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Complex Algebraic Geometry

Organised by
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Abstract. The conference focused on several classical and modern topics in the realm of complex algebraic geometry, such as moduli spaces, birational geometry and the minimal model program, Mirror symmetry and Gromov-Witten invariants, Hodge theory, curvature flows, algebraic surfaces and curves.

Mathematics Subject Classification (2000): 14xx , 18xx , 32xx , 53xx .

Introduction by the Organisers

The workshop *Complex Algebraic Geometry*, organized by Fabrizio Catanese (Bayreuth), Yujiro Kawamata (Tokyo), Bernd Siebert (Hamburg) and Gang Tian (Princeton), drew together 56 participants from all continents (...). There were several young PhD students and other PostDocs in their 20s and early 30s, together with established leaders of the fields related to the thematic title of the workshop. There were 23 50-minutes talks, each followed by a lively discussion.

As usual at an Oberwolfach Meeting, the mathematical discussions continued outside the lecture room throughout the day and the night. The Conference fully realized its purported aim, of setting in contact mathematicians with different specializations and non uniform background, of presenting new fashionable topics alongside with new insights on long standing classical open problems, and also cross-fertilizations with other research topics.

Many talks were devoted to classification theory, especially to the birational geometry of 3-folds, but also to surfaces and higher dimensional varieties, Fano manifolds, relations with mirror symmetry.

Another important direction was moduli spaces: their rationality properties, moduli spaces of curves and Gromov-Witten invariants, automorphisms.

There were also new interesting results on the classical topics of Hyperkähler manifolds, Prym varieties and maps, Hodge modules.

Tian gave a survey talk on curvature flows and on the open problem of VII_0^+ surfaces, and there were also talks on derived categories, McKay correspondences, Hurwitz type spaces.

The variety of striking results and the very interesting and challenging proposals presented in the workshop made the participation highly rewarding. We hope that these abstracts will give a clear and attractive picture, they will certainly be useful to the mathematical community.

Workshop: Complex Algebraic Geometry

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Abstracts

Rationality properties of linear group quotients

Christian Böhning
(joint work with Fedor Bogomolov, Hans-Christian Graf v. Bothmer, Gianfranco Casnati, Jakob Kröker)

This was a survey talk on recent work on the rationality problem in invariant theory. The basic set-up is the following (everything is defined over $\mathbb{C}$): let G be a linear algebraic group, V a finite-dimensional linear G-representation. One asks whether the (obviously unirational) quotient V/G is stably rational, rational, retract rational, a direct factor of a rational variety etc. It is known through examples of Saltman, Bogomolov, Peyre and others ([Sa], [Bogo2], [Pey]), all using unramified group cohomology as obstructions, that such quotients need not be stably rational if G is finite, but no examples of nonrational V/G are known for G connected. Stable rationality of V/G, where V is supposed to be generically free for the G-action, is a property of G alone by the no-name lemma. For the simply connected simple groups one knows that generically free linear quotients are stably rational except for the groups of the Spin-series and E_8 where the question remains open. Let us say that a variety X is *stably rational of level* l if $X \times \mathbb{P}^l$ is rational.

A general theorem of Bogomolov and Katsylo ([Bogo1], [Kat83], [Kat84]) says that linear $\mathrm{SL}_2(\mathbb{C})$-quotients are rational. For $\mathrm{SL}_3(\mathbb{C})$ we have the following theorem which summarizes joint work with von Bothmer and Kröker in [B-B10-1], [B-B10-2], [B-B-Kr09].

Theorem 1. *The moduli spaces of plane curves* $\mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^3)^\vee)/\mathrm{SL}_3(\mathbb{C})$ *are all rational except for* 15 *values of* d *for which this remains unknown. These exceptional* d *satisfy* $6 \leq d \leq 48$.

It turns out that, besides reductive groups, it is advantageous to consider also groups with a nontrivial unipotent radical. Their generically free representations come equipped with Jordan-Hölder filtrations which often allow one to introduce some fibration structure in V/G over a stably rational base. This gives some a priori evidence that such quotients may be amenable to more detailed study. In this direction we obtained in joint work with Bogomolov and von Bothmer [BBB10]

Theorem 2. *Let* V *be a linear, indecomposable, generically free representation of the affine group* $\mathrm{ASL}_n(\mathbb{C}) = \mathrm{SL}_n(\mathbb{C}) \ltimes \mathbb{C}^n$, *and suppose that the dimension of* V *is sufficiently large. Then* $V/(\mathrm{SL}_n(\mathbb{C}) \ltimes \mathbb{C}^n)$ *is rational.*

This can probably be substantially improved. In fact, it seems that the only very difficult point is to obtain rationality for $W/\mathrm{ASL}_n(\mathbb{C})$ where W is an extension $0 \to \mathrm{Sym}^3\mathbb{C}^n \to W \to \mathrm{Sym}^2\mathbb{C}^n \to 0$. The techniques used to prove the preceding theorem have been used to improve the bounds for levels of stable rationality of generically free linear quotients by simple groups in [BBB11]. In particular, they are linear as functions of the rank of the group (when this belongs to an infinite series) whereas previously only quadratic bounds were known.

Theorem 3. *Let G be one of the simple linear algebraic groups in the left hand column of the table below, and let $\mathcal{C}$ be the class of G-representations V of the form $V = W \oplus S^{\epsilon}$, where: W is an irreducible representation of G whose ineffectivity kernel (a finite central subgroup) coincides with the stabilizer in general position. S is a standard representation for each of the groups involved, namely $\mathbb{C}^n$ for $\mathrm{SL}_n(\mathbb{C})$, $\mathbb{C}^{2n}$ for $\mathrm{Sp}_{2n}(\mathbb{C})$, $\mathbb{C}^m$ for $\mathrm{SO}_m(\mathbb{C})$ and $\mathrm{O}_m(\mathbb{C})$, $\mathbb{C}^7$ for G_2. Here $\epsilon \in \{0,1\}$, and $\epsilon = 0$ if and only if W is already G-generically free. Thus V will always be G-generically free. Then the following table summarizes the results from* [BBB11]*:*

Group G	*Level of stable rationality N for V/G for $V \in \mathcal{C}$*
$\mathrm{SL}_n(\mathbb{C})$, $n \geq 1$	n
$\mathrm{SO}_{2n+1}(\mathbb{C})$, $n \geq 2$	$2n+1$
$\mathrm{O}_{2n+1}(\mathbb{C})$, $n \geq 2$	$2n+1$
$\mathrm{Sp}_{2n}(\mathbb{C})$, $n \geq 4$	$2n$
$\mathrm{SO}_{2n}(\mathbb{C})$, $n \geq 2$	$2n$ *except for W with highest weight $c\omega_{n-1}$ or $c\omega_n$, $c \in \mathbb{N}$, where we know only $4n$.*
$\mathrm{O}_{2n}(\mathbb{C})$, $n \geq 2$	$2n$
G_2	7

In a more visibly geometric direction, the study of groups with nontrivial unipotent radical has proven fruitful in [BBC11] to obtain

Theorem 4. *The locus of tetragonal curves $\mathfrak{M}^1_{7;4} \subset \mathfrak{M}_7$ is rational.*

Rationality of $\mathfrak{M}_7$ itself is unknown. The proof uses the realization of the moduli space of tetragonal curves as the space of pencils of cubic surfaces in $\mathbb{P}^3$ containing a given line L modulo the subgroup of projectivities of $\mathbb{P}^3$ fixing L.

At the end we mentioned very briefly group cohomological obstructions to stable rationality of quotients V/G and results in [BB11] on the stable cohomology of alternating groups.

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Birational maps of threefolds

Jungkai Alfred Chen

(joint work with Christopher Hacon, partly)

Birational maps in minimal model program consist of the following elementary ones: divisorial contractions, flips and flops. Divisorial contractions are the higher dimensional analog of the inverse of blowing up a smooth point on a surface. A flips (resp. flop) is an operation which consists of a "surgery" in codimension ≥ 2 which replaces certain K_X-negative (resp. K_X-trivial) curves by some K_X-positive (resp. K_X-trivial) curves.

Starting with a non-minimal projective variety, then one expects to obtain a minimal model of X by a finite sequence of flips and divisorial contractions. It is shown in (cf. [1]) that minimal models exist for varieties of general type. Minimal models are not unique in general. Any two minimal models are connected by a finite sequence of flops (cf. [8]).

In dimension three, many of these birational maps admit explicit description. We survey some recent advances in explicit studies of birational maps in minimal model program in dimension three. Divisorial contractions to points has been studied by Mori (cf. [15]), Cutkosky (cf. [4]), Kawamata, Kawakita, Hayakawa, and others intensively. In [7], Kawamata shows that a divisorial contraction whose image contains a terminal quotient singularity P must be a weighted blowup over P, which is called a *Kawamata blowup.* Hayakawa (cf. [5], [6]) shows that a divisorial contraction to a point of index $r > 1$ with minimal discrepancy $1/r$ can be realized as a weighted blowup. It is shown recently by Kawakita (cf. [13]) that a divisorial contraction to a point of index $r > 1$ can be realized as a weighted blowup. Indeed, in his series of works on divisorial contractions to points (cf. [9], [10], [11], [12]), Kawakita classified divisorial contractions to points in the sense that singularities and baskets are described explicitly. Moreover, almost all of the cases are known to admit embeddings so that the divisorial contractions are realized by weighted blowups. It is thus expected that a divisorial contraction to a point can always be realized as a weighted blowup.

Typical examples of divisorial contraction to curves are blowups along smooth curves or LCI curves in a smooth threefold. There are also several partial results on divisorial contractions to a curve (cf. [4], [16], [17], [18], [19] and [20]).

In the paper [3], we can factor threefold flips and divisorial contractions to curves via the simpler operations given by flops, blow-downs to LCI curves (i.e. $C \subset Y$ a local complete intersection curve in a smooth variety) and divisorial contractions to points. More precisely we prove the following:

Let X be a threefold with terminal singularity. Let $X \to W$ be a divisorial contraction to a curve (resp. a flipping contraction). There is a sequence of birational maps

$$X = X_0 \dashrightarrow \ldots \dashrightarrow X_n,$$

such that $X_n = W$ (resp. $X_n = X^+$) and each $X_i \dashrightarrow X_{i+1}$ is one of the following:

(1) a divisorial contraction to a point;
(2) a divisorial extraction over a point with minimal discrepancies, which is always a weighted blowup;
(3) a blowup along a LCI curve;
(4) a flop.

The key observation is that by taking an extremal extraction $g : Y \to X$ over a point $P \in X$ with maximal index, one can have the so-called "2-ray game" for Y over W. We also need a discrete invariant call *depth*, defined as the minimal length of Gorenstein partial resolution of a point of higher index, in order to run the inductive argument.

In fact, in [2], the similar trick also applicable for divisorial contraction to a point $P \in X$ of index r with discrepancy $a/r > 1/r$. In particular, we show that such divisorial contractions to points of index $r > 1$ can be factored into a sequence of birational maps

$$X = X_0 \dashrightarrow \ldots \dashrightarrow X_n,$$

such that $X_n = W$ (resp. $X_n = X^+$) and each $X_i \dashrightarrow X_{i+1}$ is one of the following:

(1) a divisorial contraction to a point with minimal discrepancies, which is always a weighted blowup;
(2) a divisorial extraction over a point with minimal discrepancies, which is always a weighted blowup;
(3) a flip;
(4) a flop.

Moreover, in our recent work in progress, we show that for a given three dimensional terminal singularity $P \in X$, there is a sequence of weighted blowups

$$X_n \to \ldots \to X_0 = X \ni P,$$

such that X_n is smooth and each $X_{i+1} \to X_i$ is a divisorial contraction, which is a weighted blowup, over a singular point $P_i \in X_i$ of index $r_i \geq 1$ with discrepancy $1/r_i$.

The mainly ingredient is resolving terminal singularises in a good hierarchy. The hierarchy of resolution we consider is: 1. terminal quotient singularities; 2. cA points; 3. cA/r points; 4. cD and $cAx/2$ points; 5. $cAx/4$, $cD/2$, and $cD/3$ points; 6. cE_6 points; 7. cE_7 points; 8. cE_8 points; 9. $cE/2$ points.

Problem. An optimistic guess is to expect that for any two birational threefolds $X \dashrightarrow X'$. There is a sequence of birational maps

$$X = X_0 \dashrightarrow \ldots \dashrightarrow X_n = X',$$

such that each $f_i : X_i \dashrightarrow X_{i+1}$ or f_i^{-1} is one of the following:

(1) a divisorial contraction to a point with minimal discrepancy, which is always a weighted blowup;
(2) a blowup along a smooth curve C such that the threefold is smooth near C;
(3) a flop.

By a divisorial contraction to a point with minimal discrepancies, we mean a divisorial contraction $f : Y \to X \in P$ to a point of index r with discrepancy $1/r$ if P is singular. We remark that the blowup over a smooth point which is a divisorial contraction with discrepancy 2 is considered to be a divisorial contraction with minimal discrepancy. This is because that any divisorial contraction over a smooth point is a weighted blowup with weights $(1, a, b)$ with discrepancy $a+b \geq 2$ (cf. [9]).

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On 3-folds of general type with small fractional genus: $g = \frac{1}{2}$

Meng Chen
(joint work with Jungkai A. Chen)

The motivation of this research work is the following classical problem:

> Find two optimal constants v_3 and b_3 such that, for all nonsingular projective 3-fold V of general type, $\mathrm{Vol}(V) \geq v_3$ and that φ_m is birational onto its image for all $m \geq b_3$.

Due to Chen-Chen [1, 2], one knows $v_3 \geq \frac{1}{2660}$ and $b_3 \leq 73$. The purpose of this talk is to report on some new advances on this topic.

Set $\mathbb{V}_3^* := \{X | X$ is a minimal non-Gorenstein 3-fold of general type with $p_g(X) \leq 1$, $h^1(\mathcal{O}_X) = 0\}$. $\forall\ X \in \mathbb{V}_3^*$, let $n_0(X)$ be the minimal positive integer with $P_{n_0(X)} \geq 2$. Due to Chen-Chen [1, 2] again, one has $2 \leq n_0(X) \leq 18$. We define the *fractional genus of* X, written as $g(X) := \frac{1}{n_0}$. Thus the target set $\mathbb{V}_3^*$ is roughly divided into more than a dozen of cases, which will be studied explicitly.

The main results are as follows.

Theorem 1. For all $X \in \mathbb{V}_3^*$, the following holds:

- $g(X) = \frac{1}{18}$ if and only if the weighted basket $\mathbb{B}(X) = \{B_{2a}, 0, 2\}$;
- $g(X) \neq \frac{1}{16}, \frac{1}{17}$;
- if $g(X) = \frac{1}{15}$, then $\#\{\mathbb{B}(X)\} = 8$;
- if $g(X) = \frac{1}{14}$, then $\#\{\mathbb{B}(X)\} = 78$;
- $g(X) = \frac{1}{13}$ if and only if $\mathbb{B}(X) = \{B_{41}, 0, 2\}$;
- if $p_g(X) = 1$, then $K_X^3 \geq \frac{1}{80}$ and φ_m is birational for all $m \geq 24$.

where B_{2a} and B_{41} are specified in the main table of Chen-Chen [2].

Theorem 2. For all $X \in \mathbb{V}_3^*$ with $g(X) = \frac{1}{2}$, the following holds:

- $K_X^3 \geq \frac{1}{12}$;
- φ_m is birational for all $m \geq 12$;
- either φ_{11} is birational or φ_{11} is generically finite of degree 2 and φ_{10} is birational. In the later case, $\mathbb{B}(X)$ is classified up to 19 cases in explicit.

The following examples assert the optimum of Theorem 2.

Example 3. Consider general hypersurfaces:

$$X_{22} \subset \mathbb{P}(1, 2, 3, 4, 11)$$

$$X_{6,18} \subset \mathbb{P}(2, 2, 3, 3, 4, 9)$$

$$X_{10,14} \subset \mathbb{P}(2, 2, 3, 4, 5, 7),$$

they all have the fractional genus $\frac{1}{2}$ and the volume is $\frac{1}{12}$. For the first two 3-folds, φ_m is birational for all $m \geq 11$, but φ_{10} is non-birational.

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k–gonal nodal curves on K3 surfaces

Ciro Ciliberto
(joint work with Andreas Knutsen)

This is a short report on work in progress in collaboration with A. Knutsen (University of Bergen Norway).

Let (S, H) be a (general) polarized $K3$ surface with $\mathrm{Pic}(S) \cong \mathbb{Z}[H]$. We consider the *Severi variety* $V_{|H|,\delta}$ of δ–nodal curves in $|H|$. For any integer $k \geq 2$, we define the subscheme $V^k_{|H|,\delta} \subseteq V_{|H|,\delta}$ as

$$V^k_{|H|,\delta} := \Big\{ C \in V_{|H|,\delta} \mid \text{the normalization of } C \text{ possesses a } \ g^1_k \Big\}.$$

We are interested in studying the non–emptiness and the dimension of the loci $V^k_{|H|,\delta}$.

Our first result is as follows: let (S, H) be a polarised $K3$ surface with $\mathrm{Pic}(S) \cong \mathbb{Z}[H]$ and let $p = p_a(H)$. Assume that $C \in |H|$ is a curve whose normalization possesses a g^1_k. Let g be the geometric genus of C and let $\delta = p - g$; then

$$\delta \geq \alpha\Big(g - (k-1)(\alpha+1)\Big), \tag{1}$$

where $\alpha := \Big\lfloor \frac{g}{2(k-1)} \Big\rfloor$.

This is proved with usual techniques à la Lazarsfeld [6], and extends previous results in [1].

After this, using a degeneration of S embedded in $\mathbb{P}^p$ via the linear system $|H|$, to the union of two rational normal scrolls meeting along a linearly normal elliptic curve, we prove that, if (S, H) is general, then for all δ, g verifying (1) for $k = 2$, $V^2_{|H|,\delta}$ is not empty, with a component of the *expected dimension 2*.

We have a similar result in the case $k > 2$, but at the moment it is not as strong as in the hyperelliptic case. Namely we prove that for infinitely many pairs p, k and for δ verifying (1), $V^k_{|H|,\delta}$ is not empty, with a component of the *expected dimension* $2k - 2$. However there are also infinitely many pairs p, k for which we are only able, *at the moment*, to prove the same result for δ in a slightly smaller range. We are confident we will soon be able to prove the full result also in these cases.

The existence of curves for which δ reaches the minimal value in (1) is intimately related to interesting conjectures by Hassett and Tschinkel [3, 4, 5] on the Mori cone of $\mathrm{Hilb}^k(S)$ (see [1, 2]). The rational curves in $\mathrm{Hilb}^k(S)$ naturally arising from curves in $V^k_{|H|,\delta}$ with the lowest δ predicted by (1), are the most unexpected ones form the Brill–Noether theory viewpoint. They are the closest to the boundary of the Mori cone of $\mathrm{Hilb}^k(S)$ and, according to Hassett–Tschinkel's conjecture, they are candidates to be extremal rays.

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Extremal Laurent Polynomials and Fano manifolds

Alessio Corti

(joint work with T. Coates, S. Galkin, V. Golyshev, A. Kasprzyk)

I report on *Fanosearch*, experimental work in progress with several collaborators. Our motivation is to get an idea of what the classification of smooth Fano 4-folds might look like by classifying their mirror period sequences.

Our results are published on our collaborative research blog:
`http://coates.ma.ic.ac.uk/fanosearch/`

1. Local systems in algebraic geometry I explain how to attach a rational local system $\mathbb{V}$ and a differential equation to one of the following geometric situations:

B: Let $f\colon X \to T$ be an algebraic morphism. Then the homology groups of regular fibers form a local system on the set of regular values:

$$T^{\mathrm{reg}} \ni t \to H_n(X_t, \mathbb{Q})\,.$$

This is the Picard–Fuchs local system and the associated ODE is called the Picard–Fuchs equation. (By a well-known theorem of Deligne, the Picard–Fuchs equation has regular singularities on a compactification of T^{reg}.) The special case when $f\colon \mathbb{C}^{\times\, n} \to \mathbb{C}$ is a Laurent polynomial is of special interest to us. In this case the principal period

$$\pi_f(t) = \Big(\frac{1}{2\pi\, i}\Big)^n \int \frac{1}{1 - tf(x_1, \dots, x_n)} \, \frac{d\,x_1}{x_1} \dots \frac{d\,x_n}{x_n}$$

satisfies a polynomial differential operator, which we denote by L_f, corresponding to a direct summand of $\mathrm{gr}^W_{n-1} R^{n-1} f_! \mathbb{Q}$.

A: Let X be a Fano manifold. Denote by M_m the moduli space of Kontsevich-stable morphisms $f\colon \Gamma \to X$ of genus 0 and degree $-K_X \cdot f = m$, by $\pi\colon C_m \to M_m$ the universal curve, and by $e_m\colon C_m \to X$ the evaluation morphism. Let $\psi = c_1(\omega_\pi)$ where ω_π is the relative dualising sheaf. The *quantum period* is the function

$$J_X(t) = \sum_{m\geq 0} p_m\, t^m \quad \text{where} \quad p_m = \int_{C_m} \psi^{m-2} e_m^\star(\mathrm{pt})\,.$$

It is a nontrivial fact, resting on the theory of quantum cohomology and the assumption that X is Fano, that $J(t)$ satisfies a polynomial differential operator Q_X. We prefer to work with the Fourier–Laplace transform $\widehat{J}_X(t) = \sum (m!) p_m\, t^m$ and the transformed operator $\widehat{Q}_X$ (it is a fact that $\widehat{Q}_X$ has regular singularities everywhere on $\mathbb{P}^1$ but Q_X does not because it has an order 2 pole at infinity.)

2. Mirror duality We say that f and X are *mirror dual* if $L_f = \widehat{Q}_X$ or, equivalently, if $\pi_f(t) = \widehat{J}_X(t)$. This is a very weak notion of mirror duality: if X at all has a mirror, then it has infinitely many. Nevertheless, the notion is good enough for our purposes here. Everything that follows is based on the hope that every Fano manifold X has a mirror dual Laurent polynomial. Based on this hope, we would like to classify Fano manifolds by classifying their mirror-dual Laurent polynomials. What are some necessary conditions that f must satisfy to be mirror to a Fano manifold?

Definition (Golyshev) *(1) Let $\mathbb{V}$ be a local system on $\mathbb{P}^1 \setminus S$; the ramification of $\mathbb{V}$ is the quantity*

$$rf(\mathbb{V}) = \sum_{s\in S} dim\,(\mathbb{V}_{t_0}/\mathbb{V}_{t_0}^{T_s})$$

where $t_0 \in \mathbb{P}^1$ is a general point and T_s the monodromy operator at $s \in S$.
(2) We say that $\mathbb{V}$ is extremal if $\mathbb{V}$ is irreducible, nonconstant, and $rf(\mathbb{V}) = 2rk\mathbb{V}$. (It is easy to see that this is the smallest possible value that the ramification can take on a local system that is irreducible and nontrivial.)
(3) A Laurent polynomial f is extremal if the local system $Sol\,L_f$ is extremal.

The following conjectures a new general global property of quantum cohomology.

Conjecture *If X is a Fano manifold then $Sol\widehat{Q}_X$ has small ramification. More precisely*

$$rf\big(Sol\widehat{Q}_X\big) \leq 2rk\big(Sol\widehat{Q}_X\big) + dim\,P_X^{\frac{n}{2},\frac{n}{2}}$$

where $P_X^{\frac{n}{2},\frac{n}{2}}$ is the primitive cohomology of Hodge type $n/2, n/2$. In particular, $Sol\widehat{Q}_X$ is extremal in odd dimensions.

3. Examples of Extremal Laurent polynomials In this section, P is a reflexive polytope in 3 dimensions and we always assume that 0 is the only lattice point strictly in the interior of P. We consider Laurent polynomials

$$f = \sum_{m\in P} c_m x^m$$

with Newton polytope P and, for $F \subset P$ a face, we write $f_F = \sum_{m\in F} c_m x^m$ the face term. We expect that, perhaps under mild additional conditions, for fixed

P, there are at most finitely many extremal Laurent polynomials f with Newton polytope P, and that their coefficients lie in a "small" number field.

In what follows, for simplicity, we make the following additional assumptions:

(1) $c_0 = 0$;
(2) if $v \in P$ is a vertex then $c_v = 1$;
(3) if $E \subset P$ is an edge then $f_E = x^\mu(1+x^\nu)^e$ where e is the lattice length of E.

Definition *(1) We say that f is Hodge–Tate at infinity if for all facets $F \subset P$, the curve $(f_F = 0)$ has geometric genus* 0.
(2) We say that f is Minkowski if for every facet F of P: F admits a lattice Minkowski decomposition $F = \sum F_i$ into admissible A_n-triangles and there is a corresponding factorization $f_F = \prod f_{F_i}$.

It is clear that a Minkowski polynomial is Hodge–Tate at infinity. In 3 variables, we have classified all Minkowski polynomials and we are in the process of classifying all that are Hodge–Tate at infinity. We have verified that all Minkowski polynomials are extremal and we conjecture that the same is true for Hodge–Tate at infinity.

4. Classification of Fano manifolds

In 3 dimensions we have verified that all Minkowski polynomials are mirror to a Fano 3-fold. In this way, we recover 98 of the 105 deformation families of Fano 3-folds of Fano, Iskovskikh and Mori–Mukai. (The remaining 7 families have mirror Laurent polynomials whose Newton polytope is nonreflexive. For simplicity I do not discuss the issues with these.)

By contrast there exist Laurent polynomials that are Hodge–Tate at infinity and are not mirror to any Fano manifold.

We are in the process of re-building Maximilian Kreuzer's database of reflexive polytopes in 4 dimensions. We plan to classify Minkowski polynomials in 4 variables and draw a corresponding table of Picard–Fuchs operators and Fano 4-folds.

Holomorphic Morse inequalities and the Green-Griffiths-Lang conjecture

Jean-Pierre Demailly

I. In the first part of our work, we consider asymptotic cohomology functionals : for every holomorphic line bundle $L \to X$ on a compact complex manifold of dimension $n = \dim_{\mathbb{C}} X$, define

$$\widehat{h}^q(X,L) \;:=\; \limsup_{m\to+\infty} \frac{n!}{m^n} h^q(X, L^{\otimes m}),$$
$$\widehat{h}^{\le q}(X,L) \;:=\; \limsup_{m\to+\infty} \frac{n!}{m^n} \sum_{0\le j\le q} (-1)^{q-j} h^j(X, L^{\otimes m}).$$

In case X is projective algebraic, the functional $\widehat{h}^q(X,L)$ has been introduced in [8] and further studied in [6]. It is then known that $\widehat{h}^q(X,L)$ and $\widehat{h}^{\le q}(X,L)$ only depend on $c_1(L)$, extend continuously to the real Neron-Severi space $\mathrm{NS}_{\mathbb{R}}(X)$ and are birational invariants.

Theorem (1985, [2], "Holomorphic Morse inequalities"). *Take h to be an arbitrary smooth hermitian metric on L. One has the following inequalities:*

$$\widehat{h}^q(X,L) \quad\le\quad \inf_h \int_{X(L,h,q)} (-1)^q \Theta_{L,h}^n \qquad (\textit{weak form}), \tag{1}$$

$$\widehat{h}^{\le q}(X,L) \quad\le\quad \inf_h \int_{X(L,h,\le q)} (-1)^q \Theta_{L,h}^n \qquad (\textit{strong form}), \tag{2}$$

where $\Theta_{L,h} = -\frac{i}{2\pi}\partial\overline{\partial}\log h$ the curvature form of (L,h), $X(L,h,q)$ its q-index set (an open subset of X), and $X(L,h,\le q) = \bigcup_{0\le j\le q} X(L,h,j)$:

$$X(L,h,q) = \big\{x\in X\,;\ \textit{signature}(\Theta_{L,h}(x)) = (n-q,q)\big\}.$$

The proof of the above inequalities is based on the spectral theory of complex Laplace-Beltrami operators. For the converse equalities we can prove

Theorem (2010, [3], [4]). *Equality holds in (1) and (2) whenever X is projective algebraic and $q=0$ or $q=n$ or $n=\dim_{\mathbb{C}} X \le 2$.*

When dealing with the volume ($q=0$), one can assume that L is big, i.e. $\mathrm{Vol}(X) = \widehat{h}^0(X,L) > 0$. The main argument is then to use an approximate Zariski decomposition $\mu^*L \sim E + A$ where $\mu : \widehat{X} \to X$ is a modification, E a $\mathbb{Q}$-effective divisor and A a $\mathbb{Q}$-ample divisor. A minimizing sequence of metrics on μ^*L is then obtained by taking a metric with positive curvature $\omega = \Theta_{A,h_A}$ on A and a metric with very small trace $\Theta_{E,h_E}\wedge\omega_A^{n-1}$, thanks to the orthogality estimate of [1] which tells us that $E\cdot A^{n-1}\to 0$. The case $q=n$ follows by Serre duality. When $n=2$, the picture can be completed just by considering the Euler characteristic. It should be noticed that the holomorphic Morse inequalities are transcendental in nature. However, there exist algebraic counterparts, e.g. for the strong Morse inequalities, in the form of the following inequalities:

$$\widehat{h}^{\le q}(X,L) \le \inf_h \int_{X(L,h,\le q)} (-1)^q \Theta_{L,h}^n \le \inf_{\mu^*L\equiv A-B} \sum_{0\le j\le q} (-1)^{q-j}\binom{n}{j} A^{n-j}\cdot B^j,$$

where the infimum is taken over all modifications $\mu : \widehat{X}\to X$ and all decompositions of $c_1(\mu^*L)$ as a difference $A-B$ of nef $\mathbb{Q}$-divisors A, B on $\widehat{X}$. We know that the right hand inequality is not always an equality, but nevertheless hope that such an algebraic formula exists.

II. The second part consists of applying the machinery of holomorphic Morse inequalities to study entire curves $f:\mathbb{C}\to X$ drawn in a complex irreducible n-dimensional variety X. One of the main open problems is the following

Green-Griffiths-Lang conjecture. *Let X be a projective variety of general type. Then there should exist an algebraic subvariety $Y\subset X$ such that every non constant entire curve $f:\mathbb{C}\to X$ is contained in Y.*

Arithmetic complement. *Assume moreover that X is defined over some number field $\mathbb{K}$. Then the set $X(\mathbb{K})$ should be contained in $Y \cup F$ where F is a finite set.*

These statements would give a lot of information on the geometry of X, e.g. by constraining the locus of rational of elliptic curves, or more generally of all abelian subvarieties, as all "special" subvarieties of X should be contained in Y. On the other hand, the arithmetic statement would be a vast generalization of Falting's theorem on the Mordell conjecture.

It turns out that holomorphic Morse inequalities give a partial answer in an even more general situation. Consider the category of *directed varieties*, i.e. pairs (X, V) where V is a holomorphic linear subspace of the tangent space T_X. This includes of course relative situations $X \to S$ by taking $V = T_{X/S}$, as well as the case of foliations (but here we do not require integrability of V). One says that (X, V) is of general type if the canonical sheaf K_V is big. In the presence of singularities, K_V has to be defined appropriately, namely, if X is smooth, one takes K_V to be the image of $\Omega^r_X \to (i_{X\setminus \mathrm{Sing}(V)})_*\mathcal{O}(\det V^*)$ where $r = \operatorname{rank} V$.

Generalized Green-Griffiths-Lang conjecture. *Let (X, V) be a projective directed manifold such that the canonical sheaf K_V is big. Then there should exist an algebraic subvariety $Y \subset X$ such that every non constant entire curve $f : \mathbb{C} \to X$ tangent to V is contained in Y.*

Let J_kV be the space of k-jets of curves $f : (\mathbb{C}, 0) \to X$ tangent to V. One defines the sheaf $\mathcal{O}(E^{\mathrm{GG}}_{k,m}V^*)$ of jet differentials of order k and degree m to be the sheaf of holomorphic functions $P(z; \xi_1, \ldots, \xi_k)$ on J_kV which are homogeneous polynomials of degree m on the fibers of $J^kV \to X$ with respect to derivatives $\xi_s = f^{(s)}_\nabla(0) \in V$, $1 \le s \le k$ and any local holomorphic connection ∇. The degree m considered here is the weighted degree with respect to the natural $\mathbb{C}^*$ action on J^kV defined by $\lambda \cdot f(t) := f(\lambda t)$, i.e. by reparametrizing the curve with a homothetic change of variable. One of the major tool of the theory is the following result due to [7].

Fundamental vanishing theorem. *Let (X, V) be a directed projective variety, $f : (\mathbb{C}, T_\mathbb{C}) \to (X, V)$ an entire curve tangent to V and A an ample divisor. For every section $P \in H^0(X, E^{\mathrm{GG}}_{k,m}V^* \otimes \mathcal{O}(-A))$, one has $P(f\,; f', f'', \ldots, f^{(k)}) = 0$.*

Let $X_k := (J_kV \setminus \{0\})/\mathbb{C}^*$ be the projectivized k-jet bundle and $\mathcal{O}_{X_k}(1)$ the associated tautological sheaf. By construction $\mathcal{O}(E^{\mathrm{GG}}_{k,m}V^*) = (\pi_k)_*\mathcal{O}_{X_k}(m)$ where $\pi_k : X_k \to X$ and higher direct images vanish, hence

$$H^q(X, E^{\mathrm{GG}}_{k,m}V^* \otimes \mathcal{O}(-A)) \simeq H^q(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_k^*\mathcal{O}(-A))$$

The above vanishing theorem says that the k-jet locus $f_{[k]}(\mathbb{C})$ of an entire curve sits in the base locus $Z \subset X_k$ of $H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_k^*\mathcal{O}(-A))$. The expectation is that the base locus Z projects onto a proper algebraic variety $Y = \pi_k(Z) \subset X$, whereby proving the generalized GGL conjecture. Therefore, the problem is to understand the asymptotic behavior of $h^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_k^*\mathcal{O}(-A))$ in a situation where intermediate cohomology groups H^q always appear.

Main Theorem (2010, [5]). *Let (X,V) be a directed manifold, $F\to X$ a $\mathbb{Q}$-line bundle, (V,h) and (F,h_F) smooth hermitian structures on V and F. Define*

$$L_k=\mathcal{O}_{X_k}(1)\otimes\pi_k^*\mathcal{O}\Big(\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)F\Big),\qquad \eta=\Theta_{\det V^*,\det h^*}+\Theta_{F,h_F}.$$

Then for all $q\geq 0$ and all $m\gg k\gg 1$ with m sufficiently divisible, we have

$$h^q(X_k,\mathcal{O}(L_k^{\otimes m}))\ \leq\ \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\bigg(\int_{X(\eta,q)}(-1)^q\eta^n+O((\log k)^{-1})\bigg),$$

$$h^0(X_k,\mathcal{O}(L_k^{\otimes m}))\ \geq\ \frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\bigg(\int_{X(\eta,\leq 1)}\eta^n-O((\log k)^{-1})\bigg).$$

When K_V is big, we can take $\Theta_{\det V^*,\det h^*}$ to be positive, at least in the sense of currents, and the we can choose $F=-\varepsilon A$ to be slightly negative (given by some ample divisor A on X). This shows in particular that there are many sections in

$$H^0\Big(X_k,\mathcal{O}_{X_k}(m)\otimes\pi_k^*\mathcal{O}\Big(-\frac{m\varepsilon}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)A\Big),$$

hence that all entire curves $f:\mathbb{C}\to X$ tangent to V must satisfy the corresponding algebraic differential equations $P(f\,;f',f'',\ldots,f^{(k)})=0$. In fact, the error terms $O(...)$ can be estimated accurately. This leads e.g. to an effective lower bound

$$k\geq k_n=\exp\bigg(7.38\,n^{n+1/2}\Big(\frac{\sum d_j+1}{\sum d_j-n-s-1}\Big)^n\bigg)$$

if X is a complete intersection of multi-degree $(d_1,\ldots,d_s)$ in $\mathbb{P}^{n+s}$, $\sum d_j>n+s+1$.

The proof of the Main Theorem consists of computing the Morse integrals associated with the line bundle L_k. For this, we equip L_k with the metric induced on k-jets by the Finsler metric

$$\|f_{[k]}\|_{h_k}^2:=|f'\|_h^2+\varepsilon_2\|f_\nabla^{(2)}\|_h^{2/2}+\cdots+\varepsilon_s\|f_\nabla^{(s)}\|_h^{2/s}+\cdots+\varepsilon_k\|f_\nabla^{(k)}\|_h^{2/k},$$

where ∇ is an arbitrary C^∞ connection on V and $\varepsilon_k\ll\varepsilon_{k-1}\ll\ldots\ll\varepsilon_2\ll 1$ are small rescaling factors. Modulo error terms $=O(\varepsilon_s)$, an explicit calculation yields

$$\Theta_{\mathcal{O}_{X_k}(1),h_k}(z,[\xi_1,\ldots,\xi_k])=\omega_{\rm FS}(\xi)+\frac{i}{2\pi}\bigg(\sum_{1\leq s\leq k}\frac{x_s}{s}\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}u_{s\alpha}\overline{u}_{s\beta}\bigg)dz_i\wedge d\overline{z}_j$$

where $\omega_{\rm FS}(\xi)=\frac{i}{2\pi}\partial\overline\partial\log\sum_{1\leq s\leq k}\|\xi_s\|_h^{2/s}$ is the generalized Fubini-Study metric on the fibers of $X_k\to X$ (weighted projective spaces), and $(c_{ij\alpha\beta})$ is the curvature tensor of (V^*,h^*); here we use polar coordinates $\xi_s=(x_s)^{s/2}u_s$ with $x_s=\|\xi_s\|_h^{2/s}$ and $u_s\in S(V)$, the unit sphere bundle of (V,h). The projectivization leads to take $\sum x_s=\sum\|\xi_s\|_h^{2/s}=1$. This shows that the curvature form of $\Theta_{\mathcal{O}_{X_k}(1),h_k}$ is in fact an average of the curvature tensor of Θ_{V,h^*} evaluated at the sequence of derivatives $u_s=f_\nabla^{(s)}/\|f_\nabla^{(s)}\|_h\in S(V)$. The sum $\sum_{1\leq s\leq k}$ can be considered as a Monte-Carlo evaluation of the curvature tensor when the unit derivatives u_s are considered as random variables. Almost surely with respect to $(u_s) \in S(V)^k$ and (x_s) in the $(k-1)$-simplex Δ_{k-1}, the average is equivalent as $k \to +\infty$ to

$$\frac{1}{k}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)\int_{S(V)}\sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}\big(u_{s\alpha}\overline{u}_{s\beta}\,d\sigma(u)\big)\,dz_i\wedge d\overline{z}_j.$$

However the mean value of a hermitian form on $S(\mathbb{C}^r)$ is $\frac{1}{r}$ times its trace, hence the integral is equal to $\frac{1}{r}\sum_{i,j,\alpha} c_{ij\alpha\alpha}\,dz_i\wedge d\overline{z}_j = \frac{1}{r}\Theta_{\det V^*,\det h^*}(z)$. A more precise analysis of the standard deviation implies that the Morse integrals of $\Theta_{\mathcal{O}_{X_k}(1),h_k}$ over X_k (where $\dim X_k = n+kr-1$) behave asymptotically as $k \to +\infty$ as the product of the ω_{FS}-volume of the fibers with the Morse integrals of $\eta = \Theta_{\det V^*,\det h^*} + \Theta_{F,h_F}$ on X, when we take the tensor product with π_k^*F into account. The Main Theorem follows.

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Prym map and Gaussian maps

Paola Frediani

(joint work with Elisabetta Colombo)

Consider the moduli space $\mathcal{R}_g$ parametrizing isomorphism classes of pairs $[(C,A)]$, where C is a smooth curve of genus g and $A \in Pic^0(C)[2]$ is a torsion point of order 2, or equivalently isomorphism classes of unramified double coverings $\pi : \tilde{C} \to C$. Denote by

$$Pr_g : \mathcal{R}_g \to \mathcal{A}_{g-1}$$

the Prym map which associates to a point $[(C,A)] \in \mathcal{R}_g$ the isomorphism class of the component of the origin $P(C,A)$ of the kernel of the norm map $Nm_\pi : J\tilde{C} \to JC$, with its principle polarization.

We recall that the Prym map is generically an embedding for $g \geq 7$ ([9], [11]), hence there exists an open set $\mathcal{R}_g^0 \subset \mathcal{R}_g$ where Pr_g is an embedding and such that there exists the universal family $f : \mathcal{X} \to \mathcal{R}_g^0$. If $b \in \mathcal{R}_g^0$, we have $f^{-1}(b) = (C_b, A_b)$ where C_b is a smooth irreducible curve of genus g and $A_b \in Pic^0(C_b)[2]$ is a line bundle of order 2 on C_b. Denote by $\mathcal{P} \in Pic(\mathcal{X})$ the corresponding Prym bundle and by $\mathcal{F}^{Pr} := f_*(\omega_{\mathcal{X}/\mathcal{R}_g^0} \otimes \mathcal{P})$.

On $\mathcal{A}_{g-1}$ there is a metric which is induced by the unique (up to scalar) $Sp(2g-2,\mathbb{R})$-invariant metric on the Siegel space H_{g-1} of which $\mathcal{A}_{g-1}$ is a quotient by the action of $Sp(2g-2,\mathbb{Z})$. We will call this metric the Siegel metric.

We focus on the study of the second fundamental form of the Prym map with respect to the Siegel metric.

Consider the tangent bundle exact sequence of the Prym map

$$0 \to T_{\mathcal{R}_g^0} \to T_{\mathcal{A}_{g-1}|\mathcal{R}_g^0} \to \mathcal{N}_{\mathcal{R}_g^0/\mathcal{A}_{g-1}} \to 0 \tag{1}$$

Its dual becomes

$$0 \to \mathcal{I}_2 \xrightarrow{i} S^2 f_*(\omega_{\mathcal{X}/\mathcal{R}_g^0} \otimes \mathcal{P}) \xrightarrow{m} f_*(\omega_{\mathcal{X}/\mathcal{R}_g^0}^{\otimes 2}) \to 0 \tag{2}$$

where m is the multiplication map and we denote by $\mathcal{I}_2$ the conormal bundle $\mathcal{N}^*_{\mathcal{R}_g^0/\mathcal{A}_{g-1}}$. Recall that the second fundamental form of the exact sequence (2) is defined as follows

$$II : \mathcal{I}_2 \to f_*(\omega_{\mathcal{X}/\mathcal{R}_g^0}^{\otimes 2}) \otimes \Omega^1_{\mathcal{R}_g^0}, \quad II(s) = m(\nabla(i(s))),$$

where ∇ is the metric connection on $S^2 f_*(\omega_{\mathcal{X}/\mathcal{R}_g^0} \otimes \mathcal{P}) = \Omega^1_{\mathcal{A}_{g-1}|\mathcal{R}_g^0}$. At the point $[(C,A)] \in \mathcal{R}_g^0$ the exact sequence (2) becomes

$$0 \to I_2(K_C \otimes A) \to S^2 H^0(K_C \otimes A) \xrightarrow{m} H^0(K_C^{\otimes 2}) \to 0.$$

Hence, the second fundamental form II at $[(C,A)]$ can be seen as a map

$$II : I_2(K_C \otimes A) \to H^0(2K_C) \otimes H^0(2K_C).$$

We recall now the definition in local coordinates of the second Gaussian map $\gamma^2_{K_C \otimes A}$ of the Prym-canonical line bundle $K_C \otimes A$, with $A \in Pic^0(C)[2]$,

$$\gamma^2_{K_C \otimes A} : I_2(K_C \otimes A) \to H^0(K_C^{\otimes 4}).$$

Fix a basis $\{\omega_i\}$ of $H^0(K_C \otimes A)$ and write it in a local coordinate z as $\omega_i = f_i(z) dz \otimes l$, where l is a local generator of the line bundle A. For a quadric $Q \in I_2(K_C \otimes A)$ we have $Q = \sum_{i,j} a_{ij} \omega_i \otimes \omega_j$, where $a_{ij} = a_{ji}$ and $\sum_{i,j} a_{ij} f_i f_j \equiv 0$, so we have $\sum_{i,j} a_{ij} f_i' f_j \equiv 0$. The local expression of $\gamma^2_{K_C \otimes A}$ is

$$\gamma^2_{K_C \otimes A}(Q) = \sum_{i,j} a_{ij} f_i'' f_j (dz)^4 = -\sum_{i,j} a_{ij} f_i' f_j' (dz)^4.$$

The maps $\gamma^2_{K_C\otimes A}$ glue together to give a map of vector bundles on $\mathcal{R}^0_g$,

$$\gamma^2 : \mathcal{I}_2 \to f_*((\omega_{\mathcal{X}/\mathcal{R}^0_g} \otimes \mathcal{P})^{\otimes 2} \otimes \omega_{\mathcal{X}/\mathcal{R}^0_g}{}^{\otimes 2}) \cong f_*(\omega^{\otimes 4}_{\mathcal{X}/\mathcal{R}^0_g}), \tag{3}$$

where $\mathcal{I}_2$ is as in (2). We have the following

Theorem 1. *([5]) The diagram*

$$\begin{array}{ccc} I_2(K_C \otimes A) & \xrightarrow{II} & H^0(K_C^{\otimes 2}) \otimes H^0(K_C^{\otimes 2}) \\ \downarrow{\scriptstyle \gamma^2_{K_C\otimes A}} & \swarrow{\scriptstyle \mathbf{m}} & \\ H^0(K_C^{\otimes 4}) & & \end{array} \tag{4}$$

is commutative up to a constant.

This theorem is a generalization of an analogous result of [7] on the second fundamental form of the period map $P_g : \mathcal{M}_g \to \mathcal{A}_g$. In fact in [7] it is shown that the second fundamental form of the period map lifts the second gaussian map of the canonical line bundle, as stated in an unpublished paper by Green and Griffiths (cf. [10]). With this geometrical motivation, in [3] we investigated curvature properties of $\mathcal{M}_g$ endowed with the Siegel metric. In fact, we computed the holomorphic sectional curvature of $\mathcal{M}_g$ along the tangent directions given by the Schiffer variations in terms of the second Gaussian map. The previous result also allows us to generalize the results of [3] on the holomorphic sectional curvature of $\mathcal{R}_g$ with the Siegel metric induced by $\mathcal{A}_{g-1}$ via the Prym map.

More precisely, assume that $\{Q_i\}$ is an orthonormal basis of $I_2(K_C \otimes A)$, $\{\omega_i\}$ an orthonormal basis of of $H^0(K_C \otimes A)$ and choose a local coordinate z at P and a local generator l of A such that locally $\omega_i = f_i(z)dz \otimes l$. We have the following formula for the holomorphic sectional curvature of $\mathcal{R}^0_g$ along the tangent direction given by a Schiffer variation ξ_P at the point $P \in C$.

Theorem 2. *([5]) The holomorphic sectional curvature H of $T_{\mathcal{R}^0_g}$ at $[(C, A)] \in \mathcal{R}^0_g$ computed at the tangent vector ξ_P given by a Schiffer variation in P is given by:*

$$H(\xi_P) = -1 - \frac{1}{16\alpha_P^4\pi^2} \sum_i |\mu_A(Q_i)(P)|^2$$

where $\alpha_P = \sum_i |f_i(P)|^2$.

The preceding discussion also suggested that the second Gaussian map itself of the canonical and Prym-canonical line bundles could give interesting information on the geometry of the curves, hence its rank properties have been investigated in a series of papers (see [1], [2], [4], [6]).

For the canonical line bundle the surjectivity of the second Gaussian map for general curves of high genus was proved in [4] using curves on K3 surfaces, then the sharp result for genus ≥ 18 has been shown in [1] using degeneration to binary curves, i.e. stable curves which are the union of two rational curves meeting transversally at $g + 1$ points. These degeneration techniques can be generalized to prove the surjectivity of second gaussian maps $\gamma^2_{K_C\otimes A}$ and we have the following

Theorem 3. *([5]) For the general point $[(C, A)] \in \mathcal{R}_g$, with $g \geq 20$, the Prym-canonical Gaussian map $\gamma^2_{K_C \otimes A}$ is surjective.*

In particular, this shows that the locus of curves $[C, A] \in \mathcal{R}_g$ ($g \geq 20$) for which the map $\gamma^2_{K_C \otimes A}$ is not surjective is a proper subscheme of $\mathcal{R}_g$ and one observes that for $g = 20$ it is an effective divisor in $\mathcal{R}_{20}$ of which in [5] we computed the cohomology class both in $\mathcal{R}_{20}$ and in a partial compactification $\tilde{\mathcal{R}}_{20}$ following computations developed in [8].

As concerns degeneracy loci for the second Gaussian map $\gamma^2_{K_C}$ of the canonical line bundle, in [2] we have shown that for any hyperelliptic curve of genus $g \geq 3$, the rank of $\gamma^2_{K_C}$ is $2g - 5$ and for any trigonal curve of genus $g \geq 8$, $rank(\gamma^2_{K_C}) = 4g - 18$. Finally in [6] we proved that if a curve is a hyperplane section of an abelian surface, the map $\gamma^2_{K_C}$ has corank at least 2.

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Varieties of minimal rational tangents of codimension 1

Jun-Muk Hwang

Let X be a projective manifold of dimension n. For a component $\mathcal{K}$ of the space of rational curves on X and a point $x \in X$, denote by $\mathcal{K}_x$ the subscheme of $\mathcal{K}$ consisting of members of $\mathcal{K}$ passing through x. We say that $\mathcal{K}$ is a *minimal dominating component* if $\mathcal{K}_x$ is non-empty and complete for a general $x \in X$. By bend-and-break, the dimension of $\mathcal{K}_x$ is at most $n - 1$. The following result of [2] gives a complete description of the case when $\dim \mathcal{K}_x$ is maximal.

Cho-Miyaoka-Shepherd-Barron Theorem Let $\mathcal{K}$ be a minimal dominating family with $\dim \mathcal{K}_x = n-1$ for a general point $x \in X$. Then X is biregular to $\mathbf{P}^n$ and $\mathcal{K}$ is the family of lines on $\mathbf{P}^n$.

Our aim is to describe the case when $\mathcal{K}_x$ has next-to-maximal dimension: $\dim \mathcal{K}_x = n-2$. In this case, a simple biregular classification as the above theorem is hopelss. In fact, we have the following examples.

Example Let $Z \subset \mathbf{P}^{n-1}$ be a smooth hypersurface and regard $\mathbf{P}^{n-1}$ as a hyperplane in $\mathbf{P}^n$. Let X_Z be the blow-up of $\mathbf{P}^n$ along Z and $\mathcal{K}_Z$ be the minimal dominating family on X_Z containing the proper transforms of lines on $\mathbf{P}^n$ intersecting Z. Then $\dim \mathcal{K}_{Z,x} = n-2$. Furthermore, let G be a finite group acting on $\mathbf{P}^n$ preserving $Z \subset \mathbf{P}^{n-1}$. Then G acts on X_Z. Let X be a desingularization of X_Z/G and $\Psi : X_Z \to X_Z/G \dashrightarrow X$ be the natural rational map. For suitable choices of G and X, the images of members of $\mathcal{K}_Z$ under Ψ gives rise to a minimal dominating family $\mathcal{K}$ on X with $\mathcal{K}_x = n-2$ for a general $x \in X$.

What we can hope for is a birational classification of X with a description of $\mathcal{K}$.

To state our result, we need to recall the notion of variety of minimal rational tangents (VMRT) $\mathcal{C}_x \subset \mathbf{P}T_x(X)$. This is the set of tangent vectors to members of $\mathcal{K}_x$. [4] proved that the normalization of $\mathcal{C}_x$ is smooth and biregular to $\mathcal{K}_x$. A central conjecture is

Normality Conjecture The VMRT $\mathcal{C}_x$ at a general point $x \in X$ is normal.

When $\dim \mathcal{K}_x = n-2$, Normality Conjecture implies that the VMRT $\mathcal{C}_x \subset \mathbf{P}T_x(X)$ is a smooth hypersurface. Our main result is the following.

Main Theorem Let $\mathcal{K}$ be a minimal dominating family with $\dim \mathcal{K}_x = n-2$ for a general point $x \in X$. Assume Normality Conjecture, i.e., that the VMRT is a smooth hypersurface of degree d. If $d \geq 3$, there exist Z and G as in Example and a birational map $X_Z/G \dashrightarrow X$ such that the members of $\mathcal{K}$ correspond to the images of the members of $\mathcal{K}_Z$ under the rational map $\Psi : X_Z \to X_Z/G \dashrightarrow X$.

If the degree d of the VMRT is 1, then [1] showed that X is biratonal to a $\mathbf{P}^{n-1}$-bundle over a curve. If $d = 2$, an analogue of Main Theorem follows from [5]. Thus Main Theorem gives a satisfactory birational classification under the assumption of Normality Conjecture.

To explain the key idea of the proof of Main Theorem, we need to recall the following notion.

Definition Let M be the germ of $(\mathbf{C}^n, 0)$. A hypersurface $H \subset \mathbf{P}T(M)$ is said to be *locally flat* if there exists a coordinate system $(z_1, \ldots, z_n)$ on M such that H is given by

$$\sum_{i_1+\cdots+i_n=d} a_{i_1,\ldots,i_n}(dz_1)^{i_1}\cdots(dz_n)^{i_n} = 0$$

with constants $a_{i_1,\ldots,i_n} \in \mathbf{C}$.

The key step in the proof of Main Theorem is to show that when M is the germ of (X, x) for a general point $x \in X$, the hypersurface $H \subset \mathbf{P}T(M)$ given by the union of VMRT's at points of M is locally flat. For this we need to derive a general criterion of locally flat hypersurfaces and verify that the VMRT satisfies that criterion.

The general criterion for local flatness can be stated as follows. Assume that the natural projection $\pi : H \to M$ is a smooth morphism. For a point $\alpha \in H$ and $x = \pi(\alpha)$, we can associate the 1-dimensional cone $\hat{\alpha} \subset T_x(M)$ and the hyperplane $\hat{T}_\alpha \subset T_x(M)$ given by the affine tangent space of H_x at α. Using the differential $d\pi_\alpha : T_\alpha(H) \to T_x(M)$, define

$$V_\alpha := d\pi_\alpha^{-1}(0),\ J_\alpha := d\pi_\alpha^{-1}(\hat{\alpha}),\ P_\alpha := d\pi_\alpha^{-1}(\hat{T}_\alpha).$$

This gives distributions $V \subset J \subset P$ on H with natural short exact sequences

$$0 \to V \to J \to J/V \to 0$$

$$0 \to V \to P \to P/V \to 0.$$

Definition A line subbundle $F \subset J$ on H is a *P-splitting connection* if F splits the first exact sequence and there exists a vector subbundle $W \subset P$ splitting the second exact sequence with $W \cap J = F$.

Our criterion for local flatness is the following.

Local Flatness Test Let $H \subset \mathbf{P}T(M)$ be a hypersurface such that $\pi : H \to M$ is a smooth morphism of relative degree $d \geq 3$. If there exists a P-splitting connection $F \subset J$ with $[F, P] \subset P$, then H is locally flat.

The proof of the above test uses a mixture of Kodaira-Spencer theory and E. Cartan's method of equivalence, as developed in [3].

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Special McKay correspondence and exceptional collections

Akira Ishii
(joint work with Kazushi Ueda)

1. McKay correspondence

Let G be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ and $X = \mathbb{C}^2/G$ the quotient singularity. As classified by Klein, G is either a cyclic group, a binary dihedral group, or a binary polyhedral group. Let $\tau : Y \to X$ be the minimal resolution. Then the dual graph of the exceptional locus of τ is a Dynkin graph of type A, D or E. The *McKay correspondence* is a bijective correspondence between the vertices of the Dynkin graph and the non-trivial irreducible representations of G.

The observation by McKay was a mysterious coincidence of two graphs constructed independently. Gonzalez-Sprinberg and Verdier gave a geometric explanation of the correspondence by constructing a vector bundle E_ρ on Y for each representaion ρ of G such that

$$\{c_1(E_\rho) \mid \rho : \text{ non-trivial irreducible representations of } G\}$$

is dual to the set of irreducible exceptional curves. The correspondence is extended by Kapranov and Vasserot to the equivalence

$$D^b(\operatorname{coh} Y) \cong D^b(\operatorname{coh}^G(\mathbb{C}^2))$$

of the derived category of coherent sheaves on Y and the derived category of G-equivariant coherent sheaves on $\mathbb{C}^2$. In this form, the correspondence is generalized to the case of $G \subset \mathrm{SL}(3,\mathbb{C})$ by Bridgeland, King and Reid and to the case of abelian $G \subset \mathrm{SL}(n,\mathbb{C})$ (assuming the existence of a crepant resolution) by Kawamata.

2. Special McKay correspondence

Consider the quotient $X = \mathbb{C}^2/G$ by a finite small subgroup $G \subset \mathrm{GL}(2,\mathbb{C})$. Here a finite subgroup of $\mathrm{GL}(n,\mathbb{C})$ is said to be *small* if its action on $\mathbb{C}^n$ is free in codimension one. If G is not in $\mathrm{SL}(2,\mathbb{C})$, then it turns out that the number of irreducible exceptional curves is less than the number of non-trivial representations of G. So we have to choose *special* representations of G corresponding to irreducible exceptional curves.

Definition 1 (Wunram-Riemenschneider)**.** *A representation ρ of G is special if $H^1(E_\rho^\vee)$ vanishes.*

Note that this condition holds for any ρ if $G \subset \mathrm{SL}(2,\mathbb{C})$.

Theorem 1 (Wunram)**.** *The set*

$$\{c_1(E_\rho) \mid \rho : \textit{ non-trivial special irreducible representations of } G\}$$

is dual to the set of irreducible exceptional curves.

The relation between the derived categories is described as follows.

Theorem 2. *There is a fully faithful embedding*

$$D^b(\mathrm{coh}\, Y) \hookrightarrow D^b(\mathrm{coh}^G(\mathbb{C}^2))$$

whose essential image is generated by $\{\rho \otimes \mathcal{O}_{\mathbb{C}^2} \mid \rho\text{: special}\}$.

This follows from the argument of [2]. Note that we have negative discrepancies in the resolution τ, which should imply such an embedding as conjectured by Kawamata.

3. Semiorthogonal decompositions

Put $\mathcal{A} = D^b(\mathrm{coh}\, Y)$ and $\mathcal{B} = D^b(\mathrm{coh}^G(\mathbb{C}^2))$ in the above theorem. Note that the embedding $\mathcal{A} \hookrightarrow \mathcal{B}$ has a right adjoint. In such a situation, $\mathcal{B}$ has a *semiortogonal decomposition* $\mathcal{B} = \langle \mathcal{A}^\perp, \mathcal{A}\rangle$ where

$$\mathcal{A}^\perp := \{b \in \mathcal{B} \mid \forall a \in \mathcal{A},\, \mathrm{Hom}(a, b) = 0\}$$

is the *right orthogonal complement* of $\mathcal{A}$. This means that for every object $b \in \mathcal{B}$, there are $a \in A^\perp$ and $a' \in \mathcal{A}^\perp$ (unique up to isomorphism) such that b is an extension of the form

$$a \to b \to a' \to a[1].$$

It is known that the right semiorthogonal complement $\mathcal{A}^\perp$ is generated by skyscr-paer sheaves with non-special irreducible representations:

$$\mathcal{A} = \langle \mathcal{O}_0 \otimes \rho \mid \rho\text{: non-special irreducible representations}\rangle$$

where $\mathcal{O}_0$ is the structure sheaf of $\{0\} \subset \mathbb{C}^2$.

4. Exceptional collections

An object E in a triangulated category $\mathcal{T}$ is *exceptional* if it satisfies $\mathrm{Ext}^i(E, E) = 0$ for $i \neq 0$ and $\mathrm{Hom}(E, E) \cong \mathbb{C}$. A sequence $E_1, \ldots, E_n$ of exceptional objects is an *exceptional collection* if $\mathrm{Ext}^i(E_a, E_b) = 0$ for $a > b$ and for all i. If an exceptional collection $E_1, \ldots, E_n$ generate the triangulated category $\mathcal{T}$, then we say it is *full* and write $\mathcal{T} = \langle E_1, \ldots, E_n\rangle$.

An example of an exceptional object is a line bundle on a (-1)-curve on a surface. Actually, if $\tilde{S}$ is a one point blow up of a smooth surface S, then we have a semiorthogonal decomposition $D^b(\mathrm{coh}\, \tilde{S}) = \langle \mathcal{O}_E(-1), D^b(\mathrm{coh}\, S)\rangle$, where E is the exceptional curve and $\mathcal{O}_E(-1)$ is the line bundle of degree -1 on $E \cong \mathbb{P}^1$.

Now we go back to the special McKay correspondence.

Theorem 3 ([3]). *Let G be a finite small subgroup. Then the right orthogonal complement of $D^b(\mathrm{coh}\, Y) \hookrightarrow D^b(\mathrm{coh}^G \mathbb{C}^2)$ has a full exceptional collection. In other words, there is an exceptional collection $E_1, \ldots, E_n \in D^b(\mathrm{coh}^G \mathbb{C}^2)$ such that $D^b(\mathrm{coh}^G \mathbb{C}^2) = \langle E_1, \ldots, E_n, D^b(\mathrm{coh}\, Y)\rangle$.*

The cyclic group case follows from the arguments of Kawamata [5, 6]. In [3], we gave an explicit description of an exceptional collection in terms of continued fraction expansions. The general case is reduced to the cyclic case since if we consider $G_0 := G \cap \mathrm{SL}(2,\mathbb{C})$, then we have an equivalence for G_0 and G/G_0 is cyclic. See [3] for details.

5. Invertible polynomials

Let $A = (a_{ij})$ be an $n \times n$ integer matrix with $a_{ij} \geq 0$ and $\det A \neq 0$. A determines a polynomial

$$W := \sum_{i=1}^{n} x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} \in \mathbb{C}[x_1, \dots, x_n].$$

W is called an *invertible polynomial* if it has an isolated critical point at the origin [1]. The group

$$K = \{(\alpha_1, \dots, \alpha_n) \in (\mathbb{C}^*)^n \mid \alpha_1^{a_{11}} \dots \alpha_n^{a_{1n}} = \dots = \alpha_1^{a_{n1}} \dots \alpha_n^{a_{nn}}\}$$

acts on $W^{-1}(0)$ and we consider the quotient stack $\mathcal{X} := [W^{-1}(0) \setminus \{0\}/K]$. It is conjectured that $D^b(\operatorname{coh} \mathcal{X})$ has a full exceptional collection.

Corollary 1 ([3])**.** *It* $\dim \mathcal{X} = 2$*, then* $D^b(\operatorname{coh} \mathcal{X})$ *has a full exceptional collection.*

This follows from Theorem 3 and the rationality of the coarse moduli space of $\mathcal{X}$.

We say $\mathcal{X}$ is of Fermat type if A is a diagonal matrix.

Theorem 4 ([4])**.** *If* $\mathcal{X}$ *is of Fermat type, then* $D^b(\operatorname{coh} \mathcal{X})$ *has a full strong exceptional collection consistning of line bundles.*

Note that $\mathcal{X}$ is a finite quotient of a Fermat hypersurface in a weighted projective space. Although the derived category of the Fermat hypersurface is not generated by an exceptional collection, we can show that it is recovered from a dg-enhancement of $D^b(\operatorname{coh} \mathcal{X})$ together with the action of the character group of the finite group [4].

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Du Bois singularities for the working mathematician

Sándor J. Kovács

Let X be a smooth proper variety. Then the Hodge-to-de-Rham (a.k.a. Frölicher) spectral sequence degenerates at E_1 and hence the singular cohomology group $H^i(X,\mathbb{C})$ admits a Hodge filtration

$$H^i(X,\mathbb{C}) = F^0H^i(X,\mathbb{C}) \supseteq F^1H^i(X,\mathbb{C}) \supseteq \dots \tag{1}$$

and in particular there exists a natural surjective map

$$H^i(X,\mathbb{C}) \to Gr^0_F H^i(X,\mathbb{C}) \tag{2}$$

where

$$Gr^0_F H^i(X,\mathbb{C}) \simeq H^i(X,\mathcal{O}_X). \tag{3}$$

Deligne's theory of (mixed) Hodge stuctures implies that even if X is singular, there still exists a Hodge filtration and (2) remains true, but in general (3) fails.

Du Bois singularities were introduced by Steenbrink to identify the class of singularities for which (3) remains true as well. However, naturally, one does not define a class of singularities by properties of proper varieties. Singularities should be defined by local properties and Du Bois singularities are indeed defined locally.

It is known that rational singularities are Du Bois (conjectured by Steenbrink and proved in [Kov99]) and so are log canonical singularities (conjectured by Kollár and proved in [KK10]). These properties make Du Bois singularities very important in higher dimensional geometry, especially in moduli theory (see [Kol11] for more details on applications).

Unfortunately the definition of Du Bois singularities is rather technical. The most important and useful fact about them is the consequence of (2) and (3) that if X is a proper variety over $\mathbb{C}$ with Du Bois singularities, then the natural map

$$H^i(X,\mathbb{C}) \to H^i(X,\mathcal{O}_X) \tag{4}$$

is surjective. One could try to take this as a definition, but it would not lead to a good notion; singularities should be defined by local conditions. The reasonable approach is to keep Steenbrink's original defition, after all it has been proven to define a useful class. It does satisfy the first requirement above: it is defined locally. Once that is accepted, one might still wonder if proper varieties with Du Bois singularities could be characterized with a property that is close to requiring that (4) holds.

As we have already observed, simply requiring that (4) holds is likely to lead to a class of singularities that is too large. A more natural requirement is to ask that (3) holds. Clearly, (3) implies (4) by (2), so our goal requirement is indeed satisfied.

The definition [Ste83, (3.5)] of Du Bois singularities easily implies that if X has Du Bois singularities and $H \subset X$ is a general member of a basepoint-free linear system, then H has Du Bois singularities as well. Therefore it is reasonable that in trying to give an intuitive definition of Du Bois singularities, one may assume that the defining condition holds for the intersection of general members of a fixed basepoint-free linear system.

It turns out that this is actually enough to characterize Du Bois singularities. This result is not geared for applications, it is mainly interesting from a philosophical point of view. It says that the local definition not only achieves the desired property for proper varieties, but does it in an economical way: it does not allow more than it has to.

At the same time, a side-effect of this characterization is the fact that for the uninitiated reader this provides a relatively simple criterion without the use of derived categories or resolutions directly. In fact, one can make the condition numerical. This is a trivial translation of the "real" statement, but further emphasizes the simplicity of the criterion.

In order to do this we need to define some notation: Let X be a proper algebraic variety over $\mathbb{C}$ and consider Deligne's Hodge filtration $F^{\cdot}$ on $H^i(X,\mathbb{C})$ as in (1). Let

$$Gr^p_F H^i(X,\mathbb{C}) = F^p H^i(X,\mathbb{C}) \,/\, F^{p+1} H^i(X,\mathbb{C})$$

and

$$\mathsf{f}^{\mathsf{p},\mathsf{i}}(X) = \dim_{\mathbb{C}} Gr^p_F H^i(X,\mathbb{C}).$$

We will also use the notation

$$\mathsf{h}^{\mathsf{i}}(X,\mathcal{O}_X) = \dim_{\mathbb{C}} H^i(X,\mathcal{O}_X).$$

Recall that by the construction of the Hodge filtration and the degeneration of the Hodge-to-de-Rham spectral sequence at E_1, the natural surjective map from $H^i(X,\mathbb{C})$ factors through $H^i(X,\mathcal{O}_X)$:

$$H^i(X,\mathbb{C}) \to H^i(X,\mathcal{O}_X) \to Gr^0_F H^i(X,\mathbb{C}).$$

In particular, the natural morphism

$$H^i(X,\mathcal{O}_X) \to Gr^0_F H^i(X,\mathbb{C}) \tag{5}$$

is also surjective and hence

$$\mathsf{h}^{\mathsf{i}}(X,\mathcal{O}_X) \geq \mathsf{f}^{\mathsf{0},\mathsf{i}}(X). \tag{6}$$

The main result is the following.

Theorem 1. *Let X be a proper variety over $\mathbb{C}$ with a fixed basepoint-free linear system Δ. (For instance, X is projective with a fixed projective embedding). Then X has only Du Bois singularities if and only if $\mathsf{h}^{\mathsf{i}}(L,\mathcal{O}_L) \leq \mathsf{f}^{\mathsf{0},\mathsf{i}}(L)$ for $i > 0$ for any $L \subseteq X$ which is the intersection of general members of Δ.*

Corollary 1. *Let $X \subseteq \mathbb{P}^N$ be a projective variety over $\mathbb{C}$ with only isolated singularities. Then X has only Du Bois singularities if and only if $\mathsf{h}^{\mathsf{i}}(X,\mathcal{O}_X) \leq \mathsf{f}^{\mathsf{0},\mathsf{i}}(X)$ for $i > 0$.*

Observe that (5) combined with the condition $\mathsf{h}^{\mathsf{i}}(L,\mathcal{O}_L) \leq \mathsf{f}^{\mathsf{0},\mathsf{i}}(L)$ implies that $H^i(L,\mathcal{O}_L) \to Gr^0_F H^i(L,\mathbb{C})$ is an isomorphism and hence (1) follows from the following.

Theorem 2. *Let X be a proper variety over $\mathbb{C}$ with a fixed basepoint-free linear system Δ. Then X has only Du Bois singularities if and only if for all $i > 0$ and for any $L \subseteq X$, which is the intersection of general members of Δ, the natural map,*

$$\nu_i = \nu_i(L) : H^i(L, \mathcal{O}_L) \to Gr^0_F H^i(L, \mathbb{C})$$

given by Deligne's theory [Del71, Del74, Ste83, GNPP88] *is an isomorphism for all i.*

Remark 1. It is clear that if X has only Du Bois singularities then $\nu_i(L)$ is an isomorphism for all L. Therefore the interesting statement of the theorem is that the condition above implies that X has only Du Bois singularities.

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Simply connected surfaces of general type in arbitrary characteristic via deformation theory

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(joint work with Noboru Nakayama)

One of the interesting problems in the classification of algebraic surfaces is to find a new family of simply connected surfaces of general type with the geometric genus $p_g = 0$. Surfaces with $p_g = 0$ are interesting in view of Castelnuovo's criterion: *An irrational surface with irregularity $q = 0$ must have bigenus $P_2 \geq 1$.* Simply connected surfaces of general type with $p_g = 0$ are little known and the classification is still open.

When a surface is defined over the field of complex numbers, the only known simply connected minimal surfaces of general type with $p_g = 0$ were Barlow surfaces [1] until 2006. The canonical divisor of Barlow surfaces satisfies $K^2 = 1$.

Recently, the first named author and J. Park [3] have constructed a simply connected minimal surface of general type with $p_g = 0$ and $K^2 = 2$ by using a $\mathbb{Q}$-Gorenstein smoothing and Milnor fiber of a smoothing (or rational blow-down surgery). When a surface is defined over a field of positive characteristic, the existence of algebraically simply connected minimal surface of general type with $p_g = 0$ is known only for some special characteristics.

In this talk, we shall construct such a surface of general type defined over an algebraically closed field of any characteristic applying the construction given in [3]. The following is our main result [2]:

Theorem. *For any algebraically closed field k and for any integer* $1 \leq K^2 \leq 4$*, there exists an algebraically simply connected minimal surface S of general type over k with* $p_g(S) = q(S) = \dim H^2(S, \Theta_{S/k}) = 0$ *and* $K_S^2 = K^2$ *except* $(\mathrm{char}(k), K^2) = (2, 4)$*, where* $\Theta_{S/k}$ *denotes the tangent sheaf.*

The construction in [3] is as follows in the case of $K^2 = 2$: First, we consider a special pencil of cubics in $\mathbb{P}^2$ and blow up many times to get a projective surface M ($\tilde{Z}$ in [3]) which contains a disjoint union of five linear chains of smooth rational curves representing the resolution graphs of special quotient singularities called of class T. Then, we contract these linear chains of rational curves from the surface M to produce a projective surface X with five special quotient singularities of class T and with $K_X^2 = 2$. One can prove the existence of a global $\mathbb{Q}$-Gorenstein smoothing of the singular surface X, in which a general fiber X_t of the $\mathbb{Q}$-Gorenstein smoothing is a simply connected minimal surface of general type with $p_g = 0$ and $K^2 = 2$. This method works to other types of rational elliptic surfaces which are used to construct a simply connected minimal surface of general type with $p_g = 0$ and with $K^2 = 1$, 3, or 4. We shall show that this construction of singular surface X also works in positive characteristic, but several key parts in the proof to show the existence of a global $\mathbb{Q}$-Gorenstein smoothing should be modified.

Over the field of complex numbers, the existence of a local $\mathbb{Q}$-Gorenstein smoothing of a singularity of class T is given by the index-one cover. The key idea in [3] to show the vanishing of the obstruction space for a global $\mathbb{Q}$-Gorenstein smoothing is to use the lifting property of derivations of normal surface to its minimal resolutions, the tautness of the quotient singularities, and the special configurations of resolution graphs of singular points.

In characteristic 0, the tautness holds for the quotient singularities, i.e, the minimal resolution graph of a quotient singularity determines the type of singularity. It is known in characteristic 0 that the tautness is equivalent to $H^1(\Theta_D) = 0$ for any "sufficiently large" effective divisor D supported on the exceptional divisors on its minimal resolution. However, the tautness do not hold in characteristic $p > 0$ in general. Indeed, we have some examples of rational double points when $p = 2$, 3, and 5, by Artin's classification. The lifting property of derivations of normal surface to its minimal resolution exists in characteristic 0, but this is not true in characteristic $p > 0$.

However, in our constructions, we have only two-dimensional toric singularities by contracting linear chains of smooth rational curves. We prove that the singularity obtained by contracting a linear chain of smooth rational curves is a two-dimensional toric singularity and it satisfies the tautness. Moreover, it turns out that the lifting property of derivations mentioned above is not so important for proving the vanishing of the obstruction space. We introduce the notion of toric surface singularity of class T, and construct a so-called $\mathbb{Q}$-Gorenstein smoothing explicitly by using toric description. The vanishing condition $H^2(X, \Theta_{X/k}) = 0$ for an algebraic k-variety X with only isolated singularities implies the morphism $\mathrm{Def}_X \to \mathrm{Def}_X^{(\mathrm{loc})}$ between the global and local the deformation functors of X is smooth in the sense of Schlessinger. An algebraization result is added which plays an important role when we construct an algebraic deformation. The proof uses Artin's theory of algebraization.

We shall construct a deformation of a normal projective surface X with toric singularities of class T assuming some extra conditions. As a consequence, we have a so-called $\mathbb{Q}$-Gorenstein smoothing not only over the base field k but also over a complete discrete valuation ring with the residue field k. By the smoothing over the discrete valuation ring and the Grothendieck specialization theorem, the algebraic simply connectedness of the smooth fiber is reduced to that of a smooth fiber of a $\mathbb{Q}$-Gorenstein smoothing of a reduction of X of our construction to the complex number field. Note that the simply connectedness in case $k = \mathbb{C}$ has been proved by using Milnor fiber (or rational blow-down) and by applying van-Kampen's theorem on the minimal resolution of X.

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On the K-stability and log-K-stability in the Kähler-Einstein problem

Chi Li

Let X be a Fano manifold, i.e. K_X^{-1} is ample. A basic problem in Kähler geometry is to determine whether there exists a Kähler-Einstein metric on X, i.e. whether there exists a Kähler metric ω_{KE} in the Kähler class $c_1(X)$ satisfying the equation:

$$\mathrm{Ric}(\omega_{KE}) = \omega_{KE}$$

This is a variational problem. Futaki [4] found an important invariant (now known as Futaki invariant) as the obstruction to this problem. Then Mabuchi [9] defined K-energy functional by integrating this invariant:

$$\nu_\omega(\omega_\phi) = -\int_0^1 dt \int_X (S(\omega_{\phi_t}) - \underline{S})\dot{\phi}_t \omega_\phi^n$$

The minimizer of K-energy is the Kähler-Einstein metric. Tian [12] proved that there is a Kähler-Einstein metric if and only if the K-energy is proper on the space of all Kähler metrics in $c_1(X)$. So the problem is how to test the properness of K-energy.

Tian developed a program to reduce this infinite dimensional problem to finite dimensional problems. More precisely, he proved in [11] that the space of Kähler metrics (denoted by $\mathcal{H}$) in a fixed Kähler class can be approximated by a sequence of spaces (denoted by $\mathcal{H}_k$) consisting of Bergman metrics.

The latter spaces are finite dimensional symmetric spaces. Tian ([12]) introduced K-stability condition using generalized Futaki invariant for testing the properness of K-energy on these finite dimensional spaces. Later Donaldson [2] reformulated it using algebraically defined Futaki invariant. We have the following folklore conjecture.

Conjecture 1 (Yau-Tian-Donaldson conjecture)**.** *Let (X, L) be a polarized manifold. Then there is a constant scalar curvature Kähler metric in $c_1(L)$ if and only if (X, L) is K-stable.*

In the Kähler-Einstein case, this conjecture can be explained by the following diagram:

Existence of Kähler-Einstein metric	$\Longleftrightarrow$	K-stable
$\Uparrow$ $\Downarrow$		$\Uparrow$ $\Downarrow$
Properness of K-energy on $\mathcal{H}$	$\Longleftrightarrow$	properness of K-energy on $\mathcal{H}_k$

For the left vertical equivalence, see [12]. For the right equivalence, see [12] and [10].

For the horizontal equivalence, the direction of 'existence ⇒ K-stable' is easier and was proved by Tian in [12]. The other direction is much more difficult and is still open. To prove it, one needs the uniform approximation of Kähler metrics in a minimizing sequence.

One of minimizing methods is the continuity method. That is, we fix a reference metric ω_0 and consider a family of equations with parameter t:

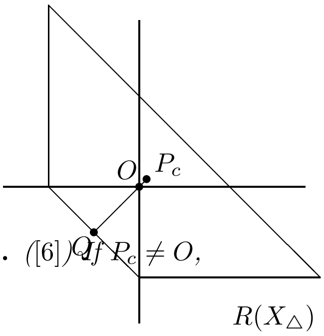
$$Ric(\omega) = t\omega + (1-t)\omega_0 \tag{1}$$

We can define

$$\begin{aligned} R(X) &= \sup\{t : \exists \text{ a Kähler metric } \omega \in c_1(X) \text{ such that } Ric(\omega) > t\omega\} \\ &= \sup\{s : (1) \text{ is solvable for } t \in [0, s]\} \end{aligned}$$

We get much information about continuity method for toric Fano manifolds. A toric Fano manifold $X_\triangle$ is determined by a reflexive lattice polytope $\triangle$.

Such polytope $\triangle$ contains the origin $O \in \mathbb{R}^n$. We denote the barycenter of $\triangle$ by P_c. If $P_c \neq O$, the ray $P_c + \mathbb{R}_{\geq 0} \cdot \overrightarrow{P_c O}$ intersects the boundary $\partial\triangle$ at point Q.



Theorem 1. *([6]) If $P_c \neq O$,*

$$R(X_\triangle) = \frac{|\overline{OQ}|}{|\overline{P_cQ}|}$$

Here $|\overline{OQ}|$, $|\overline{P_cQ}|$ are lengths of line segments $\overline{OQ}$ and $\overline{P_cQ}$. In other words,

$$Q = -\frac{R(X_\triangle)}{1 - R(X_\triangle)} P_c \in \partial\triangle$$

If $P_c = O$, then there is Kähler-Einstein metric on $X_\triangle$ and $R(X_\triangle) = 1$.

We can also study the limit behavior of the minimizing sequence along the continuity method on toric Fano manifolds. The result is compatible with Cheeger-Colding-Tian's theory [1] on compactness of Kähler-Einstein manifolds. In particular, the limit metric has conic type singulary whose information can be read out from the geometry of the polytope. For details, see [7].

There is another continuity method, which is via Kähler-Einstein metrics with conic singularities. This is equivalent to solving the following family of equations with parameter β:

$$Ric(\omega) = \beta\omega + (1 - \beta)\{D\} \tag{2}$$

where $D \in |-K_X|$ is a smooth divisor.

One can extend the theory in smooth case to the conic case. We can define the logarithmic Futaki invariant after Donaldson (cf. [3], [8]). Then we can integrate the log-Futaki invariant to get log-K-energy [8]. If we assume the log-K-energy is proper, then there exists conic Kähler-Einstein metric. (cf. [5]). We can also define log-K-stability and get the Yau-Tian-Donaldson conjecture in conic setting. See the discussion in [8].

Conjecture 2 (Logarithmic version of Tian-Yau-Donaldson conjecture)**.** *There is a constant scalar curvature conic Kähler metric on (X, Y) if and only if (X, Y) is log-K-stable.*

Donaldson made a conjecture relating the two continuity methods.

Conjecture 3. [3] *There is a cone-singularity solution* ω_β *to* (2) *for any parameter* $\beta \in (0, R(X))$. *If* $R(X) < 1$, *there is no solution for parameter* $\beta \in (R(X), 1)$.

The case of conic Riemann surface was known by the work of Troyanov, McOwen, Thurston, Luo-Tian, etc. Some evidence is provided by toric Fano manifold:

Theorem 2. [8] *Let* $X_\triangle$ *be a toric Fano variety with a* $(\mathbb{C}^*)^n$ *action. Let* Y *be a general hyperplane section of* $X_\triangle$. *When* $\beta < R(X_\triangle)$, $(X_\triangle, \beta Y)$ *is log-K-stable along any 1 parameter subgroup in* $(\mathbb{C}^*)^n$. *When* $\beta = R(X_\triangle)$, $(X_\triangle, \beta Y)$ *is semi-log-K-stable along any 1 parameter subgroup in* $(\mathbb{C}^*)^n$ *and there is a 1 parameter subgroup in* $(\mathbb{C}^*)^n$ *which has vanishing log-Futaki invariant. When* $\beta > R(X_\triangle)$, $(X_\triangle, \beta Y)$ *is not log-K-stable.*

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Rational Curves on K3 surfaces

Jun Li

(joint work with Christian Liedtke)

This talk addresses the function field version of Lang's conjectrue.

Conjectrue. *Every projective K3 surface over an algebraically closed field contains infinitely many integral rational curves.*

Bogomolov and Mumford showed that every complex projective K3 surface contains at least one rational curve. Next, Chen established existence of infinitely many rational curves on very general complex projective K3 surfaces. Since then, infinitely many rational curves have been established for elliptic K3 surfaces, and

K3 surfaces with infinite automorphism groups. In particular, this includes all K3 surfaces with $\rho \geq 5$, as well as "most" K3 surfaces with $\rho \geq 3$.

The recent breakthrough took place by the work [1].

Theorem.(Bogomolov-Tschinkel-Hassett) *A degree two projective K3 surface over a char. zero algebraic closed field and of Picard rank one contains infinitely many integral rational curves.*

Furthering their technique, Liedtke and myself proved

Theorem.([2]) *A projective K3 surface over an algebraic closed field k of char. zero with odd Picard rank contains infinitely many integral rational curves.*

Our techniques also yield the following result in positive characteristic:

Theorem.([2]) *A non-supersingular K3 surface with odd Picard rank over an algebraically closed field of char. $p \geq 5$ contains infinitely many integral rational curves.*

Here is a sketch of our proof. The initial part of the proof follows that of [1]. First, by a standard technique in algebraic geometry, we only need to prove the theorem for a K3 surface X defined over a number field L. For such an X/L, we pick a finite place $\mathfrak{p}$ and form its reduction to finite characteristic $X_\mathfrak{p}$. Then by a known result, the Picard rank of $(X_\mathfrak{p})_{\bar{\mathbb{F}}_p}$ is always even. Thus assume $X_{\bar{L}}$ has odd Picard rank, we can find a rational curve $C_\mathfrak{p} \subset (X_\mathfrak{p})_{\bar{\mathbb{F}}_p}$ that can not be lifted to $X_{\bar{\mathbb{Q}}}$. In [1], Bogomolov-Tschinkel-Hassett used the special geometry of degree two K3 surface and a lifting theorem of genus zero stable maps to show that there is another rational curve $C'_\mathfrak{p}$ in $(X_\mathfrak{p})_{\bar{\mathbb{F}}_p}$ such that the union $C_\mathfrak{p} + C'_\mathfrak{p}$ lies in a multiple of the polarization of X and can be represented by an isolated genus zero stable map, thus can be lifted to $X_{\bar{\mathbb{Q}}}$. A further argument shows that such lifts has unbounded degrees, when $\mathfrak{p}$ runs through an infinitely many places of L. This proves that $X_{\bar{\mathbb{Q}}}$ contains infinitely many integral rational curves.

For a K3 defined over a number field L with odd Picard rank $\rho(X_{\bar{L}})$, the same argument works except the existence of an irreducible $C'_\mathfrak{p}$ so that $C_\mathfrak{p} + C'_\mathfrak{p}$ lies in a multiple of the polarization of X. Instead, we can find a union of rational curves $D_\mathfrak{p}$ so that $C_\mathfrak{p} + D_\mathfrak{p}$ lies in a multiple of the polarization of X.

Our new input is to use "rigidifier" to achieve the lifting of $C_\mathfrak{p} + D_\mathfrak{p}$. Suppose we can a nodal rational curve $R_\mathfrak{p}$ in a multiple of the polarization, an elementary argument shows that

$$C_\mathfrak{p} + D_\mathfrak{p} + mR_\mathfrak{p}$$

can be represented by a rigid genus zero stable map. Because the stable map is rigid, it can be lifted fo $X_{\bar{\mathbb{Q}}}$; and if this can be done for infinitely many places $\mathfrak{p}$, we can find arbitrary high degree rational curves in $X_{\bar{\mathbb{Q}}}$. This would prove the main theorem.

In general, applying Chen's existence of nodal rational curve result, we know that there is a Zariski open subset in the moduli space of K3's so that every K3 in this open subset contains rational nodal curves in its multiple of polarization. Suppose the K3 surface X does not lie in this open subset, we deform $X_{\mathfrak{p}}$ and $C_{\mathfrak{p}} + D_{\mathfrak{p}}$ to where such $R_{\mathfrak{p}}$ exists. A routine argument guarantees that we find the desired lift of stable maps to $X_{\bar{\mathbb{Q}}}$, proving the existence theorem.

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Moduli spaces of curves with a fixed group of automorphism

Michael Lönne

(joint work with Fabrizio Catanese, Fabio Perroni)

Given a finite group G we consider smooth complex projective curves C of genus g together with an effective G-action. Such a curve can be given by a injective map ρ from G to the mapping class group Map_g and a point in the Teichmüller space T_g which is stabilised by the image.

Thus such curves come in natural families:

Theorem 1 ([1])**.** *Given $\rho : G \to Map_g$ injective the following Teichmüller space is a connected complex manifold homeomorphic to a ball*

$$T_{g,G,\rho} \quad = \quad \big\{ [C] \,|\, \rho(G) \text{ stabilizes } [C] \big\}.$$

In particular the irreducible components $M_{g,G.\rho} = T_{g,G,\rho} \big/ Map_g$ are in bijection with the orbit space of pairs (G, ρ) for the combined action of $Aut(G)$ on G and the conjugation on Map_g.

The aim of our project is to detect irreducible components by suitable geometric invariants. Such are obtained considering the quotient $C' = C/G$ with branch locus $B = \{p_1, \ldots, p_k\}$ and the factorization

$$\begin{array}{ccccc} C & \longrightarrow & C/H & & \\ & & \downarrow & & \\ & & C/G & = & C' \end{array}$$

where H is the normal subgroup of G generated by all stabilizers and the vertical map is an etale $G' = G/H$ Galois cover.

The mapping class group $Map_{g',k} := Map(C' - B)$ acts naturally on

$$\pi_1(C' - B) \;=\; \pi_{g',k} \;=\; \langle \alpha_1, \beta_1, \ldots, \alpha_{g'}, \beta_{g'}, \gamma_1, \ldots, \gamma_k \,|\, \gamma_1 \cdots \gamma_k \prod [\alpha_i, \beta_i] \,\rangle$$

The quotient map $C \to C'$ restricts to a regular topological G-cover over $C' - B$ and thus determines a surjection of $\pi_{g',k}$ onto G.

Conversely (G,ρ) is determined by such a surjection μ, via the diagram

$$\begin{array}{ccccccc} & & & & \pi_{g,k} & & \\ & & & & \downarrow & \searrow & \\ 1 & \to & \pi_g & \to & \pi^{orb}_{g,k} & \twoheadrightarrow & G \end{array}$$

where the kernel of the vertical map is normally generated by $\gamma_i^{ord(\mu(\gamma_i))}$.

In fact lifts of $g \in G$ define unambiguously $\rho : G \to Out(\pi_g)$ with image in $Map_g \subset Out(\pi_g)$. We thus get a bijection

$$\{(G,\rho)\} \big/ Aut(G) \times Map_g \quad \stackrel{1:1}{\longleftrightarrow} \quad \{G, \mu : \pi_{g,k} \twoheadrightarrow G\} \big/ Aut(G) \times Map_{g',k}$$

Definition. Consider the following datum

(1) the multiset ν of conjugacy classes of G

$$\nu(\mathcal{C}) \quad = \quad \#\{ i \mid \mu(\gamma_i) \in \mathcal{C} \}$$

(2) $\mu' : \pi_{g'} \twoheadrightarrow G'$ yields a homotopy class of classifying maps $C' \to K(G,1) = BG$ inducing

$$H_2(C',\mathbb{Z}) \to H_2(G',\mathbb{Z}), \qquad [\mu] := \mu'[C'].$$

Both parts are Map_g-invariant and we call the $Aut(G)$-orbits the *numerical types.*

These invariants are sufficient to detect the connected components in the case of abelian G by a result of Edmonds [5, 6]. Most intriguingly there is an asymptotic result in this direction independent of the group G:

Theorem 2 ([4])**.** *If G acts freely and g' is sufficiently large, then the irreducible components of moduli spaces of G-curves are in bijection with all possible numerical types i.e. with*

$$H_2(G;\mathbb{Z}) \big/ Aut(G).$$

Question: Are there asymptotic results for G-actions which are not free?

To get some clue on what to expect we studied the case of $G = D_n$ more closely. Our result is the following with the genus $g = 0$ case already published [2].

Theorem 3 ([2, 3])**.** *The irreducible components of moduli spaces of D_n-curves are detected by the numerical type if*

(1) *$g' = 0$,*
(2) *$C \to C\big/D_n$ is etale,*
(3) *n is odd,*
(4) *the stabiliser of at least one point is generated by an involution.*

In the remaining cases H is a non-trivial subgroup of rotations of D_n, n even, and the rotation by π does not generate the stabliser of some point.
Then there are two $Map_{g',k}$-orbits and the numerical type detects the irreducible component if and only if there is an automorphism of D_n which interchanges the two orbits.

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Birational automorphism groups and the movable cone theorem for Calabi-Yau manifolds of Wehler type

Keiji Oguiso

In his paper [We], Wehler observed the following pretty:

Theorem A. *Let S be a generic surface of multi-degree $(2, 2, 2)$ in $\mathbf{P}^1_1 \times \mathbf{P}^1_2 \times \mathbf{P}^1_3$. Let ι_k ($k = 1, 2, 3$) be the covering involution of the natural projection $S \to \mathbf{P}^1_i \times \mathbf{P}^1_j$ with $\{i, j, k\} = 1, 2, 3$. Then, S is a K3 surface with*

$$\mathrm{Aut}\,(S) = \langle \iota_1, \iota_2, \iota_3 \rangle \simeq \mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2 \ ,$$

the universal Coxeter group of rank 3*.*

Wehler's K3 surfaces and their variants sometimes appear in the study of complex dynamics and arithmetic dynamics as handy, concrete examples.

In my talk, I presented two generalizations of Wehler's K3 surfaces to higher dimensional Calabi-Yau manifolds ([Og]).

1st generalizatin. Let $n \geq 3$ and let X be a generic hypersurface of multi-degree $(2, \dots, 2)$ in the product $\mathbf{P}^1_1 \times \mathbf{P}^1_2 \times \dots \times \mathbf{P}^1_{n+1}$ of $n+1$ $\mathbf{P}^1_k \simeq \mathbf{P}^1$. Then X is an n-dimensional Calabi-Yau manifold. As in Wehler's K3 surface, we have $n+1$ natural projections

$$\pi_k : S \to \mathbf{P}^1_1 \times \dots \times \mathbf{P}^1_{k-1} \times \mathbf{P}^1_{k+1} \times \dots \times \mathbf{P}^1_{n+1} \ .$$

Then π_k is of degree 2. We denote the corresponding covering involution by ι_k. Then, unlike to Wehler's case, X has no non-trivial automorphism but very rich birational automorphisms:

Theorem I.
(1)$\mathrm{Aut}\,(X) = \{1\}$, *while* $\mathrm{Bir}\,(X) = \langle \iota_1\,,\, \iota_2\,,\, \dots\,,\, \iota_{n+1} \rangle \simeq \mathbf{Z}_2 * \mathbf{Z}_2 * \dots * \mathbf{Z}_2$*, the universal Coxeter group of rank* $n+1$*.*

(2) *The abstract version of the Morrison-Kawamata movable cone conjecture (*[Ka1]*) is true for X, i.e., the natural action of* $\mathrm{Bir}\,(X)$ *on the movable effective cone $\mathcal{M}^e(X)$ has a finite rational polyhedral cone as its fundamental domain.*

Here $\overline{\mathcal{M}}(X)$ is the closure of the convex cone generated by the movable divisor classes in $\mathrm{NS}(X) \otimes \mathbf{R}$ and $\mathcal{B}^e(X)$ is the convex cone generated by effective divisor classes in $\mathrm{NS}(X) \otimes \mathbf{R}$.

Our proof of (1) is based on the result of Kawamata that any birational map between minimal models can be decomposed into flops ([Ka2]) and geometric representations of Coxeter groups ([Hum]). In our proof of (2), we uses a result of Birkar, Cascini, Hacon and McKernan ([BCHM]) for big divisors, and some special geometry of X for non-big divisors.

2nd generalization. It is also interesting to find examples with rich biregular automorphisms. In our previous work, Schröer and I ([OS]) found the following new series of Calabi-Yau manifolds of any even dimension:

Theorem B. *Let S be an Enriques surface and $\mathrm{Hilb}^n(S)$ be the Hilbert scheme of n points on S, where $n \geq 2$. Let $\pi : \widetilde{\mathrm{Hilb}^n(S)} \to \mathrm{Hilb}^n(S)$ be the universal cover of $\mathrm{Hilb}^n(S)$. Then π is of degree 2 and $\widetilde{\mathrm{Hilb}^n(S)}$ is a Calabi-Yau manifold of dimension $2n$.*

Theorem II. *Let S be a generic Enriques surface. Then, for each $n \geq 2$, the biregular automorphism group of $\widetilde{\mathrm{Hilb}^n(S)}$ contains the universal Coxeter group of rank 3 as its subgroup, i.e., $\mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2 < \mathrm{Aut}\,(\widetilde{\mathrm{Hilb}^n(S)})$.*

There are many ways to see $\mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2 < \mathrm{Aut}\,(S)$ for a generic Enriques surface S. Then we have $\mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2 < \mathrm{Aut}\,(\mathrm{Hilb}^n(S))$. Essential point of the proof is the faithful lifting of the automorphism subgroup $\mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2$ of $\mathrm{Hilb}^n(S)$ to $\widetilde{\mathrm{Hilb}^n(S)}$.

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Prym varieties of triple coverings

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(joint work with Herbert Lange)

Let $f: Y \to X$ be a covering of smooth projective curves. The *Prym variety* $P(f)$ associated to this covering is, by definition, the connected component containing zero, of the kernel of the norm map Nm_f of the Jacobian JY onto the Jacobian JX. We say that $P(f)$ is a *principally polarized Prym variety* if the canonical principal polarization of JY restricts to a multiple of a principal polarization on $P(f)$. In [3, Proposition 12.3.3] it is claimed that $P(f)$ is a principally polarized Prym variety of dimension at least 2 if and only if f is a double covering ramified at most at 2 points and X is of genus ≥ 3. In this classification one case is missing, namely the Prym variety $P(f)$ associated to a non-cyclic étale triple covering f of a curve X of genus 2, is principally polarized of dimension 2. We carry out the study of these Prym varieties and their associated moduli spaces.

Given a non-cyclic, étale triple covering $f: Y \to X$ over a smooth projective genus 2 curve X, one can consider the Galois closure of f, $h: Z \to X$. The curve Z is of genus 7 admitting, not only the action of the symmetric group S_3 but also the action of the dihedral group D_6. One can prove that the composition of h with the hyperelliptic covering $X \to \mathbb{P}^1$ is a Galois covering $Z \to \mathbb{P}^1$ with Galois group D_6. The key ingredient in the description of the Prym variety $P(f)$ is the following

Theorem 1. *Any non-cyclic étale 3-fold covering Y of a curve of genus 2 is hyperelliptic.*

The proof of the above theorem is done by looking at the full tower of curves corresponding to the subgroups of D_6. The genera of all the curves in the tower can be computed and one finds an explicit degree two map $Y \to \mathbb{P}^1$. This allow us to show that $P(f)$ is the Jacobian of the smooth irreducible genus 2 curve Ξ, where Ξ gives a principal polarization in $P(f)$.
Let $\mathcal{R}_{2,3}^{nc}$ denote the moduli space of étale non-cyclic triple coverings $f: Y \to X$ over a curve X of genus 2. Let $\mathcal{A}_2$ be the moduli space of principal polarized abelian surfaces and $\mathcal{J}_2$ the locus of Jacobians in $\mathcal{A}_2$. Consider the Prym map $Pr: \mathcal{R}_{2,3}^{nc} \longrightarrow \mathcal{A}_2$, $[f: Y \to X] \mapsto P(f)$. One of the main results in [4] is the following

Theorem 2. *The Prym map $Pr: \mathcal{R}_{2,3}^{nc} \to \mathcal{J}_2$ is finite of degree 10 onto its image.*

The proof gives a constructive way of recovering the fiber of the Prym map over a general curve Ξ of genus 2, in terms of the choice of a partition (in two sets of three elements) of the Weiestrass points of Ξ. As a consequence of this construction we obtain

Proposition 1. *The moduli space $\mathcal{R}_{2,3}^{nc}$ is rational.*

The Prym map can be extended to a some family of coverings over nodal curves, in such a way that the extended map is proper onto $\mathcal{A}_2$. For this it turns out to be convenient to shift the point of view slightly. Taking the Galois closure gives a bijection between the set of connected non-cyclic étale f covers of above, and the set of étale Galois covers $h : Z \to X$, with Galois group the symmetric group S_3. Hence, if we denote by ${}_{S_3}\mathcal{M}_2$ the moduli space of étale Galois covers of smooth curves of genus 2 with Galois group S_3 as constructed for example in [2, Theorem 17.2.11], we obtain a morphism $Pr : {}_{S_3}\mathcal{M}_2 \to \mathcal{J}_2$, also call Prym map. Then we use the compactification ${}_{S_3}\overline{\mathcal{M}}_2$ of ${}_{S_3}\mathcal{M}_2$ by admissible S_3-covers as constructed in [1] to define the extended Prym map. We define the following subset of ${}_{S_3}\overline{\mathcal{M}}_2$:

$${}_{S_3}\widetilde{\mathcal{M}}_2 := \left\{ [h : Z \to X] \in {}_{S_3}\overline{\mathcal{M}}_2 \;\middle|\; \begin{array}{l} p_a(Z) = 7 \text{ and for any node } z \in Z \\ \text{the stabilizer } \mathrm{Stab}(z) \text{ is of order } 3 \end{array} \right\}.$$

Then ${}_{S_3}\widetilde{\mathcal{M}}_2$ is a non-empty open set of a component of ${}_{S_3}\overline{\mathcal{M}}_2$ containing the smooth S_3-covers ${}_{S_3}\mathcal{M}_2$. For any $[h : Z \to X] \in {}_{S_3}\widetilde{\mathcal{M}}_2$ let Y denote the quotient of Z by a subgroup of order 2 of S_3. We show that the kernel $P = P(f)$ of the map $\mathrm{Nm}_f : JY \to JX$ is a principally polarized abelian surface. The main Theorem in [5] is

Theorem 3. *The Prym map Pr extends to a proper surjective morphism $\widetilde{Pr}$: ${}_{S_3}\widetilde{\mathcal{M}}_2 \to \mathcal{A}_2$ of degree* 10.

As a consequence of this result, we have that *every principally polarized abelian surface occurs as the Prym variety of a non-cyclic degree-*3 *admissible cover f : $Y \to X$ of a stable curve X of genus* 2. Actually, we can be more precise. Consider the stratification of ${}_{S_3}\widetilde{\mathcal{M}}_2$,

$${}_{S_3}\widetilde{\mathcal{M}}_2 = {}_{S_3}\mathcal{M}_2 \sqcup R \sqcup S,$$

where R denotes the set of covers of ${}_{S_3}\widetilde{\mathcal{M}}_2$ with X singular but irreducible, and S denotes the complement of ${}_{S_3}\mathcal{M}_2 \sqcup R$ in ${}_{S_3}\widetilde{\mathcal{M}}_2$. On the other hand, let $\mathcal{E}_2$ denote the closed subset of $\mathcal{A}_2$ consisting of products of elliptic curves with canonical principal polarization. For any smooth curve C of genus 2 and any 3 Weierstrass points w_1, w_2, w_3 of C, let $\varphi_{2(w_1+w_2+w_3)}$ denote the map $C \to \mathbb{P}^1$ defined by the pencil $(\lambda(2(w_1 + w_2 + w_3)) + \mu(2(w_4 + w_5 + w_6))_{(\lambda,\mu)\in\mathbb{P}^1}$, where w_4, w_5, w_6 are the complementary Weierstrass points. The map $\varphi_{2(w_1+w_2+w_3)}$ factorizes via the hyperelliptic cover and a $3 : 1$ map $\bar{f} : \mathbb{P}^1 \to \mathbb{P}^1$. With this notation we define the following subsets of $\mathcal{J}_2$:

$$\mathcal{J}_2^u := \{JC \in \mathcal{J}_2 \mid \exists\, w_1, w_2, w_3 \text{ in } C \text{ such that } \bar{f} \text{ is simply ramified}\},$$

$$\mathcal{J}_2^r := \{JC \in \mathcal{J}_2 \mid \exists\, w_1, w_2, w_3 \text{ in } C \text{ such that } \bar{f} \text{ is not simply ramified}\}.$$

So we have the decomposition

$$\mathcal{A}_2 = \mathcal{J}_2^u \cup \mathcal{J}_2^r \sqcup \mathcal{E}_2.$$

In [5, Theorem 7.4] is proved that

$$Pr({}_{S_3}\mathcal{M}_2) = \mathcal{J}_2^u, \qquad Pr(R) \subset \mathcal{J}_2^r, \qquad Pr(S) = \mathcal{E}_2.$$

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Hodge modules, vanishing, and linearity

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(joint work with C. Schnell)

We apply M. Saito's theory of mixed Hodge modules in the case of an abelian variety A in order to produce natural classes of perverse coherent sheaves with linear support on the dual abelian variety $\widehat{A}$, and on the parameter space for Higgs line bundles $\widehat{A} \times H^0(A, \Omega_A^1)$. This can be seen as a generalization of Generic Vanishing theory, including and extending the main results of [2], [3], [4] and [6].

We first show that every mixed Hodge module on A gives rise to a collection of perverse coherent sheaves on $\widehat{A}$, namely the graded pieces of the filtration on the underlying $\mathcal{D}$-module. This uses the approach introduced in [4] for the study of generic vanishing, and the correspondence established in [7] (cf. also [6]) between GV-objects and perverse coherent sheaves. The tool which facilitates the application of these results is a vanishing theorem for the graded pieces of the de Rham complex associated to a $\mathcal{D}$-module of the type above, established in [9]. We show:

Theorem. *Let A be a complex abelian variety, and M a mixed Hodge module on A with underlying filtered $\mathcal{D}$-module $(\mathcal{M}, F)$. Then for each integer k, the Fourier-Mukai transform of* $\mathrm{Gr}_k^F \mathcal{M})$ *is a perverse coherent sheaf on $\widehat{A}$.*

It is most interesting to apply this result to push-forwards of Hodge modules on irregular smooth projective varieties via the Albanese map. Key tools are the Decomposition Theorem of Saito, extending the well-known result of Beilinson-Bernstein-Deligne, and results of Laumon and Saito on the behavior of associated graded objects under direct images. Most concrete applications, including the extension of Generic Vanishing theory mentioned above, follow already by looking at the Hodge module associated to the trivial variation of Hodge structures $\mathbb{Q}_X^H[n]$, where X is a smooth projective of dimension n.

For instance, the decomposition of the push-forward $a_* \mathbb{Q}_X^H[n]$ in the derived category of mixed Hodge modules leads to a Nakano-type generic vanishing theorem for all bundles of holomorphic forms Ω_X^p. This is quite fundamental information that has eluded previous efforts towards obtaining a complete answer, even though partial results can be found in [2] and [6]. Using standard notation, we write

$$V^q(\Omega_X^p) = \{\alpha \in \mathrm{Pic}^0(X) \mid H^i(X, \Omega_X^p \otimes P_\alpha) \neq 0\} \subset \mathrm{Pic}^0(X),$$

the q-th cohomological support locus of the bundle of holomorphic forms Ω_X^p. If $a : X \to A = \mathrm{Alb}(X)$ is the Albanese map of X, for $\ell \in \mathbb{N}$, let $A_\ell = \{y \in A \mid \dim f^{-1}(y) \geq \ell\}$, and define the *defect* of a to be

$$\delta(a) = \max_{\ell \in \mathbb{N}} \bigl(2\ell - \dim X + \dim A_\ell\bigr).$$

Theorem. *Let X be a smooth complex projective variety of dimension n. Then for each $p, q \in \mathbb{N}$ with $p + q > n$, we have*

$$\mathrm{codim}\ V^q(\Omega_X^p) \geq q + p - n - \delta(a).$$

For instance, if the Albanese map is semi-small then $\delta(a) = 0$, and one obtains generic Nakano-vanishing (known to not necessarily hold when the Albanese map is only assumed to be generically finite). The Theorem above, or its proof, imply the previously known generic vanishing statements for bundles of holomorphic forms, including the result of [2] for ω_X, and a result of [6] for arbitrary Ω_X^p.

Analogously, let $\mathrm{Char}(X) = \mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$ be the algebraic group of characters of X. We are interested in bounding the codimension of the support loci

$$\Sigma^i(X) = \{\rho \in \mathrm{Char}(X) \mid H^i(X, \mathbb{C}_\rho) \neq 0\},$$

where $\mathbb{C}_\rho$ denotes the rank 1 local system associated to a character ρ. The structure of these loci has been studied by Arapura [1] and Simpson [10], who showed that they are finite unions of torsion translates of subtori of $\mathrm{Char}(X)$. Methods similar to those described above lead us to the following quantitative result:

Theorem. *Let X be a smooth projective variety of dimension n, with Albanese map $a : X \to A$ of defect $\delta(a)$. Then, for each $k \in \mathbb{N}$,*

$$\mathrm{codim}_{\mathrm{Char}(X)} \Sigma^k(X) \geq 2|n - k| - 2\delta(a).$$

A strong version of the standard linearity results on cohomological support loci was given in [3]. Roughly speaking, it states that the standard Fourier-Mukai transform $\mathbf{R}\Phi_P \mathcal{O}_X := \mathbf{R}p_{2*}P$ in $\mathrm{D}^b_{\mathrm{coh}}(\widehat{A})$, where P is a Poincaré line bundle, is represented by a linear complex in the neighborhood of the origin of $\widehat{A}$. We extend this to the setting of the trivial $\mathcal{D}$-module $\mathcal{O}_X$. Let A be an abelian variety of dimension g, and let $A^\sharp$ be the moduli space of holomorphic line bundles with flat connection on A. The projection

$$\pi : A^\sharp \longrightarrow \widehat{A},\ (L, \nabla) \mapsto L$$

is a torsor for the trivial bundle $\mathcal{O}_{\widehat{A}} \otimes V$, where $V = H^0(A, \Omega_A^1)$. As complex manifolds, we have

$$A^\sharp \simeq H^1(A, \mathbb{C})/H^1(A, \mathbb{Z}) \qquad \text{and} \qquad \widehat{A} \simeq H^1(A, \mathcal{O}_A)/H^1(A, \mathbb{Z}),$$

and this is compatible with the exact sequence

$$0 \to H^0(A, \Omega^1_A) \to H^1(A, \mathbb{C}) \to H^1(A, \mathcal{O}_A) \to 0.$$

In particular, the universal covering space of $A^\sharp$ is isomorphic to $H^1(A, \mathbb{C})$. Recall now that Laumon [5] and Rothstein [8] have extended the Fourier-Mukai transform to $\mathcal{D}$-modules. Their generalized Fourier-Mukai transform takes bounded complexes of coherent $\mathcal{D}$-modules on A to bounded complexes of coherent sheaves on $A^\sharp$; here is a brief description following the presentation in [5]. On $A \times A^\sharp$, the pullback $P^\sharp$ of the Poincaré bundle P is endowed with a universal integrable connection $\nabla^\sharp$, relative to $A^\sharp$. Given any left $\mathcal{D}$-module $\mathcal{F}$ on A, interpreted as a quasi-coherent sheaf with integrable connection, we consider $p_1^*\mathcal{F} \otimes P^\sharp$ on $A \times A^\sharp$, endowed with the natural tensor product integrable connection ∇ relative to $A^\sharp$. We then define

$$\mathbf{R}\Phi_{P^\sharp}(\mathcal{F}) := \mathbf{R}p_{2*}\mathrm{DR}(p_1^*\mathcal{F} \otimes P^\sharp, \nabla), \tag{1}$$

where $\mathrm{DR}(p_1^*\mathcal{F} \otimes P^\sharp, \nabla)$ is the usual (relative) de Rham complex

$$\Big[p_1^*\mathcal{F} \otimes P^\sharp \xrightarrow{\nabla} p_1^*\mathcal{F} \otimes P^\sharp \otimes \Omega^1_{A\times A^\sharp/A^\sharp} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} p_1^*\mathcal{F} \otimes P^\sharp \otimes \Omega^g_{A\times A^\sharp/A^\sharp}\Big] \tag{2}$$

supported in degrees $-g, \ldots, 0$. As all of the entries in this complex are relative to $A^\sharp$, $\mathbf{R}\Phi_{P^\sharp}(\mathcal{F})$ is represented by a complex of quasi-coherent sheaves on $A^\sharp$. Restricted to coherent $\mathcal{D}$-modules, this induces an equivalence of categories

$$\mathbf{R}\Phi_{P^\sharp} : \mathrm{D}^b_{\mathrm{coh}}(\mathcal{D}_A) \longrightarrow \mathrm{D}^b_{\mathrm{coh}}(A^\sharp).$$

Now let X be a smooth projective variety with Albanese map $a\colon X \to A$. By first pushing forward to A (or equivalently by working with the pullback of $(P^\sharp, \nabla^\sharp)$ to $X \times A^\sharp$) one can similarly define

$$\mathbf{R}\Phi_{P^\sharp} : \mathrm{D}^b_{\mathrm{coh}}(\mathcal{D}_X) \longrightarrow \mathrm{D}^b_{\mathrm{coh}}(A^\sharp).$$

We prove the following linearity theorem for the Fourier-Mukai transform of the trivial $\mathcal{D}$-module $\mathcal{O}_X$.

Theorem. *Let X be a smooth projective variety of dimension n, and let $E = \mathbf{R}\Phi_{P^\sharp}(\mathcal{O}_X) \in \mathrm{D}^b_{\mathrm{coh}}(A^\sharp)$ be the Fourier-Mukai transform of the trivial $\mathcal{D}$-module on X. Then the stalk $E \otimes \mathcal{O}_{A^\sharp,0}$ is quasi-isomorphic to a linear complex.*

This can be shown to recover both the results of Green-Lazarsfeld [3] and those of Arapura [1] and Simpson [10] mentioned above. In combination with the previous result, it shows that the Fourier-Mukai transform of the trivial $\mathcal{D}$-module is a perverse coherent sheaf in $\mathrm{D}^b_{\mathrm{coh}}(A^\sharp)$, with linear support, and whose dual satisfies the same properties. The class of such perverse sheaves should be very important in the further investigation of $\mathcal{D}$-modules coming from Hodge theory.

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Lagrangian fibrations on hyperkähler manifolds

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(joint work with Daniel Greb, Christian Lehn)

Let X be a hyperkähler manifold, that is, a compact, simply-connected Kähler manifold X such that $H^0(X, \Omega_X^2)$ is spanned by a holomorphic symplectic form σ. From a differential geometric point of view hyperkähler manifolds are Riemannian manifolds with holonomy the full unitary-symplectic group $\mathrm{Sp}(n)$.

An important step in the structural understanding of a manifold is to decide whether there is a fibration $f\colon X \to B$ over a complex space of smaller dimension. For hyperkähler manifolds it is known that in case such f exists, it is a *Lagrangian fibration*: $\dim X = 2\dim B$, and the holomorphic symplectic form σ restricts to zero on the general fibre. Additionally, by the Arnold-Liouville theorem the general fibre is a smooth Lagrangian torus.

In accordance with the case of K3-surfaces (and also motivated by mirror symmetry) a simple version of the so-called Hyperkähler SYZ-conjecture asks if every hyperkähler manifold can be deformed to a hyperkähler manifold admitting a Lagrangian fibration. We studied the question of existence of a Lagrangian fibration on a given hyperkähler manifold X under a geometric assumption proposed by Beauville [Bea11, Sect. 1.6]:

Question 1 (Beauville)**.** *Let X be a hyperkähler manifold and $L \subset X$ a Lagrangian submanifold biholomorphic to a complex torus. Is L a fibre of a (meromorphic) Lagrangian fibration $f\colon X \to B$?*

We call a submanifold L as above a *Lagrangian subtorus* of X. To formulate our main results we call a pair (X, L) of a hyperkähler manifold and a Lagrangian subtorus *stably projective* if every deformation of X that preserves L is projective. It can be shown that (X, L) being stably projective is a topological property of the pair (see [GLR11a, Sect. 3]).

Theorem 1 (summarising [GLR11a])**.** *Let X be a hyperkähler manifold containing a smooth Lagrangian subtorus L.*

(1) *If X is not projective, then X admits a holomorphic Lagrangian fibration with fibre L.*
(2) *If X is projective but (X, L) is not stably projective, then X admits an almost holomorphic Lagrangian fibration $f\colon X \dashrightarrow B$ with fibre L. Moreover, there exists a holomorphic model for f on a birational hyperkähler manifold X', that is, there exists a commutative diagram*

$$\begin{array}{ccc} X & \dashrightarrow & X' \\ {\scriptstyle f}\,\vdots & & \downarrow{\scriptstyle f'} \\ B & \dashrightarrow & B' \end{array}$$

where f' is a Lagrangian fibration on X' and the horizontal maps are birational.

The proof starts from the observation that L infinitesimally behaves like the fibre of a fibration; the first item is then deduced from results of Campana, Peternell and Oguiso on non-algebraic hyperkähler manifolds. The second follows by a deformation argument while for the third we have to invoke recent advances in higher-dimensional birational geometry.

To answer Beauville's question exhaustively it remains to exclude the case of a projective hyperkähler manifold X containing a Lagrangian subtorus L such that (X, L) is stably projective. In this case some deformations of L intersect L in unexpected ways. This suggests a more geometric approach to the problem which leads to a positive answer to the strongest version of Beauville's question in dimension four.

Theorem 2 ([GLR11b])**.** *Let X be a 4–dimensional hyperkähler manifold containing a Lagrangian torus L. Then X admits a holomorphic Lagrangian fibration with fibre L.*

Unfortunately, our approach in dimension 4 does not generalise to higher dimensions at the moment. On the other hand we do not know of an explicit example of a stably projective pair (X, L) or of an almost holomorphic Lagrangian fibration which is not holomorphic.

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Mirror Symmetry in Higher Kodaira Dimensions

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(joint work with Mark Gross, Ludmil Katzarkov)

1. Homological Mirror Symmetry

In his ICM talk [Ko94], Maxim Kontsevich proposed a categorical version of mirror symmetry. He conjectured that mirror symmetry of a pair of Calabi-Yau manifolds $X, \check{X}$ should be understood as equivalences of triangulated categories

$$\mathrm{D}^b(X) \cong \mathrm{DFuk}(\check{X})$$

$$\mathrm{DFuk}(X) \cong \mathrm{D}^b(\check{X})$$

where $\mathrm{D}^b(V)$ denotes the bounded derived category of coherent sheaves of a variety V and $\mathrm{DFuk}(W)$ denotes the derived Fukaya category on a symplectic manifold W. Hori-Vafa, Givental and Auroux extended mirror symmetry to Fano varieties where the mirror of a projective Fano manifold X with a choice of effective anti-canonical divisor D and holomorphic volume form Ω^d is given by a complex manifold M together with a holomorphic function w. In this setup, the homological mirror symmetry conjecture takes the form

$$\mathrm{D}^b(X) \cong \mathrm{DFS}(M, w)$$

$$\mathrm{DFuk}(X) \cong \mathrm{D}^b(M, w)$$

where

- $\mathrm{DFS}(M, w)$ denotes the derived Fukaya-Seidel category, a conjectural concept for which a rigorous definition was given in [Se07] in the case where the critical points of w are A_1-singularities,
- $\mathrm{D}^b(M, w)$ denotes the category

  $$\mathrm{D}^b(M, w) = \prod_{t \in \mathbb{C}} \mathrm{D}^b_{\mathrm{sing}}\,(w^{-1}(t))$$

  defined in [Or11] where in turn $\mathrm{D}^b_{\mathrm{sing}}\,(w^{-1}(t))$ is the Verdier quotient of $\mathrm{D}^b(w^{-1}(t))$ by $\mathrm{Perf}(w^{-1}(t))$, the full subcategory of perfect complexes (i.e. complexes of locally free sheaves). For a non-critical value t of w, we have

  $$\mathrm{D}^b_{\mathrm{sing}}\,(w^{-1}(t)) = 0$$

  so the product above is finite.

A *Landau-Ginzburg* model (LG model) is a pair (X,w) where X is a variety (resp. complex manifold) with a flat regular (resp. holomorphic) map w to $\mathbb{A}^1$ (resp. $\mathbb{C}$). Given a LG model (X, w), we denote the critical locus of w by $\mathrm{crit}(w)$ and the critical values by $\mathrm{critval}(w)$.

We propose a mirror construction for varieties of general type following along the following steps:

Program 1. Let S be a smooth projective variety over $\mathbb{C}$.

(1) Construct a Landau Ginzburg model (X, w) such that $S = crit(w)$.
(2) Use a mirror duality of Landau-Ginzburg models to obtain another LG model $(\check{X}, \check{w})$, the mirror of (X, w).
(3) Take the pair $(\check{S}, \mathcal{F}_{\check{S}})$ as the mirror of S, where $\check{S} = \mathrm{crit}(\check{w})$ and $\mathcal{F}_{\check{S}}$ is a complex of sheaves on $\check{S}$. For a critical value t of $\check{w}$, on $\check{S} \cap \check{w}^{-1}(t)$, the sheaf $\mathcal{F}_{\check{S}}$ is given as $(\phi_{\check{w},t}\mathbb{C})[1]$ where ϕ denotes Deligne's vanishing cycle functor.

There are a couple of remarks to be made here.

First, in the construction that I will discuss later on, instead of taking all critical values into consideration as above, we will take $\check{S}$ to be only the part of the critical locus mapping to a certain subset of critical values. If the Kodaira dimension of S is non-negative, we will take $\check{S}$ as the singular locus of the critical fibre over 0. There are typically additional singular fibres of $\check{w}$ which might or might not be considered as contributing to $\check{S}$. The criterion for whether or not a critical value gives a contribution to $\check{S}$ is whether the critical value remains in a bounded region when degenerating the potential $\check{w}$ in a certain way. E.g. for the mirror of $\mathbb{P}^1$ there will be two critical values all part of $\check{S}$ whereas for the mirror of a genus two curve, there will be three critical values in our construction only the central one of which contributes to $\check{S}$, for a more detailed discussion see [GKR11].

Second, defining the mirror to be $(\check{S}, \mathcal{F}_{\check{S}})$ appears to be sufficient in order to associate a Hodge structure to $(\check{S}, \mathcal{F}_{\check{S}})$, but possibly for further invariants, e.g. the derived Fukaya-Seidel category or Gromov-Witten invariants, one might need a tubular neighbourhood of $\mathrm{crit}(\check{w})$ in $\check{X}$.

Let us discuss the homological mirror symmetry conjecture for our setup. We expect the following equivalences

$$\begin{array}{ccccc} \mathrm{D}^b(S) & \overset{1)}{\cong} & \mathrm{D}^b(X, w) & \overset{2)}{\cong} & \mathrm{DFS}(\check{X}, \check{w}) \\ \mathrm{DFuk}(S) & \overset{3)}{\cong} & \mathrm{DFS}(X, w) & \overset{2)}{\cong} & \mathrm{D}^b(\check{X}, \check{w}) \end{array}$$

where the ones marked by 2) form the generalized homological mirror symmetry conjecture for a pair of Landau-Ginzburg models. The equivalence 1) has been proved in [HW09] for the situation we will be considering. As pointed out to us by Denis Auroux, the equivalence 3) is standard to symplectic geometers using a gradient flow argument. Also the definition of $\mathrm{DFS}(X, w)$ works well for a smooth critical locus. Progress towards the equivalence $\mathrm{DFuk}(S) \cong \mathrm{D}^b(\check{X}, \check{w})$ has been made in [Se08] and [Ef09].

2. The Mirror Construction

The mirror dual of an algebraic torus is an algebraic torus. The mirror dual operation to compactification is adding a Landau-Ginzburg superpotential. The new feature of our construction is that we partially compactify $(\mathbb{C}^*)^n$ as well as its mirror dual and thus will have potentials on both sides. This builds on the most basic form of mirror symmetry: The duality of cones. We claim to have a mirror pair

$$(\tilde{X}_\sigma, w) \leftrightarrow (\tilde{X}_{\check{\sigma}}, \check{w})$$

whenever $\sigma \subseteq M \otimes_{\mathbb{Z}} \mathbb{R}$ is a full-dimensional rational polyhedral cone for $M \cong \mathbb{Z}^n$ and $\check{\sigma} = \{n \in N \otimes_{\mathbb{Z}} \mathbb{R} | \langle n, m \rangle \geq 0\}$ is its dual in $N \otimes_{\mathbb{Z}} \mathbb{R}, N = \mathrm{Hom}\,(M, \mathbb{Z})$, X_σ and $X_{\check{\sigma}}$ are the associated affine toric varieties and $\tilde{X}_\sigma, \tilde{X}_{\check{\sigma}}$ crepant toric orbifold resolutions of the blow-up of the origin respectively (which always exist). The potentials $w : X_\sigma \to \mathbb{C}$ and $\check{w} : X_{\check{\sigma}} \to \mathbb{C}$ are given as

$$w = \sum_{\check{\rho}} c_{\check{\rho}} z^{n_{\check{\rho}}} \qquad \text{and} \qquad \check{w} = \sum_{\rho} c_\rho z^{m_\rho}$$

where the sums are over the rays of the resolution of $\check{\sigma}$ and σ respectively, $c_{\check{\rho}}, c_\rho \in \mathbb{C}\backslash\{0\}$ and $z^{n_{\check{\rho}}}$ denotes the monomial function associated to the primitive integral generator of $\check{\rho}$, similarly for z^{m_ρ}. We pull back these potentials to $\tilde{X}_\sigma, \tilde{X}_{\check{\sigma}}$ respectively.

The strength of this construction is its simplicity. However, it looks naive from several points of view, e.g. the coefficients of the potentials on either side should be defined by counting holomorphic disks on the respective dual side and then possibly there are more terms to these potentials. However, we will see that for our purposes of supporting the existence of a mirror duality for varieties of any Kodaira dimension, this construction suffices. In particular we claim that

Conjecture 1. *If $\tilde{X}_\sigma$ and $\tilde{X}_{\check{\sigma}}$ are smooth, then*

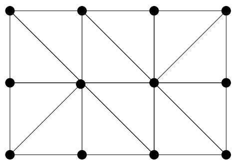
$$h^{p,q}(\tilde{X}_\sigma, w) = h^{n-p,q}(\tilde{X}_{\check{\sigma}}, \check{w}).$$

More generally, if $\tilde{X}_\sigma$ or $\tilde{X}_{\check{\sigma}}$ is an orbifold, one needs to replace the ordinary Hodge numbers by a version of the orbifold Hodge numbers. However, we haven't discussed yet, how to define $h^{p,q}(X, w)$. This will be done shortly, but let us first see how to apply our mirror construction in order to obtain the mirror of say a genus two curve:

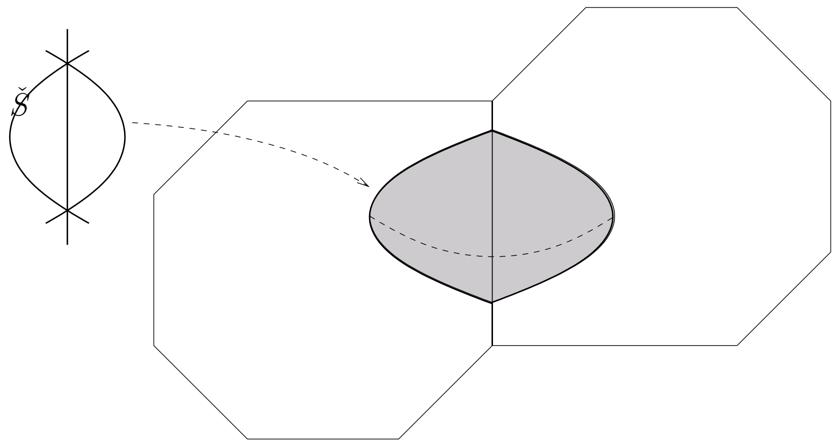
Note that our construction naturally generalizes the Batyrev-Borisov construction. This already indicates how one might use it in order to put into practice Program 1. Indeed, we can construct a mirror for a smooth variety S if S embeds as a complete intersection in a toric variety. For simplicity we will only sketch the case where S is a bidegree $(2, 3)$-hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1$ and thus S is a genus two curve: We define $\tilde{X}_\sigma$ as the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2, -3)$. It is an exercise to show that an element of $\Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 3))$ determines a map $w : \tilde{X}_\sigma \to \mathbb{C}$ such that $\mathrm{crit}(w) = S$ where S is the zero locus of the chosen section of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 3)$. Thus we have a Landau-Ginzburg model fulfilling 1) of Program 1. For the second step, we first need to observe that $\tilde{X}_\sigma$ is a toric variety given as the blow-up of the origin in an affine toric variety X_σ. Indeed, X_σ is obtained by contracting the zero-section in the total space of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2, -3)$. To obtain the mirror, we start by dualizing the cone σ to obtain a cone $\check{\sigma}$. We find this to be the cone over a rectangle Δ with edge lengths 2 and 3 and identify $X_{\check{\sigma}}$ for a fixed $t \neq 0$ as the subvariety

$$V(xy - t^2 z^2, uv - t^3 z^3)$$

of $\mathbb{A}^5$. We have $\check{w} = x + y + z + u + v$. We obtain $\tilde{X}_{\check{\sigma}}$ by triangulating Δ, e.g. as follows:



Under the pullback to $\tilde{X}_{\check{\sigma}}$, $\check{w}^{-1}(0)$ picks up two exceptional divisors given by the interior points of Δ, so 0 will be a critical value and in fact the only one we consider. We find that $\check{w}^{-1}(0)$ is a normal crossing divisor with three irreducible components meeting as follows:



We obtain $(\check{S}, \mathcal{F}_{\check{S}})$ where $\check{S}$ is a union of three projective lines all identified in 0 and ∞. The complex of sheaves $\mathcal{F}_{\check{S}}$ is just the constant sheaf $\mathbb{C}$ at a general point of $\check{S}$ and at the two special points its cohomology is $\mathbb{C}^2$ in degree 0 and $\mathbb{C}$ in degree 1. A computation of the hypercohomology of $\mathcal{F}_{\check{S}}$ exhibits the expected Hodge numbers of a mirror of S, namely:

$$\begin{array}{ccc} & 2 & \\ 1 & & 1 \\ & 2 & \end{array}$$

More generally, we enhance $\mathcal{F}_{\check{S}}$ to a cohomological mixed Hodge complex and obtain a mixed Hodge structure on its hypercohomology. We then define Hodge numbers by forgetting the weights as

$$h^{p,q}(\check{S}, \mathcal{F}_{\check{S}}) := \sum_k h^{p,q+k} \mathbb{H}^{p+q}(\check{S}, \mathcal{F}_{\check{S}})$$

and prove in [GKR11] the following:

Theorem 1. *Let S be a smooth d-dimensional ample hypersurface in a smooth toric variety $\mathbb{P}_\Delta$ and let $(\check{S}, \mathcal{F}_{\check{S}})$ be its mirror by the construction above then*

$$h^{p,q}(S) = h^{d-p,q}(\check{S}, \mathcal{F}_{\check{S}}).$$

3. Hochschild (co-)homology

Coming back to the discussion of the first section, we would like to link the hypercohomology of a mirror $(\check{S}, \mathcal{F}_{\check{S}})$ of a complete intersection S in a toric variety to the cohomology of the attached category. Recall that by a theorem of Hochschild-Kostant-Rosenberg, we have for a smooth projective manifold S that the degree k Hochschild cohomology $\mathrm{D}^b(S)$ is given by

$$HH^k(S) = \bigoplus_{p+q=k} H^q(S, \bigwedge^p \mathcal{T}_S)$$

while the degree k Hochschild homology is

$$HH_k(S) = \bigoplus_{p-q=k} H^q(S, \Omega^p_S).$$

More generally, i.e. for the mirror LG model $(\tilde{X}_{\check{\sigma}}, \check{w})$ with $\check{w}$ quasi-projective, we have by Orlov [Or11] and Lin-Pomerleano [LP11] an equivalence

$$D^b(\tilde{X}_{\check{\sigma}}, \check{w}) \cong MF(\tilde{X}_{\check{\sigma}}, \check{w})$$

where the latter is the category of matrix factorisations. It is then shown that

$$HH_{(k \bmod 2)}(MF(\tilde{X}_{\check{\sigma}}, \check{w})) \cong \mathbb{H}^{(k \bmod 2)}(\tilde{X}_{\check{\sigma}}, (\Omega^\bullet_{\tilde{X}_{\check{\sigma}}}, \wedge d\check{w})).$$

Now using a theorem by Barannikov-Kontsevich, Sabbah, Ogus-Vologodsky, we have an isomorphism

$$\mathbb{H}^k(\tilde{X}_{\check{\sigma}}, (\Omega^\bullet_{\tilde{X}_{\check{\sigma}}}, \wedge d\check{w})) \cong \bigoplus_{p \in \mathrm{critval}(\check{w})} \mathbb{H}^{k-1}(\check{w}^{-1}(p), \phi_{\check{w},p}\mathbb{C})$$

whenever $\check{w}$ is projective. We may use this by compactifying $\tilde{X}_{\check{\sigma}}$ and eventually find that we have furnished $HH_k(D^b(\tilde{X}_{\check{\sigma}}, \check{w}))$ with a mixed Hodge structure (with the caveat however that the last isomorphism given above is non-canonical).

From this perspective, Theorem 1 only considers one half of the (co-)homology groups, namely the Hochschild homology. By analyzing the sheaves of poly-vector fields, we came up with the following conjecture:

Conjecture 2. *Given a mirror pair S, $(\check{S}, \mathcal{F}_{\check{S}})$ of our construction, we have*

$$HH^k(S) \cong H^k(\check{S}, \mathbb{C}).$$

We prove this conjecture when S is a curve and exemplary for the quintic surface.

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Counting curves in surfaces and the Göttsche conjecture

Richard P. Thomas

(joint work with Martijn Kool)

Fix a nonsingular projective surface S and a homology class $\beta \in H_2(S,\mathbb{Z})$. There are various ways of counting holomorphic curves in S in class β; we focus on Gromov-Witten invariants and stable pairs. Since these are deformation invariant they must vanish in class β if there exists a deformation of S for which the Hodge type of β is not $(1,1)$. We can see the origin of this vanishing without deforming S as follows.

For simplicity work in the simplest case of an embedded curve $C \subset S$ with normal bundle $N_C = \mathcal{O}_C(C)$. As a Cartier divisor, C is the zero locus of a section s_C of a line bundle $L := \mathcal{O}_S(C)$, giving the exact sequence

$$0 \to \mathcal{O}_S \xrightarrow{s_C} L \to N_C \to 0.$$

The resulting long exact sequence describes the relationship between first order deformations and obstructions $H^0(N_C)$, $H^1(N_C)$ of $C \subset S$, and the deformations and obstructions $H^1(\mathcal{O}_S)$, $H^2(\mathcal{O}_S)$ of the line bundle $L \to S$:

$$\begin{aligned} 0 \to H^0(L)/\langle s_C\rangle \to H^0(N_C) \to H^1(\mathcal{O}_S) \to H^1(L) \\ \to H^1(N_C) \to H^2(\mathcal{O}_S) \to H^2(L) \to 0. \end{aligned} \tag{1}$$

The resulting "semi-regularity map" [1] $H^1(N_C) \to H^2(\mathcal{O}_S) = H^{0,2}(S)$ takes obstructions to deforming C to the "cohomological part" of these obstructions. Roughly speaking, if we deform S, we get an associated obstruction in $H^1(N_C)$ to deforming C with it; its image in $H^{0,2}(S)$ is the $(0,2)$-part of the cohomology class $\beta \in H^2(S)$ in the deformed complex structure. Thus it gives the obvious cohomological obstruction to deforming C: that β must remain of type $(1,1)$ in the deformed complex structure on S.

In particular, when S is fixed, obstructions lie in the kernel of $H^1(N_C) \to H^2(\mathcal{O}_S)$. More generally, if we only consider deformations of S for which β remains $(1,1)$ then the same is true. And when $h^{0,2}(S) > 0$, the existence of this trivial $H^{0,2}(S)$ piece of the obstruction sheaf guarantees that the virtual class vanishes.

So it would be nice to restrict attention to surfaces and classes (S,β) such that $\beta \in H^2(S)$ has type $(1,1)$, defining a new obstruction theory using only the kernel of the semi-regularity map. (Put differently, if we restrict to the Noether-Lefschetz locus, we know that the space of line bundles is smooth, so we can remove its obstruction space $H^2(\mathcal{O}_S)$. When the curve is an embedded divisor the result is that we consider only the obstructions $H^1(L)$ to deforming sections of L.) Checking that this kernel really defines an obstruction theory in the generality needed to define a virtual cycle – i.e. for deformations to all orders over an arbitrary base – has proved difficult; there is a hotchpotch of results in different cases due to many authors. Our general construction uses a mixture of their methods.

For stable pairs we get optimal results; this part is joint work also with Dmitri Panov. We show that the kernel of the semi-regularity map defines a reduced perfect obstruction theory whenever

$$H^2(L) = 0 \text{ for effective line bundles } L \text{ with } c_1(L) = \beta. \tag{2}$$

Equivalently the condition is that no line bundle in $\mathrm{Pic}_\beta(S)$ is *special*. (Special line bundles L are those for which both L and $K_S - L$ are effective.) This condition is necessary to ensure the semi-regularity map (1) is surjective.

For Gromov-Witten theory, multiple covers complicate the situation, but we are able to prove the same result for the moduli space of stable maps when

$$H^1(T_S) \xrightarrow{\cup\beta} H^2(\mathcal{O}_S) \text{ is surjective.} \tag{3}$$

Here $\beta \in H^1(\Omega_S)$ and we use the pairing $\Omega^1_S \otimes T_S \to \mathcal{O}_S$. Condition (3) implies (2): for any $L = \mathcal{O}(C)$ in class β, the map $\cup\beta$ factors as $H^1(T_S) \to H^1(\mathcal{O}_C(C)) \to H^2(\mathcal{O}_S)$, so surjectivity implies that $H^2(L) = 0$ by the exact sequence (1).

Our method is to consider the moduli space of curves on the fibres of a certain *algebraic twistor family* of S. This unforgivable abuse of notation is an $h^{0,2}(S)$-dimensional family $\mathcal{S}_B$ of surfaces with totally nondegenerate Kodaira-Spencer map over a *first order* Artinian base B. This relative moduli space of curves on the fibres of $\mathcal{S}_B \to B$ is equal to the moduli space of curves on the central fibre, and the natural perfect obstruction theory of the family is isomorphic to the kernel of the semi-regularity map on the standard obstruction theory. This gives a reduced obstruction theory from which we define invariants of S. These can be thought of in many ways:

(1) Reduced Gromov-Witten invariants of S, counting curves satisfying incidence conditions. These coincide with usual Gromov-Witten invariants when $h^{2,0}(S) = 0$.

(2) Reduced *residue* Gromov-Witten invariants of S defined via $\mathbb{C}^*$-localisation on the canonical bundle of S. These include the reduced Gromov-Witten invariants above, but can also involve extra classes. When K_S is trivial on β these are λ-classes pulled back from the moduli space of curves.
(3) Reduced (residue) stable pairs invariants of S. These count, in a virtual sense, pairs (C, Z) where $C \subset S$ is a divisor and $Z \subset C$ is any 0-dimensional subscheme. The curves satisfy incidence conditions, and in the residue theory the virtual counting can be replaced by a virtual Euler characteristic.
(4) If (3) holds also for $\beta' < \beta$ (i.e. β is the sum of effective classes β', β'') then the absolute moduli space of curves in $\mathcal{S}_B$ also coincides with the moduli space for S, and the reduced Gromov-Witten invariants of S become the ordinary Gromov-Witten invariants of the family $\mathcal{S}_B$.
(5) Using insertions to cut down to a δ-dimensional linear system of curves in $|L|$ for some $L \in \mathrm{Pic}_\beta(S)$, an appropriate reduced Gromov-Witten invariant of S can be shown to coincide with Göttsche's invariants counting δ-nodal curves in very ample δ-dimensional linear systems.
(6) The residue invariants are 3-fold invariants of the canonical bundle of S (a Calabi-Yau 3-fold), to which an appropriate strengthening of the MNOP conjecture of Maulik-Nekrasov-Okounkov-Pandharipande applies. In particular reduced residue Gromov-Witten and stable pair invariants are conjecturally equivalent. In the special case of the Göttsche invariants mentioned above we are able to prove this conjecture.

The MNOP conjecture is especially useful because we can really make computations on the stable pairs side. We show that the "good component" of moduli space is the zero locus of a section of a vector bundle over a bigger smooth space, and that this description induces the correct obstruction theory. This allows us to compute many invariants in terms of topological numbers: β^2, $c_1(S).\beta$, $c_1(S)^2$, $c_2(S)$ and more unusual topological invariants of β when $b_1(S) \neq 0$. (As an example of the latter consider $\beta \in H^2(S)$ to be a skew map $H^1(S) \to H^1(S)^*$ by wedging and integration on S. Now take its Pfaffian.)

In earlier work with Vivek Shende [2] we carried out the stable pairs computation in the special case related to Göttsche's invariants, giving a simple algebraic proof that they depend only on β^2, $c_1(S).\beta$, $c_1(S)^2$, $c_2(S)$. This recovers a result of Tzeng [3].

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Curvature flows on complex manifolds

Gang Tian

In this talk, I first described an on-going program on Kähler-Ricci flow on Kähler manifolds. I discussed how finite-time singularity this Kähler-Ricci flow forms and how it is related to the minimal model program in algebraic geometry. Next I explained how to construct a global solution with surgery. Then I discussed generalized Kähler-Einstein metrics and what is the limiting behavior of the global solution with surgery. The second flow discussed is the pluri-closed flow due to J. Streets and myself. I first discussed two formulations of this pluri-closed flow and show how it is related the renormalization group flow coupled with B-fields. I also propose a conjecture on sharp local existence of this flow and show how the resolution of this conjecture can be used to attack the problem of classifying mysterious class VII^+ surfaces.

Boundedness of log pairs

Chenyang Xu

(joint work with Christopher Hacon, James M^{c}Kernan)

In this joint work, we establish a general theory of boundedness of singular pairs of log general type. More precisely, fix a positive integer n and a set $I \subset [0,1]$ satisfying the descending chain condition (DCC). Now we consider the class of pairs

$$\mathfrak{D} = \{(X,\Delta) | (X,\Delta) \text{ is log canonical}, \dim X = n, K_X + \Delta \text{ is big}$$
$$\text{and the coefficients of } \Delta \text{ are in } I.\}$$

Then we show that the set $\{\mathrm{vol}(K_X+\Delta) | (X,\Delta) \in \mathfrak{D}\} \subset \mathbb{R}_{>0}$ also satisfies DCC. Furthermore, there exists a constant N which only depends on I and n such that the linear system $|r(K_X+\Delta)|$ induces a birational map for any $r \geq N$ and $(X,\Delta) \in \mathfrak{D}$.

Historically, this is first conjectured by Kollár ([5]). When $n = 2$, this is solved by Alexeev (cf. [1]). For the subclass $\mathfrak{D}^{sm}$, which consists of smooth general type varieties of no boundaries, this is proved by Tsuji, Hacon-M^{c}Kernan and Takayama (cf.[2, 6]).

To prove this, a crucial concept is the *log birational boundedness* which we obtain by refining the original idea due to Tsuji. A class of pairs $\{(X_t,\Delta_t)\}$ is called log birationally bounded if there is a scheme S of finite type with a family of pairs (Y,D) over S such that for any (X_t,Δ_t) there exists a point $s \in S$ and a birational map $f_t : X_t \dashrightarrow Y_s$ with $D_s = f_{t*}(\Delta_t) + \mathrm{Ex}(f_t^{-1})$. Then it is easy to see that if we choose a common resolution $p : W \to X$ and $q : W \to Y$ and we choose $\Delta_W = p_*^{-1}(\Delta) + \mathrm{Ex}(p)$, then

$$\mathrm{vol}(K_X+\Delta) = \mathrm{vol}(K_W+\Delta_W).$$

Therefore, we can replace (X,Δ) by (W,Δ_W) (by possibly enlarging I to $I \cup \{1\}$) and always also assume that f_t is a morphism.

Then the proof of the theorem is divided into two steps: first we show that if a subclass $\mathfrak{C}$ in $\mathfrak{D}$ is log birationally bounded, then the volumes of pairs in $\mathfrak{C}$ satisfy DCC; then we show that for any fixed constant C, if we consider the subclass $\mathfrak{D}_C \subset \mathfrak{D}$ which consists of pairs of volume at most C, then $\mathfrak{D}_C$ is log birationally bounded.

To proof the first step, we first generalize Siu's invariance of plurigenera to the case for pairs, under the assumption that its a family of log smooth pairs with one of them is of log general type. With this result, we reduce to the case that S is a point, i.e., all (X_t, Δ_t) is log birational to a fixed log smooth pair (Y, D). Then we show that after blowing up (Y, Δ) along the strata finitely many times, we indeed get a model Y with morphisms $f_t : X_t \to Y$, such that $\mathrm{vol}(K_{X_t} + \Delta_t) = \mathrm{vol}(K_Y + f_{t*}(\Delta_t))$. The conclusion now easily follows from our assumption that the coefficients satisfy DCC condition. For more details, see [3].

Then, to show the statement for the second step, we follow the strategy by Tsuji. First, we show that there exists a constant C_1, which depends on I, n and C, such that for any r, if $\mathrm{vol}(r(K_X + \Delta)) \geq C_1$, then the linear system $|\lceil r(K_X + \Delta)\rceil|$ induces a birational map. This is obtained by restricting the linear system to the log canonical centers and extending sections from the log canonical centers. Thus we need to show the restricting volumes are larger than a uniform constant by applying induction. The key point is that we can construct a boundary on the log canonical center, such that it is log big, the pluri-sections can be extended and the coefficients are contained in a DCC set. Such a boundary is obtained by using the ACC conjecture of log canonical thresholds of lower dimensions, which is also part of our induction.

The remaining subtlety is that, we need to show $|\lfloor r(K_X + \Delta)\rfloor|$ induces a birational map rather than the rounding up. This is more delicate than it first appears, since we essentially need to show that there exists a uniform constant $\epsilon > 0$, such that if $(X, \Delta) \in \mathfrak{D}$, then $K_X + (1-\epsilon)\Delta$ is also big. If this is not true, let us assume that there exists a sequence of $\epsilon_i > 0$, such that $\lim \epsilon_i = 0$, which are the pseudo-effective thresholds of a sequence of pairs $(X_i, \Delta_i) \in \mathfrak{D}$. Using MMP and the induction assumption, we can easily reduce to the case that (X_i, Δ_i) are *klt*, the underlying spaces X_i are all Fano varieties and $K_{X_i} + (1-\epsilon_i)\Delta_i$ are trivial. If the pairs (X_i, Δ_i) are log birationally bounded, then the result is easy to get by the conclusion of the first step. So we assume (X_i, Δ_i) are not log birational bounded, in which case, it just says $\mathrm{vol}(-K_{X_i})$ are unbounded. To rule out this case, we indeed show the following fact that there exists a uniform δ, such that for any $\Theta_i \geq (1-\delta)\Delta_i$, and $K_{X_i} + \Theta_i$ is trivial, then (X_i, Θ_i) is klt. For more details see the incoming paper [4].

This general result has a number of consequences. It can be applied to show that there exists a uniform C_n depending only on n, such that the order of the birational automorphism group of a n-dimensional general type variety X is less than $C_n \cdot \mathrm{vol}(K_X)$ (see [3]). As part of our induction chain, we also verify the global ACC conjecture for numerical trivial pairs as well as Shokurov's ACC conjecture on log canonical thresholds. As an immediate consequence of the last step, we prove Batyrev's conjecture on $\mathbb{Q}$-Fano varieties with bounded index. Finally, we prove the boundedness of the moduli functors of the stable schemes. Here we use the DCC property of the volumes to reduce the question to normal pairs, and then use a special type of MMP to achieve the boundedness from the log birational boundedness which we have shown.

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Compact Kähler manifolds with automorphism groups of maximal rank

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We work over the field $\mathbb{C}$ of complex numbers. For a linear transformation L on a finite-dimensional vector space V over $\mathbb{C}$ or its subfields, its *spectral radius* is defined as $\rho(L) := \max\{|\lambda|\,;\, \lambda \in \mathbb{C}$ is an eigenvalue of $L\}$. Let X be a compact complex Kähler manifold and Y a normal projective variety, and let $g \in \mathrm{Aut}(X)$ and $f \in \mathrm{Aut}(Y)$. Define the (topological) *entropy* $h(*)$ and *first dynamical degrees* $d_1(*)$ as: $h(g) := \log \rho(g^* \mid \oplus_{i \geq 0} H^i(X, \mathbb{C}))$, $d_1(g) := \rho(g^* \mid H^2(X, \mathbb{C}))$ $(= \rho(g^* \mid H^{1,1}(X)))$, $d_1(f) := \rho(f^* \mid \mathrm{NS}_{\mathbb{C}}(Y))$, where $\mathrm{NS}_{\mathbb{C}}(Y) := \mathrm{NS}(Y) \otimes_{\mathbb{Z}} \mathbb{C}$ is the *complexified Neron-Severi group.* By the fundamental work of Gromov and Yomdin, the above definition of entropy is equivalent to its original definition (cf. [1, §2.2] and the references therein). Further, when Y is smooth, the above two definitions of $d_1(*)$ coincide; for $\mathbb{Q}$-factorial Y, we have $d_1(f) = d_1(\widetilde{f})$ where $\widetilde{f}$ is the lifting of f to the one on an $\mathrm{Aut}(Y)$-equivariant resolution of Y. We call $\tau := g$ or f, of *positive entropy* (resp. *null entropy*) if $d_1(\tau) > 1$ (resp. $d_1(\tau) = 1$), or equivalently $h(\tau) > 0$ (resp. $h(\tau) = 0$) in the case of compact Kähler manifold. We say that the induced action $G \mid H^{1,1}(X)$ is *Z-connected* if its Zariski-closure in $\mathrm{GL}(H^{1,1}(X))$ is connected with respect to the Zariski topology; in this case, the *null set* $N(G) := \{g \in G \mid g$ is of null entropy$\}$ is a (necessarily normal) subgroup of G (cf. [2, Theorem 1.2]). In [2], we have proved:

Theorem 1. *Let X be an n-dimensional $(n \geq 2)$ compact complex Kähler manifold and G a subgroup of* $\mathrm{Aut}(X)$*. Then one of the following two assertions holds:*

(1) *G contains a subgroup isomorphic to the non-abelian free group $\mathbb{Z} * \mathbb{Z}$, and hence G contains subgroups isomorphic to non-abelian free groups of all countable ranks.*
(2) *There is a finite-index subgroup G_1 of G such that the induced action $G_1 \mid H^{1,1}(X)$ is solvable and Z-connected. Further, the subset $N(G_1) := \{g \in G_1 \mid g \text{ is of null entropy }\}$ of G_1 is a normal subgroup of G_1 and the quotient group $G_1/N(G_1)$ is a free abelian group of rank $r \le n-1$. We call this r* **the rank of G_1 and denote it as** *$r = r(G_1)$.*

Therefore, we are interested in the group $G \le \mathrm{Aut}(X)$ where $G \mid H^{1,1}(X)$ is solvable and Z-connected and that the rank $r(G) = \dim X - 1$ (maximal value). In the following, denote by $\mathrm{Aut}_0(X)$ the *identity connected component* of $\mathrm{Aut}(X)$. A group *virtually has a property* (P) if a finite-index subgroup of it has the property (P).

A complex torus has lots of symmetries. Conversely, our main result Theorem 2 says that the maximality $r(G) = \dim X - 1$ occurs only when X is a quotient of a complex torus T and G is mostly descended from the symmetries on the torus T.

Theorem 2. *Let X be an n-dimensional ($n \ge 3$) normal projective variety and $G \le \mathrm{Aut}(X)$ a subgroup such that the induced action $G \mid \mathrm{NS}_{\mathbb{C}}(X)$ is solvable and Z-connected and that the rank $r(G) = n-1$ (i.e., $G/N(G) = \mathbb{Z}^{\oplus n-1}$). Assume the three conditions:*

(i) *X has at worst canonical, quotient singularities.*
(ii) *X is a minimal variety, i.e., the canonical divisor K_X is nef.*
(iii) *The pair (X, G) is minimal, i.e., for every finite-index subgroup G_1 of G, every G_1-equivariant birational contraction from X onto some variety with only isolated canonical singularities, is an isomorphism.*

Then the following four assertions hold.

(1) *The induced action $N(G) \mid \mathrm{NS}_{\mathbb{C}}(X)$ is a finite group.*
(2) *$G \mid \mathrm{NS}_{\mathbb{C}}(X)$ is a virtually free abelian group of rank $n-1$.*
(3) *Either $N(G)$ is a finite subgroup of G and hence G is a virtually free abelian group of rank $n-1$, or X is an abelian variety and the group $N(G) \cap \mathrm{Aut}_0(X)$ has finite-index in $N(G)$ and is Zariski-dense in $\mathrm{Aut}_0(X)$ ($\cong X$).*
(4) *We have $X \cong T/F$ for a finite group F acting freely outside a finite set of an abelian variety T. Further, for some finite-index subgroup G_1 of G, the action of G_1 on X lifts to an action of G_1 on T.*

When $\dim X = 3$, the (i) in Theorem 2 can be replaced by: (i)' X has at worst canonical singularities (using a result of Shepherd-Barren and Wilson).

For non-algebraic manifolds, we have a result parallel to Theorem 2.

Theorem 2 answers [2, Question 2.17], assuming the conditions here. When G is abelian, the finiteness of $N(G)$ is proved in the inspiring paper of Dinh-Sibony [1, Theorem 1] (cf. also [3]), assuming only $r(G) = n-1$. For non-abelian G, the finiteness of $N(G)$ is not true and we can at best expect that $N(G)$ is virtually included in $\mathrm{Aut}_0(X)$ (as done in Theorem 2), since a larger group $\widetilde{G} := \mathrm{Aut}_0(X)\,G$ satisfies

$$\widetilde{G} \mid \mathrm{NS}_{\mathbb{C}}(X) = G \mid \mathrm{NS}_{\mathbb{C}}(X),\ N(\widetilde{G}) = \mathrm{Aut}_0(X).N(G) \geq \mathrm{Aut}_0(X),\ \widetilde{G}/N(\widetilde{G}) \cong G/N(G).$$

There are examples (X, G) with rank $r(G) = \dim X - 1$ and X complex tori or their quotients (cf. [1, Example 4.5], [3, Example 1.7]).

The conditions (i) - (iii) in Theorem 2 are quite necessary in deducing $X \cong T/F$ as in Theorem 2(4). Indeed, if $X \cong T/F$ as in Theorem 2(4), then X has only quotient singularities and $dK_X \sim 0$ (linear equivalence) with $d = |F|$, and we may even assume that X has only canonical singularities if we replace X by its global index-1 cover; thus X is a minimal variety. If the pair (X, G) is not minimal so that there is a non-isomorphic G_1-equivariant birational morphism $X \to Y$, then the exceptional locus of this morphism is G_1- and hence G-periodic, contradicting the fact that the rank $r(G) = n - 1$.

Theorem 3. *Let X be a projective minimal (terminal) variety of dimension n, with $n = 3$, and $G \leq \mathrm{Aut}(X)$ a subgroup such that $G \mid \mathrm{NS}_{\mathbb{C}}(X)$ is not virtually solvable. Then the radical $R(G) \mid \mathrm{NS}_{\mathbb{C}}(X)$ (the intersection of $G \mid \mathrm{NS}_{\mathbb{C}}(X)$ with the solvable radical of its Zariski-closure in $\mathrm{GL}(\mathrm{NS}_{\mathbb{C}}(X))$) is virtually unipotent and hence of null entropy; replacing G by a suitable finite-index subgroup, $G/R(G)$ is embedded as a Zariski-dense subgroup in an almost simple real linear algebraic group $H(\mathbb{R})$ which is either of real rank 1 or locally isomorphic to $\mathrm{SL}_3(\mathbb{R})$ or $\mathrm{SL}_3(\mathbb{C})$.*

Question 1. *In Theorem 3, if H is of real rank $n - 1$ (with n not assumed to be 3) is X birational to a quotient of an abelian variety of dimension n? (A positive answer is given by Cantat-Zeghib when G is a lattice in an almost simple Lie group of rank $n - 1$).*

References

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