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Noncommutative Geometry

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Abstract. Noncommutative Geometry applies ideas from geometry to mathematical structures determined by noncommuting variables. This meeting concentrated primarily on those aspects of Noncommutative Geometry that are related to index theory and on the connections between operator algebras and number theory.

Mathematics Subject Classification (2000): 46Lxx, 19xx, 81Txx.

Introduction by the Organisers

Noncommutative geometry applies ideas from geometry to mathematical structures determined by noncommuting variables. Within mathematics, it is a highly interdisciplinary subject drawing ideas and methods from many areas of mathematics and physics. Natural questions involving noncommuting variables arise in abundance in many parts of mathematics and theoretical quantum physics. On the basis of ideas and methods from algebraic and differential topology and Riemannian geometry, as well as from the theory of operator algebras and from homological algebra, an extensive machinery has been developed which permits the formulation and investigation of the geometric properties of noncommutative structures. This includes K-theory, cyclic homology and the theory of spectral triples. Areas of intense research in recent years include topics such as index theory, quantum groups and Hopf algebras, the Novikov and Baum-Connes conjectures as well as the study of specific noncommutative structures arising from other fields such as number theory, modular forms or topological dynamical systems. Many results elucidate important properties of fascinating specific classes of examples that arise in many applications.

The talks at this meeting covered substantial new results and insights in several of the different areas in Noncommutative Geometry. The emphasis this time was on noncommutative structures and methods related to index theory on the one hand and to number theory on the other. The workshop was attended by 55 participants.

Workshop: Noncommutative Geometry

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Abstracts

Beyond Ellipticity

Paul Baum
(joint work with Erik van Erp)

K-homology is the dual theory to K-theory. The BD (Baum-Douglas) isomorphism of Kasparov K-homology and K-cycle K-homology can be taken as providing a framework within which the Atiyah-Singer index theorem can be extended to certain non-elliptic operators. This talk will consider a class of non-elliptic differential operators on compact contact manifolds. These operators have been studied by a number of mathematicians. Working within the BD framework, the index problem will be solved for these operators. This is joint work with Erik van Erp.

Index theory for non-elliptic operators. PB and co-worker Erik van Erp have extended the Atiyah-Singer index formula to a naturally-arising class of hypoelliptic (but not elliptic) operators. A class of operators with similar analytical and topological properties was introduced by Alain Connes and Henri Moscovici. The operators of this talk occur on contact manifolds and have been investigated by a number of mathematicians. This extension of Atiyah-Singer was achieved by using the BD (Baum-Douglas) isomorphism of analytic and topological K-homology to reformulate and extend van Erp's earlier partial result.

K-homology is the dual theory to K-theory. For a finite CW-complex X, there are three ways to define its K-homology:

(1) *homotopy theory.* K-homology is the homology theory determined by the Bott (i.e. K-theory) spectrum.

(2) *functional analysis.* K-homology is the Kasparov group $KK^*(C(X), \mathbb{C})$.

(3) *K-cycles.* K-homology is the group of K-cycles on X.

In (2), $C(X)$ denotes, as usual, the commutative C^* algebra consisting of all continuous functions $f : X \to \mathbb{C}$. The Kasparov group $KK^*(C(X), \mathbb{C})$ will be referred to as the analytic K-homology of X.

For (3), a K-cycle on X is a triple (M, E, φ) with

- M is a compact Spin^c manifold without boundary.
- E is a $\mathbb{C}$ vector bundle on M.
- φ is a continuous map from M to X, $\varphi : M \longrightarrow X$.

Keeping X fixed, denote by $\{(M, E, \varphi)\}$ the collection of all K-cycles on X. On this collection impose the equivalence relation $\sim$ generated by three elementary steps:

- bordism
- direct sum - disjoint union
- vector bundle modification

Thus two K-cycles (M, E, φ) (M', E', φ') on X are equivalent if and only if it is possible to pass from (M, E, φ) to (M', E', φ') by a finite sequence of the three elementary steps. The K-cycle (or topological) K-homology of X, denoted, $K_*^{top}(X)$, is the set of equivalence classes of K-cycles:

$$K_*^{top}(X) := \{(M, E, \varphi)\}/\sim$$

Addition in $K_*^{top}(X)$ is disjoint union of K-cycles :

$$(M, E, \varphi) + (M', E', \varphi') = (M \sqcup M', E \sqcup E', \varphi \sqcup \varphi')$$

The equivalence relation $\sim$ on K-cycles (M, E, φ) preserves the dimension of M modulo 2. Therefore, as an abelian group, $K_*^{top}(X)$is the direct sum

$$K_*^{top}(X) = K_0^{top}(X) \oplus K_1^{top}(X)$$

where $K_j^{top}(X)$ is the subgroup of $K_*^{top}(X)$ consisting of those K-cycles (M, E, φ) such that:

Every connected component of M has dimension $\equiv j$ modulo 2 $\qquad j = 0, 1$

Theorem. *Let X be a finite CW complex. Then the natural map*

$$\eta : K_j^{top}(X) \longrightarrow KK^j(C(X), \mathbb{C}) \qquad j = 0, 1$$

is an isomorphism of abelian groups.

The natural map $\eta : K_j^{top}(X) \longrightarrow KK^j(C(X), \mathbb{C})$ is defined as follows. Given a K-cycle (M, E, φ) on X, denote by D_E the Dirac operator of M tensored with E. Quite directly and immediately D_E determines an element

$$[D_E] \in KK^*(C(M), \mathbb{C}).$$

Denote by

$$\varphi_* : KK^*(C(M), \mathbb{C}) \longrightarrow KK^*(C(X), \mathbb{C})$$

the map of abelian groups induced by $\varphi : M \to X$. Then the natural map $\eta : K_j^{top}(X) \longrightarrow KK^j(C(X), \mathbb{C})$ is

$$\eta(M, E, \varphi) = \varphi_*[D_E].$$

Remark. Although the natural map is an isomorphism, there is no simple direct formula for the inverse map. This is somewhat analogous to the de Rham theorem. For any C^∞ manifold X, the natural map from de Rham cohomology to singular cohomology

$$H^*_{DR}(X) \longrightarrow H^*_{sing}(X)$$

is an isomorphism, but there is no simple direct formula for the inverse map.

General Index Problem. With X a finite CW complex, the general index problem is: *Given an element $\xi \in KK^0(C(X), \mathbb{C})$, explicitly construct a K-cycle (M, E, φ) on X with*

$$\eta(M, E, \varphi) = \xi.$$

Given $\xi \in KK^0(C(X), \mathbb{C})$, suppose that the general index problem has been solved for ξ so that a K-cycle (M, E, φ) has been constructed with $\eta(M, E, \varphi) = \xi$. A corollary of such a construction is this topological formula for Index$(F \otimes \xi)$:

$$\text{Index}(F \otimes \xi) = \epsilon_*(ch(F) \cap \varphi_*(ch(E) \cup Td(M) \cap [M]))$$

In this formula, F can be any $\mathbb{C}$ vector bundle on X and Index$(F \otimes \xi)$ is the integer obtained by tensoring F with ξ and then taking the index of the resulting Fredholm operator. $\epsilon : X \to \bullet$ is the map of X to a point and $\epsilon_* : H_*(X; \mathbb{Q}) \to H_*(\bullet; \mathbb{Q}) = \mathbb{Q}$ is the induced map in rational homology. ch is Chern character. $\cup$ and $\cap$ are the standard cup and cap products of algebraic topology . $Td(M)$ is the Todd class of the Spin^c manifold M. The Spin^c structure of M orients M, so $[M] \in H_*(M; \mathbb{Z})$ is the orientation cycle of M. $\varphi_* : H_*(M; \mathbb{Q}) \to H_*(X; \mathbb{Q})$ is the map of rational homology induced by $\varphi : M \to X$.

Equivalently, set

$$H_{ev}(X; \mathbb{Q}) = \bigoplus_l H_{2l}(X; \mathbb{Q}),$$

then for each $\xi \in KK^0(C(X), \mathbb{C})$ there exists a unique $\mathcal{I}(\xi) \in H_{ev}(X; \mathbb{Q})$ such that whenever F is a $\mathbb{C}$ vector bundle on X,

$$\text{Index}(F \otimes \xi) = \epsilon_*(ch(F) \cap \mathcal{I}(\xi)).$$

To solve the index problem for ξ *rationally* is to explicitly calculate $\mathcal{I}(\xi)$. Using $KK^0(C(X), \mathbb{C})$ alone, however, there is no topological solution to the problem of calculating $\mathcal{I}(\xi)$. The above formula states that if a K-cycle (M, E, φ) on X has been constructed with $\eta(M, E, \varphi) = \xi$, then

$$\mathcal{I}(\xi) = \varphi_*(ch(E) \cup Td(M) \cap [M])$$

which is a topological formula for $\mathcal{I}(\xi)$.

A contact manifold is an odd dimensional C^∞ manifold X, dimension(X) = 2n + 1, with a given C^∞ 1-form θ such that

$$\theta(d\theta)^n \text{ is non zero at every } x \in X - i.e.\ \theta(d\theta)^n \text{ is a volume form for } X.$$

Let X be a compact contact manifold without boundary ($\partial X = \emptyset$) with given 1-form θ. Setting dimension$(X) = 2n + 1$, let

$$\gamma \colon X \longrightarrow \mathbb{C} - \{\ldots, -n-4, -n-2, -n, n, n+2, n+4, \ldots\}$$

be a complex-valued C^∞ function on X which for all $x \in X$ satisfies:

$$\gamma(x) \notin \{\ldots, -n-4, -n-2, -n, n, n+2, n+4, \ldots\}$$

Associated to γ is a hypoelliptic Fredholm operator P_γ.

P_γ is a differential operator (of order 2) and is Fredholm and hypoelliptic but not elliptic. Once the basic analytic properties of P_γ have been established it then follows that P_γ gives an element

$$[P_\gamma] \in KK^0(C(X), \mathbb{C})$$

with Index$[P_\gamma]$=Index(P_γ).

PB and co-worker Erik van Erp have constructed a K-cycle (M, E, φ) on X with $\eta(M, E, \varphi) = [P_\gamma]$, thus solving the index problem for P_γ.

Refined Analytic Torsion

Maxim Braverman
(joint work with Thomas Kappeler)

We construct a canonical element, called the *refined analytic torsion*, of the determinant line of the cohomology of a closed oriented odd-dimensional manifold M with coefficients in a flat complex vector bundle E, which depends holomorphically on the flat connection. It encodes the information about both, the Ray-Singer η-invariant of the Atiyah-Patodi-Singer odd signature operator. In particular, when the bundle E is acyclic, the refined analytic torsion is a non-zero complex number, whose absolute value is equal (up to an explicit correction term) to the Ray-Singer torsion and whose phase is expressed in terms of the η-invariant. The fact that the Ray-Singer torsion and the η-invariant can be combined into one holomorphic function allows to use the methods of complex analysis to study both invariants. We present several applications of these methods. In particular, we compute the ratio of the refined analytic torsion and the Farber-Turaev refinement of the combinatorial torsion.

Definition of the refined analytic torsion. For $\alpha \in \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ we denote by E_α the flat vector bundle over M whose monodromy is equal to α. Let ∇_α be the flat connection on E_α. We defined a canonical non-zero element

$$\rho_{\mathrm{an}}(\alpha) \;=\; \rho_{\mathrm{an}}(\nabla_\alpha) \;\in\; \mathrm{Det}\left(H^\bullet(M, E_\alpha)\right),$$

called the *refined analytic torsion*, of the determinant line $\mathrm{Det}\left(H^\bullet(M, E_\alpha)\right)$ of the cohomology $H^\bullet(M, E_\alpha)$ of M with coefficients in E_α. The construction is based on the study of the graded determinant of the Atiyah-Patodi-Singer odd signature operator. If the representation α is not unitary, this operator is not self-adjoint. To carry out the construction of the refined analytic torsion we proved several new results about determinants of non-self-adjoint operators, which have an independent interest.

Analyticity of the refined analytic torsion. The disjoint union of the lines $\mathrm{Det}\left(H^\bullet(M, E_\alpha)\right)$, $(\alpha \in \mathrm{Rep}(\pi_1(M), \mathbb{C}^n))$, forms a line bundle

$$\mathcal{D}et \to \mathrm{Rep}(\pi_1(M), \mathbb{C}^n),$$

called the *determinant line bundle*. It admits a nowhere vanishing section, given by the Farber-Turaev torsion, and, hence, has a natural structure of a trivializable holomorphic bundle.

We prove that $\rho_{\mathrm{an}}(\alpha)$ is a nowhere vanishing *holomorphic* section of the bundle $\mathcal{D}et$. It means that the ratio of the refined analytic and the Farber-Turaev torsions is a holomorphic function on $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$. For an acyclic representation α, the determinant line $\mathrm{Det}\left(H^\bullet(M, E_\alpha)\right)$ is canonically isomorphic to $\mathbb{C}$ and $\rho_{\mathrm{an}}(\alpha)$

can be viewed as a non-zero complex number. We show that $\rho_{\rm an}(\alpha)$ is a holomorphic function on the open set $\mathrm{Rep}_0(\pi_1(M), \mathbb{C}^n) \subset \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$ of acyclic representations.

Recently, Burghelea and Haller [3, 5, 4] and Cappell and Miller [6] constructed different versions of complex valued Ray-Singer torsions. Their function is different from ours and is not related to the η-invariant. The precise relationship between all three versions of torsion are established in [1, 2], we show that the Burghelea-Haller torsion can be computed in terms of the refined analytic torsion.

Comparison with the Farber-Turaev torsion. In [8, 9], Turaev constructed a refined version of the combinatorial torsion associated to a representation α, which depends on additional combinatorial data, denoted by ϵ and called the *Euler structure*, as well as on the *cohomological orientation* of M, i.e., on the orientation $\mathfrak{o}$ of the determinant line of the cohomology $H^\bullet(M, \mathbb{R})$ of M. In [7], the Turaev torsion was redefined as a non-zero element $\rho_{\epsilon,\mathfrak{o}}(\alpha)$ of the determinant line $\mathrm{Det}\,\big(H^\bullet(M, E_\alpha)\big)$.

One of our main results states that, for each connected component $\mathcal{C}$ of the space $\mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$, there exists a constant $\theta \in \mathbb{R}$, such that

$$\frac{\rho_{\rm an}(\alpha)}{\rho_{\epsilon,\mathfrak{o}}(\alpha)} = e^{i\theta} \cdot f_{\epsilon,\mathfrak{o}}(\alpha), \tag{1}$$

where $f_{\epsilon,\mathfrak{o}}(\alpha)$ is a holomorphic function of $\alpha \in \mathrm{Rep}(\pi_1(M), \mathbb{C}^n)$, given by an explicit local expression.

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Mackey Machine and Duality of Gerbes on Orbifolds

Xiang Tang
(joint work with Hsian-hua Tseng)

Motivated by the study of conformal field theory (CFT) on gerbes over orbifolds, mathematical physicists [4] formulated the so-called *Decomposition Conjecture*, which states that CFTs associated to a G-gerbe (G is a finite group) $\mathcal{Y}$ over an orbifold $\mathcal{B}$ is isomorphic to CFTs on a new orbifold $\widehat{\mathcal{Y}}$ with the twist by a B-field τ (i.e. a $U(1)$-gerbe over $\hat{\mathcal{Y}}$).

We use the groupoid formulation of gerbes developed in [1], [7], and [5] to study a G gerbe $\mathcal{Y}$ over an orbifold $\mathcal{B}$. Let $\mathcal{B}$ be an orbifold, and G be a finite group. A G-gerbe $\mathcal{Y}$ over $\mathcal{B}$ can be presented as a groupoid extension, which is an exact sequence of groupoids,

$$1 \to \mathcal{G} \xrightarrow{i} \mathcal{H} \xrightarrow{j} \mathcal{Q} \to 1,$$

where $\mathcal{Q}$ is a proper étale groupoid presenting $\mathcal{B}$, and $\mathcal{G}$ is a locally trivial bundle on $\mathcal{G}_0$ of groups isomorphic to G, and $\mathcal{H}$ is a Lie groupoid over $\mathcal{H}_0 = \mathcal{G}_0$ presenting the G-gerbe $\mathcal{Y}$. According to [5], Morita equivalence classes of G-extensions of groupoid $\mathcal{Q}$ are 1-1 correspondent to G-gerbes over $\mathcal{B}$.

Let us explain the construction of the "dual" $\widehat{\mathcal{Y}}$ and the $U(1)$-gerbe c over it. We first consider the following special case. Consider a group extension $G \to H \to Q$. Let the group H act on a manifold M such that the G action on M is trivial. The following groupoid extension

$$M \times G \rightrightarrows M \longrightarrow M \rtimes H \rightrightarrows M \longrightarrow M \rtimes Q \rightrightarrows M, \tag{1}$$

defines a G-gerbe $\mathcal{Y} = M/H$ over the orbifold $\mathcal{B} = M/Q$. Let $\widehat{G}$ be the set of isomorphism classes of finite dimensional irreducible unitary G-representations. Observe that the conjugation action of H on G defines a Q action on $\widehat{G}$. The dual orbifold $\widehat{\mathcal{Y}}$ associated to the G-gerbe $\mathcal{Y}$ over $\mathcal{B}$ is the quotient $(M \times \widehat{G})/Q$. There is a $U(1)$-gerbe on $\widehat{\mathcal{Y}}$ defined as follows. Pick a set theoretic splitting $s : Q \to H$; and for each element $[\rho]$ in $\widehat{G}$, fix a representative of a unitary G-representation $\rho : G \to GL(V_\rho)$. Given any element $q \in Q$ and $[\rho] \in \widehat{G}$, $q(\rho) := \rho \circ Ad_{s(q)} : G \to GL(V_\rho)$ defines a new irreducible representation $q(\rho)$ of G. Let $[\alpha] = [q(\rho)]$ in $\widehat{G}$. Compare the α representation of G on V_α and the $q(\rho)$ representation of G on V_ρ. As they are isomorphic, there is an isomorphism $\phi_{q,[\rho]} : V_\rho \to V_\alpha$ to intertwine $q(\rho)$ and α. Composition of the intertwiners $\phi_{q,[\rho]}$ fails to satisfy associativity. Schur's lemma tells that the difference between these two operators takes value in $U(1)$ and hence defines a $U(1)$-valued 2-cocycle c on $\widehat{G} \rtimes Q$. Accordingly c can be lifted to a $U(1)$-valued 2-cocycle c on $(M \times \widehat{G}) \rtimes Q$ determining a $U(1)$-gerbe on $\widehat{\mathcal{Y}}$. (Note that the cohomology class of c is independent of the splitting s or the family of representations $\{V_\rho\}$.)

For a G-gerbe $\mathcal{Y}$ on a general orbifold $\mathcal{B}$, we can first fix a cover of $\mathcal{B}$ such that on each local chart $\mathcal{Y}$ can be presented by the above special example (1) coming from a group extension, and define $\widehat{\mathcal{Y}}$ and a $U(1)$-gerbe on each open chart as above, and finally glue the local construction using a partition of unity.

Our viewpoint toward the *Decomposition Conjecture* in [4] is that it suggests the following claim:

Conjecture 1. *The geometry of the G-gerbe $\mathcal{Y}$ is equivalent to the geometry of $\widehat{\mathcal{Y}}$ twisted by the discrete torsion c.*

Conjecture 1 reveals a deep and highly nontrivial connection between different geometric spaces. Consider the simplest example of a G-gerbe, namely, a G-gerbe over a point, (i.e., $\mathcal{B} = pt$ and $\mathcal{Y} = [pt/G] = BG$). Then the dual orbifold $\widehat{\mathcal{Y}}$ is the discrete set $\widehat{G}$, the space of isomorphism classes of unitary irreducible G-representations, and the $U(1)$-gerbe c on $\widehat{\mathcal{Y}}$ is trivial. Conjecture (1) states that the geometry of the classifying space BG is equivalent to the geometry of the discrete set $\widehat{G}$. However, there are no clear connections between BG and $\widehat{G}$ at all at the level of spaces. For example, when $G = \mathbb{Z}_2$, the realization of $B\mathbb{Z}_2$ is $\mathbb{RP}^\infty$, and the space $\widehat{\mathbb{Z}}_2$ is a set of two points.

Inspired by the observation that both BG and $\widehat{G}$ are related to G-representations, we studied in [6] Conjecture 1 using noncommutative geometry and the Mackey machine [3]. We proved that $\mathcal{Y}$ and $(\widehat{\mathcal{Y}}, c)$ are isomorphic as noncommutative spaces. More precisely, the groupoid algebra associated to the G-gerbe $\mathcal{Y}$ is Morita equivalent to the c-twisted groupoid algebra associated to the dual orbifold $\widehat{\mathcal{Y}}$ with the discrete torsion c. We applied this noncommutative geometry development to obtain several interesting and important results in algebraic geometry and symplectic topology.

(1) The category of (coherent) sheaves on $\mathcal{Y}$ is equivalent to the category of c-twisted (coherent) sheaves on $\widehat{\mathcal{Y}}$;
(2) when $\mathcal{Y}$ is symplectic, the Chen-Ruan orbifold cohomology [2] of $\mathcal{Y}$ is isomorphic to the c-twisted orbifold cohomology of $\widehat{\mathcal{Y}}$ as graded algebras.

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A local global principle for regular operators in Hilbert C^*–modules

Matthias Lesch
(joint work with Jens Kaad)

A Hilbert C^*–module E over a C^*–algebra $\mathscr{A}$ is an $\mathscr{A}$–right module equipped with an $\mathscr{A}$–valued inner product $\langle\cdot,\cdot\rangle$ and such that E is complete with respect to the norm $\|x\| := \|\langle x,x\rangle^{1/2}\| = \|\langle x,x\rangle\|^{1/2}$. The notion was introduced by Kaplansky in the commutative case [4] and in general independently by Paschke [9], Rieffel [10] and Takahashi (for the latter cf. [10, p. 179]). Kasparov's celebrated KK–theory makes extensive use of Hilbert C^*–modules [5] and by now Hilbert C^*–modules are a standard tool in the theory of operator algebras. Our standard textbook reference will be Lance [7].

The elementary properties of Hilbert C^*–modules can be derived basically in parallel to Hilbert space theory. However, there is no analogue of the Projection Theorem which soon leads to serious obstructions and difficulties.

A Hilbert C^*–module E comes with a natural C^*–algebra $\mathscr{L}(E)$ of bounded adjointable module endomorphisms. As for Hilbert spaces one soon needs to consider unbounded adjointable operators, Baaj-Julg [1], Pal [8], Woronowicz [11]; see also [7, Chap. 9/10].

The lack of a Projection Theorem in Hilbert C^*–modules causes the theory of unbounded operators to be notoriously more complicated. To explain this let us introduce some terminology: following Pal [8] by a *semiregular* operator in a Hilbert C^*–module E over $\mathscr{A}$ we will understand an operator $T : \mathscr{D}(T) \longrightarrow E$ defined on a dense $\mathscr{A}$–submodule $\mathscr{D}(T) \subset E$ and such that the adjoint T^* is densely defined, too. One now easily deduces that T is $\mathscr{A}$–linear and closable and that T^* is closed. Besides this semiregular operators can be rather pathologic (see the discussion in Sec. 2.3 and Sec. 6 in [2]).

To have a reasonable theory (e.g. with a functional calculus for selfadjoint operators) one has to introduce the additional *axiom of regularity*: a closed semiregular operator T in E is called *regular* if $I + T^*T$ is invertible. While in a *Hilbert space* every densely defined closed operator is regular in general Hilbert C^*–modules there exist closed semiregular operators which are not regular, see [2, Sec. 6].

The regularity axiom has the considerable disadvantage that it is difficult to verify for specific unbounded operators on concrete Hilbert spaces, cf. [8, p. 332]. The aim of this project is to remedy this distressing situation.

Let us describe in non–technical terms the problem from which this paper arose. In our study of an approach to the KK–product for unbounded modules [3] we needed to study two selfadjoint regular operators S, T in a Hilbert C^*–module with "small" commutator [2, Sec. 7]. More precisely, we were looking at unbounded odd Kasparov modules (D_1, X) and (D_2, Y) together with a densely defined connection ∇. The operator S then corresponds to $D_1 \otimes 1$ whereas T corresponds to $1 \otimes_\nabla D_2$. The Hilbert C^*–module is given by the interior tensor product of X and Y over some C^*–algebra. As an essential part of forming the unbounded Kasparov product of (D_1, X) and (D_2, Y) one needs to study the selfadjointness and regularity of the unbounded product operator

$$D := \begin{pmatrix} 0 & S - i\,T \\ S + i\,T & 0 \end{pmatrix}, \quad \mathscr{D}(D) = \big(\mathscr{D}(S) \cap \mathscr{D}(T)\big)^2 \subset E \oplus E. \tag{1}$$

With some effort we could prove that this operator is selfadjoint but all efforts to prove regularity failed. For a while we even started to look for counterexamples. On the other hand, in a Hilbert space regularity comes for free and the construction of D out of S and T was more or less "functorial".

So stated somewhat vaguely, the following principle should hold true: given a "functorial" construction of an operator $D = D(S,T)$ out of two selfadjoint and regular operators S, T. If then for Hilbert spaces this construction always produces a selfadjoint operator then $D(S,T)$ is selfadjoint and regular.

To explain our result, let us consider a closed semiregular operator T in the Hilbert C^*–module E. Furthermore, let ω be a state on $\mathscr{A}$. ω gives rise to a (possibly degenerate) scalar product

$$\langle x, y\rangle_\omega := \omega(\langle x, y\rangle) \tag{2}$$

on E. $\mathscr{N}_\omega := \big\{ x \in E \;\big|\; \langle x, x\rangle_\omega = 0 \big\}$ is a subspace of E. $\langle \cdot,\cdot\rangle_\omega$ induces a scalar product on the quotient $E/\mathscr{N}_\omega$ and we denote by E^ω the Hilbert space completion of $E/\mathscr{N}_\omega$. We let $\iota_\omega : E \to E^\omega$ denote the natural map. Clearly ι_ω is continuous with dense range; it is injective if and only if ω is faithful. The Hilbert space E^ω is called the *localization* of E with respect to the state ω.

The operator T induces a densely defined operator, T_0^ω, in the Hilbert space E^ω by putting $\mathscr{D}(T_0^\omega) = \iota_\omega\big(\mathscr{D}(T)\big)$ and $T_0^\omega(\iota_\omega x) = \iota_\omega(Tx)$. It turns out that T_0^ω is closable and we call its closure, $T^\omega = \overline{T_0^\omega}$, the localization of T with respect to the state ω. T^ω is a closed densely defined operator in the Hilbert space E^ω. Furthermore,

$$(T^*)^\omega \subset (T^\omega)^*. \tag{3}$$

Instead of states, which correspond to the cyclic representations of the C^*–algebra $\mathscr{A}$, one can consider arbitrary representations π. Then E^π is given by the interior tensor product $E \hat{\otimes}_{\mathscr{A}} H_\pi$. For brevity the details are omitted here; they can be found in [2].

It turns out that equality in (3) is intimately related to regularity:

Theorem 1 (Local–Global Principle)**.**

1. *For a closed semiregular operator T in a Hilbert C^*–module the following statements are equivalent:*
   (1) *T is regular.*
   (2) *For every state $\omega \in S(\mathscr{A})$ the localizations T^ω and $(T^*)^\omega$ are adjoints of each other.*

2. *For a closed, densely defined and symmetric operator T the following statements are equivalent:*
   (1) *T is selfadjoint and regular.*
   (2) *For every state $\omega \in S(\mathscr{A})$ the localization T^ω is selfadjoint.*

The main tool for proving this Theorem is the following separation Theorem.

Theorem 2. *Let $L \subset E$ be a closed convex subset of the Hilbert C^*–module E over $\mathscr{A}$. For each vector $x_0 \in E \setminus L$ there exists a state ω on $\mathscr{A}$ such that $\iota_\omega(x_0)$ is not in the closure of $\iota_\omega(L)$. In particular there exists a state ω such that $\iota_\omega(L)$ is not dense in E^ω and hence $\iota_\omega(L)^\perp \neq \{0\}$.*

In [2] we show by a couple of examples that the Local–Global Principle can easily be checked in concrete situations. We would find it aesthetically more appealing if in Theorems 1 and 2 one could replace "state" by "pure state". We conjecture that this is true, but so far we can only prove it for Hilbert C^*–modules over a commutative C^*–algebra $\mathscr{A}$ and for the special Hilbert C^*–module $E = \mathscr{A}$ over a general C^*–algebra $\mathscr{A}$. That pure states suffice in these cases turns out to be practically useful in Sec. 5 and in the discussion of examples of nonregular operators in Sec. 6 of [2]. We therefore single out the following Conjecture:

Conjecture 3. *If L is a proper submodule of the Hilbert $\mathscr{A}$–module E then there exists a pure state ω on $\mathscr{A}$ such that $\iota_\omega(L)^\perp \neq \{0\}$.*

Consequently, a closed densely defined symmetric operator in the Hilbert C^*–module E over $\mathscr{A}$ is regular if and only if for each pure state ω on $\mathscr{A}$ the localization T^ω is selfadjoint.

Finally we mention our result that if E is a *finitely generated* Hilbert C^*–module over $\mathscr{A}$ then *every* semiregular operator is regular.

The details of the material presented here have been published in [2].

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Quantum trees and actions of free quantum groups

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(joint work with Roland Vergnioux)

In this talk we discuss K-theoretic properties of free quantum groups. In our approach certain noncommutative analogues of trees occur naturally at several points. This resembles the situation for classical free groups, and our results support the point of view that free quantum groups behave like free groups in many ways.
Let us first recall the definition of free quantum groups [13], [9].

Definition 1 (Wang-Van Daele)**.** *Let $n \in \mathbb{N}$ and $Q \in GL_n(\mathbb{C})$. The free unitary quantum group $A_u(Q)$ is the universal C^*-algebra with generators $u_{ij}, 1 \leq i, j \leq n$ such that the matrices $u = (u_{ij})$ and $Q\overline{u}Q^{-1}$ are unitary. Here $\overline{u} = (u_{ij}^*)$ denotes the transpose of the adjoint matrix u^*. The free orthogonal quantum group $A_o(Q)$ is the quotient of $A_u(Q)$ by the relation $u = Q\overline{u}Q^{-1}$.*

Both $A_u(Q)$ and $A_o(Q)$ are naturally compact quantum groups in the sense of Woronowicz. In the sequel we will use the notation $C^*_\mathsf{f}(\mathbb{F}U(Q))$ and $C^*_\mathsf{f}(\mathbb{F}O(Q))$ to emphasize that we view these C^*-algebras as the full group C^*-algebras of discrete quantum groups $\mathbb{F}U(Q)$ and $\mathbb{F}O(Q)$.
As in [11] we shall call a discrete quantum group G free iff it is of the form

$$G \cong \mathbb{F}O(P_1) * \cdots * \mathbb{F}O(P_k) * \mathbb{F}U(Q_1) * \cdots * \mathbb{F}U(Q_l)$$

for some matrices $P_j \in GL_{m_j}(\mathbb{C})$ with $m_j > 1$ for all j such that $P_j\overline{P_j} = \pm 1$ and $Q_j \in GL_{n_j}(\mathbb{C})$. Observe that the classical free group $\mathbb{F}_l$ on l generators corresponds to the case $k = 0$ and $Q_1 = \cdots = Q_l = 1 \in GL_1(\mathbb{C})$. We remark that free quantum groups can be characterized intrinsically in terms of classical Cayley graphs [11].
Meyer and Nest have developed a categorical approach to the Baum-Connes conjecture [6]. Using their framework one can formulate and study an analogue of the Baum-Connes conjecture for free quantum groups. For more information we refer to [7], [12].
Our main result is that free quantum groups satisfy a strong version of the Baum-Connes conjecture.

Theorem 2. *Let G be a free quantum group of the form*

$$G = \mathbb{F}O(P_1) * \cdots * \mathbb{F}O(P_k) * \mathbb{F}U(Q_1) * \cdots * \mathbb{F}U(Q_l)$$

for matrices $P_j \in GL_{m_j}(\mathbb{C})$ with $m_j > 2$ for all j such that $P_j\overline{P_j} = \pm 1$ and $Q_j \in GL_{n_j}(\mathbb{C})$. Then G has the strong Baum-Connes property, that is, the localizing subcategory $\langle \mathcal{CI} \rangle \subset KK^G$ generated by all algebras induced from the trivial quantum subgroup is equal to KK^G.

This may be formulated equivalently by saying that free quantum groups have a γ-element and that $\gamma = 1$. Vergnioux has defined and studied quantum Cayley trees for these quantum groups in [11]. Theorem 2 shows in particular that the

resulting Julg-Valette elements are equal to 1, which answers a question left open in [11].
As a corollary of theorem 2 one obtains the K-amenability of free quantum groups and an explicit computation of their K-theory. Let us state explicitly the result in the unitary case.

Theorem 3. *Let $n > 1$ and $Q \in GL_n(\mathbb{C})$. Then $\mathbb{F}U(Q)$ is K-amenable. In particular, the natural homomorphism $C^*_\mathfrak{f}(\mathbb{F}U(Q)) \to C^*_\mathfrak{r}(\mathbb{F}U(Q))$ induces an isomorphism in K-theory. We have*

$$K_0(C^*_\mathfrak{f}(\mathbb{F}U(Q))) = \mathbb{Z}, \qquad K_1(C^*_\mathfrak{f}(\mathbb{F}U(Q))) = \mathbb{Z} \oplus \mathbb{Z},$$

and these groups are generated by the class of 1 *in the even case and the classes of u and $\overline{u}$ in the odd case.*

We remark that the notion of K-amenability due to Cuntz [3] carries over to the setting of quantum groups in a straightforward way [10].
Let us sketch the proof of theorem 2 in the case $G = \mathbb{F}U(Q)$ where $Q \in GL_n(\mathbb{C})$ for $n > 2$ satisfies $Q\overline{Q} = \pm 1$. The basic idea is to consider the embedding $\mathbb{F}U(Q) \subset \mathbb{F}O(Q) * \mathbb{Z}$ studied by Banica [1]. It is known that the corresponding free orthogonal quantum groups satisfy the strong Baum-Connes conjecture [12], and the same holds for $\mathbb{Z}$ due to the work of Higson and Kasparov [4]. Therefore it suffices to show that the strong Baum-Connes property passes to free products of discrete quantum groups and suitable quantum subgroups. To prove inheritance for free products, we adapt constructions of Kasparov and Skandalis [5] for group actions on trees to the setting of quantum groups. An important difference to the classical situation is that one has to take into account a natural action of the Drinfeld double, compare [8].
For general matrices $Q \in GL_n(\mathbb{C})$ there is no embedding of the free unitary quantum group $\mathbb{F}U(Q)$ into a free product as above. One proceeds using monoidal equivalences in order to reduce to the case $Q\overline{Q} = \pm 1$, see [2], [12].

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Entropy and Fuglede-Kadison determinant

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(joint work with Lewis Bowen, Nhan-Phu Chung, David Kerr)

An *algebraic action* is an action of a countable discrete group Γ on a compact metrizable abelian group by continuous automorphisms. Denote by $\mathbb{Z}[\Gamma]$ the integral group ring of Γ, which consists of finitely supported $\mathbb{Z}$-valued functions on Γ and has the multiplication given by convolution. Using the Pontryagin duality, one can see that for any compact metrizable abelian group X, there is a one-to-one correspondence between algebraic actions of Γ on X and left $\mathbb{Z}[\Gamma]$-module structure of the Pontryagin dual $\widehat{X}$ of X.

Classically, when Γ is amenable, entropy is defined for continuous actions of Γ on compact metrizable spaces and measure-preserving actions of Γ on probability measure spaces [12]. It is known that for any algebraic action of amenable Γ on X, the topological entropy and the measure entropy corresponding to the Haar probability measure of X coincide [1, 6], which we shall simply call the entropy of the algebraic action.

Yuzvinskiĭ calculated the entropy for algebraic actions of $\mathbb{Z}$ [14]. Lind, Schmidt and Ward calculated the entropy for algebraic actions of $\mathbb{Z}^d$ ($1 \leq d < \infty$) [11]. The crucial case in the calculation of Lind, Schmidt and Ward is the entropy for the principal algebraic action $\mathbb{Z}^d \curvearrowright X_f = \widehat{\mathbb{Z}[\mathbb{Z}^d]/\mathbb{Z}[\mathbb{Z}^d]f}$ for $f \in \mathbb{Z}[\mathbb{Z}^d] = \mathbb{Z}[Y_1^{\pm}, \dots, Y_d^{\pm}]$. They showed that, for nonzero f the entropy is equal to $\log \mathbb{M}(f)$, for $\mathbb{M}(f)$ being the Mahler measure of f:

$$\mathbb{M}(f) = \exp(\int_{\mathbb{T}^d} \log |f(s)|\, ds)$$

for $\mathbb{T}$ being the unit circle in $\mathbb{C}$ and $\mathbb{T}^d$ being endowed with the Haar probability measure.

In general, the principal algebraic action $\Gamma \curvearrowright X_f := \widehat{\mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f}$ is defined for any countable discrete group Γ and any $f \in \mathbb{Z}[\Gamma]$. Explicitly,

$$X_f = \{x \in (\mathbb{R}/\mathbb{Z})^{\Gamma} : x \cdot f^* = 0\},$$

where the adjoint f^* of f is given by $f^*_s = f_{s^{-1}}$ for all $s \in \Gamma$ and $x \cdot f^*$ is the convolution product. The action of Γ on X_f is given by

$$(sx)_t = x_{s^{-1}t}$$

for all $x \in X_f$ and $s, t \in \Gamma$.

Given a von Neumann algebra $\mathcal{M}$ and a normal tracial state tr of $\mathcal{M}$, Fuglede and Kadison introduced a determinant $\det_{\mathcal{M}} f$ for any $f \in \mathcal{M}$ associated to tr [8]. Denote by μ the spectral measure of $|f|$ on the interval $[0, \|f\|]$ associated to tr, i.e. μ is the Borel probability measure on $[0, \|f\|]$ determined by

$$\int_0^{\|f\|} g(t)\, d\mu(t) = \mathrm{tr}(g(|f|))$$

for all complex-valued continuous functions g on $[0, \|f\|]$. Then the Fuglede-Kadison determinant is defined as

$$\det{}_{\mathcal{M}} f = \exp(\int_0^{\|f\|} \log t\, d\mu(t)).$$

When f is invertible in $\mathcal{M}$, one has the simpler formula

$$\det{}_{\mathcal{M}} f = \exp(\mathrm{tr}(\log |f|)).$$

For any countable discrete group Γ, one has the group von Neumann algebra $\mathcal{N}\Gamma$ inside the algebra of bounded linear operators on $\ell^2(\Gamma)$. Denote by δ_e the elements in $\ell^2(\Gamma)$ being 1 at the identity element e of Γ and 0 everywhere else. Then $\mathcal{N}\Gamma$ has the canonical faithful normal tracial state tr given by

$$\mathrm{tr}(a) = \langle a\delta_e, \delta_e \rangle .$$

It is well known that $\mathbb{Z}[\Gamma]$ embeds into $\mathcal{N}\Gamma$ naturally. Thus for any $f \in \mathbb{Z}[\Gamma]$, one has the Fuglede-Kadison determinant $\det_{\mathcal{N}\Gamma} f$ (associated to tr). When $\Gamma = \mathbb{Z}^d$, one has $\det_{\mathcal{N}\Gamma} f = \mathbb{M}(f)$ for every $f \in \mathbb{Z}[\mathbb{Z}^d]$. Then it is natural to ask for the relation between the entropy of the principal algebraic action $\Gamma \curvearrowright X_f$ and $\log \det_{\mathcal{N}\Gamma} f$ for general amenable Γ and $f \in \mathbb{Z}[\Gamma]$.

Theorem 1. [10] *Let Γ be a countable amenable group. Let $f \in \mathbb{Z}[\Gamma]$ be invertible in $\mathcal{N}\Gamma$. Then the entropy of $\Gamma \curvearrowright X_f$ is equal to $\log \det_{\mathcal{N}\Gamma} f$.*

Some special cases of Theorem 1 were proved in [6, 7] earlier. When Γ is amenable and $f \in \mathbb{Z}[\Gamma]$ is a left zero-divisor in $\mathbb{Z}[\Gamma]$, the entropy of $\Gamma \curvearrowright X_f$ is $+\infty$ [5], while $\log \det_{\mathcal{N}\Gamma} f = -\infty$. It is conjectured that the assumption in Theorem 1 that f is invertible in $\mathcal{N}\Gamma$ can be weakened to that f is not a left zero-divisor in $\mathbb{Z}[\Gamma]$. One consequence of Theorem 1 is the following result, answering partially a question of Deninger:

Corollary 2. [5] *Let Γ be a countable amenable group. Let $f \in \mathbb{Z}[\Gamma]$ be invertible in $\ell^1(\Gamma)$. Then $\det_{\mathcal{N}\Gamma} f = 1$ if and only if f is invertible in $\mathbb{Z}[\Gamma]$.*

The class of *sofic groups* was introduced by Gromov. For a nice survey about sofic groups, see [13]. Discrete amenable groups and residually finite groups are all sofic. Recently the entropy theory has been extended to continuous actions of countable sofic groups Γ on compact metrizable spaces and measure-preserving actions of countable sofic groups Γ on standard probability measure spaces [2, 9]. The entropy for such actions may depend on the choice of a *sofic approximation sequence* for Γ, thus one actually gets a family of invariants for the actions. When

Γ is countably infinite and residually finite, the simplest way to construct a sofic approximation sequence for Γ is to take a sequence $\Sigma = \{\Gamma_n\}_{n\in\mathbb{N}}$ of finite-index normal subgroups of Γ converging to e in the sense that $\bigcap_{n\in\mathbb{N}}\bigcup_{m\geq n}\Gamma_m = \{e\}$ (and then proceed in a standard way to obtain a sofic approximation sequence for Γ). In the rest of this abstract, we fix a countably infinite and residually finite group Γ, a sequence $\Sigma = \{\Gamma_n\}_{n\in\mathbb{N}}$ of finite-index normal subgroups of Γ converging to e, and $f \in \mathbb{Z}[\Gamma]$.

Theorem 3. [3] *Suppose that f is invertible in $\ell^1(\Gamma)$. Then the sofic measure entropy of $\Gamma \curvearrowright X_f$ corresponding to Σ and the Haar probability measure of X_f is equal to $\log \det_{\mathcal{N}\Gamma} f$.*

Theorem 4. [9] *Suppose that f is invertible in the full group C^*-algebra of Γ. Then the sofic topological entropy of $\Gamma \curvearrowright X_f$ corresponding to Σ is equal to $\log \det_{\mathcal{N}\Gamma} f$.*

Theorem 5. [4] *Suppose that f satisfy the following conditions:*

(1) $\sum_{s\in\Gamma} f_s = 0$,
(2) $f_s \leq 0$ *for every* $s \in \Gamma \setminus \{e\}$,
(3) $f = f^*$,
(4) *the support of f generates Γ.*

Then the sofic topological entropy and the sofic measure entropy of $\Gamma \curvearrowright X_f$ corresponding to Σ and the Haar probability measure of X_f are both equal to $\log \det_{\mathcal{N}\Gamma} f$.

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The algebraic K-theory Novikov conjecture for group algebras

Guoliang Yu

Let R be an H-unital ring and G be any group. The isomorphism conjecture of Farrell-Jones and Bartels-Farrell-Jones-Reich provides an algorithm for computing the algebraic K-theory of the group algebra RG in terms of the algebraic K-theory of R. More precisely, the isomorphism conjecture states that the assembly map from a certain equivariant homology associated to the non-connective algebraic K-theory spectrum of R to the algebraic K-theory $K_n(RG)$ is an isomorphism. Motivated by Connes-Moscovici's higher index theory, we consider the case when R is the ring of Schatten class operators. We prove that in this case, the assembly map is rationally injective. As a consequence, we obtain the algebraic K-theory Novikov conjecture for all group algebras over the ring of Schatten class operators. The main tool in the proof is an explicit construction of the Connes-Chern character.

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Modular curvature for noncommutative two-tori

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(joint work with Alain Connes)

We report on substantive new advances in understanding the geometry of the noncommutative two-torus $\mathbb{T}^2_\theta$, cf. [7]. The differential geometry of $\mathbb{T}^2_\theta$ equipped with the analogue of a flat metric, as well as its pseudo-differential operator calculus, was first developed in [3]. To obtain a curved geometry, one introduces (cf. [1], [2]) a Weyl factor or dilaton which modifies the metric by rescaling the volume form while keeping the same conformal structure (both these notions are explained in [4]). In [2] it was shown how to construct a twisted spectral triple (cf. [6])

representing the Dirac operator D for the conformally curved geometry associated to the dilaton h. Furthermore, a first computation was performed for the value at $s = 0$ of the zeta function $\zeta_{D^2}(a; s) = \mathrm{Trace}(a|D|^{-2s})$, $a \in C^\infty(\mathbb{T}^2_\theta)$, or equivalently for the constant term $\mathrm{a}_2(a, D^2)$ of the heat expansion. Although partial, that calculation, which was actually started in the late 1980's (cf. [1]), suffices to compute the total integral of the curvature and thus establish the analogue of the Gauss–Bonnet formula (cf. [2]), [8]). The technical obstacles for computing in full the local expression of $\mathrm{a}_2(a, D^2)$ were overcome in 2009, essentially by means of computer assisted calculations, facilitated by a generalized Rearrangement Lemma (announced at the Oberwolfach 2009 conference). The same computation was independently done in [9].

The main additional input of the present work consists in relating that computation to the Ray-Singer log-determinant of D^2, and gaining in this way new geometric insight into the meaning of the resulting expression. Moreover, computing the gradient of the Ray-Singer log-determinant in two different ways yields a deep internal consistency relation for the calculated local formula and elucidates the role of the complicated inner bi-differential terms.

We now briefly outline our main new results. As in the case of the standard torus viewed as a complex curve, the total Laplacian associated to the modular spectral triple splits into two components, one $\triangle_\varphi$ on functions and the other $\triangle_\varphi^{(0,1)}$ on $(0,1)$-forms; the two operators are isospectral outside zero. A first result is the conformal invariance of the value at 0 of the zeta functions associated to these Laplacians. In particular, this gives an priori proof for the validity of the Gauss-Bonnet formula.

The full local expression of the constant term $\mathrm{a}_2(a, D^2)$, resp. $\mathrm{a}_2(\gamma a, D^2)$, involves as a crucial ingredient the modular operator Δ of the non-tracial weight $\varphi(a) = \varphi_0(ae^{-h})$ associated to the dilaton h. Thus, $\mathrm{a}_2(a, \triangle_\varphi)$ is of the form

$$\mathrm{a}_2(a, \triangle_\varphi) = -\frac{\pi}{2\tau_2}\varphi_0(a\left(K_0(\nabla)(\triangle(h)) + \frac{1}{2}H_0(\nabla_1, \nabla_2)(\Box_\Re(h)\right) \tag{1}$$

where $\nabla = \log\Delta$ is the inner derivation implemented by $-h$,

$$\triangle(h) = \delta_1^2(h) + 2\Re(\tau)\delta_1\delta_2(h) + |\tau|^2\delta_2^2(h),$$

$\Box_\Re$ is the Dirichlet quadratic form

$$\Box_\Re(\ell) := (\delta_1(\ell))^2 + \Re(\tau)\left(\delta_1(\ell)\delta_2(\ell) + \delta_2(\ell)\delta_1(\ell)\right) + |\tau|^2(\delta_2(\ell))^2\,,$$

and ∇_i, $i = 1, 2$, signify that ∇ is acting on the ith factor. The operators $K_0(\nabla)$ and $H_0(\nabla_1, \nabla_2)$ are new ingredients, whose occurrence is a vivid manifestation of the genuinely non-unimodular nature of the conformal geometry of the noncommutative 2-torus. The functions $K_0(u)$ and $H_0(u, v)$ by which they act seem at first of a rather formidable nature, and they of course beg for a deeper understanding. Their expressions, arising from the computation, are as follows:

$$K_0(s) = \frac{-2 + s\coth\left(\frac{s}{2}\right)}{s\sinh\left(\frac{s}{2}\right)}\,, \tag{2}$$

and

(3)

$$H_0(s,t) = \frac{t(s+t)\cosh(s) - s(s+t)\cosh(t) + (s-t)(s+t+\sinh(s)+\sinh(t)-\sinh(s+t))}{st(s+t)\sinh\left(\frac{s}{2}\right)\sinh\left(\frac{t}{2}\right)\sinh\left(\frac{s+t}{2}\right)^2}$$

Our second main result is the following closed formula for the Ray-Singer determinant:

$$\log \mathrm{Det}'(\triangle_\varphi) = \log\varphi(1) + (2\log 2\pi + \log|\eta(\tau)|^4) + \frac{\pi}{8\tau_2}\varphi_0\left(\tilde{K}_0(\nabla_1)(\square_{\Re}(h))\right) \tag{4}$$

The third new result is an abstract proof of a functional relation between the functions K_0 and H_0; denoting

$$\tilde{K}_0(s) = 4\frac{\sinh(s/2)}{s}K_0(s) \quad \text{and} \quad \tilde{H}_0(s,t)) = 4\frac{\sinh((s+t)/2)}{s+t}H_0(s,t),$$

we prove, by an a priori argument, the identity

(5)

$$-\frac{1}{2}\tilde{H}_0(s_1,s_2) = \frac{\tilde{K}_0(s_2)-\tilde{K}_0(s_1)}{s_1+s_2} + \frac{\tilde{K}_0(s_1+s_2)-\tilde{K}_0(s_2)}{s_1} - \frac{\tilde{K}_0(s_1+s_2)-\tilde{K}_0(s_1)}{s_2}$$

The function $\tilde{K}_0$ is (up to the factor $\frac{1}{8}$) the generating function of the Bernoulli numbers, i.e. one has

$$\frac{1}{8}\tilde{K}_0(u) = \sum_1^\infty \frac{B_{2n}}{(2n)!}u^{2n-2}\,. \tag{6}$$

Our a priori proof of the functional relation (5) is based on the computation of the gradient of a scale-invariant version (cf. [10]) of the Ray-Singer determinant in two different ways. Using the left hand side of (4) one obtains a formula involving $\mathrm{a}_2(a,\triangle_\varphi)$, while using the right hand side of (4) gives the local expression.

As a forth fundamental result, we establish the analogue of a version of the classical uniformization theorem which asserts that in every conformal class the maximum of log-det for metrics of a fixed area is unique and attained at the constant curvature metric. The proof relies on the positivity of the function $\tilde{K}_0$. This function $\tilde{K}_0$ is, by (6), the generating function of Bernoulli numbers and is well known to play a key role in the theory of characteristic classes where it is used as a formal power series. It is quite striking that in our context the same function appears but no longer as a formal series and with a key role played by its *positivity*.

In marked contrast to the ordinary torus, for which both $\mathrm{a}_2(a,\triangle_\varphi)$ and $\mathrm{a}_2(a,\triangle_\varphi^{(0,1)})$ are constant multiples of the scalar (or Gaussian) curvature, the local curvature functionals associated to the zeta functions of the two partial Laplacians differ substantially. The function $H_1(s,t)$ of two variables involved in the expression of $\mathrm{a}_2(a, \triangle_\varphi^{(0,1)})$ is related to $H_0(s,t)$ in a simple fashion, but a new term appears, in the form of an operator $S(\nabla_1, \nabla_2)$ applied to the skew quadratic form

$$\square_{\Im}(\ell) := i\,\Im(\tau)\,(\delta_1(\ell)\delta_2(\ell) - \delta_2(\ell)\delta_1(\ell))\,, \quad \ell = 2h. \tag{7}$$

Finally, the gradient of the scale-invariant analytic torsion functional delivers in turn the appropriate analogue of the notion of scalar curvature. Furthermore, the corresponding evolution equation for the metric yields the appropriate analogue of Ricci flow.

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Absolute geometric structures and their arithmetic: an overview

Caterina Consani

(joint work with Alain Connes)

The talk reviewed the origins, the meaning and the motivations to develop an “absolute” geometric/arithmetic theory (*i.e.* geometry/arithmetic over $\mathbb{F}_1$). This research has already shed a new light on a number of important topics pertaining to the rich interconnection between the fields of noncommutative geometry and number theory. The talk focused on the description of a few recent results obtained by the speaker in an on-going research project with A. Connes. In particular, it was explained that the algebraic endomotive and the associated BC-system (quantum statistical dynamical system) which are known to give the spectral realization of the zeros of the Riemann zeta function as well as the trace formula interpretation of the Riemann-Weil explicit formulas appear from first principles by studying the algebraic closure $\bar{\mathbb{F}}_1$ of the “field with one element”. The real counting function of the hypothetical curve over $\mathbb{F}_1$ whose corresponding zeta function is the complete

Riemann zeta function exists as a distribution and can be expressed as an intersection number involving the scaling action of the idèle class group on the adèle class space.
The introduction of an elementary theory of algebraic geometry over the absolute point $\mathrm{Spec}(\mathbb{F}_1)$ reveals the role of the natural monoidal structure of the adèle class space. To understand it in more refined terms one needs to promote $\mathbb{F}_1$ to the Krasner hyperfield $\mathbb{K}$ by implementing on the set $\{0,1\}$ the structure of a hyperring (*i.e.* one endows the set $\{0,1\}$ with the obvious multiplication and a hyper-addition requiring that $1+1=\{0,1\}$). It follows that the adèle class space of a global field inherits the (algebraic) structure of a hyperring extension of $\mathbb{K}$.

In view of our recent results on the arithmetic properties of the BC-system and its connections with the theory of Witt vectors and p-adic analysis (*cfr.* A. Connes' talk) one derives a description of the geometric fibers of the sought for (absolute) curve associated to the (complete) Riemann zeta function. The fiber over a non-archimedean rational prime p is the total space of a principal bundle, with base the space $\mathrm{Val}_p(\mathbb{Q}^{\mathrm{cyc}})$ of extensions of the p-adic valuation to the maximal abelian field extension of $\mathbb{Q}$. The structure group is given by a connected, compact solenoid whose presence is due to the fact that the connected component of the identity in the idèle class group acts trivially, at the Galois level, on $\mathbb{Q}^{\mathrm{cyc}}$. The integral BC-system supplies a natural embedding of (each of) these fibers into a noncommutative space whose description matches with the definition of the adèle class space and whose algebraic structure is that of a free module of rank one over the hyperring of adèle classes.

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Drinfeld twists and cohomology of quantum groups

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(joint work with Lars Tuset)

Let G be a simply connected semisimple compact Lie group. For $q > 0$ denote by G_q the Drinfeld-Jimbo deformation of G. It is well-known that the irreducible representations of G_q are classified by the set P_+ of dominant integral weights, and the dimensions of these representations are the same as in the classical case. This implies that the von Neumann algebras $W^*(G_q)$ and $W^*(G)$ are isomorphic. Fix such an isomorphism $\varphi\colon W^*(G_q) \to W^*(G)$ extending the identification of centers of the algebras with $\ell^\infty(P_+)$. Denote by $\hat\Delta_q$ the comultiplication on $W^*(G_q)$. The isomorphism φ cannot be chosen to respect the comultiplications. Instead, by virtue of a highly nontrivial result of Drinfeld, Kazhdan and Lusztig (see [4] for a thorough discussion), there exists a unitary element $\mathcal{F} \in W^*(G)\bar\otimes W^*(G)$ such that

(i) $(\varphi\otimes\varphi)\hat\Delta_q = \mathcal{F}\hat\Delta\varphi(\cdot)\mathcal{F}^{-1}$;

(ii) $(\hat\varepsilon\otimes\iota)(\mathcal{F}) = (\iota\otimes\hat\varepsilon)(\mathcal{F}) = 1$, where $\hat\varepsilon$ is the trivial representation of G;

(iii) $(\varphi\otimes\varphi)(\mathcal{R}) = \mathcal{F}_{21}q^t\mathcal{F}^{-1}$, where $\mathcal{R}$ is the R-matrix for G_q (considered as an element affiliated with $W^*(G_q)\bar\otimes W^*(G_q)$) and $t \in \mathfrak{g}\otimes\mathfrak{g} \subset W^*(G)\bar\otimes W^*(G)$ is the element defined by a properly normalized symmetric invariant form on the complexified Lie algebra $\mathfrak{g}$ of G;

(iv) $(\iota\otimes\hat\Delta)(\mathcal{F}^{-1})(1\otimes\mathcal{F}^{-1})(\mathcal{F}\otimes 1)(\hat\Delta\otimes\iota)(\mathcal{F}) = \Phi_{KZ}$, where Φ_{KZ} is Drinfeld's associator defined via monodromy of Knizhnik-Zamolodchikov equations.

We call $\mathcal{F}$ a unitary Drinfeld twist. What can be said about these elements?

Classification of Drinfeld twists leads to the problem of computing dual invariant cohomology $H^2_{G_q}(\hat G_q;\mathbb{T})$, which is defined as follows. A unitary element $\mathcal{F} \in W^*(G_q)\bar\otimes W^*(G_q)$ is called a dual 2-cocycle on G_q if

$$(\mathcal{F}\otimes 1)(\hat\Delta_q\otimes\iota)(\mathcal{F}) = (1\otimes\mathcal{F})(\iota\otimes\hat\Delta_q)(\mathcal{F}).$$

A cocycle $\mathcal{F}$ is called invariant if it commutes with the image of $\hat\Delta_q$. Two invariant unitary cocycles $\mathcal{E}$ and $\mathcal{F}$ are called cohomologous, if there exists a central unitary element $u \in W^*(G_q)$ such that $\mathcal{E} = (u\otimes u)\mathcal{F}\hat\Delta_q(u)^{-1}$. The set of cohomology classes of dual invariant unitary 2-cocycles forms a group, denoted by $H^2_{G_q}(\hat G_q;\mathbb{T})$.

Denote by Q the root lattice of G. Let c be a $\mathbb{T}$-valued 2-cocycle on P/Q. It defines a dual invariant unitary 2-cocycle $\mathcal{E}_c$ on $\hat G_q$: the action of $\mathcal{E}_c$ on the tensor product $V_\lambda\otimes V_\nu$ of irreducible representations with highest weights λ and ν is by multiplication by $c(\lambda,\mu)$.

Theorem 1 ([2, 5])**.** *The map $c\mapsto\mathcal{E}_c$ defines an isomorphism*

$$H^2(P/Q;\mathbb{T}) \cong H^2_{G_q}(\hat G_q;\mathbb{T}).$$

Corollary 2. *If $\mathfrak{g}$ is simple and $\mathfrak{g}\not\cong\mathfrak{so}_{4n}(\mathbb{C})$ then $H^2_{G_q}(\hat G_q;\mathbb{T})$ is trivial. If $\mathfrak{g} = \mathfrak{so}_{4n}(\mathbb{C})$ then $H^2_{G_q}(\hat G_q;\mathbb{T}) \cong \mathbb{Z}/2\mathbb{Z}$.*

Corollary 3. *Suppose $\mathcal{E}$ and $\mathcal{F}$ are two Drinfeld twists for the same isomorphism φ. Then there exists a central unitary element u in $W^*(G)$ such that $\mathcal{E} = (u \otimes u)\mathcal{F}\hat{\Delta}(u)^{-1}$.*

For $q = 1$ Theorem 1 can be extended to all compact connected groups.

Theorem 4 ([3])**.** *Assume G is a compact connected group. Then there is a canonical isomorphism*

$$H^2_G(\hat{G}; \mathbb{T}) \cong H^2(\widehat{Z(G)}; \mathbb{T}).$$

Let us now consider the field of quantum groups G_q. For every $q > 0$ fix an isomorphism $\varphi_q \colon W^*(G_q) \to W^*(G)$ as above. Consider the standard generators E_i^q, F_i^q, K_i^q of $U_q\mathfrak{g}$ for $q \neq 1$, and the generators $E_i^1 = E_i, F_i^1 = F_i, H_i^1 = H_i$ of $U\mathfrak{g}$. Write $K_i^q = q^{d_i H_i^q}$. We say that the family $\{\varphi_q\}_q$ is continuous if the map $q \mapsto \pi(\varphi_q(X^q))$ is continuous for every finite dimensional representation π of G and for all $X^q = E_i^q, F_i^q, H_i^q$. It is not difficult to see that there exists a continuous family of isomorphisms such that φ_1 is the identity map [6].

Theorem 5. *There exists a strongly operator continuous family of unitary Drinfeld twists $\mathcal{F}_q$ such that $\mathcal{F}_1 = 1$. Furthermore, if $\{\psi_q \colon W^*(G_q) \to W^*(G)\}_{q>0}$ is another continuous family of $*$-isomorphisms such that $\psi_1 = \iota$, and $\{\mathcal{E}_q\}_{q>0}$ is a corresponding continuous family of unitary Drinfeld twists with $\mathcal{E}_1 = 1$, then there exists a unique continuous family of unitary elements $u_q \in W^*(G)$ such that*

$$u_1 = 1, \quad \text{and} \quad \psi_q = u_q \varphi_q(\cdot) {u_q}^* \quad \text{and} \quad \mathcal{E}_q = (u_q \otimes u_q)\mathcal{F}_q \hat{\Delta}(u_q)^* \quad \text{for all} \quad q > 0.$$

For every matrix coefficient a of a finite dimensional representation of G the element $a\varphi_q \in W^*(G_q)_*$ is a matrix coefficient of a finite dimensional representation of G_q. The C*-algebras $C(G_q)$ have a unique structure of a continuous field of C*-algebras such that the vector fields $q \mapsto a\varphi_q$ are continuous [6]. Furthermore, this structure does not depend on the choice of $\{\varphi_q\}_q$. For $0 < s < t$ denote by $C(G_{[s,t]})$ the C*-algebra of continuous sections over the interval $[s,t]$. For every $q \in [s,t]$ the evaluation maps $C(G_{[s,t]}) \to C(G_q)$ are KK-equivalences [7]. In particular, the C*-algebras $C(G_q)$ are canonically KK-equivalent to each other for all $q > 0$.

Drinfeld twists were used in [1] to construct Dirac operators D_q on G_q. Using the previous theorem we can then prove the following result.

Theorem 6 ([6])**.** *The family of Dirac operators D_q is continuous in the sense that it defines an element in $\mathcal{R}KK_n(C(G_{[s,t]}), C[s,t])$, $n = \dim G (\mathrm{mod}\, 2)$.*

This element is independent of the choice of a continuous family of isomorphisms and a continuous family of unitary Drinfeld twists.

In particular, the K-homology classes of D_q correspond to each other under the canonical KK-equivalences between the C*-algebras $C(G_q)$.

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Reconstructing global fields using noncommutative geometry

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(joint work with Matilde Marcolli)

The **general philosophy** can be described by the motto *pluralizing zeta*. Zeta functions ζ_X are counting devices associated to invariants of objects X. For example, counting ideals of a given norm in a number field, counting the spectrum of the Laplace-Beltrami operator on a Riemannian manifold, counting points of varieties over finite fields, lengths of geodesics, periodic orbits, and what not. However, such a single zeta function does not always characterize the object up to isomorphism: for number fields, this is the phenomenon of arithmetic equivalence; for Riemannian manifolds, that of isospectrality; for curves over finite fields, Tate's theory of isogenies of Jacobians, etc. We put this single zeta function into a family of zeta functions, indexed by some algebra, and the problem disappears. For number fields, the algebra is the group ring of the maximal abelian extension of the field, for Riemannian manifolds, the smooth functions on it, and so on. In this talk, I focussed on the case of number fields. For other examples, see [4], [5], [7].

I will list various **objects that do/don't determine a number field K.** The Dedekind zeta function of K is $\zeta_K(s) := \sum_{0\neq\mathfrak{a}} N_K(\mathfrak{a})^{-s}$, where the sum runs over all non-zero ideals $\mathfrak{a}$ of the ring of integers of K, and N_K is the norm from K to $\mathbf{Q}$. Knowing ζ_K is the same as knowing the inertia degree $f(\mathfrak{p}|p)$ for all prime ideals $\mathfrak{p}$. A theorem of Mihály Bauer (1903 [1]) says that if K, L are two number fields *that are Galois over* $\mathbf{Q}$, then $K \cong L$ is equivalent to $\zeta_K = \zeta_L$. However, a result of Gaßmann from 1926 [8] says that in general, there do exist non-isomorphic number fields K, L with $\zeta_K = \zeta_L$. Actually, he proves that $\zeta_K = \zeta_L$ is equivalent to the following statement: fix a common extension N of K and L that is Galois over $\mathbf{Q}$ with Galois group G, and let H_K and H_L denote the Galois groups of N/K and N/L, respectively. Then $\zeta_K = \zeta_L$ if and only if each G-conjugacy class intersects H_K and H_L in the same number of elements. A result from Perlis from 1977 [15] says that the smallest degree of a field $K/\mathbf{Q}$ with $\zeta_K = \zeta_L$ but $K \not\cong L$ is 7, and an example is given by $K = \mathbf{Q}(\alpha), L = \mathbf{Q}(\beta)$ with $\alpha^7 - 7\alpha + 3 = 0$ and $\beta^7 + 14\beta^4 - 42\beta^2 - 21\beta + 9 = 0$. Here are some further attempts at finding objects that determine isomorphism of number fields K and L: an *isomorphism of adele rings* $\mathbf{A}_K \cong \mathbf{A}_L$ is strictly stronger than equality of zeta functions, but still does not imply field isomorphism (Komatsu, 1976 [10]); an example is $K = \mathbf{Q}(\sqrt[8]{18})$and $L = \mathbf{Q}(\sqrt[8]{288})$. An isomorphism of abelian Galois groups $G_K^{\mathrm{ab}} \cong G_L^{\mathrm{ab}}$ is not enough either: Kubota [11] determined the isomorphism type of G_K^{ab} (its *Ulm invariants*) in terms of K, and Onabe (1976 [14]) gave explicit examples, such as $G^{\mathrm{ab}}_{\mathbf{Q}(\sqrt{-2})} \cong G^{\mathrm{ab}}_{\mathbf{Q}(\sqrt{-3})}$. At the other side of the spectrum, an isomorphism of absolute Galois groups $G_K \cong G_L$ does imply that $K \cong L$! This is due to Neukirch (1969 [13]) when K, L are Galois over $\mathbf{Q}$ and Uchida (1976 [17]) in general. This last theorem is the first manifestation of what Grothendieck called **an**abelian theorems. We conclude that the objects listed above, that are *internal* to a number field K (i.e., can be described in terms of ideals of K), such as $\zeta_K, \mathbf{A}_K$ or G_K^{ab} (which is internal by class field theory), lead to *failure*, whereas a mysterious object G_K, that is *external* to K (described in terms of extensions of K, or via the Langlands program in terms of automorphic forms), leads to *success* . . . Can we do better, and have "internal success"? A first example is the result of Connes and Consani [3] that the two adele class spaces $\mathbf{A}_K/K^* \cong \mathbf{A}_L/L^*$ are isomorphic as hyperrings over the Krasner hyperfield if and only if K and L are isomorphic.

We go on and look for a good topological space (rather than ring) that does it. And this space will turn out to be *noncommutative.* The method is to consider **class field theory as (noncommutative) dynamical system,** as follows. Let J_K denote the group of fractional ideals of K, J_K^+ the semigroup of integral ideals of K, $\vartheta_K \colon \mathbf{A}_K^* \to G_K^{\mathrm{ab}}$ the Artin reciprocity map and $\hat{\mathcal{O}}_K$ the integral finite adeles of K. Choose a section s of the natural map $\mathbf{A}^*_{K,f} \to J_K \colon (x_\mathfrak{p})_\mathfrak{p} \mapsto \prod \mathfrak{p}^{v_\mathfrak{p}(x_\mathfrak{p})}$. These objects were used by Ha and Paugam in 2005 [9] [12] to construct a dynamical system associated to K (for $K = \mathbf{Q}$, this is the famous Bost-Connes system [2]), as follows: we make a *topological space* $X_K = G_K^{\mathrm{ab}} \times_{\hat{\mathcal{O}}_K^*} \hat{\mathcal{O}}_K$, consisting of classes $[(\gamma, \rho)]$ for $\gamma \in G_K^{\mathrm{ab}}$ and $\rho \in \hat{\mathcal{O}}_K$, defined by the equivalence $(\gamma, \rho) \sim (\vartheta_K(u^{-1}) \cdot \gamma, u\rho)$ for all $u \in \hat{\mathcal{O}}_K^*$. Then we consider the *action* of $\mathfrak{n} \in J_K^+$ on X_K given by $\mathfrak{n} * [(\gamma, \rho)] := [(\vartheta_K(s(\mathfrak{n}))^{-1}\gamma, s(\mathfrak{n})\rho)]$. In this way, we get a dynamical system (X_K, J_K^+).

Theorem. *For two number fields K and L, an isomorphism $K \cong L$ is equivalent to a norm-preserving isomorphism of dynamical systems $(X_K, J_K^+) \cong (X_L, J_L^+)$.*

By *isomorphism of dynamical systems*, we mean a homeomorphism $\Phi \colon X_K \xrightarrow{\sim} X_L$ and a group homomorphism $\varphi \colon J_K^+ \xrightarrow{\sim} J_L^+$ such that $\Phi(\mathfrak{n} * x) = \varphi(\mathfrak{n}) * \Phi(x)$ for all $x \in X_K$ and $\mathfrak{n} \in J_K^+$; and *norm-preserving* means that $N_L(\varphi(\mathfrak{n})) = N_K(\mathfrak{n})$ for all $\mathfrak{n} \in J_K^+$. The proof is really to "hit the dynamical system with a hammer until enough isomorphic objects jump out".

The result has a reformulation using **quantum statistical mechanics**, by encoding the dynamics in Banach algebra language. We set $A_K := C(X_K) \rtimes J_K^+$ to be the semigroup crossed product C^*-algebra corresponding to the dynamical system. Physically, it corresponds to the *algebra of observables.* If we let $\mu_{\mathfrak{n}}$ and $\mu_{\mathfrak{n}}^*$ denote the partial isometries of the algebra corresponding to $\mathfrak{n} \in J_K^+$, then we also need the non-involutive subalgebra $A_K^\dagger$ of A_K generated by $C(X)$ and $\langle \mu_{\mathfrak{n}} \rangle_{\mathfrak{n} \in J_K^+}$ (but not the $\mu_{\mathfrak{n}}^*$). We also consider a one-parameter subgroup of automorphisms of A_K, denoted $\sigma_K \colon \mathbf{R} \hookrightarrow \mathrm{Aut}(A_K)$, defined by $\sigma_K(t)(f) = f$ and $\sigma_K(t)(\mu_{\mathfrak{n}}) = N_K(\mathfrak{n})^{it}\mu_{\mathfrak{n}}$. The algebra with this so-called *time evolution* is an abstract *quantum statistical mechanical system.* A slightly stronger statement than the main theorem is the following (the proof is similar to that of Davidson and Katsoulidis in [6], combined with ergodicity):

Theorem. *Two number fields K and L are isomorphic if and only if there is an isomorphism of quantum statistical mechanical systems $(A_K, \sigma_K) \xrightarrow{\sim} (A_L, \sigma_L)$ that maps $A_K^\dagger$ to $A_L^\dagger$.*

In a sense, these theorems show that a *suitable combination of failure* (ζ_K, which will be the partition function of the system, G_K^{ab} and $\mathbf{A}_K$, which occur in the system) *may lead to success.* It gives an "internal" description of the isomorphism type of a number field by a noncommutative topological space. One may replace "anabelian" by "noncommutative" ...

From the main theorem, we deduce our answer to the problems outlined before:

Theorem. *An isomorphism of number fields $K \cong L$ is equivalent to the existence of an isomorphism $\psi \colon G_K^{\mathrm{ab}} \xrightarrow{\sim} G_L^{\mathrm{ab}}$, such that **all** abelian L-series match: $L_K(\chi) = L_L((\psi^{-1})^*\chi)$ for all $\chi \in \mathrm{Hom}(G_K^{\mathrm{ab}}, S^1)$.*

The L-series of the trivial character is the zeta function, so this theorem does solve the number theoretical riddle we outlined before. What is more, we discovered this theorem because L-series occur as evaluations of low temperature equilibrium states of the system at particular test functions related to the character. Our proof of this theorem is to deduce from L-series equality an isomorphism of dynamical systems, which basically boils down to a bit of character theory, and then using the main theorem.

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Semigroup C*-algebras and their K-theory

Xin Li

(joint work with Joachim Cuntz)

We introduce a new construction called semigroup C*-algebras. As in the group case, it is possible to characterize amenability of semigroups in terms of these semigroup C*-algebras. Moreover, we compute K-theory for semigroup C*-algebras associated with certain semigroups from number theory.

Semigroup C*-algebras

By a semigroup, we mean a set P equipped with an associative binary operation. Moreover, we always assume that our semigroups have unit elements. A semigroup P is called left cancellative if for every p, q and q' in P, $pq = pq'$ implies $q = q'$. All our semigroups are supposed to be left cancellative.

Given a left cancellative semigroup P, we can construct its left regular representation: Consider the Hilbert space $\ell^2(P)$ with the canonical orthonormal basis $\{\varepsilon_q\}_{q \in P}$ (given by $\varepsilon_q(p) = \delta_{p,q}$). We define for every $p \in P$ an isometry V_p by setting $V_p \varepsilon_q = \varepsilon_{pq}$. Note that our assumption that P is left cancellative ensures that the assignment $\varepsilon_q \mapsto \varepsilon_{pq}$ indeed extends to an isometry. Now the reduced semigroup C*-algebra of P is given as the sub-C*-algebra of $\mathcal{L}(\ell^2(P))$ generated by these isometries $\{V_p \colon p \in P\}$:

Definition 1. $C_r^*(P) := C^*(\{V_p \colon p \in P\}) \subseteq \mathcal{L}(\ell^2(P))$.

We now turn to the construction of full semigroup C*-algebras. The "simplest" construction would be to define the full semigroup C*-algebra as

$$C^*\left(\{v_p\colon p \in P\} \middle| \begin{array}{c} v_p \text{ are isometries} \\ \text{satisfying } v_{pq} = v_p v_q \end{array}\right).$$

However, even in the case of very simple semigroups, this construction leads to very complicated C*-algebras which are not suited for studying amenability. For instance, for $P = \mathbb{N}_0 \times \mathbb{N}_0$, the definition above yields the universal C*-algebra generated by two commuting isometries. But this C*-algebra is not nuclear by [3], Theorem 6.2. So we need to impose more relations on the generating isometries. The idea is to make use of right ideals of our semigroups to control the range projections of the generating isometries. Let us start with some notations:

Given a subset X of P and an element $p \in P$, we set

$$pX := \{px\colon x \in X\} \text{ and } p^{-1}X := \{q \in P\colon pq \in X\}.$$

In other words, pX is the image and $p^{-1}X$ is the pre-image of X under left multiplication with p. A subset X of P is called a right ideal if it is closed under right multiplication with arbitrary semigroup elements, i.e. if for every $x \in X$ and $p \in P$, the product xp always lies in X. Let $\mathcal{J}$ be the smallest family of right ideals of P containing P and $\emptyset$, i.e.

$$P \in \mathcal{J}, \emptyset \in \mathcal{J},$$

and closed under left multiplication, taking pre-images under left multiplication,

$$X \in \mathcal{J}, p \in P \Rightarrow pX, p^{-1}X \in \mathcal{J},$$

as well as finite intersections,

$$X, Y \in \mathcal{J} \Rightarrow X \cap Y \in \mathcal{J}.$$

With the help of this family of right ideals, we can now construct the full semigroup C*-algebra of P. The idea is to ask for a projection-valued spectral measure, defined for elements in the family $\mathcal{J}$ and taking values in projections in our C*-algebra.

Definition 2. *The full semigroup C*-algebra of P is the universal C*-algebra generated by isometries $\{v_p\colon p \in P\}$ and projections $\{e_X\colon X \in \mathcal{J}\}$ satisfying the following relations:*

$$\text{I.(i) } v_{pq} = v_p v_q \quad \text{I.(ii) } v_p e_X v_p^* = e_{pX}$$
$$\text{II.(i) } e_P = 1 \quad \text{II.(ii) } e_\emptyset = 0 \quad \text{II.(iii) } e_{X\cap Y} = e_X \cdot e_Y$$

for all p, q in P and X, Y in $\mathcal{J}$. We denote this universal C-algebra by $C^*(P)$:*

$$C^*(P) := C^*\left(\{v_p\colon p \in P\} \cup \{e_X\colon X \in \mathcal{J}\} \middle| \begin{array}{c} v_p \text{ are isometries} \\ \text{and } e_X \text{ are projections} \\ \text{satisfying I and II.} \end{array}\right)$$

Of course, the question is: Where do all these relations come from? The idea is that we can think of $C^*(P)$ as a universal model of the reduced semigroup C*-algebra $C_r^*(P)$. To make this precise, let us again consider concrete operators on $\ell^2(P)$. We have already defined the isometries V_p for $p \in P$. Let X be subset of P and let E_X be the orthogonal projection onto $\ell^2(X) \subseteq \ell^2(P)$. It is now easy to check that the two families $\{V_p\colon p \in P\}$ and $\{E_X\colon X \in \mathcal{J}\}$ satisfy relations I and II (with V_p in place of v_p and E_X in place of e_X). This explains the origin of these relations. At the same time, we obtain by universal property of $C^*(P)$ a non-zero homomorphism $\lambda : C^*(P) \to C_r^*(P)$ sending v_p to V_p and e_X to E_X for every $p \in P$ and $X \in \mathcal{J}$. This homomorphism is called the left regular representation of $C^*(P)$.

We remark that if P happens to be a group, then the C*-algebras we constructed will just be the reduced and full group C*-algebras. Moreover, for special types of semigroups, the constructions we have presented have already been introduced by A. Nica in [4] and J. Cuntz (see [1]).

Amenability of semigroups

Let us now explain the connection to amenability of semigroups.

A (discrete) semigroup P is left amenable if there exists a left invariant mean on $\ell^\infty(P)$, i.e. a state μ on $\ell^\infty(P)$ such that for every $p \in P$ and $f \in \ell^\infty(P)$, $\mu(f(p\sqcup)) = \mu(f)$. We remark that in the case of left cancellative semigroups, a semigroup is left amenable if and only if it satisfies the (strong) Følner condition (see [5] for details). Here are our main results concerning amenability:

Theorem 1. *If P is cancellative and satisfies the $\bigcup$-condition, then the following are equivalent:*

- *P is left amenable*
- *The left regular representation $\lambda : C^*(P) \to C_r^*(P)$ is an isomorphism and there exists a non-zero character on $C^*(P)$.*

Here "cancellative" means left and right cancellative. The $\bigcup$-condition is a technical condition on the family $\mathcal{J}$ of right ideals of P. It says that given $X \in \mathcal{J}$ and finitely many proper subsets $X_1, \ldots, X_n \in \mathcal{J}$ of X (i.e. $X_j \subsetneq X$), then $\bigcup_{j=1}^n X_j$ is a proper subset of X (i.e. $\bigcup_{j=1}^n X_j \subsetneq X$).

Theorem 2. *Let P be cancellative and countable. If P is right amenable, then $C^*(P)$ is nuclear. Conversely, if $C^*(P)$ is nuclear and if there exists a non-zero character on $C^*(P)$, then P is left amenable.*

The reader may consult [2] for details.

K-theory

We consider special semigroups from number theory. Let K be a number field, i.e. a finite field extension of $\mathbb{Q}$, and let R be the ring of integers in K, i.e. the integral closure of $\mathbb{Z}$ in K. We are interested in the multiplicative semigroup $R^\times = R \setminus \{0\}$ and in the $ax+b$-semigroup $R \rtimes R^\times$. Here is our K-theoretic result:

Theorem 3 (joint with J. Cuntz)**.**

$$K_*(C^*(R^\times)) \cong \bigoplus_{\gamma \in Cl_K} K_*(C^*(R^*)),$$

$$K_*(C^*(R \rtimes R^\times)) \cong \bigoplus_{\gamma \in Cl_K} K_*(C^*(I_\gamma \rtimes R^*))$$

and similarly for K-homology.

Here Cl_K is the ideal class group of K. R^* is the (multiplicative) group of units in R. Moreover, in the second formula, we choose for every $\gamma \in Cl_K$ an ideal I_γ of R which represents γ.

Let us remark that for each of these semigroups, the left regular representation is an isomorphism, so that we obtain the K-theory of the reduced semigroup C*-algebras as well. Moreover, we can actually prove that $C^*(R^\times)$ is KK-equivalent to $\bigoplus_{\gamma \in Cl_K} C^*(R^*)$ and that $C^*(R \rtimes R^\times)$ is KK-equivalent to $\bigoplus_{\gamma \in Cl_K} C^*(I_\gamma \rtimes R^*)$. In addition, for every semigroup P, we can define the diagonal map $C^*(P) \to C^*(P) \otimes C^*(P)$, $v_p \mapsto v_p \otimes v_p$. This means that there is a canonical product structure on the K-homology $K^*(C^*(P))$. Our K-homology formulas for the semigroups $R^\times$ and $R \rtimes R^\times$ (which are analogous to the K-theoretic ones in the theorem above) are compatible with the product structures.

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Arithmetic of the BC-system

Alain Connes

(joint work with C. Consani, JHU)

For each prime p and each embedding σ of the multiplicative group of an algebraic closure $\bar{\mathbb{F}}_p$ of the finite field $\mathbb{F}_p$ as complex roots of unity, we construct a p-adic indecomposable representation π_σ of the integral Bost–Connes system as additive endomorphisms of the big Witt ring of $\bar{\mathbb{F}}_p$. These representations are the p-adic analogues of the complex, extremal KMS_∞ states of the BC-system. The initial motivation to seek for these representations came from the discovery that the algebraic relations fulfilled by the basic operators σ_n and $\tilde{\rho}_n$ acting on the group ring of $\mathbb{Q}/\mathbb{Z}$ of the BC-system are identical to the relations of the Frobenius endomorphisms F_n, $n \in \mathbb{N}$, and Verschiebung additive functorial maps V_n, $n \in \mathbb{N}$, in the Witt construction. This is first used at the level of the Witt ring viewed as a functor which to each commutative ring A associates the ring $W_0(A)$ classifying endomorphisms of finite projective modules over A (modulo zero endomorphisms). We then use the completion of $W_0(A)$ to the ring $W(A)$ of big Witt vectors. In our recent joint work we have pursued this analogy much further by implementing the Iwasawa theory of p-adic L-functions to construct, in the p-adic case, the partition function and the KMS_β states. The role of the Riemann zeta function, as partition function of the BC-system over $\mathbb{C}$ is replaced, in the p-adic case, by the p-adic L-functions and the polylogarithms whose values at roots of unity encode the KMS states. In particular, we have shown that the division relations for the p-adic polylogarithms at roots of unity correspond to the KMS condition. We first obtained the result for inverse temperature β in the "extended s-disk" D_p which is standard in p-adic analysis. We then use Iwasawa theory to extend the KMS theory to a covering of the completion $\mathbb{C}_p$ of an algebraic closure of $\mathbb{Q}_p$.

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Algebraic index theorem for gerbes

Ryszard Nest
(joint work with Paul Bressler, Sascha Gorokhovsky and Boris Tsygan)

This talk is about the index theorem in the context of an algebroid stack deformations of gerbes and modules over them.

An algebroid stack is a natural generalization of a sheaf of rings. It gives rise to a sheaf of categories that, in the case of a sheaf of rings, is the sheaf of categories of modules. The role of algebroid stacks in deformation theory was first emphasized in [10] and in [15]. Deformations of a sheaf of rings as such are more difficult to classify than their deformations as a stack. This is closely related to the fact that some of the most natural deformations appearing in complex analysis happen to be stacks and not sheaves. As an example, the sheaf of differential operators on a manifold gives rise to a deformation of the sheaf of functions on the cotangent bundle. If one replaces the cotangent bundle by an arbitrary holomorphic symplectic manifold, this deformation has a natural generalization which in general is an algebroid stack. The first obstruction for this stack to arise from a sheaf of algebras is the first Rozansky-Witten class in the second Dolbeault cohomology [1].

Study of deformations of algebroid stacks is being carried out from different perspectives in [1], [2], [4], [3], [12], [13], [14], [23], [22], [6], and in other works.

Analytic constructions of algebras on a manifold twisted by a gerbe appeared in [18]. In this paper the authors also prove a related index theorem. On the more algebraic side, in [13], Kashiwara and Schapira defined the Hochschild homology of an algebroid stack deformation of the sheaf of functions on a manifold, and the characteristic class $\mathrm{hh}(\mathcal{M})$ in this homology for a coherent sheaf $\mathcal{M}$. For symplectic deformations, they constructed the trace density morphism from their version of Hochschild homology to the de Rham cohomology. On the other hand, in [3] we defined the Hochschild and cyclic homologies of an algebroid stack, as well as the Chern character $\mathrm{ch}(\mathcal{M})$ of a perfect complex of modules in the negative cyclic homology. Presumably, the two definitions of the Hochschild homology coincide, and $\mathrm{hh}(\mathcal{M})$ is the image of $\mathrm{ch}(\mathcal{M})$ under the map from the negative cyclic to the Hochschild homology.

In this work we define the trace density for gerbes. It is a morphism from our versions of the Hochschild and the negative cyclic homology of a symplectic deformation of a gerbe to the de Rham cohomology. We expect that our map from the Hochschild cohomology coincides with the one defined in [13]. There is another map from the negative cyclic homology to the de Rham cohomology, namely reduction modulo $\hbar$ followed by the gerbe version of the Hochschild-Kostant-Rosenberg map.

The main result is the computation of the trace density map for gerbes in terms of the Hochschild-Kostant-Rosenberg map. Specifically, we establish that the trace density map is the Hochschild-Kostant-Rosenberg map, multiplied by the cohomology class $\sqrt{\widehat{A}(T_M)} \smile e^{\theta}$ where θ is the characteristic class of the deformation defined in [1] . From this we deduce the Riemann-Roch formula for the Chern character of a perfect complex. These results were proven for the sheaf deformations in [20] for the smooth case and in [5] for the analytic case. The index theorem for elliptic pairs conjectured in [24] follows from the partial case when the manifold is the cotangent bundle with the standard symplectic structure.

The proof goes essentially through the same steps as the one given in [5]. One needs however to replace all the constructions by the appropriate twisted versions. The development of these is the focus of the present work. Using the results from [1] and [3] we develop the analogue of Fedosov's construction [7] for gerbes. We also construct an appropriate analogue of the Gelfand-Fuks map which allows us to apply the arguments of formal differential geometry developed by Gelfand and Kazhdan [9].

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Relative pairings, eta cocycles and the Godbillon-Vey index theorem

Paolo Piazza
(joint work with Hitoshi Moriyoshi)

Let N be a closed compact manifold. Let $\Gamma \to \tilde{N} \to N$ be a Galois Γ-cover. Let T be a smooth oriented compact manifold with an action of Γ which is assumed to be by diffeomorphisms, orientation preserving and locally faithful, as in [7]. Let $Y = \tilde{N} \times_\Gamma T$ and let $(Y, \mathcal{F})$ be the associated foliation. Let D be a Γ-equivariant family of Dirac operators on the fibration $\tilde{N} \times T \to T$; such a family induces a longitudinal Dirac operator on $(Y, \mathcal{F})$.

The Godbillon-Vey index theorem of Alain Connes is a theorem on a codimension 1 foliation (thus we take $T = S^1$ in this case). Following the treatment of Moriyoshi-Natsume in [7], it can be stated in the following way:
There is a cyclic 2-cocycle τ_{GV} on $C^\infty_c(Y, \mathcal{F}) := C^\infty_c((\tilde{N} \times \tilde{N} \times S^1)/\Gamma)$ which can be paired with the (compactly supported) index class $\mathrm{Ind}^c(D) \in K_0(C^\infty_c(Y, \mathcal{F}))$; there is a holomorphically closed subalgebra $\mathfrak{A}$, $C^\infty_c(Y, \mathcal{F}) \subset \mathfrak{A} \subset C^*(Y, \mathcal{F})$, containing the C^*-index class $\mathrm{Ind}(D)$ and such that τ_{GV} extends to $\mathfrak{A}$; the pairing $\langle \mathrm{Ind}(D), [\tau_{GV}] \rangle$ can be written down explicitly and it involves the Godbillon-Vey class of the foliation, $GV \in H^3(Y)$.

One might wonder if Connes index theorem for the Godbillon-Vey cocycle can be extended to foliated bundles with boundary, in the spirit of the seminal work of Atiyah-Patodi-Singer [1].

The main goal of my talk was to explain recent results, in collaboration with Hitoshi Moriyoshi, establishing such a result. See the announcement [5] and the complete paper [6]. Notice that our index formula constitutes the first instance of a higher APS index theorem on type III foliations. Notice also that, consequently, we define *a Godbillon-Vey eta invariant* on the boundary-foliation; this is a *type III eta invariant*. In tackling this specific index problem we develop a new approach to index theory on geometric structures with boundary, heavily based on the interplay between absolute and relative pairings.

Let us give a brief description of our main results.

It is clear from the structure of the classic Atiyah-Patodi-Singer index formula that one of the basic tasks in the theory is to split in a precise way the interior contribution from the boundary contribution in the higher index formula. We look at operators on the boundary through the translation invariant operators on the associated infinite cyclinder; by Fourier transform these two pictures are equivalent. We solve the Atiyah-Patodi-Singer higher index problem on a foliated bundle with boundary $(X_0, \mathcal{F}_0)$, $X_0 = \tilde{M} \times_\Gamma T$, by solving the associated L^2-problem on the associated foliation with cylindrical ends $(X, \mathcal{F})$. With the goal of splitting the interior contribution from the boundary contribution in mind, we define a short exact sequence of C^*-algebras

$$0 \to C^*(X, \mathcal{F}) \to A^*(X, \mathcal{F}) \to B^*(\mathrm{cyl}(\partial X), \mathcal{F}_{\mathrm{cyl}}) \to 0\,.$$

This is an extension by the foliation C^*-algebra $C^*(X, \mathcal{F})$ of a suitable algebra of *translation invariant operators* on the cylinder; we call it the Wiener-Hopf extension. We briefly denote the Wiener-Hopf extension as $0 \to C^*(X, \mathcal{F}) \to A^* \to B^* \to 0$. These C^*-algebras are the receptacle for the two C^*-index classes we will be working with. Thus, given a Γ-equivariant family of Dirac operators $(D_\theta)_{\theta \in T}$ with invertible boundary family $(D^\partial_\theta)_{\theta \in T}$ we prove that there exist an index class $\mathrm{Ind}(D) \in K_*(C^*(X, \mathcal{F}))$ and a relative index class $\mathrm{Ind}(D, D^\partial) \in K_*(A^*, B^*)$. The higher Atiyah-Patodi-Singer index problem for the Godbillon-Vey cocycle consists in proving that there is a well defined paring $\langle \mathrm{Ind}(D), [\tau_{GV}] \rangle$ and giving a formula for it, with a structure similar to the one displayed by the Atiyah-Patodi-Singer index formula. Now, as in the case of Moriyoshi-Natsume, τ_{GV} is initially defined on the small algebra $J_c(X, \mathcal{F})$ of Γ-equivariant smoothing kernels of Γ-compact support; however, because of the structure of the parametrix on manifolds with cylindrical ends, there does *not* exist an index class in $K_*(J_c(X, \mathcal{F}))$. Hence, even *defining* the index pairing is not obvious. We solve this problem by showing that there exists a holomorphically closed intermediate subalgebra $\mathfrak{J}$ containing the index class $\mathrm{Ind}(D)$ but such that τ_{GV} extends. This point involves elliptic theory on manifolds with cylindrical ends in an essential way.

Once the higher Godbillon-Vey index is defined, we search for an index formula for it. Our main idea is to show that such a formula is a direct consequence of the equality

$$\langle \mathrm{Ind}(D), [\tau_{GV}] \rangle = \langle \mathrm{Ind}(D, D^\partial), [(\tau^r_{GV}, \sigma_{GV})] \rangle \tag{1}$$

where on the right hand side a new mathematical object, the *relative* Godbillon-Vey cocycle, appears. The relative Godbillon-Vey cocycle is built out of the usual Godbillon-Vey cocycle by means of a very natural procedure. First, we proceed algebraically. Thus we first look at a subsequence of $0 \to C^*(X, \mathcal{F}) \to A^* \to B^* \to 0$ made of small algebras, call it $0 \to J_c(X, \mathcal{F}) \to A_c \to B_c \to 0$; $J_c(X, F)$ are, as above, the Γ-equivariant smoothing kernels of Γ-compact support; B_c is made of $\Gamma \times \mathbb{R}$-equivariant smoothing kernels on the cylinder of $\Gamma \times \mathbb{R}$-compact support. The A_c cyclic 2-cochain τ^r_{GV} is obtained from τ_{GV} through a regularization à la Melrose. The B_c cyclic 3-cocycle σ_{GV} is obtained by *suspending* τ_{GV} on the cylinder with Roe's 1-cocycle. We call this σ_{GV} the *eta cocycle* associated to τ_{GV}. One proves that $(\tau^r_{GV}, \sigma_{GV})$ is a relative cyclic 2-cocycle for $A_c \to B_c$. We obtain in this way a relative cyclic cohomology class $[\tau^r_{GV}, \sigma_{GV}] \in HC^2(A_c, B_c)$.

We remark here that for technical reasons having to do with the extension of these cocycles to suitable smooth subalgebras, see below, we shall have to consider the cyclic cocycle and the relative cyclic cocycle obtained from τ_{GV} and $(\tau^r_{GV}, \sigma_{GV})$ through the S operation in cyclic cohomology, see [2]: thus we consider $S^{p-1}\tau_{GV}$ and $(S^{p-1}\tau^r_{GV}, \frac{3}{2p+1} S^{p-1}\sigma_{GV})$ obtaining in this way a class in $HC^{2p}(J_c)$ and a relative class in $HC^{2p}(A_c, B_c)$. With a small abuse of notation we still denote these cyclic $2p$-cocycles by τ_{GV} and $(\tau^r_{GV}, \sigma_{GV})$.

Once the algebraic theory is clarified, we need to pair the class $[\tau_{GV}] \in H^{2p}(J_c)$ and the relative class $[\tau^r_{GV}, \sigma_{GV}] \in HC^{2p}(A_c, B_c)$ with the corresponding index classes $\mathrm{Ind}(D) \in K_*(C^*(X,\mathcal{F}))$ and $\mathrm{Ind}(D, D^\partial) \in K_*(A^*, B^*)$. To this end we construct an *intermediate* short exact subsequence $0 \to \mathfrak{J} \to \mathfrak{A} \to \mathfrak{B} \to 0$ of Banach algebras, sitting half-way between $0 \to C^*(X,\mathcal{F}) \to A^* \to B^* \to 0$ and $0 \to J_c(X,\mathcal{F}) \to A_c \to B_c \to 0$. Much work is needed in order to define such a subsequence, prove that $\mathrm{Ind}(D) \in K_*(\mathfrak{J}) \cong K_*(C^*(X,\mathcal{F}))$ and $\mathrm{Ind}(D, D^\partial) \in K_*(\mathfrak{A},\mathfrak{B}) \cong K_*(A^*, B^*)$ and establish that the Godbillon-Vey cyclic $2p$-cocycle τ_{GV} and the relative cyclic $2p$-cocycle $(\tau^r_{GV}, \sigma_{GV})$ extend for p large enough from J_c and $A_c \to B_c$ to $\mathfrak{J}$ and $\mathfrak{A} \to \mathfrak{B}$, thus defining elements

$$[\tau_{GV}] \in HC^{2p}(\mathfrak{J}) \quad \text{and} \quad [\tau^r_{GV}, \sigma_{GV}] \in HC^{2p}(\mathfrak{A},\mathfrak{B}).$$

We have now made sense of both sides of the equality (1) $\langle \mathrm{Ind}(D), [\tau_{GV}]\rangle = \langle \mathrm{Ind}(D, D^\partial), [(\tau^r_{GV}, \sigma_{GV})]\rangle$. The equality itself is proved by establishing and using the excision formula: if $\alpha_{\mathrm{ex}} : K_*(\mathfrak{J}) \to K_*(\mathfrak{A},\mathfrak{B})$ is the excision isomorphism, then $\alpha_{\mathrm{ex}}(\mathrm{Ind}(D)) = \mathrm{Ind}(D, D^\partial) \quad \text{in} \quad K_*(\mathfrak{A},\mathfrak{B})$. The index formula is obtained by explicitly writing the relative pairing $\langle \mathrm{Ind}(D, D^\partial), [(\tau^r_{GV}, \sigma_{GV})]\rangle$ in terms of the graph projection e_D, multiplying the operator D by $s > 0$ and taking the limit as $s \downarrow 0$. The final formula in the 3-dimensional case (always with an invertibility assumption on the boundary family) reads:

$$\langle \mathrm{Ind}(D), [\tau_{GV}]\rangle = \int_{X_0} \omega_{GV} - \eta_{GV}\,, \tag{2}$$

with ω_{GV} equal, as in the closed case, to (a representative of) the Godbillon-Vey class GV and

$$\eta_{GV} := \frac{(2p+1)}{p!} \int_0^\infty \sigma_{GV}([\dot{p}_t, p_t], p_t, \ldots, p_t, p_t)dt\,, \tag{3}$$

with $p_t := e_{tD^{\mathrm{cyl}}}$ the graph projection associated to the cylindrical Dirac family tD^{cyl}. Observe that by Fourier transform *the Godbillon-Vey eta invariant* η_{GV} only depends on the boundary family $D^\partial \equiv (D^\partial_\theta)_{\theta \in T}$.

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KO Invariants and Topological Insulators

Terry A. Loring
(joint work with Matthew B. Hastings, Adam Sørenson)

Topological Insulators. A topological insulator is a state of matter that is insulating in the interior and allows movement of charge along its boundary. A related phenomenon is the quantum Hall effect, but a topological insulator exists without an externally applied magnetic field. Of course, the quantum Hall effect has been investigated using noncommutative geometry by Bellissard [1]. Both 2D and 3D topological insulators are observed to be robust, often explained as being "topologically protected." A system in a topological insulating state that is being deformed to an ordinary insulator is expected to first go through a metallic state.

Finite Models. After many reductions, the transitions between ordinary insulators, metallic phases and topological insulators can studied via a finite lattice model, with closed geometry, for noninteracting Fermions. Conclusions made to real world systems with boundary will be more convincing if based on multiple closed geometries, at least including the sphere and torus of the appropriate dimension.

We consider a tight binding model for non-interacting fermions moving within sites on a compact subset of $\mathbb{R}^n$ or $\mathbb{C}^n$. Thus we consider n commuting operators, or just n diagonal matrices, to specify position. For the geometry of an n-torus we consider scaled unitaries $\hat{U}_1, \ldots, \hat{U}_n$. We also need a Hamiltonian $\mathcal{H}^* = \mathcal{H}$ to specify dynamics. The standard basis vector $\mathbf{e}_s$ has position

$$p(\mathbf{e}_s) = \left(\left\langle \hat{U}_1 \mathbf{e}_s, \mathbf{e}_s \right\rangle, \ldots, \left\langle \hat{U}_n \mathbf{e}_s, \mathbf{e}_s \right\rangle \right)$$

and, using Euclidean metric, we obtain a distance $d(\mathbf{e}_s, \mathbf{e}_t)$ defined between two standard basis vectors.

Band-Compressed Position Operators. For 2D investigations, we tend to assume

$$\hat{U}_1^{\,*} \hat{U}_1 = \hat{U}_2^{\,*} \hat{U}_2 = L^2$$

in $\mathbf{M}_d(\mathbb{C})$, so are using the geometry of a torus. System size should be allowed to grow, meaning $L \to \infty$ and $n \to \infty$, and we need technical conditions to reflect the fact that this is a 2D system, thus disallowing clumping near a circle. Assume $\mathcal{H}$ has bounded strength $\|\mathcal{H}\| \leq J$ and acts locally: $\exists R$,

$$d(\mathbf{e}_s, \mathbf{e}_t) > R \implies \langle \mathcal{H}\mathbf{e}_s, \mathbf{e}_\mathbf{t} \rangle = 0.$$

This implies$\left\| \left[\mathcal{H}, \hat{U}_r \right] \right\| \leq CRJ$ (or with $\hat{X}_r$ if we work on a sphere, meaning $\hat{X}_r^* = \hat{X}_r$ and $\hat{X}_1^2 + \hat{X}_2^2 + \hat{X}_3^2 = L^2 I$).

If modeling an insulator, $\mathcal{H}$ will have a spectral gap:

$$(E_F, E_F + \Delta) \cap \sigma(\mathcal{H}) = \emptyset.$$

The "Band projector" P is spectral projection of $\mathcal{H}$ for $(-\infty, E_F]$. From here we obtain

$$\left\| \left[P, \hat{X}_r\right] \right\| \leq 2C\frac{RJ}{\Delta}.$$

If modeling a semimetal or semiconductor, we still expect $\left[P, \hat{X}_r\right] \approx 0$ will be caused by a "mobility gap." Even in the lab, one freely varies E_F. (Doping or gate insulators.) We form $\frac{1}{L}P\hat{U}_1P$ and $\frac{1}{L}P\hat{U}_2P$ to get matrices that almost commute. Working in a smaller Hilbert space (low-energy) we end up with matrices U_1 and U_2 that describe $\frac{1}{L}P\hat{U}_1P$ and $\frac{1}{L}P\hat{U}_2P$ and are *almost unitary* and *almost commuting*. We find, with $O(L^{-2})$ convergence, $\|[U_1, U_2]\| \to 0$ and $\|U_r^*U_r - I\| \to 0$. The following are equivalent (more-or-less):

(1) approximating these by commuting matrices
(2) $\mathcal{H}$ can be deformed to a trivial Hamiltonian respecting locality and spectral gap.
(3) localized basis for band subspace exists (generalized Wannier functions).

There is a potential obstruction.

(1) What physicists call the *Chern number* of the band.
(2) What mathematicians call the *Bott element*, in $K_0(\mathbf{M}_d(\mathbb{C})) \cong \mathbb{Z}$.

One way to compute the Bott element is as

$$\operatorname{Sig}\left(\begin{bmatrix} f(U_2) & g(U_2) + h(U_2)U_1 \\ g(U_2) + U_1^*h(U_2) & I - f(U_2) \end{bmatrix}\right)$$

where $f, g, h : S^1 \to \mathbb{R}$ are some specific functions with $f^2 + g^2 + h^2 = 1$. Here Sig is "signature" meaning half the number of positive eigenvalues minus the number of negative eigenvalues.

Spin Chern numbers. If we have time reversal (TR) invariance, the Chern number vanishes. However, we find $U_1^\sharp = U_1$ where

$$X^\sharp = -ZX^{\mathrm{T}}Z,$$

using $Z = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \in \mathbf{M}_{2N}(\mathbb{C})$, and then

$$Q^*\begin{bmatrix} f(U_2) & g(U_2) + \frac{1}{2}\{h(U_2), U_1\} \\ g(U_2) + \frac{1}{2}\{h(U_2), U_1^*\} & I - f(U_2) \end{bmatrix}Q$$

is pure imaginary and hermitian. (Q is a specific unitary.) The spin Chern number lives in the group $\mathbb{Z}_2 = \{-1, 1\}$, and is computed in $\mathcal{O}(N^3)$ time with the Pfaffian,

$$\operatorname{Sign}\left(\operatorname{Pf}\left(iQ^*\begin{bmatrix} f(U_2) & g(U_2) + \frac{1}{2}\{h(U_2), U_1\} \\ g(U_2) + \frac{1}{2}\{h(U_2), U_1^*\} & I - f(U_2) \end{bmatrix}Q\right)\right).$$

In less computational terms, the spin Chern number is created using the "almost homomorphism"

$$\left(C(\mathbb{T}^2), \mathrm{id}\right) \to \left(\mathbf{M}_{2N}(\mathbb{C}), \sharp\right)$$

determined by the band-projected position matrices. It is an element of

$$\hom\left(K_{-2}\left(\left(C(\mathbb{T}^2),\mathrm{id}\right)\right),K_{-2}\left(\left(\mathbf{M}_{2N}(\mathbb{C}),\sharp\right)\right)\right)$$

and leads to an element in

$$K_{-2}\left(\left(\mathbf{M}_{2N}(\mathbb{C}),\sharp\right)\right)\cong K_2\left(\left(\mathbf{M}_{2N+2}(\mathbb{C}),\mathrm{T}\right)\right)\cong\mathbb{Z}_2.$$

The spin Chern number is the only obstruction to TR-invariant Wannier functions, [6]. The proof uses the quaternionic version of Lin's theorem [5]. We prove results for S^2 and $\mathbb{T}^2$ and many other geometries, such as an n-hole torus. However, our results are nonquantitative in the direction

$$\text{trivial invariant} \implies \text{localized TR-invariant basis.}$$

Going 3D. We finished by discussing a 3D invariant, that lives in

$$K_{-3}\left(\left(\mathbf{M}_{2N}(\mathbb{C}),\sharp\right)\right).$$

We also discussed the data from 2D [4], [7] and 3D [2] numerical studies.

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The Chern-Connes character is not rationally injective

Michael Puschnigg

The Chern-Connes character is the unique nontrivial multiplicative natural transformation

$$ch_{biv}:\,KK(-,-)\longrightarrow HC^*_{loc}(-,-)$$

from Kasparov's bivariant K-theory of separable C^*-algebras [1] to bivariant local cyclic cohomology [3]. It is known to be rationally injective on the bootstrap category of C^*-algebras KK-equivalent to commutative ones. The first examples of C^*-algebras not in this class were provided by G. Skandalis [5], who showed that the reduced group C^*-algebra of a word hyperbolic group with Kazhdan's *Property* (T) cannot be KK-equivalent to a nuclear C^*-algebra. As a consequence, he showed that the image of Kasparov's "*Gamma*"-element [1] under descent is not equal to one (even rationally) for such groups:

$$j_r(\gamma) \neq 1 \in KK(C^*_r(\Gamma), C^*_r(\Gamma)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The main result announced in my talk is a complete calculation of the bivariant local cyclic cohomology of the reduced C^*-algebra of a word-hyperbolic group:

$$HC^*_{loc}(C^*_r(\Gamma), C^*_r(\Gamma)) \simeq Hom(H_*(\Gamma, \mathbb{C}\Gamma_{fin}), H_*(\Gamma, \mathbb{C}\Gamma_{fin})),$$

where $\mathbb{C}\Gamma_{fin}$ is the linear span of the set of torsion elements in Γ. This result in conjunction with the axiomatic characterization of the "*Gamma*"-element permits to calculate its Chern-Connes character:

$$ch_{biv}(j_r(\gamma)) = 1 = ch_{biv}(1) \in HC^*_{loc}(C^*_r(\Gamma), C^*_r(\Gamma)).$$

In particular, one learns from Skandalis' work that the Chern-Connes character

$$ch_{biv} : KK(C^*_r(\Gamma), C^*_r(\Gamma)) \longrightarrow HC^*_{loc}(C^*_r(\Gamma), C^*_r(\Gamma))$$

is not rationally injective for word-hyperbolic groups with Kazhdan's *Property* (T).

There are two approaches to these results. The first uses Lafforgue's recent breakthrough proof of the Baum-Connes conjecture with coefficients for word hyperbolic groups [2]. The homotopy between the "*Gamma*"-element and the unit element, which he constructs among Fredholm representations of weakly exponential growth, passes to local cyclic cohomology under the Chern-Connes character and leads to the above claims. The second approach, which is considerably shorter and simpler, is based on the explicit calculation of the local cyclic cohomology groups involved. It seems to yield however only the slightly weaker result, that the Chern-Connes character of the "Gamma"-element equals one in reduced local cyclic cohomology. The calculation is carried out as follows. As local cyclic cohomology is stable under passage to holomorphically closed subalgebras (at least under some technical conditions which are satisfied in our context), one may pass from the group C^*-algebra to an unconditional isospectral Banach subalgebra. For such algebras one may consider an unconditional version of the analytic cyclic bicomplex, which can be identified explicitely in the derived ind-category [3] with the complex calculating the homology of Γ with coefficients in $\mathbb{C}\Gamma_{fin}$. This unconditional cyclic bicomplex is then identified with the original analytic cyclic bicomplex in the appropriate triangulated category.

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Graphs with large girth and the (coarse) Baum-Connes conjecture

Rufus Willett
(joint work with Guoliang Yu)

Let (X_n) be a sequence of finite graphs, with $\text{girth}(X_n) \to \infty$ as $n \to \infty$ (recall that the girth of a graph is the length of its shortest non-trivial cycle). We study such sequences from the point of view of coarse (metric) geometry and the coarse Baum-Connes conjecture. To do this, let $X = \sqcup X_n$ be the disjoint union of the X_n, and metrize X using any metric d that restricts to the edge metric on each individual graph, and is such that the distance between each X_n and its complement tends to infinity as n tends to infinity.

Such spaces X have been of interest recently, as Gromov has shown [2, 1] that (under additional assumptions) there exists a finitely presented group 'containing' X in its Cayley graph, where 'containing' is to be understood in the sense of coarse geometry. The assumption that girth tends to infinity is necessary for Gromov's methods.

It is possible to make Gromov's ideas work in cases where the sequence (X_n) is an expander (and has girth tending to infinity); in this case the resulting group is called a *Gromov monster group*, and provides a counterexample to the Baum-Connes conjecture with coefficients. Gromov monster groups provide the only known such counterexamples.

Using a comparison of a sequence of graphs (X_n) with girths tending to infinity with the corresponding sequence of universal covers, Yu and I study the coarse Baum-Connes conjecture for (X_n). We obtain a fairly complete analysis.

Theorem. *Let (X_n) be a sequence of graphs with bounded degrees such that girth$(X_n) \to \infty$, and $X = \sqcup X_n$ the space built above. Let*

$$\mu : \lim_{R\to\infty} K_*(P_R(X)) \to K_*(C^*(X))$$

be the coarse Baum-Connes assembly map for X. Then:

- *μ is injective;*
- *if X is also an expander, μ is not surjective;*
- *the* maximal *version of μ is an isomorphism.*

On the other hand, based on work of Oyono-Oyono and Yu [4], we introduce a strengthening of the notion of 'expander' called *geometric property (T)*; if (X_n) has geometric property (T) (for example if (X_n) is a Margulis-type expander built as a sequence of quotients of a property (T) group), then the third part above fails.

The theorem above has the following corollary, which can be thought of as an extension of results of Higson–Lafforgue–Skandalis [3] on counterexamples to the Baum-Connes conjecture with coefficients.

Corollary. *Let Γ be a Gromov monster group. Then there exists a (commutative) Γ-C^*-algebra A such that if*

$$\mu : K_*^{top}(\Gamma; A) \to K_*(A \rtimes_r \Gamma)$$

is the Baum-Connes assembly map for Γ *with coefficients in* A *then:*

- μ *is injective;*
- μ *is not surjective;*
- *the* maximal *version of* μ *is an isomorphism.*

The third parts of the theorem and corollary can be thought of as a mitigation of monstrosity for Gromov's groups.

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Propagation and controlled *K*-theory

Hervé Oyono-Oyono
(joint work with Guoliang Yu)

The study of elliptic differential operators from the point of view of index theory and its generalisations to higher order indices gives rise to C^*-algebras where propagation makes sense and encodes the large scale geometry of the underlying space. Prominent examples for such C^*-algebras are Roe algebras, group C^*-algebras and cross product C^*-algebras. The locallity of these differential operators implies that these (generalised) indices can be defined as classes of idempotent with finite propagation. For instance, let D be an elliptic differential operator on a compact manifold M and let Q be a parametrix for D. Then $S_0 := Id - QD$ and $S_1 := Id - DQ$ are smooth kernel operators on $M \times M$ and

$$P_D = \begin{pmatrix} {S_0}^2 & S_0(Id + S_0)Q \\ S_1 D & Id - {S_1}^2 \end{pmatrix} \tag{1}$$

is an idempotent. Since Q can be choosen with support arbitrary close to the diagonal i.e with arbitrary small propagation, then so are the coefficients of P_D. The index of the Fredholm operator D is then

$$\operatorname{Ind} D = [P_D] - \left[\begin{pmatrix} 0 & 0 \\ 0 & Id \end{pmatrix}\right] \in K_0(\mathcal{K}(L^2(M)) \cong \mathbb{Z}.$$

Unfortunately, K-theory for C^*-algebras does not keep track of propagation. Our purpose is to develop a quantitative K-theory that takes into account propagation phenomena. C^*-algebras with propagation are modelized in the following setting:

Definition 1. *A filtered C^*-algebra A is a C^*-algebra equipped with a family $(A_r)_{r>0}$ of linear subspaces such that $A_r \subset A_{r'}$ if $r \leqslant r'$, A_r is stable by involution, $A_r \cdot A_{r'} \subset A_{r+r'}$ and $\bigcup_{r>0} A_r$ is dense in A. If A is unital, we also require that the identity 1 is an element of A_r for every positive number r.*

Example 1.

(1) *If X is a metric space and μ a borelien measure, then $\mathcal{K}(L^2(X))$ is filtered by $(\{T \in \mathcal{K}(L^2(X))$ with support of kernel of diameter less than $r\})_{r>0}$;*
(2) *Let Σ be a proper discrete metric space, and let H be a separable Hilbert space. Let $C[\Sigma]_r$ be the space of locally compact operators on $\ell^2(\Sigma)\otimes H$ with propagation less than r, i.e such that when written $T = (T_{x,y})_{(x,y)\in\Sigma^2}$ as operator blocks on H, then $T_{x,y}$ is a compact operator on H and $T_{x,y} = 0$ if $d(x,y) > r$. The Roe algebra of Σ is then $C^*(\Sigma) = \overline{\cup_{r>0} C[\Sigma]_r} \subset \mathcal{L}(\ell^2(\Sigma)\otimes H)$ and is by definition filtered by $(C[\Sigma]_r)_{r>0}$.*
(3) *If Γ is a discrete finitely generated group equipped with a word metric, let $B(e,r)$ be for any $r > 0$ the ball of radius r centered at the neutral element. Let us set $\mathbb{C}[\Gamma]_r = \{x \in \mathbb{C}[\Gamma]$ with support in $B(e,r)\}$. Then $C^*_{red}(\Gamma)$ and $C^*_{max}(\Gamma)$ are filtered by $(\mathbb{C}[\Gamma]_r)_{r>0}$.*
(4) *More generally, if Γ acts on a C^*-algebra A by automorphisms, then $A\rtimes_{red}\Gamma$ and $A\rtimes_{max}\Gamma$ are in the same way filtered C^*-algebras.*

Definition 2. *For ε in $(0, 1/4)$ and $r > 0$, an element q in a filtered C^*-algebra $A = (A_r)_{r>0}$ is an ε-r projection if q is in A_r, $q = q^*$ and $\|q^2 - q\| < \varepsilon$.*

If q is a ε-r-projection, then q gives rise by continuous functional calculus to a projection $\kappa(q)$ such that $\|\kappa(q) - q\| < 2\varepsilon$.

Example 2. *Recall that a family of graphs $(X_n)_{n\in\mathbb{N}}$ is a family of expanders if $|X_n| \to \infty$ and there exists $k \in \mathbb{N}$ and $0 < c \leqslant 2$ such that each X_n has valence at most k and (normalized) Laplacian Δ_{X_n} with spectrum in $\{0\} \cup [c, 2]$. Set then*

$$P_{X_n} = \frac{1}{|X_n|}\begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix}$$

for the spectral projection on $\ker \Delta_{X_n}$. Because of the uniform spectral gap, we get that for every $0 < \varepsilon < 1/4$, there exist $r > 0$ and a family of ε-r-projections $(Q_{X_n})_{n\in\mathbb{N}}$ such that $\kappa(Q_{X_n}) = P_{X_n}$ for all integer n.

We are now in the position to define quantitative K-theory. We proceed indeed as for usual K-theory. Let $A = (A_r)_{r>0}$ be a unital filtered C^*-algebras and let $\mathrm{P}^{\varepsilon,r}(A)$ be the set of ε-r-projections of A. We set $\mathrm{P}^{\varepsilon,r}_\infty(A) = \bigcup_{n\in\mathbb{N}} \mathrm{P}^{\varepsilon,r}(M_n(A))$. We define on $\mathrm{P}^{\varepsilon,r}_\infty(A)\times\mathbb{N}$ the equivalence relation: $(p,l) \sim (q,l')$ if there is $k \in \mathbb{N}$ and $h \in \mathrm{P}^{\varepsilon,r}_\infty(C([0,1],A))$ such that $h(0) = \mathrm{diag}(p, I_{k+l'})$ and $h(1) = \mathrm{diag}(q, I_{k+l})$.

Definition 3. *For a unital filtered C^*-algebras $A = (A_r)_{r>0}$, then $K_0^{\varepsilon,r}(A) = \mathrm{P}^{\varepsilon,r}(A)/\sim$ and $[p,l]_{\varepsilon,r}$ is the class of (p,l) mod. $\sim$.*

We can check that $[p,l]_{\varepsilon,r} + [p',l']_{\varepsilon,r} = [\mathrm{diag}(p,p'), l+l']_{\varepsilon,r}$ provides $K_0^{\varepsilon,r}(A)$ with an abelien group structure. If we equip $\mathbb{C}$ with its obvious filtration, then we can show that we have an isomorphism $K_0^{\varepsilon,r}(\mathbb{C}) \stackrel{\cong}{\to} \mathbb{Z}$; $[p,l]_{\varepsilon,r} \mapsto \mathrm{rank}\,\kappa(p) - l$. This allows to define the quantitative K-theory in the non-unital case as follows:

Definition 4. *If A is a non-unital filtered C^*-algebra and A^+ is its unitalization, then $K_0^{\varepsilon,r}(A) = \ker : K_0^{\varepsilon,r}(\tilde{A}) \to K_0^{\varepsilon,r}(\mathbb{C}) \cong \mathbb{Z}$ for the map induced by $A^+ \to \mathbb{C}$; $(a,\lambda) \mapsto \mathbb{C}$*

In the same way, we can define $K_1^{\varepsilon,r}(A)$ in term of ε-r-unitaries ($u \in A_r$, $\|u^* \cdot u - 1\| < \varepsilon$ and $\|u \cdot u^* - 1\| < \varepsilon$). Althought $K_1^{\varepsilon,r}(A)$ is only an abelian semigroup, it can be turned into a group by enlarging control and propagation in the homotopy. We have for $0 < \varepsilon \leqslant \varepsilon' < 1/4$ and $0 < r \leqslant r'$ structure homomorphisms

- $\iota_0^{\varepsilon,r} : K_0^{\varepsilon,r}(A) \longrightarrow K_0(A)$; $[p,l]_{\varepsilon,r} \mapsto [\kappa(p)] - [I_l]$.
- $\iota_1^{\varepsilon,r} : K_1^{\varepsilon,r}(A) \longrightarrow K_1(A)$; $[u]_{\varepsilon,r} \mapsto [u]$.
- $\iota_0^{\varepsilon,\varepsilon',r,r'} : K_0^{\varepsilon,r}(A) \longrightarrow K_0^{\varepsilon',r'}(A)$; $[p,l]_{\varepsilon,r} \mapsto [p,l]_{\varepsilon',r'}$.
- $\iota_1^{\varepsilon,\varepsilon',r,r'} : K_1^{\varepsilon,r}(A) \longrightarrow K_1^{\varepsilon',r'}(A)$; $[u]_{\varepsilon,r} \mapsto [u]_{\varepsilon',r'}$.
- $\iota_*^{\varepsilon,r} = \iota_0^{\varepsilon,r} \oplus \iota_1^{\varepsilon,r}$ and $\iota_*^{\varepsilon,\varepsilon',r,r'} = \iota_0^{\varepsilon,\varepsilon',r,r'} \oplus \iota_1^{\varepsilon,\varepsilon',r,r'}$.

Then we can check that for any $0 < \varepsilon < 1/4$ and any y in $K_*(A)$, there exists $r > 0$ and x in $K_*^{\varepsilon,r}(A)$ such that $\iota_*^{\varepsilon,r}(x) = y$.

Back to our index formula (1), if D is an elliptic differential operator on a compact manifold M, then for every $0 < \varepsilon < 1/4$ and $r > 0$, we can by choosing the parametrix Q with very small propagation and approximating the projection $((2P_Q^* - 1)(2P_Q - 1) + 1)^{1/2} P_Q ((2P_Q^* - 1)(2P_Q - 1) + 1)^{-1/2}$ (equivalent to P_Q) by a power series, construct a ε-r-projection $q_D^{\varepsilon,r}$ in $\mathcal{K}(L^2(M))$ such that $\mathrm{Ind}\, D = [\kappa(q_D^{\varepsilon,r})] - \left[\begin{pmatrix} 0 & 0 \\ 0 & Id \end{pmatrix}\right]$. We can define in this way a controlled index $\mathrm{Ind}^{\varepsilon,r} D = [Q_D^{\varepsilon,r}, 1]_{\varepsilon,r}$ in $K_0^{\varepsilon,r}(\mathcal{K}(L^2(M)))$ such that $\mathrm{Ind}\, D = \iota_0^{\varepsilon,r}(\mathrm{Ind}^{\varepsilon,r} D)$ in $K_0(\mathcal{K}(L^2(X))) \cong \mathbb{Z}$. More generally, we have

Lemma 1. *Let X be a compact metric space. For any $0 < \varepsilon < 1/4$ and any $r > 0$, there exists a controlled index map $\mathrm{Ind}_X^{\varepsilon,r} : K_0(X) \to K_0^{\varepsilon,r}(\mathcal{K}(L^2(X)))$ such that*

(1) $\iota_0^{\varepsilon,\varepsilon',r,r'} \circ \mathrm{Ind}_X^{\varepsilon,r} = \mathrm{Ind}_X^{\varepsilon',r'}$*;*

(2) *the composition $K_0(X) \to K_0^{\varepsilon,r}(\mathcal{K}(L^2(X))) \stackrel{\iota_0^{\varepsilon,r}}{\to} K_0(\mathcal{K}(L^2(X))) \cong \mathbb{Z}$ is the index map.*

For small propagation, this controlled index map turns out to be an isomorphism.

Theorem 1. *Let X be a finite simplicial complex equipped with a metric. Then there exists $0 < \varepsilon_0 < 1/4$ such that the following holds :*

For every $0 < \varepsilon < \varepsilon_0$, there exists $r_0 > 0$ such that for any $0 < r < r_0$ then $\mathrm{Ind}_X^{\varepsilon,r} : K_0(X) \to K_0^{\varepsilon,r}(\mathcal{K}(L^2(X)))$ is an isomorphism.

We focus now on large propagation properties for reduced cross product C^*-algebras.

Theorem 2. *Let* Γ *be a finitely generated discrete group. Assume that*

- Γ *satisfies the Baum-Connes conjecture with coefficients;*
- Γ *admits a cocompact universal example for proper actions.*

Then for some universal $\lambda > 1$, *any* $\varepsilon \in (0, \frac{1}{4\lambda})$ *and any* $r > 0$, *there exists* $R > r$ *such that for any action of* Γ *on a* C^**-algebra* A *by automorphisms, the following holds:*

Let x *and* x' *be elements in* $K_*^{\varepsilon,r}(A \rtimes_{red} \Gamma)$ *such that* $\iota_*^{\varepsilon,r}(x) = \iota_*^{\varepsilon,r}(x')$ *in* $K_*(A \rtimes_{red} \Gamma)$ *then* $\iota_*^{\varepsilon,\lambda\varepsilon,r,R}(x) = \iota_*^{\varepsilon,\lambda\varepsilon,r,R}(x')$ *in* $K_*^{\lambda\varepsilon,R}(A \rtimes_{red} \Gamma)$.

By a recent result of V. Lafforgue, Gromov hyperbolic groups satisfy the assumptions of the theorem . If we choose $A = C_0(\Gamma)$, equipped with the action of Γ by left translations, we can identify $C_0(\Gamma) \rtimes_r \Gamma$ and $\mathcal{K}(\ell^2(\Gamma))$ as filtered C^*-algebras. We get under the assumptions of theorem 2 the linear algebra statement:

Corollary 1. *For some universal* $\lambda > 1$, *any* $\varepsilon \in (0, \frac{1}{4\lambda})$ *and any* $r > 0$, *there exists* $R > r$ *such that the following hold:*

Let $[q, l]_{\varepsilon,r}$ *and* $[q', l']_{\varepsilon,r}$ *be elements in* $K_0^{\varepsilon,r}(\mathcal{K}(\ell^2(\Gamma)))$ *such that* $\operatorname{rank} \kappa(q) + l' = \operatorname{rank} \kappa(q') + l$ *then* $[q, l]_{\lambda\varepsilon,R} = [q', l']_{\lambda\varepsilon,R}$ *in* $K_0^{\lambda\varepsilon,R}(\mathcal{K}(\ell^2(\Gamma)))$.

The so-called Atiyah conjecture on rationality of L^2-Betti numbers

Thomas Schick
(joint work with Mikaël Pichot, Andrzej Zuk)

In the seventies, Atiyah defined L^2-Betti numbers of a compact manifold M in terms of harmonic L^2-forms on the universal covering. A priori, these could be arbitrary non-negative real numbers. However, their alternating sum is the Euler characteristic of M. This led Atiyah to the question about the possible values these L^2-Betti numbers can assume, in particular whether they always have to be rational. Various conjectures in this direction have been popularized as the "strong Atiyah conjecture". These conjectures predict in particular that the numbers are rational.

The L^2-Betti numbers can (by Dodziuk's L^2-Hodge de Rham theorem) also be computed from the cellular chain complex of the universal covering, which is a chain complex of free $\mathbb{Z}[\Gamma]$-modules —this way, their definition extends to finite CW-complexes. It turns out that the strong Atiyah conjecture is equivalent to the following purely algebraic statement about elements of the integral group ring $\mathbb{Z}[\Gamma]$.

Let A be a $d \times d$-matrix over $\mathbb{Z}\Gamma$. It acts by left convolution multiplication as bounded operator on the Hilbert space $l^2(\Gamma)^n$. Let p_A be the orthogonal projection onto the null space of A. Let $\delta_e \in l^2(\Gamma)$ be the characteristic function of the identity element and $\delta_e^i \in l^2(\Gamma)^n$ the vector with entry δ_e at position i and 0 at all other positions. Then $b^{(2)}(A) := \sum_{i=1}^{d} \langle p_A \delta_e^i, \delta_e^i \rangle_{l^2(\Gamma)^n}$. This is the normalized von Neumann trace of p_A. The set of possible values of L^2-Betti numbers of manifolds with fundamental group Γ coincides with the set of possible values of $b^{(2)}(A)$, where A varies over matrices over $\mathbb{Z}\Gamma$. Note that Γ must be finitely presented to be the fundamental group of a compact manifold.

The question now is to find groups Γ and elements $A \in \mathbb{Z}[\Gamma]$ where $\ker(A)$ and $b^{(2)}(A)$ are explicitly calculable and have transcendental values.

First calculations in this direction for the random walk operator on the lamplighter group have been carried out in [1], where a complete eigenspace decomposition is derived. This has been taken up for free lamplighter groups, i.e. the restricted wreath product of $\mathbb{Z}/2\mathbb{Z}$ by non-abelian free groups. Recently an explicit irrational (algebraic over $\mathbb{Q}$) L^2-Betti number for a random walk operator on a free lamplighter group has been computed by Lehner and Wagner [2]. The main point here is to get hold on the combinatorial difficulties of understanding all finite connected subgraphs in the Cayley graph of a free group (a regular tree) and the kernel of the graph Laplacian on these.

In a different direction and slightly earlier, Austin [7] uses suitable quotients of the free lamplighter group and taylor-made (very much generalized) relatives of the random walk operator to produce an uncountable set of L^2-Betti numbers, so that transcendental ones have to exist. The drawback, however, is that this method is not explicit and does not give finitely presented groups as examples.

The talk cumulated in a report on [8]. The main thread there is the refinement of the work of Austin in such a way as to arrive at explicit calculations. For this, a different class of operators is used.

We arrive at the following results:

- there are explicit finitely presented groups and elements in their group ring with transcendental L^2-Betti numbers; therefore also closed manifolds with such L^2-Betti numbers.
- every algebraic number is an L^2-Betti number of a closed manifold, moreover every real number which admits a Turing machine producing its decimal expansion (in correct order), e.g. π.
- purely algebraically, every element of $\mathbb{R}_{\geq 0}$ is a $b^{(2)}(A)$ for a suitable matrix over A the integral group ring of a suitable discrete group, but here, one has to allow for groups which are not finitely presented.

Very similar results have been obtained independently, using a way to implement Turing machines into groups and elements of their group ring, by Lukasz Grabowski in [4].

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Relative Cyclic Cohomology and Geometric Invariants

Markus J. Pflaum
(joint work with Henri Moscovici, Matthias Lesch)

Pairings in relative cyclic (co)homology allow for a conceptually clear construction of geometric invariants such as the divisor flow originally defined by Melrose [11] and its higher dimensional versions [9] or the η-cochains which appear in the study of the Atiyah-Patodi-Singer index theorem within the framework of noncommutative geometry (cf. [5, 14]). In a series of papers [6, 7, 8] we explained this philosophy, applied it to the above mentioned examples and set up the foundations for further applications (see also [12, 13, 15, 16] for related work).

The fundamental ingredient in our approach to pairings in relative cyclic cohomology is the following result which essentially is based on the cone construction in homological algebra.

Proposition 1 ([7]). *Assume to be given a short exact sequences of algebras*

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{A} \xrightarrow{\sigma} \mathcal{B} \longrightarrow 0. \tag{1}$$

Then the relative cyclic cohomology $HC^\bullet(\mathcal{A},\mathcal{B})$ coincides with the cohomology of the total complex

$$\big(\operatorname{Tot}^\bullet_\oplus \mathcal{B}C^{\bullet,\bullet}(\mathcal{A}) \oplus \operatorname{Tot}^{\bullet+1}_\oplus \mathcal{B}C^{\bullet,\bullet}(\mathcal{B}), \widetilde{b+B}\big),$$

where the differential is given by

$$\widetilde{b+B} = \begin{pmatrix} b+B & -\sigma^* \\ 0 & -(b+B) \end{pmatrix}.$$

Moreover, each class in $HC^k(\mathcal{A},\mathcal{B})$ has a representative $(\varphi,\psi) \in C^k_\lambda(\mathcal{A}) \oplus C^{k+1}_\lambda(\mathcal{B})$ with $b\varphi = \sigma^*\psi$, where $C^\bullet_\lambda$ stands for the subcomplex of cyclic cochains [2, 3].

Likewise, the relative cyclic homology $HC_\bullet(\mathcal{A},\mathcal{B})$ is the homology of the complex $\big(\operatorname{Tot}^{\oplus}_\bullet \mathcal{B}C_{\bullet,\bullet}(\mathcal{A},\mathcal{B}), \widetilde{b}+\widetilde{B}\big)$, *where* $\mathcal{B}C_{p,q}(\mathcal{A},\mathcal{B}) = \mathcal{B}C_{p,q}(\mathcal{A}) \oplus \mathcal{B}C_{p,q+1}(\mathcal{B})$,

$$\widetilde{b} = \begin{pmatrix} b & 0 \\ -\sigma_* & -b \end{pmatrix}, \quad \text{and } \widetilde{B} = \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}.$$

Finally, the natural pairing between cyclic chains and cochains gives rise to a pairing in the relative case,

$$\begin{aligned} \langle -,- \rangle : C^\bullet_\lambda(\mathcal{A},\mathcal{B}) \times C^\lambda_\bullet(\mathcal{A},\mathcal{B}) \to \mathbb{C}, \\ ((\varphi_k,\psi_{k+1}),(a_k,b_{k+1})) \mapsto \langle \varphi_k, a_k\rangle + \langle \psi_{k+1}, b_{k+1}\rangle \end{aligned} \tag{2}$$

which ultimately induces a pairing $HC^k(\mathcal{A},\mathcal{B}) \times HC_k(\mathcal{A},\mathcal{B}) \to \mathbb{C}$.

An analogous result holds true for pairings in relative periodic cyclic (co)homology.

For short exact sequences of algebras as in (1), both the homological and cohomological Connes–Chern characters [3] have natural extensions to the relative setting:

$$\mathrm{ch}_\bullet : K_{\mathrm{ev/odd}}(\mathcal{A},\mathcal{B}) \to HC^{\mathrm{per}}_{\mathrm{ev/odd}}(\mathcal{A},\mathcal{B}),\ \mathrm{ch}^\bullet : K^{\mathrm{ev/odd}}(\mathcal{A},\mathcal{B}) \to HC_{\mathrm{per}}^{\mathrm{ev/odd}}(\mathcal{A},\mathcal{B}).$$

By excision in periodic cyclic (co)homology, one knows that the relative Connes–Chern character for the pair $(\mathcal{A},\mathcal{B})$ has to coincide with the Connes–Chern character of the ideal $\mathcal{J}$. One of the goals of the papers [6, 7, 8] is to provide a representation of the homological and cohomological relative Connes–Chern characters not only in terms of the ideal $\mathcal{J}$ but in terms of $\mathcal{A}$ plus correction terms coming from $\mathcal{B}$, and then study the corresponding pairings in relative cyclic cohomology. This could be dubbed the "relative philosophy" within noncommutative geometry.

Pairings in relative cyclic (co)homology give new insight to certain geometric, possibly secondary, invariants. For example, in [7], we succeeded to reinterpret Melrose's divisor flow and its higher variants as relative pairings of the relative Chern character of a path of elliptic elements in the suspended algebra of pseudodifferential operators over a closed manifold with some odd relative cyclic cocycle coming from a regularized trace on the suspended algebra of pseudodifferential operators. By the interpretation of the division flow as a relative pairing one immediately derives its fundamental properties such as integrality, additivity and homotopy invariance.

In [8] the "relative philosophy" in noncommutative geometry has been illustrated in detail by the example of a b-Dirac operator D on a compact manifold with boundary M of dimension m within the framework of the b-calculus [10]. Let us briefly sketch the fundamental idea. By the work of Baum–Douglas–Taylor [1], the Dirac operator D defines a Fredholm module over the pair $\big(\mathcal{C}^\infty(M), \mathcal{C}^\infty(\partial M)\big)$, hence an element $[\mathsf{D}] \in K_m(M,\partial M)$.

Under certain assumptions on the spectrum of D, Getzler [5] constructed the Connes–Chern character of $[\mathsf{D}]$ with values in entire cyclic cohomology. It can be understood as a b-version of the JLO-cocycle:

$$ {}^{\mathrm{b}}\mathrm{Ch}^k(\mathsf{D})(a_0,\ldots,a_k) := {}^{\mathrm{b}}\langle a_0,[\mathsf{D},a_1],\ldots,[\mathsf{D},a_k]\rangle_{\mathsf{D}}, \quad \text{where} $$

$$ {}^{\mathrm{b}}\langle A_0,A_1,\ldots,A_k\rangle_{\mathsf{D}} := \int_{\Delta_k} {}^{\mathrm{b}}\operatorname{Str}\big(A_0 e^{-\sigma_0\mathsf{D}^2}A_1\cdot\ldots\cdot A_k e^{-\sigma_k\mathsf{D}^2}\big)\,d\sigma. $$

By extending a retraction procedure of Connes–Moscovici [4] to the relative setting, we succeeded to convert the entire Connes–Chern character into the periodic one within the relative setting. More precisely, we put for $t > 0$

$$ {}^{\mathrm{b}}\mathrm{ch}^k_t(\mathsf{D}) := \sum_{j\geq 0}\mathrm{Ch}^{k-2j}(t\mathsf{D}) + BT\,\not{c}\mathrm{h}^{k+1}_t(\mathsf{D}), $$

$$ \mathrm{ch}^k_t(\mathsf{D}^\partial) := \sum_{j\geq 0}\mathrm{Ch}^{k-2j+1}(t\mathsf{D}^\partial) + BT\,\not{c}\mathrm{h}^{k+2}_t(\mathsf{D}^\partial), $$

where D^∂ is the Dirac operator induced on the boundary, $\not{c}\mathrm{h}^\bullet$ the transgressed Connes–Chern character, and $T\,\not{c}\mathrm{h}^{k+1}_t(\mathsf{D}) := \int_0^t {}^{\mathrm{b}}\not{c}\mathrm{h}^k(s\mathsf{D},\mathsf{D})ds$.

Theorem 2 ([8])**.** *Under the assumptions as above and the assumption that m is even, the pair of cochains $\big({}^{\mathrm{b}}\mathrm{ch}^k_t(\mathsf{D}),\mathrm{ch}^{k+1}_t(\mathsf{D}^\partial)\big)$ with $t>0$, $k>\dim M+4$, and $k \equiv \dim M \mod 2$ is a cocycle in the relative total complex $\mathrm{Tot}^\bullet_\oplus \mathcal{B}C^{\bullet,\bullet}\big(\mathcal{C}^\infty(M),\mathcal{C}^\infty(\partial M)\big)$ of the pair $\big(\mathcal{C}^\infty(M),\mathcal{C}^\infty(\partial M)\big)$. Its periodic cyclic cohomology class is independant of $k >> \dim M$ and of $t>0$ and represents the Connes–Chern character of the class $[\mathsf{D}]\in K_0(M,\partial M)$. Moreover, the limit $t\searrow 0$ exists, is local, and also represents the Connes–Chern character of $[\mathsf{D}]$:*

$$ \lim_{t\searrow\infty}\big({}^{\mathrm{b}}\mathrm{ch}^k_t(\mathsf{D}),\mathrm{ch}^{k+1}_t(\mathsf{D}^\partial)\big) = \Big(\int_{{}^{\mathrm{b}}M}{}^{\mathrm{b}}\omega_{\mathsf{D}}\wedge -, \int_{\partial M}\omega_{\partial\mathsf{D}}\wedge -\Big) $$

If D^∂ is invertible, the limit $\lim_{t\to\infty}\big({}^{\mathrm{b}}\mathrm{ch}^k_t(\mathsf{D}),\mathrm{ch}^{k+1}_t(\mathsf{D}^\partial)\big)$ is computable, involves the relative APS-index and represents the Connes–Chern character of $[\mathsf{D}]$ as well.

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On the Radul cocycle

Denis Perrot

Let M be a closed, not necessarily orientable, smooth manifold and denote by $\mathrm{CL}(M)$ the algebra of classical, one-step polyhomogeneous pseudodifferential operators on M. The space of smoothing operators $\mathrm{CL}^{-\infty}(M)$ is a two-sided ideal in $\mathrm{CL}(M)$, and we call the quotient $\mathrm{CS}(M) = \mathrm{CL}(M)/\mathrm{CL}^{-\infty}(M)$ the algebra of *formal symbols* on M. The cyclic homology of $\mathrm{CS}(M)$ has been studied by Wodzicki [6]. In this talk we are interested in a particular cyclic cocycle, originally introduced by Radul in the context of Lie algebra cohomology [4]: it is the bilinear functional defined by

$$c(a_0, a_1) = \fint a_0[\log q, a_1]$$

for any two formal symbols $a_0, a_1 \in \mathrm{CS}(M)$. The above integral denotes the Wodzicki residue [5], which is a trace on $\mathrm{CS}(M)$, and q is a fixed positive elliptic symbol of order one. One can show that c is a cyclic 1-cocycle over $\mathrm{CS}(M)$, and that its cyclic cohomology class does not depend on the choice of q. Hence the class $[c] \in HC^1(\mathrm{CS}(M))$ is completely canonical. Therefore a natural question is to identify this class. We give the answer for the image of $[c]$ in the periodic cyclic cohomology of the subalgebra $\mathrm{CS}^0(M) \subset \mathrm{CS}(M)$, the formal symbols of order ≤ 0. This goes as follows. First, the leading symbol map gives rise to an algebra homomorphism $\lambda : \mathrm{CS}^0(M) \to C^\infty(S^*M)$, where S^*M is the cosphere bundle of M. This allows to pullback any homology class of S^*M (with complex coefficients) to the periodic cyclic cohomology of the symbol algebra:

$$\lambda^* : H_\bullet(S^*M, \mathbb{C}) \to HP^\bullet(\mathrm{CS}^0(M)) .$$

In fact, Wodzicki shows that λ^* is an *isomorphism*, provided that the natural locally convex topology of $\mathrm{CS}^0(M)$ is taken into account [6]. Our main result is the following theorem, which holds in the algebraic setting or the locally convex setting regardless to Wodzicki's isomorphism.

Theorem 1: *Let M be a closed manifold. The periodic cyclic cohomology class of the Radul cocycle $[c] \in HP^1(\mathrm{CS}^0(M))$ is*

$$[c] = \lambda^*\big([S^*M] \cap \pi^*\mathrm{Td}(T_{\mathbb{C}}M)\big) \ , \tag{1}$$

where $\mathrm{Td}(T_{\mathbb{C}}M) \in H^\bullet(M, \mathbb{C})$ is the Todd class of the complexified tangent bundle, and $\pi : S^*M \to M$ is the cosphere bundle endowed with its canonical orientation and fundamental class $[S^*M] \in H_\bullet(S^*M, \mathbb{C})$.

This is a statement of pure algebraic topology, hence we give a purely algebraic proof. The central idea is to consider the algebra of formal symbols $\mathrm{CS}(M)$ as a bimodule over itself. We develop a formalism of generalized Dirac operators and graded traces within the algebra $\mathrm{CS}(M) \otimes \mathrm{CS}(M)^{\mathrm{op}}$, and use it to construct cyclic cocycles over $\mathrm{CS}^0(M)$. The latter are given by algebraic analogues of JLO formulas [2]. By choosing genuine Dirac operators we obtain equality (1). Note that our algebraic JLO formula actually provides an explicit representative of the Todd class as a closed differential form over M involving the curvature of an affine torsion-free connexion.

As an immediate corollary we obtain the Atiyah-Singer index formula for elliptic pseudodifferential operators [1]. It calculates the index of such an operator in terms of its leading symbol. The latter is a (matrix-valued) invertible function g over the cosphere bundle, thus defines a class in the algebraic K-theory $K_1(C^\infty(S^*M))$.

Corollary (Index theorem): *Let Q be an elliptic classical pseudodifferential operator of order ≤ 0 on a closed manifold M, with leading symbol class $[g] \in K_1(C^\infty(S^*M))$. The index of Q is the integer*

$$\mathrm{Ind}(Q) = \langle [S^*M], \pi^*\mathrm{Td}(T_{\mathbb{C}}M) \cup \mathrm{ch}([g]) \rangle \tag{2}$$

where $\mathrm{ch}([g]) \in H^\bullet(S^*M, \mathbb{C})$ is the Chern character of the K-theory class $[g]$.

The corollary is proven using the well-known fact (see for example [3]) that the Radul cocycle is the image of the operator trace under the excision map of the fundamental extension

$$0 \to \mathrm{CL}^{-\infty}(M) \to \mathrm{CL}^0(M) \to C^\infty(S^*M) \to 0 \ .$$

In fact, at least in the locally convex setting, (1) and the index formula are equivalent. Hence our approach gives a new, purely algebraic approach of the index theorem. The interesting fact is that our formalism can be applied to more general geometric situations, including smooth groupoids. As a first step in this direction, using the same methods, we state the following equivariant index theorem on the circle.

Theorem 2: *Let $M = S^1$ be the circle, and G be a discrete group acting on M by diffeomorphisms. The periodic cyclic cohomology class of the equivariant Radul cocycle $[c] \in HP^1(\mathrm{CS}^0(M) \rtimes G)$ is*

$$[c] = \lambda^*\big(\Phi(\mathrm{Td}(T_{\mathbb{C}}M))\big) \ , \tag{3}$$

where $\Phi : H^\bullet(S^*M \times_G EG) \to HP^\bullet(C^\infty(S^*M) \rtimes G)$ *is Connes' map from equivariant cohomology to periodic cyclic cohomology.*
This provides a non-trivial result since the first Chern class of the equivariant tangent bundle already appears in dimension one.

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Computing the modular curvature of the noncommutative two torus

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(joint work with Farzad Fathizadeh)

In this work we give a local expression for the *modular curvature* of the noncommutative two torus $A_\theta = C(\mathbb{T}_\theta^2)$ equipped with an arbitrary translation invariant complex structure and Weyl factor. More precisely, for any complex number τ in the upper half plane, representing the conformal class of a metric on $\mathbb{T}_\theta^2$, and a Weyl factor given by a positive invertible element $k \in C^\infty(\mathbb{T}_\theta^2)$, we give an explicit formula for an element $R = R(\tau, k) \in C^\infty(\mathbb{T}_\theta^2)$ that is the scalar curvature of the underlying noncommutative Riemannian manifold $\mathbb{T}_\theta^2$. This is achieved by evaluating the value of the (analytic continuation of the) *spectral zeta functional* $\zeta_a(s) := \text{Trace}(a\triangle^{-s})$ at $s = 0$ as a linear functional in $a \in C^\infty(\mathbb{T}_\theta^2)$. A new, purely noncommutative, feature here is the appearance of the *modular automorphism group* from the theory of type III factors and quantum statistical mechanics in the final formula for curvature. This result will ppear in our forthcoming paper [17]. This formula exactly reproduces the formula that was recently obtained independently by Connes and Moscovici in their forthcoming paper [14]. It also reduces, for $\tau = \sqrt{-1}$, to a formula that was earlier obtained by Alain Connes for the scalar curvature of the noncommutative two torus.

Our main result extends and refines the recent work on *Gauss-Bonnet theorem* for the noncommutative two torus that was initiated in the pioneering work of Connes and Tretkoff in [15] (cf. also [5, 4] for a preliminary version) and its later generalization in our paper [16]. In fact after applying the standard trace of the noncommutative torus to the scalar curvature R one obtains, for all values of τ and k, the value 0. This is the Gauss-Bonnet theorem for the noncommutative two torus and, in the commutative case, is equivalent to the classical Gauss-Bonnet theorem for a surface of genus 1.

The backbone of the present work is *Connes' noncommutative differential geometry program* [6, 7, 9, 11]. According to parts of this theory that is relevant here the metric information on a noncommutative space is fully encoded as a *spectral triple* on the noncommutative algebra of coordinates on that space. Various technical results corroborates, in fact fully justifies, this vision. First of all, *Connes' reconstruction theorem* [10] guarantees that in the commutative case, the notion of spectral triple is strong enough to fully recover the Riemannian (spin) manifold from its natural spectral triple data defined using the Dirac operator acting on spinors. Secondly, as it is shown by Connes [8, 9, 11], ideas of spectral geometry, in particular formulation of several invariants of a Riemannian manifold like, volume and scalar curvature in terms of asymptotics of the trace of the heat kernel of Laplacians and Dirac operators, have very natural extensions in the noncommutative setting and recover the classical results in the commutative case. Other relevant results are the Connes-Moscovici local index formula [12] and Chamseddine-Connes spectral action principle [2]. In passing to the noncommutative case, sooner or later one must face the prospect of type III algebras and the lack of trace on them. It was exactly for this reason that *twisted spectral triples* were introduced by Connes and Moscovici in [13]. The spectral triple at the foundation of the present paper was defined in [15] and is in fact, via the right action corresponding to the Tomita anti-linear unitary map, a twisted spectral triple.

One of the main technical tools employed in this work [17] is Connes' pseudodifferential operators and their symbol calculus on the noncommutative torus [6] and the use of the asymptotic expansion of the heat kernel in computing zeta values. This, however, by itself is not enough and, similar to [5, 15, 16], one needs an extra and intricate argument to express $\zeta_a(0)$ in terms of the modular operator defined by the Weyl factor. As a first step, the calculation of the asymptotic expansion of the heat operator for arbitrary values of the conformal class is quite involved and must be performed by a computer. We found it impossible to carry this step without the use of symbolic calculations. Finally we should mention that, as is explained in [15, 16], there is a close relationship between the subject of this paper and scale invariance in spectral action [2, 3] on the one hand, and non-unimodular (or twisted) spectral triples [13] on the other hand.

The following formula in our paper [17] gives the local expression for the modular curvature of the noncommuative two torus. It was also independently obtained by Alain Connes and Henri Moscovici [14].

Theorem 1. *Let θ be an irrational number, τ a complex number in the upper half plane representing the conformal class of a metric on T_θ^2, and k an invertible positive element in A_θ^∞ playing the role of the Weyl factor. Then the scalar curvature R of the perturbed spectral triple attached to (T_θ^2, τ, k), up to an overall factor of*

$-\frac{\pi}{\tau_2}$, *is equal to*

$$\begin{aligned} & R_1(\log\Delta)\big(\delta_1^2(\log k)+|\tau|^2\delta_2^2(\log k)+2\tau_1\delta_1\delta_2(\log k)\big) \\ + \quad & R_2(\log\Delta_{(1)},\log\Delta_{(2)})\Big(\delta_1(\log k)\delta_1(\log k)+|\tau|^2\delta_2(\log k)\delta_2(\log k)+ \\ & \qquad \tau_1\big(\delta_1(\log k)\delta_2(\log k)+\delta_2(\log k)\delta_1(\log k)\big)\Big) \\ - \quad & iW(\log\Delta_{(1)},\log\Delta_{(2)})\Big(\tau_2\big(\delta_1(\log k)\delta_2(\log k)-\delta_2(\log k)\delta_1(\log k)\big)\Big), \end{aligned}$$

where

$$R_1(x) := K(x)+S(x) = -\frac{2\coth(x/4)}{x}+\frac{1}{2\sinh^2(x/4)} = \frac{\frac{1}{2}-\frac{\sinh(x/2)}{x}}{\sinh^2(x/4)},$$

$$R_2(s,t) := H(s,t)+T(s,t) = -(1+\cosh((s+t)/2))\times$$
$$\frac{-t(s+t)\cosh s+s(s+t)\cosh t-(s-t)(s+t+\sinh s+\sinh t-\sinh(s+t))}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)},$$

and

$$W(s,t) = \frac{-s-t+t\cosh s+s\cosh t+\sinh s+\sinh t-\sinh(s+t)}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}.$$

Remark 2. *We note that the above local expression R for the scalar curvature of $(\mathbb{T}_\theta^2,\tau,k)$, reduce to the scalar curvature of the ordinary two torus when $\theta=0$. Namely, since*

$$\lim_{x\to 0} R_1(x) = -\frac{1}{3},$$
$$\lim_{x\to 0} R_1^\gamma(x) = 1,$$
$$\lim_{s,t\to 0} R_2(s,t) = \lim_{s,t\to 0} R_2^\gamma(s,t) = 0,$$

and

$$\lim_{s,t\to 0} W(s,t) = -\frac{2}{3},$$

in the commutative case, the expression for R stated in the above theorem, reduces to constant multiples of

$$\frac{1}{\tau_2}\delta_1^2(\log k)+\frac{|\tau|^2}{\tau_2}\delta_2^2(\log k)+2\frac{\tau_1}{\tau_2}\delta_1\delta_2(\log k).$$

It is a great pleasure to thank and to express our indebtness to Alain Connes for motivating and enlightening discussions and for much help during the various stages of the work on this paper. At several crucial stages he generously shared his insight and ideas with us and communicated their relevant joint results in [14] with us. This gave us a good chance of finding potential errors in the computations. In fact the idea of using the *full Laplacian*, on functions and 1-forms, as opposed to just functions, was suggested to us by him. While in the commutative case one can recover the curvature from zeta functionals from the Laplacian on functions, this is no more the case in the noncommutative case. We would also like to heartily thank Henri Moscovici for a push in the right direction at an early stage. After the

appearance of our Gauss-Bonnet paper [16], he and Alain Connes kindly pointed out to us that the calculations in that paper might be quite relevant for computing the scalar curvature of the noncommutative two torus.

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Verlinde modules and quantization

Varghese Mathai

This work is partly inspired by a theorem of Freed, Hopkins and Teleman [1, 2], which identifies the twisted G-equivariant K-homology group $\mathrm{K}^G(G, \eta)$ of a compact Lie group G, with the Verlinde algebra $R_\ell(G)$ of G at a level ℓ determined by the twist η. Firstly, they show that $\mathrm{K}^G(G, \eta)$ is an algebra, with product induced by the multiplication $m : G \times G \longrightarrow G$ on the group G, and that their isomorphism "explains" the combinatorially complicated fusion product in the Verlinde algebra.

Another consequence of their theorem is that the Verlinde algebra $R_\ell(G)$ has the same functorial properties as $\mathrm{K}^G(G,\eta)$. Recall that the importance of the Verlinde algebra is that it encodes the selection rules for the operator product expansion in certain rational conformal field theories such as the WZW-model. That is, they encode the dimensions of spaces of conformal blocks of these rational conformal field theories, i.e. dimensions of certain spaces of generalized theta functions (cf. [4]). These dimensions and their polynomial behaviour are of fundamental importance in conformal field theory.

Another source of inspiration is the recent work by Meinrenken [3] on the relation of quasi-Hamiltonian manifolds to the work of Freed, Hopkins and Teleman. To every compact quasi-Hamiltonian manifold (M,ω,Φ) with group-valued moment map $\Phi : M \to G$ (which satisfies $\Phi^*(\eta) = d\omega$), Meinrenken defines the quantization $Q(M)$ to be the element of the Verlinde algebra $\Phi_*([M]) \in \mathrm{K}^G(G,\eta) \cong R_\ell(G)$, where $[M]$ denotes the equivariant fundamental class of the compact G-manifold M, which is an element in $\mathrm{K}^G(M, \mathrm{Cliff}(TM))$, since he shows that M has an equivariant twisted Spinc structure, (explained later in the section). He then establishes several very interesting properties of his quantization procedure, as well as calculations of it.

Here we outline the following, full details to be given elsewhere. Given a compact simple Lie group G and a primitive degree 3 twist η, we define a braided, balanced, strict monoidal category $\mathcal{C}(G,\eta)$ with a May structure given by disjoint union and fusion product. An object in the category $\mathcal{C}(G,\eta)$ is a pair (X,f), where X is a compact G-manifold and $f : X \to G$ a smooth G-map with respect to the conjugation action of G on itself. Such an object determines a module, the twisted equivariant K-homology group $\mathrm{K}^G(X, f^*(\eta))$, for the Verlinde algebra, termed a Verlinde module, where the module action is induced by the G-action on X. In order to understand which objects in $\mathcal{C}(G,\eta)$ can be quantized, we define the closely related monoidal category $\mathcal{D}(G,\eta)$ consisting of equivariant twisted geometric K-cycles, which also has a May structure given by disjoint union and fusion product. There is a forgetful functor $\mathcal{D}(G,\eta) \to \mathcal{C}(G,\eta)$, showing that an object in $\mathcal{D}(G,\eta)$ determines a Verlinde module. Every object in the category $\mathcal{D}(G,\eta)$ also has a quantization, valued in the Verlinde algebra. Finally, the quantization functor induces an isomorphism between the geometric equivariant twisted K-homology ring $\mathrm{K}^G_{geo}(G,\eta)$ and the Verlinde algebra.

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Hopf cyclic cohomology, Weil algebra, and characteristic classes

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(joint work with Serkan Sütlü)

We report on the recent developments in Hopf cyclic cohomology. Admitting coefficients is one of the most significant properties of this theory [1, 3, 2, 4]. The one dimensional coefficients are called modular pairs in involution. An algebra map $\delta : H \to \mathbb{C}$ is called a character. An element $\sigma \in H$ is called a group-like if $\Delta(\sigma) = \sigma \otimes \sigma$. The pair (δ, σ) is called modular pair in involution (MPI) if $\delta(\sigma) = 1$ and $\widetilde{S}_\delta^2(h) = \sigma h \sigma^{-1}$. Here $\widetilde{S}_\delta(h) = \sum \delta(h_{(1)}) S(h_{(2)})$ and we have used the Sweedler notation $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$.

In [1], Connes-Moscovici prove:

(1) For any $n \geq 1$ and any oriented flat manifold M^n, there is a canonical Hopf algebra $\mathcal{H}_n$ acting on the algebra $\mathcal{A}_n := C_c^\infty(FM) \rtimes \mathrm{Diff}(M)$.
(2) There is a cyclic cohomology theory for Hopf algebras. The cyclic cohomology of $\mathcal{H}_n$ is canonically isomorphic to the Gelfand Fuks cohomology of the Lie algebra of formal vector fields on $\mathbb{R}^n$.
(3) There is a characteristic map from the mentioned cyclic cohomology of $\mathcal{H}_n$ to the cyclic cohomology of the algebra $\mathcal{A}_n$ such that the index cocycle is trapped in its image.

The general coefficients for Hopf cyclic cohomology are called stable-anti-Yetter-Drinfeld (SAYD) modules [3]. An SAYD module is a right $\mathcal{H}$-module and a left $\mathcal{H}$-comodule M such that

$$\blacktriangledown(m \cdot h) = S(h_{(3)}) m_{<-1>} h_{(1)} \otimes m_{<0>} \cdot h_{(2)}, \qquad m_{<0>} \cdot m_{<-1>} = m.$$

To the datum $(\mathcal{H}, M)$ one associates a cocyclic module and hence defines its Hopf cyclic cohomology which is denoted by $HC^\bullet(\mathcal{H}, M)$ [3].

A bicrossed product Hopf algebra $\mathcal{F} \blacktriangleright\!\!\lhd \mathcal{U}$ is made of a pair of Hopf algebras $(\mathcal{U}, \mathcal{F})$ satisfying certain conditions which guarantees that $\mathcal{F} \otimes \mathcal{U}$ is a Hopf algebra via $\mathcal{F} \rtimes \mathcal{U}$ as an algebra and $\mathcal{F} \blacktriangleright\!\!< \mathcal{U}$ as a coalgebra [5]. As a working example we focus on the bicrossed product Hopf algebra associated to a matched pair of Lie algebras $(\mathfrak{g}_1, \mathfrak{g}_2)$. Here the commutative Hopf algebra $\mathcal{F} := R(\mathfrak{g}_1)$ is the Hopf algebra of representative functions on the enveloping algebra $U(\mathfrak{g}_1)$, and the cocommutative Hopf algebra $\mathcal{U}$ is $U(\mathfrak{g}_1)$.

There is a canonical MPI for such bicrossed product Hopf algebras [9]. Any representation of the total Lie algebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ induces a YD module over the associated bicrossed product Hopf algebra [9]. The tensor product of an MPI and a YD module produces an AYD module [3]. In our case, the resulting module is also stable. These SAYD modules are called induced modules [9]. The Hopf cyclic cohomology of $\mathcal{F} \blacktriangleright\!\!\lhd \mathcal{U}$ is isomorphic to the Lie algebra cohomology of the Lie algebra $\mathfrak{g}_1 \bowtie \mathfrak{g}_2$ with coefficients in the original representation.

SAYD modules over Lie algebras were defined and studied in [10]. It was observed that the corresponding cyclic complex is known by different names in literature.

It is proved that the (truncated) polynomial algebra of a Lie algebra is a SAYD module and the corresponding cyclic complex is identified with the (truncated) Weil algebra [10]. The category of SAYD modules over the enveloping algebra of a Lie algebra is identified with those on the Lie algebra. It is shown that such comodules over $U(\mathfrak{g})$ are in one to one correspondence with the nilpotent modules over the symmetric algebra $S(\mathfrak{g}^*)$. Using this fact, we identify AYD modules over $U(\mathfrak{g})$ with modules over the semidirect sum Lie algebra $\widetilde{\mathfrak{g}} = \mathfrak{g}^* \rtimes \mathfrak{g}$. Here $\mathfrak{g}^* = \mathrm{Hom}(\mathfrak{g}, \mathbb{C})$ is considered to be a commutative Lie algebra and to be acted upon by $\mathfrak{g}$ via the coadjoint representation. SAYD modules over $U(\mathfrak{g})$ correspond to SAYD modules over $\mathfrak{g}$. Furthermore, their cyclic cohomologies are identified.

As the most general case, we show that SAYD modules over $\mathcal{F} \blacktriangleright\!\!\triangleleft \mathcal{U}$ and SAYD modules over $\mathfrak{g}_1 \bowtie \mathfrak{g}_2$ are the same. We upgrade the van Est isomorphism between Hopf cyclic cohomology and Lie algebra cohomology evolved in [1, 7, 8, 9]. Precisely we prove that the Hopf cyclic cohomology of $\mathcal{F} \blacktriangleright\!\!\triangleleft \mathcal{U}$ with coefficients in ${}^{\sigma}M_{\delta} = M \otimes {}^{\sigma}\mathbb{C}_{\delta}$ and the Lie algebra cohomology of $\mathfrak{g}_1 \bowtie \mathfrak{g}_2$ relative to a Levi subalgebra with coefficients in the $\mathfrak{g}_1 \bowtie \mathfrak{g}_2$-module M [11].

From [7], we know that $\mathcal{H}_n$ is a bicrossed product Hopf algebra $\mathcal{F}(N) \blacktriangleright\!\!\triangleleft \mathcal{U}$ endowed with a canonical MPI [1]. We prove in [11] that this is the only AYD module over $\mathcal{H}_n$.

Following [11] we illustrate our theory with a nontrivial example. We produce a SAYD module over the Schwarzian Hopf algebra $\mathcal{H}_{1\mathrm{S}}$ introduced in [1]. By definition, $\mathcal{H}_{1\mathrm{S}}$ is a quotient Hopf algebra of $\mathcal{H}_1$ by the Hopf ideal generated by $\delta_2 - \frac{1}{2}\delta_1^2$. Here δ_i are generators of $\mathcal{F}(N)$. So the Hopf algebra $\mathcal{H}_{1\mathrm{S}}$ is generated by X, Y, and δ_1.

As we know, $\mathcal{H}_{1\mathrm{S}}^{\mathrm{cop}}$ is isomorphic to $R(\mathbb{C}) \blacktriangleright\!\!\triangleleft U(g\ell_1^{\mathrm{aff}})$ [6]. Our theory guarantees that any SAYD module M over $s\ell_2 = g\ell_1^{\mathrm{aff}} \bowtie \mathbb{C}$ will produce a SAYD module M_δ over $\mathcal{H}_{1\mathrm{S}}^{\mathrm{cop}}$. We take the truncated polynomial algebra $M = S(s\ell_2)_{[2]}$. The resulting 4-dimensional SAYD module M_δ is then generated by $\mathbf{1}$, $\mathbf{R}^X$, $\mathbf{R}^Y$, and $\mathbf{R}^Z$. The action and coaction of $\mathcal{H}_{1\mathrm{S}}^{\mathrm{cop}}$ on M_δ are defined by

$\triangleleft$	X	Y	δ_1
$\mathbf{1}$	0	0	$\mathbf{R}^Z$
$\mathbf{R}^X$	$-\mathbf{R}^Y$	$2\mathbf{R}^X$	0
$\mathbf{R}^Y$	$-\mathbf{R}^Z$	$\mathbf{R}^Y$	0
$\mathbf{R}^Z$	0	0	0

$\blacktriangledown : M_\delta \longrightarrow \mathcal{H}_{1\mathrm{S}}\,\mathrm{cop} \otimes M_\delta$
$\mathbf{1} \mapsto 1 \otimes \mathbf{1} + X \otimes \mathbf{R}^X + Y \otimes \mathbf{R}^Y$
$\mathbf{R}^X \mapsto 1 \otimes \mathbf{R}^X$
$\mathbf{R}^Y \mapsto 1 \otimes \mathbf{R}^Y + \delta_1 \otimes \mathbf{R}^X$
$\mathbf{R}^Z \mapsto 1 \otimes \mathbf{R}^Z + \delta_1 \otimes \mathbf{R}^Y + \frac{1}{2}\delta_1^2 \otimes \mathbf{R}^X.$

The surprises here are the nontriviality of the action of δ_1 and the appearance of X and Y in the coaction. In other words this is not an induced module [9].

We apply the machinery developed in [7] by Moscovici and one of the authors to prove that the following two cocycles generates the Hopf cyclic cohomology of $\mathcal{H}_{1\mathrm{S}}^{\mathrm{cop}}$ with coefficients in M_δ.

$$c^{\text{odd}} = \mathbf{1} \otimes \delta_1 + \mathbf{R}^Y \otimes X + \mathbf{R}^X \otimes \delta_1 X + \mathbf{R}^Y \otimes \delta_1 Y + 2\mathbf{R}^Z \otimes Y,$$

$$\begin{aligned}c^{\text{even}} &= \mathbf{1} \otimes X \otimes Y - \mathbf{1} \otimes Y \otimes X + \mathbf{1} \otimes Y \otimes \delta_1 Y - \mathbf{R}^X XY \otimes X \\ &- \mathbf{R}^X \otimes Y^2 \otimes \delta_1 X - \mathbf{R}^X \otimes Y \otimes X^2 + \mathbf{R}^Y \otimes XY \otimes Y + \mathbf{R}^Y \otimes Y^2 \otimes \delta_1 Y \\ &+ \mathbf{R}^Y \otimes X \otimes Y^2 + \mathbf{R}^Y \otimes Y \otimes \delta_1 Y^2 - \mathbf{R}^Y \otimes Y \otimes X - \mathbf{R}^X \otimes XY^2 \otimes \delta_1 \\ &- \frac{1}{3}\mathbf{R}^X \otimes Y^3 \otimes {\delta_1}^2 + \frac{1}{3}\mathbf{R}^Y \otimes Y^3 \otimes \delta_1 - \frac{1}{4}\mathbf{R}^X \otimes Y^2 \otimes {\delta_1}^2 - \frac{1}{2}\mathbf{R}^Y \otimes Y^2 \otimes \delta_1.\end{aligned}$$

As can be seen by the inspection, the expression of the above cocycles cannot be easily found directly. The machinery in [7] is used to arrive at this elaborate formulae.

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Fredholm modules and boundary actions of hyperbolic groups

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(joint work with Heath Emerson)

The boundary $\partial\Gamma$ of a non-elementary hyperbolic group Γ is a compact space on which Γ acts by homeomorphisms. In this report, we sketch the construction of certain finitely summable Fredholm modules for the crossed product C*-algebra $C(\partial\Gamma) \rtimes \Gamma$. These Fredholm modules enjoy the following features:

- homologically relevant: they represent a distinguished K-homology class, which is typically non-trivial;
- meaningful summability: roughly speaking, they are p-summable for every p greater than the Hausdorff dimension of the boundary;
- very simple form, quite unlike any other Fredholm modules known so far.

It should be noted that we are in a Type III situation - $C(\partial\Gamma)\rtimes\Gamma$ is purely infinite simple [6, 2] - so there are no finitely summable spectral triples.

The boundary extension class. The action of Γ on $\partial\Gamma$ is amenable [1]. Therefore the maximal and the reduced crossed products for the action coincide, and $C(\partial\Gamma)\rtimes\Gamma$ is a nuclear C*-algebra. For unital nuclear C*-algebras, we may identify the K^1-group of homotopy classes of odd Fredholm modules with the Ext-group of extensions by compacts. There is a natural extension of $C(\partial\Gamma)\rtimes\Gamma$ by compacts, given by the boundary compactification $\overline{\Gamma}=\Gamma\cup\partial\Gamma$:

$$0\to\mathcal{K}(\ell^2\Gamma)\to C(\overline{\Gamma})\rtimes\Gamma\to C(\partial\Gamma)\rtimes\Gamma\to 0$$

The corresponding odd homology class, denoted $[\partial_\Gamma]$, is called the boundary extension class.

Assume that Γ is torsion-free. On the one hand, from [4] we know that there is a Poincaré duality isomorphism $\mathrm{K}^*(C(\partial\Gamma)\rtimes\Gamma)\cong\mathrm{K}_{*+1}(C(\partial\Gamma)\rtimes\Gamma)$, and that the Poincaré dual of the K^1-class $[\partial_\Gamma]$ is the K_0-class of the unit [1]. (The proof from [4] - though most likely not Poincaré duality itself - needs a mild symmetry condition on $\partial\Gamma$, but we shall disregard this minor technical point in what follows.) On the other hand, from [5] we know that the order of $[1]\in\mathrm{K}_0(C(\partial\Gamma)\rtimes\Gamma)$ is determined by the Euler characteristic of Γ as follows: [1] has finite order $|\chi(\Gamma)|$ if $\chi(\Gamma)\neq 0$, and infinite order otherwise. Combining these two facts, we obtain:

Theorem 1 (from [4] & [5])**.** *Let Γ be torsion-free. Then $[\partial_\Gamma]$ is non-trivial, unless $\chi(\Gamma)=\pm 1$. Furthermore, $[\partial_\Gamma]$ has infinite order if and only if $\chi(\Gamma)=0$.*

Naive Fredholm modules for crossed products. Let us consider the general situation of a discrete group G acting by homeomorphisms on a compact space X. In order to construct a Fredholm module for the reduced crossed product $C(X)\rtimes_{\mathrm{r}} G$, we need a representation of $C(X)\rtimes_{\mathrm{r}} G$ on a Hilbert space, and a projection in that Hilbert space. For the representation, we make the obvious choice: a regular representation. If μ is a fully supported Borel probability measure on X, then $C(X)$ is faithfully represented on $L^2(X,\mu)$ by multiplication, which in turn defines a faithful representation of $C(X)\rtimes_{\mathrm{r}} G$ on $\ell^2\big(G,L^2(X,\mu)\big)$. This is the regular representation of $C(X)\rtimes_{\mathrm{r}} G$ defined by μ, and we denote it by λ_μ. Next, the choice of a projection is again the obvious one: we consider the projection of $\ell^2\big(G,L^2(X,\mu)\big)$ onto ℓ^2G.

In order to describe the Fredholmness and the summability of $(\lambda_\mu,P_{\ell^2G})$, we define dynamical versions of two standard probabilistic notions, expectation and standard deviation. The *G-expectation* and the *G-deviation* of $\phi\in C(X)$ are the maps $\mathrm{E}\phi : G \to \mathbb{C}$ and $\sigma\phi : G \to [0, \infty)$ given by the formulas

$$\mathrm{E}\phi(g) = \int_X \phi \,\mathrm{d}(g_*\mu), \qquad \sigma\phi = \sqrt{\mathrm{E}|\phi|^2 - |\mathrm{E}\phi|^2}.$$

Now the Fredholmness and the summability of $(\lambda_\mu, P_{\ell^2 G})$ can be characterized by decay conditions for the G-deviation, as follows:

Proposition 2. *$(\lambda_\mu, P_{\ell^2 G})$ is a Fredholm module for $C(X) \rtimes_{\mathrm{r}} G$ if and only if $\sigma\phi \in C_0(G)$ for all $\phi \in C(X)$. Furthermore, $(\lambda_\mu, P_{\ell^2 G})$ is p-summable if and only if $\sigma\phi \in \ell^p G$ for all ϕ in a dense subalgebra of $C(X)$.*

Alternately, and interestingly, the Fredholmness of $(\lambda_\mu, P_{\ell^2 G})$ can be described by a kind of "pure proximality" à la Furstenberg:

Proposition 3. *$(\lambda_\mu, P_{\ell^2 G})$ is a Fredholm module for $C(X) \rtimes_{\mathrm{r}} G$ if and only if $g_*\mu$ only accumulates to point masses in Prob(X) as $g \to \infty$ in G.*

We need two further properties in what follows. The first is an independence result motivated by the fact that, in general, there is no canonical measure on the boundary of a hyperbolic group. The second is a multiplicativity property motivated by the desire to extend Theorem 1 to virtually torsion-free Γ. We say that two measures are *comparable* if one is between constant multiples of the other.

Proposition 4. *Let μ' be a fully supported Borel probability measure on X which is comparable to μ. Then $(\lambda_{\mu'}, P_{\ell^2 G})$ enjoys the same Fredholmness and summability as $(\lambda_\mu, P_{\ell^2 G})$. If $(\lambda_\mu, P_{\ell^2 G})$ and $(\lambda_{\mu'}, P_{\ell^2 G})$ are Fredholm modules, then they are K^1-homologous.*

Proposition 5. *Assume that $(\lambda_\mu, P_{\ell^2 G})$ is a Fredholm module for $C(X) \rtimes_{\mathrm{r}} G$, and that the measures $\{g_*\mu\}_{g\in G}$ are mutually comparable. Let $H \leq G$ be a subgroup of finite index. Then $[(\lambda_\mu, P_{\ell^2 G})] = [G : H] \cdot [(\lambda_\mu, P_{\ell^2 H})]$ in $\mathrm{K}^1(C(X) \rtimes_{\mathrm{r}} H)$.*

Back to hyperbolic groups. The boundary of a non-elementary hyperbolic group carries certain natural measures induced by "hyperbolic fillings". Namely, if Γ acts geometrically - that is, isometrically, properly, and cocompactly - on a (hyperbolic) space X, then ∂X is a topological incarnation of $\partial\Gamma$. A visual metric on ∂X is any metric comparable with $\exp(-\epsilon(\cdot,\cdot)_\bullet)$, where $(\cdot,\cdot)_\bullet$ stands for the extended Gromov product. It turns out that such metrics exist for small enough $\epsilon > 0$, and any two visual metrics are Hölder equivalent. The visual probability measures on ∂X are the normalized Hausdorff measures induced by visual metrics. Any two visual probability measures are comparable. Most importantly, visual measures are Ahlfors regular [3]: if d is a visual metric on ∂X, then the corresponding visual probability measure μ has the property that $\mu(R\text{-ball}) \asymp R^{\,\mathrm{hdim}(\partial X, d)}$. The point is that, roughly speaking, Ahlfors regularity implies that the Γ-deviation of Lipschitz maps on $(\partial X, d)$ is in $\ell^p\Gamma$ for $p > \mathrm{hdim}(\partial X, d)$. By Proposition 2, this means that $(\lambda_\mu, P_{\ell^2\Gamma})$ is a p-summable Fredholm module for $p > \mathrm{hdim}(\partial X, d)$.

However, since the summability is independent of the choice of visual probability measure (Proposition 4), we are led to considering the "minimal Hausdorff dimension" of ∂X with respect to the visual metrics:

$$\mathrm{visdim}\,\partial X = \inf\{\mathrm{hdim}(\partial X, d) : d \text{ visual metric}\}.$$

We may now state our main result:

Theorem 6. *Let Γ act geometrically on X. Then, for every visual probability measure μ on ∂X, the following hold:*

i) *$(\lambda_\mu, P_{\ell^2\Gamma})$ is a Fredholm module for $C(\partial\Gamma)\rtimes\Gamma$ which is p-summable for every $p > \max\{\mathrm{visdim}\,\partial X, 2\}$. In the case when $\mathrm{visdim}\,\partial X > 2$ and it is attained, $(\lambda_\mu, P_{\ell^2\Gamma})$ is in fact $(\mathrm{visdim}\,\partial X)^+$-summable;*
ii) *$(\lambda_\mu, P_{\ell^2\Gamma})$ represents $[\partial_\Gamma]$.*

The last point is based on the fact that extending $\phi \in C(\partial\Gamma)$ by $\mathrm{E}\phi$ on Γ yields a function, denoted $\overline{\mathrm{E}}\phi$, on $\overline{\Gamma}$ which is continuous. Hence $\overline{\mathrm{E}}$ is a Γ-equivariant cp-section for $0 \to C_0(\Gamma) \to C(\overline{\Gamma}) \to C(\partial\Gamma) \to 0$, and then $\overline{\mathrm{E}}$ can be promoted to a cp-section for $0 \to \mathcal{K}(\ell^2\Gamma) \to C(\overline{\Gamma})\rtimes\Gamma \to C(\partial\Gamma)\rtimes\Gamma \to 0$. One concludes by a Stinespring dilation argument.

From Proposition 5 we deduce a multiplicativity property for the boundary extension class: if $\Lambda \leq \Gamma$ is a subgroup of finite index, then the natural map $\mathrm{K}^1(C(\partial\Gamma)\rtimes\Gamma) \to \mathrm{K}^1(C(\partial\Lambda)\rtimes\Lambda)$ sends $[\partial_\Gamma]$ to $[\Gamma : \Lambda]\cdot[\partial_\Lambda]$. For virtually torsion-free groups, which have a well-defined notion of rational Euler characteristic, Theorem 1 and the above multiplicativity property imply the following criterion:

Corollary 7. *Let Γ be virtually torsion-free. If $\chi(\Gamma) \notin 1/\mathbb{Z}$ then $[\partial_\Gamma]$ is non-trivial. If $\chi(\Gamma) = 0$ then $[\partial_\Gamma]$ has infinite order.*

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Arithmetic models of Bost-Connes sytems

Bora Yalkinoglu

The definition of the Bost-Connes system $\mathcal{A}_K = (A_K, \sigma_t)$ for a number field K is based on a natural action of the monoid of integral ideals I_K of K on the balanced product $Y_K = \widehat{\mathcal{O}}_K \times_{\widehat{\mathcal{O}}_K^\times} Gal(K^{ab}/K)$ of the finite integral adeles and the maximal abelian Galois group of K. Namely, the BC-system $\mathcal{A}_K$ is defined as the semigroup crossed product C^*-algebra $A_K = C(Y_K) \rtimes I_K$ together with a natural time evolution σ_t. BC-systems for arbitrary number fields were first defined by Ha and Paugam [8] inspired by important foundational work of Bost and Connes [3] and Connes, Marcolli and Ramachandran [6]. The systems $\mathcal{A}_K$ satisfy the following remarkable properties.

Theorem 1 (cf., [8] and [9])**.**

(i) *The partition function of $\mathcal{A}_K$ is given by the Dedekind zeta function of K.*

(ii) *The maximal abelian Galois group $Gal(K^{ab}/K)$ acts as symmetries on $\mathcal{A}_K$.*

(iii) *For each inverse temperature $0 < \beta \leq 1$ there is a unique KMS$_\beta$-state.*

(iv) *For each $\beta > 1$ the action of the symmetry group $Gal(K^{ab}/K)$ on the set of extremal KMS$_\beta$-states is free and transitive.*

Now, an **arithmetic model** of $\mathcal{A}_K$ is a K-rational subalgebra A_K^{arith} of A_K such that

(v) For every extremal KMS$_\infty$-state ϱ and every $f \in A_K^{arith}$ we have

$$\varrho(f) \in K^{ab}$$

and further K^{ab} is generated over K by these values.

(vi) If we denote by ${}^\nu\varrho$ the action of a symmetry $\nu \in Gal(K^{ab}/K)$ on an extremal KMS$_\infty$-state ϱ (given by pull-back) we have for every element $f \in A_K^{arith}$ the following compatibility relation

$${}^\nu\varrho(f) = \nu^{-1}(\varrho(f))$$

(vii) The $\mathbb{C}$-algebra $A_K^{arith} \otimes_K \mathbb{C}$ is dense in A_K.

The existence of an arithmetic model implies in particular that the class field theory of K is realized through the dynamics of $\mathcal{A}_K$.

Beforehand, the existence of an arithmetic subalgebra was known in the case of $\mathbb{Q}$ [3] and in the case of imaginary quadratic fields $\mathbb{Q}(\sqrt{-D})$ [6] for D a square-free positive integer and relied on the explicit class field theory for the corresponding number fields.

Despite the fact that for other number fields an explicit class field theory is not known, our main result reads as follows.

Theorem 2. *For every K there exists an arithmetic model A_K^{arith} of the BC-system $\mathcal{A}_K$.*

Moreover, it was shown by Sergey Neshveyev (see Appendix [10]) that the arithmetic model A_K^{arith} is in fact unique.
Our construction of the arithmetic model A_K^{arith} relies on two main ingredients, the theory of endomotives introduced by Connes, Consani and Marcolli [4] and a Galois correspondence for finite, étale Λ-rings developed by Borger and de Smit [2]. This Galois correspondence allows us to encompass the missing explicit class field theory in that it provides us with an inductive system of finite, étale K-algebras $(E_{\mathfrak{f}})_{\mathfrak{f}\in I_K}$ together with an action of I_K on the direct limit $E_K = \varinjlim_{\mathfrak{f}\in I_K} E_{\mathfrak{f}}$ by Frobenius lifts. Here $E_{\mathfrak{f}}$ is isomorphic to the finite product $\prod_{\mathfrak{d}|\mathfrak{f}} K_{\mathfrak{f}/\mathfrak{d}}$ of strict ray class fields of K of conductor dividing $\mathfrak{f}$. Our arithmetic model A_K^{arith} is then defined by the (algebraic) crossed product $E_K \rtimes I_K$. More precisely we prove

Theorem 3. *A_K^{arith} is an algebraic endomotive whose associated measured analytic endomotive is naturally isomorphic the BC-system $\mathcal{A}_K$.*

Further, there are two important corollaries from our construction, first, using the results of [2] it is clear that A_K^{arith} can be expressed in terms of generalized Witt vectors [1] and, second, the space Y_K used in the definition of the BC-system is in fact naturally isomorphic to the so called Deligne-Ribet monoid DR_K used in [7] to construct p-adic L-functions for totally real number fields. These two observations pave the way for generalizing recent work of Connes and Consani [5] on p-adic representations of the classical BC-system $\mathcal{A}_{\mathbb{Q}}$ to more general number fields.

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Transfer in differential algebraic K-theory

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(joint work with David Gepnerr)

We define the $j+1$'th polylogarithm, $j \geq 1$ by

$$\mathtt{Li}_{j+1}(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^j} \ .$$

We consider the real $\mathtt{Gal}(\mathbb{C}/\mathbb{R})$-module $\mathbb{R}(j) := \mathbb{C}/i^{j+1}\mathbb{R}$ and the $\mathtt{Gal}(\mathbb{C}/\mathbb{R})$ set of p'th roots of unity $\mu_p \subset \mathbb{C}$ for a prime $p \geq 3$. The polylogarithm on roots of unity gives rise to an invariant function

$$\mathtt{Li}_{j+1} \in [\mathbb{R}(j)^{\mu_p}]^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})} \ .$$

We consider the cyclotomic ring $R := \mathbb{Z}[\xi]/(1+\xi+\cdots+\xi^{p-1})$ and the Borel regulator map

$$b : K_{2j-1}(R) \to [\mathbb{R}(j)^{\mu_p}]^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})} \ .$$

Theorem 1 (Borel, [Bor74])**.** *The image of b is a lattice of full rank, and the kernel of b is the torsion subgroup.*

This opens the question about the image of the regulator map.

Theorem 2 (Beilinson, [Beĭ86])**.** *There exists an element $x \in K_{2j-1}(R)$ and $q \in \mathbb{Q}$ such that $b(x) = q\ \mathtt{Li}_{j+1}$.*

In the talk I explaind how this result can been seen as a consequence of a conjectural index theorem for the Becker-Gottlieb transfer in differential algebraic K-theory.

I start with a description of differential cohomology via the differential function spectrum [HS05]. As data we take a spectrum E, a chain complex A of real vector spaces, and an equivalence of spectra $c : E \otimes \mathbb{R} \to H(A)$, where $H(A)$ is the Eilenberg-MacLane spectrum associated to A. We have a natural spectrum-level de Rham isomorphism

$$j : H(\Omega A(M)) \xrightarrow{\sim} H(A)^M \ ,$$

where $\Omega A(M)$ is the de Rham complex of a smooth manifold M with coefficients in A, and $H(A)^M$ denotes the function spectrum. The differential function spectrum is now defined as a homotopy pull-back in spectra

$$\begin{array}{ccc} \mathtt{Diff}(E)(M) & \longrightarrow & H(\sigma\Omega A(M)) \\ \downarrow & & \downarrow \\ E^M & \longrightarrow & H(\Omega A(M)) \end{array} \ .$$

Here $\sigma\Omega A(M)$ is the cut-off to non-negative degrees, and the lower horizontal and the left vertical maps are given by c and j^{-1}, and the natural inclusion, respectively. The differential E-cohomology of M is by definition

$$\hat{E}^0(M) := \pi_0(\mathtt{Diff}(E)(M)) \ .$$

It comes with natural transformations

$$\begin{array}{ccccc} & & & & \Omega A^0_{cl}(M) \\ & & & \overset{R}{\nearrow} & \\ \Omega A^{-1}(M)/\mathtt{im}(d) & \xrightarrow{a} & \hat{E}^0(M) & & \\ & & & \underset{I}{\searrow} & \\ & & & & E^0(M)\ . \end{array}$$

Let now $\pi : W \to B$ be a proper submersion. The Becker-Gottlieb transfer [BG75] is a natural homotopy class of maps $E^W \to E^B$ which induces a homomorphism $\mathtt{tr} : E^0(W) \to E^0(B)$ on the zero'th homotopy group. We show:

Theorem 3. *Given a Riemannian structure (vertical metric and horizontal distribution) on π we can define a natural differential refinement $\mathtt{Diff}(E)(W) \to \mathtt{Diff}(E)(B)$. The induced map $\hat{\mathtt{tr}} : \hat{E}^0(W) \to \hat{E}^0(B)$ is compatible with $\mathtt{tr}$ (via I) and*

$$\Omega A^0_{cl}(W) \ni \omega \mapsto \int_{W/B} e(\pi) \wedge \omega \in \Omega A^0_{cl}(B)$$

(via R), where $e(\pi)$ is the Euler form of π.

We now consider a number ring R and define the complex A with trivial differential by $A_i := K_i(R) \otimes \mathbb{R}$. For $i \geq 2$ its dimension is determined by Theorem 1. A geometry on a local system $\mathcal{V}$ of finitely generated projective R-modules on a manifold M is a collection of metrics $h^{\mathcal{V}}$ on the collection of flat complex vector bundles induced by $\mathcal{V}$ for all the complex embeddings of R. To such a geometry, following ideas of [BL95], we can associate a characteristic form

$$\omega(h^{\mathcal{V}}) \in \Omega A^0_{cl}(M)\ .$$

Let KR denote connective algebraic K-theory spectrum of R and $c : KR \otimes \mathbb{R} \to H(A)$ be the canonical equivalence. We have the following result:

Theorem 4. *There exists a natural additive cycle map which associates to an isomorphism class of pairs $(\mathcal{V}, h^{\mathcal{V}})$ an element*

$$\mathtt{cycle}(\mathcal{V}, h^{\mathcal{V}}) \in \widehat{KR}^0(M)$$

such that

$$R(\mathtt{cycle}(\mathcal{V}, h^{\mathcal{V}})) = \omega(h^{\mathcal{V}})\ , \quad I(\mathcal{V}, h^{\mathcal{V}}) = [\mathcal{V}] \in KR^0(M)\ .$$

Let now $\pi : W \to B$ be a proper submersion with a Riemannian structure g and $(\mathcal{V}, h^{\mathcal{V}})$ be given on W. We define the topological index by

$$\widehat{\mathtt{index}}^{top}(\mathcal{V}, h^{\mathcal{V}}) := \hat{\mathtt{tr}}(\mathtt{cycle}(\mathcal{V}, h^{\mathcal{V}})) \in \widehat{KR}^0(B)\ .$$

We further define

$$\widehat{\mathtt{index}}^{an}(\mathcal{V}, h^{\mathcal{V}}) := \mathtt{cycle}(R\pi_*\mathcal{V}, h^{R\pi_*\mathcal{V}}_{L^2}) - a(T(\pi, g, \mathcal{V}, h^{\mathcal{V}}))\ ,$$

where the L^2 metric is given by fibrewise Hodge theory and $T(\pi, g, \mathcal{V}, h^{\mathcal{V}})$ is the Bismut-Lott torsion form. We define

$$\widehat{\mathtt{index}}^{top}(\mathcal{V}, h^{\mathcal{V}}) - \widehat{\mathtt{index}}^{an}(\mathcal{V}, h^{\mathcal{V}}) =: \delta \in \widehat{KR}^0(B) \ .$$

The conjectural index theorem is now

Conjecture 5.

$$\delta = 0 \ .$$

The following consequences are known to be true:

Theorem 6 (Bismut-Lott, [BL95])**.** $R(\delta) = 0$

Theorem 7 (Dwyer-Weiss-Williams, [DWW03])**.** $I(\delta) = 0$.

Theorem 8 (Cheeger/Müller, [Che79], [Mül78])**.** $\delta = 0$ *if* $B = *$.

Finally we assume that R is the cyclotomic number ring above, $W \to B$ is the $U(1)$-bundle on $\mathbb{C}P^n$ with Chern class pc_1, and $\mathcal{V}$ is one-dimensional with holonomy $\xi \in R$. Then, after some calculations, the following theorem is equivalent to Theorem 2:

Theorem 9. *In the case described above, δ is a torsion element.*

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