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Correlations and Interactions for Random Quantum Systems

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Abstract. Random quantum systems cover a broad range of mathematical models from random Schrödinger operators to random matrices and quantum spin models with random parameters. Their understanding requires techniques which combine functional analysis and probability. The workshop brought together researchers from these various branches which discussed new results, methods and future challenges. This is a report on the meeting containing extended abstracts of the lectures.

Mathematics Subject Classification (2000): 47N50, 60K40, 81V70.

Introduction by the Organisers

The half-size workshop on *Correlations and Interactions for Random Quantum Systems* was organised by Peter D. Hislop (Lexington, KY), Werner Kirsch (Hagen), Peter Müller (München) and Simone Warzel (München). It was attended by 30 participants from Canada, Japan, the US and various European countries. The program consisted of 22 lectures covering new results, recent developments and future challenges in the field. Special attention was paid throughout to providing a platform for younger researchers. This report contains extended abstracts of these lectures. On behalf of all participants, the organisers would like to thank the staff and the director of the Mathematisches Forschungsinstitut Oberwolfach for providing such a stimulating and inspiring atmosphere.

Mathematical research on random quantum systems covers various branches: the theory of random Schrödinger operators, random matrices and the analysis of models in quantum statistical mechanics with random parameters. The common theme is to describe localization phenomena arising in different branches of quantum physics from condensed matter theory to quantum information processing. While being motivated by physics, these models pose interesting mathematical challenges in themselves which call for a combination of ideas from functional analysis to probability. The interplay of mathematical disciplines in this area is nicely illustrated by the topics covered in this workshop. For examples, A. Klein presented a joint work with J. Bourgain on a 30-year-old problem concerning the regularity of the density of states of rather general Schrödinger operators which need not be random. On the other hand, B. Virag gave a lecture on his recent results pertaining to the Brownian corousel, a stochastic processes which is related to the random process of eigenvalues of the Anderson model in dimension $d = 1$ in a weak disorder limit.

The Anderson model is the prototypical example of a random Schrödinger operator describing a single quantum particle in a random potential. One of its striking features is the occurrence of a dense point spectrum and the absence of transport for $d = 1$ and in higher dimension close to band edges or for strong disorder. For low disorder and $d > 2$, physicists expect a region with absolutely continuous spectrum and diffusive transport. This remains an open challenge for mathematicians. Recent progress related to proofs of transport in albeit different models than the Anderson model were presented at the meeting (e.g. in the talks by A. Joye and J. Schenker).

One other challenge in the field is to step beyond the framework of single-particle operators and investigate the localization properties of systems of many interacting particles. Some progress has been made in this direction: for a fixed, but finite number of particles the localization regime in the Anderson model was proven to be stable under finite-range interactions. However, for systems of infinitely many particles with a positive density, the precise formulation and stability of the localization regime generally remains an open problem. All the more remarkable are therefore recent results for special systems such as the one related to the Quantum Hall Effect and certain integrable models of quantum statistical mechanics in $d = 1$. The latter were presented at the workshop in the talks of B. Sims and G. Stolz. Moreover, these talks were conveniently framed by survey talks of B. Nachtergaele and L. Pastur explaining the greater challenges and questions to the community emerging from the field of quantum information theory and quantum statistical mechanics.

Workshop: Correlations and Interactions for Random Quantum Systems

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Abstracts

Dissipative Dynamics for Electrons in Semiconductors

Jean Bellissard
(joint work with G. Androulakis and C. Sadel)

Semiconductors are made of perfectly crystalline materials with a large gap. They are doped with a very low concentration of impurities. The groundstate of the valence electrons of the impurities have energy in a very narrow band close to either the valence band or the conduction band. At room temperature these electrons can thermally be activated to fill the bottom of the conduction band and produce electric current. This phenomenon is the reason why semiconductors can be used to built electronic devices such as transistors or diodes. At very low temperature, however, below the evaporation point of liquid Helium, the impurity electrons are confined in the impurity bands and are localized on the impurities. The problem is: *what mechanism permits to understand how these electrons are transported if a small electric field is switched on ?*. The answer was provided by physicists between 1960 and 1974. Namely, phonons are produced at a very low rate but can kick an electron away from its *mother-impurity*. It has to find another impurity free from electrons. This may happen at the price of crossing an energy barrier by tunneling effect. The Mott theory shows that the distance such electrons should cross is enormous, at least ten times the average distance between impurities. This is called *variable range hopping*. Mott, using this scheme, predicted that the conductivity must behave as

$$\sigma \sim e^{(T_0/T)^{1/(d+1)}} \qquad \textbf{(Mott's law)}$$

where d is the space dimension and T_0 a constant temperature characteristic of the material.

In order to construct a mathematically rigorous model liable to lead to a proof of the Mott law, several difficulties have to be taken into account:

(i) The impurity electrons see only the location of the impurities, which lie on a random sublattice. These sublattices break the translation invariance.

(ii) The impurity electrons must be treated in the second quantization scheme to take into account the correlation due to the Fermi statistics. The theory must be made in the infinite volume limit to account for the absence of surface phenomena. In particular this leads to use a quasilocal C^*-algebra of observables like the ones used for quantum spin systems or in algebraic quantum field theory.

(iii) The lack of translation invariance can be treated by using the formalism of *groupoids* through the ensemble of possible impurity lattices. This lead to a *covariant field* of C^*-algebras which has to be proved to be continuous.

(iv) The thermal equilibrium is described through a continuous covariant field of KMS-states of these algebras.

(v) The interaction electron-phonons is weak enough to be treated in the so-called *Markov approximation*, obtained after integrating out the phonon degrees of freedom. The generator of the Markov semigroup is a Lindbladian, with various jump operators that are built from the scheme proposed by Mott to describe the electron transport.

The main results are

(A) a precise description of this dynamics and various theorems proving the convergence of the model in the infinite volume limit,
(B) the proof of the existence of a unique equilibrium state for this dynamics,
(C) the proof that this model has a gap above the groundstate, showing that the return to equilibrium is exponentially fast in time.

References

[1] G. Androulakis, J. Bellissard, C. Sadel, "Dissipative dynamics in semiconductors at low temperature", `arXiv:1107.1248`, submitted to *J. Stat. Phys.*.

Dynamical localization for generic deterministic operators

Victor Chulaevsky

We study localization properties of parametric families of lattice Schrödinger operators (LSO) of the form

$$H(\omega) = \Delta + gV(x;\omega,\theta),\ x \in \mathbb{Z}^d,$$

where Δ is the nearest-neighbor lattice Laplacian and the potential V is generated by the values of a function $v : \Omega \times \Theta \to \mathbb{R}$ (called *the hull* of the potential V) along the trajectories of a dynamical system $T : \mathbb{Z}^d \times \Omega \to \Omega$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

$$V(x;\omega,\theta) = v(T^x\omega,\theta),\ \ \omega \in \Omega = \mathbb{T}^\nu,\ \theta \in \Theta.$$

The hulls are parameterized by elements θ of an auxiliary probability space (parameter space) $(\Theta, \mathfrak{B}, \mu)$.

Principal example: incommensurable shifts of a torus $\mathbb{T}^\nu$.

Our approach naturally extends to a more general situation where Ω is a compact Riemannian manifold with distance $\mathrm{dist}(\cdot,\cdot)$.

Principal assumption. "Uniformly slow returns" (USR):

$$\exists\, C, A \in (0,+\infty)\ \ \forall\,\omega \in \Omega\,\forall\, x \in \mathbb{Z}\quad \mathrm{dist}(T^x\omega,\omega) \geq C(1+|x|)^{-A}.$$

Additional assumption. Tempered local divergence of trajectories:

$$\exists\, C', B \in (0,+\infty)\ \ \forall\, x \in \mathbb{Z}\quad \mathrm{dist}(T^x\omega, T^x\omega') \leq C'(1+|x|)^B \mathrm{dist}(\omega,\omega')$$

for all $\omega, \omega' \in \Omega$. This assumption rules out hyperbolic systems, such as hyperbolic toral automorphisms, e.g., the "Arnold's cat map". Indeed, our method is adapted to dynamical systems with very weak disorder properties: speaking informally, we work with systems of zero entropy. In fact, even the ergodicity is not required *per se*, although it follows from the (USR) condition, e.g. for translations of the torus $\mathbb{T}^1$. As a result, a sufficiently rich auxiliary parameter space Θ is required to avoid 'abnormally small denominators' appearing in the scaling procedure which is a variant of the Multi-Scale Analysis (MSA).

Naturally, hyperbolic dynamical systems possess intrinsic mechanisms which should enhance localization phenomena, but in the author's opinion, these mechanisms should be exploited in a more direct way in the course of the MSA induction.

The key to an adaptation of the MSA to deterministic potentials with very weak disorder properties (including quasi-periodic potentials) is a special construction of the parameter space Θ or, more precisely, of the parametric families of hulls (referred to as "grand ensembles"). We give two different constructions.

1. "Randelette" expansions.

$$v(\omega,\theta) = \sum_{n=1}^{\infty} a_n \sum_{k=1}^{K_n} \theta_{n,k}\varphi_{n,k}(\omega)$$

where $\{\theta_{n,k}, n \geq 1, k \in [1, K_n\}$ are IID on $(\Theta, \mathfrak{B}, \mu)$ e.g., $\sim Unif([0,1])$, and $\operatorname{diam}\operatorname{supp}\varphi_{n,k} = O(2^{-n})$.

- Piecewise constant randelettes ("haarsh"; inspired by Haar's wavelets): $\theta_{n,k}(\omega) = \mathbf{1}_{[2^{-n}k, 2^{-n}(k+1)]}(\omega)$.
- C^M-randelettes (e.g., B-splines); possibly C^∞, but **not analytic**.

2. Gaussian random fields on Ω.

$v : \Omega \times \Theta \to \mathbb{R}$ is a Gaussian random field on Ω, relative to the probability space $(\Theta, \mathfrak{B}, \mu)$, with a.s. continuous samples and satisfying the following condition:

$\exists C'', b \in (0, +\infty)$ *such that for any ball* $B_\epsilon(\omega_0) \subset \Omega$ *of radius* $\epsilon > 0$ *the conditional variance of the random variable* $v(\omega_0, \cdot)$ *given all the values* $\{v(\omega, \cdot), \omega \notin B_\epsilon(\omega_0)\}$ *outside the ball admits a lower bound by* $C''\epsilon^b$.

In other words, the random field v is 'locally non-deterministic': the local interpolation problem must not admit an a.s. exact (i.e., unique) solution. This property can be expressed for homogeneous Gaussian random fields on a torus $\Omega = \mathbb{T}^\nu$ in terms of a tempered decay of their Fourier coefficients, playing here the role of spectral measure. Clearly, this **rules ou**t the case where samples are **analytic** and makes our approach complementary to that developed earlier by Bourgain, Goldstein and Schlag (cf. [1, 2, 3]).

Note that Chan [4] has proven Anderson localization for a family of 1D single-frequency quasi-periodic LSO with C^3-hulls, using a topological parameter exclusion technique different from ours.

Note also that despite the fact that the auxiliary parameter space Θ is constructed in probabilistic terms, this does not mean that one adds to the resulting potential $V(x;\omega,\theta)$, with fixed θ conserved by the dynamical system, a 'hidden' random noise. It is to be stressed that, for example, quasi-periodic operators $H(\omega,\theta)$, forming an ergodic ensemble for each fixed θ, are 'genuinely quasi-periodic'. Yet, the construction of the grand ensembles of deterministic potentials allows to adapt, in a very natural way, the probabilistic MSA techniques used so far only for 'genuinely random' (i.e., non-deterministic) potentials.

Theorem 1 (Main Theorem)**.** *$\exists\, g^* < \infty$ such that for $|g| \geq g^*$ $\exists\, \widetilde{\Theta}(g) \subset \Theta$ of measure $\geq 1 - Cg^{-1/2}$ such that $\forall\, \theta \in \widetilde{\Theta}(g)$ and for $\mathbb{P}$-a.e. ω, operator $H(\omega,\theta)$ has p.p. spectrum with exponentially decaying eigenfunctions:*

$$|\psi_j(x;\omega,\theta)| \leq C_j(\omega,\theta)\, e^{-m(g)\|x\|}$$

with $\lim\limits_{g\to+\infty} m(g) = +\infty$. Furthermore, for any finite set $\mathcal{K} \subset \mathbb{Z}^d$ and $\forall\, s > 0$

$$\mathbb{E}\left[\sup_t \|X^s e^{itH(\omega,\theta)}\mathbf{1}_{\mathcal{K}}\|\right] < \infty,$$

where X is defined by $(Xf)(x) = |x|f(x)$.

Minami-type estimates.

Theorem 2. *For $B' \in (0,+\infty)$ large enough, for any finite interval $I \subset \mathbb{R}$ and any $L > 0$*

$$\mu\left\{\operatorname{tr}\Pi_I\big(H_{\Lambda_L(u)}(\omega,\theta)\big) \geq J\right\} \leq C_J\, L^{B'J}|I|^J, \quad J \geq 1.$$

References

[1] J. Bourgain, and M. Goldstein, *On nonperturbative localization with quasiperiodic potentials.* Annals of Math., 2000, **152**, no. 3, 835–879.

[2] J. Bourgain, and W. Schlag, *Anderson localization for Schrödinger operators on $\mathbb{Z}$ with strongly mixing potential.* Commun. Math. Phys., 2001, **215**, 143–175.

[3] J. Bourgain, M. Goldstein, and W. Schlag, *Anderson localization for Schrödinger operators on $\mathbb{Z}$ with potential generated by skew-shift.* Commun. Math. Phys., 2001, **220**, 583–621.

[4] J. Chan, *Method of variations of potential of quasi-periodic Schrödinger equations.* Geom. Funct. Anal., **17**, 2007, 1416–1478.

[5] V. Chulaevsky, *Anderson localization for generic deterministic operators.* To appear in J. Funct. Anal.; cf. also arXiv/math-ph:1101.4892.

Snakes, Hybrids and non-classical edge modes, a semi-classical analysis

Nicolas Dombrowski

(joint work with P. Hislop and E. Soccorsi)

In this talk, we make a brief synthesis of previous results from [DGR] and recent one from [DHS]. We give a descriptions of edge-transport recently pointed out by physicists [RP]. Edge transport has received a lot of attention since a decade. The two main motivations coming from, first, the so-called quantum wave-guides and secondly, that the origin of transport in Quantum Hall systems (QHS) is believed, and proved in many cases, as being these Edge-transport.
Until now, all the models of edge considered in the past are of electrical nature having variations only in one direction implying the one dimensionality of the problem (soft wall) or considering Dirichlet restriction (hardwall).
Recently, physicist address the question of look at the magnetic counterpart of this phenomenon having in mind some magneto-electrical hybrid edge-mode.

In this work we are focus on precise analysis of the sole magnetic variation influence on the transport. First, we were interested in global proper characterization of such systems, namely we were looking for the quantization conductance. We have exhibit two classes of variations implying two different quantum topological numbers, what we call type I and type II magnetic profile.
We have proved that if we look at a translationally-invariant magnetic field, two kind of magnetic profiles exhibit qualitatively different behavior. More precisely, we call an edge-magnetic profile a magnetic field such that $B \in C^2(\mathbb{R})$, B is monotone and $\lim_{x \to \pm\infty} B(x) = B_\pm$. In this case we call type I potential such magnetic field adding the condition that $B_+ > B_- > 0$ and type II potential if the profile is antisymmetric i.e $B_+ = -B_- = B$. First, in [DGR], we have proved that the edge-conductance, namely σ_e^I with I a spectral window taken in the $n-^{th}$ so-called Landau-gap, has the same type of quantization than in the electrical edge-model and therefore (as proved in [SBKR]) equals to the Bulk conductance (model without edge). But, surprisingly for the type II potential we proved that this quantization are two times the bulk-quantization, pointing out a an new kind of transport phenomenon of which we now make the spectral analysis. We compare the spectral and transport-band estimates for both of these limit cases. In order to do this we choose a Toy model. This model was chosen for three reasons, first because it is the simplest one, avoiding us precise computations, secondly, being a limiting case it is very illustrative of what is happening really, and finally because it is the one used by physicists to perform numerical computations enabling us to compare the theoretical and numerical results. (furthermore our analysis provides us to perform more accurate numerical approximations.)
We would point out that for the Type I potential everything happen like we would hope from classical mechanic (by example: localization of wave functions near the magnetic one dimensional edge in a strip of width proportional to the square roots of the magnetic modulus), whereas for the type II there arise states spreading current in one direction having pure quantum nature and non-equivalent classical counterpart.
More precisely, we prove that considering a sharp case with $B(x) = B_+\chi_{\mathbb{R}^+} + B_-\chi_{\mathbb{R}^-}$. We look at the Toy model given by $H_\bullet(B_-, B_+) := p_x^2 + (p_y - \beta_\bullet(x))^2$ on $L^2(\mathbb{R}^2)$ with $p_x = -i\nabla_x$ and $\beta_\bullet$ being for $\bullet = I$ (respc. for $\bullet = II$) with $B_+ > B_- > 0$ (respc. for $B_+ = B_- = B$). We can decompose the two dimensional problem into a one dimensional one by partial Fourier Transform. Thus, we are looking for $h_\bullet(k, B_\pm) = p_x^2 + (k + \beta_\bullet(x))^2$ acting on $L^2(\mathbb{R})$. We denote $\{\omega_j(k, B_\pm)\}_{j\in\mathbb{N}}$ the dispersive curve e.g the eigenvalues of $h_\bullet(k, B_\pm)$ and $\phi_j(k, B_\pm)$ the corresponding wave functions.
We then make the spectral analysis in order to get a better understanding of the local behavior in the momentum space.

Type I potential: In this case the effective potential is a simple-well with a singularity coming from the magnetic jump at zero, we call this jump the magnetic singularity. Being in a harmonic-well like potential, we can get estimate *à la De Bièvre-Pullé*, (see [DBP]) as follow. More precisely, we get a lower bound on the

band-velocity, the current spreading and finally get an estimation on the exponential localization of these edge-states near the magnetic singularity in a region delimited by the magnetic radius, namely $\sqrt{B_-}$ (*respc.* $\sqrt{B_+}$) on the left (*respc.* on the right).
We recall that the current operator in this case is given by $\mathbf{v}_y := p_y - \beta(x)$ and we denote the associated functional given by $\mathcal{J}_y(\varphi) := \langle \mathbf{v}_y\varphi, \varphi\rangle$. We proved that for $N \in \mathbb{N}^*$, there exist positive constants $C_{N,j}, \tilde{C}_{N,j}$ and such that for any $j \in \{1, \cdots, \mathbb{N}\}$ and some $\Delta_j \subset \Big((2j-1-\tilde{C}_{N,j})B_-, (2j+1-\tilde{C}_{N,j})B_+\Big)$ we have for φ such that $\varphi = \mathbf{P}_{\Delta_j}(H)\varphi$ with $\mathbf{P}$ the spectral projector associated to the Hamiltonian. We then get that for φ such that $\varphi = \mathbf{P}_{\Delta_j}(H)\varphi$ that $\mathcal{J}_y(\varphi) \geq C(B_\pm)\|\varphi\|^2$ and finally that $\int_{\mathbb{R}^2} \chi_{I_{\epsilon,\eta}}|\varphi|^2 dxdy \geq 1 - C(\tilde{B}_\pm)e^{\eta B_-^2 \epsilon}$ with $I_{\epsilon,\eta} := [-c_\eta B_-^{-\frac{1}{2}+\epsilon}; c_\eta B_+^{-\frac{1}{2}+\epsilon}]$.
We have then finally all the expected properties of these edge-modes adding to the usual quantization coming from that the spectral flows are topologically equivalents. What is surprising is that it is not the case for the Type II potential as we can conclude by its semi-classical analysis.

Type II potential: This case is more subtle and more rich of applications in a new type of technological devices (see physicist paper [RP]).
In this case we will use semi-classical analysis, being more adapted to this local analysis. In such a way we recall that perform an analysis with fixed Planck constant and strong magnetic field is equivalent (by scaling) to perform an semi-classical analysis (i.e small $\hbar$) with fixed magnetic field. Arguing of that, we now consider the following local operator, $h(\hbar, k) = \hbar^2 p_x^2 + (k - |x|)^2$. First of all, we can get some information on the global behavior using the symmetries of the hamiltonian and get a Bolley-Dauge-Helffer formula, and that $w'_{2n+1} < 0$ for any $k \in \mathbb{R}$ and that $w'_{2n}(k) < 0$ for $k \in (-\infty, k_0)$ and $w'_{2n}(k) > 0$ for $k \in (k_0, +\infty)$ which shows that we have a qualitatively very different behavior of the dissipative curves. In order to get a local behavior we must perform a semi-classical analysis. We distinguish two different behaviors, for k negative or positive.
First for $k \leq 0$, the potential is a one-well having only one minimum at the origin with value k^2. Thus, this is the linear term of the potential which is dominating the spectral behavior. And so on, performing the semi-classical approximation of the spectrum (see [BDPR]) we get an asymptotic given by

$$\omega_{2n}(\hbar, k) \sim_{\hbar\sim 0} k^2 - \mathcal{Z}_n^{'}(\hbar|k|)^{\frac{2}{3}} + \mathcal{O}(\hbar k)^{\frac{4}{3}}$$
$$\omega_{2n+1}(\hbar, k) \sim_{\hbar\sim 0} k^2 - \mathcal{Z}_n(\hbar|k|)^{\frac{2}{3}} + \mathcal{O}(\hbar k)^{\frac{4}{3}}$$

with $\mathcal{Z}_n$ (*respc.* $\mathcal{Z}'_n$) the $n-^{th}$ zero of the Airy function (*respc.* the $n-^{th}$ zero of the derivative of the Airy function).
Secondly, for k positive we have that the effective potentials is given by a symmetric double-well leading to tunneling. We have to use the Helffer-Söjstrand technology of interaction matrix and we have that the decay of eigenvalue is controlled by the Agmon-like estimate. And so we get that the splitting between the two eigenvalues

around the Landau level is given by $\omega_1(\hbar,k)-\omega_0(\hbar,k) \sim_{\hbar\sim 0} C(\hbar,k)\mathcal{O}(e^{\frac{-k^2}{\hbar}})$, where the even eigenvalues are corresponding to the Von-Neumann case and the odd case corresponds to the Dirichlet one and with $C(\hbar,k)$ growing at most polynomially w.r.t. $\hbar$ and k.
In conclusion, in the first paper we have, by the study of conductance, exhibit two types of edge profiles. After that we have done the local analysis of dispersive curve giving spectral interpretation to this difference of behaviors. In the same way we have pointed out the pure quantum nature of certain of these magnetic edge-states called non-classical edge-states. We see that the topological nature of the band spectrum is strictly non-equivalent to the usual Quantum Hall one, justifying that the conductance has a different quantization relying on the idea that the characterization of Hall systems by topological invariant take its origin in the spectral flow topology.

References

[BDPR] V. Bonnaillie-Noël, M. Dauge, N. Popoff, and N. Raymond, *Eigenpairs of a momdel Schrödinger Operator with Neumann boundary conditions.* Zeitschrift für Angewandte Mathematik (2011).

[CHS] J.-M. Combes, P.D. Hislop, and E. Soccorsi, *Edge states for quantum Hall Hamiltonians.* Mathematical results in quantum mechanics (Taxco, 2001), 69–81, Contemp. Math. **307**, Amer. Math. Soc., Providence, RI, 2002

[DBP] S. De Bièvre and J. Pulé, *Propagating edge states for a magnetic Hamiltonian.* Math. Phys. Elec. J. **5** (1999), paper 3.

[DGR] N. Dombrowski, F. Germinet, and G. Raikov, *Quantization of the edge currents for magnetic models.* Ann. Henri-Poincaré **12** (2011), 1169–1197.

[DHS] N. Dombrowski, P. Hislop, and E. Soccorci, In preparation

[Iw] A. Iwatsuka, *Examples of absolutely continuous Schrödinger operators in magnetic fields.* Publ. Res. Inst. Math. Sci. **21** (1985), 385–401.

[RP] J. Reijniers and F.M. Peeters, *Snake orbits and related magnetic edge states.* J. Phys.: Condens. Matter **12** (2000), 9771–9786.

[SBKR] H. Schulz-Baldes, J. Kellendonk, and T. Richter, *Simultaneous quantization of edge and bulk Hall conductivity.* J. Phys. A **33** (2000), L27-L32.

Fractional moment analysis for random multiparticle Schrödinger operators

Michael Fauser

(joint work with S. Warzel)

We report on recent work concerning localization properties of systems of n interacting quantum particles in d-dimensional space subject to an alloy-type random potential. The Hamiltonian under consideration acts on $L^2(\mathbb{R}^{dn})$ and has the form

$$H^{(n)}(\omega) = \sum_{j=1}^{n}(-\Delta_j + V_0(x_j) + V(\omega, x_j)) + \alpha \sum_{j<k} w(x_j - x_k)\,,$$

where Δ_j is the Laplacian acting on the coordinates x_j of the j-th particle, V_0 is a $\mathbb{Z}^d$-periodic bounded background potential and $w \in L_c^\infty(\mathbb{R}^d)$ corresponds to a two-body interaction whose strength is controlled by the parameter $\alpha \geq 0$. The random potential $V(\omega)$ is of the form

$$V(\omega, x_j) = \sum_{\zeta \in \mathbb{Z}^d} \eta_\zeta(\omega) U(x_j - \zeta)\,,$$

where the single-site potential $U \geq 0$ is bounded and compactly supported and the random variables $(\eta_\zeta)_{\zeta \in \mathbb{Z}^d}$ are independent and identically distributed with a Lebesgue density $\rho \in L_c^\infty(\mathbb{R})$. In addition, we require the single-site potential to satisfy a covering condition, i.e., $\inf_{x_j} \sum_\zeta U(x_j - \zeta) > 0$.

The goal of our work is to prove localization in an interval $I = [E_0^{(n)}, E_0^{(n)} + \eta^{(n)}]$ at the bottom $E_0^{(n)} = \inf \sigma(H^{(n)})$ of the deterministic spectrum of the n-particle Hamiltonian. More specifically, we obtain dynamical localization in the sense of estimates of the form

$$\mathbb{E}\big[\sup_{t \in \mathbb{R}} \|\chi_{\mathbf{x}} e^{-itH^{(n)}} P_I(H^{(n)}) \chi_{\mathbf{y}}\|\big] \leq Ce^{-\mu \mathrm{dist}_H(\mathbf{x},\mathbf{y})},$$

where $P_I(H^{(n)})$ denotes the spectral projection of $H^{(n)}$ onto the interval I, $\chi_{\mathbf{x}}$ and $\chi_{\mathbf{y}}$ are characteristic functions of the unit balls around the configurations $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{dn}$ and $\mathrm{dist}_H(\mathbf{x}, \mathbf{y})$ denotes the Hausdorff distance of the sets $\{x_j | j \in \{1, \ldots, n\}\}$ and $\{y_j | j \in \{1, \ldots, n\}\}$.

In the one-particle case, results of this type have been obtained for certain models satisfying the above assumptions, cf. [1] and references therein. For our multiparticle results, i.e., $n \geq 2$, we will assume that an interval $[E_0^{(1)}, E_0^{(1)} + \eta^{(1)}]$ of localization for the one-particle operator exists.

Two different notions of localization in an interval I play an important role in our analysis. For this purpose, it is convenient to consider the Hamiltonian on finite volumes Ω^n, where $\Omega \subset \mathbb{R}^d$ is open and bounded. The restriction of $H^{(n)}$ to Ω^n with Dirichlet boundary conditions is denoted by $H_\Omega^{(n)}$.

Definition 1. An interval I is a regime of *n-particle fractional moment localization* if there are $C, \mu > 0$ and $s \in (0, 1)$ such that

$$\sup_{\substack{\mathrm{Re} z \in I \\ |\mathrm{Im} z| \leq 1}} \mathbb{E}\big[\|\chi_{\mathbf{x}}(H_\Omega^{(n)} - z)^{-1} \chi_{\mathbf{y}}\|^s\big] \leq Ce^{-\mu \mathrm{dist}_H(\mathbf{x},\mathbf{y})}$$

holds for all open and bounded sets $\Omega \subset \mathbb{R}^d$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{dn}$ satisfying $B_1(\mathbf{x}) \cap \Omega^n, B_1(\mathbf{y}) \cap \Omega^n \neq \emptyset$.
An interval I is a regime of *n-particle eigenfunction correlator localization* if there exist $C, \mu > 0$ such that

$$\mathbb{E}\Bigg[\sum_{E \in \sigma(H_\Omega^{(n)}) \cap I} \|\chi_{\mathbf{x}} P_{\{E\}}\big(H_\Omega^{(n)}\big) \chi_{\mathbf{y}}\|\Bigg] \leq Ce^{-\mu \mathrm{dist}_H(\mathbf{x},\mathbf{y})}$$

holds for all open and bounded sets $\Omega \subset \mathbb{R}^d$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{dn}$ satisfying $B_1(\mathbf{x}) \cap \Omega^n, B_1(\mathbf{y}) \cap \Omega^n \neq \emptyset$.

These two concepts of localization are closely related: Fractional moment localization in a bounded interval I implies eigenfunction correlator localization in I, which in turn implies fractional moment localization in intervals $J \subset I$ satisfying $\operatorname{dist}(J, \partial I) > 0$. In addition, dynamical localization in the sense discussed above follows easily from eigenfunction correlator localization.

Our main result is the following:

Theorem 1. *Assume that the interval* $[E_0^{(1)}, E_0^{(1)}+\eta^{(1)}]$ *is a regime of one-particle fractional moment localization. Then the following holds:*

- *For any* $\eta^{(n)} \in (0, \eta^{(1)})$*, there exists* $\alpha^{(n)} > 0$ *such that for all* $\alpha \in [0, \alpha^{(n)}]$ *the interval* $[E_0^{(n)}, E_0^{(n)} + \eta^{(n)}]$ *is a regime of* n*-particle fractional moment localization.*
- *If* $w \geq 0$*, then for any* $\alpha \geq 0$ *there exists* $\eta^{(n)} \in (0, \eta^{(1)})$ *such that the interval* $[E_0^{(n)}, E_0^{(n)} + \eta^{(n)}]$ *is a regime of* n*-particle fractional moment localization.*

The first part of the theorem focuses on the case of weak interactions, whereas the second part addresses the case of repulsive interactions of arbitrary strength, where we obtain localization in a smaller interval. An extension of this result to the case of attractive interactions, i.e., $w \leq 0$, will be the focus of further research.

The proof of our result is an induction on the number of particles that makes use of the close relation between fractional moment localization and eigenfunction correlator localization. It is very similar to what was done in [2], where localization in an Anderson model for n particles in $\mathbb{Z}^d$ was proved. In particular, one studies the implications of localization in subsystems. In this regard, it is important to note that if $J \dot\cup K = \{1, \dots, n\}$ is a partition of the n-particle system into two subsystems and $H^{(J,K)}$ is the n-particle Hamiltonian with interactions between the clusters J and K removed, then $\sigma(H^{(J,K)}) \subseteq \sigma(H^{(n)})$ almost surely. Apart from these considerations specific to the multiparticle nature of the problem, the proof relies heavily on techniques that were developed for the one-particle continuum model in [1].

There is also a different approach to multiparticle localization via a multiscale analysis instead of the analysis of fractional moments. It has been developed for continuum as well as discrete models, cf. [3, 4, 5] and references therein.

References

[1] M. Aizenman, A. Elgart, S. Naboko, J. H. Schenker and G. Stolz, *Moment analysis for localization in random Schrödinger operators*, Invent. Math. **163** (2006), 343–413.

[2] M. Aizenman and S. Warzel, *Localization bounds for multiparticle systems*, Comm. Math. Phys. **290** (2009), 903–934.

[3] V. Chulaevsky, A. Boutet de Monvel and Y. Suhov, *Dynamical localization for a multi-particle model with an alloy-type external random potential*, Nonlinearity **24** (2011), 1451–1472.

[4] V. Chulaevsky and Y. Suhov, *Eigenfunctions in a two-particle Anderson tight binding model*, Comm. Math. Phys. **289** (2009), 701–723.

[5] V. Chulaevsky and Y. Suhov, *Multi-particle Anderson localisation: induction on the number of particles*, Math. Phys. Anal. Geom. **12** (2009), 117–139.

Spectral concentration estimates for trees

Richard Froese
(joint work with W. Kirsch, M. Krishna and W. Spitzer)

This talk is a preliminary report on work in progress. Let T be a $(d+1)$-regular infinite tree and let

$$H = \Delta + \kappa q$$

be an Anderson Hamiltonian acting on $\ell^2(T)$. Specifically, Δ is the adjacency operator for T, q is an i.i.d. random potential whose single site distribution has bounded support, and $\kappa \geq 0$ is a coupling constant.

Let X_n be a $(d+1)$-regular labeled graph with $|X_n| = n$, chosen uniformly at random and let

$$H_n = \Delta_n + \kappa q$$

acting on $\ell^2(X_n)$, where Δ_n is the adjacency matrix for X_n.

We wish to determine how well H_n approximates H as n tends to infinity by comparing the density of states for these operators. Pick vertices $0 \in T$ and $0 \in X_n$ and define e_0 in $\ell^2(T)$ and also in $\ell^2(X_n)$ by

$$e_0(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0. \end{cases}$$

Let χ_I denote the indicator function of the interval I. For H, the density of states at the site 0 for the energy interval I is defined as

$$\begin{aligned} \mathbb{E}\left[\,\mathrm{tr}\,(|e_0\rangle\langle e_0|\chi_I(H))\,\right] &= \mathbb{E}\left[\,\langle e_0, \chi_I(H)e_0\rangle\,\right] \\ &= \mathbb{E}\left[\int \chi_I(x)d\mu(x)\right] \end{aligned}$$

where $d\mu$ is the spectral measure for e_0. The density of states does not depend on the choice of 0. For H_n the density of states is defined as

$$\begin{aligned} \mathbb{E}\left[\,\mathrm{tr}\,(|e_0\rangle\langle e_0|\chi_I(H_n))\,\right] &= \mathbb{E}\left[\,\langle e_0, \chi_I(H_n)e_0\rangle\,\right] \\ &= \frac{1}{n}\mathbb{E}\left[\,\mathrm{tr}\chi_I(H_n)\,\right] \\ &= \frac{1}{n}\#\{\sigma(H_n) \cap I\} \\ &= \mathbb{E}\left[\int \chi_I(x)d\rho_n(x)\right]. \end{aligned}$$

Here $d\rho_n(x) = \frac{1}{n}\sum_{\lambda\in\sigma(H_n)} \delta(x-\lambda)$ is the empirical counting measure.

To compare the density of states, we introduce the respective Green functions

$$G(E+i\epsilon) = \langle e_0, (H-E-i\epsilon)^{-1}e_0\rangle$$
$$G_n(E+i\epsilon) = \langle e_0, (H_n-E-i\epsilon)^{-1}e_0\rangle,$$

whose expectations are the Stieltjes transforms of the density of states measures $\mathbb{E}d\mu$ and $\mathbb{E}d\rho_n$. In both cases their distribution does not depend on the choice of 0.

Our main observation is that since a point in X_n typically has a tree neighbourhood with $\log(n)/\log(d)$ levels [4], we can compare the Green functions for X_n and T using contraction estimates similar to those that have been used to prove the existence of absolutely continuous spectrum for H [2, 3]. For fixed ϵ and small κ an ϵ dependent contraction estimate shows that

$$\mathbb{E}\,|G_n(E+i\epsilon) - G(E+i\epsilon)| \to 0$$

as $n\to\infty$. This implies that the density of states measure for H_n converges vaguely to that of H. When $\kappa = 0$ this is a classical result of McKay [5]. Our goal is to go beyond this and show that for a sequence $\epsilon_n \to 0$ we have

$$\mathbb{E}\,|G_n(E+i\epsilon_n) - G(E+i\epsilon_n)| \to 0,$$

for $|E| < 2\sqrt{d}$ and κ small. This would imply spectral concentration estimates. We are able to do this if we let the coupling constant $\kappa = \kappa_n$ also depend on n with $\kappa_n \to 0$. In this case G can be replaced by Green function of the tree without a potential. This is the (deterministic) Stieltjes transform of the Kesten-McKay law. We can also consider the case where the co-ordination number increases with n. If $d = d_n$ with $d_n\to\infty$ and if we scale H_n by $1/\sqrt{d_n}$ a similar estimate is true where G is replaced by Stieltjes transform of the semi-circle law. The random graph case where $\kappa = 0$ and $d_n \to \infty$ has been the subject of recent activity and stronger results are known [1].

References

[1] I. Dumitriu and S. Pal, *Sparse regular random graphs: Spectral density and eigenvectors.* arXiv 0910:5306v4 (2011).

[2] R. Froese, D. Hasler, and W. Spitzer, *Absolutely continuous spectrum for the Anderson Model on a tree: a geometric proof of Klein's theorem.* Comm. in Math. Phys. **269** (2007), 239–257.

[3] R. Froese, D. Hasler, and W. Spitzer, *A geometric approach to absolutely continuous spectrum for discrete Schrödinger operators.* Progress in Probability **64** (2011), 201–226.

[4] E. Makover and J. McGowan, *Regular trees in random regular graphs.* arXiv 0610:858v2 (2010).

[5] B. D. McKay, *The expected eigenvalue distribution of a random labelled regular graph.* Linear Algebra and its Applications **40** (1981), 203–216.

Random block operators

Martin Gebert
(joint work with P. Müller)

We study spectral properties of random block operators which arise in the modeling of mesoscopic disordered systems such as dirty superconductors. More precisely we consider a system of fermions and approximate the second quantized Hamiltonian by a self consistent field method introduced by the BCS theory of superconductivity. This yields the Boguliobov-de Gennes equations [dG66] and, moreover, restricting ourselves to the symmetry class $C1$ in [AZ97], the following eigenvalue problem for a quasi-particle excitation in block operator form

$$\begin{pmatrix} H & B \\ B & -H \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = E \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}.$$

Here $H := H_0 + V$ is the one-particle Hamiltonian and B the so called pair potential. From now on we assume the Hilbert space to be $l^2(\mathbb{Z}^d) \oplus l^2(\mathbb{Z}^d)$. We introduce randomness by letting $V := V_\omega$ and $B := B_\omega$ be multiplication operators by two independent *iid* sequences of random variables with single site distribution μ_V, respectively μ_B. In particular, $H := H_\omega$ is the Hamiltonian of the Anderson model. Throughout we denote the random block operator defined in this way by $\mathbb{H}$.

Considering this model, Kirsch, Metzger and Müller [KMM11] proved that the integrated density of states $\mathbb{N}$ of $\mathbb{H}$ exists and is self-averaging. They also found that there is a robust spectral gap of $\mathbb{H}$ around energy 0 if H or B is bigger than a positive constant and that under these circumstances one can prove a Wegner estimate, i.e. we have a Lipschitz continous integrated density of states. Here one has to cope with a non monotone dependence of the eigenvalues on the potential. First, we extend this Wegner estimate to operators without a gap in the spectrum. However, we still we have to assume that H and B are non-negative.

Theorem 1 (Wegner estimate [Geb11])**.** *Assume that $B \geq 0$ and $V \geq 0$. Moreover, suppose that μ_V and μ_B are absolutely continuous with piecewise continuous Lebesgue densities ϕ_V, ϕ_B of bounded variation and compact support.*
Then the integrated density of states $\mathbb{N}$ of $\mathbb{H}$ is Lipschitz continuous with a bounded density $\mathbb{D} := d\mathbb{N}/dE$ satisfying

$$\|\mathbb{D}\|_\infty \leq 2\left(\|\phi_V\|_{BV} + \|\phi_B\|_{BV}\right). \tag{1}$$

Here, $\|\cdot\|_{BV}$ denotes the bounded-variation norm.

In [KMM11] one can also find an approach to deduce Lifschitz tails of the integrated density of states near the spectral gap around 0 assuming once again H bounded away from 0. We improve this statement again to non-negative H. Before stating the result precisely, we note that the spectral gap around energy 0 is precisely given by $(-\sqrt{\lambda^2+\beta^2}, \sqrt{\lambda^2+\beta^2})$ provided the bounds $H \geq \lambda$ and $B \geq \beta$ are sharp for $\lambda, \beta \geq 0$ [KMM11].

Theorem 2 (Lifschitz Tails [Geb11]). *Assume $H \geq \lambda$ for some $\lambda \geq 0$ and that at least one of the following conditions is fulfilled:*

(1) *$B \geq \beta$, for some $\beta \geq 0$,*
(2) *$-B \geq \beta$, for some $\beta \geq 0$,*
(3) *$0 \in \sigma(B)$, in which case we set $\beta := 0$.*

Then we have for the integrated density of states $\mathbb{N}$ the following Lifschitz-tail behaviour

$$\limsup_{\epsilon \searrow 0} \frac{\ln \left| \ln \left[\mathbb{N}(\sqrt{\lambda^2+\beta^2}+\epsilon) - \mathbb{N}(\sqrt{\lambda^2+\beta^2}) \right] \right|}{\ln \epsilon} \leq -\frac{d}{2}. \tag{2}$$

Remark 1. (1) The proof of the Lifschitz-tails result relies heavily on a variational principle for block operators [Tre08]. Simplified to our case the variational principle tells us that once we have $H > 0$ we obtain for the n-th positive eigenvalue of $\mathbb{H}$ the formula

$$\lambda_n = \min_{\substack{\mathcal{L} \subset \mathcal{H} \\ dim\mathcal{L}=n}} \max_{\substack{f \in \mathcal{L} \\ \|f\|=1}} \max_{\substack{g \in \mathcal{H} \\ \|g\|=1}} \frac{(\langle f, Hf\rangle - \langle g, Hg\rangle)}{2} + \sqrt{\left(\frac{\langle f, Hf\rangle + \langle g, Hg\rangle}{2}\right)^2 + |\langle f, Bg\rangle|^2}.$$

(2) The assumptions of the theorem include cases where the spectral gap around 0 is closed but nevertheless we infer Lifschitz tails at energy 0.

Having those results, it is natural to ask if we can infer localization in some spectral regime near the inner band edges. The answer is given by

Theorem 3 (Localization [Geb11]). *Assume $H \geq \lambda$ for some $\lambda \geq 0$ and that the single site distributions μ_V and μ_B fulfill the hypotheses of the Wegner estimate. Then there exists some interval $I = [-a, a]$, $a > 0$, with $\sigma(\mathbb{H}) \cap I \neq \emptyset$ such that a.s.*

$$\sigma(\mathbb{H}) \cap I = \sigma_{pp}(\mathbb{H}) \cap I, \tag{3}$$

where σ_{pp} denotes the pure point spectrum.

Remark 2. The proof is done by the multiscale analysis, and also yields exponential decay of the eigenfunctions. In order to adapt the multiscale analysis to the block operator case without changing its formal structure it is advantageous to formulate it in terms of Green's functions with a 2×2-matrix-valued kernel

$$G_E(n,m) := (\mathbb{H} - E)^{-1}(n,m)$$

$$= \begin{pmatrix} \left\langle \begin{pmatrix} \delta_n \\ 0 \end{pmatrix}, (\mathbb{H}_{\Lambda_L} - E)^{-1} \begin{pmatrix} \delta_m \\ 0 \end{pmatrix} \right\rangle & \left\langle \begin{pmatrix} \delta_n \\ 0 \end{pmatrix}, (\mathbb{H}_{\Lambda_L} - E)^{-1} \begin{pmatrix} 0 \\ \delta_m \end{pmatrix} \right\rangle \\ \left\langle \begin{pmatrix} 0 \\ \delta_n \end{pmatrix}, (\mathbb{H}_{\Lambda_L} - E)^{-1} \begin{pmatrix} \delta_m \\ 0 \end{pmatrix} \right\rangle & \left\langle \begin{pmatrix} 0 \\ \delta_n \end{pmatrix}, (\mathbb{H}_{\Lambda_L} - E)^{-1} \begin{pmatrix} 0 \\ \delta_m \end{pmatrix} \right\rangle \end{pmatrix}$$

where $n, m \in \mathbb{Z}^d$ and δ_n denotes the canonical basis vector of $l^2(\mathbb{Z}^d)$ associated with site n. The only remaining change is then to replace the modulus of the Green's function's kernel in the standard case by the matrix norm of the 2×2-matrix kernel above.

References

[AZ97] A. Altland, and M. Zirnbauer, *Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures.* Phys. Rev. B **55** (1997), 1142–1161.
[dG66] P. G. de Gennes, *Superconductivity of Metals and Alloys.* Benjamin, New York, 1966.
[Geb11] M. Gebert, *Anderson Localization for random block operators.* Diploma thesis, LMU München, 2011.
[KMM11] W. Kirsch, B. Metzger, and P. Müller, *Random block operators.* Journal of Statistical Physics **143** (2011), 1035–1054.
[Tre08] C. Tretter, *Spectral theory of block operator matrices and applications.* Imperial College Press, London, 2008.

Enhanced Wegner and Minami estimates and applications to eigenvalue statistics for the Anderson model

François Germinet
(joint work with F. Klopp)

We consider the discrete Anderson Hamiltonian $H_\omega := H_0 + V_\omega$, acting on $\ell^2(\mathbb{Z}^d)$, where H_0 is a convolution matrix with exponentially decaying off-diagonal coefficients i.e. exponential off-diagonal decay that is $H_0 = ((h_{k-k'}))_{k,k' \in \mathbb{Z}^d}$ such that, $h_{-k} = \overline{h_k}$ for $k \in \mathbb{Z}^d$ and such that for some $c > 0$, $|h_k| \leq \frac{1}{c} e^{-c|k|}$, $k \in \mathbb{Z}^d$. Next V_ω is an Anderson potential: $V_\omega(x) := \sum_{j \in \mathbb{Z}^d} \omega_j \Pi_j$, where Π_j is the projection onto site j, and $\omega = \{\omega_j\}_{j \in \mathbb{Z}^d}$ is a family of independent identically distributed random variables whose common probability distribution μ is non-degenerate and has a bounded density g.
We denote by $\Sigma \subset \mathbb{R}$ the almost sure spectrum of H_ω, and by $\Sigma_{SDL} \subset \Sigma \subset \mathbb{R}$ the set of energies where strong dynamical localization holds (see e.g. [GK06]). We denote by $H_\omega(\Lambda)$ the restriction of H_ω to a given cube Λ, and by $1_I(H)$ the spectral projection associated to the operator H onto the interval I. The Integrated Density of States (IDS) is denoted by $N(I) = \mathbb{E}\mathrm{tr}(1_I(H_\omega)\Pi_0)$ where I is an interval, and we set $N(E) = N(]-\infty, E])$ when $E \in \mathbb{R}$. We have the following results.

Fix $\xi \in (0,1)$. Let $L > 1$ and $I \subset \Sigma_{SDL}$ be a compact interval so that $|N(I)| \geq C \exp(-cL^\xi)$ then, the following estimates holds:

$$\text{(1)} \qquad \text{(Wegner)} \qquad \mathbb{E}(\mathrm{tr} 1_I(H_\omega(\Lambda))) \leq 2N(I)|\Lambda|,$$

and

$$\text{(2)} \qquad \text{(Minami)} \qquad \mathbb{E}\left[\mathrm{tr} 1_I(H_\omega(\Lambda))(\mathrm{tr} 1_I(H_\omega(\Lambda)) - 1)\right] \leq 2N(I)|I||\Lambda|^2.$$

Let E_0 be an energy in Σ_{SDL}. As in [Mol82, Min96] we investigate the local statistics of the eigenvalues, and extend results of [GKl10] to situations where the

IDS is very small. The *unfolded local level statistics* near E_0 is the point process defined by

$$\Xi(\xi; E_0, \omega, \Lambda) = \sum_{j\geq 1} \delta_{|\Lambda|(N(E_j(\omega,\Lambda))-N(E_0))}(\xi). \tag{3}$$

Pick E_0 be an energy in Σ_{SDL} such that the integrated density of states satisfies, for some $\rho \in (0, 1/d)$, $\exists\, a_0 > 0$ s.t. $\forall a \in (-a_0, a_0) \cap (\Sigma - E_0)$,

$$|N(E_0 + a) - N(E_0)| \geq e^{-|a|^{-\rho}}. \tag{4}$$

When $|\Lambda| \to +\infty$, the point process $\Xi(E_0, \omega, \Lambda)$ converges weakly to

- a Poisson point process on the real line with intensity 1 if $E_0 \in \overset{\circ}{\Sigma}$, the interior of Σ.
- a Poisson point process on the half line with intensity 1 if $E_0 \in \partial\Sigma$, the half-line being $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) if $(E_0-\varepsilon, E_0)\cap\Sigma = \emptyset$ (resp. $(E_0, E_0+\varepsilon)\cap\Sigma = \emptyset$) for some $\varepsilon > 0$.

As a direct consequence, we obtain Poisson statistics at the edge of the spectrum in dimension 1. To our best knowledge, this is the first such result.

Next, the enhanced Wegner and Minami estimates (1)-(2) enable us to extend results obtained in [GKl10, Kl10] on the asymptotic ergodicity of the local eigenvalue distribution and on eigenlevel spacings statistics.

Finally, in some regimes, we also can improve upon the deviation estimate obtained for the eigenvalue counting function in [GKl10] for which we also prove a central limit theorem:

For $L > 1$, let $\Lambda = \Lambda_L$. Pick a sequence of compact intervals $I_\Lambda \subset \Sigma_{SDL}$ so that, for some $1 \leq \beta \leq \beta' < \alpha' \leq \alpha < \infty$, for all L, one has

$$|I_\Lambda|^{-\alpha'} \lesssim |\Lambda| \lesssim |I_\Lambda|^{-\alpha} \quad \text{and} \quad |I_\Lambda|^{\beta'} \lesssim N(I_\Lambda) \lesssim |I_\Lambda|^{\beta}. \tag{5}$$

Set $\nu_0 = \dfrac{1}{\alpha - \beta} \min\left(\alpha' - \beta', \dfrac{1}{d+1}\right)$.

(1) **Deviation estimate.** For $\varepsilon > 0$ small enough (depending on ν_0), we have

$$\mathbb{P}\left\{|\mathrm{tr}1_{I_\Lambda}(H_\omega(\Lambda)) - N(I_\Lambda)|\Lambda|| \geq (N(I_\Lambda)|\Lambda|)^{\max(\frac{1}{2}, 1-\nu_0)+\varepsilon}\right\} \leq \exp\left(-(N(I_\Lambda)|\Lambda|)^{\varepsilon}\right).$$

(2) **Central limit theorem.** Assume $\nu_0 > \frac{1}{2}$. Then the random variable

$$\frac{\mathrm{tr}1_{I_\Lambda}(H_\omega(\Lambda)) - N(I_\Lambda)|\Lambda|}{(N(I_\Lambda)|\Lambda|)^{\frac{1}{2}}}$$

converges in law to the standard Normal distribution.

References

[CGK10] Jean-Michel Combes, François Germinet, and Abel Klein. Poisson statistics for eigenvalues of continuum random Schrödinger operators. *Anal. PDE*, 3(1):49–80, 2010.

[GK06] Francois Germinet and Abel Klein. New characterizations of the region of complete localization for random Schrödinger operators. *J. Stat. Phys.*, 122(1):73–94, 2006.

[GKl10] François Germinet and Frédéric Klopp. Spectral statistics for random Schrödinger operators in the localized regime. ArXiv http://arxiv.org/abs/1011.1832, 2011.
[Kl10] Frédéric Klopp. Asymptotic ergodicity of the eigenvalues of random operators in the localized phase. ArXiv: http://fr.arxiv.org/abs/1012.0831, 2010.
[Min96] Nariyuki Minami. Local fluctuation of the spectrum of a multidimensional Anderson tight binding model. *Comm. Math. Phys.*, 177(3):709–725, 1996.
[Mol82] Stanislav A. Molchanov. The local structure of the spectrum of a random one-dimensional Schrödinger operator. *Trudy Sem. Petrovsk.*, (8):195–210, 1982.

Anderson Localization for Random Magnetic Schrödinger Operators

David Hasler
(joint work with L. Erdös)

We consider an electrically charged quantum mechanical particle moving in two dimensions and interacting with a perpendicular magnetic field which is random and stationary. We show Wegner estimates which can be used to prove Anderson localization.

First we discuss the continuous model with Hilbert space $L^2(\mathbb{R}^2)$ and Hamiltonian

$$H(A) = (-i\nabla - A)^2, \tag{1}$$

where $A = (A_1, A_2) : \mathbb{R}^2 \to \mathbb{R}^2$ denotes a vector potential for the magnetic field $B : \mathbb{R}^2 \to \mathbb{R}$, that is, $B = \nabla \times A$. We consider a random magnetic field of the following form

$$B_\omega(x) = b_0 + \sum_{k=0}^{\infty} \sum_{z \in \Lambda^{(k)}} \omega_z^{(k)} u(2^k(x - z)), \quad x \in \mathbb{R}^2, \tag{2}$$

where $b_0 > 0$ denotes a constant magnetic field, $u \in C_0^1(\mathbb{R}^2; [0, 1])$ is a profile function, and the randomness, living on arbitrarily small scales $\Lambda^{(k)} = \frac{1}{2^k}\mathbb{Z}^2$, is given by the a collection of independent random variables, $\{\omega_z^{(k)} : k \in \mathbb{N}_0, z \in \Lambda^{(k)}\}$, with mean zero.

Let A_ω denote a vector potential for the magnetic field B_ω, and let $H_L(A_\omega)$ denote the Hamiltonian restricted to the square $[-\frac{L}{2}, \frac{L}{2}]^2$ with Dirichlet boundary conditions. With some additional assumptions on the profile function u and the probability distributions of the random variables $\omega_z^{(k)}$ one can prove the following theorem [1].

Theorem 1. *For $K > 0$, there exist positive constants C_0, C_1, and L_0 such that*

$$\mathbb{E}\mathrm{Tr}\mathbf{1}_{[E-\eta/2, E+\eta/2]}(H_L(A_\omega)) \leq C_0 \eta L^{C_1}, \tag{3}$$

for all $E \in [0, K]$, $0 \leq \eta \leq 1$, and $L \geq L_0$.

Inequality (3) can be shown by analyzing the derivatives of the eigenvalues of $H_L(A_\omega)$ with respect to the random variables. This does not necessarily yield sign definite expressions. However, a strictly positive expression can be obtained by summing the squares of the derivatives with respect to each random variable.

In [1] this positivity is used to prove (3). Previously, Wegner estimates for the operator (1) have been obtained for stationary random vector potentials [4, 8].

The Wegner estimate (3) can be used to prove Anderson localization at band edges. It is well known that the Hamiltonian for a constant magnetic field of magnitude b_0 has pure point spectrum. The spectrum is given by the so called Landau-levels $\{(2n+1)b_0 : n \in \mathbb{N}_0\}$ and the eigenvalues are infinitely degenerate. Under small perturbations of the constant magnetic field, b_0, the spectrum remains in bands around the Landau levels. The spectral type is unstable under perturbations and arbitrarily small perturbations can lead to purely absolutely continuous spectrum [5]. However, for a random magnetic field of the form (2) one observes Anderson localization. In [3] it is shown that there exists a subset $\Sigma \subset \mathbb{R}$ such that almost surely $\sigma(H(A_\omega)) = \Sigma$. Moreover, there exist intervals $\{\Sigma_n\}$ such that Σ_n contains the n-th Landau level, $(2n+1)b_0$, and

$$\Sigma = \bigcup_{n=0}^{\infty} \Sigma_n.$$

Furthermore, for each $n \in \mathbb{N}_0$ the operator $H(A_\omega)$ has dense pure point spectrum at both edges of Σ_n with exponentially localized eigenfunctions, provided b_0 is sufficiently large.

Let us now consider the discrete case where the particle moves on the lattice $\mathbb{Z}^2$. In this model the magnetic field is described by a function

$$B : \mathcal{F} \to \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z}),$$

where $\mathcal{F}$ denotes the set of so called plaquets defined by

$$\mathcal{F} = \{\{x, x+e_1, x+e_1+e_2, x+e_2\} : x \in \mathbb{Z}^2\},$$

with unit vectors $e_1 = (1,0)$ and $e_2 = (0,1)$. A vector potential for B is a function $A : \mathcal{E} \to \mathbb{T}$ on the set of directed edges

$$\mathcal{E} = \{(x,y) \in \mathbb{Z}^2 \times \mathbb{Z}^2 : |x-y| = 1\}$$

with the following properties: $A(x,y) = -A(y,x)$, and $B(f) = \frac{1}{2}\sum_{e\in\partial f} A(e)$ for all plaquets $f \in \mathcal{F}$, where the boundary, ∂f, is defined as the set consisting of all directed edges for which both endpoints are in f. For the square $\Lambda_L = [-\frac{L}{2}, \frac{L}{2}]^2 \cap \mathbb{Z}^2$ we define the Hamiltonian acting on $\psi \in l^2(\Lambda_L)$ by

$$[h_L(A)\psi](x) = 4\psi(x) - \sum_{y\in\Lambda_L : |x-y|=1} e^{iA(x,y)}\psi(y), \quad \forall\ x \in \Lambda_L.$$

Let $\{\omega_f\}_{f\in\mathcal{F}}$ denote i.i.d. random variables with values in $\mathbb{T}$ and with density function $v \in C^2(\mathbb{T})$, such that for some $c_0 \in (0, \pi/2)$ one has $\pm c_0 \in \operatorname{supp} v$ and almost surely

$$\omega_f \in [c_0, \pi - c_0] + \mathbb{Z}\pi.$$

We define the random magnetic field, B_ω, by $B_\omega(f) = \omega_f$. Let A_ω denote a vector potential for B_ω. The following Wegner estimate can be proven by invoking similar ideas as in the continuous case [2].

Theorem 2. *Let $E^* < 4 - \sqrt{8}$. Then there exists a finite constant C such that*

$$\mathbb{E}\mathrm{Tr}\mathbf{1}_{[E-\eta/2,E+\eta/2]}(h_L(A_\omega)) \leq C\eta L^8,$$

for all $\eta \geq 0$ and $E \geq 0$ with $E + \eta/2 \leq E^*$.

Theorem 2 together with initial length scale estimates obtained in [7, 6] imply that the infinite volume Hamiltonian exhibits Anderson localization near the edges of the spectrum with pure point spectrum and exponentially localized eigenfunctions.

References

[1] L. Erdös and D. Hasler, *Wegner estimate and Anderson localization for random magnetic fields.* To appear in Commun. Math. Phys., arXiv:1012.5185.
[2] L. Erdös and D. Hasler, *Anderson localization for random magnetic Laplacian on Z^2.* arXiv:1101.2139.
[3] L. Erdös and D. Hasler, *Anderson Localization at Band Edges for Random Magnetic Fields.* To appear in J. Stat. Phys. arXiv:1103.3744.
[4] P.D. Hislop and F. Klopp, *The integrated density of states for some random operators with non-sign definite potentials.* J. Funct. Anal. **195** (2002), 12–47.
[5] A. Iwatsuka, *Examples of absolutely continuous SchrOdinger operators in magnetic fields.* Publ. RIMS. Kyoto Univ. **21** (1985), 385–401.
[6] F. Klopp, S. Nakamura, F. Nakano, and Y. Nomura, *Anderson localization for 2D discrete Schrödinger operators with random magnetic fields.* Ann. Henri Poincaré **4** (2003), 795–811.
[7] S. Nakamura, *Lifshitz tail for 2D discrete Schrödinger operator with random magnetic field.* Ann. Henri Poincaré **1** (2000), 823–835.
[8] N. Ueki, *Wegner estimates and localization for random magnetic fields.* Osaka J. Math. **45** (2008), 565–608.

Correlated Markov Quantum Walks

Alain Joye
(joint work with E. Hamza)

We consider the discrete time unitary dynamics given by a quantum walk on $\mathbb{Z}^d$ performed by a particle with internal degree of freedom, a spin, according to the following iterated rule: a unitary update of the spin takes place, followed by a shift on the lattice, conditioned on the spin state of the particle. We study the large time behavior of the quantum mechanical probability distribution of the position observable in $\mathbb{Z}^d$ for random updates of the spin states of the following form. The random sequences of unitary updates are given by a site dependent function of a Markov chain in time, with the following properties: on each site, they share the same stationary Markovian distribution and, for each fixed time, they form a deterministic periodic pattern on the lattice.

This choice is motivated by a crude analogy with the time one evolution generated by the lattice Anderson model. The shift is viewed as the effect of the discrete Laplacian and the random unitary update as the effect of the random onsite potential. The choice of time dependent unitary updates corresponds to the case of a time dependent potential given by a space periodic function of a discrete time Markov process. Such time dependent Anderson operators were tackled in [4, 3, 1] in various degrees of generality. The situation we consider corresponds in our unitary setup to the case addressed in [1]. The case where the unitary updates are independent of the space variable was dealt with in [2].

We prove a Feynman-Kac formula to express the characteristic function of the averaged distribution over the randomness at time n in terms of the n^{th} power of an operator M. By analyzing the spectrum of M, we show that this distribution possesses a drift proportional to the time and its centered counterpart displays a diffusive behavior with a diffusion matrix we compute. Moderate and large deviations principles are also proven to hold for the averaged distribution and the limit of the suitably rescaled corresponding characteristic function is shown to satisfy a diffusion equation.

An example of random updates for which the analysis of the distribution can be performed without averaging is worked out. The random distribution displays a deterministic drift proportional to time and its centered counterpart gives rise to a random diffusion matrix whose law we compute.

References

[1] E. Hamza, Y. Kang, J. Schenker, Diffusive propagation of wave packets in a fluctuating periodic potential, *Lett. Math. Phys.*, **95**, 53-66, (2011).

[2] A. Joye, Random Time-Dependent Quantum Walks, *Commun. Math. Phys.*, **307**, 65-100, (2011).

[3] Y. Kang, J. Schenker, Diffusion of wave packets in a Markov random potential, *J. Stat. Phys.*, **134**, 1005-1022, (2009).

[4] C.A. Pillet, Some Results on the Quantum Dynamics of a Particle in a Markovian Potential, *Commun. Math. Phys.*, **102**, 237-254, (1985).

Absolutely continuous spectrum on trees: random potentials, random hopping and Galton-Watson trees

Matthias Keller

(joint work with D. Lenz and S. Warzel)

We study operators on rooted trees with an underlying substitution structure. These trees are often called trees of finite cone type and their graph Laplacians exhibit finitely many bands of purely absolutely continuous spectrum. This absolutely continuous spectrum is shown to be stable under various random perturbations - small but extensive as well as large but rare. These include small random potentials and hopping terms and on the other hand multi-type Galton-Watson trees with a distribution close to a deterministic one.

Trees of finite cone type are defined by a finite set $\mathcal{A}$ and a matrix $M : \mathcal{A}\times\mathcal{A} \to \mathbb{N}$. To each $j \in \mathcal{A}$ we associate a rooted tree $\mathbb{T} = \mathbb{T}(M, j)$ with root $o = o(j)$ and labeling in $\mathcal{A}$ by the following rules: The root carries label j and each vertex of label k has $M_{k,l}$ forward neighbors of label l for $k, l \in \mathcal{A}$.

We consider the Laplacian $\Delta : \ell^2(\mathbb{T}) \to \ell^2(\mathbb{T})$ with a boundary condition at the root

$$\Delta\varphi(x) = \sum_{y\sim x}(\varphi(x) - \varphi(y)) + 1_{\{x=o\}}\varphi(o).$$

It is well known that the Green functions $z \mapsto G_x(z,\Delta) = \langle \delta_x, (\Delta - z)^{-1}\delta_x\rangle$, $x \in \mathbb{T}$, are analytic functions from the upper half plane $\mathbb{H}$ into itself. In order to study the spectrum $\sigma(\Delta)$ of Δ, we investigate the Green functions in the limits $\Im z \downarrow 0$. For the Laplacian we get the following:

Theorem 1 ([5, 7])**.** *For all $x \in \mathbb{T}$, the functions $E \mapsto G_x(E + i\varepsilon, \Delta)$ stay uniformly bounded as $\varepsilon \downarrow 0$ and there is a finite set $\Sigma_0 \subset \mathbb{R}$ such that $G_x(E + i0, \Delta) = \lim_{\varepsilon\downarrow 0} G_x(E + i\varepsilon, \Delta)$ exists all for $E \in \mathbb{R} \setminus \Sigma_0$. Moreover, the function*

$$\mathbb{R} \setminus \Sigma_0 \to \mathbb{R} \cup \mathbb{H}, \quad E \mapsto G_x(E + i0, \Delta)$$

is continuous and takes values in $\mathbb{H}$ on finitely many intervals. In particular, $\sigma(\Delta)$ consists of finitely many bands of purely absolutely continuous spectrum.

The absolutely continuous spectrum on non-regular trees turns out to be very stable under radially symmetric perturbations.

Theorem 2 ([5, 7])**.** *Let $\mathbb{T}$ be non-regular and $I \subset \sigma(\Delta) \setminus \Sigma_0$ be compact. Then there is $\lambda > 0$ such that for all radially symmetric $v : \mathbb{T} \to [-\lambda, \lambda]$ the map*

$$I \to \mathbb{H}, \quad E \mapsto G_x(E + i0, \Delta + v)$$

is continuous. In particular, the spectrum of $\Delta + v$ is purely absolutely continuous on I. If additionally $v(x) \to 0$ as $|x| \to \infty$, then $\sigma_{ac}(\Delta) = \sigma_{ac}(\Delta + v)$.

We now turn to the stability of the absolutely continuous spectrum under random perturbations. In particular, we look at three models: random potentials, random hopping and multi-type Galton-Watson trees.

(a) **Random potentials.** Let $(v_x)_{x\in\mathbb{T}}$ be independent identically distributed random variables with support in $[-1, 1]$. For $\lambda \geq 0$, let

$$H^\omega = \Delta + \lambda v^\omega, \qquad \omega \in \Omega.$$

Stability of absolutely continuous spectrum for small λ on regular trees was first proven in [9] which was followed by [1, 2, 3].

(b) **Random hopping.** Let $(t_{xy})_{x\sim y}$ be independent identically distributed random variables on the edges with support in $(-1, 1)$. Then, for $\lambda \in [0, 1]$ the operators $T^\omega : \ell^2(\mathbb{T}) \to \ell^2(\mathbb{T})$ given by

$$T^\omega\varphi(x) = \sum_{y\sim x}(1 + \lambda t^\omega_{xy})(\varphi(x) - \varphi(y))$$

define the random hopping model. It sometimes goes under the name of first passage percolation, where transit times of random walks are studied.

(c) **Multi-type Galton-Watson trees.** Finally, we consider a randomization of the geometry. Let b be a multi-type Galton-Watson branching process with types in $\mathcal{A}$. Denote the set of realizations by Θ. We are interested in the absolutely continuous spectrum of the operators $\Delta^\theta : \ell^2(\theta) \to \ell^2(\theta)$ given by

$$\Delta^\theta \varphi(x) = \sum_{y \sim x} (\varphi(x) - \varphi(y)), \qquad \theta \in \Theta.$$

For $s \in \mathbb{N}_0^{\mathcal{A}}$, $j \in \mathcal{A}$, denote by $\mathbb{P}_j^{(b)}(s)$ the probability that a vertex of label j has s_k, $k \in \mathcal{A}$, forward neighbors. We impose two assumptions on b:

(i) Every vertex has at least one forward neighbor: $\mathbb{P}_j^{(b)}(s \equiv 0) = 0$, $j \in \mathcal{A}$.

(ii) $\sum_{s \in \mathbb{N}_0^{\mathcal{A}}} \mathbb{P}_j^{(b)}(s) \|s\|^2 < \infty$, $j \in \mathcal{A}$, where $\|s\| = \sum_{k \in \mathcal{A}} s_k$.

Note that in order to expect purely absolutely continuous spectrum, assumption (i) is vital. Dropping (i) immediately yields eigenvalues with compactly supported eigenfunctions spread all over the spectrum. Furthermore, for b_1, b_2 satisfying (ii) we can define the metric

$$d(b_1, b_2) = \max_{j \in \mathcal{A}} \sum_{s \in \mathbb{N}_0^{\mathcal{A}}} \left| \mathbb{P}_j^{(b_1)}(s) - \mathbb{P}_j^{(b_2)}(s) \right| \|s\|^2.$$

If a process b satisfies $\mathbb{P}_j^{(b)}(s = M_{j,\cdot}) = 1$ for all $j \in \mathcal{A}$, then the set of realizations consists exactly of the elements $\mathbb{T}$ which are given by substitution matrix M. In this case, we denote $b = b_M$. Apart from the deterministic case, the simplest example is the one of a binary tree, where one of the forward edges of each vertex is deleted with probability $1 - p$, $p \in (0, 1)$. For small p this model is discussed in [4].

For the models (a), (b) and (c) we have the following theorem:

Theorem 3. *Let $I \subset \sigma(\Delta) \setminus \Sigma_0$ be compact. There is $\lambda > 0$ such that*

(a) ([5, 8]) *H^ω has purely absolutely continuous spectrum in I a.s.,*

(b) ([5, 8]) *T^ω has purely absolutely continuous spectrum in I a.s.,*

(c) ([6]) *Δ^θ has purely absolutely continuous spectrum in I for a.e. $\theta \in \Theta^b$ whenever b is such that $d(b, b_M) < \lambda$ for some M.*

References

[1] M. Aizenman, R. Sims, and S. Warzel, *Stability of the absolutely continuous spectrum of random Schrödinger operators on tree graphs.* Probability Theory and Related Fields **136** (2006), 363–394.

[2] M. Aizenman, and S. Warzel, *Resonant delocalization for random Schrödinger operators on tree graphs.* Preprint (2011), arXiv:1104.0969.

[3] R. Froese, D. Hasler, and W. Spitzer, *Absolutely continuous spectrum for the Anderson model on a tree: a geometric proof of Klein's theorem.* Comm. Math. Phys. **269** (2007), 239–257.

[4] R. Froese, D. Hasler, and W. Spitzer, *A geometric approach to absolutely continuous spectrum for discrete Schrödinger operators.* Boundaries and Spectra of Random Walks, Progress in Probability, Birkhaeuser, 2011.

[5] M. Keller, *On the spectral theory of operators on trees.* Ph.D. Thesis 2010.

[6] M. Keller, *Absolutely continuous spectrum for multi-type Galton Watson trees.* Preprint (2011), arXiv:1109.1959.

[7] M. Keller, D. Lenz, and S. Warzel, *On the spectral theory of trees with finite cone type.* To appear in Israel Journal of Mathematics, arXiv:1001.3600.
[8] M. Keller, D. Lenz, and S. Warzel, *Absolutely continuous spectrum for random operators on trees of finite cone type.* To appear in Journal d'Analyse Mathematique, arXiv:1108.0057
[9] A. Klein, *Absolutely continuous spectrum in the Anderson model on the Bethe lattice.* Math. Res. Lett. **1** (1994), 399–407.

Bounds on the density of states for Schrödinger operators

Abel Klein
(joint work with J. Bourgain)

We consider the Schrödinger operator

$$H = -\Delta + V \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d) \quad \left(\text{or} \quad \ell^2(\mathbb{Z}^d)\right),$$

where Δ is the Laplacian operator and V is a bounded potential.

We let

$$\Lambda_L(x) := x + \left]-\tfrac{L}{2}, \tfrac{L}{2}\right[^d$$

denote the (open) box of side L centered at $x \in \mathbb{R}^d$ (or $\mathbb{Z}^d$). By a box Λ_L we will mean a box $\Lambda_L(x)$ for some $x \in \mathbb{R}^d$ (or $\mathbb{Z}^d$).

Given a finite open box $\Lambda \subset \mathbb{R}^d$ (or $\mathbb{Z}^d$) we let $H_\Lambda^\sharp$ and $\Delta_\Lambda^\sharp$ be the restriction of H and Δ to $\mathrm{L}^2(\Lambda)$ ($\ell^2(\Lambda)$) with $\sharp$ boundary condition, where $\sharp = D$ (Dirichlet), N (Neumann), or P (periodic). (Our results will be independent of the boundary condition.)

We define finite volume (normalized) density of states measures $\eta_\Lambda^\sharp$ on Borel subsets B of $\mathbb{R}^d$ by

$$\eta_\Lambda(B) = \eta_{\Lambda,\infty}(B) := \tfrac{1}{|\Lambda|} tr\left\{\chi_B(H)\chi_\Lambda\right\},$$

$$\eta_{\Lambda,\sharp}(B) := \tfrac{1}{|\Lambda|} tr\left\{\chi_B(H_\Lambda^\sharp)\right\} \quad \text{for} \quad \sharp = D, N, P.$$

Note that for $\sharp = \infty, D, N, P$ and $B \subset]-\infty, E]$ we have

$$\eta_{\Lambda,\sharp}(B) \le C_{d,V_\infty,E} < \infty.$$

We define outer-measures on Borel subsets B of $\mathbb{R}^d$ for $\sharp = \infty, D, N, P$ by

$$\eta_{L,\sharp}^*(B) := \sup_{x \in \mathbb{R}^d} \eta_{\Lambda_L(x),\sharp}(B),$$

$$\eta_\sharp^*(B) := \limsup_{L\to\infty} \eta_{L,\sharp}^*(B).$$

If for some value of $\sharp$ we have

$$\lim_{\varepsilon\to 0} \eta_\sharp^*([E-\varepsilon, E+\varepsilon]) = 0 \quad \text{for all} \quad E \in \mathbb{R},$$

it is known that for all $E_1, E_2 \in \mathbb{R}$, $E_1 \le E_2$, we have

$$\eta^*([E_1,E_2]) := \eta_\infty^*([E_1,E_2]) = \eta_D^*([E_1,E_2]) = \eta_N^*([E_1,E_2]) = \eta_P^*([E_1,E_2]).$$

We prove the following results:

Theorem 1 (Discrete Schrödinger operators)**.** *Let H be a Schrödinger operator on $\ell^2(\mathbb{Z}^d)$. Then for all $E \in \mathbb{R}$ and $\varepsilon \leq \frac{1}{2}$ we have*

$$\eta^*\left([E, E+\varepsilon]\right) \leq \frac{C_{d,\|V\|_\infty}}{\log \frac{1}{\varepsilon}}.$$

Theorem 2 (Continuous Schrödinger operators)**.** *Let H be a Schrödinger operator on $\mathrm{L}^2(\mathbb{R}^d)$, where $d = 1, 2, 3$. Then, given $E_0 \in \mathbb{R}$, for all $E \leq E_0$ and $\varepsilon \leq \frac{1}{2}$ we have*

$$\eta^*\left([E, E+\varepsilon]\right) \leq \frac{C_{d,\|V\|_\infty, E_0}}{\left(\log \frac{1}{\varepsilon}\right)^{\kappa_d}}, \quad \textit{where} \quad \kappa_1 = 1,\ \kappa_2 = \tfrac{1}{4},\ \kappa_3 = \tfrac{1}{8}.$$

The Integrated Density of States for the Wilson Dirac Operator

Carolin Kurig
(joint work with V. Bach)

The strong nuclear force is described by Quantum Chromodynamics (QCD) and the associated Lagrangian density is given by

$$(1) \qquad \mathcal{L}(x) \;=\; -\frac{1}{4}\, F_{\mu\nu}(x)\, F^{\mu\nu}(x) \;+\; \overline{\psi}(x)\big(iD(\underline{A}) - m\big)\psi(x),$$

with $D(\underline{A}) := \gamma_\mu(\partial_\mu + iA_\mu)$ being the Dirac operator of the fermion field $\overline{\psi}(x)$, $\psi(x)$, which depends on the gauge field $\underline{A} := (A_\mu(x))_{\mu,x}$, and $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - [A_\mu(x), A_\nu(x)]$ being the field tensor. Concrete quantitative predictions are difficult to derive from (1), and one often resorts either to calculations in perturbation theory or numerical simulations on a discretized (Euclidean, after Wick rotation) space-time, known as *lattice QCD*.

We fit the formulation of the model in LQCD into the mathematical framework of ergodic operator families. In contrast to random Schrödinger operators, however, the randomness in LQCD models lies on the lattice bonds.

The gauge fields associated to the bonds are elements of a compact Lie group $\mathcal{G}$, the gauge group. In LQCD we have $\mathcal{G} = SU(N)$. We need the product of the gauge fields along a plaquette $p = p(x; \mu, \nu)$,a collection of four bonds that form a plane square in $\mathbb{Z}^d$,

$$(2) \qquad U_p \;:=\; U_{x,\nu}^{-1}\, U_{x+\hat{e}_\nu,\mu}^{-1}\, U_{x+\hat{e}_\mu,\nu}\, U_{x,\mu},$$

where the orientation of the bonds leads to the inverse gauge fields.

A priori, we let the gauge field U_b on a single bond be a random variable that is uniformly distributed with respect to the Haar measure μ_H of $\mathcal{G}$. The product measure $\widetilde{\mathbb{P}}$ is modified by a weight function that represents the gauge action. Formally, the measure $\mathbb{P}$ is defined as

$$(3) \qquad d\mathbb{P}(\underline{U}) \;=\; Z^{-1} e^{-S(\underline{U})}\, d\widetilde{\mathbb{P}}(\underline{U})$$

with Z being a normalization factor. We assume the gauge action to be of the Wilson action used in lattice QCD calculations,

$$S_\Lambda(\underline{U}) \;=\; \beta \sum_{p\subset\Lambda} \Re\,\mathrm{Tr}(1-U_p) \tag{4}$$

with $\beta > 0$, $\Lambda \subset \mathbb{Z}^d$ and U_p the plaquette variable as defined in (2). We use the Gibbsian formalism to establish the existence of the Gibbs measure $\mathbb{P}$ on $\mathcal{G}^{\mathbb{Z}^d}$ and Dobrushin's uniqueness criterion ensures its uniqueness, provided

$$0 \;<\; \beta \;<\; \frac{1}{12N(d-1)}. \tag{5}$$

Using a result of Föllmer [1] we show that the measure $\mathbb{P}$ is even ergodic, provided (5) holds true.

In lattice gauge theories the Wilson Dirac operator D is used, which is a discretized version of the QCD-Dirac operator $D_{QCD} = \gamma_\mu(\partial_\mu + iA_\mu) + m$ with gauge fields A_μ. The corresponding matter fields are defined on the hypercubic lattice $\mathbb{Z}^4$ and are assumed to have a Dirac structure labeled by Dirac indices $\alpha \in \{1,2,3,4\}$, as well as a colour structure with labels $c \in \{1,\ldots,N_c\}$. The Dirac structure is represented by the 4×4 Euclidean Dirac matrices $\{\gamma_\mu\}_{\mu=1,\ldots,4}$. The Wilson Dirac operator $D_{\underline{U}}$ is defined by

$$[D_{\underline{U}}\phi](x) = \gamma_5\big[\phi(x) - \kappa \sum_{\mu=1}^{4}\sum_{\sigma=\pm1}(r-\sigma\gamma_\mu)U_{x,\sigma\mu}\;\phi(x+\sigma\hat{e}_\mu)\big]\;. \tag{6}$$

The parameter $r \in (0,1]$ is the Wilson parameter and $\kappa > 0$ the hopping parameter. We remark that $\{D_{\underline{U}}\}_{\underline{U}\in\mathcal{G}^{\mathcal{B}}}$ is an ergodic operator family provided $\mathbb{P}$ is ergodic.

Our aim is the definition of the integrated density of states for $\{D_{\underline{U}}\}_{\underline{U}\in\Omega}$. Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of nested cubes in $\mathbb{Z}^d$ and let $N_{\Omega_n,\underline{U}}(E)$ denote the number of eigenvalues of $D_{\underline{U}}$ restricted to Ω_n smaller than E. As it turns out, the boundary condition imposed is immaterial. Provided $\mathbb{P}$ is ergodic, we proof that the limit

$$\rho(E) = \rho_{\underline{U}}(E) \;:=\; \lim_{n\to\infty}\frac{1}{|\Omega_n|}N_{\Omega_n,\underline{U}}(E) \tag{7}$$

exists and is independent of the chosen sequence and gauge field configuration, $\mathbb{P}$-almost surely.

As on outlook we present the idea that the distribution of the low-lying eigenvalues of the fermion Dirac operator is very close to the one of the corresponding (i.e., respecting symmetries) random matrix ensemble, as was put forward in [2, 3, 4] and affirmed by numerous numerical studies, for a review see for example [5].

References

[1] H. Föllmer, *A covariance estimate for Gibbs measures.* Journal of Functional Analysis **46** (1982), 387–395.

[2] E. V. Shuryak, and J. J. M. Verbaarschot, *Random matrix theory and spectral sum rules for the Dirac operator in QCD.* Nuclear Physics A **560** (1993), 306–320.

[3] J. J. M. Verbaarschot, and I. Zahed, *Spectral density of the QCD Dirac operator near zero virtuality.* Phys. Rev. Lett. **70** (1993), 3852–3855.
[4] J. J. M. Verbaarschot, *Spectrum of the QCD Dirac operator and chiral random matrix theory.* Phys. Rev. Lett. **72** (1994), 2531–2533.
[5] J. J. M. Verbaarschot, and T. Wettig, *Random matrix theory and chiral symmetry in QCD.* Ann. Rev. Nuc. Part. Sci. **50** (2000), 343–410.

Periodic and quasiperiodic approximations for slowly oscillating potential

Asya Metelkina

This is a report about some new results on Schrödinger operators with a slowly oscillating potential obtained in [4].

We consider a family $\{\mathbb{H}_\theta\}$ of self-adjoint operators in $L_2(\mathbb{R}_+)$, indexed by $\theta \in [0,\pi)$. For each θ, $\mathbb{H}_\theta$ is defined by $(\mathbb{H}_\theta f)(x) = -f''(x) + (V(x) + W(x^\alpha))f(x)$ for all $f \in D(\mathbb{H}_\theta) = \{f \in H^2(\mathbb{R}_+) \quad | \quad f(0)\sin\theta + f'(0)\cos\theta = 0\}$. We assume that the periodic potential $V : \mathbb{R} \to \mathbb{R}$ is a function in $L_{2,loc}(\mathbb{R})$ and $V(x+1) = V(x)$ and the slowly oscillating potential $W(x^\alpha)$ is composition of a function $W : \mathbb{R} \to \mathbb{R}$ in C^1 with $W(x+2\pi) = W(x)$ and x^α with $\alpha \in (0,1)$.

It is known [7, 6] that this family $\{\mathbb{H}_\theta\}$ exhibits interesting spectral phenomena such as a sharp mobility edge separating the absolutely continuous spectrum from the dense pure point spectrum (delocalization-localization transition). Moreover, there is a G_δ-dense set of values of θ for which $\mathbb{H}_\theta$ exhibits the singular continuous spectrum.
In [5], Simon-Zhu studied the particular family $\{\mathbb{H}_\theta\}$ corresponding to $V \equiv 0$ and $W \equiv \cos$. They proved the existence and obtained the formula for the integrated density of states $k(E)$ and for the Lyapunov exponent $\gamma(E)$.

Put $l_n := (2\pi n)^\alpha$, $n \in \mathbb{N}$, and $(\mathbb{H}_\theta(n)f) = -f''(x) + (V(x) + W(x^\alpha)f(x)$ for $f \in D(\mathbb{H}_\theta(n)) = \{f \in H^2([0,l_n]) \quad | \quad f(0)\cos\theta + f'(0)\sin\theta = 0, \quad f(l_n) = 0\}$. For a semibounded self-adjoint operator A with a discrete spectrum denote $\mathbb{P}_I(A)$ its spectral projection to the interval $I \subset \mathbb{R}$ and define the eigenvalue counting function $N(A,E) = \dim \text{Ran } \mathbb{P}_{(-\infty,E]}(A)$. Define $k_n(E,\theta) = \frac{N(\mathbb{H}_\theta(n),E)}{l_n}$.

Definition 1. When the limit exists, we call $k(E) := \lim_{n\to\infty} k_n(E,\theta)$ the *integrated density of states* for $\mathbb{H}_\theta$.

Denote $\mathbb{H}^0$ the self-adjoint periodic operator in $L_2(\mathbb{R})$ defined by $(\mathbb{H}^0 f) = -f''(x) + V(x)$ for all $f \in D(\mathbb{H}^0) = H^2(\mathbb{R})$.

Theorem 1. *The integrated density of states $k(E)$ exists for all $E \in \mathbb{R}$, is independent of θ and is given by the formula: $k(E) = \frac{1}{2\pi}\int_0^{2\pi} k_0(E - W(x))dx$. Moreover, one has an estimate of the convergence: $|k(E) - k_n(E)| \le Cn^{-\nu}$ with $\nu = \min\left(\frac{1-\alpha}{3\alpha}, 1\right)$, where $k_0(E) = \frac{1}{\pi}\Re\kappa_p(E)$ is the integrated density of states for the periodic operator $\mathbb{H}^0$.*

One proof of this theorem follows the ideas of Simon-Zhu [5], using periodic approximations.

In the case of real analytic W and $\alpha > \frac{1}{2}$, the rate of convergence can be improved ($n^{-\frac{1-\alpha}{\alpha}}$ instead of $n^{-\nu}$ in theorem 1) using different method based on quasiperiodic approximations and a rotation number approach.

Let $\{f_1, f_2\}$ be a basis of solutions of $(\mathbf{H}_\theta f)(x,E) = Ef(x,E)$ and $\mathbf{F}(x,E) = \begin{pmatrix} f_1 & f_2 \\ (f_1)' & (f_2)' \end{pmatrix}(x,E)$. The matrix $T(x,y,E) := \mathbf{F}(x,E)\mathbf{F}^{-1}(y,E)$ is called the *transfer matrix* for $\mathbb{H}_\theta$.

Definition 2. We call the *Lyapunov exponent* $\gamma(E)$ for $\mathbb{H}_\theta$ the following limit when it exists $\gamma(E) = \lim_{n\to\infty} \frac{\ln \|T(l_n,0,E)\|}{l_n}$.

Theorem 2. *The Lyapunov exponent $\gamma(E)$ exists for all $E\backslash D_\infty$ where D_∞ is a set of capacity zero (and thus of Hausdorff dimension zero). For $E \in \mathbb{R}\backslash D_\infty$ we have a formula: $\gamma(E) = \frac{1}{2\pi}\int_0^{2\pi} \gamma_0(E - W(x))dx$, where $\gamma_0(E) = \Im\kappa_p(E)$, the integrated density of states for periodic operator the $\mathbb{H}^0$.*

One proof uses the Sturm oscillations theory and theorem 1 to prove the Thouless formula $\gamma(E) - \gamma_0(E) = \int_{\mathbb{R}} \ln|E - E'|d(k - k_0)(E')$. Proving this formula, one needs to eliminate the bad set D_∞ of resonant energies.

For real analytic W and $\alpha > \frac{1}{2}$ there is an other proof of this result leading to a more accurate description of the set D_∞. One approaches the associated Schrödinger equation by $-f''_{z,\varepsilon}(x,E) + (V(x) + W(\varepsilon x + z))f_{z,\varepsilon}(x,E) = Ef_{z,\varepsilon}(x,E)$ with parameters z and ε and uses a version of the complex WKB method of Fedotov-Klopp [1, 2] in order to obtain the asymptotic of transfer matrices and compute the Lyapunov exponent.

Let $\tilde{l}_n$ be the entire part of l_n and $\varepsilon_n = \frac{2\pi}{l_{n+1} - l_n}$. In order to obtain the asymptotic of transfer matrix $T(\tilde{l}_{n+1}, \tilde{l}_n, E)$, we use the approach based on quasiperiodic approximations of the form $-f''_{z,\varepsilon}(x,E) + (V(x) + W(\varepsilon x + z))f_{z,\varepsilon}(x,E) = Ef_{z,\varepsilon}(x,E)$ of the equation $\mathbb{H}_\theta f(x,E) = f(x,E)$. This leads to the following result:

Theorem 3. *Suppose $E \in \sigma_{ac}(\mathbb{H}_\theta)$, W real analytic and $\alpha > \frac{1}{2}$. The transfer matrix $T(\tilde{l}_{n+1}, \tilde{l}_n, E)$ has the following asymptotic as n tends to infinity:*

$$T(\tilde{l}_{n+1}, \tilde{l}_n, E) = (\mathbf{\Phi}_0(E) + o_n(1))diag\{e^{i\frac{\phi(E)}{\varepsilon_n} + i\lambda_n}, e^{-i\frac{\phi(E)}{\varepsilon_n} - i\lambda_n}\}(\mathbf{\Phi}_0(E) + o_n(1))^{-1}.$$

Here the matrix $\mathbf{\Phi}_0(E) = \begin{pmatrix} f_+ & f_- \\ f'_+ & f'_- \end{pmatrix}(0, E - W(0))$ is formed by values in $x = 0$ of Bloch solutions $f_\pm$ of $\mathbb{H}^0 f(x,E) = (E - W(0))f(x,E)$ and their derivatives. $\lambda_n = iq(E)(\alpha_n - \alpha_{n+1})$ with $\phi(E) = \pi\sigma(k(E) + 2\pi C)$, $C = 2[\frac{m}{2}]$, $\sigma = (-1)^m$ and $m \in \mathbb{N}$; $q(E) = \sigma(\kappa_p(E - W(0)) + 2\pi C) \in (0, \pi)$ where $\kappa_p(E)$ is the main branch of Bloch quasimomentum; α_n is the fractional part of l_n. The 2×2 matrix $o_n(1)$ satisfies $\|o_n(1)\| \leq cn^{-\min(\frac{1}{\alpha} - 2, 1 - \frac{1}{\alpha})}$.

On each interval $[\tilde{l}_n, \tilde{l}_{n+1}]$ one chooses good parameters z_n and ε_n. The proof is based on the analysis of transfer matrices $T_{z,\varepsilon}$ for quasiperiodic approximations using the relationship between $T_{z,\varepsilon}$ and the monodromy matrix $M_{z,\varepsilon}(E)$ associated to a consistent basis of solutions (satisfying $f_{z,\varepsilon}(x+1, E) = f_{z+\varepsilon,\varepsilon}(x, E)$) of the quasiperiodic equation. One gets the asymptotics of $M_{z,\varepsilon}(E)$ as ε tends to zero from [3].

References

[1] A. Fedotov, and F. Klopp, *A complex WKB method for adiabatic problems.* Asymptot. Anal. **27** (2001), 219–264.

[2] A. Fedotov, and F. Klopp, *Geometric tools of the adiabatic complex WKB method.* Asymptot. Anal. **39** (2004), 309–357.

[3] A. Fedotov, and F. Klopp, *On the absolutely continuous spectrum of one-dimensional quasi-periodic Schrödinger operators in the adiabatic limit.* Trans. Amer. Math. Soc. **357** (2005), 4481–4516 (electronic).

[4] A. Metelkina, *Slowly oscillating perturbations of periodic Schrödinger equation.* PhD Thesis, Universite Paris 13, 2011.

[5] B. Simon, and Y. Zhu, *The Lyapunov exponents for Schrödinger operators with slowly oscillating potentials.* J. Funct. Anal. **140** (1996), 541–556.

[6] G. Stolz, *Shrödinger operators.* Habilitation Thesis, Bermingem, 1996.

[7] G. Stolz, *Spectral theory for slowly oscillating potentials. II. Schrödinger operators.* Math. Nachr. **183** (1997), 275–294.

What is a gapped ground state phase?

Bruno Nachtergaele

(joint work with S. Bachmann, S. Michalakis and R. Sims)

Let (Γ, d) be an sufficiently regular infinite metric graph such as, e.g., $\mathbb{Z}^\nu$ with the standard lattice distance. The C^*-algebra of quasi-local observables, $\mathcal{A}$, of a quantum spin system on Γ is the completion of the algebra of local observables $\mathcal{A}_{\mathrm{loc}}$, which itself is defined as the inductive limit:

$$\mathcal{A}_{\mathrm{loc}} = \bigcup_{X \subset \Gamma} \mathcal{A}_X,$$

where the union is over finite subsets X and

$$\mathcal{A}_X = \bigotimes_{x \in X} \mathbb{C}^d.$$

A quantum spin model on Γ is defined by specifying an interaction Φ, which is a map from the finite subsets $\Lambda \subset \Gamma$ to the algebra of local observables $\mathcal{A}_{\mathrm{loc}}$ with the property that $\Phi(X) = \Phi(X)^* \in \mathcal{A}_X$. The local Hamiltonian associated with any finite $\Lambda \subset \Gamma$ is given by

$$H_\Lambda = \sum_{X \subset \Lambda} \Phi(X).$$

The Heisenberg dynamics generated by H_Λ given by $\tau_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}$, is a group of automorphisms that we can regard as acting on $\mathcal{A}_\Lambda$, or $\mathcal{A}_{\mathrm{loc}}$ or $\mathcal{A}$ as desired. Under suitable conditions, the limit $\Lambda \uparrow \Gamma$ of $\tau_t^\Lambda(A)$ defines a strongly continuous group of automorphisms $\{\tau_t \mid t \in \mathbb{R}\}$ on $\mathcal{A}$.

We are interested in the ground states ψ_Λ of H_Λ and in the set $\mathcal{S}$ of all their possible thermodynamic limits as linear functionals ω on $\mathcal{A}$:

$$\omega(A) = \lim_{\Lambda \uparrow \Gamma} \langle \psi_\Lambda, A\psi_\Lambda \rangle.$$

More specifically, we are interested in understanding the classification of the so-called "gapped" ground state phases which are equivalence classes of sets of ground states defined as follows. Suppose $\mathcal{S}_0$ and $\mathcal{S}_1$ are two sets of ground states for a given quantum spin system on Γ, obtained as the thermodynamic limits of the ground states of two models with interactions Φ_0 and Φ_1, respectively. If there exists a differentiable family of interactions Φ_s, for $s \in [0,1]$, that satisfy suitable conditions (in particular that they are sufficiently short-ranged) and such that there exists a constant $\gamma > 0$ that is a common lower bound for the gap between the ground state and first excited state energies of the corresponding local Hamiltonians $H_\Lambda^{(s)} = \sum_{X \subset \Lambda} \Phi_s(X)$, then we say that $\mathcal{S}_0$ and $\mathcal{S}_1$ belong to the same phase.

For the situation described in the previous paragraph, we constructed in [1] a strongly continuous co-cycle of automorphisms $\alpha_{s,t}$ of the algebra of quasi-local observables $\mathcal{A}$, such that $\mathcal{S}_1 = \mathcal{S}_0 \circ \alpha_{1,0}$. The automorphisms satisfy a Lieb-Robinson bound with uniformly bounded velocity and posses all the symmetries of the family of models $H^{(s)}$. In particular, it follows that if $\mathcal{S}_0$ and $\mathcal{S}_1$ are *not* isomorphic as convex sets of states on the observable algebra, the gap must close somewhere on any curve of Hamiltonians connection the two. Lieb-Robinson bounds [2, 3, 4] and a version of Hastings' quasi-adiabatic evolution [5], which we call the *spectral flow*, play an essential role in this work.

References

[1] S. Bachmann, S. Michalakis, B. Nachtergaele, and R. Sims, *Automorphic Equivalence within Gapped Phases of Quantum Lattice Systems.* To appear in Commun. Math. Phys. (2011), `arXiv:1102.0842`.

[2] E.H. Lieb and D.W. Robinson, *The finite group velocity of quantum spin systems.* Comm. Math. Phys. **28** (1972), 251–257.

[3] B. Nachtergaele and R. Sims, *Lieb-Robinson Bounds and the Exponential Clustering Theorem.* Commun. Math. Phys. **265** (2006), 119–130.

[4] M. Hastings and T. Koma, *Spectral Gap and Exponential Decay of Correlations.* Commun. Math. Phys. **265** (2006), 781–804.

[5] M. Hastings and X. Wen., *Quasi-adiabatic continuation of quantum states: The stability of topological ground-state degeneracy and emergent gauge invariance.* Phys. Rev. B. **72** (2005), 045141.

[1] Based on work supported by the U.S. National Science Foundation under grants # DMS-0757581, DMS-0757424, and DMS-1009502. Work carried out in part during stay at the Institut Mittag-Leffler, Djursholm, Sweden (Fall 2010)

Remarks on the spectral shift function and the Friedel sum rule

Shu Nakamura
(joint work with M. Kohmoto and T. Koma)

We consider Schrödinger operator $H = H_0 + V(x)$ on $L^2(\mathbb{R}^d)$ with $d \geq 1$, where $H_0 = -\triangle$. We study the spectral shift function and a related quantity, called the excess charge in the solid state physics, and then we show they are equivalent. This implies a formula, called the Friedel sum rule.

We suppose

$$|V(x)| \leq C \langle x \rangle^{-\alpha}, \quad x \in \mathbb{R}^d,$$

where $\alpha > d+3$ and $\langle x \rangle = \sqrt{|x|^2+1}$. Let us fix $v \in C_0^\infty(\mathbb{R}^d)$ such that $v = 1$ in a neighborhood of 0, and we set $v_R(x) = v(x/R)$ for $x \in \mathbb{R}^d$, $R > 0$. Then we define

$$Z_R(\lambda) = \mathrm{Tr}[v_R(E_H(\lambda) - E_{H_0}(\lambda))v_R], \quad \lambda > 0,$$

where $E_A(\lambda)$ denotes the spectral projection for a self-adjoint operator A. Here λ corresponds to the Fermi energy in the solid state physics. It is natural to consider

$$Z(\lambda) = \lim_{R\to\infty} Z_R(\lambda)$$

is a rigorous expression of $\mathrm{Tr}[E_H(\lambda) - E_{H_0}(\lambda)]$, which is often called the *excess charge*.

The spectral function (SSF) (of Lifshitz, Krĕin) is defined as a function $\xi(\lambda)$ on $\mathbb{R}$ such that

$$\mathrm{Tr}[f(H) - f(H_0)] = -\int_{-\infty}^{\infty} f'(\lambda)\xi(\lambda)d\lambda$$

for any $f \in C_0^\infty(\mathbb{R})$, and it is also considered as another regularization of $\mathrm{Tr}[E_H(\lambda) - E_{H_0}(\lambda)]$ (see [BY] and references therein). We show

Theorem 1. *On* $(0,\infty) \setminus \sigma_p(H)$, $\xi(\lambda)$ *is continuous, and* $\xi(\lambda) = \lim_{R\to\infty} Z_R(\lambda)$ *for* $\lambda \in (0,\infty) \setminus \sigma_p(H)$.

We recall the Birman-Krĕin formula for the SSF:

$$e^{2\pi i \xi(\lambda)} = \det S(\lambda), \quad \lambda > 0,$$

where $S(\lambda)$ is the scattering matrix for the pair (H, H_0). If we denote

$$\theta(\lambda) = \frac{1}{2\pi i} \log \det S(\lambda)$$

(with suitable choice of the branch), we have $\theta(\lambda) = \xi(\lambda)$. Combining these observations, we learn:

Theorem 2. (The Friedel sum rule [F]) $Z(\lambda) = \theta(\lambda)$ for $\lambda \in (0,\infty) \setminus \sigma_p(H)$.

The proof of Theorem 1 relies on the construction of the SSF due to M. Krĕin, but instead of the L^1-convergence, we show the pointwise convergence for $\lambda \in (0,\infty) \setminus \sigma_p(H)$ under our stronger assumption on V. In order to show $Z(\lambda) = \xi(\lambda)$, we also use a microlocal propagation estimate due to Isozaki and Kitada [IK].

References

[BY] Birman, M. Sh., Yafaev, D. R.: The spectral shift function. The papers of M. G. Kreĭn and their further development. St. Petersburg Math. J. **4** (1993), no. 5, 833–870.
[F] Friedel, J.: The Distribution of Electrons Round Impurities in Monovalent Metals, Phil. Mag. **43** (1952), 153–189; Electric Structure of Primary Solid Solutions in Metals, Adv. Phys. **3** (1954), 446–507; Structure Électronique des Impuretés dans les Métaux, Annales de Physique. **9** (1954) 158–202; Metallic Alloys. Nuovo Cimento, Suppl. **7** (1958), 287–311.
[IK] Isozaki, H., Kitada, H.: A remark on the micro-local resolvent estimates for two body Schrödinger operators. Publ. RIMS. **21** (1985), 889–910.

Entanglement, Gibbs Distribution, Etc.

Leonid Pastur

We discuss several topics of quantum informatics, statistical physics, and spectral theory, related to the notion of the entanglement and its quantifications.

We start from the short history and basic definitions: reduced density matrix, its von Neumann entropy and related quantities (linear entropy, Renyi entropy, entanglement spectrum). We then present important theoretical physics findings on the validity of the area law for the entanglement entropy, the corresponding rigorous results as well as certain models, where the area law is not valid. This is, in particular, the Ising model in transverse external field (known also as the quantum Ising model) and the the typical (random) states.

In both cases the leading term of the entanglement entropy is proportional to the logarithm of the size of the system, although the mathematical mechanisms of these asymptotics is quite different as well as the constants in front of the logarithm. In the case of the the quantum Ising model the mechanism is due to the quantum phase transition at zero temperature, i.e., due the spins interactions, resulting in strong quantum correlations and slow decay of corresponding correlation functions (zero gap in the excitation spectrum). In the second model the mechanism is the linearity of the density of states and the analog of the corresponding partition function in the size of the system. We give a simple proof of the validity with probability 1 of the Page conjecture for the model for any random pure state, whose components are arbitrary i.i.d. random variables whose mean is not necessarily zero. Note that the initial page conjecture was for the Gaussian states and for the mean entropy.

We then present the links of the entanglement topics, the reduced density matrix and its von Neumann entropy in particular. with those on the validity and derivation of the Gibbs distribution in the traditional setting of the equilibrium statistical mechanics, i.e., as the distribution of the subsystem of the large system, which are allowed to have the energy exchange (the large system is usually called the reservoir or the thermal bath and determines via its entropy the temperature of the subsystem). We discuss first physics arguments leading to the Gibbs distribution and then several models, where it can be proved: boson-boson models both classical (Bogolyubov) and quantum (Glauber), spin boson (Caldeira-Legget and others), and spin-random matrix (Lebowitz-Pastur) as a result of certain asymptotic procedures. We then present our recent results on the derivation of the Gibbs distribution for a certain generalization of the last model, in which the spin (two-level system or qubit) is replaced by an arbitrary finite dimensional matrix.

We conclude by discussing another scheme of derivation of the Gibbs distribution as the large time limit (stationary measure) of the time dependent reduced density matrix of the subsystem, the time dependence results from from the joint dynamics of the system and reservoir. We outline conditions under which this derivation can be implemented in various models, in particular the Bogolyubov-van Hove limit and various assumptions and asymptotic regimes, guarantying the macroscopic nature of the reservoir.

Dynamical localization for Delone-Anderson operators

Constanza Rojas-Molina
(joint work with F. Germinet and P. Müller)

We study dynamical localization in Delone-Anderson operators, a particular case of non-ergodic random Schrödinger operators. A discrete point set $D \subset \mathbb{R}^d$ is called an *(r,R)-Delone* set for $r, R > 0$, if it satisfies the following properties,

- D is *uniformly discrete*: for any $x \in \mathbb{R}^d$, $\sharp(D \cap \Lambda_r(x)) \leq 1$,
- D is *relatively dense*: for any $x \in \mathbb{R}^d$, $\sharp(D \cap \Lambda_R(x)) \geq 1$,

where $\Lambda_y(x)$ is a box of side length y centered in x and $\sharp$ stands for cardinality. Then there exist constants $C_{r,d}$ and $C_{R,d}$ such that $C_{R,d}L^d \leq \sharp(D \cap \Lambda_L(x)) \leq C_{r,d}L^d$ for all $L > 0$. Specific examples are the lattice (periodic Delone set) and the Penrose tiling (aperiodic Delone set).

We consider the operator $H_\omega = H_0 + \lambda V_\omega$, on $L^2(\mathbb{R}^d)$, where $\lambda > 0$ is a parameter of disorder, the free Hamiltonian $H_0 = -\Delta$, and the random *Delone-Anderson* potential is defined as

$$V_\omega(x) = \sum_{\gamma \in D} \omega_\gamma u(x - \gamma),$$

where D is a (r, R)-Delone set, $\{\omega_\gamma\}$ are independent identically distributed random variables with absolutely continuous probability density μ, and $\text{supp}\mu = [0, M]$, $0 < M$. The probability space is $\Omega = [0, M]^D$. The single-site potential $u \geq 0$ is assumed to be bounded and not "too small".

Then H_ω is a self-adjoint, lower semi-bounded operator with spectrum $\sigma_\omega = [0, \infty)$,for $a.e.\ \omega \in \Omega$. Since the Delone set D is non-periodic in general, H_ω is non-ergodic, in the sense that there is no family of unitary operators $\{U_\gamma\}$ associated to a group of translations τ_γ acting on Ω such that the usual covariance relation $H_{\tau_\gamma(\omega)} = U_\gamma H_\omega U_\gamma^*$ holds.

Dynamical localization for similar models has been discussed in [BdMNSS] using the Fractional Moment Method, where the localized regime lies below the fluctuation boundary level. However, since the Delone set is non-periodic, there is actually no information on how the fluctuation boundary depends on the parameters of the model. On the other hand, [G] considered Bernoulli random variables and proved related results using Bourgain and Kenig's multiscale analysis [BoK].

In our approach, we aim to obtain a more quantitative description of the localization regime and, in particular, to get explicit information for the length of the localization interval in terms of the parameters of the model. We use the Bootstrap Multiscale Analysis for non-ergodic operators (MSA), developed in [RM], to draw stronger conclusions than [G, GK11]. As a result, we show the existence of an energy $E(R,\lambda) = C_{d,\lambda} R^{-(2d+2)}(\ln R)^{-2/d} > 0$, where $C_{d,\lambda}$ is a positive constant, such that for any subinterval $I \subset [0, E(R,\lambda)]$, the operator H_ω exhibits, among other features, strong dynamical localization in I. Furthermore, we prove the almost-sure existence of the Integrated Density of States (IDS) for these models under some assumptions on the "extent of aperiodicity" of the Delone set.

In order to start the MSA, we need to prove a uniform Wegner estimate and an initial length scale estimate for the finite-volume operator $H_{\omega,x,L} = -\Delta_{x,L} + V_{\omega,x,L}$, that is, H_ω restricted to a box $\Lambda_L(x)$ of side L centered in $x \in \mathbb{R}^d$. In order to do this we use a spatial averaging trick used in [BoK] and [G] that consists in considering an auxiliary potential $\bar{V}_{x,L}$ defined as

$$\bar{V}_{\omega,x,L}(\cdot) := \frac{1}{R^d} \int_{\Lambda_R(0)} \tilde{V}_{\omega,x,L}(\cdot - a) da$$

where $V_{\omega,x,L}$ denotes the restriction of V_ω to the box $\Lambda_L(x)$. It enables us to derive a simple quantitative unique continuation principle type estimate which is the key for the needed uniform Wegner estimate (see e.g. [RMV]).

On the other hand, for the Hamiltonian $\bar{H}_{\omega,x,L} = -\Delta_{x,L} + \bar{V}_{\omega,x,L}$, we estimate the probability that there is a gap in the spectrum above zero, as done in [G]. The gap is shown to be big enough, which enables us to use Combes-Thomas estimates to prove the initial length scale estimate.

In order to study the IDS, the lack of ergodicity prevents us from applying Birkhoff's Ergodic Theorem, and therefore there is no straight answer to the question on the existence of the IDS. However, this can be overcome by treating H_ω as a function of a randomly coloured Delone set, that encodes the information of the random potential V_ω. We consider the randomly coloured version of D, $D^\omega = \{(\gamma, \omega_\gamma) : \gamma \in D, \omega_\gamma \in [0, M]\}$ and the closed $\mathbb{R}^d$-orbits of D and D^ω: $X_D = \overline{\{D + t : t \in \mathbb{R}^d\}}$, $X_D^\omega = \overline{\{D^\omega + t : t \in \mathbb{R}^d\}}$, where the closure is taken with respect to the vague topology in the space of all Delone sets in $\mathbb{R}^d$.

In this context, we use an ergodic theorem for randomly coloured point sets proved by [MR] and show that, for any energy $E \in \mathbb{R}$, if the dynamical system X_D is uniquely ergodic, that is, there exists a unique ergodic measure μ on X_D, then the eigenvalue counting function

$$\nu_{\omega,x,L}(E) = \frac{1}{L^d} \sharp\{\text{eigenvalues of } H_{\omega,x,L} \leq E\}$$

has a limit when L tends to infinity, everywhere in X_D and for *a.e.* $\omega \in \Omega$, which we denote by $\nu(E)$. In particular, for the Delone set $D \in X_D$ we started with, this limit exists almost surely, and it is called the IDS of H_ω. Since a uniform Wegner estimate is satisfied at the bottom of the spectrum, the IDS is shown to be Lipschitz continuous. Moreover, it exhibits a Lifshitz-tail behavior, as expected near a fluctuation boundary, that is

$$\lim_{E\searrow 0} \frac{\ln|\ln(\nu(E))|}{\ln(E)} = -\frac{d}{2}.$$

In the magnetic case, dynamical localization was proved in [RM] for Delone random Landau Hamiltonians, where the Delone set is aperiodic.

References

[BoK] J. Bourgain and C. Kenig, *On localization in the continuous Anderson-Bernoulli model in higher dimension*. Invent. Math. **161** (2005), 389–426.

[BdMNSS] A. Boutet de Monvel, S. Naboko, P. Stollmann, and G. Stolz, *Localization near fluctuation boundaries via fractional moments and applications*. J. Anal. Math. **100** (2006), 83–116.

[G] F. Germinet, *Recent advances about localization in continuum random Schrödinger operators with an extension to underlying Delone sets*. In Mathematical results in quantum mechanics, *Proceedings of the QMath10 Conference*, 2008.

[GK1] F. Germinet and A. Klein, *Bootstrap multiscale analysis and localization in random media*. Comm. Math. Phys. **222** (1998), 415–448.

[GK11] F. Germinet and A. Klein, *A comprehensive proof of localization for continuous Anderson models with singular random potentials*. To appear in JEMS, aXiv:1105.0213v1.

[MR] P. Müller and C. Richard, *Ergodic properties of randomly coloured point sets*. To appear in Can. J. Math., arXiv:1005.4884v2.

[RM] C. Rojas-Molina, *Characterization of the Anderson metal-insulator transition for non ergodic operators and application*. Submitted, arXiv:1110.4652v1.

[RMV] C. Rojas-Molina and I. Veselic, *A quantitative continuation principle for solutions of the Schrödinger equation and application*. Work in progress.

Diffusion for Markov Random Schrödinger Equations

Jeffrey Schenker

(joint work with Y. Kang, E. Hamza and B. C. Musselman)

It is generally expected that over long times wave packets in a disordered material will propagate diffusively, at least in dimension $d \geq 3$. This expectation stems from a picture of wave propagation as a multiple scattering process. However, so far this heuristic argument has not been turned into rigorous analysis, at least without restricting the time scale over which the wave evolution is followed, as in the work of Erdös *et. al.* [1, 2]. One major obstacle is a lack of control over recurrence: the wave packet may return often to regions visited previously, denying us the independence needed to carry out the central limit argument.

A natural way to avoid recurrence difficulties is to bring a time dependence into the disordered background — to suppose that the environment evolves as the packet propagates. With Yang Kang, we studied the unitary evolution

$$\mathrm{i}\partial_t \psi_t(x) \;=\; H_0\psi_t(x) + \omega_t(x)\psi_t(x) \tag{1}$$

on $\ell^2(\mathbb{Z}^d)$ where H_0 is a translation invariant hopping term satisfying a non-degeneracy condition spelled out in [5], and $\omega_t(x)$ is a time dependent random potential, which evolves in time as a Markov process. The key assumption is that the process has a "spectral gap," which means roughly that

$$\left|\mathbb{E}f(\omega_t) - \int_\Omega f(\omega)d\mu(\omega)\right| \leq e^{-t/T}, \tag{2}$$

for some $T > 0$ with μ a non-trivial invariant measure. For this evolution, we obtained the following result which shows that the density of the wave converges, in a scaling limit, to a solution of a heat equation:

Theorem 1 (Kang and Schenker 2009 [5])**.** *For solutions to* (1)*, with the assumptions of* [5]*, we have*

$$\lim_{\tau\to\infty} \sum_{x\in\mathbb{Z}^d} e^{-\mathrm{i}\frac{\vec{k}}{\sqrt{\tau}}\cdot x}\mathbb{E}|\psi_{\tau t}(x)|^2 \ = \ e^{-t\sum_{i,j} D_{i,j}k_ik_j}, \tag{3}$$

with $D_{i,j}$ a positive definite matrix. Furthermore

$$\lim_{T\to\infty} \frac{1}{T} \sum_{x\in\mathbb{Z}^d} |x|^2\mathbb{E}|\psi_T(x)|^2 = \sum_i D_{i,i}.$$

A key assumption in [5] was that potentials at various sites of the lattice should be distinct. Specifically,

$$\inf_{x\neq y\in\mathbb{Z}^d} \mathbb{E}\left(|\omega(x)-\omega(y)|^2\right) > 0.$$

With Eman Hamza and Yang Kang we studied an equation in which this assumption is violated in an extensive way:

$$\omega_t(x+L\xi) = \omega_t(x) \tag{4}$$

for any $x, \xi \in \mathbb{Z}^d$ for some $L > 1$. That is ω_t is a random periodic function on $\mathbb{Z}^d$ with period L. For this equation we determined that the motion was diffusive after an additional Fourier transform to account for the additional conserved quasi-momentum associated with the symmetry under translations by multiples of L:

Theorem 2 (Hamza, Kang and Schenker 2011 [4])**.** *For solutions to* (1) *with the potential of the form* (4) *but under the remaining assumptions of* [5]*, we have*

$$\lim_{\tau\to\infty} \sum_{x\in\mathbb{Z}^d} e^{-\mathrm{i}\frac{\vec{k}}{\sqrt{\tau}}\cdot x}\mathbb{E}|\psi_{\tau t}(x)|^2 \ = \ \int_{\mathbb{T}^d} e^{-t\sum_{i,j} D_{i,j}(\mathbf{p})k_ik_j} w(\mathbf{p})d\mathbf{p},$$

where $\mathbb{T}^d = (-\pi,\pi]^d$ is the d-torus, $D_{i,j}(\mathbf{p})$ is positive definite for each $\mathbf{p} \in \mathbb{T}^d$ and $w(\mathbf{p})$ is a positive weight depending on the initial condition ψ_0.

The proofs of each of these results rely on an "augmented space" representation, similar to that used in the study of random walk in a random environment [6, 7, 3]. It is remarkable to note that there is no weak disorder or high dimension assumption here – the time dependence of the potential eliminates recurrence and produces diffusion even in a regime for which a static potential would produce localization.

Each of these results rely crucially on the gap condition (2) to allow for a spectral analysis of the generator in the "augmented space." To examine how crucial this gap condition is we have examined with B. Clark Musselman a Schrd̈ingier equation with divergence form generator

$$\mathrm{i}\partial_t \psi_t(x) \;=\; \sum_{<y,x>} \theta_{<y,x>}(\omega_t)\,(\psi_t(y) - \psi_t(x))\,. \tag{5}$$

Here, for each nearest neighbor bond $< x, y >$ of the lattice $\mathbb{Z}^d$, the variable $\theta_{<x,y>}(\omega_t)$ is non-negative and evolves as an independent Markov process. Because of the divergence form coupling, even with a gap condition on the Markov generator there is no gap in the spectrum of the relevant generator in the augmented space. We have not been able to derive diffusion as in (3), however we do obtain a result with time averaging:

Theorem 3 (Musselman and Schenker 2011 [8])**.** *For solutions to* (5)*, we have*

$$\lim_{\tau\to\infty} \int_0^\infty e^{-\eta} t \sum_{x\in\mathbb{Z}^d} e^{-\mathrm{i}\frac{\vec{k}}{\sqrt{\tau}}\cdot x} \mathbb{E}|\psi_{\tau t}(x)|^2 dt \;=\; \frac{1}{\eta + \sum_{i,j} D_{i,j} k_i k_j}$$

with $D_{i,j}$ *a positive definite matrix.*

References

[1] L. Erdős, M. Salmhofer, and H.-T. Yau, *Quantum diffusion for the Anderson model in the scaling limit.* Ann. Henri Poincaré **8** (2007), 621–685.

[2] L. Erdős, M. Salmhofer, and H.-T. Yau, *Quantum diffusion of the random Schrödinger evolution in the scaling limit.* Acta Math. **200** (2008), 211–277.

[3] S. Goldstein, *Antisymmetric functionals of reversible Markov processes.* Ann. H. Poin. **31** (1995), 177–190.

[4] E. Hamza, Y. Kang, and J. Schenker, *Diffusive Propagation of Wave Packets in a Fluctuating Periodic Potential.* Lett. Math. Phys. **95** (2011), 53–66.

[5] Y. Kang and J. Schenker, *Diffusion of wave packets in a Markov random potential.* J. Stat. Phys. **134** (2009, 1005–1022.

[6] C. Kipnis and S. R. S. Varadhan, *Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions.* Comm. Math. Phys. **104** (1986), 1–19.

[7] A. De Masi, P. Ferrari, S. Goldstein, and W. Wick, *An invariance principle for reversible Markov processes. Applications to random motions in random Environments.* J. Stat. Phys. **55** (1989), 787–855.

[8] B. C. Musselman and J. Schenker, *Diffusion for a Markov Schrödinger equation with divergence form generator.* In preparation.

Orbital magnetization for disordered media

Hermann Schulz-Baldes
(joint work with S. Teufel)

The orbital magnetization of a system of independent electrons is defined as the derivative of the pressure w.r.t. the magnetic field. In recent years there were several works showing that for Bloch electrons it can be expressed in terms of the Berry curvature and the Rammal-Wilkinson tensor. We present a general formula holding for any space-homogeneous Hamiltonian. In the situation where the Fermi level μ lies in a region of dynamical localization, the zero-temperature magnetization in the direction $j = 1, 2, 3$ takes a form similar to the Chern invariants:

$$M_j \;=\; \frac{1}{2\,\imath}\,\mathcal{T}\big((\mu - H)(1-2P)[[X_{j+1}, P], [X_{j+2}, P]]\big)\;.$$

Here $\mathcal{T}$ is the trace per unit volume, H the Hamiltonian on $\ell^2(\mathbb{Z}^3, \mathbb{C}^L)$, P is the Fermi projection and X_j are the components of the position operator with index j calculated cyclically. The derivative of M_j w.r.t. μ is then equal to the Chern invariant, a fact that is simply related to Streda's formula. It is also shown that these Chern invariants in turn determine the surface currents when the system is restricted to a half-space. An example where all the above is non-trivial is the Haldane model, a prime example of a topological insulator. The main technique leading to a proof of the above formula is Bellissard's Ito derivative w.r.t. the magnetic field and a new generalized DuHamel formula for this Ito calculus (at complex times).

Locality Bounds and Correlation Estimates

Robert Sims
(joint work with E. Hamza and G. Stolz)

1. Quantum Lattice Systems

Consider a collection of quantum systems labeled by $x \in \mathbb{Z}^\nu$. By this, we mean that corresponding to each site $x \in \mathbb{Z}^\nu$ there is a Hilbert space $\mathcal{H}_x$ and a densely defined, self-adjoint operator H_x acting on $\mathcal{H}_x$. The operator H_x is typically referred to as the on-site Hamiltonian. For finite $\Lambda \subset \mathbb{Z}^\nu$, the Hilbert space of states corresponding to Λ is given by

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x\,, \tag{1}$$

and the algebra of observables is

$$\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{B}(\mathcal{H}_x) = \mathcal{B}(\mathcal{H}_\Lambda)\,, \tag{2}$$

where $\mathcal{B}(\mathcal{H})$ is the set of bounded linear operators over the Hilbert space $\mathcal{H}$. In this case, an observable $A \in \mathcal{A}_\Lambda$ depends only on those degrees of freedom in Λ.

In general, these collections of quantum systems are used to describe many interesting physical phenomena e.g., the moments associated with atoms in a magnetic material, a lattice of coupled oscillators, or an array of qubits in which quantum information is stored. For simplicity, we have described them on a lattice $\mathbb{Z}^\nu$. In general, this is not necessary.

The systems described above are of particular interest when they are allowed to interact. In general, a *bounded interaction* for such quantum systems is a mapping Φ from the set of finite subsets of $\mathbb{Z}^\nu$ into the algebra of observables which satisfies

$$\Phi(X)^* = \Phi(X) \in \mathcal{A}_X \quad \text{for all finite } X \subset \mathbb{Z}^\nu. \tag{3}$$

A *model* over $\mathbb{Z}^\nu$ is defined by a collection of quantum systems $\{(\mathcal{H}_x, H_x)\}_{x\in\mathbb{Z}^\nu}$ and an interaction Φ.

Associated to a given model there is a family of *local Hamiltonians*, $\{H_\Lambda\}$, parameterized by the finite subsets of $\mathbb{Z}^\nu$. In fact, to each finite $\Lambda \subset \mathbb{Z}^\nu$,

$$H_\Lambda = \sum_{x\in\Lambda} H_x + \sum_{X\subset\Lambda} \Phi(X) \tag{4}$$

is a densely defined, self-adjoint operator. Here the second sum is over all finite subsets of Λ, and is therefore finite. By Stone's theorem, the corresponding *Heisenberg dynamics*, τ_t^Λ, given by

$$\tau_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda} \quad \text{for all } A \in \mathcal{A}_\Lambda \text{ and } t \in \mathbb{R}, \tag{5}$$

is a well-defined, one-parameter group of automorphisms on $\mathcal{A}_\Lambda$; see [1] for more of the general theory.

2. Locality Bounds

Locality bounds, also known as Lieb-Robinson bounds [4], hold for a large class of models comprised of bounded interactions with sufficiently short range. To be more precise, fix $a > 0$ and denote by $\mathcal{B}_a(\mathbb{Z}^\nu)$ the set of all those interactions Φ for which

$$\|\Phi\|_a = \sup_{x,y\in\mathbb{Z}^\nu} e^{a|x-y|} \sum_{\substack{X\subset\mathbb{Z}^\nu:\\ x,y\in X}} \|\Phi(X)\| < \infty. \tag{6}$$

It is easy to see that all finite range, uniformly bounded interactions Φ are in $\mathcal{B}_a(\mathbb{Z}^\nu)$ for all $a > 0$.

A typical Lieb-Robinson bound, as proven e.g. in [7, 3, 5, 6], can be stated as follows.

Theorem 1. *Fix a collection of on-site Hamiltonians $\{H_x\}_{x\in\mathbb{Z}^\nu}$. Let $a > 0$, $\Phi \in \mathcal{B}_a(\mathbb{Z}^\nu)$, and take finite subsets $X, Y \subset \mathbb{Z}^\nu$. For any finite $\Lambda \subset \mathbb{Z}^\nu$ with $X \cup Y \subset \Lambda$, any $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, and $t \in \mathbb{R}$, one has that*

$$\|[\tau_t^\Lambda(A), B]\| \le C e^{-a(d(X,Y) - v_\Phi|t|)},$$

where

$$C = c_\nu \|A\| \|B\| \min[|X|, |Y|] \text{ and } v_\Phi = \frac{2\|\Phi\|_a c_\nu'}{a}.$$

3. Correlation Estimates

It is an interesting fact, see [7, 3, 5] for a proof, that locality bounds imply correlation estimates for many systems. The phrase *exponential clustering* refers to a proof that the existence of a spectral gap implies exponential decay of spatial correlations in the ground state.

Fix $a > 0$, let $\Phi \in \mathcal{B}_a(\mathbb{Z}^\nu)$, and take a finite set $\Lambda \subset \mathbb{Z}^\nu$. Suppose that $H_\Lambda \geq 0$ and assume that $\Omega_0 \in \mathcal{H}_\Lambda$ is the unique normalized vector such that $H\Omega_0 = 0$. Let $\gamma = \gamma(\Lambda) > 0$ denote the spectral gap, i.e.,

$$\gamma = \sup\{\delta > 0 \mid \sigma(H_\Lambda) \cap (0, \delta) = \emptyset\}.$$

Theorem 2. *Under the assumptions above, there exists $\mu > 0$ and a constant $C < \infty$ such that for any $X, Y \subset \Lambda$ with $X \cap Y = \emptyset$,*

$$|\langle AB\rangle_0 - \langle A\rangle_0 \langle B\rangle_0| \leq C\|A\| \|B\| \min[|X|, |Y|] e^{-\mu d(X,Y)}.$$

holds for all $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$. Here

$$\langle A\rangle_0 = \langle \Omega_0, A\Omega_0\rangle \quad \textit{for all } A \in \mathcal{A}_\Lambda$$

One can take

$$\mu = \frac{a\gamma}{\gamma + 4\|\Phi\|_a}.$$

General review articles for the above information can be found in [8, 9].

Our recent result (see [2]) is a proof that, under certain additional assumptions, one can prove a clustering result with a decay rate *independent* of the size of the gap.

Definition 1. Let Φ be an interaction on a quantum spin system over $\mathbb{Z}^\nu$. We say that Φ satisfies a **zero-velocity** Lieb-Robinson bound if there exists $\mu > 0$ such that given any finite $X, Y \subset \mathbb{Z}^\nu$ there exists $C(X, Y) < \infty$ with

$$\|[\tau_t^\Lambda(A), B]\| \leq C(X, Y) \min[|t|, 1]\|A\|\|B\|e^{-\mu d(X,Y)} \tag{7}$$

for all finite Λ with $X, Y \subset \Lambda$, $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, and $t \in \mathbb{R}$.

In [2] we prove that dynamical localization, in the form of a zero-velocity Lieb-Robinson bound, implies exponential clustering with a decay rate independent of the gap size. In fact, we show the following.

Theorem 3. *Let Φ satisfy a zero-velocity Lieb-Robinson bound. Let X and Y be finite, disjoint subsets of $\mathbb{Z}^\nu$ and take Λ finite with $X \cup Y \subset \Lambda$. Suppose $H_\Lambda \geq 0$ and has a unique normalized ground state $\Omega_0 = \Omega_0(\Lambda)$ satisfying $H_\Lambda\Omega_0 = 0$. Then for any $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$, the estimate*

$$|\langle AB\rangle_0 - \langle A\rangle_0\langle B\rangle_0| \leq C\|A\|\|B\|e^{-\mu d(X,Y)}$$

where one may take

$$C = 1 + \frac{C(X,Y)}{\pi}\left[2 - \ln\left(\frac{\gamma}{\sqrt{\pi\mu d(X,Y)}}\right)\right]$$

Here, as before, $\langle\cdot\rangle_0$ denotes ground state expectations, μ is as in (7), and $\gamma = \gamma(\Lambda)$ is the gap.

In [2], we demonstrate that the above result applies to a random xy model.

References

[1] O. Bratteli and D. Robinson, *Operator Algebras and Quantum Statistical Mechanics 2*, 2nd Edition (Springer-Verlag, New York, 1997).

[2] E. Hamza, R. Sims and G. Stolz, Preprint 2011, arXiv:1108.3811.

[3] M. Hastings and T. Koma, *Commun. Math. Phys.* **265**, 781–804, (2006).

[4] E. Lieb and D. Robinson, *Commun. Math. Phys.* **28**, 251–257, (1972).

[5] B. Nachtergaele, Y. Ogata, and R. Sims, *J. Stat. Phys.* **124**, no.1, 1–13, (2006).

[6] B. Nachtergaele, H. Raz, B. Schlein, and R. Sims, *Commun. Math. Phys.* **286**, 1073–1098, (2009).

[7] B. Nachtergaele and R. Sims, *Commun. Math. Phys.* **265**, 119–130, (2006).

[8] B. Nachtergaele and R. Sims, *Contemp. Math.* **529** 141–176, (2010).

[9] R. Sims, *Mathematical Results in Quantum Physics*, ed. Pavel Exner (World Scientific, New Jersey 2011) pgs 95-106.

A multi-component Sherrington–Kirkpatrick spin-glass in a transverse magnetic field

Wolfgang Spitzer

(joint work with R. Ruder)

We consider a classical mean-field spin-glass model of multi-component spins in a transverse magnetic field. Let $2 \le N \in \mathbb{N}$ be the number of spins (or sites). For every $i \in \{1, \ldots, N\}$ the single-site spin-configuration $S_i = (S_i^x, S_i^y) \in \mathbb{S}^1$, where $\mathbb{S}^1$ is the Euclidean unit sphere in $\mathbb{R}^2$. By $S := (S_1, \ldots, S_N)$ we denote the spin-configuration of all N spins.

Furthermore, let $(g_{ij})_{1\le i<j\le N}$ be a family of independent Gaussian random variables of mean zero and variance one. The Sherrington–Kirkpatrick spin-glass Hamiltonian that we consider is of the form

$$S \mapsto H_N(S) := -N^{-1/2} \sum_{1\le i<j\le N} g_{ij} S_i^x S_j^x - h \sum_{1\le i\le N} S_i^y \,. \tag{1}$$

Here, $h \in \mathbb{R}$ is the strength of an exterior magnetic field transversal to the direction of the spin-glass mean-field pair interaction.

For inverse temperature $\beta > 0$ we define the partition function and free energy (density)

$$Z_N(\beta, h) := \operatorname{tr} \exp\left(-\beta H_N\right), \quad f_N(\beta, h) := -(\beta N)^{-1} \log Z_N(\beta, h)\,,$$

where for a function $A : (\mathbb{S}^1)^N \to \mathbb{R}$ and a fixed measure μ on $\mathbb{S}^1$,

$$\operatorname{tr} A := \int_{(\mathbb{S}^1)^N} \prod_{i=1}^N \mathrm{d}\mu(S_i)\, A(S)\,.$$

If $\mu = \frac{1}{2}\delta_{(-1,0)} + \frac{1}{2}\delta_{(1,0)}$ then H_N reduces to the mean-field Ising spin-glass Hamiltonian (with zero magnetic field, see (4)) introduced by Sherrington and Kirkpatrick [7] in 1975. In the following we assume μ to be the uniform (probability) measure on $\mathbb{S}^1$.

In 2002/3, Guerra and Toninelli [2, 3] proved the existence of the free energy in the thermodynamic limit $N \to \infty$ for one and multi-component Sherrington–Kirkpatrick models, which marked the beginning of a renewed interest and very successful activities in this field. So let us denote by $f(\beta, h) := \lim_{N\to\infty} f_N(\beta, h)$ the limiting free energy.

In their seminal paper [7], Sherrington and Kirkpatrick used a replica-symmetric ansatz for the computation of the free energy, which we denote by f^{SK}. Our main result is that for large β or for large h, the free energy, f, equals f^{SK}. In order to write down f^{SK} we define the function $L_{r,q}$ on the single-site spin space, $\mathbb{S}^1$, as

$$S \mapsto L_{r,q}(S) := -\sqrt{g}\, q S^x - \frac{\beta}{2}(r-q)(S^x)^2 - hS^y \,, \tag{2}$$

where g is a Gaussian random variable with mean zero and variance one. f^{SK} is given by the formula

$$f^{\mathrm{SK}}(\beta, h) := \inf_{r\in\mathbb{R}} \sup_{q\in\mathbb{R}} \left\{ \frac{\beta}{4}(r^2 - q^2) - \frac{1}{\beta}\mathbb{E} \log \operatorname{tr} \exp\left(-\beta L_{r,q}\right) \right\}, \tag{3}$$

where $\mathbb{E}$ denotes the expectation with respect to g.

Here is our main result [6]:

Theorem 1 (High-temperature paramagnetic phase)**.**

(i) *If* $\beta \le 1$ *then for all* $h \in \mathbb{R}$ *we have that* $f(\beta, h) = f^{\mathrm{SK}}(\beta, h)$.
(ii) *If* $\beta > 1$ *then there exists an* h *large enough so that* $f(\beta, h) = f^{\mathrm{SK}}(\beta, h)$.
(iii) *There exists an* h_c *with* $1.5 < h_c < 2$ *such that* $\lim_{\beta\to\infty} f(\beta, h) = -|h|$ *for* $|h| > h_c$.

The free energy, f, is, of course, a random variable but, by a result of Pastur and Shcherbina [5], almost surely equal to its expectation value. All three above statements are understood in the almost sure sense.

The method of proof is an extension of Guerra and Toninelli's proof [4] of a similar result for the one-component Sherrington–Kirkpatrick model in a *longitudinal* magnetic field of strength γ with Hamiltonian,

$$S \in \{-1, 1\}^N \mapsto \tilde{H}_N(S) := -N^{-1/2} \sum_{1\le i<j\le N} g_{ij} S_i S_j - \gamma \sum_{1\le i\le N} S_i \,. \tag{4}$$

Concerning statement (iii) in Theorem 1, there is now no critical γ_c at zero temperature that separates the paramagnetic from the spin-glass phase.

In 1987, Aizenman, Lebowitz, and Ruelle [1] proved (among other things) the existence of a low-temperature spin-glass phase for the above Hamiltonian $\tilde{H}_N$ (with $\gamma = 0$) in terms of a spin-glass parameter, Q. To this end, let $\langle\cdot\rangle$ be the

Gibbs-expectation value with respect to H_N from (1) and let $\mathbb{E}$ be the expectation value with respect to the random variables (g_{ij}). Then we define

$$Q := \lim_{N\to\infty} \frac{2}{N(N-1)} \sum_{1\le i<j\le N} \mathbb{E}(\langle S_i^x S_j^x \rangle^2)\,.$$

In the high-temperature phase described in the above theorem $Q = 0$.

Following the line of arguments of [1] we can show the following [6]:

Theorem 2 (Low-temperature spin-glass phase)**.** *For β large enough and $|h|$ small enough we have $Q > 0$.*

References

[1] M. Aizenman, J. L. Lebowitz, and D. Ruelle, *Some rigorous results on the Sherrington–Kirkpatrick spin glass model.* Commun. Math. Phys. **112** (1987), 3–20.

[2] F. Guerra and F. L. Toninelli, *The thermodynamic limit in mean field spin glass models.* Commun. Math. Phys. **230** (2002), 71–79.

[3] F. Guerra and F. L. Toninelli, *The infinite volume limit in generalized mean field disordered models.* Markov Proc. Rel. Fields **9** (2003), 195–207.

[4] F. Guerra and F. L. Toninelli, *Quadratic replica coupling in the Sherrington–Kirkpatrick mean field spin glass model.* J. Math. Phys. **43**, (2002) 3704–3716.

[5] L.A. Pastur und M. V. Shcherbina, *The absence of the selfaverageness of the order parameter in the Sherrington–Kirkpatrick model.* J. Stat. Phys. **62** (1991), 1–19.

[6] R. Ruder and W. Spitzer, *A multi-component Sherrington–Kirkpatrick spin-glass in a transverse magnetic field.* Preprint in preparation.

[7] D. Sherrington and K. Kirkpatrick, *Solvable model of a spin-glass.* Phys. Rev. Lett. **35** (1975), 1792–1795.

Dynamical Localization for a Random XY-Spin Chain

Günter Stolz
(joint work with E. Hamza and R. Sims)

The anisotropic xy spin chain in an exterior magnetic field is given by the Hamiltonian

$$H_n = \sum_{j=1}^{n-1} \mu_j[(1+\gamma_j)\sigma_j^x\sigma_{j+1}^x + (1-\gamma_j)\sigma_j^y\sigma_{j+1}^y] + \sum_{j=1}^{n} \nu_j\sigma_j^z\,. \tag{1}$$

acting on $\mathcal{H} = \bigotimes_{j=1}^n \mathbb{C}^2$. Here σ^x, σ^y, and σ^z are the Pauli matrices. The real-valued sequences $\{\mu_j\}$, $\{\gamma_j\}$, and $\{\nu_j\}$ will be considered random variables and $\mathbb{E}$ will denote the expectation with respect to the disorder.

We denote the Heisenberg evolution of an observable $A \in B(\mathcal{H})$ under H_n by

$$\tau_t^n(A) := e^{iH_nt}A\,e^{-iH_nt}.$$

For a subset $N_0 \subset \{1,\dots,n\}$ we identify the linear operators $A \in \mathcal{A}_{N_0} := \bigotimes_{j\in N_0} B(\mathbb{C}^2)$ only acting on the spins at sites $j \in N_0$ with the operator $A' = A \otimes I \in B(\mathcal{H}) = \bigotimes_{j=1}^n B(\mathbb{C}^2)$, where I acts on the spins at sites $\{1,\dots,n\} \setminus N_0$.

Define a $2n \times 2n$-matrix by

$$M^{(n)} = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix},$$

where

$$A = \begin{pmatrix} \nu_1 & -\mu_1 & & \\ -\mu_1 & \ddots & \ddots & \\ & \ddots & \ddots & -\mu_{n-1} \\ & & -\mu_{n-1} & \nu_n \end{pmatrix},$$

,

$$B = \begin{pmatrix} 0 & -\mu_1\gamma_1 & & \\ \mu_1\gamma_1 & \ddots & \ddots & \\ & \ddots & \ddots & -\mu_{n-1}\gamma_{n-1} \\ & & \mu_{n-1}\gamma_{n-1} & 0 \end{pmatrix}.$$

Definition: *We say the matrices $M^{(n)}$ are dynamically localized if there exist numbers $C > 0$ and $\eta > 0$ such that for any integers $j, k, n \geq 1$ with $j, k \in [1, n]$,*

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} |M^{(n)}_{j,k}(t)| + \sup_{t\in\mathbb{R}} |M^{(n)}_{j,n+k}(t)|\right) \leq Ce^{-\eta|j-k|}, \tag{2}$$

where $M^{(n)}_{j,k}(t) = \left(e^{-iM^{(n)}t}\right)_{j,k}$.

Dynamical localization of $M^{(n)}$ implies the following *zero velocity Lieb-Robinson bound* for the xy chain after averaging over the disorder:

Theorem 1: *Assume that $M^{(n)}$ is dynamically localized. Then there exists $C > 0$ such that for η as in (2), any integers $1 \leq j < k$ and any $n \geq k$, the bound*

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \|[\tau^n_t(A), B]\|\right) \leq C\|A\|\|B\|e^{-\eta|k-j|} \tag{3}$$

holds for all $A \in \mathcal{A}_j$ and $B \in \mathcal{A}_{[k,n]}$.

A special case of (1) where the assumption of the Theorem are fulfilled is the isotropic xy chain in random exterior magnetic field given by

$$H_{n,\mathrm{iso}} = \mu \sum_{j=1}^{n} [\sigma^x_j \sigma^x_{j+1} + \sigma^y_j \sigma^y_{j+1}] + \sum_{j=1}^{n} \nu_j \sigma^z_j, \tag{4}$$

where μ is a non-zero constant. We assume that the magnetic field ν_j, $j \in \mathbb{N}$, in (4) is given by i.i.d. random variables, whose common distribution is absolutely continuous with bounded and compactly supported density.

In this case dynamical localization of $M^{(n)}$ reduces to dynamical localization for the Anderson model, which is known under this assumption. Thus we have

Corollary: *The Heisenberg evolution of $H_{n,\mathrm{iso}}$ satisfies the zero-velocity Lieb-Robinson bound (3).*

For deterministic quantum spin systems, under mild assumptions and up to a correction term which depends logarithmically on the ground state gap, zero-velocity Lieb-Robinson bounds imply exponential decay of ground state correlations, see the talk of R. Sims at this workshop. With similar arguments one shows

Theorem 2: *The groundstate ψ_0 of (4) is almost surely non-degenerate and there exist $C' < \infty$ and $\eta' > 0$ such that*

$$\mathbb{E}\left(|\langle\psi_0, AB\psi_0\rangle - \langle\psi_0, A\psi_0\rangle\langle\psi_0, B\psi_0\rangle|\right) \leq C'\|A\|\|B\|\, n\, e^{-\eta'|j-k|}$$

for all $A \in \mathcal{A}_j$, $B \in \mathcal{A}_{[k,n]}$ and all $1 \leq j < k \leq n$.

More details and proofs of the above results as well as references to related works can be found in E. Hamza, R. Sims and G. Stolz, Dynamical Localization for Disordered Quantum Spin Systems, Preprint 2011, `arXiv:1108.3811`.

Random matrices and random Schrödinger operators

Bálint Virág
(joint work with E. Kritchevski and B. Valkó)

We consider the one-dimensional discrete random Schrödinger operators

$$(H_n\psi)_\ell = \psi_{\ell-1} + \psi_{\ell+1} + v_\ell\psi_\ell,$$

$\psi_0 = \psi_{n+1} = 0$ where $v_k = \sigma\omega_k/\sqrt{n}$. and the ω_k are independent random variables with mean 0, variance 1 and bounded third absolute moment.

The matrix H_n is a perturbation of the adjacency matrix of a path. When the variance of v_k does not depend on n, eigenvectors are localized and the local statistics of eigenvalues are Poisson (see [1, 3], from which this abstract was distilled, for detailed references). Our regime, where the variance of the random variables v_ℓ are of order $n^{-1/2}$ captures the transition between localization an delocalization.

If there is no noise (i.e. $\sigma = 0$) then the eigenvalues of the operator are given by $2\cos(\pi k/(n+1))$ with $k = 1, \dots, n$. The asymptotic density near $E \in (-2, 2)$ is given by $\frac{\rho}{2\pi}$ with $\rho = \rho(E) = 1/\sqrt{1 - E^2/4}$. We will study the spectrum Λ_n of the scaled operator $\rho n(H_n - E)$. By the well-known transfer matrix description the eigenvalue equation $H_n\psi = \mu\psi$ is written as

$$(1) \qquad \begin{pmatrix}\psi_{\ell+1}\\ \psi_\ell\end{pmatrix} = T(\mu - v_\ell)\begin{pmatrix}\psi_\ell\\ \psi_{\ell-1}\end{pmatrix} = M_\ell^\lambda\begin{pmatrix}\psi_1\\ \psi_0\end{pmatrix},$$

where $T(x) := \begin{pmatrix} x & -1\\ 1 & 0\end{pmatrix}$ and with $\mu = E + \frac{\lambda}{\rho n}$ and $\varepsilon_\ell = \frac{\lambda}{\rho n} - \frac{\sigma\omega_\ell}{\sqrt{n}}$, we have

$$(2) \qquad M_\ell^\lambda = T(E+\varepsilon_\ell)T(E+\varepsilon_{\ell-1})\cdots T(E+\varepsilon_1) \text{ for } 0 \leq \ell \leq n.$$

Then μ is an eigenvalue of H_n if and only if $M_n\binom{1}{0} = c\binom{0}{1}$. The scaling of $v_\ell = \sigma\omega_\ell/\sqrt{n}$ ensures that, with high probability, the transfer matrices M_ℓ^λ are bounded and the eigenfunctions are delocalized.

The starting observation is that M_ℓ^λ cannot have a continuous limit, since for large n the transfer matrix $T(E+\varepsilon_k)$ in (2) is not close to I but to $T(E)$. Thus we are led to consider, instead of M_ℓ^λ, the regularly-evolving matrices

$$X_\ell^\lambda = T^{-\ell}(E) M_\ell^\lambda, \qquad 0 \le \ell \le n. \tag{3}$$

To control the correction factor $T^{-\ell}(E)$, we diagonalize $T(E) = ZDZ^{-1}$ with

$$D = \begin{pmatrix} \overline{z} & 0 \\ 0 & z \end{pmatrix}, \quad Z = \begin{pmatrix} \overline{z} & z \\ 1 & 1 \end{pmatrix}, \quad z = E/2 + i\sqrt{1-(E/2)^2}. \tag{4}$$

Theorem 1. *Assume $0 < |E| < 2$. Let $\mathcal{B}(t), \mathcal{B}_2(t), \mathcal{B}_3(t)$ be independent standard Brownian motions in $\mathbb{R}$, $\mathcal{W}(t) = \frac{1}{\sqrt{2}}(\mathcal{B}_2(t) + i\mathcal{B}_3(t))$. Then the stochastic differential equation*

$$dX^\lambda = \frac{1}{2} Z\left(\begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix} dt + \begin{pmatrix} id\mathcal{B} & d\mathcal{W} \\ d\overline{\mathcal{W}} & -id\mathcal{B} \end{pmatrix}\right) Z^{-1} X^\lambda, \qquad X^\lambda(0) = I \tag{5}$$

has a unique strong solution $X^\lambda(t) : \lambda \in \mathcal{C}, t \ge 0$, which is analytic in λ. Moreover with $\tau = (\sigma\rho)^2$

$$(X^\lambda_{\lfloor nt/\tau \rfloor}, 0 \le t \le \tau) \Rightarrow (X^\lambda(t), 0 \le t \le \tau),$$

in the sense of finite dimensional distributions for λ and uniformly in t. Also, for any given $0 \le t \le \tau$ the random analytic functions $X^\lambda_{\lfloor nt/\tau \rfloor}$ converge in distribution to $X^\lambda(t)$ with respect to the local uniform topology.

Moreover the shifted eigenvalue process $\Lambda_n - \arg(z^{2n+2})$ converges in distribution to a point process Sch_τ.

The point process Sch_τ is only invariant under translation by integer multiples of 2π. A translation-invariant version (shifted by an independent uniform random variable) $\mathsf{Sch}^*_\tau = \mathsf{Sch}_\tau + U[0, 2\pi]$ can be described through a variant of the the Brownian carousel introduced in [2].

The Brownian carousel. Let $(\mathcal{V}(t), t \ge 0)$ be Brownian motion on the hyperbolic plane $\mathbb{H}$. Pick a point on the boundary $\partial\mathbb{H}$ and let $x^\lambda(0)$ equal to this point for all $\lambda \in \mathbb{R}$. Let $x^\lambda(t)$ be the trajectory of this point rotated continuously around $\mathcal{V}(t)$ at speed λ. Recall that Brownian motion in $\mathbb{H}$ converges to a point $\mathcal{V}(\infty)$ in the boundary $\partial\mathbb{H}$. Then we have $\mathsf{Sch}^*_\tau \overset{d}{=} \{\lambda : x^{\lambda/\tau}(\tau) = \mathcal{V}(\infty)\}$.

The following properties of Sch_τ help compare it to random matrices.

Theorem 2 (Eigenvalue repulsion)**.** *For $\mu \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\{\mathsf{Sch}_\tau[\mu, \mu+\varepsilon] \ge 2\} \le 4 \exp\left(-(\log(\tau/\varepsilon) - \tau)^2/\tau\right). \tag{6}$$

whenever the squared expression is nonnegative.

Theorem 3 (Probability of large gaps)**.** *The probability that Sch_τ has a large gap is*

$$\mathbb{P}(\mathsf{Sch}_\tau[0,\lambda] = 0) = \exp\left\{-\frac{\lambda^2}{4\tau}(1 + o(1))\right\}$$

where $o(1) \to 0$ for a fixed τ as $\lambda \to \infty$.

The above results show that the eigenvalue statistics of 1D random Schrödinger operators are *not* universal. However, GOE statistics appear for very thin boxes in $\mathbb{Z}^2$. The proof first establishes a fixed higher dimensional version of Theorem 1 and then uses recent results in universality of Wigner matrices.

Theorem 4. [3] *There exists a sequence of weighted boxes on $\mathbb{Z}^2$ with diameter converging to ∞ so that the rescaled eigenvalue process of the adjacency matrix plus diagonal noise converges to the bulk point process limit of the GOE ensemble.*

References

[1] E. Kritchevski, B. Valkó, and B. Virág, *The scaling limit of the critical one-dimensional random Schrödinger operator*. In preparation.

[2] B. Valkó and B. Virág, *Continuum limits of random matrices and the Brownian carousel.* Inventiones Math. **177** (2009), 463–508.

[3] B. Valkó and B. Virág, *Random Schrödinger operators on long boxes, noise explosion and the GOE.* (2010), arXiv:0912.0097.

Quantitative unique continuation principle and Wegner estimates

Ivan Veselić

(joint work with C. Rojas-Molina)

Let us denote boxes centered at the origin by $\Lambda_L := \left[-\frac{L}{2}, \frac{L}{2}\right]^d$. For a Schrödinger operator $-\Delta+V$ we denote its restriction to Λ_L with periodic boundary conditions by $(-\Delta - V)_{\Lambda_L}$.

The main result is the following **scale-free quantitative unique continuation principle**: Let $K, E_0, \delta \in (0,\infty)$. There is a constant $C_{UCP} \in (0,\infty)$ depending only on K, E_0 and δ such that for any sequence $\{y_k\}_{k\in\mathbb{Z}^d}$ satisfying $B_\delta(y_k) \subset \Lambda_1$ for all $k \in \mathbb{Z}^d$, any scale $L \in \mathbb{N}$, any measurable potential $V\colon \mathbb{R}^d \to [-K,K]$, any energy $E \leq E_0$, any solution $\psi \in W^{2,2}(\Lambda_L)$ with periodic boundary conditions of the equation

$$(-\Delta - V)_{\Lambda_L}\psi = E\psi$$

the following bound holds:

$$\int_{\Lambda_L} |\psi(x)|^2 dx \leq C_{UCP} \int_{S_L} |\psi(x)|^2 dx \qquad (1)$$

where $S_L := \bigcup_{k\in\mathbb{Z}^d} B_\delta(y_k + k) \ \cap \ \Lambda_L$.

The main point of the statement is that the constnt C_{UCP} does not depend on the scale $L \in \mathbb{N}$. Note also that the constant does not depend on the specific shape of the potential V, but only on its supremum $\|V\|_\infty \leq K$.

If all y_k are equal to zero, we will say that the balls $S = \bigcup_{k\in\mathbb{Z}^d} B_\delta(k)$ are arranged periodically. This is a very important special case.

Similar (and in some cases and some aspects even stronger) statements have been obtained previously:

Energies near the spectral minimum: The earliest results of the type above have been obtained in the situation where E_0 is small, where the smallness depends on the parameter $\delta > 0$, and for more specific choices of the structure of the potential. For a proof of the statement in the case that the points y_k are all equal to the origin, see e.g. [7]. For general sequences $B_\delta(y_k) \subset \Lambda_1$ of balls see e.g. [3]. The last result holds actually for functions ψ with sufficiently small expectaion value $\langle \psi, -\Delta\psi \rangle$.

Energies near spectral edges: Properties of eigenfunctions associated to eigenvalues near spectral edges have been considered e.g. in [8, Prop.3.2]. The potential there consists of a periodic part plus a random alloy-type part. For this types of potentials there is a well defined notion of spectral edges. The considered arrangement of balls S is again periodic in this paper.

One space dimension: In one space dimension the scale free quantitative unique continuation principle holds for all functions ψ belonging to the spectral subspace associated to the interval $(-\infty, E_0]$ and the operator $(-\Delta + V)_{\Lambda_L}$. This has been proven in [10, 9]. Actually the formulation there refers to periodic arrangements of balls, but the proof holds also in the non-periodic situation, as has been stated explicitly in [6].

Periodic arrangement of balls: In the case that the arrangement of balls S and the potential V is periodic, the scale-free unique continuation principle holds for all functions ψ belonging to the spectral subspace associated to the interval $(-\infty, E_0]$ of the Schrödinger operator. This has been proven in [2]. Here there is no restriction on the value of E_0. An alternative proof for this result with more explicit control of constants has been derived in [4].

The above stated scale-free unique continuation principle has several consequences for random Schrödinger operators with non-negative potentials of Delone-alloy-type.

- A Wegner estimate holds (under the usual assumptions on the regularity of the coupling constants) which is linear in the volume of the box Λ_L and almost linear in the length 2ε of the considered energy interval $[E-\varepsilon, E+\varepsilon] \subset (-\infty, E_0]$. This result uses [5].
- The bottom of the spectrum of a Schrödinger operator is lifted by the addition of a non-negative Delone-alloy-type potential.
- Using [1] it follows that the scale-free quantitative unique continuation principle holds for all functions ψ belonging to the spectral subspace associated to the interval $(-\infty, E_0]$ of the Schrödinger operator (for sufficiently small E_0).

References

[1] A. Boutet de Monvel, D. Lenz, and P. Stollmann, *An uncertainty principle, wegner estimates and localization near fluctuation boundaries*, Math. Zeit., online first.

[2] J.-M. Combes, P. D. Hislop, and F. Klopp, *Hölder continuity of the integrated density of states for some random operators at all energies*, Int. Math. Res. Not., no. 4, 179–209, 2003.

[3] Germinet, F., *Recent advances about localization in continuum random Schrödinger operators with an extension to underlying Delone sets*, Mathematical results in quantum mechanics, World Sci. Publ., pp. 79–96, 2008.

[4] F. Germinet, A. Klein : A comprehensive proof of localization for continuous Anderson models with singular random potentials, preprint 2011.

[5] D. Hundertmark, R. Killip, Shu Nakamura, P. Stollmann, and I. Veselić, *Bounds on the spectral shift function and the density of states*, Comm. Math. Phys. **262** , no. 2, 489–503, 2006

[6] M. Helm and I. Veselić, *Linear Wegner estimate for alloy-type Schrödinger operators on metric graphs*, J. Math. Phys. **48** no. 9, 092107, 7, 2007.

[7] W. Kirsch, *Wegner estimates and Anderson localization for alloy-type potentials*, Math. Z. **221**, 507–512, 1996

[8] W. Kirsch, P. Stollmann, and G. Stolz, *Localization for random perturbations of periodic Schrödinger operators*, Random Oper. Stochastic Equations **6**, no. 3, 241–268, 1998.

[9] W. Kirsch and I. Veselić, *Existence of the density of states for one-dimensional alloy-type potentials with small support*, Contemp. Math., vol. 307, pp. 171–176, 2002.

[10] I. Veselić, *Lokalisierung bei zufällig gestörten periodischen Schrödingeroperatoren in Dimension Eins*, Diplomarbeit, Ruhr-Universität Bochum, available at http://www.ruhr-uni-bochum.de/mathphys/ivan/diplomski-www-abstract.htm, 1996.

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