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Mini-Workshop: Manifolds with Lower Curvature Bounds

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January 1st – January 7th, 2012

Abstract. The purpose of the meeting was to relate and study new developments in the geometry and topology of Riemannian manifolds with lower curvature bounds. Special emphasis was given to lower Ricci curvature bounds in the sense of Lott-Villani and Sturm and to the gradient flow on metric spaces as well as to manifolds with a lower sectional curvature bound.

Mathematics Subject Classification (2000): 53: Differential Geometry, 53C20: Global Riemannian geometry, including pinching, 53C21: Methods of Riemannian geometry, including PDE methods; curvature restrictions.

Introduction by the Organisers

The workshop *Manifolds with Lower Curvature Bounds*, organised by Anand Dessai (Fribourg), Wilderich Tuschmann (Karlsruhe) and Burkhard Wilking (Münster), was held January 1st–January 7th, 2012. The meeting was attended by 17 participants, ranging from second year graduate students to senior researchers.

The purpose of the meeting was to relate and study new developments in the geometry and topology of Riemannian manifolds with lower curvature bounds. Special emphasis was given to manifolds with given lower bounds on Ricci curvature as well as to manifolds of nonnegative/positive sectional curvature.

The meeting was organised around ten one-hour talks, two mini courses, and two short talks by young PhD students, leaving plenty of time between and after talks for informal discussions.

The workshop started off with a talk by Guofang Wei surveying various characterizations of lower Ricci curvature bounds. Later on in the week Nicola Gigli (Oberwolfach Prize Winner 2010) and Giuseppe Savaré each gave 2 hour mini courses on optimal transport, lower Ricci curvature bounds and their work on gradient flows on metric spaces thereby also providing a gentle introduction to the recent work of Lott-Villani and Sturm. David Wraith described new methods to construct invariant metrics of positive Ricci curvature on G-manifolds with finitely many non-principal orbits.

Other talks were related to lower bounds on sectional curvature (curvature for short). In his talk Igor Belegradek considered the space $\mathcal{R}(N)$ of complete metrics of nonnegative curvature on an open connected manifold N and showed in particular that the complement of any countable subset of $\mathcal{R}(\mathbb{R}^2)$ is path connected. Dmitri Panov discussed a polyhedral analogue of Frankel's conjecture and supporting evidence for it in low dimensions. Comparison theorems for integrals and Hölder norms under curvature bounds in the absence of injectivity radius bounds were addressed in Patrick Ghanaat's talk. Marco Radeschi showed that singular Riemannian foliations of the round sphere with leaves of dimension ≤ 3 are homogeneous. Computations for the algebra of stable polynomial invariants of Riemannian manifolds and speculations on how to describe it by graphs using a Rozansky-Witten approach were discussed by Gregor Weingart.

Another bulk of talks was devoted to the topology of Riemannian manifolds with nonnegative/positive sectional curvature and large symmetry. Fernando Galaz-Garcia and Wolfgang Spindeler discussed the (equivariant) classification of nonnegatively curved manifolds of dimension 4 and 5 with isometric action of the 2-torus and nonnegatively curved fixed point homogeneous 5-manifolds. Lee Kennard explained his use of the Steenrod algebra to prove generalized Hopf conjectures for positively curved manifolds under logarithmic lower bounds on the symmetry rank.

The meeting also included two short talks by young PhD students. Martin Herrmann presented a criterion for the total space of principal bundles to admit almost nonnegative curvature operators. Nicolas Weisskopf discussed a conjecture on the strong rigidity of the elliptic genus for positively curved spin manifolds with symmetry.

Mini-Workshop: Manifolds with Lower Curvature Bounds

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Abstracts

Characterizations of Lower Ricci Curvature Bound

Guofang Wei

We survey several characterizations of lower Ricci curvature bound for Riemannian manifolds, including Bochner inequality, comparison theorems, gradient estimate of heat kernel and entropy. This motivated the definition of lower Ricci curvature bound for general metric measures spaces defined by Lott-Villani [2] and Sturm [4, 5] independently.

The fundamental tools for studying Ricci curvature are the Bochner formula and the Jacobian determinant of the Exponential map. The Bochner formula for functions is

Theorem 1 (Bochner's Formula)**.** *For a smooth function u on a Riemannian manifold (M^n, g),*

$$\frac{1}{2}\Delta|\nabla u|^2 = |Hess\, u|^2 + \langle \nabla u, \nabla(\Delta u)\rangle + Ric(\nabla u, \nabla u). \tag{1}$$

When $\mathrm{Ric} \geq (n-1)H$, combining with the Cauchy-Schwarz inequality $|\mathrm{Hess}\, u|^2 \geq \frac{(\Delta u)^2}{n}$, we obtain the Bochner inequality

$$\frac{1}{2}\Delta|\nabla u|^2 \geq \frac{(\Delta u)^2}{n} + \langle \nabla u, \nabla(\Delta u)\rangle + (n-1)H|\nabla u|^2. \tag{2}$$

This gives us the first characterization:

Theorem 2. *A Riemannian manifold (M^n, g) has $Ric_M \geq (n-1)H$ iff the Bochner inequality (2) holds for all $u \in C^3(M)$.*

Now we give some comparison characterizations. If $\mathrm{Ric}_M \geq (n-1)H$, one can apply the Bochner formula to the distance function to deduce

- The Laplacian comparison: For $r(x) = d(p, x)$
$$\Delta r \leq \Delta_H r,$$
where Δ_H is the Laplacian in the model space M^n_H, the n-dimensional simply connected space with constant sectional curvature H.
- Write the volume element in polar coordinate: $dvol = \mathcal{A}(r,\theta)drd\theta, dvol_H = \mathcal{A}_H(r)drd\theta$. Since $(\log \mathcal{A})' = \Delta r$, the Laplacian comparison yields the volume element comparison:
$$\mathcal{A}(r,\theta) \leq \mathcal{A}_H(r).$$
- Fix $p \in M^n$. For any measurable set $B \subset M$, connect every point $y \in B$ to p with a minimal geodesic γ_y such that $\gamma_y(0) = p, \gamma_y(1) = y$. For $t \in [0,1]$, put $B_t = \{\gamma_y(t) | y \in B\}$. Integrating the volume element comparison, one gets, for $0 \leq t \leq 1$, localized Bishop-Gromov comparison:
$$\mathrm{vol}(B_t) \geq t \int_B \frac{\mathcal{A}_H(td(x,y))}{\mathcal{A}_H(d(x,y))} dvol_y.$$

This comparison was formulated by Ohta [3] and Sturm [5] independently.

Theorem 3. *$Ric_M \geq (n-1)H$ is equivalent to each of the above comparisons.*

On the other hand, Bakry-Emery [1] gives the following characterization in terms of gradient of heat kernel, which follows from the Bochner formula.

Theorem 4. *$Ric_M \geq (n-1)H$ is equivalent to the inequality*

$$|\nabla(E_t f)|^2(x) \leq e^{-2(n-1)Ht} E_t(|\nabla f|^2)(x),$$

for all $f \in C_c^\infty(M), t > 0, x \in M$. Here $E_t = e^{t\Delta}$ is the heat operator.

Now we focus on the Jacobian determinant of the exponential map. Let ξ be a vector field on a Riemannian manifold M^n, $T_t(x) = \exp_x(t\xi(x)) : M^n \to M^n$. Denote $J(t,x) = dT_t(x) : T_xM \to T_{T_t(x)}M$. Then for each $v \in T_xM$, $J(t,x)(v)$ is a Jacobi field along $T_t(x)$ and satisfies the Jacobi equation

$$J''(t,x)(v) + R(J(t,x)(v), \dot{T}_t(x))\dot{T}_t(x) = 0.$$

In addition, $J(0,x) = Id, J'(0,x) = \nabla\xi(x)$.

Let $U(t,x) = \dot{J} \cdot J^{-1}$. Then the Jacobi equation becomes the first order Riccati type equation

$$\dot{U} + R(\cdot, \dot{T}_t(x))\dot{T}_t(x) + U^2 = 0.$$

Taking trace, we have

$$\frac{d}{dt}(\operatorname{tr} U) + \operatorname{tr}(U^2) + \operatorname{Ric}(\dot{T}_t(x), \dot{T}_t(x)) = 0,$$

which is another important formula involving Ricci curvature. This is in fact equivalent to the Bochner formula. The two are dual to each other, with the above equation being the Lagrangian viewpoint while the Bochner formula Eulerian.

When $\xi = \nabla\psi$ is a gradient vector field, $\nabla\xi = \operatorname{Hess}\psi$ is symmetric, which implies that $U(t,x)$ is symmetric. Therefore we can use the Cauchy-Schwartz inequality, $\operatorname{tr}(U^2) \geq \frac{(\operatorname{tr} U)^2}{n}$, to obtain

$$\frac{d}{dt}(\operatorname{tr} U) + \frac{(\operatorname{tr} U)^2}{n} + \operatorname{Ric}(\dot{T}_t(x), \dot{T}_t(x)) \leq 0.$$

Recall $U(t,x) = \dot{J} \cdot J^{-1}$. Therefore $\operatorname{tr} U = \frac{\dot{\mathcal{J}}(t,x)}{\mathcal{J}(t,x)}$, where $\mathcal{J}(t,x) = \det J(t,x)$. Hence

$$\frac{\ddot{\mathcal{J}}}{\mathcal{J}} - (1 - \frac{1}{n})(\frac{\dot{\mathcal{J}}}{\mathcal{J}})^2 \leq -\operatorname{Ric}(\dot{T}_t(x), \dot{T}_t(x)).$$

Let $\mathcal{D}(t) = (\mathcal{J}(t))^{\frac{1}{n}}$, $l(t) = -\log \mathcal{J}(t)$. Then the above rewrites as

$$\begin{aligned} \frac{\ddot{\mathcal{D}}}{\mathcal{D}} &\leq -\frac{\operatorname{Ric}(\dot{T}_t(x), \dot{T}_t(x))}{n}, \\ \ddot{l}(t) &\geq \frac{\dot{l}(t)^2}{n} + \operatorname{Ric}(\dot{T}_t(x), \dot{T}_t(x)). \end{aligned}$$

We arrive at the ODE characterizations of lower Ricci curvature bound.

Theorem 5. *$Ric_M \geq H$ is equivalent to either of the inequalities*

$$\begin{aligned} \frac{\ddot{\mathcal{D}}}{\mathcal{D}} &\leq -\frac{H|\dot{T}_t(x)|^2}{n}, \\ \ddot{l}(t) &\geq \frac{\dot{l}(t)^2}{n} + H|\dot{T}_t(x)|^2. \end{aligned}$$

These ODEs has equivalent integral versions. For example, $\operatorname{Hess} F \geq kg$ is equivalent to

$$F(\gamma(t)) \leq tF(\gamma(1)) + (1-t)F(\gamma(0)) - \frac{k}{2}t(1-t)d^2(\gamma(0),\gamma(1))$$

for each geodesic $\gamma : [0,1] \to X$, and all $t \in [0,1]$. We refer to these functions as k-convex.

These considerations motivate the entropy characterization of lower Ricci curvature bound. Given a metric measure space (X, d, μ), let $\mathcal{P}_2(X)$ be the space of Borel probability measures equipped with Wasserstein distance W_2, then it is a length (geodesic) space. Set

$$\mathcal{P}_2(X,\mu) = \{\nu \in \mathcal{P}_2(X) \,|\, \nu = \rho\mu, \int_X d^2(x_0,x)\rho(x)d\mu(x) < \infty\}.$$

The relative (Shannon) entropy of $\nu \in \mathcal{P}_2(X)$ with respect to μ is defined by

$$H_\mu(\nu) = \begin{cases} \lim_{\epsilon\to 0}\int_{\rho>\epsilon}\rho\log\rho\,d\mu & \text{if } \nu \in \mathcal{P}_2(X,\mu) \\ +\infty & \text{otherwise} \end{cases}.$$

Von Renesse & Sturm [6] gave the following characterization.

Theorem 6. *$Ric_M \geq k$ is equivalent to that the Shannon entropy $H_\mu(\nu)$ is k-convex on $\mathcal{P}_2(M)$.*

Remark. Among all the characterizations the local Bishop-Gromov comparison and entropy characterization do not use differential structure. Though the local Bishop-Gromov comparison is much simpler and more geometric, it is not stable under the measured Gromov-Hausdorff limit. Therefore one uses entropy characterization to define lower Ricci bound for non-smooth metric measure spaces.

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On G-manifolds with finitely many non-principal orbits

David J. Wraith

(joint work with Stefan Bechtluft-Sachs)

We consider a compact Lie group G acting smoothly on a compact manifold M. The *cohomogeneity* of such an action is the dimension of the space of orbits M/G.

In recent years, the geometry of cohomogeneity-one manifolds has been extensively studied. (See for example [7].) Our motivating aim is to ask what can be said about topology and geometry in cohomogeneity greater than one? This question is deliberately vague, and allows many possible interpretations depending on the additional assumptions one makes in order to create a reasonable problem. Our approach is to focus on G-manifolds with only finitely many non-principal orbits. Thus each non-principal orbit is isolated in the sense that it has a tubular neighbourhood (with respect to some G-invariant background metric) in which all other orbits are principal. For an alternative interpretation of the above question, see [2].

In the following, K will denote the principal isotropy of the G-action, and H_1,...,H_p will denote the non-principal isotropy groups. Let N_i denote a tubular neighbourhood of the non-principal orbit G/H_i. Then $M - \cup_{i=1}^p N_i$ has the structure of a principal-orbit bundle. Let B denote the base of this bundle, so $B = (M - \cup_{i=1}^p N_i)/G$. It is clear that B is a manifold with p boundary components. We note that $T_i := \partial N_i$ has two fibration structures: it is fibered by principal orbits, and is also fibered by normal spheres S^r.

We first consider the structure of the orbit space M/G. The key to understanding this is the following result:

Theorem. *([1]; chapter 4, §6) Let L be a compact Lie group acting locally smoothly, effectively and with one orbit type on S^r. If $\dim L > 0$ then L acts transitively or freely, and if L acts freely, we must have $L \cong U(1)$, $N_{SU(2)}U(1)$ or $SU(2)$. If $\dim L = 0$ then $S^r \to S^r/L$ is the universal covering, so L must also act freely.*

In our case, if any H_i acts transitively then the cohomogeneity must be one. As we wish to study cohomogeneities greater than one, we will assume the H_i-action is not transitive. We deduce:

Corollary. *If the cohomogeneity is greater than one, then K is ineffective kernel of the H_i action on S^r, so K is normal in H_i and $H_i/K \cong U(1)$, $N_{SU(2)}U(1)$, $SU(2)$, or is finite, and acts freely and linearly on S^r.*

In turn we deduce:

Corollary. *If the cohomogeneity is greater than one, then T_i/G is either a complex or quaternionic projective space, or a $\mathbb{Z}_2$ quotient of an odd dimensional complex projective space in the case of a singular orbit, or in the case of an exceptional orbit a real projective or lens space. Also, each N_i/G is a cone over one of these spaces.*

The structure of M/G is then given by:

Theorem. *M/G is the union of a manifold with boundary B, where each boundary component is one of the above listed spaces, together with cones over the boundary components.*

Notice that if there is at least one singular orbit, this forces the cohomogeneity to be odd. For more details about the topology of these objects, see [3].

We now consider the geometry of these objects, and in particular we consider the existence of invariant metrics with positive Ricci curvature. To provide some motivation for this, let us recall the following result for cohomogeneity one:

Theorem. *([5]) A compact cohomogeneity one manifold admits an invariant metric with positive Ricci curvature if and only if its fundamental group is finite.*

There is little possibility of proving a result as strong as this in the current context: the space of orbits in cohomogeneity one is either a circle or an interval. Either way, this makes no contribution to the curvature. However, in higher cohomogeneities, it is to be expected that the geometry of the space of orbits will play some role in determining the global geometric properties.

In the statement of the theorem below, g_i denotes a metric on the appropriate boundary component induced via the standard submersion from the round metric of radius one. For more details, see [4].

Theorem. *Suppose that $\pi_1(G/K)$ is finite. Then if B admits a Ricci positive metric such that*
i) the metric on boundary component i is $\lambda_i^2 g_i$, and
ii) the principal curvatures (with respect to the inward normal) at boundary component i are greater than $-1/\lambda_i$,
then M admits an invariant Ricci positive metric.

Corollary. *All G-manifolds with two singular orbits, orbit space a suspension $\Sigma\mathbb{C}P^m$ or $\Sigma\mathbb{H}P^m$, and principal orbit G/K with $\pi_1(G/K) < \infty$ admit invariant metrics with positive Ricci curvature.*

To illustrate this, given any two Aloff-Wallach spaces W_{p_1,p_2} and W_{q_1,q_2}, there is a 11-dimensional $SU(3)$-manifold $M^{11}_{p_1p_2q_1q_2}$ of cohomogeneity 3 with two singular orbits W_{p_1,p_2} and W_{q_1,q_2} and orbit space $\Sigma\mathbb{C}P^1 = S^3$. This family contains infinitely many homotopy types, and all manifolds in this family admit invariant metrics of positive Ricci curvature.

The above corollary raises the question of whether there are Ricci positive examples with more than two non-principal orbits. The key to answering this is the following result, which relies on a construction in [6].

Proposition. *For each $n \geq 3$, $m \geq 1$ and sufficiently small $\rho > 0$, there is a $\delta_0 = \delta_0(\rho) > 0$ such that for all $0 < \delta < \delta_0$ there is a Ricci positive metric on $S^n - \coprod_{i=1}^m D^n$ such that each boundary component is a round sphere of radius δ, and the principal curvatures at the boundary (with inward normal) are all equal to $-\rho/\delta$.*

Using this Proposition in conjunction with the above Theorem yields the following:

Theorem. *In cohomogeneities 3 and 5, there are G-manifolds with any given number of isolated singular orbits admitting an invariant metric of positive Ricci curvature.*

For example, in cohomogeneity 3, there is an 11-dimensional $SU(3)$-manifold with an invariant Ricci positive metric having isolated singular orbits equal to any given (finite) collection of Aloff-Wallach spaces.

Open question. *Can we find manifolds with more than two non-principal orbits and an invariant Ricci positive metric in cohomogeneities $\neq 3, 5$?*

The problem in this case is to understand the geometry of the space of orbits. We know for example that $\mathbb{H}P^{2k+1}$, $\mathbb{C}P^{2k+1}$ and $\mathbb{R}P^{2k+1}$ are boundaries, and so we can create manifolds with boundary (by a connected sum on the interior of the bounding manifolds) having any selection of these spaces as boundary components. These are all candidates for the manifold B. Geometrically, what can be said about such manifolds? Do any admit Ricci positive metrics?

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4- and 5-dimensional simply connected manifolds with nonnegative curvature and torus actions

Fernando Galaz-Garcia

(joint work with Martin Kerin, Catherine Searle)

The study of Riemannian manifolds with positive and, more generally, nonnegative (sectional) curvature is a field in which metric aspects of differential geometry, such as comparison arguments, play a central role (cf. [15, 16]). Despite the existence of general structure results (e.g., the Cheeger-Gromoll soul theorem [2]) and of obstructions to nonnegative curvature (e.g., Gromov's Betti number theorem [7]), examples of positively and nonnegatively curved manifolds are scarce, as well as techniques for their construction. Thus, finding new spaces in this class remains a central problem in the field. In this context, considering manifolds with a "large" isometry group provides a systematic approach to the study of both positively and nonnegatively curved manifolds (see [8]).

Let M be a compact Riemannian manifold and G its isometry group, which is a compact Lie group. Observe that G acts on M by isometries. One possible measure for the size of G is the *symmetry rank* of M, denoted by $\mathrm{symrank}(M)$, and defined as the rank of G. Here "large" is interpreted as $\mathrm{symrank}(M)$ being big. Grove and Searle [9] showed that the maximal symmetry rank of a compact positively curved n-manifold M is $\lfloor (n+1)/2 \rfloor$ and that, if M has maximal symmetry rank, then it must be diffeomorphic to a sphere, a lens space, or a complex projective space.

If M is assumed to have nonnegative curvature, an upper bound on the symmetry rank smaller than the dimension of the manifold, as in the positively curved case, cannot be achieved in full generality, since the n-dimensional flat torus has maximal symmetry rank n. Under the additional hypothesis of simple connectivity, it has been conjectured (cf. [5]) that if M is a compact, simply connected nonnegatively curved n-manifold, then $\mathrm{symrank}(M) \leq \lfloor 2n/3 \rfloor$. In joint work with Searle, this conjectural bound on the symmetry rank has been verified in dimensions at most 9 (cf. [5]) and compact, simply connected manifolds of nonnegative curvature and maximal symmetry rank have been classified up to diffeomorphism in dimensions at most 6:

Theorem 1 ([5]). *Let M^n be a compact, simply connected Riemannian n-manifold with nonnegative curvature and an effective, isometric torus action of maximal rank.*

1. *If $n = 4$, then M^4 is diffeomorphic to $\mathbb{S}^4$, $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$.*
2. *If $n = 5$, then M^5 is diffeomorphic to $\mathbb{S}^5$, $\mathbb{S}^2 \times \mathbb{S}^3$ or $\mathbb{S}^2 \tilde{\times} \mathbb{S}^3$, the non-trivial $\mathbb{S}^3$-bundle over $\mathbb{S}^2$.*
3. *If $n = 6$, then M^6 is diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^3$.*

The classification program for compact, simply connected nonnegatively curved manifolds of maximal symmetry rank consists of two parts. The first one is the *topological classification*, whose goal is to determine, up to diffeomorphism, all possible compact, simply connected nonnegatively curved manifolds supporting an (effective) isometric torus action of maximal rank. The second part is the *equivariant classification*, where the goal is to determine, up to equivariant diffeomorphism, all possible isometric torus actions of maximal rank on a given compact, simply connected nonnegatively curved manifold. Thus, the natural next step after Theorem 1 is the equivariant classification of compact, simply connected nonnegatively curved manifolds of maximal symmetry rank in dimensions 4, 5 and 6. This classification was carried out in joint work with Kerin:

Theorem 2 ([4]). *Let M^n, $n \in \{4, 5, 6\}$, be a compact, simply connected n-manifold which admits a Riemannian metric with nonnegative curvature and maximal symmetry rank. Then every smooth, effective action on M^n by a torus T^{n-2}*

is equivariantly diffeomorphic to an effective, isometric action on a normal biquotient.

Recall that a *biquotient* is a quotient of a Lie group G by the two-sided, free action of a subgroup $U \subset G \times G$. If G is equipped with a bi-invariant metric, then the action of U is by isometries and the quotient $G/\!/U$ equipped with the induced metric (of nonnegative curvature) is called a *normal biquotient*.

When considering the classification of compact, simply connected manifolds with nonnegative curvature equipped with an effective isometric torus action, one can systematically decrease the amount of symmetry, starting with those with maximal symmetry rank, in the hope of eventually reaching a complete classification without any symmetry assumptions. Thus, in dimension 4, one considers compact, simply connected, nonnegatively curved 4-manifolds with an isometric S^1 action, and, in dimension 5, compact, simply connected 5-manifolds of nonnegative curvature with an isometric T^2 action. In dimension 4 both the topological and the equivariant classification problems have been solved (cf. [3, 4, 10]). The topological classification in dimension 5 was carried out in joint work with Searle:

Theorem 3 ([6]). *Let M^5 be a compact, simply connected, nonnegatively curved 5-manifold. If T^2 acts isometrically and effectively on M^5, then M^5 is diffeomorphic to one of $\mathbb{S}^5$, $\mathbb{S}^3 \times \mathbb{S}^2$, $\mathbb{S}^3 \tilde{\times} \mathbb{S}^2$ (the non-trivial $\mathbb{S}^3$-bundle over $\mathbb{S}^2$) or the Wu manifold* $\mathrm{SU}(3)/\mathrm{SO}(3)$.

The proofs of these results rest on the determination, via Alexandrov geometry, of the possible orbit spaces of the actions and on classification results for compact, simply connected smooth 4-, 5- and 6-manifolds with smooth torus actions, in the case of Theorems 1 and 2 (cf. [11, 12, 13]), and on the Barden-Smale classification of compact, simply connected smooth 5-manifolds (cf. [1, 14]), in the case of Theorem 3.

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Nonnegatively curved fixed point homogeneous 5-manifolds

Wolfgang Spindeler
(joint work with Fernando Galaz-Garcia)

The classification of closed Riemannian manifolds with positive or nonnegative sectional curvature is a fundamental open problem in Riemannian geometry. In this context, the classification of these manifolds in the presence of a non-trivial symmetry group can be regarded as a first step towards more general classification results. One is led in this way to consider positively or nonnegatively curved closed Riemannian manifolds with an isometric action of a compact Lie group.

Let M be a smooth manifold with a smooth action of a compact Lie group G. One possible measure for the size of the action $\mathrm{G} \times M \to M$ is its *cohomogeneity*, defined as the dimension of the orbit space M/G. Under this interpretation, the largest actions will be those for which $\dim M/\mathrm{G} = 0$, i.e., the action is transitive and M is a homogeneous space. If the action has fixed points, $\dim M/\mathrm{G}$ is bounded below by the dimension of the fixed point set $\mathrm{Fix}(M, \mathrm{G})$ and

$$\dim M/\mathrm{G} \geq \dim \mathrm{Fix}(M, \mathrm{G}) + 1$$

for any non-trivial action. In this case the *fixed point cohomogeneity* of the action, denoted by $\mathrm{cohomfix}\,(M, \mathrm{G})$, is defined by

$$\mathrm{cohomfix}\,(M, \mathrm{G}) = \dim M/\mathrm{G} - \dim \mathrm{Fix}(M, \mathrm{G}) - 1 \geq 0.$$

For an action with fixed points, "large" may be interpreted as having low fixed point cohomogeneity. If the fixed point cohomogeneity of the action is 0, we say that the action is *fixed point homogeneous* and M is a *fixed point homogeneous manifold* (cf. [3]).

Grove and Searle [3] classified closed Riemannian manifolds with positive curvature and a fixed point homogeneous isometric Lie group action, along with the possible actions. In the simply connected case one has the following topological classification:

Theorem (Grove, Searle). Any closed, simply connected fixed point homogeneous manifold with positive curvature is diffeomorphic to either $\mathbb{S}^n, \mathbb{CP}^m, \mathbb{HP}^k$, or $\mathbb{OP}^2$.

In dimensions at most 5, where fixed point set components have dimension at most 3, a complete topological classification can be given in the simply connected case. In dimensions 4 and below the topological classification was carried out in [2]. Here we address the topological classification in dimension 5. Our main result is the following theorem.

Theorem. Let M^5 be a closed, simply connected 5-dimensional nonnegatively curved fixed point homogeneous G-manifold. Then G is SO(5), SO(4), SU(2), SO(3) or S^1 and we have the following classification.

(a) If G = SO(5), SO(4) or SU(2), then M^5 is diffeomorphic to $\mathbb{S}^5$.

(b) If G = SO(3) or S^1, then M^5 is diffeomorphic to $\mathbb{S}^5$ or to one of the two bundles over $\mathbb{S}^2$ with fiber $\mathbb{S}^3$.

Observe that the list of fixed point homogeneous 5-manifolds in the Main Theorem contains every known closed, simply connected 5-manifold of nonnegative sectional curvature except for the Wu manifold SU(3)/SO(3).

When G is one of SO(5), SO(4), SU(2) or SO(3) the result follows easily from results in the literature. When G equals S^1 the argument is more complicated; the hypothesis of nonnegative curvature allows us to show, by looking at the orbit space structure, that M^5 decomposes as a union of two disc bundles over smooth submanifolds of M^5, one of which is a 3-dimensional component of $\mathrm{Fix}(M^5, \mathrm{S}^1)$. This in turn allows us to show, after some work, that $H_2(M^5, \mathbb{Z})$ is either 0 or $\mathbb{Z}$, whence the conclusion follows from the Barden-Smale classification of smooth, closed simply connected 5-manifolds [1, 4].

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Integral norms and Hölder estimates under curvature bounds

Patrick Ghanaat

We discuss comparison theorems for volume normalized integrals or L^p–norms and for Hölder norms of functions on distance balls in a Riemannian manifold M. Specifically, we show that the norm of a function u defined on a distance ball $B(p, r) \subseteq M$ is comparable to the corresponding euclidean norm of its lift $u \circ \exp_p$ on the ball $B_p(0, r')$ in the tangent space T_pM, sometimes with a modified radius r'. The radius r is required to satisfy an inequality $r \leq \mathrm{const}/\sqrt{\kappa}$, where κ is a curvature bound, but no assumption on the injectivity of $\exp_p$ is made.

The estimates can be used to transplant inequalities of analysis, such as estimates of the Sobolev type or regularity theorems on differential operators, from euclidean space to Riemannian manifolds in a geometrically controlled manner. We illustrate this via Morrey's Sobolev inequality and an L^p-estimate for the Hodge-Dirac operator $d + \delta$. This is an extract from joint work with B. Colbois [1]. In the present account, we restrict attention to the comparison of integrals.

Consider a complete Riemannian manifold (M, g) of dimension n. For measurable subsets $\Omega \subseteq M$, let

$$\fint_\Omega u\, d\mu := \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega u\, d\mu$$

denote the average value of a function $u : \Omega \to \mathbf{R}$ with respect to the Riemannian volume element $d\mu$. On balls $B(p, r) \subseteq M$ of radius r, these averages have the following monotonicity property.

Lemma. *Let $u : M \to \mathbf{R}$ be a non-negative measurable function, $p \in M$, and $0 < r_1 \leq r_2$. Suppose $\mathrm{Ric} \geq (n-1)\rho$ on the ball $B(p, r_2)$ for some real number ρ. Then*

$$\fint_{B(p,r_1)} u\, d\mu \leq \frac{v_n(r_2, \rho)}{v_n(r_1, \rho)} \fint_{B(p,r_2)} u\, d\mu.$$

Here $v_n(r, \rho)$ is the volume of a ball of radius r in the simply connected Riemannian n–manifold of constant sectional curvature ρ.

This is an immediate consequence of the volume comparison theorem of Bishop–Günther–Gromov. The following lemma, obtained from a standard packing argument, serves to globalize local estimates of integral norms.

Lemma. *Let $u : M \to \mathbf{R}$ be non-negative and measurable, $p \in M$ and $0 < r_1 \leq r_2$. Suppose $\mathrm{Ric} \geq (n-1)\rho$ on the ball $B(p, r_1/2 + 2r_2)$. Then*

$$\fint_{B(p,r_2)} u\, d\mu \leq \frac{v_n(r_1/2 + 2r_2, \rho)}{v_n(r_1/2, \rho)} \sup_{q \in B(p,r_2)} \frac{\mathrm{vol}\, B(q, r_1)}{\mathrm{vol}\, B(p, r_2)} \fint_{B(q,r_1)} u\, d\mu.$$

In particular, if M is compact with diameter d, then for $r \leq d$,

$$\fint_M u\, d\mu \leq \frac{v_n(d, \rho)}{v_n(r/2, \rho)} \sup_{p \in M} \fint_{B(p,r)} u\, d\mu.$$

For $p \in M$, let $B_p(0,r)$ denote the euclidean ball of radius r in the tangent space T_pM. The exponential map $\exp_p : T_pM \to M$ maps $B_p(0,r)$ onto $B(p,r) \subseteq M$. The next result compares the average value of a function u on $B(p,r)$ to the average of its lift $u \circ \exp_p$ on $B_p(0,r)$ with respect to the Lebesgue measure $d\lambda$.

Theorem. *Assume* $\mathrm{Ric} \geq (n-1)\rho$ *and a sectional curvature bound* $K \leq \kappa$ *on* $B(p,3r)$ *for real numbers* κ *and* ρ. *Let* $r \leq \pi/(3\sqrt{\kappa})$ *in case* $\kappa > 0$. *Then, for every measurable function* $u \geq 0$ *on* M,

$$a_1 \fint_{B(p,r/3)} u \, d\mu \leq \fint_{B_p(0,r)} u \circ \exp_p \, d\lambda \leq a_2 \fint_{B(p,r)} u \, d\mu.$$

Here

$$a_1 := 3^{-n} \mathbf{s}_\rho(r)^{1-n} \frac{v_n(r/3,\kappa)}{v_n(r/3,0)} \quad \text{and} \quad a_2 := 3^n \mathbf{s}_\kappa(r)^{1-n} \frac{v_n(3r,\rho)}{v_n(3r,0)}$$

if $\rho < 0$ and $\kappa > 0$. If $\rho \geq 0$, $\mathbf{s}_\rho(r)$ is replaced by 1, and if $\kappa \leq 0$, then $\mathbf{s}_\kappa(r)$ is replaced by 1. The functions $\mathbf{s}_\rho$ and $\mathbf{s}_\kappa$ are defined by $\mathbf{s}_\rho(r) = \sinh(\sqrt{-\rho}\, r)/(\sqrt{-\rho}\, r)$, 1 or $\sin(\sqrt{\rho}\, r)/(\sqrt{\rho}\, r)$ depending on whether $\rho < 0$, $\rho = 0$ or $\rho > 0$, respectively. In any case, the constants a_1 and a_2 depend only on κr^2, ρr^2, and n.

Proof. We prove the second inequality. For $q \in B(p,r)$ let

$$N_p(q,r) := \sharp \, \exp_p^{-1}(q) \cap B_p(0,r)$$

denote the number of inverse images of q in $B_p(0,r)$ under the exponential map. Let $\tilde{u} := u \circ \exp_p$. The area formula yields

$$\begin{aligned} \int_{B_p(0,r)} \tilde{u}(x) \, | \det T_x \exp_p | \, d\lambda(x) &= \int_{B(p,r)} N_p(q,r) u(q) \, d\mu(q) \\ &\leq \max_{q \in B(p,r)} N_p(q,r) \int_{B(p,r)} u(q) \, d\mu(q). \end{aligned}$$

The sectional curvature bound $K \leq \kappa$ implies the lower bound

$$| \det T_x \exp_p | \geq \mathbf{s}_\kappa(|x|)^{n-1}$$

for the Jacobian $| \det T_x \exp_p |$. If $\kappa \leq 0$, we use $\mathbf{s}_\kappa(t) \geq 1$ for $t \geq 0$. If $\kappa > 0$, then the function $\mathbf{s}_\kappa(t)$ is decreasing for $0 \leq t \leq \pi/\sqrt{\kappa}$. We obtain

$$\int_{B_p(0,r)} \tilde{u}(x) \, | \det T_x \exp_p | \, d\lambda(x) \geq \mathbf{s}_\kappa(r)^{n-1} \int_{B_p(0,r)} \tilde{u}(x) \, d\lambda(x)$$

and therefore

$$\int_{B_p(0,r)} \tilde{u}(x) \, d\lambda(x) \leq \mathbf{s}_\kappa(r)^{1-n} \max_{q \in B(p,r)} N_p(q,r) \int_{B(p,r)} u(q) \, d\mu(q) \,. \tag{1}$$

Using the area formula again, we have

$$\begin{aligned} \int_{B_p(0,3r)} | \det T_x \exp_p | \, d\lambda(x) = \int_{B(p,3r)} N_p(q,3r) \, d\mu(q) &\geq \int_{B(p,r)} N_p(q,3r) \, d\mu(q) \\ &\geq \min_{q \in B(p,r)} N_p(q,3r) \, \mathrm{vol}\, B(p,r) \,. \end{aligned}$$

The integral on the left hand side of this inequality is the volume of $B_p(0,3r)$ with respect to pullback Riemannian metric $\tilde{g} = \exp_p^* g$. Since $\tilde{g}$ satisfies $\mathrm{Ric} \geq (n-1)\rho$, volume comparison yields

$$\int_{B_p(0,3r)} |\det T_x \exp_p| \, d\lambda(x) \leq v_n(3r,\rho) = \frac{v_n(3r,\rho)}{v_n(3r,0)} \, 3^n \mathrm{vol}\, B_p(0,r)$$

and we obtain

$$\frac{1}{\mathrm{vol}\, B_p(0,r)} \leq \frac{v_n(3r,\rho)}{v_n(3r,0)} \, 3^n \frac{1}{\min_{q\in B(p,r)} N_p(q,3r)} \, \frac{1}{\mathrm{vol}\, B(p,r)} \, . \tag{2}$$

We now claim that

$$\frac{\max_{q\in B(p,r)} N_p(q,r)}{\min_{q\in B(p,r)} N_p(q,3r)} \leq 1 \, . \tag{3}$$

To show this, we define a one to one map

$$\phi_\gamma : \exp_p^{-1}(q) \cap B_p(0,r) \to \exp_p^{-1}(q') \cap B_p(0,3r)$$

for q and q' in $B(p,r)$ as follows. Let $\gamma : [0,1] \to M$ be a geodesic of length less than $2r$ joining q to q'. For $x \in \exp_p^{-1}(q) \cap B_p(0,r)$ let $\bar{\gamma}_x : [0,1] \to T_m M$ denote the lift of γ with initial point x, so that $\exp \circ \bar{\gamma}_x = \gamma$ and $\gamma(0) = x$. Then we define $\phi_\gamma(x) = \bar{\gamma}_x(1)$. This map is easily seen to have the required properties. Inequalities (1),(2) and (3) together imply the upper bound in the theorem. The proof of the lower bound is similar.

Discussion. After the talk, Guofang Wei pointed out the utility of smoothing methods for geometric estimates on analytic constants. The counting argument employed in the proof above is not restricted to exponential coordinates. Non–injective “coordinate charts” are used in [2].

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The space of complete nonnegatively curved metrics on the plane

Igor Belegradek

In the last decade there has been considerable progress in studying spaces of Riemannian metrics that satisfy various curvature assumptions. In this report we are interested in the set $\mathcal{R}(N)$ of complete metrics of nonnegative sectional curvature on a fixed open connected manifold N. Here $\mathcal{R}(N)$ is given the topology of C^∞-uniform convergence on compact subsets, and more generally, this topology is given to all function spaces discussed below. Let $\mathcal{M}(N)$ denote the associated moduli space, i.e. the quotient space of $\mathcal{R}(N)$ by the pullback $\mathrm{Diff}(N)$-action.

Recall that any open complete manifold of nonnegative sectional curvature is diffeomorphic to a normal bundle of a compact totally convex submanifold called a soul. A soul is not unique but all souls of a given metric are isometric. Thus the isometry class of the soul is a basic invariant of the metric.

Kapovitch-Petrunin-Tuschmann [3] proved that if the normal bundle to a soul of some metric in $\mathcal{R}(N)$ has nonzero Euler class, then the diffeomorphism type of the soul defines a locally constant function on $\mathcal{R}(N)$ and $\mathcal{M}(N)$. More recently Belegradek-Kwasik-Schultz [1] showed that the result still holds when the "diffeomorphism type" of the soul is replaced by its "ambient isotopy type". These results lead to examples of manifolds for which $\mathcal{M}(N)$ has infinitely many path-components [3, 1, 2, 4].

If N admits a metric with a codimension one soul, then the topology of $\mathcal{M}(N)$ can be easily described in terms of the topology of the corresponding moduli spaces of its souls, of which there could be more than one [1].

The simplest case in which the methods of [3, 1] fail is when N has a codimension two soul with trivial normal bundle. To study the spaces of metrics for such manifolds it seems necessary to understand what happens for $N = \mathbb{R}^2$. It is easy to see that $\mathcal{R}(\mathbb{R}^2)$ is path-connected, and more generally, the following is true, which is the main result of this report.

Theorem 1. *Any countable (or finite) subset of $\mathcal{R}(\mathbb{R}^2)$ has the path-connected complement. The same holds for $\mathcal{M}(\mathbb{R}^2)$ in place of $\mathcal{R}(\mathbb{R}^2)$.*

The proof is based on the uniformization theorem, properties of subharmonic functions, and infinite-dimensional topology. The starting point is a classical result of Huber that any complete metric g on $\mathbb{R}^2$ of nonnegative curvature is conformal to the standard flat metric g_0. Thus g can be written as $\phi^*(e^{-2u}g_0)$ where u is a smooth function on $\mathbb{R}^2$, and ϕ is a self-diffeomorphism of $\mathbb{R}^2$. Nonnegativity of the curvature is equivalent to subharmonicity of u. Deciding which subharmonic functions give rise to complete metrics is more subtle, and is crucial for the proof. One can normalize ϕ so that it fixes two points of $\mathbb{R}^2$, say the complex numbers 0 and 1, so the map $(u, \phi) \to \phi^*(e^{-2u}g_0)$ defines a continuous bijection $C \times \mathrm{Diff}_{0,1}(\mathbb{R}^2) \to \mathcal{R}(\mathbb{R}^2)$, where C is a certain star-shaped set of subharmonic functions in the Fréchet space of smooth functions on $\mathbb{R}^2$, and $\mathrm{Diff}_{0,1}(\mathbb{R}^2)$ is the subgroup of $\mathrm{Diff}(\mathbb{R}^2)$ that fixes 0 and 1. There is also a continuous surjection $C \to \mathcal{M}(\mathbb{R}^2)$ whose fibers are closed subgroups of $\mathrm{Aff}(\mathbb{R}^2)$, the group of conformal automorphisms of $(\mathbb{R}^2, g_0)$. The topological group $\mathrm{Diff}_{0,1}(\mathbb{R}^2)$ is homeomorphic to the separable Hilbert space l_2. Also we shall make use of the classical result of infinite dimensional topology that the complement of any countable union of compact subsets of a separable Fréchet space is homeomorphic to l_2. Unfortunately, the homeomorphism type of C is unclear (to the author). There is a well-known topological classifications of closed convex subsets of separable Fréchet spaces, e.g. such a subset is homeomorphic to l_2 if and only if it is not locally compact. This classification does not seem to apply to C because it is probably neither closed nor convex; nevertheless, combining the classfication with a more detailed description of C allows one to show that the complement in C of any countable union of compact sets is path-connected, which easily implies Theorem 1.

Similar techniques yield Theorem 1 for S^2 in place of $\mathbb{R}^2$, and in fact, even stronger results hold in the S^2 case, which will be discussed elsewhere.

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Polyhedral analogue of Frankel conjecture.

Dmitri Panov

(joint work with Misha Verbitsky)

In this talk we propose a conjecture that can be seen as a polyhedral analogue of the celebrated Frankel's conjecture in Kähler geometry proved by Mori [3] and Siu-Yau [6]. Frankel conjecture states that a Kähler manifold with positive bisectional curvature is biholomorphic to a complex projective space. We explain an approach to our conjecture based on the theory of polyhedral Kähler manifolds, developed in [4]. Before stating the conjecture we need to give some definitions.

Definition. A polyhedral manifold is a manifold that is glued from a collection of Euclidean simplexes by identifying their hyperfaces via isometry.

Example. The surface of a tetrahedron in $\mathbb{R}^3$ represents a two-sphere with flat metric that has four singularities, namely conical points.

The singularities of a polyhedral metric happen in real codimension 2 (at codimension two faces) and at generic points the singularity is locally isometric to a product of $\mathbb{R}^{d-2}$ with a two-dimensional cone. A polyhedral manifold is called *non-negatively curved* if the cone angle at each face of codimension two is at most

2π. An important result about non-negatively curved manifolds was obtained by Cheeger [2].

Theorem (Cheeger). *Suppose that a compact polyhedral manifold M^n is non-negatively curved. Then any harmonic form h in $H^i(M^n)$ is parallel, i.e., $\nabla h = 0$.*

It follows from this theorem that the holonomy of the non-negatively curved metric on M^n satisfies non-trivial constraints, provided some non-trivial Betti number b_i of M^n, $0 < i < n$ is non-zero. In particular in the case $b_2 > 0$ and the holonomy of the metric is irreducible, a simple lemma shows that the manifold is even dimensional and the holonomy of the metric belongs to $U(\frac{n}{2}) \subset SO(n)$. This observation was one of the main motivations for the following definition and the conjecture:

Definition ([4]). An even-dimensional polyhedral manifold M^{2n} is called polyhedral Kähler if the holonomy of the metric belongs to $U(n)$.

Now we formulate our conjecture:

Conjecture 1. *Consider a non-negatively curved non-flat polyhedral manifold M. Suppose that the holonomy of the metric on M is irreducible and $b_2(M) > 0$. Then M has a natural holomorphic structure with respect to which it is biholomorphic to $\mathbb{C}P^n$ and the original polyhedral metric on M is a singular Kähler metric with respect to this natural complex structure.*

It follows immediately from the results of [4] that Conjecture 1 holds in case $dim(M) \leq 4$. Below we will outline the strategy of the proof of Conjecture 1 that is a work in progress with Misha Verbitsky.

Step 1. By Cheeger's theorem, any manifold satisfying the conditions of Conjecture 1 is polyhedral Kähler. Even though in Conjecture 1 we start with a space that does not have complex structure, the following conjecture (proven in [4] in dimension four and in [5] in dimension six) claims that polyhedral Kähler manifolds are naturally complex spaces. This justifies the formulation of Conjecture 1.

Conjecture 2. Every polyhedral Kähler manifold M^{2n} has a natural complex structure. More precisely, M^{2n} is PL diffeomeorphic to a complex analytic space with complex singularities in complex co-dimension 3; the diffeomorphism is a biholomorphism outside of metric singularities. The singularities of the metric form a collection of divisors on M^{2n}.

The following example explains why complex singularities can appear in a polyhedral Kähler manifold even though the underlying PL structure is smooth.

Example. Consider the complex hypersurface $z_0^n + z_1^2 + z_2^2 + z_3^2 = 0$, $n = 3$ in $\mathbb{C}^4$. The link of the isolated singularity at 0 is the so-called *Kervaire sphere* and it is diffeomorphic to S^5; more generally we can replace 3 by $n = 3, 5 \mod 8$. These manifolds admit a polyhedral Kähler metric whose underlying PL structure is smooth. Indeed, to construct such a metric just consider the ramified cover of the hyperplane $\sum_i z_i = 0$ in $\mathbb{C}^4$ given by the map $(z_0, z_1, z_2, z_3) \to (z_0^n, z_1^2, z_2^2, z_3^2)$ and take the pullback of a flat metric on $\sum_i z_i = 0$.

Step 2. It might happen that proving Conjecture 2 in full generality will be too complicated, but in [5] we are going to prove this conjecture provided the polyhedral Kähler manifold is non-negatively curved. Then the main task will be to prove that in fact in the case when the metric in non-negatively curved, the complex structure does not have singularities at all. Note, that the example given above is very far from been non-negatively curved.

Step 3. In the case one knows that the underlying complex structure of the manifold is smooth the proof can be finished by applying Mori's result [3].

Theorem (Mori). A compact complex manifold with ample tangent bundle is biholomorphic to $\mathbb{C}P^n$.

One proves immediately, that the tangent bundle of the manifold from Conjecture 1 is nef. Since the holonomy of the metric on the manifold is irreducible, $b_2 = 1$. Moreover, since the manifold is assumed to be non-flat, the manifold is Fano type. The total space of the projectivization of the tangent bundle has $b_2 = 2$ and one needs to check that $O(1)$ bundle on the projectivisation restricts positively on any complex curve. For vertical curves this is obvious, for non-vertical this holds since their projection to the base intersects the singular locus of the metric on the base.

As a final remark we note that the following question ([1]) seems to be open in dimensions higher than three.

Question [1]. Does there exist an algebraic torus quotient $Y = A/G$ by a freely acting finite group G, such that $b_2(Y) = 1$?

If one can rule out existence of such flat complex manifolds with $b_2 = 1$, one can remove the condition on non-flatness from Conjecture 1.

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Introduction to optimal transport and relations with Ricci curvature bounds

Nicola Gigli

This lecture is not aimed to present new research achievements, but rather to give a quick overview on the Wasserstein distance and its relation with Ricci curvature bounds. We refer to [7] and [1] for a more comprehensive introduction and detailed references.

The first step is the introduction of the distance W_2 on the space $\mathcal{P}(X)$ of Borel probability measures on a metric space (X, d), which for simplicity we assume compact. Given $\mu, \nu \in \mathcal{P}(X)$ it is defined as

$$W_2^2(\mu,\nu) := \inf \int d^2(x,y)\, d\gamma(x,y),$$

where the inf is taken among all measures $\gamma \in \mathcal{P}(X \times X)$ such that

$$\pi^1_\sharp \gamma = \mu,$$
$$\pi^2_\sharp \gamma = \nu.$$

The basic properties of such distance are analyzed, in particular: the fact that it is indeed a distance, that $(\mathcal{P}(X), W_2)$ is complete and separable and that if (X,d) has geodesic, then the same is true for $(\mathcal{P}(X), W_2)$.

Then the relative entropy functional is introduced. Assume that (X,d) is also endowed with a reference measure $\mathrm{m} \in \mathcal{P}(X)$. Then $\mathrm{Ent}_\mathrm{m} : \mathcal{P}(X) \to [0,\infty]$ is defined by

$$\mathrm{Ent}_\mathrm{m}(\mu) := \begin{cases} \displaystyle\int \rho \log \rho \, d\mathrm{m} & \text{if } \mu = \rho \mathrm{m}, \\ +\infty & \text{if } \mu \text{ is not absolutely continuous w.r.t. m.} \end{cases}$$

A key theorem of Sturm-von Renesse states that for a compact and smooth Riemannian manifold M endowed with the Riemannian distance and the normalized volume measure vol the following two are equivalent:

i) The Ricci curvature of M is uniformly bounded from below by $K \in \mathbb{R}$.
ii) The relative entropy functional $\mathrm{Ent}_\mathrm{vol}$ is K-geodesically convex on the space $(\mathcal{P}(M), W_2)$.

The latter condition means that for any two $\mu, \nu \in \mathcal{P}(M)$ with finite entropy, there exists a (constant speed and minimizing) geodesic (μ_t) connecting them such that

$$\mathrm{Ent}_\mathrm{vol}(\mu_t) \le (1-t)\mathrm{Ent}_\mathrm{vol}(\mu_0) + t\mathrm{Ent}_\mathrm{vol}(\mu_1) - \frac{K}{2} t(1-t) W_2^2(\mu_0,\mu_1), \qquad \forall t \in [0,1].$$

An intuition of Lott-Villani on one side, and of Sturm on the other has been to revert such theorem and propose condition (ii) as abstract definition of 'Ricci curvature bounded from below by K' for metric measure spaces. The key features of this definition are the compatibility with the Riemannian case and the stability w.r.t. the Gromov-Hausdorff convergence.

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Heat flow in metric measure spaces

Giuseppe Savaré
(joint work with Luigi Ambrosio and Nicola Gigli)

It is well known that in a complete Riemannian manifold (X, g) the heat equation

$$\partial_t u - \Delta_{\mathsf{g}} u = 0, \tag{1}$$

associated to the Laplace-Beltrami operator Δ_{g}, is the gradient flow in the Hilbert space $L^2_{\mathfrak{m}}(X)$ of the Dirichlet energy

$$\mathsf{D}(u) := \frac{1}{2}\int_X |\nabla u|^2_{\mathsf{g}} \,\mathrm{d}\mathfrak{m}, \quad u \in W^{1,2}_{\mathfrak{m}}(X, \mathsf{g}), \tag{2}$$

where $\mathfrak{m}$ is the Riemannian volume measure and $W^{1,2}_{\mathfrak{m}}(X, \mathsf{g})$ is the usual Sobolev space of $L^2_{\mathfrak{m}}(X)$ functions whose (distributional) gradient is square summable.

Entropy and Wasserstein distance. In their seminal paper [9], Jordan, Kinderlehrer, and Otto showed that the Heat equation in the Euclidean space $X := \mathbb{R}^d$ (endowed with the canonical metric and the d-dimensional Lebesgue measure) can also be viewed as the gradient flow of the Entropy functional

$$\mathsf{Ent}_{\mathfrak{m}}(\rho) := \int_X u \log u \,\mathrm{d}\mathfrak{m} \quad \text{if } \rho = u\mathfrak{m} \ll \mathfrak{m}, \qquad \mathsf{Ent}_{\mathfrak{m}}(\rho) = +\infty \quad \text{if } \rho \not\ll \mathfrak{m}, \tag{3}$$

with respect to the so-called L^2-Wasserstein distance W_2 in the space of the Borel probability measures $\mathcal{P}(X)$: the (extended, since it can take the value $+\infty$) distance $W_2(\mu, \nu)$ between two measures in $\mathcal{P}(X)$ can be defined as

$$W_2^2(\mu, \nu) := \min \int \mathsf{d}^2(x, y) \,\mathrm{d}\pi(x, y) \in [0, \infty], \tag{4}$$

where the minimum is taken among all the couplings $\pi \in \mathcal{P}(X \times X)$ having marginals μ and ν and $\mathsf{d}(x, y)$ denotes the Riemannian distance induced by g.

This new characterization of the heat flow (in fact stated for the more general class of Fokker-Planck equations) has been further extended to Hilbert spaces [1, 3], Riemannian manifolds [6, 13], Finsler [11] and Alexandrov spaces [7].

Metric-Measure spaces with lower Ricci curvature bounds. The talk has been devoted to present the recent results of [2] concerning the identification of these flows in a wide class of metric-measure spaces $(X, \mathsf{d}, \mathfrak{m})$, including in particular those satisfying the lower Ricci curvature bound $\mathrm{CD}(K, \infty)$ recently introduced by Sturm [12] and Lott-Villani [10].

Here (X, d) is a complete metric space and $\mathfrak{m}$ is a Borel measure on X such that

$$\int_X \mathrm{e}^{-L\,\mathsf{d}^2(x, x_0)} \,\mathrm{d}\mathfrak{m} < \infty \quad \text{for some } x_0 \in X,\ L \geq 0.$$

$\mathrm{CD}(K, \infty)$ spaces naturally arise in the Wasserstein approach, since they are defined by assuming the geodesic K-convexity of the entropy functional (3). More precisely, $(X, \mathsf{d}, \mathfrak{m})$ has Ricci curvature bounded from below by $K \in \mathbb{R}$ according to [12, 10] if every couple $\mu_0, \mu_1 \in D(\mathsf{Ent}_{\mathfrak{m}}) = \{\rho \in \mathcal{P}(X) : \mathsf{Ent}_{\mathfrak{m}}(\rho) < \infty\}$ with

$W_2(\mu_0,\mu_1)<\infty$ can be connected by a *minimal geodesic* curve $s\mapsto\mu_s\in D(\mathsf{Ent}_{\mathfrak m})$, $s\in[0,1]$, such that for every $s,t\in[0,1]$

$$(5)\qquad \begin{aligned} &W_2(\mu_s,\mu_t)=|s-t|W_2(\mu_0,\mu_1),\\ &\mathsf{Ent}_{\mathfrak m}(\mu_s)\le(1-s)\mathsf{Ent}_{\mathfrak m}(\mu_0)+s\mathsf{Ent}_{\mathfrak m}(\mu_1)-\frac{K}{2}s(1-s)W_2^2(\mu_0,\mu_1).\end{aligned}$$

The gradient flow of the Entropy functional in the Wasserstein space. Let us recall (see e.g. [1]) that a *curve of maximal slope* for the Entropy functional in the Wasserstein space $(\mathcal P(X),W_2)$ is a curve $\rho:[0,\infty)\to D(\mathsf{Ent}_{\mathfrak m})$ satisfying

(a) $\rho\in AC^2_{\rm loc}([0,\infty);\mathcal P(X),W_2)$, i.e. there exists a function $v\in L^2_{\rm loc}(0,\infty)$ s.t.

$$W_2(\rho_s,\rho_t)\le\int_s^t v(r)\,\mathrm dr\quad\text{whenever } 0\le s\le t<\infty;$$

its metric velocity $|\dot\rho|$ is then defined by

$$|\dot\rho_t|:=\lim_{h\to0}\frac{W_2(\rho_t,\rho_{t+h})}{|h|}\quad\text{for a.e. } t\in(0,\infty).$$

(b) For every $0\le s\le t<\infty$ we have

$$\mathsf{Ent}_{\mathfrak m}(\rho_t)+\frac12\int_s^t\Big(|\dot\rho_r|^2+|\nabla^-\mathsf{Ent}_{\mathfrak m}|^2(\rho_r)\Big)\,\mathrm dr=\mathsf{Ent}_{\mathfrak m}(\rho_s),$$

where $|\nabla^-\mathsf{Ent}_{\mathfrak m}|(\rho):=\limsup_{\sigma\to\rho}\frac{(\mathsf{Ent}_{\mathfrak m}(\rho)-\mathsf{Ent}_{\mathfrak m}\sigma)_+}{W_2(\rho,\sigma)}$ denotes the descending slope of the entropy.

If $(X,\mathsf d,\mathfrak m)$ is a $\mathrm{CD}(K,\infty)$ space, then adapting the arguments of [1, 8] it is possible to prove [2] that *for every initial datum $\rho_0=u_0\mathfrak m$ in $D(\mathsf{Ent}_{\mathfrak m})$ there exists a unique curve of maximal slope $\rho_t=u_t\mathfrak m$ for the Entropy functional* according to (a) and (b) above.

Cheeger energy in $(X,\mathsf d,\mathfrak m)$. The L^2-approach can also be settled in a metric measure space $(X,\mathsf d,\mathfrak m)$, by introducing a weak notion of "modulus of the gradient" and of Sobolev space $W^{1,2}_{\mathfrak m}(X,\mathsf d)$, inspired by Cheeger [5]. The *Cheeger energy* is defined by

$$(6)\quad \mathsf{Ch}(u):=\frac12\inf\left\{\liminf_{h\to\infty}\int|\nabla u_h|^2\,\mathrm d\mathfrak m: u_h\in\mathrm{Lip}(X),\ \int_X|u_h-u|^2\,\mathrm d\mathfrak m\to0\right\},$$

where for a Lipschitz function $u:X\to\mathbb R$ we set

$$|\nabla u|(x):=\limsup_{y\to x}\frac{|u(y)-u(x)|}{\mathsf d(x,y)}.$$

It is not difficult to prove that $\mathsf{Ch}:L^2_{\mathfrak m}(X)\to[0,\infty]$ is a convex (but no more quadratic, in general) and lower semicontinuous functional, whose proper domain $\{u\in L^2_{\mathfrak m}(X):\mathsf{Ch}(u)<\infty\}$ provides a natural definition for the space $W^{1,2}_{\mathfrak m}(X,\mathsf d)$. When $\mathrm{Ch}(u)<\infty$, the Cheeger energy admits the integral representation

$$(7)\qquad \mathsf{Ch}(u)=\frac12\int_X|\nabla u|_w^2\,\mathrm d\mathfrak m,$$

where $|\nabla u|_w$ denotes the relaxed gradient of u, a quantity that satisfies various calculus properties similar to the classical ones for $|\nabla u|_{\mathsf{g}}$ in a smooth Riemannian setting; it also turns out to be extremely useful to estimate the derivative of u along suitable collections of absolutely continuous curves in X.

The (possibily nonlinear) Laplace operator $\Delta_{\mathfrak{m},\mathsf{d}}$ can be defined in a dense subset of $L^2_{\mathfrak{m}}(X)$ as the minimal selection of the subdifferential of Ch; it generates a unique flow (see e.g. [4]) $\mathsf{H}_t : L^2_{\mathfrak{m}}(X) \to W^{1,2}_{\mathfrak{m}}(X, \mathsf{d})$ such that $u_t := \mathsf{H}_t(u_0)$ satisfy the Cauchy problem for the evolution equation

$$\partial_t u - \Delta_{\mathfrak{m},\mathsf{d}} u = 0, \quad \lim_{t\downarrow 0} u_t = u_0 \quad \text{in } L^2_{\mathfrak{m}}(X). \tag{8}$$

H_t is a contraction w.r.t. any $L^p_{\mathfrak{m}}(X)$-norm and it is order and mass preserving: in particular,

$$u_0 \geq 0, \int_X u_0 \,\mathrm{d}\mathfrak{m} = 1 \quad \Rightarrow \quad u_t \geq 0, \quad \int_X u_t \,\mathrm{d}\mathfrak{m} = 1 \quad \text{for every } t \geq 0.$$

Theorem [2]. *If $\rho_0 = u_0\mathfrak{m} \in \mathcal{P}(X)$ with $u_0 \in L^2_{\mathfrak{m}}(X)$ and $\int_X \mathsf{d}^2(x, x_0)\,\mathrm{d}\rho_0 < \infty$, the curve $\rho_t = u_t\mathfrak{m}$ is a curve of maximal slope for the Entropy functional in $\mathcal{P}(X)$ if and only if $u_t = \mathsf{H}_t u_0$ is the L^2-gradient flow of the Cheeger energy solving* (8).

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Metric measure spaces with Riemannian Ricci curvature bounded from below

Nicola Gigli
(joint work with Luigi Ambrosio, Giuseppe Savaré)

Two observations are the basis for the results presented in this talk.

- The curvature dimension condition $CD(K, N)$, in the original version of Sturm ([6]) and Lott-Villani ([5]) does not exclude from the analysis Finsler geometries. For instance, it is possible to check that $(\mathbb{R}^d, \|\cdot\|, \mathcal{L}^d)$, where $\|\cdot\|$ is any norm, is always a $CD(0, d)$ space (see the last theorem in [7] for a proof of this fact).
- A Finsler manifold cannot arise as limit of Riemannian manifolds with Ricci curvature uniformly bounded from below and dimension uniformly bounded from above, unless it is Riemannian (as a consequence of the analysis made by Cheeger and Colding in [2], [3], [4]).

It is therefore natural to ask whether there exists an abstract definition of Ricci curvature bound which is still stable w.r.t. measured Gromov-Hausdorff convergence (like the $CD(K, N)$ one) and rules out Finsler geometries.
In this talk I present a proposal ([1]) in this direction for the particular case of no dimensionality constraint, i.e. $N = \infty$. Up to technicalities, this new notion is a reinforcement of the standard $CD(K, \infty)$ condition with the requirement that the heat flow is linear. The motivation for such proposal comes from the well known fact that the Laplacian on a Finsler manifold is linear if and only if the manifold is Riemannian. What is a priori non obvious, is the fact that this condition is stable w.r.t. mGH convergence: such stability is achieved by looking at the heat flow as gradient flow of the relative entropy in the Wasserstein space, and then using the Γ-convergence of the entropies along sequences of spaces which are mGH converging to a limit space.
We called such spaces, spaces with Riemannian Ricci curvature bounded from below by K.
This strengthening of the $CD(K, \infty)$ condition carries some new properties of the spaces, in particular there is:

- exponential contractivity of the Wasserstein distance W_2 along two heat flows,
- full compatibility with the theory of Dirichlet forms,
- existence of a Brownian motion with continuous sample paths,
- validity of the Bakry-Emery curvature condition,
- in case the measure is doubling and supports a local Poincaré inequality: Lipschitz continuity of the heat kernel.

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On the Hopf conjectures with symmetry

Lee Kennard

The study of Riemannian manifolds with positive sectional curvature is old, yet there exist few known examples. Beyond spheres and projective spaces, all known simply connected examples have dimension at most 24 (see [4]) .

Moreover, there exist few topological obstructions to a closed manifold admitting a positively curved metric. In fact, there are no known obstructions to positive curvature for simply connected spaces that are not already obstructions to nonnegative curvature. One well-known conjectured obstruction was stated by Hopf: In even dimensions, a closed Riemannian manifold with positive sectional curvature has positive Euler characteristic. The conjecture is true in dimensions 2 and 4 by the the theorems of Gauss-Bonnet or Bonnet-Myers. It is open in general.

Progress in the study of positive curvature has been made in the last two decades by restricting attention to metrics with a large isometry group. See [3, 1] for surveys. Our first result falls into this category:

> **Theorem 1:** Let M^n be a closed Riemannian manifold with positive sectional curvature and an effective, isometric T^r-action. If $n \equiv 0 \bmod 4$ and $r \geq 2\log_2(n)$, then $\chi(M) > 0$.

Another well-known conjecture of Hopf is that $S^2 \times S^2$ admits no metric of positive sectional curvature. More generally, one might conjecture that no symmetric space of rank greater than one admits a positively curved metric. We provide evidence for this conjecture in the presence of symmetry:

> **Theorem 2:** Suppose M^n has the rational cohomology ring of a compact, simply connected Riemannian symmetric space N. Assume M admits a metric with positive sectional curvature invariant under a r-torus action. If $r \geq 2\log_2(n)+8$, then N is a product of Grassmannians.
>
> Moreover, at most one factor in the product is not a sphere, and the only candidates for that factor are $\mathbb{C}P^m$, $\mathbb{H}P^m$, or the rank l Grassmannian $SO(l+m)/SO(l) \times SO(m)$ with $l \in \{2,3\}$.

A key tool is Wilking's connectedness theorem (see [2]), which has proven to be fundamental in the study of positively curved manifolds with symmetry. The theorem relates the cohomology of a closed, positively curved manifold with that of a totally geodesic submanifold. An important consequence of the theorem is a certain periodicity in cohomology. By using the action of the Steenrod algebra on cohomology, we refine this periodicity. Specifically, we prove:

> **Theorem 3:** Let N^n be a closed, simply connected, positively curved manifold that contains a pair of totally geodesic, transversely intersecting submanifolds of codimensions $k_1 \leq k_2$. If $2k_1 + 2k_2 \leq n$, then the rational cohomology rings of N, N_1, N_2, and $N_1 \cap N_2$ are $\gcd(4, k_1)$-periodic.
>
> In particular, if $n \not\equiv 2 \bmod 4$, then N has the rational cohomology ring of S^n, $\mathbb{C}P^{n/2}$, $\mathbb{H}P^{n/4}$, or $S^3 \times \mathbb{H}P^{(n-3)/4}$.

For a closed, simply connected manifold N^n and a coefficient ring R, we say that $H^*(N; R)$ is k-periodic if there exists $x \in H^k(N; R)$ such that the maps $H^i(N; R) \to H^{i+k}(N; R)$ induced by multiplication by x are surjective for $0 \leq i < n - k$ and injective for $0 < i \leq n - k$.

The main step in the proof of Theorem 3 is the following topological result:

> **Theorem 4:** If M^n is a closed, orientable manifold such that $H^*(M; \mathbb{Z})$ is k-periodic with $3k \leq n$, then $H^*(M; \mathbb{Q})$ is $\gcd(4, k)$-periodic.

We conclude this report by addressing an open question related to this work, namely, whether Theorem 1 holds in all even dimensions. This is related to the question of whether a closed, simply connected n-manifold can have 4-periodic rational cohomology with nonvanishing odd Betti numbers. For $n \equiv 0 \bmod 4$, Poincaré duality implies no such space exists. However, for $n = 6$, the space $(S^2 \times S^4)\#(S^3 \times S^3)$ is an example, and further examples are obtained by adding additional $S^3 \times S^3$ components. For $n \equiv 2 \bmod 4$ and $n \geq 10$, however, the author does not know whether such a space exists. If one could prove that there is no such space, then one would have a proof of Theorem 1 in all even dimensions.

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Elliptic genera and positive sectional curvature

Nicolas Weisskopf

In the 1960s Lichnerowicz [3] showed that the index of the Dirac operator, the $\hat{A}$–genus, vanishes on a closed Riemannian Spin manifold admitting a metric of positive scalar curvature. In this short talk we discussed the behaviour of the elliptic genus in the presence of positive sectional curvature.

For a compact, oriented $4k$-dim. smooth manifold M the elliptic genus $\phi(M)$ is a modular function over the group $\Gamma_0(2) = \{A \in SL(2,\mathbb{Z}) | A \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod 2\}$, which expands in one of the cusps as

$$\begin{aligned} \phi(M) &= sign(M, \bigotimes_{n=1}^{\infty} \Lambda_{q^n} T_{\mathbb{C}}M \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T_{\mathbb{C}}M) \\ &= sign(M) + sign(M, T_{\mathbb{C}}M) \cdot q + \ldots \in \mathbb{Z}[[q]]. \end{aligned}$$

This power series of indices of twisted signature operators can be best thought of as the equivariant signature of the free loop space LM with respect to the natural S^1–action. The main property of the elliptic genus is that it transforms in the other cusp of $\Gamma_0(2)$ as the following power series

$$\begin{aligned} \phi_0(M) &= q^{-k/2} \cdot \hat{A}(M, \bigotimes_{n=2m+1>0}^{\infty} \Lambda_{-q^n} T_{\mathbb{C}}M \otimes \bigotimes_{n=2m>0}^{\infty} S_{q^n} T_{\mathbb{C}}M) \\ &= q^{-k/2} \cdot (\hat{A}(M) - \hat{A}(M, T_{\mathbb{C}}M) \cdot q \ \pm \ldots) \in q^{-k/2}\mathbb{Q}[[q]]. \end{aligned}$$

If M is Spin, each coefficient of this power series can be interpreted as the index of a twisted Dirac operator. In this case, the elliptic genus reveals a beautiful identity between the Spin and signature geometry of a manifold.

Suppose now that the manifold admits a metric of positive sectional curvature. Then the elliptic genus seems to feature further remarkable properties. Recently, Dessai [1] raised the following question.

Question ([1]). *Let (M, g) be a closed Riemannian Spin manifold with $sec(M) > 0$. Is the elliptic genus constant as a modular function, i.e. $\phi(M) = sign(M)$?*

Since the $\hat{A}$–genus is a coefficient of the elliptic genus, a positive answer to this question would generalize the Lichnerowicz theorem and more important, exhibit new obstructions to positive sectional curvature. We now give some evidence that indicate a positive answer.

Theorem ([2]). *Let M be a Spin homogeneous space. Then $\phi(M) = sign(M)$.*

In particular, the quaternionic projective spaces $\mathbb{H}P^n$ and the spheres S^n have constant elliptic genus. Note that this theorem does not use any curvature assumption at all. However, many of the positively curved manifolds known happen to be homogeneous spaces. Another piece of evidence is provided by the following classification result.

Theorem ([4]). *Let (M^n, g) be a simply–connected Riemannian manifold with $n \geq 10$ and $sec(M) > 0$. Suppose that a d-dim. torus T^d acts effectively and isometrically on M with $d \geq \frac{n}{4} + 1$. Then M is either homeomorphic to S^n or $\mathbb{H}P^{n/4}$ or homotopy equivalent to $\mathbb{C}P^{n/2}$.*

The elliptic genus of the standard complex projective spaces $\mathbb{C}P^n$ is not constant. Nevertheless, they can be ruled out, since they are not Spin for dimension divisible by 4. Finally, we mention the following result, which uses a smaller symmetry rank.

Theorem ([1]). *Let (M^n, g) be a closed Riemannian Spin manifold with $n > 12r - 4$ and $sec(M) > 0$. Suppose that a $2r$-dim. torus T^{2r} acts effectively and isometrically on M. Then the first $(r + 1)$ coefficients of $\phi_0(M)$ vanish.*

In particular, the elliptic genus vanishes on a 12–dim. positively curved Spin manifold admitting an effective and isometric T^2–action.

We concluded this talk by pointing out that the question raised above is wrong, if one weakens the curvature assumption to positive Ricci curvature. One can then find complete intersections, which possess a positive Ricci curvature metric, but whose elliptic genus is not constant.

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Almost nonnegative curvature operator on certain principal bundles

Martin Herrmann

In my talk I presented a criterion for a certain family of metrics on principal bundles to give almost nonnegative curvature operator.

A compact manifold M is said to admit *almost nonnegative curvature operator* if there is a family $(g_n)_{n\in\mathbb{N}}$ of Riemannian metrics on M such that (M, g_n) Gromov-Hausdorff converges to a point and (M, g_n) admits a uniform lower bound for (the eigenvalues of) the curvature operator, or, equivalently, if for every $\varepsilon > 0$ there is a Riemannian metric g on M such that

$$\hat{R}_{(M,g)}\mathrm{diam}(M, g)^2 > -\varepsilon.$$

The inequality is to be understood as an inequality for the eigenvalues of $\hat{R}_{(M,g)}$ in p for every $p \in M$.

In [2] J.-P. Bourguignon and H. Karcher proved pinching estimates for the curvature operator in terms of bounds on the sectional curvature. From these one easily deduces that almost flat manifolds admit almost nonnegative curvature operator, in fact they admit almost flat curvature operator. Other examples can be constructed using a theorem of D. Sebastian mentioned below. For example, the Witten manifolds carry metrics of almost nonnegative curvature operator.

A special case of a result by P. H. Bérard in [1] is, that manifolds M admitting almost nonnegative curvature operator have their total betti number bounded above by $2^{\dim(M)}$, which has been conjectured for almost nonnegatively curved manifolds by M. Gromov.

On a principal bundle $\pi : P \to M$ over a compact manifold M with a compact Lie group G as fibre we can construct metrics as follows: Let g^M be a Riemannian metric on M, b a biinvariant metric on G and γ a connection form on P. Then, for every $t > 0$, we can define the metric

$$g^t(X, Y) = g^M(\pi_* X, \pi_* Y) + t^2 b(\gamma(X), \gamma(Y)),$$

for which the fibers are totally geodesic. Fukaya and Yamaguchi used these metrics in [3] to show that a fiber bundle with a compact Lie group as structure group, invariantly nonnegatively curved fibre and almost nonnegatively curved base space has metrics of almost nonnegative sectional curvature.

Computing the curvature operator in an orthonormal basis adapted to the decomposition of the tangent space into horizontal and vertical vectors and using a simple lemma from linear algebra one gets the following result:

Proposition. *Let Ω denote the curvature form of γ.*

(1) Let (M, g^M) have nonnegative curvature operator. Then (P, g^t) has almost nonnegative curvature operator for $t \to 0$ if and only if $\mathrm{Im}(\Omega) \subset [\mathfrak{g}, \mathfrak{g}]^\top$.

(2) Let M admit almost nonnegative curvature operator. If $\mathrm{Im}(\Omega) \subset [\mathfrak{g}, \mathfrak{g}]^\top$, then P admits almost nonnegative curvature operator.

From this proposition one can easily deduce the following corollary.

Corollary. *Let G be semisimple. Then (P, g^t) has almost nonnegative curvature operator for $t \to 0$ if and only if (P, g^t) is locally isometric to the Riemannian product $(M, g^M) \times (G, t^2 b)$.*

One also obtains the theorem previously proven by D. Sebastian

Theorem (D. Sebastian, [4])**.** *Let P be a principal bundle with an abelian Lie group as fiber and a compact base space admitting almost nonnegative curvature operator. Then P admits almost nonnegative curvature operator.*

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Low dimensional Singular Riemannian Foliations in spheres

Marco Radeschi

A partition $\mathcal{F}$ of a complete Riemannian manifold M by connected immersed submanifolds (the *leaves*) is called a *Riemannian Foliation* if it satisfies condition (1) and (2) below:

(1) $\mathcal{F}$ is a *singular foliation*, i.e. for each leaf L and each $v \in TL$ with foot point p, there is a smooth vector field X with $X(p) = v$ that is tangent at each point to the corresponding leaf.
(2) $\mathcal{F}$ is *transnormal*, i.e., every geodesic perpendicular to one leaf is perpendicular to every leaf it meets.

One does not ask for the leaves to have all the same dimension. If the dimension of the leaves is constant, that we talk about a *Regular Riemannian Foliation*. Otherwise, the foliation is a *Singular Riemannian Foliation* (SRF for short). In either case, one defines the *dimension of* $\mathcal{F}$ to be the maximal dimension of the leaves. A typical example of a SRF is the partition of a Riemannian manifold into the orbits of an isometric action. Any such foliation is called *homogeneous*. Another example is the partition into the fibers of a Riemannian submersion $\pi : M \to B$.

Given a Riemannian foliation $(M, \mathcal{F})$, and a point $p \in M$, there is a process of linearizing the folitation $\mathcal{F}$ around p. This process gives rise to a SRF $\mathcal{F}_p$ on T_pM, which is called *infinitesimal foliation* at p, and such a foliation is completely determined by its restriction to the unit sphere of the normal space $\nu_p^1 L \subseteq \nu_p L \subseteq T_pM$. Therefore the study of $\mathcal{F}_p$ is reduced to the study of a SRF on a round sphere $(\nu_p^1 L, \mathcal{F}_p|_{\nu_p^1 L})$.

What's known about Riemannian foliations in spheres depends radically on whether all the foliation is regular or singular. In the regular case, a series of works by Haefliger [5], Ghys [2] and Browder [4] shows that the dimension of the leaves is $1, 3, 7$. Moreover, Gromoll and Grove proved in [1] that any foliation of dimension 1 and 3 is homogeneous, i.e. it comes from a group action. If the foliation has dimension 7, then the sphere needs to be 15-dimensional, and Wilking [3] proved that the only such foliation that comes from a Riemannian submersion is the Hopf fibration $S^7 \to S^{15} \to S^8$, and it's conjectured that no other 7-dimensional regular foliations exist. This would complete the classification of Riegular Riemannian Foliations in spheres.

In the singular case very little is known. First of all, there is a way to compose two SRF's $\mathcal{F}_1$, $\mathcal{F}_2$ on (possibly different) spheres, obtaining a new foliation $\mathcal{F}_1 \star \mathcal{F}_2$ in the spherical join of the two original spheres. One says that a foliation is *irreducible* if it cannot be decomposed in this way. The case of irreducible foliations of codimension 1, is equivalent to the problem of classifying irreducible isoparametric hypersurfaces in sphere. This problem has long been studied, and the classification is almost complete. It's worth mentioning that this case contains non homogeneous foliations; together with the Hopf fibration mentioned above, these are all the non homogeneous irreducible foliations known in round spheres.

Our main results classifies low dimensional SRF's in spheres, generalizing the result of Grove and Gromoll [1]

Theorem 1 (-). *Any SRF* $(S^n, \mathcal{F})$ *with* $\dim \mathcal{F} \leq 3$ *is homogeneous.*

Toward the proof of this result, a first step was to understand the low dimensional stratification of SRF's in spheres.

The *r-dimensional stratum* of a foliation $\mathcal{F}$ is the collection of all the leaves of dimension r, and it's denoted Σ_r. It was already proved by Molino that any component of a stratum is a (possibly noncomplete) submanifold.

Theorem 2. *Let* $(S^n, \mathcal{F})$ *be a SRF. Then:*

(1) $\Sigma_0 = S^h$ *is totally geodesic in* S^n. *Moreover* $(S^n, \mathcal{F})$ *splits as* $(S^h, \mathcal{F}_0) \star (S^{n-h-1}, \mathcal{F}_1)$ *where* $\mathcal{F}_0$ *consists of points, and* $\mathcal{F}_1$ *does not contain any 0-dimensional leaves.*

(2) If $\Sigma_0 = \emptyset$, *then* $\Sigma_1 = \cup_i S^{n_i}$, *where each component* S^{n_i} *is a totally geodesic sphere, and* $d(p_i, p_j) = \pi/2$ *for any* $p_i \in S^{n_i}$, $p_j \in S^{n_j}$, *for* $i \neq j$.

(3) For any r, *any component of* Σ_r *is a minimal submanifold.*

The theorem above takes immediately care of 1-dimensional and two dimensional foliations. In fact, for one dimensional foliations the only singular leaves are o-dimensional, and by the theorem above $(S^n, \mathcal{F}) = (S^h, \mathcal{F}_0) \star (S^{n-h-1}, \mathcal{F}_1)$. The foliation $\mathcal{F}_1$ has now only 1-dimensional leaves, and by the result of Grove and Gromoll it is homogeneous, with group $\mathbb{R}$. Since $\mathcal{F}_0$ is trivially homogeneous, and the join of homogeneous foliations is again homogeneous, it follows that $\mathcal{F}$ itself is homogeneous.

For 2-dimensional foliations, again one has the splitting $(S^n, \mathcal{F}) = (S^h, \mathcal{F}_0) \star (S^{n-h-1}, \mathcal{F}_1)$, where $\mathcal{F}_0$ has only 0-dimensional leaves, and the only singular leaves in $\mathcal{F}_1$ are 1-dimensional. Applying point 2 in the theorem above, one shows that $\mathcal{F}_1$ splits further as $(S^{k_2}, \mathcal{F}_2) \star (S^{k_3}, \mathcal{F}_3)$. Now $\mathcal{F}_2$, $\mathcal{F}_3$ have to be both 1-dimensional, and therefore homogeneous with group $\mathbb{R}$. Once again, $\mathcal{F}_1 = \mathcal{F}_2 \star \mathcal{F}_3$ is homogeneous as well, with group $\mathbb{R}^2$. Finally, the original $\mathcal{F} = \mathcal{F}_0 \star \mathcal{F}_1$ is homogeneous with group $\mathbb{R}^2$.

It remains to consider 3-dimensional foliations. Here another important tool is the presence of tensors, namely the shape operator $S_x : T_pL \to T_pL$ and the O'Neill tensor $A_x : \nu_pL \to T_pL$, where $x \in \nu_pL$. These are tensors defined on the *regular part* of $\mathcal{F}$, i.e. the union of leaves of maximal dimension, and are related in the following way:

$$
\begin{aligned}
(\nabla^v_x A)_x y &= 2 S_x A_x y \\
(\nabla^v_x S)_x u &= S_x^2 u + R^v(u, x)x - 3 A_x A_x^* u
\end{aligned}
$$

Another important tool, is the following theorem of Grove and Gromoll, that gives condiitons for homogeneity of a SRF in a sphere, given the existence of a special class of vector fields:

Theorem 3 (Homogeneity Theorem, [1])**.** *Let $(S^n, \mathcal{F})$ be a SRF in a round sphere. Then $\mathcal{F}$ is homogeneous if there is a regular leaf L_0, and a subsheaf $\mathcal{E} \in \mathfrak{X}(L_0)$ such that the following conditions hold:*

a) *$\mathcal{E}$ is a locally constant sheaf of Lie algebras, and it's finite dimensional.*
b) *The elements of $\mathcal{E}$ span T_pL_0, for all $p \in L_0$,*
c) *If $V, W \in \mathcal{E}$, then $\langle V, W\rangle$ is constant,*
d) *For every basic vector field X, Y and $V \in \mathcal{E}$, $S_XV \in \mathcal{E}$, $A_XY \in \mathcal{E}$.*

The proof of Theorem 1 in the 3-dimensional case starts by considering several cases separately, depending on the possible ranks of the O'Neill tensor. In each case we find the vector fields of $\mathcal{E}$, by first understanding the singular stratification of $\mathcal{F}$, and then reducing the problem to linear algebra, using the S tensor and A tensor above.

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Polynomial Invariants of Riemannian Manifolds

Gregor Weingart

In the talk I wanted to sketch a research project on polynomial invariants of Riemannian manifolds I have started some time ago following a suggestion of Ch. Bär. Every $\mathbf{SO}(\mathbb{R}^m)$–invariant polynomial homogeneous of degree k on the space $\mathrm{Kr}\,\mathfrak{so}(\mathbb{R}^m)$ of algebraic curvature tensors in dimension m gives rise to an invariant of oriented Riemannian manifolds of dimension m via integration

$$[\,\mathrm{Sym}^k\mathrm{Kr}\,\mathfrak{so}(\mathbb{R}^m)\,]^{\mathbf{SO}(\mathbb{R}^m)} \longrightarrow \mathbb{R}, \qquad p \longmapsto \int_M p(\,R^g\,)\,\mathrm{vol}_g$$

where $R^g \in \Gamma(\,\mathrm{Kr}\,\mathfrak{so}(TM, g)\,)$ is the Riemannian curvature tensor and vol_g the oriented volume form for the given Riemannian metric g on M. In general the algebra of invariant polynomials on $\mathrm{Kr}\,\mathfrak{so}(\mathbb{R}^m)$ is rather complicated as illustrated by the following table showing the dimension of the space of invariant polynomials of degree $k = 0, 1, \ldots, 8$ in dimensions $m = 2, 3, \ldots, 12$:

	2	3	4	5	6	7	8	9	10	11	12
0	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	4	3	3	3	3	3	3	3	3
3	1	3	9	7	8	8	8	8	8	8	8
4	1	4	19	20	24	25	28	26	26	26	26
5	1	5	39	51	83	84	101	89	90	90	90
6	1	7	82	150	361	359	509	403	409	403	412
7	1	8	151	431	1697	1761	3125	2194	2407	2240	2281
8	1	10	291	1318	8719	10552	24236	15775	20030	16343	17124

Looking at the dimensions given in the table above it is interesting to observe that the dimensions of the spaces $[\,\mathrm{Sym}^k\mathrm{Kr}\,\mathfrak{so}(\,\mathbb{R}^m\,)\,]^{\mathbf{SO}(\,\mathbb{R}^m\,)}$ stabilize with $k \geq 0$ fixed for $m > 2k$, and the resulting algebra of stably invariant polynomials is definitely a much nicer object of study:

$$\mathfrak{A}^\bullet(\,\mathfrak{so}\,) \;:=\; \overleftarrow{\lim_m}\,[\,\mathrm{Sym}^\bullet\mathrm{Kr}\,\mathfrak{so}(\,\mathbb{R}^m\,)\,]^{\mathbf{SO}(\,\mathbb{R}^m\,)}$$

Needless to say the most important examples of polynomial invariants of Riemannian manifolds arise from the theory of characteristic classes in topology. Specifically all Pontryagin numbers of a Riemannian manifold M of dimension m divisible by 4 arise from invariant polynomials of degree $\frac{m}{2}$ on the space of algebraic curvature tensors. Instead of being stably invariant it appears as if these polynomials account *exactly* for the drop from the last unstable to the stable dimension for even polynomial degrees in the table above. Contrary to the intuition however the Euler characteristic of an oriented Riemannian manifold M of dimension m divisible by 2 is a polynomial invariant arising from a stably invariant polynomial of degree $\frac{m}{2}$ on the space of algebraic curvature tensors.

Other stably invariant polynomials on the space of algebraic curvature tensors arise from the moments of the sectional curvature. Considering the sectional curvature as a random variable defined on the Grassmannian Gr_2TM of 2–planes in TM we may define its k–th moment for $k \in \mathbb{N}_0$ by

$$E_M[\,x^k\,] \;:=\; \frac{1}{\mathrm{Vol}(\,\mathrm{Gr}_2TM\,)}\int_{\mathrm{Gr}_2TM}\mathrm{Sec}(\,X \wedge Y\,)^k\,\mathrm{vol}_{FS}(\,X \wedge Y\,),$$

where FS denotes the Riemannian metric on Gr_2TM which makes the projection to M a Riemannian submersion with the Fubini–Study metric in the fibers. Using a couple of integration tricks these moments can be integrated over the fibers in order to write them as polynomial invariants of Riemannian manifolds arising from stably invariant polynomials homogeneous of degree k. The main result reads

$$E_M[\,x^k\,] \;=\; \frac{1}{\mathrm{Vol}\,M}\int_M \frac{(-\Delta)^k}{[m\,+\,2k\,-\,2]_{2k}}\Big|_0\left(X \longmapsto \exp\Big(\sum_{\mu>0}\frac{1}{2\mu}\mathrm{tr}[\,R^{\mu}_{\cdot,X}X\,]\Big)\right),$$

where for given $X \in T_xM$ the notation $R^{\mu}_{\cdot,X}X$ refers to the Jacobi endomorphism on T_xM in the direction of X raised to the μ–th power, Δ denotes the metric

Laplace operator acting on the resulting formal power series in $X \in T_xM$ and $[z]_r = z(z-1)\dots(z-r+1)$ the falling factorial polynomial.

On a Riemannian symmetric space M of rank 1 the Jacobi endomorphism $R_{\cdot,X}X$ is independent of the direction X so that the above formula can be used in fact to calculate the full probabilty density of the sectional curvature on these spaces. Normalizing the Riemannian metric to have sectional curvature in the interval $[1,4]$, the final result reads for all $f \in C^\infty(\mathbb{R})$:

$$\begin{aligned}
E_{\mathbb{C}P^n}[\,f\,] &= \frac{1}{6}\binom{n-\frac{3}{2}}{n-2}\int_1^4 f(s)\left(\frac{4-s}{3}\right)^{n-2}\left(\frac{s-1}{3}\right)^{-\frac{1}{2}}ds\\
E_{\mathbb{H}P^n}[\,f\,] &= \frac{3}{6}\binom{2n-\frac{3}{2}}{2n-3}\int_1^4 f(s)\left(\frac{4-s}{3}\right)^{2n-3}\left(\frac{s-1}{3}\right)^{+\frac{1}{2}}ds\\
E_{\mathbb{O}P^2}[\,f\,] &= \frac{7}{6}\binom{\frac{13}{2}}{3}\int_1^4 f(s)\left(\frac{4-s}{3}\right)^{3}\left(\frac{s-1}{3}\right)^{+\frac{5}{2}}ds
\end{aligned}$$

Interestingly these probability measures converge to the Dirac measure in 1 for dimension $n \to \infty$. It should be noted that the calculation of moments can be used in principle to calculate the pinching constant of a Riemannian metric without ever having to calculate a single sectional curvature.

The main idea of the presented research project is to describe the algebra $\mathfrak{A}^\bullet(\mathfrak{so})$ of stably invariant polynomials on the space of algebraic curvature tensors by means of a suitable graph algebra similar to the graph algebra defining the Rozansky–Witten invariants of hyperkähler and quaternionic Kähler manifolds. The adequate graph algebra uses trivalent graphs with edges colored red and black such that at every vertex there exists exactly one red flag. A particular merit of this graph algebra description of stably invariant polynomials on the space of algebraic curvature tensors is that it makes the dependencies between stably invariant polynomials on Einstein manifolds completely explicit.

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