Z_1(s, w; \psi, \psi') := \sum_{\mathfrak{a} \in I(S)} \frac{L^*(s, \psi, \mathfrak{a})\psi'(\mathfrak{a})}{\mathcal{N}\mathfrak{a}^w} \tag{2}$$

where $I(S)$ is the set of nonzero integral ideals coprime to S. This function has been studied in detail in [2].

Our first main result is a subconvexity result for the function $Z(1/2, w)$ on the critical line $\Re w = 1/2$. Specializing to the central value $s = 1/2$, the function Z_1 has a functional equation in w. The completed L-function looks roughly like $\Gamma(nw)^{d/2} Z_1(1/2, w; \psi, \psi')$ and is essentially invariant under $w \leftrightarrow 1-w$. A standard convexity argument shows

$$Z_1(1/2, w, \psi, \psi') \ll |w|^{\frac{nd}{4}+\varepsilon}, \quad \Re w = 1/2.$$

Our first result is a power saving of this estimate and establishes a subconvexity result for this L-function.

Theorem 1. *With the notation developed above we have*

$$Z_1(1/2, w, \psi, \psi') \ll |w|^{\frac{nd}{4} - \frac{d}{12} + \varepsilon}, \quad \Re w = 1/2.$$

for any $\varepsilon > 0$.

We emphasize that here and henceforth all implied constants may depend on n, K (including S, $\mathfrak{c}$ etc.) and ε even if this is not explicitly mentioned. With more work this bound can be improved; our aim was to provide a subconvexity result with as little technical work as possible.

The functional equation for $Z_1(1/2, w; \psi, \psi')$ and the bound of Theorem 1 are shadows of the general situation when $Z_1(s, w; \psi, \psi')$ is treated as a function of two complex variables s and w. This double Dirichlet series has two obvious functional equations: one arising from the functional equation of $L(s, \chi_{\mathfrak{a}})$, and another from reciprocity of the character $\chi_{\mathfrak{a}}$. These two functional equations generate a finite group containing $(s, w) \leftrightarrow (1-s, 1-w)$. In fact, we prove rather more: viewed as a function of two variables s, w, we deduce a *uniform* subconvexity bound for $Z(s, w)$ on the lines $\Re s = \Re w = 1/2$. There is no obvious concept of analytic conductor or convexity bound in the context of multiple Dirichlet series, but in view of the functional equations of $Z_1(s, w)$, it seems reasonable to measure the complexity of $Z_1(s, w)$ for $\Re s = \Re w = 1/2$ in terms of the quantity

$$C := (|s||s + w|^{n-1}|w|)^d. \tag{3}$$

We expect a "convexity" bound

$$Z_1(s, w) \ll C^{1/4+\varepsilon}, \quad \Re s = \Re w = 1/2. \tag{4}$$

These notions agree with the classical ones if we specialize to the one variable case $s = 1/2$ or $w = 1/2$. We are now ready to state the following result, which improves on the convexity bound (4) and generalizes Theorem 1.

Theorem 2. *With* C *as in* (3) *we have*

$$Z_1(s, w; \psi, \psi') \ll C^{\frac{1}{4} - \frac{1}{12(n+1)} + \varepsilon}$$

for $\Re s = \Re w = 1/2$ *and any* $\varepsilon > 0$.

This provides a *uniform* subconvexity result in all ranges of $\Im s$ and $\Im w$, even in "exceptional" ranges like $\Im s = -\Im w$ where the conductor drops considerably. Again Theorem 2 may be improved with somewhat more work.

One of the main new ingredients of the present paper is a tool of independent interest, a large sieve inequality for n-th order characters. Such results for $n = 2$ and $n = 3$ have been obtained by Heath-Brown [4, 5]. The following theorem, which is valid for all $n \geq 3$ (for $n = 2$ better bounds are proved in [3].), is of the same quality as [5].

Theorem 3. *Let $M, N \geq 1/2$, and let $\lambda_{\mathfrak{a}} \in \mathbb{C}$ be a sequence of complex numbers indexed by $\mathfrak{a} \in I(S)$. Then*

$$\sum_{\substack{\mathfrak{a}\in I(S)\\ \mathcal{N}\mathfrak{a}\leq M}}^{\times} \Big|\sum_{\substack{\mathfrak{b}\in I(S)\\ \mathcal{N}\mathfrak{b}\leq N}}^{\times} \lambda_{\mathfrak{b}}\chi_{\mathfrak{a}}(\mathfrak{b})\Big|^2 \ll (MN)^{\varepsilon}(M+N+(MN)^{2/3}) \sum_{\substack{\mathfrak{b}\in I(S)\\ \mathcal{N}\mathfrak{b}\leq N}}^{\times} |\lambda_{\mathfrak{b}}|^2$$

for any $\varepsilon > 0$. Here $\sum^{\times}$ indicates that the summation is restricted to squarefree ideals.

To further illustrate the utility of this tool, we deduce several results concerning the behavior of the Hecke L-function $L(s, \chi_{\mathfrak{a}})$. We obtain an estimate of the second moment on the critical line

Corollary 4. *For $N \geq 1$, $t \in \mathbb{R}$ and $\varepsilon > 0$, one has with the notation developed so far*

$$\sum_{\substack{\mathfrak{a}\in I(S)\\ \mathcal{N}\mathfrak{a}\leq N}} |L(1/2+it,\chi_{\mathfrak{a}})|^2 \ll \Big(N(1+|t|)^{d/2}\Big)^{1+\varepsilon}.$$

This as strong as Lindelöf with respect to $\mathrm{cond}(\chi_{\mathfrak{a}})$ together with the convexity bound with respect to t. For fixed t, this result is not new. In fact, an asymptotic formula for a smoothed second moment has been established in [1]. In principle this can certainly be made uniform in t, but the outcome is not clear a priori. Our method is comparatively short and elementary and gives a uniform upper bound. We obtain a non-vanishing result at the central point

Corollary 5. *For $N \geq 1$ sufficiently large and $\varepsilon > 0$ we have*

$$\#\{\mathfrak{a} \in I(S) \mid \mathcal{N}\mathfrak{a} \leq N,\ L(1/2, \chi_{\mathfrak{a}}) \neq 0\} \gg N^{1-\varepsilon}.$$

For $n = 3$ this was proved in [6]. We obtain a zero-density theorem for zeros off the critical line

Corollary 6. *For $1/2 < \sigma \leq 1$, $T \geq 1$ and a squarefree integral ideal $\mathfrak{a}$ coprime to S let $N(\sigma, T, \mathfrak{a})$ be the number of zeros $\rho = \beta + i\gamma$ of $L(s, \chi_{\mathfrak{a}})$ in the rectangle $\sigma \leq \beta \leq 1$, $|\gamma| \leq T$. Then*

$$\sum_{\substack{\mathfrak{a}\in I(S)\\ \mathcal{N}\mathfrak{a}\leq N}}^{\times} N(\sigma,T,\mathfrak{a}) \ll N^{g(\sigma)} T^{1+d\frac{1-\sigma}{3-2\sigma}} (NT)^{\varepsilon},$$

$$g(\sigma) := \begin{cases} \frac{2(10\sigma-7)(1-\sigma)}{24\sigma-12\sigma^2-11}, & 5/6 < \sigma \leq 1, \\ \frac{8(1-\sigma)}{7-6\sigma}, & 1/2 < \sigma \leq 5/6. \end{cases}$$

This is relatively weak in the T aspect (and we have not optimized the T-exponent), but non-trivial in the N-aspect for any $\sigma > 1/2$. In particular, this shows that for any $1/2 < \sigma \leq 1$ there are at most $N^{g(\sigma)+\varepsilon}$ characters $\chi_{\mathfrak{a}}$ with $\mathcal{N}\mathfrak{a} \leq N$ such that $L(s, \chi_{\mathfrak{a}})$ has a *real* zero in the segment $[\sigma, 1]$. A similar zero density result in the special case $n = 3$ that is weaker in the N-aspect and stronger in the T-aspect was proved in [7].

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Torus periods of automorphic functions for GL_2 and meromorphic continuation of certain Dirichlet series

Andre Reznikov

We consider automorphic functions for the group $PSL_2(\mathbb{Z})$, and their periods associated to a quadratic field. We prove meromorphic continuation for a Dirichlet series build from these periods. We consider L-functions and Hecke characters of quadratic fields. Although arguments presented here are valid for a general quadratic number field E, in order to simplify the presentation we present the simplest case of Gauss numbers, and describe the general case in [2]. Hence, let $E = \mathbb{Q}(i)$ be the Gauss field. For an integer $n \in \mathbb{Z}$, consider the Hecke character of E given by $\chi_n(a) = \big(a/|a|\big)^{4n}$. The set of all such characters $\{\chi_n\}_{n\in\mathbb{Z}}$ could be described as the set $\mathcal{X}_{un}(E)$ of all (maximally) unramified Hecke characters of E. The corresponding Hecke L-function is given by the series $L(s,\chi) = \sum\limits_{a\in I^*(\mathcal{O}_E)} \chi(a)N(a)^{-s}$, for $Re(s) > 1$ (where the summation is over all non-zero integer ideals of E, i.e., over $\mathbb{Z}[i]/\{\pm 1, \pm i\}$ for $E = \mathbb{Q}(i)$).

We consider a double Dirichlet series given by

$$D_E(s,w) = L(s,\chi_0) + \sum_{n\in\mathbb{Z}\setminus 0} L(s,\chi_n)|n|^{-w} \ ,$$

for $(s, w) \in \mathbb{C}^2$, $Re(s) \gg 1$, $Re(w) \gg 1$. For what follows, one can omit the first term $L(s, \chi_0)$ from the sum, or take the sum over positive n only.

Theorem 1. *The series $D_E(s,w)$ defines the function which extends to a meromorphic function on $\mathbb{C}^2$.*

It turns out that it is more natural to consider the function

$$\tilde{D}_E(s,w) = \frac{2^{w/2}}{\Gamma\left(\frac{1-w}{2}\right)} \cdot D_E(s,w) \ , \tag{1}$$

and we prove the meromorphic continuation for this function.

0.1. **Torus periods.** One quickly recognizes that the above theorem is related to periods of automorphic functions. In fact, our proof of the meromorphic continuation is based on two well-known facts. First, we invoke classical results of E. Hecke (also discussed by H. Maass and by C. Siegel) about torus periods of Eisenstein series. Namely, we consider the automorphic representation E_s corresponding to the (normalized) Eisenstein series $E(s,x)$ for PGL_2 over $\mathbb{Q}$. Let $T_E \subset PGL_2$ be the torus corresponding to E. The (Fourier) expansion along the orbit $T_E(\mathbb{Q}) \setminus T_E(\mathbb{A}_\mathbb{Q}) \subset PGL_2(\mathbb{Q}) \setminus PGL_2(\mathbb{A}_\mathbb{Q})$ is given in terms of Hecke characters of E, and naturally leads to Hecke L-functions $L(s,\chi)$ over E. This allows us to realize the series $D_E(s,w)$ as the spectral expansion of the value at (the class of) the identity element $\bar{e} \in T_E(\mathbb{Q}) \setminus T_E(\mathbb{A}_\mathbb{Q})$ for a special vector v_w in the space of the Eisenstein series representation E_s. We note that the vector v_w is not a smooth vector, and belongs to an appropriate Sobolev space completion of E_s. In particular, we invoke the meromorphic continuation of smooth Eisenstein series, although in this particular case an elementary treatment based on Fourier expansion of $E(s)$ is also available.

To prove the meromorphic continuation of the value at the identity for the special vector v_w, we use Hecke operators and the classical technique going back to at least M. Riesz of the analytic continuation strip by strip (which the author learned from the seminal paper [1]). The main observation allowing us to apply this technique is the fact that modulo higher Sobolev spaces, the vector v_w is an approximate eigenvector of (appropriately understood) Hecke operators.

0.2. **Cusp forms.** One can apply the same argument to a Hecke-Maass cusp form ϕ instead of the Eisenstein series $E(s)$. The resulting series is a usual Dirichlet series in *one variable* build from the coefficients a_n which are (twisted) torus periods of the cusp form ϕ. We now define these coefficients.

Let ϕ be a Hecke-Maass form for the group $PGL_2(\mathbb{Z})$ (one can easily extend our arguments to a congruence subgroup). In particular, ϕ is an eigenfunction of the Laplace-Beltrami operator Δ on $PSL_2(\mathbb{Z}) \setminus \mathcal{H}$ with the eigenvalue which we denote by $\Lambda(\phi) = (1-\lambda)/4$ for $\lambda = \lambda(\phi) \in i\mathbb{R} \cup (0,1)$. We normalize ϕ by its L^2-norm. We denote by (V_π, π_λ) the isomorphism class of the (smooth) automorphic representation of $PGL_2(\mathbb{R})$ generated by ϕ. The structure of such a representation of $PGL_2(\mathbb{R})$ is well known, and in particular V_π has an orthonormal basis of K-types $\{e_n\}_{n\in 2\mathbb{Z}}$ which we fix (here $K = PSO(2,\mathbb{R}) \simeq S^1$). We consider the Taylor-like expansion of ϕ at $z = i$ (generally one considers a CM-point $\mathfrak{z} \in \mathcal{H}$; in fact such an expansion exists at any point of $\mathcal{H}$). Denote by $F_\lambda(n,g) = \langle \pi_\lambda(g)e_0, e_n \rangle_{\pi_\lambda}$, $n \in 2\mathbb{Z}$, matrix coefficients in the representation π_λ. Functions $F_\lambda(n,g)$ are right K-invariant and hence could be viewed as functions of $z \in \mathcal{H} \simeq PGL_2^+(\mathbb{R})/K$. Functions $F_\lambda(n,z)$ are eigenfunctions of Δ on $\mathcal{H}$ with the same eigenvalue as ϕ, and spherically equivariant $F_\lambda\left(n, \left(\begin{smallmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{smallmatrix}\right) z\right) = e^{in\theta}F_\lambda(n,z)$. We have the following well-known expansion (first considered by H. Petersson for holomorphic forms and also by A. Good in general)

$$\phi(z) = \sum_{n\in 2\mathbb{Z}} a_n F_\lambda(n,z) \ , \tag{2}$$

where $a_n = a_n(\phi) \in \mathbb{C}$. Of course coefficients a_n depend on the normalization of functions $F_\lambda(n,z)$, which we explain in [2]. We note that the analogous expansion is valid for the Eisenstein series $E(s,z)$ as well, and gives $a_n(E(s)) = L(s,\chi_{-n})$.

We consider the Dirichlet series

$$D_E(\phi,w) = a_0 + \sum_{n\in 2\mathbb{Z}\setminus 0} a_n|n|^{-w} \tag{3}$$

defined for $|w| \gg 1$. (The coefficient a_0 is included out of an esthetic reason only.) As with the Eisenstein series we consider the function

$$\tilde{D}_E(\phi,w) = \frac{2^{w/2}}{\Gamma\left(\frac{1-w}{2}\right)} \cdot D_E(\phi,w) \ , \tag{4}$$

Theorem 2. *The Dirichlet series $\tilde{D}_E(\phi,w)$ extends to a holomorphic function on $\mathbb{C}$.*

We note that analogous results are valid for periods along closed geodesics (i.e., those associated to real quadratic fields).

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Multiple Dirichlet series ans shifted convolution sums

Jeffrey Hoffstein

The objective of this talk was to present progress made in the past three years in the area of expanding the concept of multiple Dirichlet series to include shifted Rankin-Selberg convolutions. The original notion, from 1939 (Rankin) and 1940 (Selberg) was as follows: If

$$L(s,f) = \sum_{m\geq 1} \frac{a(m)}{m^s} \text{ and } L(s,g) = \sum_{m\geq 1} \frac{b(m)}{m^s}$$

are two L-series associated to modular forms, each convergent when $\Re s > 1$, with a functional equation in s, and a meromorphic continuation to $\mathbb{C}$, the Rankin-Selberg convolution provides a functional equation and a meromorphic continuation to $\mathbb{C}$ of the new L-series

$$L(s,f\otimes g) = \sum_{m\geq 0} \frac{a(m)\bar{b}(m)}{m^s}.$$

In 1965, in a paper summing up progress in modular forms, Selberg introduced the notion of a shifted convolution. If h is a positive integer, his shifted convolution was the series

$$\sum_{m\geq 1} \frac{a(m+h)\bar{b}(m)}{(2m+h)^s}. \tag{1}$$

Selberg mentioned that the meromorphic continuation of this could be obtained. (It converges absolutely for $\Re s > 1$.) He went on to remark: "We cannot make much use of this function at the moment...".

Since then it has been generally recognized that the analytic properties of series such as (1) play a very important role in estimating the size of certain shifted sums, such as

$$\sum_{m<x} a(m)\bar{b}(m+h),$$

and that these in turn play an important role in progress toward the Lindelöf Hypothesis, that is, the bounding from above of automorphic L-series at points inside the critical strip.

A major factor hindering progress in this area has been a difficulty in applying the meromorphic continuation of the series (1) to the analysis of shifted sums. Although non-trivial results can be obtained from any continuation past the trivial line $\Re s = 1$, technical difficulties have interfered with more refined applications.

One purpose of this talk was to suggest that the series (1) may not be the correct one to look at, or at any rate, might not tell the whole story. In particular, the series

$$D(s;h) = \sum_{m\geq 1} \frac{a(m+h)\bar{b}(m)}{m^s},$$

and generalizations of it, can be meromorphically continued to all of $\mathbb{C}$. In addition, the notion of a multiple shifted Dirichlet series, in which a sum is taken over the shifts h as well, was presented:

$$Z(s,w) = \sum_{h\geq 1} \frac{D(s;h)}{h^{w+(k-1)/2}} = \sum_{\substack{h\geq 1, m_2\geq 1 \\ m_1 = m_2 + h}} \frac{a(m_1)\bar{b}(m_2)}{(m_2)^{s+k-1} h^{w+(k-1)/2}}.$$

These as well can be meromorphically continued. One very interesting aspect of these shifted multiple Dirichlet series is that just like their L-series and multiple Dirichlet series counterparts they have an analog of a critical strip where all the deep number theoretic information is buried. They also have two incarnations: a Dirichlet series that converges absolutely in one range, and a spectral expansion that converges absolutely in a disjoint region. As one might expect, all the interesting information seems to be hidden in between. The spectral expansions have another curious aspect. The terms decay polynomially in the spectral parameter t_j, rather than exponentially, and the ratio of gamma factors governing this decay seems to act in a manner analogous to the usual $m^{-\sigma}$ indexing of a Dirichlet series (polynomial decay corresponding to the size of σ.)

A sample application of this method to a second moment problem is:

Theorem 1. *Let f, g be cusp forms of even weight k for $\Gamma_0(N_0)$ and let χ be a Dirichlet character modulo Q. Let*

$$S_{f,g}(Q) = \frac{1}{\varphi(Q)} \sum_{\chi \mod Q} L(1/2, f, \chi)\overline{L(1/2, g, \chi)}.$$

The smoothed sum

$$S_{f,g}(X, y) = \sum_{Q \geq 1} S_{f,g}(Q) e^{-y^2(\log(X/Q))^2/(4\pi)}$$

defines an average of $S_{f,g}(Q)$ over the interval of length X/y centered at X. For any y in the interval

$$1 \leq y \ll X^{1-\epsilon}$$

we have the following estimates for $S_{f,g}(X, y)$: When $f = g$

$$(2) \quad S_{f,f}(X, y) = \frac{C_1(f)L(1, f, \vee^2)e^{\pi/y^2} X \log X}{y} + \frac{C_2(f)X}{y} + \mathcal{O}\left(\frac{X^{1/2}}{y}\right) + \mathcal{O}(X^\epsilon)$$

When $f \neq g$

$$S_{f,g}(X, y) = \frac{c_{f,g}e^{\pi/y^2}}{y} L(1, f \otimes g)X + \mathcal{O}\left(\frac{X^{1/2}}{y}\right) + \mathcal{O}(X^\epsilon).$$

Here $C_1(f), C(f, g)$ are non-zero constants that can be made explicit.

There is a power savings in the exponent of size exactly the interval of Q we average over, and there are no restrictions on Q other than being in this interval.

Graph Eigenfunctions and Quantum Unique Ergodicity

Elon Lindenstrauss
(joint work with Shimon Brooks)

The purpose of this talk is to report on joint work with Shimon Brooks which gives an application of the techniques we developed in the context of our work [5] on eigenfunctions of large graphs to some Quantum Unique Ergodicity problems. This abstract is based in our research announcement [4].

Let H be a quaternion division algebra over $\mathbb{Q}$, split over $\mathbb{R}$, and R an order in H. Fix an isomorphism $\Psi : H(\mathbb{R}) \cong \mathrm{Mat}_2(\mathbb{R})$. For $\alpha \in R$ of positive norm $n(\alpha)$, we write $\underline{\alpha} = n(\alpha)^{-1/2}\Psi(\alpha) \in SL_2(\mathbb{R})$. Set Γ to be the image under Ψ of the subgroup of norm 1 elements of R. As is well known, Γ is discrete and co-compact in $SL_2(\mathbb{R})$, and so the quotient $X = \Gamma\backslash SL_2(\mathbb{R})$ is a 2-to-1 cover of the unit cotangent bundle of a compact hyperbolic surface $M = \Gamma\backslash\mathbb{H}$.

Write $R(m)$ for the set of elements of R of norm m, and define the Hecke operator

$$T_m : f(x) \mapsto \frac{1}{\sqrt{m}} \sum_{\alpha \in R(1) \backslash R(m)} f(\underline{\alpha} x)$$

as the operator averaging over the Hecke points

$$T_m(x) = \{\underline{\alpha} x : \alpha \in R(1) \backslash R(m)\}$$

We will be interested in the case where $m = p^k$ are powers of a fixed prime p. It is well known that T_{p^k} is a polynomial in T_p; so in particular, eigenfunctions of T_p are eigenfunctions of all T_{p^k}. For all but finitely many primes, the points $T_{p^k}(x)$ form a $p+1$-regular tree as k runs from 0 to ∞; we will always assume that p is such a prime. We denote by S_{p^k} the sphere of radius k in this tree, given by Hecke points corresponding to the primitive elements of R of norm p^k.

For any eigenfunction ϕ_j of the Laplacian Δ on M, normalized by $||\phi_j||_2 = 1$, one can construct a measure μ_j on S^*M (which we view as a measure on the double cover $\Gamma \backslash SL_2(\mathbb{R})$) called the **microlocal lift** of ϕ_j which is asymptotically invariant under the geodesic flow as the Laplace eigenvalue of ϕ_j tends to infinity. We shall use the variant of this construction used in [6] due to Wolpert, where $\mu_j = |\Phi_j|^2 dVol$ for suitably chosen $\Phi_j \in L^2(S^*M)$ in the irreducible representation of $SL_2(\mathbb{R})$ on $\Gamma \backslash SL_2(\mathbb{R})$ generated by translates of ϕ_j. The construction satisfies that Φ_j is an eigenfunction of T_p when ϕ_j is. Since Δ commutes with all of the T_p, we may consider sequences $\{\phi_j\}$ of joint eigenfunctions, whereby each Φ_j is also an eigenfunction of T_p (with the same eigenvalue as ϕ_j).

Theorem 1. *Let p be a prime (outside the finite set of bad primes for M), and let $\{\phi_j\}_{j=1}^{\infty}$ be a sequence of L^2-normalized joint eigenfunctions of Δ and T_p on M. Then any weak-* limit point μ of the microlocal lifts μ_j has positive entropy on almost every ergodic component.*

Note that even without any Hecke operators, Anantharaman [1] has shown (for general negatively curved compact manifolds) that any quantum limit has positive entropy, and this has been further sharpened in her joint work with Nonnenmacher [2]. Hence the point of Theorem 1 is that it gives information on *almost all ergodic components* of a quantum limit. Such a pointwise entropy bound (of a more explicit and uniform nature) was given by Bourgain and L. in [3] assuming ϕ_j are eigenfunctions of *all* Hecke operators.

In view of the measure classification results of [6], this implies the following:

Corollary 2. *Let $\{\phi_j\}$ as above be a sequence of joint eigenfunctions of Δ and T_p. Then the sequence μ_j converges weak-* to Liouville measure on S^*M.*

Both Theorem 1 and Corollary 2 remain valid even if the assumptions on ϕ_i being eigenfunctions of Δ and T_p are replaced by an assumption of being an approximate eigenfunction in a surprisingly weak sense.

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Logarithm laws for one parameter unipotent flows

Dubi Kelmer
(joint work with Amir Mohammadi)

Let G denote a semisimple Lie group and $\Gamma \subseteq G$ a non-uniform irreducible lattice. Consider the action of an unbounded one parameter subgroup $\{g_t | t \in \mathbb{R}\} \subseteq G$ on the space $X = \Gamma\backslash G$ endowed with the probability G-invariant Haar measure σ. By Moore's Ergodicity Theorem this action is ergodic on $\Gamma\backslash G$ and hence the orbit of σ-a.e. $x \in X$ becomes equidistributed. In particular, these orbits make excursions far out into the cusps. A natural way of measuring the rate of these excursions is considering the distances $\mathrm{dist}(o, xu_t)$ from a fixed point $o \in X$ and asking what is the fastest rate at which they grow for a typical point $x \in X$; note that the ergodicity of the action implies $\overline{\lim}_{t\to\infty} \mathrm{dist}(o, xg_t) = \infty$ for σ-a.e. $x \in X$.

This problem can be treated as an instance of a shrinking target problem and, as such, an upper bound for the rate of excursions follows from the Borel-Cantelli Lemma. In particular, if we denote by

$$B_r = \{x \in X | \mathrm{dist}(o, x) > r\},$$

a family of sets shrinking to infinity, then under an appropriate normalization of the distance function we have that $\sigma(B_r) \asymp e^{-r}$ and the first half of the Borel-Cantelli Lemma together with a standard continuity argument imply that $\overline{\lim}_{t\to\infty} \frac{\mathrm{dist}(o,xg_t)}{\log(t)} \leq 1$ for σ-a.e. $x \in X$. We say that the flow $\{g_t\}_{t\in\mathbb{R}}$ satisfies the logarithm law if this upper bound is sharp, that is if $\overline{\lim}_{t\to\infty} \frac{\mathrm{dist}(o,xg_t)}{\log(t)} = 1$ for σ-a.e. $x \in X$.

The case of the geodesic flow on finite volume non-compact hyperbolic manifolds (that is, $\Gamma\backslash\mathbb{H}^{m+1}$ with $\Gamma \subseteq \mathrm{SO}(m+1, 1)$ a non-uniform lattice) was studied by Sullivan [4]. Sullivan utilized a geometric proof Khinchin's theorem on approximation of reals by rationals to prove logarithm laws for the geodesic flow. The general case of locally symmetric spaces of noncompact type and $\{g_t | t \in \mathbb{R}\}$ a one parameter diagonalizable subgroup was proved by Kleinbock and Margulis [3]. Using the exponential rate of mixing of such flows they show that the events $xg_\ell \in B_{r_\ell}$ and $xg_k \in B_{r_k}$ are (exponentially close to being) independent. Then, using an effective version of the second half of the Borel-Cantelli Lemma they proved that diagonalizable flows satisfy the logarithm law.

More recently the case of one parameter unipotent groups has attracted some attention. For unipotent flows, the (polynomial) rate of mixing is not fast enough to obtain the desired effective independence used in the case of diagonalizable flows. Nevertheless, in [1] Athreya and Margulis proved logarithm laws for one parameter unipotent groups acting on $X_n = \mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})$, with respect to a distance like function $\alpha_1 : X_n \to [0,\infty)$, given in terms of the length of the shortest vector in $\Lambda = \mathbb{Z}^n g$. To obtain this result they use the interpretation of this space as the space of lattices in $\mathbb{R}^n$ and prove a random version of Minkowski's theorem, showing that a large set in $\mathbb{R}^n$ intersects most lattices (with respect to normalized Haar measure on X_n).

A natural question which arises is whether the result in [1] is a consequence of the arithmetic nature of this space or rather a general phenomena coming from the geometry of such spaces in general. To understand this, we try to generalize their approach to prove logarithm laws for one parameter unipotent flows on more general homogenous spaces $\Gamma\backslash G$ which are non necessarily arithmetic. Though it seems that this approach should work in general (at least for lattices of $\mathbb{Q}$-rank one) it relies on estimates of certain theta functions that we were able to establish so far only for the case where Γ is an irreducible lattice in $G = \mathrm{SL}_2(\mathbb{R})^{r_1} \times \mathrm{SL}_2(\mathbb{C})^{r_2}$. In particular we show

Theorem 1. *Let $G = \mathrm{SL}_2(\mathbb{R})^{r_1} \times \mathrm{SL}_2(\mathbb{C})^{r_2}$, $\Gamma \subseteq G$ an irreducible lattice, and $K \subset G$ a maximal compact. Let* dist *denote a distance function on $X = \Gamma\backslash G$ obtained from a left G-invariant, bi K-invariant Reimannian metric on G, normalized so that $\sigma(B_r) \asymp e^{-r}$. Then, for any one-parameter unipotent group $\{u_s\}_{s\in\mathbb{R}} \subseteq G$*

$$\forall o \in X, \quad \text{for } \sigma\text{-a.e. } x \in X, \quad \overline{\lim_{s\to\infty}} \frac{\mathrm{dist}(o, xu_s)}{\log s} = 1. \tag{1}$$

Remark 2. We note that the only non-arithmetic cases are when there is only one copy, and it is in these cases that the proof is also the hardest.

As mentioned above, a crucial ingredient in the proof is a bound on norms of theta functions in $L^2(\Gamma\backslash G)$. Specifically, to any compactly supported function $f \in C_c(\Gamma_\infty\backslash G)$ the corresponding theta function $\Theta_f \in L^2(\Gamma\backslash G)$ is defined by

$$\Theta_f(g) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\gamma g),$$

where Γ_∞ is the stabilizer of a cusp at infinity. What we need is an estimate for the L^2-norm $\|\Theta_f\|_2$ in terms of the L^1-norm $\|f\|_1$ that is uniform for a certain sequence of functions f_k approximating the indicator functions of specific sets $\mathfrak{D}_k \subseteq \Gamma_\infty\backslash G$ becoming very long and narrow with area going to infinity as $k \to \infty$. We note that the known bounds for $\|\Theta_f\|_2$, coming from the work of Harishchandra [2], involve certain Sobolev norms of f. For our functions these norms grow much faster the the L^1 norm and hence are not sufficient for our needs. To overcome this difficulty we prove the following bounds

Theorem 3. *Let $G = \mathrm{SL}_2(\mathbb{R})^{r_1} \times \mathrm{SL}_2(\mathbb{C})^{r_2}$ and $\Gamma \subseteq G$.*

- *For Γ an arithmetic irreducible lattice, for any positive $f \in C_c^\infty(\Gamma_\infty\backslash G)$ we have*
$$\|\Theta_f\|^2 \leq C_\Gamma(\|f\|_2^2 + \|f\|_1^2).$$
- *For any Γ and f_k approximating the indicator functions of $\mathfrak{D}_k$ we have*
$$\|\Theta_{f_k}\|^2 \leq C_\Gamma \|f_k\|_1^2,$$

The proof of Theorem 3 relies on a formula for $\|\Theta_f\|^2$ in terms of the poles of corresponding Eisenstein series, together with a comparison of the norms of theta functions constructed with respect to different lattices. In particular, for $\Gamma = \mathrm{SL}_2(\mathcal{O}_K)$ the Eisenstein series has no exceptional poles and this bound easily follows from the formula. We can then show that if this bound holds for Γ, then it holds for any finite index subgroup. This concludes the proof for the arithmetic setting. Next, utilizing the fact that the are also arithmetic lattices for which there are poles arbitrarily close to one, we can boost the result from the arithmetic case to get bounds also in the non-arithmetic setting.

It is an interesting open question for what groups G and lattices Γ can one prove such bounds for all theta functions. We note that a formula for the norms of theta functions in terms of poles of Eisenstein series can be obtained in general. Moreover, the fact that such a bound for a lattice implies the same bound for any finite index subgroup, can also be proved in this generality. The main ingredient that is missing in order to prove Theorem 3, and hence also Theorem 1, for hyperbolic manifolds in higher dimensions, is the existence of a nice family of lattices for which the Eisenstein series is known not to have exceptional poles.

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Automorphic forms of higher order

Anton Deitmar

A classical automorphic form is a function φ in the space $L^2(\Gamma\backslash G)$, where Γ is a lattice in a reductive Lie group G. If Γ is cocompact, such a function φ can be viewed as a Γ-invariant element of the space $V = L^2_{\mathrm{loc}}(G)$ of locally integrable function on the group G. The space of invariants, $V^\Gamma = H^0(\Gamma, V)$ is the space of all $v \in V$ with $Jv = 0$, where J is the augmentation ideal of the group ring $\mathbb{Z}\Gamma$. Higher order forms are functions $\varphi \in V$ such that $J^q\varphi = 0$ for some natural number q. We call these higher order invariants and write the space of such as
$$H^0_q(\Gamma, V) = \{v \in V; J^q v = 0\}.$$

In recent years these came up in various contexts, such as percolation theory, Eisenstein series twisted with modular symbols, and converse theorems. The talk is a report on the project of understanding these forms from a representation theoretic viewpoint.

One first problem is to extend the definition to non-cocompact lattices. Previously, classical cusp forms of higher order have been defined as higher order invariants in the space of holomorphic functions on the upper half space which decay rapidly as one approaches any cusp. This following observation motivates our definition of higher order forms in the general case. If $D \subset G$ is a measurable set of representations of the quotient $\Gamma\backslash G$, then any Γ-invariant function φ is uniquely determined by its restriction to D. Let $S \subset \Gamma$ be a finite symmetric set of generators, then every function φ of G which satisfies $J^q = 0$ for some $q \geq 2$ is uniquely determined by its restriction to the set

$$S^{q-1}D = \bigcup_{s_1,\dots,s_{q-1}\in S} s_1 \cdots s_{q-1} D.$$

So one defines higher order forms as measurable functions φ on G, which satisfy $J^q\varphi = 0$ and which are in $L^2(S^{q-1}D)$. The definition does not depend on the choice of S. It depends on D, but only very mildly. The inner product depends on choices, but not the induced topology.

In the case of a cocompact lattice, there is a complete spectral theory, for non-cocompact lattices only partial results are known. For instance, one knows that cusp forms decompose discretely and that the continuous spectrum can be described in terms of classical Eisenstein series. A main Theorem asserts that higher order forms can be embedded into a tensor product of adelic automorphic representations. As such a tensor product has no irreducible sub-representations, this explains why higher order L-functions have no Euler products.

When it comes to constructing higher order forms, one often uses iterated integrals in the sense of Chen. These are defined as follows. Let X be a smooth manifold and $p : [0,1] \to X$ a path. For 1-forms $\omega_1, \dots, \omega_s$ on X one defines the iterated integral as

$$\int_p \omega_1 \dots \omega_s = \int_0^1 \int_0^{t_s} \dots \int_0^{t_2} p^*\omega_1(t_1) \dots p^*\omega_s(t_s).$$

Suppose that the forms ω_j are such that the induced map on the path space is homotopy invariant. Then one gets a function $f(x) = \int_{x_0}^x \omega_1 \dots \omega_s$ on the universal covering $\tilde{X}$. A central result of Ivan Horozov and the author states that f is a higher order invariant for the group $\Gamma = \pi_1(X)$ and that all higher order invariants in $C(\tilde{X})$ are obtained in this way after multiplying by functions from $C(X)$.

Central values of the symmetric square L-functions

Wenzhi Luo

Let $S_k(\Gamma_0(1))$ denote as usual the space of holomorphic cusp forms of weight k with respect to the modular group $\Gamma_0(1)$, and let H_k be the normalized Hecke basis for $S_k(\Gamma_0(1))$. It's well known that the associated symmetric square L-function admits analytic continuation to the whole complex plane $\mathbf{C}$, and satisfies the functional equation

$$\begin{aligned}\Lambda(s,\ \mathrm{sym}^2(f)) \quad &=: \quad \pi^{-3s/2}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)L(s,\ \mathrm{sym}^2(f)) \\ &= \quad \Lambda(1-s,\ \mathrm{sym}^2(f)) \ .\end{aligned}$$

We show, using the Zagier's kernel function for $L(s,\ \mathrm{sym}^2(f))$, that for any $\epsilon > 0$ and as $K \to \infty$,

$$\sum_{2|k,\ K \le k \le K+K^{1/2}} \ \sum_{f \in H_k} |L(1/2,\ \mathrm{sym}^2(f))|^2 \ll_\epsilon K^{3/2+\epsilon} \ . \tag{1}$$

Using Zagier's kernel function, the left hand side of (1) is reduced to an explicit and interesting sum involving highly oscillating exponential functions twisted by the central L-values of the quadratic Diriclet L-series. We derive (1) by using the bound due to M.Jutila for the second moment of the quadratic Dirichlet L-functions at the central point.

Subconvexity of $GL(3) \times GL(2)$ L-functions

Rizwanur Khan

An outstanding problem in the analytic theory of L-functions is the subconvexity problem. Let $L(s)$ be an L-function of degree d from the Selberg Class which has a functional equational of analytic conductor C, relating values at s and $1-s$. Assuming the Ramanujan-Petersson conjecture, it is quite straightforward to prove the convexity, or trivial, bound at the central point: $L(1/2) \ll_{d,\epsilon} C^{1/4+\epsilon}$, for any $\epsilon > 0$. The subconvexity problem is to prove that $L(1/2) \ll C^{1/4-\delta}$ for some $\delta > 0$, while it is expected that in fact $L(1/2) \ll_\epsilon C^\epsilon$ (the Lindelöf hypothesis). The problem is completely solved for degree 1 and 2 automorphic L-functions, but remains a challenging open problem in higher degree. We study the subconvexity problem for certain degree 6 L-functions.

Let $H_k^\star(q)$ denote the set of holomorphic cusp forms of weight k which are newforms of level q with trivial nebentypus in the sense of Atkin-Lehner Theory. Every $f \in H_k^\star(q)$ has a Fourier expansion of the type

$$f(z) = \sum_{n=1}^{\infty} a_f(n) n^{\frac{k-1}{2}} e(nz) \tag{1}$$

for $\Im z > 0$, where $e(z) = e^{2\pi i z}$, $a_f(n) \in \mathbb{R}$ and $a_f(1) = 1$.

Fix g a self-dual Hecke-Maass form for $SL_3(\mathbb{Z})$ which is unramified at infinity. We write $A(n,m) = A(m,n)$ for the Fourier coefficients of g in the Fourier expansion (6.2.1) of [2], normalized so that $A(1,1) = 1$. Note that $A(n,m) \in \mathbb{R}$.

The Rankin-Selberg L-function $L(s, g \times f)$ is defined as

$$L(s, g \times f) = \sum_{n,r \geq 1} \frac{A(r,n)a_f(n)}{(r^2 n)^s} \tag{2}$$

for $\Re(s) > 1$. It satisfies the functional equation

$$q^{\frac{3s}{2}} G(s) L(s, g \times f) = \epsilon_{g\times f} q^{\frac{3(1-s)}{2}} G(1-s) L(1-s, g \times f), \tag{3}$$

where $\epsilon_{g\times f} = -i^k a_f(q) q^{\frac{1}{2}} = \pm 1$ and

$$\begin{aligned} G(s) = \pi^{-3s} &\Gamma\Big(\frac{s + \frac{k+1}{2} + 3\nu - 1}{2}\Big)\Gamma\Big(\frac{s + \frac{k+1}{2}}{2}\Big)\Gamma\Big(\frac{s + \frac{k+1}{2} + 1 - 3\nu}{2}\Big) \\ &\times \Gamma\Big(\frac{s + \frac{k-1}{2} + 3\nu - 1}{2}\Big)\Gamma\Big(\frac{s + \frac{k-1}{2}}{2}\Big)\Gamma\Big(\frac{s + \frac{k-1}{2} + 1 - 3\nu}{2}\Big). \end{aligned} \tag{4}$$

We see that the analytic conductor of $L(s, g \times f)$ in the q-aspect is q^3 and the convexity bound is $L(1/2, g \times f) \ll q^{3/4+\epsilon}$. We are interested in improving this bound. Subconvexity for $GL(3) \times GL(2)$ L-functions in other aspects has been considered by Li [6] and Blomer [5].

We prove

Theorem 1. *Fix $k > 10^6$ an even number. Let q be a prime number. Suppose that for some $f_0 \in H_k^\star(q)$ we have*

$$\sum_{n<L} \frac{a_{f_0}(n)^2}{\sqrt{n}} \gg_\epsilon L^{1/2-\epsilon} \tag{5}$$

for $L = q^{1/4+1/2000}$. Then we have

$$L(1/2, g \times f_0) \ll_{g,k} q^{3/4-1/2001}. \tag{6}$$

The assumption (5), that $a_{f_0}(n)^2 \gg 1$ on average in a short interval, is known to be true for almost all $f_0 \in H_k^\star(q)$. It is expected to be true for all $f_0 \in H_k^\star(q)$ and all $L > q^\epsilon$. The tools involved in the proof of the theorem are the amplifier method, Lapid's theorem [1], the approximate functional equation, the Petersson trace formula, and the $GL(3)$ Voronoi summation formula [4, 3].

The theorem is proved by bounding the first moment of $L(1/2, g \times f_0)$ multiplied by an amplifier:

$$\sum_{f \in H_k^\star(q)} L(1/2, g \times f)\Big(\sum_{n<L} \frac{a_{f_0}(n)a_f(n)}{\sqrt{n}}\Big)^2 \ll_{g,k,\epsilon} q^{1+\epsilon} \tag{7}$$

By a result of Lapid, we have that $L(1/2, g \times f) \geq 0$. Thus by (5) and (7), we have

$$L(1/2, g \times f_0) L^{1-\epsilon} \ll_\epsilon L(1/2, g \times f_0)\Big(\sum_{n<L} \frac{a_{f_0}(n)^2}{\sqrt{n}}\Big)^2 \ll_{g,k,\epsilon} q^{1+\epsilon}, \tag{8}$$

which implies (6).

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On the non-vanishing of *L*-functions inside the critical strip

Goran Muić

In [1], Kohnen showed that completed L–functions for normalized eigenforms of $SL_2(\mathbb{Z})$ satisfy the generalized Riemann hypothesis as the weight of eigenforms approaches ∞. Using the same approach, this was later generalized by Raghuram (see the references in [4]) to cover the cases of newforms for $\Gamma_0(N)$ and primitive Dirichlet characters as the weight of newforms approaches ∞ or N approaches ∞. They both concentrate on a single point (or a segment with a fixed imaginary part) inside the critical strip and use Fourier expansion of the kernel function to study non–vanishing.

We use (see [4]) a slightly different form of (and approach to) the kernel function which is more convenient for our non–vanishing criterion (given by [2], Lemma 3-1). In fact, we work in a rather general set–up assuming that Γ is a Fuchsian group of the first kind (meaning that it is discrete and it has a finite covolume in $SL_2(\mathbb{R})$) having a regular cusp at ∞. The groups $\Gamma_0(N)$ are the most important examples. We consider the usual space of cuspidal modular forms $S_m(\Gamma, \chi)$ where $m \geq 3$ and χ is a character $\Gamma \to \mathbb{C}^\times$ of finite order satisfying

$$\chi \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} = 1, \quad \chi(-1) = (-1)^m \text{ if } -1 \in \Gamma, \tag{1}$$

where $h > 0$ is the width of the cusp ∞ of Γ. We remind the reader that $h = 1$ when $\Gamma = \Gamma_0(N)$. In the set–up of (1) and for $m \geq 3$, we can define the classical Poincaré series in $S_m(\Gamma, \chi)$ in the following way:

$$\psi_{n,m,\chi}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e(n(\gamma.z)/h)\mu(\gamma, z)^{-m}\chi(\gamma)^{-1}, \quad n \geq 1.$$

We remind the reader that for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, we let $\mu(g, z) = cz + d$. The following theorem is the key step of our approach:

Theorem 1. *Assume that $m \geq 3$ is an integer such that (1) hold. Assume that $\alpha > 0$ satisfies the following: n–th Fourier coefficient of any $f \in S_m(\Gamma, \chi)$ is $O(n^\alpha)$ as $n \to \infty$. Then*

$$\sup_{\xi \in \mathbb{H}} Im(\xi)^{k+m/2} \left| d^k \psi_{n,m,\chi}(\xi)/d\xi^k \right| \ll n^{\alpha - m + 1},$$

where the implied constant depends only on k and α. [1]

The proof of Theorem 1 depends on the another set of cuspidal modular forms constructed in [3] which we recall now. First, the space $S_m(\Gamma, \chi)$ is a finite–dimensional Hilbert space under the Petersson inner product:

$$\langle f_1, \ f_2 \rangle = \int_{\Gamma \backslash \mathbb{H}} y^m f_1(z) \overline{f_2(z)} \ \frac{dxdy}{y^2}.$$

Then, for $m \geq 3$ and $\xi \in \mathbb{H}$, we define the cuspidal modular forms $\Delta_{k,m,\xi,\chi}$ in the following way:

$$\langle f, \ \Delta_{k,m,\xi,\chi} \rangle = \frac{d^k f}{dz^k}\Big|_{z=\xi}, \quad f \in S_m(\Gamma, \chi), \quad k \geq 0. \tag{2}$$

Those cuspidal modular forms had been known to Petersson and Selberg at least when $k = 0$. Based on this in [3] we prove the following expansion:

$$\begin{aligned} \Delta_{k,m,\xi,\chi}(z) &= \epsilon_\Gamma^{-1} 2^{m-2} \pi^{-1} (\sqrt{-1})^m \times \\ &\times \prod_{i=0}^{k} (m-1+i) \sum_{\gamma \in \Gamma} \frac{1}{\left(\gamma.z - \overline{\xi}\right)^{k+m}} \mu(\gamma, z)^{-m} \chi(\gamma)^{-1}. \end{aligned} \tag{3}$$

The construction is also valid when Γ has no cusps at all, but in our set–up (i.e., when ∞ is a regular cusp for Γ) is more elementary and it gives the following Fourier expansion:

$$\Delta_{k,m,\xi,\chi}(z) = \frac{(4\pi)^{m-1}}{(m-2)! h^m} \sum_{l=1}^{\infty} \left(l^{m-1} \overline{d^k \psi_{l,m,\chi}(\xi)/d\xi^k} \right) e(lz/h).$$

This expansion along with a variant of the trivial estimate for Fourier coefficients of $\Delta_{k,m,\xi,\chi}$ implies Theorem 1.

We proceed with a more general construction of cuspidal modular forms via Poincaré series and their non–vanishing.

Theorem 2. *Let $m \geq 3$. Then, for a holomorphic function $\varphi : \mathbb{H} \to \mathbb{C}$ such that $\varphi(z + h) = \varphi(z)$ and $\int_0^h \int_0^\infty y^{m/2} |\varphi(z)| \frac{dxdy}{y^2} < \infty$, the series $P_\Gamma(\varphi)(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \varphi(\gamma.z) \chi^{-1}(\gamma) \mu(\gamma, z)^{-m}$ converges absolutely and uniformly on compact sets in $\mathbb{H}$ to an element of $S_m(\Gamma, \chi)$. Moreover, the series is not identically zero provided that there exists a compact set $C \subset [0, \ h] \times \ ]0, \ \infty[$ such that the following two conditions hold:*

(1) *$\gamma.C \cap C \neq \emptyset$ implies $\gamma \in \Gamma_\infty$, and*
(2) *$\int\int_C y^{m/2} |\varphi(z)| \frac{dxdy}{y^2} > \frac{1}{2} \int_0^h \int_0^\infty y^{m/2} |\varphi(z)| \frac{dxdy}{y^2}$.*

[1] This is in particular true for $\alpha = m/2$ by the trivial estimate.

The cuspidality result is standard while the non–vanishing is given by [2], Lemma 3-1. We use Theorems 1 and 2 to study the non–vanishing of L–functions attached to modular forms $f \in S_m(\Gamma, \chi)$, for $m \geq 3$. We recall the definition of a L–function. If $f(z) = \sum_{n=1}^{\infty} a_n(f) e(nz/h)$ is the Fourier expansion of $f \in S_m(\Gamma, \chi)$. Then, by standard theory, we have $|a_n(f)| = O\left(n^{m/2}\right)$. This implies that the L–series $L(s, f) = \sum_{n=1}^{\infty} \frac{a_n(f)}{n^s}$ defines a holomorphic function in the region $Re(s) > \frac{m}{2} + 1$. Also, by the standard properties of the classical Poincaré series we can write (as in [1]) $L(s, f) = \frac{(4\pi)^{m-1}}{(m-2)!h^m} \langle f,\ \sum_{n=1}^{\infty} \frac{1}{n^{\overline{s}-m+1}} \psi_{n,m,\chi} \rangle$, for $Re(s) > \frac{m}{2} + 1$. The fact is that Theorem 1 assures that $\sum_{n=1}^{\infty} \frac{1}{n^{\overline{s}-m+1}} \psi_{n,m,\chi}$ is a well–defined element of $S_m(\Gamma, \chi)$. The Lipschitz's identity helps us to move from $Re(s) > \frac{m}{2} + 1$ to $Re(s) > \frac{m}{2}$: we define $\varphi(z, s) = \sum_{n=1}^{\infty} n^{s-1} e(nz/h)$ for $s \in \mathbb{C}$. Then, for $1 < Re(s) < m/2$ or $Re(s) < m/2 - 1$, $\varphi(\cdot, s)$ satisfies the assumptions of Theorem 2 in place of φ. We let $\Psi_m(\cdot, s, \chi) = P_\Gamma(\varphi(\cdot, s))$, and we have the following:

Theorem 3. *Let $m \geq 5$. Then, for $f \in S_m(\Gamma, \chi)$, we have that $L(s, f)$ can be analytically continued to the region $Re(s) > m/2$ such that*

$$L(s, f) = \frac{(4\pi)^{m-1}}{(m-2)!h^m} \langle f,\ \Psi_m(z, m - \overline{s}, \chi) \rangle.$$

Finally, we can use the non–vanishing criterion given by Theorem 2 along with Theorem 3 to study the non–vanishing of L–functions. We define ϵ_Γ to be the infimum of all $|c|$, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ranges over $\Gamma - \Gamma_\infty$.

Theorem 4. *Let $\epsilon, \nu, \eta > 0$, $\epsilon \leq \nu$. We assume that $1/2 \notin]\epsilon, \nu[$ if we consider odd integers m. Then, for*

$$m \geq \frac{4\pi}{h\epsilon_\Gamma} + \begin{cases} 2\nu + 4 + \left(\frac{2\pi}{h\epsilon_\Gamma}\right)^{2\nu} \frac{4e^{3\pi\eta+5}}{\epsilon^2}; & \text{if } m \text{ is even or } \epsilon \geq 1/2 \\ 4\nu + 4 + \left(\frac{2^{3/2} e^{\frac{5}{2} + \frac{3\pi\eta}{2}}}{\epsilon}\right)^{1/\epsilon}; & \text{if } m \text{ is odd and } \epsilon < 1/2, \end{cases}$$

and s in the region $m/2 + \nu \geq Re(s) \geq m/2 + \epsilon$, $Im(s) \leq \eta$, there exists $f \in S_m(\Gamma, \chi)$ satisfying $L(s, f) \neq 0$, for any χ satisfying (1). In particular, if χ is primitive modulo $N \geq 1$, then $S_m(\Gamma_0(N), \chi)$ consists of newforms, and consequently f can be taken to be a newform.

Since it is well–known that $h\epsilon_\Gamma \geq 1$, we can make m independent of Γ. Also, using the trick that $f(z) = \sum_{n=1}^{\infty} a_n e(nz) \in S_m(\Gamma_0(N), \chi)$ implies that $f^\vee(z) = \overline{f(-\overline{z})} = \sum_{n=1}^{\infty} \overline{a}_n e(nz) \in S_m(\Gamma_0(N), \chi^{-1})$ we can get rid η.

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Large values of cusp forms on GL_n

Farrell Brumley
(joint work with Nicolas Templier)

We study the large values attained by cusp forms in the transition range on non-compact locally symmetric spaces in higher rank. Our most complete results are for self-dual GL_3 Maass forms.

1. The compact picture

Let M be a compact Riemannian manifold of dimension d. Let f be an L^2-normalized eigenfunction of the Laplacian: $\Delta f + \lambda f = 0$, $||f||_2 = 1$. A basic question in the spectral geometry of M is,

How large can the sup norm $||f||_\infty = \sup_{x\in M} |f(x)|$ *be?*

For compact manifolds, it is known [8] that there are infinitely many eigenfunctions f. Our interest is then in the large λ aspect.

1.1. **Compact manifolds.** The first results toward this question are due to Hormander [2] and Sogge [11]. They show, by local considerations, that $||f||_\infty \ll \lambda^{\frac{d-1}{4}}$. This bound is sharp for the d-sphere, where the zonal spherical harmonics realize the upper bound. It is expected to be far from the truth for M of strictly negative sectional curvature. Indeed, in this case the ergodicity of the geodesic flow for such M should tend to delocalize eigenstates.

For example, in the case when $d = 2$ and M is a compact hyperbolic surface (meaning, of constant curvature -1), a conjecture of Iwaniec-Sarnak [4] states a Lindelöf-type bound of $||f||_\infty \ll_\varepsilon \lambda^\varepsilon$. They gave evidence towards their conjecture by showing that for M arithmetic and f an eigenfunction of all Hecke correspondences then $||f||_\infty \ll_\varepsilon \lambda^{5/24+\varepsilon}$, an improvement over the local bound.

The higher dimensional case is more delicate. Take $d = 3$ and M a compact hyperbolic 3 manifold. In this situation the Lindelf type bound is not true (nor was it conjectured). In fact, Rudnick and Sarnak, in one of their first investigations into quantum chaos and scarring [9], found arithmetic M admitting sequences of eigenfunctions f_j with $||f_j|| \gg \lambda_j^{1/4}$. These f_j come from a base change lifting. More recently, Milicevic [7] identified the precise class of arithmetic hyperbolic 3-folds M for which such sequences exist. Furthermore, the meaning of the power $1/4$ has been clarified by Sarnak in his purity conjecture [10].

1.2. **Compact locally symmetric spaces.** In the presence of more symmetries, one can do better. For example, consider the torus $M = \mathbb{T}^d$. Take f to be an eigenfunction of the linear differeential operators $i\partial/\partial x_j$, $j = 1, \ldots, d$. Then f is a product of exponentials in d variables, from which it follows that $||f||_\infty \leq 1$. Clearly, this is far from the Hormander-Sogge bound.

The proper generalization of the above situation is to compact locally symmetric spaces $M = \Gamma\backslash G/K$, where G is a real semi-simple Lie group, Γ is a uniform lattice in G, and K is a maximal compact subgroup of G. If G has rank r, then the ring $\mathcal{D}$ of G-invariant differential operators on M is a polynomial algebra in r variables, containing Δ. Then a theorem of Sarnak [10] states that if f is an L^2-normalized $\mathcal{D}$-eigenfunction on a compact locally symmetric space M (of dimension d and rank r) then $||f||_\infty \ll \lambda^{\frac{d-r}{4}}$. The exponent $(d-r)/4$ recovers the uniform bound in the torus example. Sarnak's theorem should be thought of as the analog of the Hormander-Sogge bound for this particular class of M, in that it is purely local.

2. The non-compact picture

If M is non-compact, it is of course easy to adapt the aforementioned techniques to prove the same bounds when the eigenfunctions are restricted to a fixed bounded subset $\Omega \subset M$. One obtains the same numerical exponents, with the constants now depending on Ω. For sup norms over the entire manifold, however, few of the bounds listed in the previous paragraph remain true, once the compactness assumption is removed.

2.1. **Dependence on M.** To see why this should be the case, let us consider the dependence on the manifold M in the two local bounds referenced above, that of Hormander-Sogge and of Sarnak.

Donnelly [1] showed that, once the dimension of M is fixed, the implied constant in the local bound of Hormander and Sogge depends only on

(1) an upper bound on the absolute value of the sectional curvature of M, and
(2) a lower bound on the injectivity radius R of M.

One can infer from this that the local methods of Hormander and Sogge should extend to complete non-compact manifolds M of *bounded geometry.* But this class is quite small among all complete non-compact manifolds, where the injectivity radius will often vanish.

On the other hand, when M is a compact locally symmetric space, the dependence of the implied constant on R in Sarnak's theorem is much easier to extricate. Indeed a direct inspection of Sarnak's proof yields $||f||_\infty \leq C\mathrm{vol}(B(o,R))^{-1}\lambda^{\frac{d-r}{4}}$, where $o \in M$ is arbitrary and $B(o,R)$ is the ball of radius R. We conclude that this line of reasoning cannot be made to apply to non-compact locally symmetric M, since the injectivity radius of a such spaces is always zero.

2.2. **The work of Iwaniec-Sarnak.** So what *is* known for non-compact M? Very little, in fact. It seems that the only non-trivial results for sup norms on non-compact M are again due to Iwaniec-Sarnak [4], and a follow-up article by

Koyama [5]. For example, in [4] the authors show that for M a non-compact arithmetic hyperbolic surface and f a Hecke-Maass L^2-normalized cusp form, one has $||f||_\infty \ll_\varepsilon \lambda^{5/24+\varepsilon}$, precisely generalizing their bound for compact arithemetic M. In [10] Iwaniec and Sarnak show something more: with the above assumptions one has $||f||_\infty \gg_\varepsilon \lambda^{\frac{1}{12}-\varepsilon}$. Hence one has *power growth* for all f!

The reason for this power growth is easy to see (and has considerably less arithmetic significance than the upper bound). High in the cusp, the size of f is determined by the first Fourier-Bessel coefficient. Indeed, if f is an even L^2-normalized Hecke-Maass form with eigenvalue $\lambda = 1/4+r^2$, then for y large enough ($2\pi y = r + O(1)$ works), one has

$$f(x+iy) = 2y^{1/2}\rho_f(1)K_{ir}(2\pi y)\cos(2\pi x) + O(e^{-r}).$$

Now one must insert two ingredients:

(1) the arithmetic bound $\rho_f(1) \gg_\varepsilon r^{-\varepsilon}\exp(\pi r/2)$ due to Iwaniec [3], and
(2) the local estimate $K_{ir}(2\pi r) \asymp r^{-1/3}\exp(-\pi r/2)$ which is classical.

So up to an arithmetical factor, the behavior in the cusp is modeled by that of the Bessel function. Now the Bessel function oscillates for $0 < y < r - r^{1/3}$, decays rapidly for $y > r + O(1)$, and during the transition between these two regimes attains its highest value ($\asymp r^{-1/3}$). This spike is due to a degenerate singularity in the oscillatory integral defining K_{ir}. This singularity is of fold type, giving rise to the Airy function, which then governs the asymptotics of K_{ir} in the transition region and accounts for the lower bound $||f||_\infty \gg_\varepsilon \lambda^{1/12-\varepsilon}$.

3. The big picture

One would like to understand this phenomenon for cusp forms on more general non-compact locally symmetric spaces, especially in higher rank.

There are essentially two objectives, closely related to each other.

(1) Obtain lower bounds on $||f||_\infty$ in terms of the Laplace eigenvalue λ, or more generally, in terms of its spectral parameters (the eigenvalues of $\mathcal{D}$).
One would then like to compare the resulting power growth against the $\lambda^{\frac{d-r}{4}}$ bound for the compact locally symmetric case and to interpret the exponent in terms of notions in semi-classical analysis.
(2) Quantify the old observation, credited to Gelfand and Piatetski-Shapiro, that a cusp form decays rapidly in the cusp.
There are two equivalent ways of expressing this more formally.
   (a) The first is to determine the "essential support" of f; this is the bulk of the space on which f oscillates and begins its transition to rapid decay. This notion is especially well-adapted for cusp forms on groups defined over function fields, where the support is compact (modulo the center).
   (b) Alternatively one could explicate the dependence on f in the decay estimate $f(g) \ll_A ||g||^{-A}$, where $||\cdot||$ is, say, the Frobenius norm on PGL_n.

In each case, one is aiming for uniformity with respect to the spectral parameters of f.

4. Our results

Our main result answers the above two questions in the first non-trivial higher rank case.

Theorem 1 (B. - Templier)**.** *Let f be an L^2-normalized self-dual Maass cusp form for $\mathrm{PGL}_3(\mathbb{Z})$ having Laplacian eigenvalue λ. Then*

$$||f||_\infty \gg_\varepsilon \lambda^{1/3-\varepsilon}.$$

Moreover,

$$f(g) \ll_A \lambda^{A/2}||g||^{-A}$$

for g in a Siegel set for $\mathrm{PGL}_3(\mathbb{Z})$, the implied constant being independent of λ.

Remark 2*.* The second statement can be interpreted to mean that the "essential support" of f is $\mathfrak{S}_{1/2,\lambda^{1/2}}$, the truncation of the Siegel domain for $\mathrm{PGL}_3(\mathbb{Z})$ to height less than $\lambda^{1/2}$.

Remark 3*.* We have some preliminary results on PGL_n which indicate that for n large enough the convexity bound $||f||_\infty \ll \lambda^{(d-r)/4}$ is far from the truth on non-compact quotients.

Sketch of Proof: The proof of Theorem 1 follows the same strategy as outlined in the GL_2 case. Taking the place of the arithmetical bound is the convexity bound on the residue of the Rankin-Selberg L-function at $s = 1$, which in the context of GL_n is due to Li [6]. More substantial is the local input, which requires uniform asymptotics (or lower bounds, at least) of the Jacquet-Whittaker integral. To obtain these, one must identify the critical points of the phase function, and classify their degeneracy. Interestingly, it turns out that there is a caustic curve of degenerate singularities, all of them being of fold type. In the vicinity of the caustic, the Jacquet-Whittaker function is therefold modeled by the Airy function, just as in the GL_2 case.

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On the sup-norm of Maass cusp forms

Gergely Harcos
(joint work with Nicolas Templier)

Comparing the various Sobolev norms of automorphic forms is useful in the theory of quantum chaos and subconvexity of L-functions, which in turn have deep arithmetic applications. We consider the following special case.

Problem. *Let f be a Hecke–Maass cuspidal newform of level N and Laplacian eigenvalue λ. Assume that $\|f\|_2 = 1$ with respect to $dxdy/y^2$. Bound $\|f\|_\infty$ in terms of N and λ.*

In the λ-aspect the first nontrivial (and so far unsurpassed) bound is due to Iwaniec and Sarnak [6]: $\|f\|_\infty \ll_{N,\epsilon} \lambda^{5/24+\epsilon}$ for any $\epsilon > 0$. In the N-aspect the trivial bound is $\|f\|_\infty \ll_{\lambda,\epsilon} N^{\epsilon}$, while the most optimistic bound would be $\|f\|_\infty \ll_{\lambda,\epsilon} N^{-1/2+\epsilon}$. Here and later, the dependence on λ is understood to be continuous. The breakthrough in the N-aspect was recently achieved by Blomer–Holowinsky [2] who proved $\|f\|_\infty \ll_{\lambda,\epsilon} N^{-25/914+\epsilon}$, at least for square-free N. The restriction on N seems difficult to remove as it is needed for a certain application of Atkin–Lehner theory. By a systematic use of geometric arguments Templier [7] derived $\|f\|_\infty \ll_{\lambda,\epsilon} N^{-1/22+\epsilon}$, and Helfgott–Ricotta [3] improved this to $\|f\|_\infty \ll_{\lambda,\epsilon} N^{-1/20+\epsilon}$. As we shall explain below, an efficient use of Atkin–Lehner theory leads to a short and clean proof of the following result [5]:

Theorem 1. *Let f be an L^2-normalized Hecke–Maass cuspidal newform of square-free level N, trivial nebentypus, and Laplacian eigenvalue λ. Then for any $\epsilon > 0$ we have a bound*

$$\|f\|_\infty \ll_{\lambda,\epsilon} N^{-1/6+\epsilon},$$

where the implied constant depends continuously on λ.

The theorem improves our earlier bound [4] with exponent $-1/12+\epsilon$. A hybrid version can also be established, improving significantly on [2, Theorem 2].

We turn to an informal discussion of our method. Very vaguely, the idea of proving a result as above has been like this:

(1) Pick any $z \in \mathfrak{H}$ where $|f(z)|$ needs to be estimated.
(2) Apply an Atkin–Lehner operator on z to ensure that $\operatorname{Im} z$ is not too small.

(3) Use the amplification method and some trace formula to reduce the problem to a counting problem depending on z.
(4) Do the counting based on the diophantine properties of z.

Our improvement results mainly from the following shortcut:

(2') Apply an Atkin–Lehner operator on z to maximize $\operatorname{Im} z$.
(4') Observe that z has good diophantine properties automatically, allowing a more efficient counting.

For a square-free level N the Atkin–Lehner operators can be represented by matrices of the form

$$W_M = \frac{1}{\sqrt{M}} \begin{pmatrix} a & a \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}), \qquad M \mid N,$$

where $a, b, c, d \in \mathbb{Z}$ are integers satisfying

$$ad - bc = M, \qquad a \equiv 0\,(M), \qquad d \equiv 0\,(M), \qquad c \equiv 0\,(N).$$

A key feature is the multiplication rule

$$W_M W_{M'} = W_{M''} \qquad \text{with} \qquad M'' = \frac{MM'}{(M,M')^2},$$

which shows that the W_M's form a group $A_0(N)$ containing $\Gamma_0(N)$ as a normal subgroup. As a result, Atkin–Lehner operators induce an action on $\Gamma_0(N)\backslash\mathfrak{H}$ by the finite group $A_0(N)/\Gamma_0(N) \cong (\mathbb{Z}/2\mathbb{Z})^{\omega(N)}$, where $\omega(N)$ is the number of distinct prime factors of N.

By Atkin–Lehner theory [1], a Hecke–Maass cuspidal newform f of level N is an eigenvector for $A_0(N)$ with eigenvalues ± 1, therefore in examining the sup-norm of f we can restrict to the following fundamental domain for $A_0(N)$:

$$\mathcal{F}(N) := \{z \in \mathfrak{H} \mid \operatorname{Im} z \geq \operatorname{Im} \delta z \text{ for all } \delta \in A_0(N)\}.$$

Our starting point was the observation that the elements of $\mathcal{F}(N)$ have good diophantine properties (we assume that N is square-free):

Lemma 2. *Let $z = x + iy \in \mathcal{F}(N)$. Then the lattice $\langle 1, z\rangle$ has minimal distance at least $N^{-1/2}$ and covolume $y \gg N^{-1}$.*

The usefulness of this lemma becomes apparent when we relate $|f(z)|$ to a lattice counting problem depending on z. By combining the amplification method of Duke–Friedlander–Iwaniec with the pretrace formula of Selberg we obtain

$$\Lambda^2 \left|f(z)\right|^2 \ll_{\lambda,\epsilon} N^\epsilon \sum_{l \geq 1} \frac{y_l}{\sqrt{l}} M(z,l,N),$$

where $M(z,l,N)$ denotes the number of integral matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$(*) \qquad \det(\gamma) = l, \qquad c \equiv 0\,(N), \qquad \left|-cz^2 + (a-d)z + b\right|^2 \leq ly^2 N^\epsilon,$$

Λ is a large parameter (the amplifier length), and

$$y_l := \begin{cases} \Lambda, & l = 1; \\ 1, & l = l_2 \text{ or } l_1 l_2 \text{ or } l_1 l_2^2 \text{ or } l_1^2 l_2^2 \text{ with } \Lambda < l_1, l_2 < 2\Lambda \text{ primes}; \\ 0, & \text{otherwise}. \end{cases}$$

Our second key observation is that for each range $l \asymp L$ and for each fixed c the inequality in $(*)$ can be used to bound the number of choices for the pair $(a-d, b)$. Indeed, the pairs correspond to the lattice points in a disk of radius $\ll L^{1/2} y N^{\epsilon}$, hence the Lemma together with some geometry of numbers yields the bound

$$\#(a-d, b) \ll_{\epsilon} N^{\epsilon} \left(1 + L^{1/2} N^{1/2} y + L y\right)$$

for any c. A simple manipulation of $(*)$ also shows $c \ll_{\epsilon} L^{1/2} N^{\epsilon}/y$. Finally for any triple $(c, a-d, b)$ we regard the identity

$$(a+d)^2 - 4l = (a-d)^2 + 4bc$$

as an equation for the pair $(a+d, l)$. By the sparsity of potential l's we can bound the number of quadruples $(c, a-d, b, a+d)$ efficiently, which of course is the same as bounding the sum of $M(z, l, N)$ over the l's considered.

Along these lines we obtain

$$\Lambda^2 \left|f(z)\right|^2 \ll_{\lambda,\epsilon} N^{\epsilon} \left(\Lambda + \Lambda^{5/2} N^{-1/2} + \Lambda^4 N^{-1}\right),$$

at least when $y < N^{-2/3}$ and $\Lambda^4 < y^{-2} N^{-\epsilon}$. The latter is automatic for $y < N^{-2/3}$ under the choice

$$\Lambda := N^{1/3-\epsilon},$$

which incidentally also balances the terms in the previous display. Hence by amplification we really see that

$$f(x+iy) \ll_{\lambda,\epsilon} N^{-1/6+\epsilon}, \qquad y < N^{-2/3}.$$

For the remaining range $y \geq N^{-2/3}$ we use a simple bound based on the Fourier expansion at the cusp ∞ (see [7, § 3.2] or [2, (92) & (27)]):

$$f(x+iy) \ll_{\lambda,\epsilon} N^{-1/2+\epsilon} y^{-1/2} \leq N^{-1/6+\epsilon}, \qquad y \geq N^{-2/3}.$$

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Trace formulas, character sums and multiple Dirichlet series

Adrian Diaconu
(joint work with Vicentiu Pasol)

For a root system Φ and a field F containing the n-th roots of unity, one associates a multiple Dirichlet series,

$$Z_\Psi(s_1,\ldots,s_r;\Phi) \;=\; \sum_{c_1,\ldots,c_r\neq 0} H(c_1,\ldots,c_r)\Psi(c_1,\ldots,c_r)\mathrm{N}(c_1)^{-s_1}\cdots\mathrm{N}(c_r)^{-s_r},$$

where the sum is over ideals (c_i) of the ring $\mathfrak{o}_S$ of S–integers for some sufficiently large set S of places; it is assumed that the finite set S contains all archimedean places, and that $\mathfrak{o}_S$ is a principal ideal domain. The product $H\Psi$ remains unchanged if c_i, $1\le i\le r$, is multiplied by a unit, i.e., it is a function of ideals in $\mathfrak{o}_S$. The function H is very important, giving the structure of the multiple Dirichlet series. It is completely determined, via a *twisted* multiplicativity, by the function field analog of $Z_\Psi(s_1,\ldots,s_r;\Phi)$, which turns out to be a rational function. The factor Ψ is less important, and represents a technical device chosen from a finite-dimensional vector space of functions on $F_S=\prod_{v\in S}F_v$, constant on cosets of an open subgroup, such that, together with H, it gives a multiple Dirichlet series possessing meromorphic continuation to $\mathbb{C}^r$ and satisfying a finite group of functional equations isomorphic to the Weyl group of Φ.

For very important applications in number theory, it is necessary, however, to establish relevant meromorphic continuation of multiple Dirichlet series possessing infinite groups of functional equations. In 1996, Bump, Friedberg and Hoffstein noticed that this type of object, *cannot* be continued in all its variables $s_1,\ldots,s_r$ to $\mathbb{C}^r$, and must have, in fact, a wall of singularities.

For concreteness, consider the series over $\mathbb{Q}$ associated to the r-th moment of quadratic Dirichlet L–functions

$$\begin{aligned}\sum_{d\neq 0}\frac{L(s_1,\chi_d)\cdots L(s_r,\chi_d)}{|d|^{s_{r+1}}} &= \sum_{m_1,\ldots,m_r\neq 0}\frac{L(s_{r+1},\chi_{m_1\cdots m_r})}{|m_1|^{s_1}\cdots|m_r|^{s_r}}\\ &= \sum_{\substack{m_1,\ldots,m_r,\,d\neq 0\\ d=d_0d_1^2,\ d_0\text{ square-free}}}\frac{\chi_{d_0}(\hat{m}_1)\cdots\chi_{d_0}(\hat{m}_r)}{|m_1|^{s_1}\cdots|m_r|^{s_r}|d_0|^{s_{r+1}}d_1^{2s_{r+1}}}\cdot a(m_1,\ldots,m_r,d).\end{aligned}$$

Here $\hat{m}_i$, $i=1,\ldots,r$, denotes the part of m_i coprime to d_0. This multiple Dirichlet series is symmetric with respect to $s_1,\ldots,s_r$, and satisfies a group W_r of functional equations isomorphic to a Coxeter group with the distinguished generators $\alpha_1,\ldots,\alpha_{r+1}$,

$$\alpha_i: \quad s_i \to 1 - s_i, \quad s_j \to s_j \text{ for } j \neq i, \quad s_{r+1} \to s_{r+1} + s_i - \tfrac{1}{2} \quad (1 \leq i \leq r)$$
$$\alpha_{r+1}: \quad s_j \to s_j + s_{r+1} - \tfrac{1}{2} \text{ for } 1 \leq j \leq r, \quad s_{r+1} \to 1 - s_{r+1}.$$

The group W_r is associated to a symmetrizable Kac-Moody Lie algebra attached to the generalized $(r+1) \times (r+1)$ Cartan matrix

$$A = \begin{pmatrix} 2 & & & & -1 \\ & 2 & & & -1 \\ & & & & \vdots \\ & & & 2 & -1 \\ -1 & -1 & \cdots & -1 & 2 \end{pmatrix}.$$

In particular, W_r is finite for $r \leq 3$, and infinite for $r \geq 4$. When $r = 4$, it is a Coxeter group of type affine D_4 associated with the extended Dynkin diagram $D_4^{(1)}$. The coefficients $a(m_1, \ldots, m_r, d)$ above are multiplicative, in the sense that

$$a(m_1, \ldots, m_r, d) \;=\; \prod_{\substack{p^{k_i} || m_i \\ p^\ell || d}} a(p^{k_1}, \ldots, p^{k_r}, p^\ell),$$

the product being over primes.

When $r \leq 3$, one deals with this multiple Dirichlet series in two steps:

- The coefficient $a(p^{k_1}, \ldots, p^{k_r}, p^\ell)$ is *uniquely* determined by the above properties imposed on the multiple Dirichlet series; it can be computed from the corresponding coefficient in the power series expansion of the function field analog of this multiple Dirichlet series.
- The meromorphic continuation to $\mathbb{C}^r$ is an immediate consequence of Bochner's theorem on analytic continuation of functions in several complex variables, applied to a union of translates by functional equations of an initial region of absolute convergence.

By contrast, the nature of the coefficients $a(p^{k_1}, \ldots, p^{k_r}, p^\ell)$, for $r \geq 4$, is not at all understood. The above constraints imposed on the multiple Dirichlet series are no longer sufficient to determine these coefficients. Needless to say, all methods used so far to establish meromorphic continuation to $\mathbb{C}^r$ of multiple Dirichlet series satisfying finite groups of functional equations, either did not adapt to the present situation when $r \geq 4$, or fell short of producing any significant result.

While the present work *does not* offer any new insight in obtaining meromorphic continuation for such objects, it reveals that the structure of the above multiple Dirichlet series when $r \geq 4$ should ultimately be understood using methods from algebraic geometry. Thus, the following natural questions arise:

Is there a canonical choice of the coefficients $a(p^{k_1}, \ldots, p^{k_r}, p^\ell)$ *when* $r \geq 4$? *If so, do these coefficients have any connection with the theory of automorphic forms?*

In joint work with Chinta and Gunnells, it was noticed that the global object over $\mathbb{Q}$ (or any number field) can be constructed from its function field analog, in the following sense. Given such a multiple Dirichlet series over a finite field with q elements, we can present it by a power series

$$\sum_{k_1,\ldots,k_r,\,\ell\geq 0} \lambda(k_1,\ldots,k_r,\ell;q)\, x_1^{k_1}\cdots x_r^{k_r} x_{r+1}^{\ell} \qquad (x_i = q^{-s_i} \text{ for } i=1,\ldots,r+1).$$

We take this series normalized by $\lambda(0,\ldots,0,0;q)=1$. We remark that it suffices to pick a power series satisfying the correct functional equations (see [2]) of the multiple Dirichlet series associated to the r-th moment over $\mathbb{F}_q(T)$; when $r\leq 3$ there is a unique such series, and for $r\geq 4$ there are *infinitely* many. In addition, we can choose a series for which the coefficients $\lambda(k_1,\ldots,k_r,\ell;1/q)$ are defined. Setting

$$a(p^{k_1},\ldots,p^{k_r},p^{\ell}) := p^{k_1+\cdots+k_r+\ell}\lambda(k_1,\ldots,k_r,\ell;1/p),$$

one verifies that the resulting multiple Dirichlet series over $\mathbb{Q}$ satisfies indeed the correct functional equations. However, to have any chance of progress, it is necessary to eliminate the *arbitrariness* of the coefficients λ by exploiting the fact that these multiple Dirichlet series are number-theoretic objects.

In light of this, understanding the structure of the above multiple Dirichlet series over function fields, from number-theoretic viewpoint, is a very important problem. The starting point of our investigation is based on our personal belief that there should exist a Langlands type philosophy attaching the original multiple Dirichlet series over $\mathbb{Q}$ (or any number field) to an automorphic representation. There are reasons for this belief, especially through the work of Beilinson, Braverman, Drinfeld, Gaitsgory, Kapranov, Kazhdan and others. This would also explain the local-global compatibility already mentioned.

It turns out that the local components (i.e., the coefficients $\lambda(k_1,\ldots,k_r,\ell;q)$) are completely determined what they ought to be by connecting moments of character sums of hyperelliptic curves of genus g over finite fields to traces on spaces of automorphic forms –a comparison of Arthur-Selberg and Grothendieck-Lefschetz trace formulas. When $g=1$, this reduces to a well-known result of Birch [1]. (We recall that one side of Birch's identity involves moments of character sums of elliptic curves over a fixed finite field.)

Unlike the genus one case, when $g=2$ the moment side is of the form

$$\textit{moment side} \;=\; \textit{moments of genus two curves} + \textit{degenerate part}$$

with degenerate part consisting of moments of incomplete character sums attached to elliptic curves over $\mathbb{F}_q$ (i.e., the degeneracy corresponds to lower genus curves). Although with growing complexity, this structure of the moment side persists in general. It also appears in general that the moment side is explicitly related to alternating sums of traces of Frobenius on compactly supported ℓ–adic étale cohomology of certain local systems on classical moduli spaces. (When $g=2$, we work with the Baily-Borel compactification of the moduli space of principally polarized abelian surfaces.) The cohomology of these local systems on moduli $\mathcal{M}_g$, $\mathcal{A}_g$, $\mathcal{M}_{g,n}$ etc. is intimately connected with (vector-valued) Siegel modular forms (see [3], [5], [6], [7], [4]).

Accordingly, we expect the coefficients $\lambda(k_1, \ldots, k_r, \ell; q)$ to be determined from moment identities

moment side = combination of traces of Hecke operators on spaces of Siegel modular forms

with right hand side incorporating contributions from Eisenstein cohomology, and (eventual) endoscopic contributions (see [8], [9] and [10]). We carried out completely this program when $g = 2$.

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Statistics of wave functions for a point scatterer on the torus

Zeev Rudnick

(joint work with Henrik Ueberschär)

Quantum systems whose classical counterpart have ergodic dynamics satisfy Schnirelman's theorem, which asserts that almost all eigenstates are uniformly distributed in phase space in an appropriate sense [17, 4, 18]. In contrast, when the classical dynamics is integrable, there is concentration of eigenfunctions on invariant structures in phase space. In joint work with Henrik Ueberschär [14], we study eigenfunction statistics for an intermediate system, that of a point scatterer on the standard flat torus.

The use of point scatterers, or δ-potentials, goes back to the Kronig-Penney model [9] which is an idealized solvable model used to explain conductivity in a solid crystal and the appearance of electronic band structure. They have also been studied in the mathematical literature to explain the spurious occurrence of the Riemann zeros in a numerical experiment [6]. Billiards with a point scatterer have been used extensively in the quantum chaos literature to model quantum systems strongly perturbed in a region smaller than the the wavelength of the particle (the Planck scale). Much of that literature dealt with spectral statistics of such systems, finding intermediate statistics in the case of perturbations of integrable systems [15, 16, 2, 3, 13].

We consider the standard flat torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ with the Laplacian $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, which has as its spectrum the set of integers $\mathcal{N}$ which are sums of two squares, with multiplicities $r(n)$ which is the number of representations of $n = a^2 + b^2$ with $a, b \in \mathbb{Z}$ integers, the corresponding orthogonal basis of eigenfunctions being $e^{i(ax+by)}$. The geodesic flow on the torus is completely integrable. Placing a point scatterer at $x_0 \in \mathbb{T}^2$ does not change the classical dynamics except for a measure zero set of trajectories, and gives a quantum system whose dynamics is generated by an operator formally written as

$$-\Delta + \alpha\delta_{x_0} \tag{1}$$

with δ_{x_0} being the Dirac mass at x_0 and α being a coupling parameter. Mathematically this corresponds to picking a self-adjoint extension of the Laplacian $-\Delta$ acting on functions vanishing near x_0 Such extensions are parameterized by a phase $\phi \in (-\pi, \pi]$, with $\phi = \pi$ corresponding to the standard Laplacian ($\alpha = 0$ in (1)). We denote the corresponding operator by $-\Delta_{x_0,\phi}$, whose domain consists of a suitable space of functions $f(x)$ whose behavior near x_0 is given by

$$f(x) = C\left(\cos\frac{\phi}{2} \cdot \frac{\log|x - x_0|}{2\pi} + \sin\frac{\phi}{2}\right) + o(1), \quad x \to x_0 \tag{2}$$

for some constant C. For $\phi \neq \pi$ ($\alpha \neq 0$) the resulting spectral problem still has the eigenvalues from the unperturbed problem, with multiplicity decreased by one, as well as a new set Λ_ϕ of eigenvalues interlaced between the sequence of sums of two squares, each appearing with multiplicity one, with the corresponding eigenfunction being the Green's function $G_\lambda(x; x_0) = (\Delta + \lambda)^{-1}\delta_{x_0}$. Consequently, by using Landau's theorem [10] on sums of two squares we can determine the asymptotics for the counting function of the spectrum

$$\#\{\lambda \in \Lambda_\phi : \lambda \leq x\} \sim B\frac{x}{\sqrt{\log x}} \tag{3}$$

where $B = \frac{1}{\sqrt{2}}\prod(1 - p^{-2})^{-1/2} = 0.764\ldots$, the product over primes $p = 3 \bmod 4$. The empirical distribution of the normalized nearest-neighbor spacings in the perturbed spectrum, plotted in Figure 1, appears to display level repulsion similar to that observed in [15, 16, 2, 3, 13].

Our main result is that for almost all $\lambda \in \Lambda_\phi$, the perturbed eigenfunctions $G_\lambda(\bullet; x_0)$ are uniformly distributed in position space. To formulate the result precisely, we denote by $g_\lambda(x) := G_\lambda(x; x_0)/||G_\lambda||_2$ the L^2-normalized Green's function:

Theorem 1 ([14]). *Fix $\phi \in (-\pi, \pi)$. There is a subset $\Lambda_{\phi,\infty} \subset \Lambda_\phi$ of density one so that for all observables $a \in C^\infty(\mathbb{T}^2)$,*

$$\int_{\mathbb{T}^2} a(x)g_\lambda(x)^2 dx \to \frac{1}{\text{area}(\mathbb{T}^2)}\int_{\mathbb{T}^2} a(x)dx \tag{4}$$

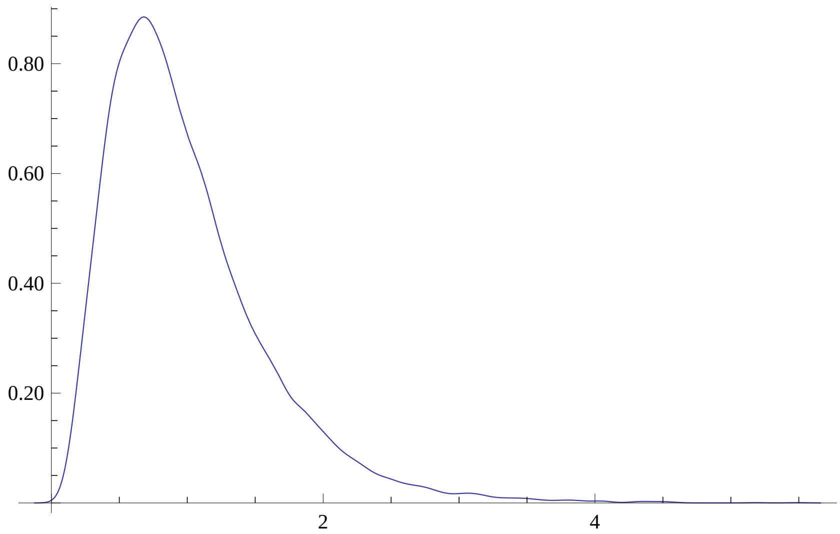

Figure 1. The level spacing distribution for the perturbed eigenvalues on the rational torus. Displayed are about 13,000 levels around $\lambda = 10^{10}$. (Numerics by Maja Rudolph)

as $\lambda \to \infty$ *along the subsequence* $\Lambda_{\phi,\infty}$

A key arithmetic ingredient in the proof of Theorem 1 is information on the number of sums of two squares in short intervals [5, 11, 12].

We note that for the eigenfunctions of the unperturbed Laplacian, there is a variety of possible limits in the position representation, which were investigated by Jakobson [7].

A related, and in some sense complementary, issue was studied by Berkolaiko, Keating and Winn [1] who predict that for an irrational torus with a point scatterer there is a subsequence of eigenfuctions which "scar" in momentum space, and this was proved by Keating, Marklof and Winn [8] to be the case assuming that the eigenvalues of the Laplacian on the unperturbed irrational torus have Poisson spacing distribution, as is predicted by the Berry-Tabor conjecture.

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The field of coefficients of a modular form

V. Kumar Murty

Let f be a holomorphic cusp form of weight $k \geq 2$, level N, and character ϵ which is a normalized eigenform for the Hecke operators. The Fourier coefficients $\{a_n\}$ of f generate a number field E_f. It is natural to ask how many Fourier coefficients are needed to generate E_f. Suppose that f is not of CM type. We show that if N is squarefree and $\epsilon = 1$, then one coefficient suffices. More precisely, we show the following.

Theorem 1. *With hypotheses as above, for a set of primes p of density 1, we have*

$$E_f = \mathbb{Q}(a_p).$$

In fact, the number of primes $p \leq x$ for which a_p fails to generate E_f is

$$\ll x(\log\log x)^2(\log x)^{-22/21}.$$

If we assume the GRH, this estimate can be improved to

$$\ll x^{13/14}.$$

Without these hypotheses on N and ϵ, we can show that one coefficient suffices to generate a distinguished subfield of E_f. We also give an effective bound for the least p for which $E_f = \mathbb{Q}(a_p)$.

On Zaremba's conjecture

ALEX KONTOROVICH
(joint work with Jean Bourgain)

We describe joint work with Jean Bourgain on Zaremba's Conjecture, which is available at [3].

For given $A > 0$, let $\mathcal{C}_A \subset [0,1]$ be the Cantor-like set of real numbers x in the unit interval, whose partial quotients are bounded by A. Thus, writing x in its continued fraction expansion

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots \cfrac{1}{a_k + \ddots}}} = [a_1, a_2, \dots, a_k, \dots],$$

we have that all partial quotients a_k are bounded by A. The Hausdorff dimension δ_A of $\mathcal{C}_A$ is asymptotically

$$\delta_A = 1 - \frac{6}{\pi^2 A} - \frac{72 \log A}{\pi^4 A^2} + O\Big(\frac{1}{A^2}\Big)$$

as $A \to \infty$ [6]. In particular, $\delta_A \to 1$ as $A \to \infty$.

Further, let $\mathcal{R}_A$ denote the set of all partial convergents $\frac{b}{d}, (b,d) = 1$ of numbers in $\mathcal{C}_A$ and let $\mathcal{D}_A$ be the set of all continuants d.

Zaremba's conjecture [10] states that

$$\mathcal{D}_A = \mathbb{Z}_+$$

for sufficiently large A (possibly $A = 5$ is enough). The conjecture arises naturally in studying statistics of the linear congruential pseudorandom number generator, and in good lattice points for quasi-Monte Carlo methods for numerical multi-dimensional integration, see e.g. [8]. It is related to Markoff and Lagrange spectra, as well as families of divergent low-lying geodesics on the modular surface, see e.g. [7].

It was shown by Niederreiter [9] that Zaremba's conjecture holds for small powers, in fact

$$\{2^j\} \subset \mathcal{D}_3.$$

On the other hand, a result due to Hensley [5] states that there are constants $0 < c < C < \infty$ so that for N sufficiently large,

$$c \cdot N^{2\delta_A} < \#\Big\{\frac{b}{d} \in \mathcal{R}_A; (b,d) = 1 \text{ and } 1 \le b < d \le N\Big\} < C \cdot N^{2\delta_A}.$$

Note also that $b+d \in \mathcal{D}_A$ whenever $\frac{b}{d} \in \mathcal{R}_A$. An easy consequence the previous two facts is that

$$\#\mathcal{D}_A \cap [1,N] \gg N^{\delta_A}.$$

Thus by allowing A to get very large, one can produce $N^{1-\epsilon}$ continuants up to N.

Our main result is the following

Theorem 1. *For $A \geq 50$, almost every integer satisfies Zaremba's conjecture. That is,*

$$\#\mathcal{D}_A \cap [1,N] = N\big(1+o(1)\big),$$

as $N \to \infty$.

The main observation (made long ago) leading to the above Theorem is that

$$\begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_k \end{pmatrix} = \begin{pmatrix} * & b \\ * & d \end{pmatrix}$$

is equivalent to $[a_1,\dots,a_k] = \frac{b}{d}$. It is therefore natural to consider the semi-group $\mathcal{G}_A$ generated by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix} \quad \text{with} \quad 1 \leq a \leq A.$$

Then the orbit $\mathcal{G}_A \cdot (0,1)^t$, where ^{t} denotes transpose, consists precisely of the set of coprime pairs $(b,d)^t$ with $\frac{b}{d} \in \mathcal{R}_A$. Moreover, we have the correspondence

$$\mathcal{D}_A = (0,1) \cdot \mathcal{G}_A \cdot (0,1)^t,$$

so Zaremba's problem is reduced to the study of $(2,2)$-entries of the matrices in $\mathcal{G}_A$.

Bourgain and the author attacked just such a problem in [2], studying the sequence of integers produced by an orbit of a subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, under the assumption that the dimension $0 < \delta < 1$ of the limit set of Γ close enough to 1. The technique introduced there is to proceed by the Hardy-Littlewood circle method, analyzing certain relevant exponential sums on 'minor' and 'major' arcs. While this approach is quite standard in number theoretical problems (for instance in the Goldbach problem), the ingredients involved in the present situation are special.

In [2], the analysis on the minor arcs is achieved using Vinogradov-type multilinear estimates, depending essentially on the group structure. Then a precise evaluation of the exponential sum on the major arcs is obtained by relying on the spectral and representation theory of $\Gamma \backslash \mathrm{SL}_2$, as developed in [4]. The outcome is the usual local-to-global representation formula, with a small exceptional set.

The main difference between [2] and Zaremba's problem is that instead of the group Γ, we have only the semi-group $\mathcal{G}_A$. It turns out however that this distinction has essentially no effect on the minor arcs analysis. On the other hand to proceed with the description of the exponential sum on the major arcs, the automorphic approach from [4] is no longer applicable. Instead we rely on the thermodynamic formalism based on the Ruelle transfer operator (which actually is already exploited in [5]). The necessary quantitative resonance-free region for "congruence" transfer operators is provided by [1] in a form applicable to our problem.

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A reciprocity formula for a cotangent sum

Brian Conrey

(joint work with Sandro Bettin)

For a rational number $x = h/k \neq 0$ with $(h,k) = 1$ and $k > 0$ define

$$c(x) = -\sum_{a=1}^{k-1} \frac{a}{k} \cot \frac{\pi a h}{k}.$$

The value of $c(x)$ is an algebraic number, i.e. $c : \mathbb{Q}^* \to \overline{\mathbb{Q}}$. Notice that c is odd and is periodic with period 1. Here is a table of the first few values:

x	$c(x)$
1	0
$\frac{1}{2}$	0
$\frac{1}{3}$	$\frac{1}{3\sqrt{3}}$
$\frac{1}{4}$	$\frac{1}{2}$
$\frac{1}{5}$	$\frac{(\sqrt{5}-1)\sqrt{5-\sqrt{5}}+3(\sqrt{5}+1)\sqrt{5+\sqrt{5}}}{10\sqrt{10}}$
$\frac{2}{5}$	$\frac{3(\sqrt{5}-1)\sqrt{5-\sqrt{5}}-(\sqrt{5}+1)\sqrt{5+\sqrt{5}}}{10\sqrt{10}}$
$\frac{1}{6}$	$\frac{7}{3\sqrt{3}}$
$\frac{5}{6}$	$\frac{-7}{3\sqrt{3}}$

It is not hard to see that $c(h,k)$ is contained in the maximal real subfield of the cyclotomic field of kth roots of unity.

In turns out that $c(x)$ satisfies a reciprocity formula, somewhat analogous to the Dedekind sum, but apparently deeper.

Theorem 1. *There exists a function g which is analytic on the complex plane minus the negative real axis such that*

$$xc(x) + c(1/x) - \frac{1}{\pi \operatorname{Den}(x)} = g(x)$$

for all rational numbers $x \neq 0$, where $\operatorname{Den}(x) > 0$ is the denominator of x. For $|x| < 1$, the function $g(x)$ may be expressed as

$$g(1+x) = \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n g_n x^n$$

where $g_0 = -1$, $g_1 = -1/2$ and for $n \geq 2$,

$$g_n = \frac{1}{n(n+1)} + 2b_n + 2\sum_{j=0}^{n-2} \binom{n-1}{j} b_{j+2}$$

where

$$b_n = \frac{\zeta(n)B(n)}{n}$$

with B_n denoting the nth Bernoulli number.

The coefficients are rational polynomials in π; for example, the coefficient of x^{20} is

$$\frac{1}{420} + \frac{\pi^2}{36} - \frac{19\pi^4}{600} + \frac{646\pi^6}{19845} - \frac{323\pi^8}{13500} + \frac{4199\pi^{10}}{343035} - \frac{154226363\pi^{12}}{36569373750} + \frac{1292\pi^{14}}{1403325}$$
$$- \frac{248571091\pi^{16}}{2170943775000} + \frac{1924313689\pi^{18}}{288905366499750} - \frac{30489001321\pi^{20}}{252669361772953125};$$

numerically, this is

$$0.00238095 + 0.274156 - 3.08462 + 31.2954 - 227.022 + 1146.32 - 3897.98$$
$$+8398.51 - 10308.6 + 5918.59 - 1058.25 = 0.0499998087\ldots.$$

Notice how close this number is to $1/20$.

Theorem 2. *We have*

$$g_n - \frac{1}{n} \sim 2^{5/4}\pi^{3/4}n^{-3/4}e^{-2\sqrt{\pi n}} \sin(2\sqrt{\pi n} + 3\pi/8)$$

as $n \to \infty$.

The remarkable asymptotics for g_n stated here were conjectured by Don Zagier. Also, $c(x)$ is (nearly) an example of what Zagier calls a “quantum modular form.”

Corollary 3. *The numbers $c(x)$ can be computed to within a prescribed accuracy in a time that is polynomial in $\log Den(x)$.*

It transpires that the reason for the reciprocity formula for $c(x)$ is because of its connection with what Lewis and Zagier call period functions for Maass forms and Eisenstein series.

We also mention that the function $c(x)$ arises in the Nyman–Beurling, Baez-Duarte reformulation of the Riemann Hypothesis. Specifically, with $\{x\} = x - [x]$ denoting the fractional part of x, the sum

$$V(h,k) = \sum_{a=1}^{k-1} \left\{ \frac{ah}{k} \right\} \cot \frac{\pi a}{k} = -c(\overline{h}/k),$$

known as the Vasyunin sum, arises in the study of the Riemann zeta-function by virtue of the formula (valid for $(h,k) = 1$):

$$\frac{1}{2\pi\sqrt{hk}} \int_{-\infty}^{\infty} |\zeta(1/2+it)|^2 (h/k)^{it} \frac{dt}{\frac{1}{4}+t^2}$$
$$= \frac{\log 2\pi - \gamma}{2} \left(\frac{1}{h} + \frac{1}{k} \right) + \frac{k-h}{2hk} \log \frac{h}{k} - \frac{\pi}{2hk} \left(V(h,k) + V(k,h) \right).$$

The Riemann Hypothesis is true if and only if $\lim_{N\to\infty} d_N = 0$ where

$$d_N^2 = \inf_{A_N} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|1 - \zeta A_N(1/2+it)\right|^2 \frac{dt}{\frac{1}{4}+t^2}$$

where the inf is over all Dirichlet polynomials $A_N(s) = \sum_{n=1}^N a_n n^{-s}$ of length N.

As a further application of this circle of ideas, we give an exact formula for a weighted second moment of the Riemann zeta-function on the critical line.

Theorem 4. *For $0 < \delta < \pi$, we have*

$$\int_0^\infty |\zeta(1/2+it)|^2 e^{-\delta t}\, dt \quad = \quad \frac{\gamma - \log 2\pi\delta}{2\sin\frac{\delta}{2}} + \frac{\pi i}{\sin\frac{\delta}{2}} S_0 \left(\frac{-1}{1-e^{-i\delta}} \right) + h(\delta) + k(\delta),$$

where $k(\delta)$ is analytic in $0 < \Re(\delta) < \pi$ and $h(\delta)$ is C^∞ in $\mathbb{R}$ with

$$h(\delta) = i \sum_{n\ge 0} h_n e^{-i(n+\frac{1}{2})\delta},$$

where

$$h_n \ll e^{-2\sqrt{\pi n}}$$

as $n \to \infty$.

Previously, for this second moment there was an asymptotic, but divergent, series in powers of δ.

Lebedev-Whittaker transform for $GL(n)$

Dorian Goldfeld
(joint work with Alex Kontorovich)

Let $\pi = \otimes\pi_v$ be an irreducible cuspidal automorphic representation for $GL(n, \mathbb{A})$ where $\mathbb{A}$ denotes the adele group of $\mathbb{Q}$. We shall assume that π is unramified at ∞. Let W_∞ denote a Whittaker model for π_∞. Then there exists exist spectral parameters $\nu = (\nu_1, \ldots, \nu_{n-1}) \in \mathbb{C}^{n-1}$. and a spherical Whittaker function $W_\nu \in W_\infty$ which is characterized by the fact that it is an eigenfunction of all the $GL(n, \mathbb{R})$-invariant differential operators and, in addition, satisfies

$$W_\nu(uz) = \psi(u) \cdot W_\nu(z)$$

for all $z \in GL(n, \mathbb{R})\big/ \left(O(n, \mathbb{R}) \cdot R^\times\right)$, all $u \in U(n, \mathbb{R})$ = upper triangular unipotent group of $n \times n$ real matrices, and some character ψ of $U(n, \mathbb{R})$. The Whittaker function W_ν is given explicitly in [1, Section 5.9].

The Lebedev-Whittaker transform for $GL(n, \mathbb{R})$ is defined as follows. Let $f : \mathbb{R}_+^{n-1} \to \mathbb{C}$ be a smooth function and let $t = (t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1}$. Then we define the Lebedev-Whittaker transform

$$f^\sharp(t) := \int_{\mathbb{R}_+^{n-1}} f(y) W_{it}(y)\, d^\times y.$$

provided the above integral converges absolutely.

When $n = 2$, the above transform is precisely the Kontorovich-Lebedev Bessel transform [2], [3] which was originally introduced to solve certain boundary-value problems and has since found many applications in modern analytic number theory (see [4]. The natural generalization to reductive Lie groups was worked out in [5], but an explicit version of the transform involving variables on the torus was still lacking.

In recent work, we have found that the inverse Lebedev-Whittaker transform on $GL(n, \mathbb{R})$ takes the following very explicit form. Let $h : \mathbb{R}^{n-1} \to \mathbb{C}$ be a function on the torus of $GL(n, \mathbb{R})$ which is symmetric under the Weyl group. Then the inverse Lebedev-Whittaker transform is defined by

$$h^\flat(y) = \frac{1}{(4\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} h(t)\overline{W_{it}(y)}\, \frac{dt}{\prod\limits_{1 \le k \ne \ell \le n} \Gamma\left(\frac{\alpha_k - \alpha_\ell}{2}\right)},$$

assuming the above integral converges absolutely, and where α_k are determined in [1, (11.6.15)].

It was essentially proved in [5] that under suitable growth and regularity conditions that

$$\left(f^\sharp\right)^\flat = f, \qquad \left(h^\flat\right)^\sharp = h.$$

In [6] the growth and regularity conditions are made explicit for the case of $GL(3, \mathbb{R})$. For $\eta > 0$ and $A \gg 1$ let $H_{\eta,A}$ denote the class of functions $h(t_1, t_2)$

which are Weyl group invariant and have a holomorphic extension to the horizontal strip $\Im(t_1), \Im(t_2) \in (-\eta, \eta)$, and, in addition, satisfy the growth estimate

$$h(t_1, t_2) \ll \exp\left(-\frac{3\pi}{4}\sum_{k=1}^{3}|t_k|\right)\prod_{k=1}^{e}(1+|t_k|)^{-A}, \qquad (t_3 = t_1 + t_2).$$

Then it is shown that for $\eta > 0, A \gg 1$, and $h \in H_{\eta,A}$ that the integral defining the inverse transform $h^\flat(y)$ converges absolutely and satisfies

$$h^\flat(y) \ll (y_1 y_2)^{1+\frac{\eta}{2}}.$$

Further, under the above conditions, we also have

$$\left(h^\flat\right)^\sharp = h.$$

The above explicit transform proves useful in the Kuznetsov trace formula for $GL(3, \mathbb{R})$. The Kuznetsov trace formula can be obtained by computing the inner product of two Poincaré series in two different ways. The spectral decomposition of a Poincaré series immediately leads to sums of the Lebedev-Whittaker transform (weighted by arithmetic Fourier coefficients) over the spectrum. The Lebedev-Whittaker transform that appears involves a test function associated to the Poincaré series. With a geometric computation, the inner product of two Poincaré series can also be realized as a sum of Kloosterman sums weighted by the inverse Lebedev-Whittaker transform of the same test function. Since we now have such an explicit version of the Lebedev-Whittaker trasform and its inverse, it is possible to choose Poincaré series, and hence test functions with good properties on both sides of the trace formula.

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Sampling the Lindelöf hypothesis

Jörn Steuding

The value-distribution of the Riemann zeta-function $\zeta(s)$ is of great interest in number theory. The famous open Riemann hypothesis claims that there are no zeros of $\zeta(s)$ to the right of the critical line $s = \frac{1}{2} + i\mathbb{R}$ which is equivalent to find all nontrivial (non-real) zeros on this line. The yet unproved Lindelöf hypothesis states that $\zeta(\frac{1}{2} + it) \ll t^\epsilon$ for any positive ϵ, as $t \to \infty$. The Riemann hypothesis implies the Lindelöf hypothesis.

Recently, Lifshits & Weber [2] published a paper entitled "*Sampling the Lindelöf Hypothesis with the Cauchy Random Walk*" which describes the content of their interesting paper very well. If (X_m) is an infinite sequence of independent Cauchy distributed random variables, the Cauchy random walk is defined by $\mathcal{C}_n = \sum_{m \leq n} X_m$. Lifshits & Weber proved among other things (in slightly different notation) that almost surely

$$\lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq n \leq N} \zeta(\tfrac{1}{2} + i\mathcal{C}_n) = 1 + o(N^{-\frac{1}{2}} (\log N)^b) \tag{1}$$

for any $b > 2$. It should be noted that the expectations $\mathbf{E}X_m$ and $\mathbf{E}\mathcal{C}_n$ do not exist, and, indeed, the values of $\mathcal{C}_n$ provide a sampling of randomly distributed real numbers of unpredictable size. Hence, the almost sure convergence theorem of Lifshits & Weber shows that the expectation value of $\zeta(s)$ on the Cauchy random walk $s = \frac{1}{2} + i\mathcal{C}_n$ equals one, which indicates that the values of the zeta-function are small on average.

Similar to the approach of Lifshits & Weber we study the distribution of values of the zeta-function on vertical lines with respect to the ergodic transformation $T : \mathbb{R} \to \mathbb{R}$ given by $T0 := 0$ and $Tx := \frac{1}{2}(x - \frac{1}{x})$ for $x \neq 0$. We use the abbreviation Tx for $T(x)$ and $T^n x$ is defined by $T \circ T^{n-1} x$ and $T^0 x = x$. Since T is ergodic, almost all orbits lie dense in $\mathbb{R}$ and thus contain arbitrarily large positive and arbitrarily large negative real numbers. This provides ergodic samples for testing the Lindelöf hypothesis. The following theorem yields the almost sure convergence of the related Cesàro means:

Theorem 1. *Let s be given with $\Re s > -\frac{1}{2}$. Then*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{0 \leq n < N} \zeta(s + iT^n x) = \frac{1}{\pi} \int_{\mathbb{R}} \zeta(s + i\tau) \frac{\mathrm{d}\tau}{1 + \tau^2}. \tag{2}$$

for almost all $x \in \mathbb{R}$. Define

$$\ell(s) = \frac{1}{\pi} \int_{\mathbb{R}} \zeta(s + i\tau) \frac{\mathrm{d}\tau}{1 + \tau^2}.$$

If $\Re s < 1$, then

$$\ell(s) = \zeta(s + 1) - \frac{2}{s(2 - s)}; \tag{3}$$

here the case of $s = 0$ *is included as* $\ell(0) = \lim_{s\to 0} \ell(s) = \gamma - \frac{1}{2}$ *where* $\gamma := \lim_{M\to\infty}(\sum_{m=1}^{M} \frac{1}{m} - \log M) = 0.57721\ldots$ *denotes the Euler-Mascheroni constant. If* $s = 1 + it$ *with some real number* t*, then*

$$\ell(s) = \zeta(s+1) - \frac{1}{s(2-s)} = \zeta(2+it) - \frac{1}{1+t^2}.$$

Finally, if $\Re s > 1$*, then*

$$\ell(s) = \zeta(s+1).$$

The proof of Theorem 1 relies on the pointwise ergodic theorem of Birkhoff and calculus of residues. It is easy to see that T is measurable and, using the substitution $\tau = Tx$, $\mathrm{d}\tau = \frac{1}{2}(1 + \frac{1}{x^2})\mathrm{dx}$, it turns out that

$$\int_{\mathbb{R}} f(Tx)\frac{\mathrm{d}x}{1+x^2} = \int_{\mathbb{R}} f(\tau)\frac{\mathrm{d}\tau}{1+\tau^2}$$

for any Lebesgue integrable function f. It follows that T is measure preserving with respect to the probability measure $\mathbf{P}$ defined by

$$\mathbf{P}((\alpha,\beta)) = \frac{1}{\pi}\int_{(\alpha,\beta)} \frac{\mathrm{d}\tau}{1+\tau^2}. \tag{4}$$

Moreover, the only T-invariant sets A with respect to the related probability measure $\mathbf{P}$ are $A = \{0\}$ and $A = \mathbb{R}$ for which $\mathbf{P}(A) = 0$ or $= 1$. Hence, T is indeed ergodic and $(\mathbb{R}, \mathcal{B}, \mathbf{P}, T)$ is an ergodic system, where $\mathcal{B}$ denotes the Borel sigma-algebra associated with $\mathbb{R}$. This example of an ergodic transformation on the real line is due to Lind (cf. [1], Example 2.9). Using the Phragmén-Lindelöf principle and the functional equation for $\zeta(s)$, it turns out that the function $\tau \mapsto \frac{\zeta(s+i\tau)}{1+\tau^2}$ is Lebesgue integrable on $\mathbb{R}$ for fixed $\Re s > -\frac{1}{2}$. Now (2) follows from the pointwise ergodic theorem. For the evaluation of this ergodic limit in the case $s = \frac{1}{2}$ we can use another interpretation of the integral on the right-hand side of (2) which is implied by the work of Lifshits & Weber [2] mentioned above. Note that the density function of a Cauchy distributed random variable X is given by $\tau \mapsto \frac{1}{\pi(1+\tau^2)}$, hence, (4) is the associated probability measure and the integral in question equals the expectation of $\zeta(\frac{1}{2} + iX)$. In their account to prove (1) Lifshits & Weber computed several expectation values by elementary means, in particular, the latter one by $\zeta(\frac{3}{2}) - \frac{8}{3}$, which yields (3) in the case $s = \frac{1}{2}$. However, to give an independent analytic evaluation of $\ell(s)$ which is valid not only for $s = \frac{1}{2}$ or real values of s one may use contour integration and Cauchy's theorem.

The work of Lifshits & Weber has been extended by Shirai [3] to a subclass of Lévy processes for which a similar phenomenon was observed. His class consists of so-called symmetric α-stable processes $\mathcal{S}_n$ which includes besides the Cauchy random walk ($\alpha = 1$) also the Brownian motion ($\alpha = 2$). Shirai succeeds in proving an analogue of the theorem of Lifshits & Weber for $1 \le \alpha \le 2$. Also in his theorem the expectation of $\frac{1}{N}\sum_{1\le n\le N} \zeta(\frac{1}{2} + i\mathcal{S}_n)$ equals one, so is independent of α, and the only impact of α is visible in the remainder term.

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Discrete Mean Values of Dirichlet L-functions

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(joint work with Niko Laaksonen)

The values of two distinct L-functions at a given point s_0 can be equal for arithmetic reasons, e.g. let f_i, $i = 1, 2$ be two Hecke eigenforms of weight 2 for $\Gamma_0(N_i)$ with odd functional equation (centered at 1/2). Then $L(f_1, 1/2) = L(f_2, 1/2) = 0$. We investigate how often it can happen that the values of the two L-functions are distinct, or, even more, linearly independent over $\mathbb{Q}$ or $\mathbb{R}$. We concentrate only on Dirichlet L-functions in this note. The work is motivated by the Grand Simplicity Hypothesis [7] that states that the set

$$\{\gamma \geq 0, L(1/2 + i\gamma, \chi) = 0, \chi \text{ primitive}\}$$

is linearly independent over $\mathbb{Q}$. Significant amount of work has been done on common zeros of two L-functions. In 1976 Fujii [2] looked at the product $L(s, \chi_1)L(s, \chi_2)$ with χ_i primitive, and proved that a positive proportion of the zeros are distinct, with a zero called distinct, if it is a zero of only one of the $L(s, \chi_i)$ or a zero of both with different multiplicity. In 1988 Conrey, Ghosh and Gonek [1] proved that $L(s, \chi_1)L(s, \chi_2)$ has a least $T^{6/11}$ simple zeros up to height T, while under GRH this can be improved to a positive proportion. R. Murty and K. Murty [5] looked at the zero sets $Z_{F_i}(T) = \{s, F_i(s) = 0, |\Im s| \leq T, \Re s \geq 1/2\}$ of two L-functions in the Selberg class. If the two L-functions have many common zeros, then the symmetric difference $Z_{F_1}(T) \triangle Z_{F_2}(T)$ should have small size. They proved that, if $|Z_{F_1}(T) \triangle Z_{F_2}(T)| = o(T)$, then $F_1 = F_2$.

We investigate when $L_1(s_j) \neq L_2(s_j)$ for many points s_j. We look at infinitely many points in the complex plane, selected in a canonical way, e.g. $\sigma + i\gamma$, where $\sigma > 1/2$ and $\zeta(1/2 + i\gamma) = 0$. For simplicity we assume RH, although this is not necessary.

Theorem 1. *Let χ_i, $i = 1, 2$ be two primitive real nonprincipal characters of conductors q_i, $q_1 \neq q_2$. Let $\sigma \in (1/2, 1]$. Then for a positive proportion of γ's the values $L(\sigma + i\gamma, \chi_1)$ and $L(\sigma + i\gamma, \chi_2)$ are linearly independent over $\mathbb{R}$, i.e. they do not have the same or opposite arguments.*

Remark 2. We currently work on extending the result to the critical line $\Re(s) = 1/2$. Similar results should hold for $L(s, f_i)$, where f_i are holomorphic cusp forms with, say, real coefficients. A first such result (for the abscissa of convergence) was proved in [6] in relation to the perturbation theory of scattering poles for $\Gamma_0(p)$.

1. Outline of the proof

The idea is to set up the question in terms of discrete moments of L-functions. The two values $L(\sigma+i\gamma,\chi_1)$ and $L(\sigma+i\gamma,\chi_2)$ are linearly independent over $\mathbb{R}$ iff

$$L(\sigma+i\gamma,\chi_1)\overline{L(\sigma+i\gamma,\chi_2)}-L(\sigma+i\gamma,\chi_2)\overline{L(\sigma+i\gamma,\chi_1)}\neq 0.$$

We define

(1)
$$A(\gamma)=B(\sigma+i\gamma,P)\left(L(\sigma+i\gamma,\chi_1)\overline{L(\sigma+i\gamma,\chi_2)}-L(\sigma+i\gamma,\chi_2)\overline{L(\sigma+i\gamma,\chi_1)}\right),$$

where $B(s,P)$ is an appropriate Dirichlet polynomial of the form

$$B(s,P)=\prod_{p\leq P}(1-\chi_1(p)p^{-s})(1-\chi_2(p)p^{-s}).$$

By the Cauchy-Schwarz inequality we have

$$\sum_{\substack{0<\gamma\leq T,\\ A(\gamma)\neq 0}}1\geq\frac{\left|\sum_{0<\gamma\leq T}A(\gamma)\right|^2}{\sum_{0<\gamma\leq T}|A(\gamma)|^2}.$$

To prove Theorem 1, it suffices to prove the following propositions:

Proposition 3. *For $\sigma>1/2$ we have*

$$\sum_{0<\gamma\leq T}B(\sigma+i\gamma,P)L(\sigma+i\gamma,\chi_1)\overline{L(\sigma+i\gamma,\chi_2)}\sim\sum_{n=1}^{\infty}\frac{d_n\chi_2(n)}{n^{2\sigma}}N(T),\tag{2}$$

$$\sum_{0<\gamma\leq T}B(\sigma+i\gamma,P)L(\sigma+i\gamma,\chi_2)\overline{L(\sigma+i\gamma,\chi_1)}\sim\sum_{n=1}^{\infty}\frac{\chi_1(n)e_n}{n^{2\sigma}}N(T),\tag{3}$$

where the coefficients d_n and e_n are defined through

$$B(s,P)L(s,\chi_1)=\sum_{n=1}^{\infty}\frac{d_n}{n^s},\quad B(s,P)L(s,\chi_2)=\sum_{n=1}^{\infty}\frac{e_n}{n^s}.$$

Proposition 4. *With the definition of $A(\gamma)$ in (1) we have*

$$\sum_{0<\gamma\leq T}|A(\gamma)|^2\ll N(T),\quad T\to\infty.$$

Using (2) and (3) we can deduce that

$$\sum_{0<\gamma\leq T}A(\gamma)\sim cN(T)$$

with

$$c=\sum_{n=1}^{\infty}\frac{d_n\chi_2(n)}{n^{2\sigma}}-\sum_{n=1}^{\infty}\frac{\chi_1(n)e_n}{n^{2\sigma}}.$$

Proposition 5. *There exists a prime P such that $c\neq 0$.*

Remark 6. The introduction of the Dirichlet polynomial $B(s, P)$ desymmetrizes the Rankin-Selberg convolutions in (2) and (3). Without it, the discrete mean values in these equations would be equal, i.e. $c = 0$.

The proofs of Propositions 3 and 4 are based on (i) use of the approximate functional equation (Lavrik's form [4] with sharp cutoff suffices), and (ii) Landau's formula as strengthened by Gonek in [3]. Landau proved that for any $x > 0$ we have

$$\sum_{0<\gamma\leq T} x^{\rho} \sim -\frac{T}{2\pi}\Lambda(x) + O(\log T),$$

where $\Lambda(x) = 1$ if x is a prime power, and $\Lambda(x) = 0$ otherwise. Here ρ are the nontrivial Riemann zeros. Gonek's improvement provides uniformity, which is useful when applying the result to estimate discrete means over the Riemann zeros. The improvement is in the error term, which can be written as:

$$O(x\log(2xT)\log\log(3x)) + O(\log x \min(T, x/\langle x\rangle)) + O(\log(2T)\min(T, 1/\log x)),$$

with $\langle x\rangle$ the distance of x to the nearest prime power, other than x. In Propositions 3 and 4 the main contribution comes from the diagonal. For the non-diagonal terms $\sum_{0<\gamma\leq T}\left(\frac{n}{m}\right)^{i\gamma}$ with $n \neq m$ we use Gonek's formula. As far as Proposition 5 is concerned, we compare Euler factors. For $p > P$ the Euler factors agree and are equal to $(1-\chi_1(p)\chi_2(p)p^{-2\sigma})^{-1}$. Note that χ_i are real characters.

For $p \leq P$ the Euler factors are $(1-\chi_2^2(p)p^{-2\sigma})^{-1}$ and $(1-\chi_1^2(p)p^{-2\sigma})^{-1}$. If for all P we had

$$\sum_{n=1}^{\infty}\frac{d_n\chi_2(n)}{n^{2\sigma}} = \sum_{n=1}^{\infty}\frac{e_n\chi_1(n)}{n^{2\sigma}},$$

then we would have for all P, by successive division,

$$\prod_{p\leq P}(1-\chi_2^2(p)p^{-2\sigma}) = \prod_{p\leq P}(1-\chi_1^2(p)p^{-2\sigma}).$$

This contradicts the fact that the conductors q_1 and q_2 are different.

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Selberg's small eigenvalue conjecture and residual eigenvalues

Morten S. Risager

For a general cofinite discrete subgroup Γ of $\mathrm{PSL}_2(\mathbb{R})$ and a unitary character $\chi : \Gamma \to S^1$ the spectrum of the corresponding hyperbolic Laplacian $L(\Gamma, \chi)$ consists of a

- continuous part $\sigma_c = [1/4, \infty[$ with multiplicity equal to the number of open cusps, and a
- discrete part σ_d consisting of eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ (If $\chi \neq 1$ λ_0 should be removed) which in the congruence case satisfy a Weyl law, but for general (Γ, χ) are believed to be small, or even finite.

The discrete spectrum consists 2 parts: $\sigma_{d,r}$ – the residual eigenvalues $s_j(1 - s_j)$ where s_j is a pole of an Eisenstein series $E(z, s)$ with $1/2 < \Re(s) \leq 1$, and $\sigma_{d,c}$ – cuspidal eigenvalues where the corresponding eigenfunction is cuspidal. In the congruence case $\Gamma = \Gamma_0(N)$, and $\chi = \chi_D$ arising from a Dirichlet character the residual spectrum is easy to understand: Either $\sigma_{d,r} = \emptyset$ or $\sigma_{d,r} = \{1\}$. In this case we can compute the scattering matrix explicitly and from this we can find all the residual eigenvalues: As an example we have

$$\phi_{PSL_2(\mathbb{Z})}(s) = \frac{\xi(2s-1)}{\xi(2s)},$$

where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed zeta-function. Therefore there are no residual eigenvalues in the interval $]0, 1/4[$. Selberg conjectured in 1965 that this interval also contains no cuspidal eigenvalues. More precisely

Conjecture 1 (Selberg [6])**.**

$$\lambda_1(\Gamma_0(N), \chi_D) \geq 1/4$$

Selberg proved that $\lambda_1 \geq 3/16$ using Weil's bound on Kloosterman sums. This has later been improved several times with $\lambda_1 \geq \frac{975}{4096} \simeq 0.238\ldots$ as the current best bound due to Kim and Sarnak [2]. The most recent bounds has all been proved using automorphicity of symmetric powers. In fact if all the symmetric powers of automorphic representations are automorphic Selberg's conjecture would follow (along with the full Ramanujan Petersson conjecture). In a different direction Selberg's conjecture for $N = 1$ goes back to Roelcke. It is now known to be true for $\Gamma_1(N)$ with squarefree $N < 857$ by the work of Booker and Strömbergsson [1].

In formulating his conjecture Selberg was motivated by trying to get bounds on sums of Kloosterman sums.

Conjecture 2 (Selberg[6]-Linnik [3])**.** [1]

$$\sum_{\substack{c \equiv 0(N) \\ 0 \leq c \leq x}} \frac{S(m, n, c)}{c} = O((x|mn|)^{\varepsilon}) \tag{1}$$

[1]See also [5]

In fact the Selberg-Linnik conjecture implies Selberg's conjecture along with the rest of the Ramanujan-Petersson conjecture for $\mathrm{GL}_2/\mathbb{Q}$.

We want to try to understand Selberg's conjecture in terms of character perturbations of the Laplacian. Consider the set $S_2^\infty(N)$ of modular forms of weight 2 for $\Gamma_0(N)$ which are cuspidal at infinity. For $N = q_1q_2$, $q_i > 1$

$$G_{q_1,q_2}(z) := E_2(z) - q_1E_2(q_1z) - E_2(q_2z) - q_1E_2(q_1q_2z))$$

lies in this space. Here $E_2(z) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)e(nz)$ is the weight 2 (quasi-modular) Eisenstein series.

Consider now the family of unitary characters defined for $f \in S_2^\infty(N)$ by

$$\chi_\epsilon(\chi) = \chi_D(\gamma)\exp(2\pi i\epsilon \int_{z_0}^{\gamma z_0} \Re(f(z)dz))$$

If we consider the discrete spectrum of $L(\epsilon) := L(\Gamma_0(N), \chi_\epsilon)$, this will consist of eigenvalues $\lambda_0(\epsilon) \leq \lambda_1(\epsilon) \leq \ldots \leq \lambda_i(\epsilon) \leq \ldots$. For f not cuspidal this perturbation is not regular but an application of Selbergs trace formula shows the following lemma:

Lemma 3 ([4])**.** *The spectrum in* $]0, 1/4[$ *is continuous in* ϵ *at* $\epsilon = 0$.

For $\epsilon = 0$ Selberg's conjecture predicts that the intersection of the spectrum of $L(\epsilon)$ with $]0, 1/4[$ is empty. Combining this with the above lemma we find that for every $\delta_0 > 0$ there exists $\epsilon_0 > 0$ such that for $|\epsilon| < \epsilon_0$ there are no eigenvalues of $L(\epsilon)$ in $[0+\delta, 1/4-\delta]$. In particular there are no *residual* eigenvalues. Surprisingly this weaker result can be reversed:

Theorem 4 ([4])**.** *Selberg's eigenvalue conjecture is true if and only if the residual spectrum of* $L(\epsilon)$ *is continuous in* ϵ *at* $\epsilon = 0$.

This can be proved by investigating the Phillips-Sarnak integral

$$I(s) = \int_{\Gamma\backslash\mathcal{H}} L^{(1)}\psi E_\infty(z, s)d\mu(z)$$

where $L^{(1)}h = 4\pi iy^2(f(z)\frac{\partial h}{\partial \overline{z}} + \overline{f}(z)\frac{\partial h}{\partial z})$ and ψ is a newform corresponding to a cuspidal eigenvalue $s_1(1-s_1) = \lambda_1 \in]0, 1/4[$. By twisting with Dirichlet characters we may assume the the corresponding standard L-function is non-zero at $s_1 - 1/2$. By unfolding (and assuming $f = G_{q_1,q_2}$) we find after some computations that $I(s_1) \neq 0$.

On the other hand perturbation theory allows one to conclude, assuming that the residual spectrum behaves sufficiently nice under perturbation that $I(s_1) = 0$. Hence the eigenvalue λ_1 cannot exist. For details we refer to [4].

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Zeros of the Selberg zeta-function

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(joint work with Markus Fraczek, Dieter Mayer)

I have reported on joint work in progress with Markus Fraczek and Dieter Mayer.

Markus Fraczek has computed zeros of the Selberg zeta-function with a transfer operator as the tool. The group is $\Gamma_0(4)$, with the one-parameter family of characters $\alpha \mapsto \chi_\alpha$ given by

$$\chi_\alpha \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = e^{2\pi i\alpha}, \qquad \chi_\alpha \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} = 1 .$$

In the lecture I have discussed resonances in the region $\mathrm{Im}\beta > 0$, $\mathrm{Re}\beta < \frac{1}{2}$ for the spectral parameter β. (Resonances correspond to singularities of an Eisenstein series.) In the results of the computations one can observe several phenomena concerning the behavior of the resonances as $\alpha \downarrow 0$. We could confirm some of the phenomena by theoretical methods:

Theorem 1. *There is a neighborhood U of $(0, \frac{1}{2})$ in $(-1, 1) \times \mathbb{C}$ such that all resonances in*

$$\big\{(\alpha, \beta) \in U \ : \ \alpha > 0 \,, \mathrm{Im}\beta > 0\big\}$$

occur on countably many curves

$$\alpha \mapsto \big(\alpha, \beta_k(\alpha)\big)$$

parametrized by the natural numbers $k \in \mathbb{N}$. Uniformly in $k \in \mathbb{N}$ we have as $\alpha \downarrow 0$

$$\begin{aligned} \beta_k(\alpha) \quad &= \quad \frac{1}{2} + \frac{\pi i k}{|\log \pi\alpha|} + \frac{\pi i k \log \pi}{|\log \pi\alpha|^2} + \frac{\pi i k(\log \pi)^2 - 2(\pi k \log 2)^2}{|\log \pi\alpha|^3} \\ &\qquad + \mathrm{O}\big(|\log \pi\alpha|^{-4}\big) . \end{aligned}$$

Theorem 2. *Let $t > 0$ be such that $\frac{1}{2} + it$ is not the spectral parameter of an unperturbed eigenvalue. Then there is a sequence of resonances $\big(\alpha_k, \sigma_k + it\big)$, with $k \geq k_1$ for some k_1, for which*

$$\begin{aligned} \alpha_k \quad &= \quad C_1\, e^{-\pi k/t}\Big(1 + \mathrm{O}\big(\frac{1}{k}\big)\Big) , \\ \sigma_k \quad &= \quad \frac{1}{2} - C_2\, \frac{1}{k} + \mathrm{O}\big(\frac{1}{k^2}\big) , \end{aligned}$$

with explicit constants C_1 and C_2.

Theorem 3. *Let $I \subset (0, \infty)$ be an interval containing $t_\ell = \frac{\pi \ell}{\log 2}$, $\ell \in \mathbb{N}$, such that the unperturbed Selberg zeta-function satisfies $Z\big(0, \frac{1}{2} + it\big) \neq 0$ for $t \in I$. Then there are curves*

$$t \mapsto \big(\alpha_k(t), \sigma_k(t) + it\big) \qquad (t \in I)\,,$$

for all integral $k \geq k_1$ for some $k_1 \in \mathbb{N}$, that carry resonances, such that the curve $t \mapsto \sigma_k(t) + it$ is tangent to the central line $\frac{1}{2} + i\mathbb{R}$ in a point $\frac{1}{2} + it_\ell + i\delta_{\ell,k}$ for which

$$\delta_{\ell,k} = \eta_\ell \, e^{-2\pi k / t_\ell}\big(1 + \mathrm{O}(1/k)\big)$$

for some, not explicitly known, $\eta_\ell \in \mathbb{R}$.

The asymptotic results in these theorems are confirmed by the computational data.

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