Mathematisches Forschungsinstitut Oberwolfach

Report No. 03/2012

DOI: 10.4171/OWR/2012/03

Mini-Workshop: Thermodynamic Formalism, Geometry and Stochastics

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January 1st – January 7th, 2012

Abstract. Thermodynamic formalism and all its branches and applications in conformal dynamics, probability theory, stochastics and fractal geometry represent highly important fields in modern Mathematics, which are currently very active and rapidly growing. The workshop brought leading experts in these fields together with junior researchers to provide them with the opportunity to exchange their knowledge and experience. It led to various new insights as well as promising new research collaborations.

Mathematics Subject Classification (2000): Primary 37Axx; Secondary 28Dxx, 37Dxx, 37Fxx, 37Hxx.

Introduction by the Organisers

The workshop *Thermodynamic Formalism, Geometry and Stochastics*, organised by Bernd Otto Stratmann (Bremen), Mariusz Urbański (Denton, Texas) and Anna Zdunik (Warsaw), was held January 1st – January 7th, 2012. This meeting was well attended with 17 participants with broad geographic representation from almost all continents. Every participant gave a 45 minutes presentation and additionally, there was a round table discussion about thermodynamic formalism as well as an *Open Problem Session* to which almost every participant contributed. The workshop was a splendid blend of researchers with various different backgrounds in thermodynamic formalism, conformal dynamics, probability theory, stochastics and fractal geometry. These areas are central in the theory of dynamical systems. They have their origins in the pioneering work of Ruelle, Sinai, Bowen, Dobroushin and others, who from the late sixties until the mid seventies adapted key methods from statistical physics, in particular the theory of gas lattices, to the context of continuous dynamical systems on compact metric spaces. It was also during this period that important concepts such as topological pressure, the variational principle, equilibrium states and Gibbs states made their decisive breakthrough.

The first areas in which the principles of thermodynamic formalism have successfully been applied include Axiom A diffeomorphisms and smooth expanding endomorphisms of Riemannian manifolds and the foundations for these important applications were laid by Bowen and Ruelle. Since then, thermodynamic formalism has flourished in many different directions and found fruitful applications in various fields in Pure and Applied Mathematics. The recent book *Conformal Fractals: Ergodic Theory Methods* by Przytycki and Urbański, which both were present at the workshop, provides a systematic account of the current state of the art.

In conformal dynamics and fractal geometry, the relevance of thermodynamic formalism became apparent through groundbreaking work of Rufus Bowen, which established a relationship between the Hausdorff dimension of the limit set of a quasi-Fuchsian group and the unique zero of the associated pressure function. This was expressed in one formula, now called Bowen's formula. Shortly after the appearance of this formula, it became clear that Bowen's approach is also applicable in many other situations, most notably in the study of Julia sets of conformal expanding maps. Also, thermodynamic formalism has been employed in differential geometry to derive deep new insights into the nature of geodesic flows on compact Riemannian manifolds with negative curvature. In number theory, prominent applications of thermodynamic formalism were given in the realm of continued fractions and the Gauss map. This has been done, for instance, in a series of papers by D. Mayer and also via the theory of conformal graph directed Markov systems with an infinite set of edges, whose general theory has been developed by Mauldin and Urbański. A further generalisation of the latter theory to the case in which additionally the set of vertices is infinite has been obtained in joint work by Stratmann and Urbański. Moreover, a closely related multifractal analysis of the Gauss map and the Farey map has been obtained in a series of papers by Kesseböhmer and Stratmann. The graph directed approach to continued fractions has also led to various important theorems about real numbers whose continued fraction expansions have entries restricted to some fixed infinite subset of positive integers.

Besides the above-mentioned applications of thermodynamic formalism in the study of limit sets of Kleinian groups, thermodynamic formalism has also had great impact in the rigorous exploration of fractal phenomena of Julia sets of rational functions on the Riemann sphere. For the most classical of these maps, namely those which are expanding on their Julia sets, the results for conformal expanding repellers can be applied directly. Moreover, the case of parabolic rational maps has been almost completely dealt with in a cycle of classical papers by Denker and Urbański. The non-recurrent case has been extensively discussed by Urbański, whereas the important class of Collet-Eckmann maps has been explored thoroughly by Graczyk and Smirnov, as well as by Rivera-Letelier and Przytycki. The work of Denker and Urbański has also been significant in the comprehensive further development of the theory of equilibrium states for Hölder continuous potential functions, which also served as one of the main motivations for the important work of Haydn, as well as for the very recent research of Urbański and Zdunik concerning fine inducing.

There are numerous further areas where thermodynamic formalism has turned out to be an indispensable tool. For instance, thermodynamic formalism has always been fruitfully inspired by multifractal formalism, which is part of fractal geometry and a special case of thermodynamic formalism. Moreover, in finer studies of parabolic phenomena, recent work by Thaler and Zweimüller, using thermodynamic formalism, has shed new and surprising light on Darling-Kac type theorems. Similarly, recent work of Melbourne significantly clarifies statistical properties of these parabolic maps. Another large area where thermodynamic formalism has been applied to, is the field of random dynamical systems. Here, the work of Rugh has to be mentioned, which is based on pioneering research by Kifer, Bogenschütz and Gundlach. However, this area is far from being complete. For instance, Simmons works on random iterates of general rational functions and Hölder continuous potentials, and V. Mayer, Skorulski and Urbański are currently working on random parabolic Cantor sets as well as random transcendental meromorphic functions. Also, thermodynamic formalism for transcendental entire and meromorphic functions has flourished since the seminal work of Barański, which in turn has been intensively developed further by Kotus, V. Mayer, Urbański and Zdunik.

Mini-Workshop: Thermodynamic Formalism, Geometry and Stochastics

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Abstracts

Stochastics and thermodynamics for equilibrium measures of holomorphic endomorphisms of complex projective spaces

Anna Zdunik
(joint work with M. Szostakiewicz and M. Urbański)

Fix an integer $k \geq 1$. Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of degree $d \geq 2$ of the complex projective space $\mathbb{P}^k$. Denote by $J = J(f)$ the Julia set of the map $f : \mathbb{P}^k \to \mathbb{P}^k$, i. e. the topological support of the measure of maximal entropy. The map $f : \mathbb{P}^k \to \mathbb{P}^k$ is called regular if its exceptional set $E = E(f)$ does not intersect the Julia set $J = J(f)$. Recall that the exceptional set $E = E(f)$ is a proper algebraic, totally invariant subset contained in the critical set, such that, given $a \in \mathbb{P}^k$, the sequence of point measures $d^{-kn}(f^n)^*\delta_a$ is equally distributed on the preimages of the point a converges to the measure of maximal entropy if and only if $a \notin E$. For $k = 1$ the set $E(f)$ is either empty, of cardinality 2 ($z \mapsto z^{\pm d}$), or of cardinality 1 (polynomials). Obviously, for $k = 1$ the set $E(f)$ never intersects $J(f)$, so in dimension 1, every map is regular. In dimension $k > 1$, we take this as an additional assumption, although we do not know any example of a holomorphic map $f : \mathbb{P}^k \to \mathbb{P}^k$ for which the intersection $E(f) \cap J(f)$ is nonempty. Moreover, it is known that the set $E(f)$ is empty for a generic holomorphic map $f : \mathbb{P}^k \to \mathbb{P}^k$.

Let $\phi : J(f) \to \mathbb{R}$ be a continuous function, in the sequel frequently referred to as a potential. By $P(\phi)$ we denote the (classical) topological pressure of the potential ϕ with respect to the dynamical system $f : J(f) \to J(f)$. If μ is a Borel probability f-invariant measure on $J(f)$, we denote by $h_\mu(f)$ its Kolmogorov–Sinai metric entropy. The relation between pressure and entropy is given by the following celebrated Variational Principle.

$$P(\varphi) = \sup\left\{h_\mu(f) + \int \varphi d\mu\right\},$$

where the supremum is taken over all Borel probability f-invariant measures μ, or equivalently, over all Borel probability f-invariant ergodic measures μ. The measures μ for which

$$h_\mu(f) + \int \phi\, d\mu = P(\phi)$$

are called equilibrium states for the potential ϕ. The main theorem obtained in a previous joint work with M.Urbański is the following.

Theorem 1. *For every regular holomorphic endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ of a complex projective space $\mathbb{P}^k$, $k \geq 1$, there exists a positive number $\kappa_f > 0$ such that if $\phi : J(f) \to \mathbb{R}$ is a Hölder continuous function with $\sup(\phi) - \inf(\phi) < \kappa_f$ (we then say that ϕ has a pressure gap), then ϕ admits a unique equilibrium state μ_ϕ on J. This equilibrium state is equivalent to a fixed point of the normalized dual Perron-Frobenius operator. In addition the dynamical system (f, μ_ϕ) is K-mixing, whence ergodic. In the case when the Julia set J does not intersect any* *periodic irreducible algebraic variety contained in the critical set of f, we have that $\kappa_f = \log d$.*

The main object of study in our next paper (joint work with Michal Szostakiewicz and Mariusz Urbański) was the dynamical system (f, μ_ϕ). We show that this system has exponential decay of correlations of Hölder continuous observables as well as the Central Limit Theorem and the Law of Iterated Logarithm for the class of these variables that, in addition, satisfy a natural co-boundary condition. We also show real analyticity of the topological pressure function.

More precisely, we have

Theorem 2. *For the dynamical system (f, μ_ϕ) the following hold.*

(1) *For every $\alpha \leq 1$, every α–Hölder continuous function $g : J(f) \to \mathbb{R}$ and every bounded measurable function $\psi : J(f) \to \mathbb{R}$, we have that*

$$\left| \int \psi \circ f^n \cdot g d\mu_\phi - \int g d\mu_\phi \int \psi d\mu_\phi \right| = O(\theta^n)$$

for some $0 < \theta < 1$ depending on α.

(2) *The Central Limit Theorem holds for every Hölder continuous function $g : J(f) \to \mathbb{R}$ that is not cohomologous to a constant in $L^2(\mu_\phi)$, i.e. for which there is no square integrable function η for which $g = \mathrm{const} + \eta \circ f - \eta$. Precisely this means that there exists $\sigma > 0$ such that*

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} g \circ f^j \to \mathcal{N}(0, \sigma)$$

in distribution.

(3) *The Law of Iterated Logarithm holds for every Hölder continuous function $g : J(f) \to \mathbb{R}$ that is not cohomologous to a constant in $L^2(\mu_\phi)$. This means that there exists a real positive constant A_g such that such that μ_ϕ almost everywhere*

$$\limsup_{n \to \infty} \frac{S_n g - n \int g d\mu}{\sqrt{n \log \log n}} = A_g.$$

Theorem 3. *The topological pressure function*

$$t \mapsto P(t\phi) \in \mathbb{R}$$

is real–analytic is real–analytic throughout the open set of all parameters t for which the potentials $t\phi$ have a pressure gap.

Bowens formula for meromorphic functions

Krzysztof Barański

The thermodynamical formalism has provided a number of useful tools to study the geometry and ergodic properties of conformal repellers in the Julia set $J(f)$ of a rational map f on the Riemann sphere. In this setting a conformal repeller is a compact set $X \subset J(f)$, such that $f(X) \subset X$ and $|(f^k)'|_X > 1$ for some $k > 0$. In particular, the celebrated Bowen formula asserts that the Hausdorff dimension of a conformal repeller X (e.g. the Julia set for a hyperbolic rational map) is equal to the unique zero of the pressure function $t \mapsto P(f|_X, t)$, where

$$P(f|_X, t) = \lim_{n\to\infty} \frac{1}{n} \ln \sum_{w \in f^{-n}(z)\cap X} |(f^n)'(w)|^{-t}$$

for $z \in X$ is the topological pressure of $f|_X$ for the potential $\varphi = -t \ln |f'|$.

In recent years, there have been more and more attempts to generalise the results of the thermodynamical formalism theory to the case of transcendental meromorphic maps. However, this encounters some difficulties, due to lack of compactness, infinite degree of the map and more complicated geometry.

A transcendental meromorphic map f is hyperbolic, if the closure (in $\overline{\mathbb{C}}$) of the post-singular set $\mathcal{P}(f)$, i.e. the union of forward trajectories of the singular (critical and asymptotic) values of f is disjoint from $J(f) \cup \{\infty\}$.

In [1], K. Barański developed some elements of the thermodynamical formalism (in particular Bowen's formula) for certain hyperbolic meromorphic maps of the form $f(z) = h(\exp(az))$, where $a \in \mathbb{C}$ and h is a rational function, in particular for the hyperbolic maps from the tangent family $\lambda \tan(z)$, $\lambda \in \mathbb{C}$. In [8, 9], M. Urbański and A. Zdunik created the thermodynamical formalism theory for hyperbolic maps in the exponential family $f(z) = \lambda \exp(z)$, $\lambda \in \mathbb{C}$. In particular, they discovered that the unique zero of the pressure function is equal not to the Hausdorff dimension of the Julia set $J(f)$ (which is equal to 2 for all parameters λ), but to the Hausdorff dimension of the radial Julia set $J_r(f)$. The set $J_r(f)$ is, by definition, the set of $z \in J(f)$ for which there exists $r = r(z) > 0$ and a sequence $n_j \to \infty$, such that a holomorphic branch of f^{-n_j} sending $f^{n_j}(z)$ to z is well-defined on the disc in the spherical metric, centred at $f^{n_j}(z)$ of radius r. In [3, 4], V. Mayer and M. Urbański developed the thermodynamical formalism theory for hyperbolic transcendental meromorphic maps of finite order with the so-called balanced derivative growth condition. Also in this case the unique zero of the pressure function is equal to the Hausdorff dimension of $J_r(f)$. Among examples of maps satisfying the balanced derivative growth condition are hyperbolic functions of the form $f(z) = P(\exp(Q(z)))$, where P, Q are polynomials.

In this work we show that Bowen's formula in its new form is actually satisfied for all transcendental meromorphic maps in the class $\mathcal{S}$ and for a wide class of maps from the class $\mathcal{B}$. What is more, our proof works for non-hyperbolic maps as well. Recall that the class $\mathcal{S}$ (resp. $\mathcal{B}$) consists of transcendental meromorphic maps for which the set of singular values is finite (resp. bounded).

We obtained the following two results.

Theorem 1. *For every transcendental entire or meromorphic map f in the class $\mathcal{S}$ and every $t > 0$ the topological pressure*

$$P(f,t) = P(f,t,z) = \lim_{n\to\infty} \frac{1}{n} \ln \sum_{w\in f^{-n}(z)} |(f^n)^*(w)|^{-t}$$

(where * denotes the derivative with respect to the spherical metric) exists (possibly equal to $+\infty$) and is independent of $z \in \mathbb{C}$ up to an exceptional set of Hausdorff dimension zero (consisting of points quickly approximated by the forward orbits of singular values of f). We have

$$P(f,t) = P_{hyp}(f,t),$$

where $P_{hyp}(f,t)$ is the supremum of the pressures $P(f|_X,t)$ over all transitive isolated conformal repellers $X \subset J(f)$. The function $t \mapsto P(f,t)$ is non-increasing and convex when it is finite and satisfies $P(f,2) \le 0$. The following version of Bowen's formula holds:

$$\dim_H J_r(f) = \dim_{hyp} J(f) = \delta(f),$$

where $\delta(f) = \inf\{t > 0 : P(t) \le 0\}$.

A conformal repeller X is called transitive if for all non-empty sets U and V, both open in X, we have $f^n(U) \cap V \neq \emptyset$ for some $n \ge 0$. X is called isolated if there exists a neighbourhood W of X such that for every $z \in W \setminus X$ there exists $n > 0$ with $f^n(z) \notin W$. The hyperbolic dimension of the Julia set $J(f)$ (denoted $\dim_{hyp}$) is defined as the supremum of the Hausdorff dimensions (denoted $\dim_H$) of all conformal repellers contained in $J(f)$.

Theorem 2. *For every non-exceptional transcendental entire or meromorphic map f in the class $\mathcal{B}$, such that $J(f) \setminus \overline{\mathcal{P}(f)} \neq \emptyset$ (in particular, for every hyperbolic map in $\mathcal{B}$) and every $t > 0$ the topological pressure*

$$P(f,t) = P(f,t,z) = \lim_{n\to\infty} \frac{1}{n} \ln \sum_{w\in f^{-n}(z)} |(f^n)^*(w)|^{-t}$$

exists (possibly equal to $+\infty$) and is independent of $z \in J(f) \setminus \overline{\mathcal{P}(f)}$, which is an open dense subset of $J(f)$. We have

$$P(f,t) = P_{hyp}(f,t).$$

The function $t \mapsto P(f,t)$ is non-increasing and convex when it is finite and satisfies $P(f,2) \le 0$. Bowen's formula holds:

$$\dim_H J_r(f) = \dim_{hyp} J(f) = \delta(f).$$

If, additionally, f is hyperbolic, then $P(f,t) > 0$ for every $0 < t < \delta(f)$ and $P(f,t) < 0$ for every $t > \delta(f)$.

We call f called non-exceptional if there are no (Picard) exceptional values a of f such that $a \in J(f)$ and f has a non-logarithmic singularity over a.

We were inspired by the papers by F. Przytycki, J. Rivera Letelier and S. Smirnov [5, 6], where the authors developed the theory of pressure for arbitrary (not necessarily hyperbolic) rational maps. Another source of inspiration was the paper [7] by G. Stallard containing ideas which are very close to the notion of the pressure for hyperbolic transcendental meromorphic maps in the class $\mathcal{B}$.

References

[1] K. Barański, *Hausdorff dimension and measures on Julia sets of some meromorphic maps*, Fund. Math. 147 (1995), 239–260.

[2] K. Barański, B. Karpińska and A. Zdunik, *Bowen's formula for meromorphic functions*, 2010, to appear in Ergodic Theory Dynam. Systems.

[3] V. Mayer and M. Urbański, *Geometric thermodynamic formalism and real analyticity for meromorphic functions of finite order*, Ergodic Theory Dynam. Systems 28 (2008), 915–946.

[4] V. Mayer and M. Urbański, *Thermodynamical formalism and multifractal analysis for meromorphic functions of finite order*, Mem. Amer. Math. Soc. 954 (2010), 1–105.

[5] F. Przytycki, *Conical limit set and Poincaré exponent for iterations of rational functions*, Trans. Amer. Math. Soc. 351 (1999), 2081-2099.

[6] F. Przytycki, J. Rivera Letelier and S. Smirnov, *Equality of pressures for rational functions*, Ergodic Theory Dynam. Systems 24 (2004), 891–914.

[7] G. M. Stallard, *The Hausdorff dimension of Julia sets of hyperbolic meromorphic functions*, Math. Proc. Camb. Phil. Soc. 127 (1999), 271–288.

[8] M. Urbański and A. Zdunik, *The finer geometry and dynamics of the hyperbolic exponential family*, Michigan Math. J. 51 (2003), 227–250.

[9] M. Urbański and A. Zdunik, *Real analyticity of Hausdorff dimension of the finer Julia sets of exponential family*, Ergodic Theory Dynam. Systems 24 (2004), 279–315.

Stability and bifurcation in random complex dynamics

Hiroki Sumi

Since nature has many random terms, it is natural and important to investigate random dynamical systems. Many physicists are investigating "noise-induced phenomena" (new phenomena caused by noise and randomness). Regarding the dynamics of a rational map h with $\deg(h) \geq 2$ on the Riemann sphere $\hat{\mathbb{C}}$, we always have the **chaotic part** in $\hat{\mathbb{C}}$. **However**, we show that in the (i.i.d.) random dynamics of polynomials on $\hat{\mathbb{C}}$, **generically**, (1) **the chaos of the averaged system disappears**, due to the automatic cooperation of many kinds of maps in the system (**cooperation principle**), and (2) the limit states are **stable** under perturbations of the system.

Moreover, we investigate the **bifurcation** of 1-parameter families of random complex dynamical systems.

Definition 1. *(1)* We denote by $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ the Riemann sphere and denote by d the spherical distance on $\hat{\mathbb{C}}$. *(2)* We set

$$Rat := \{h : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid h \text{ is a non-const. rational map}\}$$

endowed with the distance η defined by $\eta(f,g) := \sup_{z\in\hat{\mathbb{C}}} d(f(z),g(z))$. We set $Rat_+ := \{h \in Rat \mid \deg(h) \geq 2\}$. *(3)* We set

$$\mathcal{P} := \{h : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid h \text{ is a polynomial map}, \deg(h) \geq 2\}$$

endowed with the relative topology from *Rat*. *(4)* For a metric space X, we denote by $\mathfrak{M}_1(X)$ the space of all Borel probability measures on X. From now on, we take a $\tau \in \mathfrak{M}_1(Rat)$ and we consider the **(i.i.d.) random dynamics** on $\hat{\mathbb{C}}$ such that at every step we choose a map $h \in Rat$ according to τ. This determines a time-discrete Markov process with time-homogeneous transition probabilities on the phase space $\hat{\mathbb{C}}$ such that for each $x \in \hat{\mathbb{C}}$ and for each Borel measurable subset A of $\hat{\mathbb{C}}$, the transition probability $p(x,A)$ from x to A is defined as $p(x,A) = \tau(\{h \in Rat \mid h(x) \in A\})$. *(5)* Note that *Rat* and $\mathcal{P}$ are **semigroups** where the semigroup operation is functional composition. A subsemigroup of *Rat* is called a **rational semigroup**. *(6)* For a rational semigroup G, we set $F(G) := \{z \in \hat{\mathbb{C}} \mid \exists \text{ nbd } U \text{ of } z \text{ s.t. } G \text{ is equicontinuous on } U\}$. This $F(G)$ is called the **Fatou set** of G. Moreover, we set $J(G) := \hat{\mathbb{C}} \setminus F(G)$. This $J(G)$ is called the **Julia set** of G. *(7)* (**Key**) For a rational semigroup G, we set $J_{\ker}(G) := \bigcap_{h\in G} h^{-1}(J(G))$. This is called the **kernel Julia set** of G.

Remark: Let $\tau \in \mathfrak{M}_1(\mathcal{P})$ be such that $\operatorname{supp}\tau$ is compact. If there exists an $f_0 \in \mathcal{P}$ and a non-empty open subset U of $\mathbb{C}$ s.t. $\{f_0 + c \mid c \in U\} \subset \operatorname{supp}\tau$, then $J_{\ker}(G_\tau) = \emptyset$, where G_τ denotes the rational semigroup generated by $\operatorname{supp}\tau$. Thus, for **most** $\tau \in \mathfrak{M}_1(\mathcal{P})$ with compact support, $J_{\ker}(G_\tau) = \emptyset$.

Theorem 1 (**Cooperation Principle and Disappearance of Chaos).** *Let* $\tau \in \mathfrak{M}_1(\mathrm{Rat})$ *be such that* $\operatorname{supp}\tau$ *is compact. Let* G_τ *be the rational semigroup generated by* $\operatorname{supp}\tau$. *Suppose* $J_{\ker}(G_\tau) = \emptyset$ *and* $J(G_\tau) \neq \emptyset$. *(*note*: if there exists an* $h \in \operatorname{supp}\tau$ *with* $\deg(h) \geq 2$, *then* $J(G_\tau) \neq \emptyset$.*) Then, we have all of the following* (1)(2)(3).

(1) We say that a non-empty compact subset K *of* $\hat{\mathbb{C}}$ *is a* **minimal set** *of* G_τ *in* $\hat{\mathbb{C}}$ *if* K *is minimal in* $\{L \subset \hat{\mathbb{C}} \mid \emptyset \neq L \text{ is compact}, \forall h \in G_\tau, h(L) \subset L\}$ *with respect to the inclusion. Moreover, we set* $\mathrm{Min}(G_\tau,\hat{\mathbb{C}}) := \{L \mid L \text{ is a minimal set of } G_\tau \text{ in } \hat{\mathbb{C}}\}$. *Then,* $1 \leq \sharp\mathrm{Min}(G_\tau,\hat{\mathbb{C}}) < \infty$.

(2) For **each** $z \in \hat{\mathbb{C}}$, *there exists a Borel subset* $\mathcal{A}_z$ *of* $(\mathrm{Rat})^{\mathbb{N}}$ *with* $(\Pi_{j=1}^{\infty}\tau)(\mathcal{A}_z) = 1$ *such that for each* $\gamma = (\gamma_1,\gamma_2,\ldots) \in \mathcal{A}_z$, *the following* (a)*and* (b) *hold.* (a) *There exists a* $\delta = \delta(z,\gamma) > 0$ *such that* $\mathrm{diam}\gamma_n \cdots \gamma_1(B(z,\delta)) \to 0$ *as* $n \to \infty$. (b) $d(\gamma_n,\cdots\gamma_1(z), \bigcup_{L\in\mathrm{Min}(G_\tau,\hat{\mathbb{C}})} L) \to 0$ *as* $n \to \infty$.

(3) We set $C(\hat{\mathbb{C}}) := \{\varphi : \hat{\mathbb{C}} \to \mathbb{C} \mid \varphi \text{ is conti.}\}$ *endowed with the sup. norm* $\|\cdot\|_\infty$. *Let* $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$ *be the operator defined by* $M_\tau(\varphi)(z) := \int_{\mathrm{Rat}} \varphi(h(z))\, d\tau(h)$, $\forall\varphi \in C(\hat{\mathbb{C}}), \forall z \in \hat{\mathbb{C}}$. *Let* $\mathcal{U}_\tau$ *be the space of all finite linear combinations of unitary eigenvectors of* $M_\tau : C(\hat{\mathbb{C}}) \to C(\hat{\mathbb{C}})$, *where an eigenvector is said to be* **unitary** *if the absolute value of the corresponding eigenvalue is* 1. *Then,* $1 \leq \dim_{\mathbb{C}}\mathcal{U}_\tau < \infty$ *and*

$$C(\hat{\mathbb{C}}) = \mathcal{U}_\tau \oplus \{\varphi \in C(\hat{\mathbb{C}}) \mid M_\tau^n(\varphi) \to 0 \text{ as } n \to \infty\}.$$

Moreover, each $\varphi \in \mathcal{U}_\tau$ *is locally constant on* $F(G_\tau)$ *and is Hölder continuous on* $\hat{\mathbb{C}}$.

Remark 2. Theorem 1 describes **new phenomena** which **cannot hold in the usual iteration dynamics** of a single $h \in \text{Rat}$ with $\deg(h) \geq 2$.

Definition 2. Let $\tau \in \mathfrak{M}_1(Rat)$ be such that $supp\,\tau$ is compact. We say that τ is **mean stable** if there exist non-empty open subsets U, V of $F(G_\tau)$ and a number $n \in \mathbb{N}$ such that all of the following *(1)(2)(3)* hold. *(1)* $\overline{V} \subset U \subset F(G_\tau)$. *(2)* For all $\gamma = (\gamma_1, \gamma_2, \ldots) \in (supp\,\tau)^{\mathbb{N}}$, $(\gamma_n \circ \cdots \circ \gamma_1)(U) \subset V$. *(3)* For all $z \in \hat{\mathbb{C}}$, there exists an $h \in G_\tau$ such that $h(z) \in U$.

Definition 3. Let $\mathcal{Y}$ be a closed subset of *Rat*. Let $\mathfrak{M}_{1,c}(\mathcal{Y}) := \{\tau \in \mathfrak{M}_1(\mathcal{Y}) \mid supp\,\tau \text{ is compact}\}$. Let $\mathcal{O}$ be the topology in $\mathfrak{M}_{1,c}(\mathcal{Y})$ such that $\tau_n \to \tau$ in $(\mathfrak{M}_{1,c}(\mathcal{Y}), \mathcal{O})$ if and only if *(1)* $\int \varphi\, d\tau_n \to \int \varphi\, d\tau$ for each bounded continuous function $\varphi : \mathcal{Y} \to \mathbb{R}$, and *(2)* $supp\,\tau_n \to supp\,\tau$ with respect to the Hausdorff metric in the space of all non-empty compact subsets of $\mathcal{Y}$.

Theorem 3 (**Density of Mean Stable Systems).** *The set* $\{\tau \in \mathfrak{M}_{1,c}(\mathcal{P}) \mid \tau \textit{ is mean stable}\}$ *is* **open and dense** *in* $(\mathfrak{M}_{1,c}(\mathcal{P}), \mathcal{O})$.

Theorem 4 (**Stability).** *Suppose* $\tau \in \mathfrak{M}_{1,c}(\text{Rat})$ *is mean stable and* $J(G_\tau) \neq \emptyset$. *Then there exists a neighborhood* Ω *of* τ *in* $(\mathfrak{M}_{1,c}(\text{Rat}), \mathcal{O})$ *such that all of the following* (1)(2)(3) *hold.* (1) *For each* $\nu \in \Omega$, ν *is mean stable,* $J_{\ker}(G_\nu) = \emptyset$ *and* $J(G_\nu) \neq \emptyset$ (*thus* Theorem A *for* ν *holds*). (2) *The map* $\nu \mapsto \mathcal{U}_\nu$ *is continuous on* Ω. (3) *The map* $\nu \mapsto \sharp\text{Min}(G_\nu, \hat{\mathbb{C}})$ *is constant on* Ω.

Theorem 5 (**Bifurcation).** *For each* $t \in [0,1]$, *let* μ_t *be an element of* $\mathfrak{M}_{1,c}(\text{Rat}_+)$. *Suppose that all of the following* (1)–(4) *hold.* (1) $t \mapsto \mu_t \in (\mathfrak{M}_{1,c}(\text{Rat}_+), \mathcal{O})$ *is continuous on* $[0,1]$. (2) *If* $t_1, t_2 \in [0,1]$ *and* $t_1 < t_2$, *then* $\text{supp}\,\mu_{t_1} \subset \text{int}(\text{supp}\,\mu_{t_2})$ *with respect to the topology of* Rat_+. (3) $\text{int}(\text{supp}\,\mu_0) \neq \emptyset$ *and* $F(G_{\mu_1}) \neq \emptyset$. (4) $\sharp(\text{Min}(G_{\mu_0}, \hat{\mathbb{C}})) \neq \sharp(\text{Min}(G_{\mu_1}, \hat{\mathbb{C}}))$.

Let $B := \{t \in [0,1] \mid \mu_t \textit{ is not mean stable}\}$. *Then, we have all of the following* (a)(b)(c)(d).

- *(a) For each* $t \in [0,1]$, *we have* $J_{\ker}(G_{\mu_t}) = \emptyset$ *and* $J(G_{\mu_t}) \neq \emptyset$, *and all statements in Theorem A (with* $\tau = \mu_t$*) hold.*
- *(b)* $1 \leq \sharp(B \cap [0,1)) \leq \sharp\text{Min}(G_{\mu_0}, \hat{\mathbb{C}}) - \sharp\text{Min}(G_{\mu_1}, \hat{\mathbb{C}}) < \infty$.
- *(c) For each* $t \in [0,1] \setminus B$ *and for each* $L \in \text{Min}(G_{\mu_t}, \hat{\mathbb{C}})$, $L \subset F(G_{\mu_t})$ *and* L *is an "***attractor***" in a neighborhood of* L *for* G_{μ_t}.
- *(d) For each* $t \in B$, *there exists an* $L \in \text{Min}(G_{\mu_t}, \hat{\mathbb{C}})$ *such that either* (a) $L \cap J(G_{\mu_t}) \neq \emptyset$ *or* (b) $L \subset F(G_{\mu_t})$ *and* L *meets a Siegel disc or a Hermann ring of some element of* G_{μ_t}.

Summary and Remarks: (1) Regarding the random dynamics of polynomials, **generically**, the chaos of the averaged system **disappears** and the limit states are **stable** under perturbations of the system. (2) In order to prove the above result, we need the **classification of minimal sets**. (3) We can investigate the

bifurcation of the 1-parameter family of random complex dynamical systems. (4) There exist a lot of examples of $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ such that $J_{\ker}(G_\tau) = \emptyset$ (thus the chaos disappears) but τ is **not mean stable**. At such a τ, a kind of **bifurcation** occurs. (5) There exists an example of means stable $\tau \in \mathfrak{M}_{1,c}(\mathcal{P})$ with $\sharp\, \mathrm{supp}\, \tau < \infty$ such that there exists a $\varphi \in \mathcal{U}_\tau$ whose Hölder exponent is strictly less than 1 ("**Devil's Coliseum**", which is the function of probability of tending to ∞. To prove this result, we use ergodic theory and potential theory). Therefore, even if the chaos disappears in the "C^0" sense, the chaos may remain in the "C^1" sense (or in the space of Hölder continuous functions with some exponent $\alpha_0 < 1$). Thus, in random dynamics, we have a kind of gradation between non-chaos and choas. It is interesting to investigate the pointwise Hölder exponent of the above φ. The above φ is a continuous function on $\hat{\mathbb{C}}$ which varies precisely on the Julia set $J(G_\tau)$, which is a thin fractal set. Thus it is important to estimate the Hausdorf dimension $\dim_H(J(G_\tau))$ of $J(G_\tau)$. By using the thermodynamical formalisms, we can show that $\dim_H(J(G_\tau))$ is equal to the zero of the pressure function, under certain conditions. Also, in order to investigate the pointwise Hölder exponent of this function φ in detail, we can sometimes apply the thermodynamical formalisms. We are very interested in studying the poitwise-Hölder-exponent spectrum of this function $\varphi \in \mathcal{U}_\tau$.

References: (The content of this talk is included in the following list of references).

REFERENCES

[1] H. Sumi, *Random complex dynamics and semigroups of holomorphic maps*, Proc. London Math. Soc. (2011) 102(1), pp 50–112.

[2] H. Sumi, *Cooperation principle, stability and bifurcation in random complex dynamics*, preprint 2010, http://arxiv.org/abs/1008.3995.

Thermodynamical formalism for a modified shift map

STEVEN MUIR
(joint work with M. Urbański)

In a nutshell, our project was to introduce a transfer operator and use it to prove some theorems of a classical thermodynamical formalism flavor in a novel setting: the "alphabet" E is a compact metric space equipped with an *apriori* probability measure ν and an endomorphism T and the dynamical action S is defined on the product space $E^{\mathbb{N}}$ by the rule $(x_1, x_2, x_3, \dots) \mapsto (T(x_2), x_3, \dots)$. The greatest novelty is found in the variational principle, where a term must be added to the entropy to reflect the transformation of the first coordinate by T after shifting. Our motivation is that this system, in its full generality, cannot be treated by the existing methods of either rigorous statistical mechanics of lattice gases (where only the true shift action is used, see [3] or [7]) or dynamical systems theory (where the apriori measure is always implicitly taken to be the counting measure).

Now we shall provide precise definitions and statements of results. Let (E, d_0) be a compact metric space equipped "apriori" with a Borel probability measure $\nu : \mathcal{B}(E) \to [0, 1]$. Let $T : E \to E$ continuously and surjectively with the additional "not-too-contracting-at-short-range" property that there exist constants $\kappa > 0$ and $\delta > 0$ for which $d_0(a, b) < \delta$ implies $d_0(T(a), T(b)) \geq \kappa \, d_0(a, b)$. If $\kappa > 1$ then this property is called *distance expanding*, but distance expanding isn't required for any of our present theorems. From these assumptions it follows that T is a local homeomorphism. Therefore T preserves the Borel sets of E in *both* directions. Further assume that T is quasi-invariant with respect to ν in both directions, i.e. $\nu \circ T^{-1} << \nu$ and $\nu \circ T << \nu$. Unless T injects the function $\nu \circ T$ is not additive on the whole $B(E)$ (though it is subadditive). So the statement $\nu \circ T << \nu$ must be understood to mean that if $\nu(B) = 0$ then $\nu(T(B)) = 0$, too. This implies the Radon-Nikodym derivative $\frac{d\nu \circ T}{d\nu}$ can be defined at every point of E by restricting T to a local set on which it is a homeomorphism and therefore $\nu \circ T$ is a true measure. Then integration against the "global measure" $\nu \circ T$ can be defined via the density.

In the language of shift spaces, E serves as the *alphabet* or *state space* and the product space $X = E^{\mathbb{N}}$ serves as the (full) *shift space* or *configuration space*. For any $0 < q < 1$ the distance function $d = \frac{1-q}{q} \sum_{k=1}^{\infty} q^k \, d_0 \circ (\pi_k \times \pi_k)$, wherein π_k is the *kth* coordinate projection, makes X into a compact metric space (where the constant $\frac{1-q}{q}$ guarantees $diam(X) = diam(E)$) and the product measure $\nu^{\mathbb{N}}$ makes X into a Borel probability space. Our aim is to apply ideas of thermodynamical formalism to produce an invariant measure with good stochastic properties for the titular *modified shift map* $S : X \to X$ defined by $S(x_1, x_2, x_3, \dots) = (T(x_2), x_3, \dots)$. It is continuous under the metric d by the continuity of T under the metric d_0, but in the cases where (E, d_0) contains a proper limit point the map S cannot be expansive, much less distance expanding.

If $\phi : X \to \mathbb{R}$ continuously then the *transfer operator associated to* ϕ is the bounded linear operator mapping $C(X) \to C(X)$ by the rule

$$\mathfrak{L}f(x) = \int_{a \in E} \sum_{b \in T^{-1}(x_1)} f e^{\phi}(abx_2x_3 \dots) \, d\nu(a).$$

We call a function weighting a transfer operator a *potential function* and a function the transfer operator acts on an *observable function*.

For continuous potentials ϕ we use a standard kind of subadditivity argument to show that the *pressure* $p(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \mathfrak{L}^n \mathbb{1}(x)$ exists independently of x and then we use the Schauder-Tichonov fixed point theorem to show that there is a Borel probability measure γ on X for which $\mathfrak{L}^* \gamma = e^{p(\phi)} \gamma$.

Whenever $f : (X, d) \to (X', d')$ is a map between metric spaces we use the *modulus of continuity* notation $m(f, t) = \sup\{d'(f(x), f(y)) : d(x, y) \leq t\}$. and say f satisfies *Dini's condition* if $\sum_{n=0}^{\infty} m(f, q^n) < \infty$ (which is either true for all $0 < q < 1$ or false for all $0 < q < 1$).

For ϕ satisfying Dini's condition we let $\mathfrak{L}_0 = e^{-p(\phi)} \mathfrak{L}$ we show equicontinuity and uniform boundedness of the sequence of functions $\{\frac{1}{n} \sum_{m=0}^{n-1} \mathfrak{L}_0^m \mathbb{1}\}_{n \geq 1}$. By the Arzela-Ascoli theorem this sequence has a uniform limit point ρ. This function ρ is bounded away from 0 and is a fixed point of the operator $\mathfrak{L}_0$. Moreover the weighted measure $\rho\gamma$ is invariant under the modified shift S.

We define a *Gibbs measure* for $\phi \in C(X)$ as a Borel probability measure μ on X for which there exist constants $c_1, c_2 > 0$ such that for every $n \geq 1$, every set $A \in \mathcal{B}(E^n)$ with $\nu^n(A) > 0$, and every point $x \in X$

$$\frac{1}{c_1 c_2^n} \leq \frac{\mu([A])}{\mathfrak{L}^n \mathbb{1}_{[A]}(x)} \leq \frac{c_1}{c_2^n}.$$

Note that $\nu^n(A) = 0$ if and only if $\mathfrak{L}^n \mathbb{1}_{[A]}(x) = 0$, so there is no danger of dividing by 0 in this definition. We check that any Gibbs measure μ has projections $\mu \circ \pi_{1\ldots n}^{-1}$ equivalent to the apriore product measure ν^n.

We also see that if a Gibbs measure exists the the constant c_2 in the definition must be $= e^{p(\phi)}$, so that satisfying the estimates $c_1^{-1} \leq \left(\mathfrak{L}_0^n \mathbb{1}_{[A]}(x)\right)^{-1} \mu([A]) \leq c_1$ provides an equivalent, but cleaner looking definition of a Gibbs measure.

We show that the measure γ introduced above is a Gibbs measure for ϕ, and the normalized measure $\frac{1}{\rho\gamma(\mathbb{1})}\rho\gamma$ is an S invariant Gibbs measure for ϕ.

Using the martingale convergence theorem applied to sequences of Radon-Nikodym derivatives relative to finer and finer σ-algebras (as in [2] Theorem 35.7) we show that an S invariant Gibbs measure is totally ergodic. Because all Gibbs measures for a given potential are boundedly equivalent, this implies that a potential ϕ that satisfies Dini's condition has a unique S-invariant Gibbs measure η as described above.

Now we turn to a description of equilibrium measures and our variational principle. For an S invariant Borel probability measure μ let

$$H(\mu \circ \pi_{1\ldots n}^{-1} | \nu^n) = \mu\left(-\log\left(\frac{d\mu \circ \pi_{1\ldots n}^{-1}}{d\nu^n}\right) \circ \pi_{1\ldots n}\right)$$

if the n^{th} projection of μ is absolutely continuous with respect to ν^n, and $-\infty$, otherwise. Basic entropy theory (see [7], section III.4) states that

$$h(\mu|\nu^{\mathbb{N}}) \equiv \lim_{n\to\infty} \frac{1}{n} H(\mu \circ \pi_{1\ldots n}^{-1} | \nu^n) \in [-\infty, 0]$$

exists. At this point the thermodynamical formalism for the modified shift diverges from the usual formulation. We add a term involving the transfer of ν under T and call it the *modified entropy*:

$$h_S(\mu|\nu^{\mathbb{N}}) \equiv h(\mu|\nu^{\mathbb{N}}) + \int_E \log \frac{d\nu \circ T}{d\nu}\, d\mu \circ \pi_2^{-1}.$$

This entropy could be positive or negative, though we assume it to be $< +\infty$. Define the *modified Gibbs free energy* for ϕ as $G_\phi(\mu) \equiv \mu(\phi) + h_S(\mu|\nu^{\mathbb{N}})$. If G_ϕ attains its supremum at some measure μ, then we call μ an *equilibrium measure* for ϕ. Our variational principle can be stated as follows:

For all continuous $\phi : X \to \mathbb{R}$ the number $p(\phi)$ is an upper bound on G_ϕ, and if $\phi : X \to \mathbb{R}$ satisfies Dini's condition then $p(\phi) = \sup G_\phi$ and the S-invariant Gibbs state η for ϕ is an equilibrium state for ϕ. Moreover, if ϕ satisfies Dini's condition then η is the unique equilibrium measure for ϕ.

We show that this equilibrium measure varies continuously with the apriori measure ν, where measures are treated with the weak topology. The pressure $p(\phi)$ also varies continuously with ν.

The last set of results requires the stronger modulus of continuity requirement that ϕ be Hölder continuous. Under this assumption we derive probability laws for the stochastic process $\{f \circ S^n\}_{n\geq 1}$ on the probability space $(X, \mathcal{B}(X), \eta)$, where f is any other Hölder continuous "observable" function. For stochastic laws the operator $\mathfrak{L}_1 = \frac{1}{\rho}\mathfrak{L}_0\rho$ is useful. The method is to show that if ϕ is Hölder continuous then $\mathfrak{L}_1$ acts with the Romanian or two-norm inequality on the Banach subspace of Hölder continuous functions of any order less than or equal to the order of ϕ. This is the main conditon needed to invoke the famous Ionescu Tulcea - Marinescu theorem (of [5]) that provides a decomposition of $\mathfrak{L}_1$ into $Q + P$ where P is a projection orthogonal to Q and (using the fact that it's spectral radius is < 1) the norm of Q^n shrinks to 0 exponentially fast with n. Along the way we also see that $\mathfrak{L}_1^n(f) \to \eta(f)$ exponentially fast as $n \to \infty$.

After that we can quickly obtain that there is an exponential decay of correlations (under η) for any two Hölder observables f and g as they are right-composed with iterates of S whose difference goes to infinity. From this we can appeal to Gordin's old result (of [4]) to see that the central limit theorem holds for $\{f \circ S^n\}_{n\geq 1}$. Finally, in the cases where T is injective (and hence by the open mapping theorem a homeomorphism), we appeal to Philipps and Stout ([1], c.f. [6]) to conclude that the law of the iterated logarithm applies as well.

References

[1] Philipps, Stout: Almost sure invariance principles for partial sums of weakly dependent random variables. Mem. AMS 161 (1975)

[2] Billingsley, Patrick.: Probability and Measure. Wiley Interscience, (1995)

[3] Georgii, H-O.: Gibbs Measures and Phase Transitions. DeGruyter, (1981)

[4] Gordin, M.I.: The central limit theorem for stationary processes. Soviet Math Dokl. Vol. 10 (1969) No.5 p.1174-1176

[5] Ionescu Tulcea, C.T., Marinescu, G.: Théorie ergodique pour des classes d'opérations non complètement continues. Ann. of Math. (2) 52, 140-147 (1950)

[6] Przytycki, F., Urbański, M. Conformal Fractals - Ergodic Theory Methods. Cambridge University Press, (2009)

[7] Simon, B.: The Statistical Mechanics of Lattice Gases, Vol. I. Princeton University Press, Princeton (1993).

[8] Urbański, M., Zdunik, A.: Ergodic Theory for Holomorphic Endomorphisms of Complex Projective Spaces. Preprint (2009).

Geometric pressure for complex rational maps and multimodal maps of the interval

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(joint work with J. Rivera-Letelier)

We present some results obtained in a paper which will soon be submitted for publication. The aim is to transfer certain results from the theory of iterations of rational functions on the Riemann sphere ([6], [7], [8], [9]) to maps of the interval. In this transfer some parts are harder to prove than in the complex setting, since unlike in the complex setting, these maps need not be open (at turning critical points and at the end points).

In the recent, remarkable paper [2] the authors allow only a small interval for parameter t, which denotes the inverse of temperature, and assume additionally a weak growth of the absolute value of the derivatives of iterates at critical values. In our paper the latter assumption is not needed and our results on the analyticity of the geometric pressure as a function of the temperature and the equilibria hold for the whole interval of possible temperatures between the freezing and the condensation point.

Definition 1. We call a C^2 map $f : I \mapsto I$, for $I = [0, 1]$ the unit interval, *multimodal* if it is has only a finite number of non-flat critical points.

Consider an arbitrary forward f-invariant compact subset K of I, such that $f|_K$ is topologically transitive, the topological entropy $h_{\mathrm{top}}(f|_K)$ is positive, K is *weakly isolated* that is there exists $\epsilon > 0$ such that for every f-periodic orbit $O(p)$, if it is in $B(K, \epsilon)$ then it is in K, and else f satisfies Darboux property on K. The latter means that for every interval $T \subset I$ there exists an interval $T' \subset I$ (open, with one end, or with both ends) such that $f(T \cap K) = T' \cap f(K)$, compare [5]. Seemingly such maps satisfy *Hölder bounded distortion* condition, that is there exist $C, \alpha > 0$ such that for every intervals $T, S \subset I$ intersecting K such that there is $n > 0$ for which f^n maps diffeomorphically T onto S and T is short enough, for the interval $S' \subset S$ twice shorter with the same center and any $x, y \in S'$ for $g := (f|_T)^{-n}$ it holds $|g'(x)/g'(y) - 1| \leq C|x - y|^\alpha$. (This condition holds with $\alpha = 1$ for f having negative Schwarzian derivative; for smooth maps see e.g. [1]). We call such sets K, *basic sets*. Basic sets in the spectral decomposition, [4, Section 3.4], are examples.

Let $M(f, K)$ be the set of all probability measures supported on K that are invariant by f. For $\mu \in M(f, K)$ we denote by $h_\mu(f)$ the *measure theoretic entropy of* μ, and by $\chi_\mu(f) = \int \log |f'| d\mu$ the *Lyapunov exponent of* μ.

For every real number t we define the *pressure of* $f|_K$ *for the potential* $-t \log |f'|$ (called also *geometric pressure*), see [10], by

$$P(K, t) = \sup \{h_\mu(f) - t\chi_\mu(f) \mid \mu \in M(f, K)\}. \tag{1}$$

For each $t \in R$ we have $P(K, t) < +\infty$ since $\chi_\mu(f) \geq 0$ for each $\mu \in M(K, f)$. Sometimes we call $P(K, t)$ *variational geometric pressure* and denote it by $P_{\mathrm{var}}(K, t)$.

A measure μ is called an *equilibrium state of f on K for the potential* $-t\log|f'|$, if the supremum in (1) is attained at μ.

Define the numbers,

$$\chi_{\inf}(f,K) = \inf\{\chi_\mu(f) \mid \mu \in M(K,f)\},$$
$$\chi_{\sup}(f,K) = \sup\{\chi_\mu(f) \mid \mu \in M(K,f)\},$$
$$t_- = \inf\{t \in R \mid P(t) > -t\chi_{\sup}(f)\},\quad t_+ = \sup\{t \in R \mid P(t) > -t\chi_{\inf}(f)\}.$$

We have:

- $t_- < 0 < t_+$;
- for all $t \in R \setminus (t_-, t_+)$ we have $P(K,t) = \max\{-t\chi_{\sup}(f), -t\chi_{\inf}(f)\}$;
- for all $t \in (t_-, t_+)$ we have $P(K,t) > \max\{-t\chi_{\inf}(f), -t\chi_{\sup}(f)\}$.

Theorem 1. *Let $f : I \to I$ be a C^2 multimodal interval map and let K be its basic subset containing no parabolic periodic orbits. Then the following properties hold.*

Analyticity of the pressure function: *The pressure function $t \mapsto P(K,f)$ is real analytic on (t_-, t_+), and linear with slope $-\chi_{\sup}(f)$ (resp. $-\chi_{\inf}(f)$) on $(-\infty, t_-)$ (resp. $[t_+, +\infty)$).*

Conformal measure: *There is a unique $(t, P(K,t))$-conformal probability measure $\mu_{(K,t)}$ positive on open sets in K. Moreover, this measure is non-atomic, ergodic, and is supported on the conical limit set $K_{\rm con}(f)$.*

Equilibrium states: *For each $t \in (t_-, t_+)$ there is a unique equilibrium state of f for the potential $-t\log|f'|$. Furthermore, this measure is absolutely continuous with respect to $\mu_{(K,t)}$, with density bounded away from 0, ergodic and mixing and satisfies the CLT for Lipschitz gauge functions, provided $f|_K$ is topologically mixing.*

Note that without the topological transitivity assumption, the assertion concerning the analyticity of $t \mapsto P(K,t)$ can be false (see [3]).

The item **Conformal measure** needs some definitions:

$$K_{\rm con}(f) := \{x \in K \mid (\exists r > 0, n_j \to \infty, r_j \to 0)\ (\forall j)\ f^{n_j} \text{ maps}$$
$$K - \text{diffeomorphically } B(x, r_j) \text{ onto } B(f^{n_j}(x), r)\}.$$

K-diffeomorphically means $f^s(B(x,r_j))$ do capture neither critical points nor end points of K for $s = 0, ..., n_j - 1$. The term $(t, P(K,t))$-conformal means that for every Borel set $E \subset K$ on which f is injective $\mu(f(E)) = \exp(P(K,t)) \int_E |f'|^t d\mu$. A point $x \in K$ is called *an end point* of K if it is an end point of a maximal K-interval on which f is strictly monotone. K-interval means a set of the form $L \cap K$, where L is an interval.

Denote the set of all critical points and end points in K by $S(f,K)$. The idea of the proof is to find a *nice couple* of open sets $(V, \hat{V})$, that is unions of $(V^c, \hat{V}^c)$, where $\hat{V}^c$ containing the closure of V^c, each V^c contains exactly one point $c \in S(f,K)$, $V \supset S(f,K)$ and all V^c are pairwise disjoint, and to consider the induced map $F(x) := f^{m(x)}(x)$, where $m(x)$ is the first time for which $x \in V$

returns to V such that $f^{m(x)}$ on the component $f^{-m(x)}(\hat{V})$ containing x is a K-diffeomorphism onto a component of $\hat{V}$. (The number $m(x)$ need not be the first return time !). Then one finds conformal and F-invariant measures for F on $K(F)$, the limit set of the graph-directed Markov system F^{-1}, and spreads them to K. To make this formally correct one extends f to a neighbourhood of K in the real line if K contains 0 or 1.

Similarly to the complex case ([9], [10]) one can define and prove the following equalities of pressures

Theorem 2. *$P_{\mathrm{Per}}(K,t) = P_{\mathrm{tree}}(K,t) = P_{\mathrm{hyp}}(K,t) = P_{\mathrm{varhyp}}(K,t) = P_{\mathrm{var}}(K,t) = P_{\mathrm{conf}}(K,t)$.*

Remarks. In the complex case the first equality needs additional assumptions. $P_{\mathrm{tree}}(K,t) = P_{\mathrm{tree}}(K,t,z)$ for z safe (approached by $f^n(\mathrm{Crit}(f))$ not faster than subexponentially) and expanding. In the complex case the 'expanding' assumption (i.e. belonging to a hyperbolic subset) can be skipped. The last equality holds for $t_- < t < t_+$.

References

[1] H. Bruin, J. Rivera-Letelier, W. Shen, S. van Strien, *Large derivatives, backward contraction and invariant densities for interval maps*, Inventiones Mathematicae **172** (2008), 509–533.

[2] H. Bruin, M. Todd, *Equilibrium states for interval maps: the potential $-t\log|Df|$*, Ann. Scient. Éc. Norm. Sup. 4 serié **42** (2009), 559–600.

[3] N. Dobbs, *Renormalisation-induced phase transitions for unimodal maps*, Commun. Math. Phys. **286** (2009), 377–387.

[4] W. de Melo, S. van Strien, *One-dimensional Dynamics*, Springer-Verlag, 1994.

[5] M. Misiurewicz, W. Szlenk, *Entropy of piecewise monotone mappings*, Studia Math. **67** (1980), 45–63.

[6] F. Przytycki, J. Rivera-Letelier, and S. Smirnov, *Equivalence and topological invariance of conditions for non-uniform hyperbolicity in the iteration of rational maps*, Inventiones Mathematicae **151** (2003), 29–63.

[7] F. Przytycki, J. Rivera-Letelier, and S. Smirnov, *Equality of pressures for rational functions*, Ergodic Theory Dynam. Systems **24** (2004), 891–914.

[8] F. Przytycki, J. Rivera-Letelier, *Statistical properties of topological Collet-Eckmann maps*, Annales Scientifiques de l'École Normale Superieure 4^e série, **40** (2007), 135–178.

[9] F. Przytycki, J. Rivera-Letelier, *Nice inducing schemes and the thermodynamics of rational maps*, Communications in Math. Phys. **301.3** (2011), 661–707.

[10] F. Przytycki, M. Urbański, *Conformal Fractals: Ergodic Theory Methods.* London Math. Society: Lecture Notes Series, 371. Cambridge University Press, 2010.

On thermodynamic formalism for group extensions

Manuel Stadlbauer

In this talk, we study aspects of thermodynamic formalism for extensions of topological Markov chains by discrete groups. In the context of topological Markov chains, the dynamical system

$$T : X \times G \to X \times G,\ (x,g) \mapsto (\theta(x), g\psi(x))$$

is referred to as a group extension, where $\theta : \Sigma_A^+ \to \Sigma_A^+$ is a topologically mixing, topological Markov chain, G a discrete group G and $\psi : \Sigma_A^+ \to G$ a locally constant map. We now aim to relate amenability of G with equality of the Gurevič pressures of θ and T with respect to a given weakly Hölder continuous potential $\varphi : \Sigma_A^+ \to \mathbb{R}$ (or its canonical lift, respectively).

The motivation for the analysis of a possible change of pressure under group extensions stems from the attempt to relate two classical results from probability theory and geometry. The probabilistic result was obtained by Kesten in [5] and characterises amenability in terms of the spectral radius of the Markov operator associated to a symmetric random walk, that is a group G is amenable if and only if the spectral radius of the operator acting on $\ell^2(G)$ is equal to 1. The following counterpart in geometry was discovered by Brooks ([2]) using a completely different method. Assume that Γ is a Kleinian group acting on hyperbolic space $\mathbb{H}^{n+1}$ with exponent of convergence $\delta(G)$ bigger than $n/2$ and that $N \lhd \Gamma$ is a normal subgroup. Then the bottoms of the spectra of the Laplacians on $\mathbb{H}/\Gamma$ and $\mathbb{H}/N$ are equal if and only if Γ/N is amenable. Or equivalently, using the characterisation of the bottom of the spectrum in terms of the exponents of convergence, Γ/N is amenable if and only if $\delta(\Gamma) = \delta(N)$. More recently, these results were partially improved. Roblin ([8]) used conformal densities to prove that amenability implies $\delta(\Gamma) = \delta(N)$ if Γ is of divergence type and Sharp obtained in [9] the same statement for convex-cocompact Schottky groups using Grigorchuk's results on the co-growth of shortest representations (see [4]) applied to the Cayley graph of G.

In order to extend these results to group extensions, it is then necessary to introduce the following natural notion of symmetry for T. That is, we assume that ψ is constant on cylinders of length 1 and that there exists an involution κ acting on the alphabet A such that $\psi([a]) = \psi([\kappa a])^{-1}$. This involution then can be extended to finite words, leading to the notion of a weakly symmetric potential by requiring that there exists a sequence (D_n) with $\lim_{n\to\infty} D_n/n = 0$ such that

$$\sup_{x\in[w],y\in[\kappa w]} \left| \sum_{j=0}^{n-1} \left(\varphi \circ \theta^j(x) - \varphi \circ \theta^j(y)\right) \right| \le D_n,$$

for all $w \in \mathcal{W}^n$ with $\mathcal{W}^n$ referring to the words of length n. Note that this notion of symmetry is a natural object since applications of group extensions to random walks or geodesic flows always share this property. A random walk can be recovered by assuming that Σ_A is a symmetric full shift equipped with a locally constant, summable symmetric potential whereas the relation to the geodesic flow is obtained through a group extension of the coding map associated with the covered manifold (as, e.g., in [6]). Note that weak symmetry also allows to treat systems with some parabolicity without an additional inducing process. The main results in [10] relate pressures and amenability as follows.

Theorem 1. *Assume that $P_G(\theta) < \infty$ and T is transitive and weakly symmetric. Then $P_G(\theta) = P_G(T)$ if G is amenable.*

Theorem 2. *Assume that φ is a Hölder continuous and summable potential and Σ_A has the big images and preimages property. Then G is amenable if $P_G(\theta) = P_G(T)$.*

It is worth noting that also the proofs and not only the assumptions of the theorems above reveal a strong asymmetry. The proof of the first theorem relies on a straightforward application of Kesten's theorem by associating a symmetric random walk to T. Furthermore, the construction of this random walk is sufficiently stable to only require that the potential satisfies a weak bounded distortion property as defined above.

The proof of the second theorem then is more involved and relies on a careful analysis of the action of the Ruelle operator on the embedding of $\ell^2(G)$ into a certain subspace of $C(\Sigma_A \times G)$. Namely, if the spectral radius is equal to 1, then it is possible to prove that the embedded space contains almost eigenfunctions for the eigenvalue 1, even though this space is not invariant under the operator. The result then essentially follows from an argument of Day (see [3]) using uniform rotundity.

These two theorems have the following immediate implications. Assume that $\mathcal{G} := \{g_1^{\pm 1}, g_2^{\pm 1}, \ldots\}$ is an at most countable set of generators of G and P is a symmetric probability measure on $\mathcal{G}$. Then Kesten's theorem follows by applying the theorems to the full shift with alphabet $\mathcal{G}$ and $\varphi|_{[g]} := \log P(g)$. Furthermore, the results on co-growth of groups by Grigorchuk and Cohen (see [4]) can be recovered by considering the case $|\mathcal{G}| < \infty$, $\varphi = 0$ and the subshift of finite type where the only forbidden transitions are gg^{-1}, for all $g \in \mathcal{G}$.

Also note that both theorems can be rewritten in terms of Gibbs-Markov maps as implicitly introduced in [1]. So assume that (Σ_A, θ, μ) is a topologically mixing Gibbs-Markov map with big images and preimages and that (X, T) is a topologically transitive G-extension. It then follows from Theorem 2 that G is amenable if

$$\limsup_{n\to\infty} \frac{1}{n} \log \big(\mu(\{x \in \Sigma_A :\ \psi(x) \circ \cdots \circ \psi(\theta^{n-1}(x)) = \mathrm{id}\}\big) = 0. \tag{1}$$

If in addition the potential $d\mu/d\mu\circ\theta$ is weakly symmetric, then Theorem 1 implies that condition (1) is equivalent to amenability. Note that this might also be seen as an extension of Kesten's result to random walks on groups with stationary increments.

Furthermore, there is an application to hyperbolic geometry through the coding of the geodesic flow for an essentially free Kleinian group Γ (see [11]). Note that this class of groups contains all geometrically finite Fuchsian groups and could be described in higher dimensions as Schottky-type groups with parabolic elements of arbitrary rank. Then, for a normal subgroup $N \lhd \Gamma$, we have $\delta(N) = \delta(\Gamma)$ if and only if Γ/N is amenable (see [10]).

References

[1] J. Aaronson, M. Denker, and M. Urbański. Ergodic theory for Markov fibred systems and parabolic rational maps. *Trans. Am. Math. Soc.*, 337(2):495–548, 1993.

[2] R. Brooks. The bottom of the spectrum of a Riemannian covering. *J. Reine Angew. Math.*, 357:101–114, 1985.
[3] M. M. Day. Convolutions, means, and spectra. *Illinois J. Math.*, 8:100–111, 1964.
[4] R. I. Grigorchuk. Symmetrical random walks on discrete groups. In *Multicomponent random systems*, volume 6 of *Adv. Probab. Related Topics*, pages 285–325. Dekker, New York, 1980.
[5] H. Kesten. Full Banach mean values on countable groups. *Math. Scand.*, 7:146–156, 1959.
[6] F. Ledrappier and O. Sarig. Fluctuations of ergodic sums for horocycle flows on $\mathbb{Z}^d$-covers of finite volume surfaces. *Discrete Contin. Dyn. Syst.*, 22(1-2):247–325, 2008.
[7] R. Ortner and W. Woess. Non-backtracking random walks and cogrowth of graphs. *Can. J. Math.*, 59(4):828–844, 2007.
[8] T. Roblin. Un théorème de Fatou pour les densités conformes avec applications aux revêtements galoisiens en courbure négative.. *Isr. J. Math.*, 147:333–357, 2005.
[9] R. Sharp. Critical exponents for groups of isometries. *Geom. Dedicata*, 125:63–74, 2007.
[10] M. Stadlbauer An extension of a theorem of Kesten to topological Markov chains. *Preprint*, arXiv:1110.2942.
[11] M. Stadlbauer and B. O. Stratmann. Infinite ergodic theory for Kleinian groups. *Ergod. Th. Dynam. Sys.*, 25(4):1305–1323, 2005.

Regularity and irregularity of fiber dimensions in non-autonomous dynamics

Volker Mayer
(joint work with B. Skorulski and M. Urbański)

This talk concerns non-autonomous dynamics of rational functions, and, in particular, the variation of the fractal dimensions of the Julia set under perturbation of a non-autonomous hyperbolic maps.

The deterministic hyperbolic case is completely understood by now. Indeed in 1979, R. Bowen [1] showed that the Hausdorff dimension of the Julia set can be expressed by the zero of a pressure function. There are generalizations of this formula to various contexts and we also establish a corresponding formula in the non-autonomous case. D. Ruelle [2] completed the deterministic picture in establishing real analytic dependence of the dimension for hyperbolic Julia sets. This fact has been generalized recently by H. Rugh [3] to random repellers.

Let $\mathcal{F} = \{f_\tau\,;\ \tau \in \Lambda_0\}$ be a family of rational functions depending analytically on a parameter $\tau \in \Lambda_0$, Λ_0 being some open subset of $\mathbb{C}^d$, $d \geq 1$. We investigate the dynamics of functions

$$f_{\lambda_n} \circ f_{\lambda_{n-1}} \circ ... \circ f_{\lambda_1} \quad , \quad n \geq 1\,.$$

Hence the dynamics and the corresponding Julia sets $\mathcal{J}_\lambda$ depend on the arbitrary choice of $\lambda = (\lambda_1, \lambda_2, ..) \in \Lambda = \Lambda_0^{\mathbb{N}}$. Such a dynamical system is called *non-autonomous*. In our work we first provide an if and only if condition for holomorphic stability of non-autonomous maps. As a corollary we get Hölder continuity of the dimensions for hyperbolic non-autonomous Julia sets.

Theorem 1. *Suppose Λ is equipped with the sup-topology $\|.\|_\infty$ and suppose that $\eta \in \Lambda$ is a hyperbolic (in fact stable) parameter. Then $\lambda \mapsto \mathcal{J}_\lambda$ is continuous and*

$$\lambda \mapsto Hdim(\mathcal{J}_\lambda) \quad \textit{and} \quad \lambda \mapsto Pdim(\mathcal{J}_\lambda)$$

are Hölder continuous in some neighborhood of η with Hölder exponent

$$\alpha = \alpha(\lambda) \to 1 \quad \textit{if} \quad \lambda \to \eta\,.$$

But, contrary to the deterministic and random case, the dimension functions are not smooth. Indeed, inspecting a particular family greater in detail, we show that the dimension functions are not differentiable at any point.

Theorem 2. *Consider the quadratic family $\mathcal{F} = \left\{ f_\tau(z) = \frac{\tau}{2}(z^2-1)+1\,,\ \ \tau \in \Lambda_0 \right\}$ where $\Lambda_0 = \{|\tau| > 40\}$ and let $\Lambda = \Lambda_0^{\mathbb{N}}$ be equipped with the sup-topology. Then $\Lambda = \Lambda^{hyp}$ and the functions*

$$\lambda \mapsto Hdim(\mathcal{J}_\lambda) \quad \textit{and} \quad \lambda \mapsto Pdim(\mathcal{J}_\lambda)$$

are not differentiable at any point $\eta \in \Lambda$.

The following result completes the picture of the rather intriguing properties of the finer fractal structure of the Julia sets related to this particular family.

Theorem 3. *There exist an open and dense set $\Omega \subset \Lambda$ for the sup-topology such that*

$$Hdim(\mathcal{J}_\lambda) < Pdim(\mathcal{J}_\lambda) \quad \textit{for every} \quad \lambda \in \Omega\,.$$

References

[1] Rufus Bowen. Hausdorff dimension of quasicircles. *Inst. Hautes Études Sci. Publ. Math.*, (50):11–25, 1979.

[2] David Ruelle. Repellers for real analytic maps. *Ergodic Theory Dynamical Systems*, 2(1):99–107, 1982.

[3] Hans Henrik Rugh. On the dimension of conformal repellors. randomness and parameter dependency. Preprint 2005, Ann of Math., vol 168, no 3, 695-748 (2008).

The Schottky-Klein function and Fenchel double crosses

Mark Pollicott

Let C_i, C_i' for $i = 1, \cdots, d$ be finite sets of disjoint circles in $\mathbb{C}$ with disjoint interiors. Let $g_i : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the Möbius transform which maps the interior C_i to the exterior of C_i'. The associated Schottky group $\Gamma = \langle g_1, \cdots, g_d \rangle$ is the free group they generate. We denote by F the Fundamental region in the complex plane exterior to all of the circles. We can then write $\widehat{\mathbb{C}} = \cup_g^d gF$.

Definition 1. The Schotty-Klein prime function is given by

$$w(z,\xi) := (z-\xi) \prod_{g \in \Gamma_0} R(gz, g\xi, \xi, z)$$

where

$$R(gz, g\xi, \xi, z) = \frac{(gz-\xi)(g\xi-z)}{(gz-z)(g\xi-\xi)}. \tag{1.1}$$

are cross ratios and $z, \xi \in F$ whenever it converges, where the product is restricted to those elements of the Schottky group generated by $g_1, \cdots, g_k$, excluding the identity and taking only an element or its inverse (but not both).

The function is a classical object, although interest was recently revived due to its application to generalizations of the classical Chrstoffel-Schwarz theorem.

A basic problem to address is whether the product defining $w(z, \xi)$ actually converges. In particular, convergence of the series (to a non-zero value) easily follows providing these converge to 1 quickly enough. For example, if we denote

$$M(x) := \text{Card}\left\{g \in \Gamma :\ |R(gz, g\xi, \xi, z)| \not\in \left[1 + \frac{1}{x^2}, 1 - \frac{1}{x^2}\right]\right\}$$

then convergence of $w(z, \xi)$ would follow easily follow from

$$d := \limsup_{x \to +\infty} \frac{\log M(x)}{\log x} < 1 \tag{1.2}$$

The problem of convergence was discussed by Baker in his 1897 book "Abelian functions", Schottky in an 1887 article and the famous book of Fricke and Klein. Heuristically, there is convergence if the circles are not too close together. It is more convenient to formulate this in modern parlance in terms of the limit set $\Lambda = \Lambda(\Gamma)$ (i.e., accumulation points of $\{g0 :\ g \in \Gamma\}$). In particular, we can identify $d = \dim_H(\Lambda)$, the Hausdorff dimension of Λ.

The following result is neither surprising nor difficult to prove.

Theorem 1. *The product converges whenever the dimension d of the limit set is strictly smaller than* 1.

However, we have the following more delicate result.

Theorem 2. *There exists $C > 0$ such that*

$$M(T) \sim CT^d \text{ as } T \to +\infty.$$

(i.e., $\lim_{T \to +\infty} \frac{M(T)}{T^d} = C$).

This can be nicely interpreted in terms of the geometry of hyperbolic three manifolds in terms of the distance between a geodesic in hyperbolic space and its images under the Schottky group, as we will now explain.

1. The hyperbolic circle problem and counting double crosses

We begin with a classical result.

1.1. **Classical circle problem.** Let $\Gamma = \langle g_1, \cdots, g_d \rangle$ be a Schottky group. We let

$$\mathbb{H}^3 = \{z + jt :\ z = x + iy \in \mathbb{C}, t > 0\}$$

denote the upper half-space with metric $ds^2 = (dx^2 + dy^2 + dt^2)/t^2$. The group Γ acts on $\mathbb{H}^3$ by linear fractional transformations.

Definition 2. Let $p_0 \in \mathbb{H}^3$. We denote $N_0(T) = \text{Card}\{g \in \Gamma :\ d(gp_0, p_0) \le T\}$ for $T > 0$.

We recall the hyperbolic circle problem for the orbit of the reference point $p_0 \in \mathbb{H}^3$.

Theorem 3 (Hyperbolic circle problem)**.** *There exists* $C > 0$ *such that*

$$N_0(T) \sim Ce^{\delta T} \text{ as } T \to +\infty.$$

We would next like to consider instead the orbits of geodesics rather than of points.

1.2. **Counting double crosses.** We can consider a geodesic γ in $\mathbb{H}^3$ with end points $z_1 = \gamma(-\infty)$ and $z_2 = \gamma(+\infty)$ in $\mathbb{C}$. Given $g \in \Gamma$ we can consider the image $g\gamma$, which is again a geodesic, this time with endpoints $z_3 = g\gamma(-\infty)$ and $z_4 = g\gamma(+\infty)$. We denote the distance between γ and $g\gamma$ by

$$d(\gamma, g\gamma) = \inf_{t_1, t_2 \in \mathbb{R}} d(\gamma(t_1), g\gamma(t_2)).$$

Definition 3. We denote $N(T) = \text{Card}\{g \in \Gamma : d(\gamma, g\gamma) \leq T\}$ for $T > 0$.

We have the following analogue of the previous theorem.

Theorem 4 (Hyperbolic cross problem)**.** *There exists* $C > 0$ *such that* $N(T) \sim Ce^{dT}$ *as* $T \to +\infty$.

1.3. **Cross ratios and double crosses.** Following Fenchel and Ahlfors, the connection between the cross ratio and the geometry is the following.

Lemma 5. $|R(z_1, z_2, z_3, z_4)| = \tanh(d(\gamma, g\gamma))$

This gives rise to Theorem 4 as a corollary:

Corollary 6 (Theorem 4)**.** *There exists* $C > 0$ *such that*

$$M(T) \sim CT^d$$

as $T \to +\infty$.

2. Idea of the proof of Theorem 4

As usual, the points in the limit set can be coded by infinite sequences of generators $(x_n)_{n=0}^{\infty}$ in s shift space. Following an approach of Lalley, we can augment the symbol space (of generators) with an extra symbol 0 to get a one sided subshift of finite type $\sigma : \Sigma \to \Sigma$. Moreover, we can associate a Hölder continuous function $r : \Sigma \to \mathbb{R}$ such that the natural complex function

$$\eta(s) = \sum_{g \in \Gamma} e^{-sd(\gamma, g\gamma)},$$

which converges for $Re(s) > d$, can written as

$$\eta(s) = \sum_{n=0}^{\infty} \mathcal{L}_{sr}^n 1(\dot{0}),$$

where $\mathcal{L}_{sr} : C^\alpha(\Sigma) \to C^\alpha(\Sigma)$ is the usual transfer operator defined by

$$\mathcal{L}_{sr} w(x) = \sum_{\sigma y = x} e^{-sr(y)} w(y), \text{ for } w \in C^\alpha(\Sigma).$$

The usual spectral properties of the operators $\mathcal{L}_{sr}$ show that $\eta(s)$ has a simple pole at $s = d$ and an analytic extension to a neighbourhood of $\{s \in \mathbb{C} : Re(s) = d\}\backslash\{d\}$. The asymptotic result follows by standard tauberian methods.

A variation of Bowen's Formula for (non-necessarily irreducible) graph directed Markov systems

Mario Roy

The last 25 years have been a period of extensive study for finite conformal iterated function systems (abbreviated to CIFSs). Over the last 15 years, Mauldin and Urbański have, in addition to making new discoveries, generalized many results to infinite CIFSs (see [2] and [3], for instance) and later extended the theory to finite and infinite conformal graph directed Markov systems (CGDMSs) (see [4]). In their work, they investigated all CGDMSs that are finitely irreducible, that is, whose irreducibility can be witnessed by a finite set of words. Recently, Ghenciu and Mauldin [1] have studied systems that are irreducible, though not finitely irreducible (see also [5] for earlier results). In particular, they showed that the Bowen formula that Mauldin and Urbański derived for finitely irreducible systems does extend to all irreducible systems. They also proved that this formula holds for all finite systems, irreducible and non-irreducible alike. However, there are some simple, infinite systems that do not obey Bowen's formula. Accordingly, Ghenciu and Mauldin raised questions that remained open until now (see [1], page 22). Let me reformulate the main one and add one:

(1) Does Bowen's formula hold for some "large" classes of non-irreducible systems?
(2) Is there a variation of Bowen's formula whose scope encompasses an even broader class of systems?

The aim of this talk is to answer those questions, and propose a variation of Bowen's formula (for a more complete exposé, see [6]). This variation boils down to the natural idea of replacing the pressure function of the entire system by the supremum of the pressure functions of its strongly connected components. This variation applies to a large collection of non-irreducible systems and is shown to coincide with the original Bowen formula which holds for all irreducible systems. More precisely, our variation holds for all CGDMSs whose strongly connected components form chains that each have a maximal element (maximal in terms of a natural partial ordering of the strongly connected components) and which do not admit infinite words consisting only of isolated edges. These CGDMSs form a wide class which not only comprises all irreducible CGDMSs (which consist of a single strongly connected component without isolated edges), but also all conformal graph directed systems (CGDSs).

We further show that the original version of Bowen's formula does not hold for some classes of non-irreducible systems, including some CGDMSs whose strongly connected components are IFSs. By investigating more carefully the pressure function, we show that the isolated edges of some non-irreducible systems may create disruption on their own. Indeed, though they have no impact on the Hausdorff dimension of the limit set, they may catapult the pressure of the system to infinity for some parameter values. For this reason, the classical Bowen formula sometimes does not hold. We present a simple example of the occurrence of such a situation. Given that the system in question has infinitely many isolated edges, one might be tempted to believe that finitely many isolated edges cannot create such a problem. In a subsequent example, we show that even the association of a unique isolated edge with a strongly connected component which is not finitely irreducible can propel the pressure to infinity for some parameters and make the classical Bowen formula break apart.

References

[1] A. Ghenciu and R.D. Mauldin, *Conformal graph directed Markov systems*, preprint, arXiv:0711.1182v1.

[2] R. D. Mauldin and M. Urbański, *Dimensions and measures in infinite iterated function systems*, Proc. London Math. Soc. **73** (3) (1996), no. 1, 105–154.

[3] R. D. Mauldin and M. Urbański, *Conformal iterated function systems with applications to the geometry of continued fractions*, Trans. Amer. Math. Soc. **351** (1999), 4995–5025.

[4] R. D. Mauldin and M. Urbański, *Graph Directed Markov Systems: Geometry and Dynamics of Limit Sets*, Cambridge 2003.

[5] R. D. Mauldin and S. C. Williams, *Hausdorff dimension in graph directed constructions*, Trans. Amer. Math. Soc. **309** (2) (1988), 811–829.

[6] M. Roy, *A new variation of Bowen's formula for graph directed Markov systems*, to appear in Discrete and Continuous Dynamical Systems.

Distance expanding random maps, thermodynamical formalism, Gibbs measures, and fractal geometry

Mariusz Urbański

In this monograph we develop the thermodynamical formalism for *measurable expanding random mappings.* This theory is then applied in the context of conformal expanding random mappings where we deal with the fractal geometry of fibers.

Distance expanding maps have been introduced for the first time in Ruelle's monograph "Thermodynamic Formalism". A systematic account of the dynamics of such maps, including the thermodynamical formalism and the multifractal analysis, can be found in the book by Prztycki and Urbański. One of the main features of this class of maps is that their definition does not require any differentiability or smoothness condition. Distance expanding maps comprise symbol systems and expanding maps of smooth manifolds but go far beyond. This is also a characteristic feature of our approach.

We first define measurable expanding random maps. The randomness is modeled by an invertible ergodic transformation θ of a probability space $(X, \mathcal{B}, m)$. We investigate the dynamics of compositions

$$T_x^n = T_{\theta^{n-1}(x)} \circ ... \circ T_x \ \ , \ n \geq 1,$$

where the $T_x : \mathcal{J}_x \to \mathcal{J}_{\theta(x)}$ $(x \in X)$ is a distance expanding mapping. These maps are only supposed to be measurably expanding in the sense that their expanding constant is measurable and a.e. $\gamma_x > 1$ or $\int \log \gamma_x \, dm(x) > 0$.

In so general setting we build the thermodynamical formalism for arbitrary Hölder continuous potentials φ_x. We show, in particular, the existence, uniqueness and ergodicity of a family of *Gibbs measures* $\{\nu_x\}_{x\in X}$. Following ideas of Kifer, these measures are first produced in a pointwise manner and then we carefully check their measurability. Often in the literature all fibres are contained in one and the same compact metric space and symbolic dynamics plays a prominet role. Our approach does not require the fibres to be contained in one metric space neither we need any Markov partitions or, even auxiliary, symbol dynamics.

Our results contain those Bogenschütz, Gundlach and Kiefer on related topics.

Throughout the entire monograph where it is possible we avoid, in hypotheses, absolute constants. Our feeling is that in the context of random systems all (or at least as many as possible) absolute constants appearing in deterministic systems should become measurable functions. With this respect the thermodynamical formalism developed in here represents also, up to our knowledge, new achievements in the theory of random symbol dynamics or smooth expanding random maps acting on Riemannian manifolds.

Unlike recent trends aiming to employ the method of Hilbert metric our approach to the thermodynamical formalism stems primarily from the classical method presented by Bowen and Kifer. Developing it in the context of random dynamical systems we demonstrate that it works well and does not lead to too complicated (at least to our taste) technicalities. The measurability issue mentioned above results from convergence of the Perron-Frobenius operators. We show that this convergence is exponential, which implies exponential decay of correlations. These results precede investigations of a pressure function $x \mapsto P_x(\phi)$ which satisfies the property

$$\nu_{\theta(x)}(T_x(A)) = e^{P_x(\phi)} \int_A e^{-\phi_x} d\nu_x$$

where A is any measurable set such that $T_x|_A$ is injective. The integral, against the measure m on the base X, of this function is a central parameter $\mathcal{E}P(\varphi)$ of random systems called the *expected pressure*. If the potential ϕ depends analytically on parameters, we show that the expected pressure also behaves real analytically. We would like to mention that, contrary to the deterministic case, the spectral gap methods do not work in the random setting. Our proof utilizes the concept of complex cones introduced by Rugh, and this is the only place, where we use the projective metric.

We then apply the above results mainly to investigate fractal properties of fibers of *conformal random systems*. They include Hausdorff dimension, Hausdorff and packing measures, as well as multifractal analysis. First, we establish a version of Bowen's formula (obtained in a somewhat different context by Bogenschütz) showing that the Hausdorff dimension of almost every fiber $\mathcal{J}_x$ is equal to h, the only zero of the expected pressure $\mathcal{E}P(\phi_t)$, where $\phi_t = -t \log |f'|$ and $t \in \mathbb{R}$. Then we analyze the behavior of h–dimensional Hausdorff and packing measures. It turned out that the random dynamical systems split into two categories. Systems from the first category, rather exceptional, behave like deterministic systems. We call them, therefore, *quasi-deterministic*. For them the Hausdorff and packing measures are finite and positive. Other systems, called *essentially random*, are rather generic. For them the h–dimensional Hausdorff measure vanishes while the h-packing measure is infinite. This, in particular, refutes the conjecture stated by Bogenschütz and Ochs that the h–dimensional Hausdorff measure of fibers is always positive and finite. In fact, the distinction between the quasi-deterministic and the essentially random systems is determined by the behavior of the Birkhoff sums

$$P_x^n(\phi) = P_{\theta^{n-1}(x)}(\phi) + ... + P_x(\phi)$$

of the pressure function for potential $\phi_h = -h \log |f'|$. If these sums stay bounded then we are in the quasi-deterministic case. On the other hand, if these sums are neither bounded below nor above, the system is called essentially random. The behavior of P_x^n, being random variables defined on X, the base map for our skew product map, is often governed by stochastic theorems such as the law of the iterated logarithm whenever it holds. This is the case for our primary examples, namely conformal DG-systems and classical conformal random systems. We are then in position to state that the quasi-deterministic systems correspond to rather exceptional case where the asymptotic variance $\sigma^2 = 0$. Otherwise the system is essential.

The fact that Hausdorff measures in the Hausdorff dimension vanish has further striking geometric consequences. Namely, almost all fibers of an essential conformal random system are not bi-Lipschitz equivalent to any fiber of any quasi-deterministic or deterministic conformal expanding system. In consequence almost every fiber of an essentially random system is not a geometric circle nor even a piecewise analytic curve. We then show that these results do hold for many explicit random dynamical systems, such as conformal DG-systems, classical conformal random systems, and, perhaps most importantly, Brück and Büger polynomial systems. As a consequence of the techniques we have developed, we positively answer the questions of Brück and Büger of whether the Hausdorff dimension of almost all naturally defined random Julia set is strictly larger than 1. We also show that in this same setting the Hausdorff dimension of almost all Julia sets is strictly less than 2.

Concerning the multifractal spectrum of Gibbs measures on fibers, we show that the multifractal formalism is valid, i.e. the multifractal spectrum is Legendre conjugated to a temperature function. As usual, the temperature function is implicitly given in terms of the expected pressure. Here, the most important, although perhaps not most strikingly visible, issue is to make sure that there exists a set X_{ma} of full measure in the base such that the multifractal formalism works for all $x \in X_{ma}$.

If the system is in addition uniformly expanding then we provide real analyticity of the pressure function. This part is based on work by Rugh's work and it is the only place where we work with the Hilbert metric. As a consequence and via Legendre transformation we obtain real analyticity of the multifractal spectrum.

Hyperbolic dynamics on folded saddle sets

Eugen Mihailescu

We are concerned with the case when f is a smooth endomorphism on a manifold M, having a *basic set with overlaps* $\Lambda \subset M$. We shall assume also that the endomorphism f is *hyperbolic* on Λ; however in this non-invertible case the unstable directions depend on full backwards trajectories $\hat{x} \in \hat{\Lambda}$ ([14], [15], [4]). The map f is not assumed expanding on Λ. Through a given point x from Λ there may pass many (even uncountably many) local unstable manifolds corresponding to different prehistories of x in the natural extension $\hat{\Lambda}$; these unstable manifolds may intersect also outside Λ. In general in this non-invertible case we *do not* have a Markov partition on Λ.

The non-invertible case has different techniques and phenomena than the diffeomorphism or the expanding cases (see [1], [11], [6] and the references there). For instance in this case the stable dimension does *not* vary continuously on perturbation. There are many examples of interesting and/or unexpected dynamical behaviour for endomorphisms, for instance: examples of perturbations of linear toral endomorphisms, for which the unstable manifolds through a given point depend on prehistories (Przytycki, [14]); horseshoes with overlaps (Bothe, [1]); hyperbolic non-expanding fractal repellers with overlaps (Mihailescu,[5]); skew products with overlaps, having in fibers Cantor sets of points with uncountably many unstable directions associated to them (Mihailescu, [4]); examples from higher dimensional complex dynamics, etc.

For equilibrium measures on folded basic sets we have the following approximation with discrete measures supported on those n-preimages remaining in Λ (notice that Λ is *not* totally invariant):

Theorem 1 ([8]). *Let $f : M \to M$ be a smooth map (say $\mathcal{C}^2$) on a Riemannian manifold M, so that f is hyperbolic on a saddle basic set Λ; assume also that the critical set $\mathcal{C}_f$ of f does not intersect Λ. Let ϕ be a Hölder continous potential on Λ and μ_ϕ the equilibrium measure of ϕ on Λ. Then $\forall g \in \mathcal{C}(\Lambda, \mathbb{R})$,*

$$\int_\Lambda | < \frac{1}{n} \sum_{y \in f^{-n}(x)\cap\Lambda} \frac{e^{S_n\phi(y)}}{\sum\limits_{z\in f^{-n}(x)\cap\Lambda} e^{S_n\phi(z)}} \cdot \sum_{i=0}^{n-1} \delta_{f^i y} - \mu_\phi, g > | d\mu_\phi(x) \underset{n\to\infty}{\longrightarrow} 0.$$

Let now Λ be a connected hyperbolic *repeller* for a smooth endomorphism $f : M \to M$ defined on a Riemannian manifold M, and assume f has no critical points in Λ. Let V be a neighbourhood of Λ in M and for any $z \in V$ define the measures

$$\mu_n^z := \frac{1}{n} \sum_{y\in f^{-n}(z)\cap V} \frac{1}{d(f(y))\dots d(f^n(y))} \sum_{i=1}^{n} \delta_{f^i y},$$

where $d(y)$ is the number of f-preimages belonging to V of a point $y \in V$. Then we proved in [5] that there exists an f-invariant measure μ^- on Λ, a neighbourhood V of Λ and a borelian set $A \subset V$ with $m(V \setminus A) = 0$ (where m is the Lebesgue measure on M) and a subsequence $n_k \to \infty$ s.t for any $z \in A$, $\mu^z_{n_k} \underset{k\to\infty}{\to} \mu^-$. The measure μ^- is called the *inverse SRB measure* of the non-invertible hyperbolic repeller. We showed that μ^- is the equilibrium measure of the stable potential $\Phi^s(x) := \log|\det Df_s(x)|, x \in \Lambda$, with respect to f. The difficulty is that f is non-invertible, hence μ^- is **not** simply the SRB measure for the inverse f^{-1} (the inverse does not even exist). We proved that μ^- is the unique f-invariant measure μ satisfying an inverse Pesin entropy formula; if f is d-to-1 on Λ we have:

$$h_{\mu^-}(f) = \log d - \sum_{i,\lambda_i(\mu^-)<0} \lambda_i(\mu^-) m_i(\mu^-).$$

Theorem 2 (Mihailescu, arxiv.org 2011)**.** *Let f be a hyperbolic toral endomorphism on $\mathbb{T}^m, m \geq 2$ given by an integer-valued matrix A without zero eigenvalues, and let g be a $\mathcal{C}^1$ perturbation of f. Consider μ_g^- the inverse SRB measure of g and μ_g^+ the (usual forward) SRB measure. Then the entropy production $e_g(\mu_g^-)$ of μ_g^- and the entropy production $e_g(\mu_g^+)$ satisfy the following:*
a) $e_g(\mu_g^-) \leq 0$ and $F_g(\mu_g^-) = \log d$. Moreover $e_g(\mu_g^+) \geq 0$.
b) $e_g(\mu_g^-) = 0$ if and only if $|detDg|$ is cohomologous to a constant on $\mathbb{T}^m$.

One can find a measurable partition ξ of Λ, subordinated to the stable manifolds W^s and can define the *lower pointwise stable dimension* of μ as $\underline{\delta}^s(\mu, x, \xi) := \liminf_{r\to 0} \frac{\log \mu_x^\xi(B(x,r))}{\log r}$, where $\{\mu_x^\xi\}_x$ is the system of conditional measures of μ associated to ξ. Similarly define $\overline{\delta}^s(\mu, x, \xi)$.

Theorem 3 ([7])**.** *Let $f : M \to M$ be a smooth endomorphism which is hyperbolic on a saddle basic set Λ and conformal on its stable manifolds. Assume f is d-to-1 on Λ and let $\Phi^s(y) := \log|Df_s(y)|, y \in \Lambda$, δ^s the stable dimension of Λ, and μ_s the equilibrium measure of $\delta^s\Phi^s$. Then the conditional measures $\mu^s_{s,A}$ of μ_s associated to ξ are geometric probabilities, i.e for $(\mu_s)_\xi$-a.a $\pi_\xi(A)$ of Λ/ξ there is a constant $C_A > 0$ such that*

$$C_A^{-1}\rho^{\delta^s} \leq \mu^s_{s,A}(B(y,\rho)) \leq C_A\rho^{\delta^s}, y \in A \cap \Lambda, 0 < \rho < \frac{r(A)}{2}.$$

In particular the lower and upper pointwise stable dimensions of μ_ϕ are equal a.e to δ^s.

For automorphisms Ornstein proved a famous result, namely that two invertible Bernoulli shifts on Lebesgue spaces are isomorphic if and only if they have the same measure theoretic entropy. However, as Parry and Walters showed, for measure-preserving *endomorphisms* $f : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$, the entropy $h_\mu(f)$ alone *does not* determine the conjugacy class (see [13]). So the problem of coding for endomorphisms of Lebesgue spaces (in particular for *1-sided Bernoulli shifts*) is subtle and there are no exhaustive classifications (see also [2] and [3]). If f_A is a linear toral endomorphism given by a hyperbolic matrix A with $|\det(A)| > 1$, then Katznelson showed that its natural extension is 2-sided Bernoulli. However, this does not mean that $(\mathbb{T}^m, f_A, m)$ is 1-sided Bernoulli.

Theorem 4 ([10]). *Let f_A be a toral endomorphism on $\mathbb{T}^m, m \geq 2$, given by the integer-valued matrix A, all of whose eigenvalues are strictly larger than 1 in absolute value. Then the endomorphism f_A on the torus $\mathbb{T}^m$, when equipped with its Lebesgue (Haar) measure μ_m, is isomorphic to a uniform model 1-sided Bernoulli shift.*

We now consider the general case of equilibrium measures of endomorphisms on folded fractals.

Theorem 5 ([9]). *Let f be a smooth hyperbolic endomorphism on a connected basic set Λ; also, let ϕ be a Hölder continuous potential on Λ and μ_ϕ the unique equilibrium measure of ϕ. If (Λ, f, μ_ϕ) is 1-sided Bernoulli, then: a) either f is distance-expanding on Λ; or b) the stable dimension of μ_ϕ is zero, i.e $HD^s(\mu_\phi, x) = 0$ for μ_ϕ-a.e $x \in \Lambda$.*

Theorem 6 ([9]). *a) Let f be hyperbolic on the saddle set Λ such that $\mathcal{C}_f \cap \Lambda = \emptyset$. If the system (Λ, f, μ_0) given by the measure of maximal entropy is 1-sided Bernoulli, then f is distance expanding on Λ.*

b) Assume f is an expanding endomorphism on Λ. If μ_ϕ is the equilibrium measure of the Hölder potential ϕ and if (Λ, f, μ_ϕ) is 1-sided Bernoulli, then $\mu_\phi = \mu_0$, where μ_0 is the unique measure of maximal entropy for f on Λ.

The family of skew products with variable overlaps of Cantor sets in fibers given in [4] consists of maps which are hyperbolic (see also [12]) and strongly non-invertible on their respective basic sets, and on the other hand they are not constant-to-one; one can apply the above results for them too.

There also exist relations between the *stable dimension* and the *geometry* of the fractal Λ.

Theorem 7 ([11]). *Let $f : M \to M$ be a smooth endomorphism which is hyperbolic on a basic set Λ with $C_f \cap \Lambda = \emptyset$ and which is conformal on local stable manifolds. Assume that d is the maximum possible value of the preimage counting function $d(\cdot)$ on Λ, and that $\exists x \in \Lambda$ with $\delta^s(x) := HD(W^s_r(x) \cap \Lambda) = t_d = 0$. Then it follows that $d(\cdot) \equiv d$ on Λ and $\delta^s(y) = 0,\ y \in \Lambda$.*

Theorem 8 ([6]). *In the setting of of the previous theorem, if d is the maximum possible value of $d(\cdot)$ on Λ and if $\delta^s = t_d = 0$, then there exist finitely many global unstable manifolds that contain Λ, and $f|_\Lambda$ is expanding. In particular if Λ is connected, then there exists one global unstable manifold containing Λ.*

References

[1] H. G. Bothe, Shift spaces and attractors in non-invertible horseshoes, Fund. Math., **152**, 1997, 267-289.

[2] H. Bruin and J. Hawkins, Rigidity of smooth 1-sided Bernoulli endomorphisms, New York J. Math., **15**, 2009, 1-33.

[3] C. Hoffman and D. Rudolph, Uniform endomorphisms which are isomorphic to a Bernoulli shift, Annals of Math., **156**, 2002, 79-101.
[4] E. Mihailescu, Unstable directions and fractal dimensions for a family of skew products with overlaps, Math. Zeitschrift, **269**, 2011, 733-750.
[5] E. Mihailescu, Physical measures for multivalued inverse iterates near hyperbolic repellors, J. Stat. Physics, **139**, 2010, 800-819.
[6] E. Mihailescu, Local geometry and dynamical behavior on folded basic sets, J. Stat. Physics **142**, 2011, 154-167.
[7] E. Mihailescu, On a class of stable conditional measures, Ergod. Th. and Dynam. Syst., **31**, 2011, 1499-1515.
[8] E. Mihailescu, Approximations for Gibbs states of arbitrary Holder potentials on hyperbolic folded sets, Discrete and Cont. Dynam. Syst., **32**, 2012, 961-975.
[9] E. Mihailescu, On some coding and mixing properties for a class of chaotic systems, Monatshefte Math., online 2011, DOI 10.1007/s00605-011-0347-8.
[10] E. Mihailescu, Higher dimensional expanding maps and toral extensions, to appear Proceed. Amer. Math. Soc., 2012.
[11] E. Mihailescu and M. Urbański, Relations between stable dimension and the preimage counting function on basic sets with overlaps, Bull. London Math. Soc., **42**, 2010, 15-27.
[12] S. Newhouse, Cone fields, domination and hyperbolicity, Modern Dynamical Systems and Applications. M. Brin, B. Hasselblatt and Y. Pesin (eds.), Cambridge Univ. Press, 2004.
[13] W. Parry and P. Walters, Endomorphisms of a Lebesgue space, Bull. AMS **78**, 1972, 272-276.
[14] F. Przytycki, Anosov endomorphisms, Studia Math., **58**, 1976, 249-285.
[15] D. Ruelle, Elements of differentiable dynamics and bifurcation theory, Academic Press, 1989.
[16] L. S Young, What are SRB measures and which dynamical systems have them?, J. Stat. Physics, **108**, 2002, 733-754.

A Fréchet law and an Erdős-Philipp law for maximal cuspidal windings

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(joint work with J. Jaerisch and B. O. Stratmann)

The talk presented recent results obtained in [6], stated in Theorem 1–3 below. In this paper we establish a Fréchet law and an Erdős-Philipp law for maximal cuspidal windings of the geodesic flow on $\mathbb{H}/G$, for a finitely generated, essentially free Fuchsian group G acting on the upper half-space model $(\mathbb{H}, d)$ of 2-dimensional hyperbolic space. Recall that to each ξ in the radial limit set $L_r(G)$ of G one can associate an infinite word expansion in the symmetric set G_0 of generators of G. Namely, with F referring to the Dirichlet fundamental domain of G at $i \in \mathbb{H}$, the images of F under G tesselate $\mathbb{H}$ and each side of each of the tiles is uniquely labeled by an element of G_0. The hyperbolic ray s_ξ from i towards $\xi \in L_r(G)$ has to traverse infinitely many of these tiles, and the infinite word expansion associated with ξ is then obtained by progressively recording, starting at i, the generators of the sides at which s_ξ exits the tiles. In this way we derive an infinite reduced word on the alphabet G_0. We then form blocks as follows. Each hyperbolic generator in this word has block length 1. Further, if there is a block in which the same parabolic generator appears n times and if there is no larger block of this parabolic generator containing that block, then this block is of length n. This allows us to define the process (X_k) by setting X_k to be equal to the length of the k-th block. By construction, such a block of length n corresponds to the event that the projection of s_ξ onto $\mathbb{H}/G$ spirals precisely $n-1$ times around a cusp of $\mathbb{H}/G$. The main results of this paper will establish asymptotic estimates and strong distributional convergence for the process (Y_n), given by

$$Y_n := \max_{k=1,\dots,n} X_k.$$

Theorem 1. Fréchet law for maximal cuspidal windings. *For each essentially free, finitely generated Fuchsian group G with parabolic elements and with exponent of convergence $\delta = \delta(G)$, the following holds. For every $s > 0$ and for each probability measure ν absolutely continuous with respect to the Patterson measure m_δ of G, we have that*

$$\lim_{n\to\infty} \nu\left(\left\{Y_n^{2\delta-1}/n \le s\right\}\right) = \exp\left(-\kappa\left(G\right)/s\right).$$

Here, the constant $\kappa\left(G\right)$ is explicitly given by

$$\kappa(G) := \sum_{\gamma\in\Gamma_0} \left(\Phi\left(p_\gamma\right) w_\gamma^{-\delta}\right)^2 / ((2\delta-1)\ \mu_\delta\left(\{X_1 = 1\}\right)),$$

where Γ_0 refers to the set of parabolic generators of G, p_γ to the parabolic fixed point and w_γ to the width of the cusp associated with $\gamma \in \Gamma_0$, μ_δ to the unique measure absolutely continuous with respect to m_δ which is invariant under the Bowen-Series map, and $\Phi : L(G) \to \mathbb{R}$ to a version of the Radon-Nikodym derivative $d\mu_\delta/dm_\delta$ given by $\Phi\left(\xi\right) := \int \chi_{\mathcal{L}(G)}\left(\xi,\eta\right) \left|\xi-\eta\right|^{-2\delta} dm_\delta(\eta)$, where

$$\mathcal{L}(G) := \{(\xi,\eta) : \xi,\eta \in L(G), \xi \neq \eta \text{ and there exists } t \in \mathbb{R} \text{ such that } \ell_{\xi,\eta}(t) \in \overline{F}\}$$

and F denotes a Dirichlet fundamental region for G at i and $\ell_{\xi,\eta}$ denotes the geodesic connecting ξ and η. Let us remark that the above constant $\kappa(G)$ is strictly positive, since by a result of Beardon [1] we have that if G has parabolic elements, then $\delta(G) > 1/2$. Also, note that our Fréchet law (cf. [3]) gives an answer to a question asked by Pollicott in [9], where he shows that a result by Galambos [4, 5] can be rephrased in terms of the modular group (see also [2]). The reader might like to recall that this result of Galambos states that for all $s > 0$ and for each probability measure ν absolutely continuous with respect to the Lebesgue measure on $(0,1)$, we have that $\lim_{n\to\infty} \nu\left(\{(\max_{k=1,\dots,n} a_k)/n \le s\}\right) = \exp\left(-1/(s\log 2)\right)$. Here, $a_k(x)$ refers to the k-th entry in the regular continued fraction expansion of $x \in (0,1)$. Let us remark that a straightforward adaptation of our proof of our Fréchet law gives an alternative proof of this result of Galambos.

We will also give some interesting applications of our Fréchet law. These include the following Erdős-Philipp law, whose first assertion extends a result of Philipp in [8, Theorem 1], who showed that for Lebesgue almost every $x \in (0,1)$ we have that $\liminf_{n\to\infty} \max_{k=1,\dots,n} a_k(x)(\log\log n)/n = 1/\log 2$. This settled a conjecture by Erdős (see [8]), who had previously conjectured that the above *Limes inferior* is equal to 1. Also, note that the second assertion of our Erdős-Philipp law extends [8, Corollary to Theorem 3].

Theorem 2. Erdős-Philipp law for maximal cuspidal windings. *For G as above, we have m_δ almost everywhere that*

$$\liminf_{n\to\infty} Y_n^{2\delta-1}(\log\log n)/n = \kappa(G).$$

Moreover, for each sequence (ℓ_n) of positive reals we have m_δ almost everywhere that

$$\limsup_{n\to\infty} Y_n/\ell_n \in \{0,\infty\}.$$

The Erdős-Philipp law has some further interesting consequences. Namely, it permits the derivation of the following Khintchine-type results, where ξ_t denotes the unique point on the hyperbolic ray from $i \in \mathbb{H}$ towards an element $\xi \in L_r(G)$ such that $d(i,\xi_t) = t$, for $t \geq 0$.

Theorem 3. Khintchine type laws. *For G as above, we have for m_δ almost every $\xi \in L_r(G)$ that*

$$\lim_{n\to\infty} \frac{\log Y_n(\xi)}{\log n} = \frac{1}{2\delta-1} \quad \textit{and} \quad \lim_{T\to\infty} \max_{0\leq t\leq T} \frac{d(\xi_t, G(i))}{\log T} = \frac{1}{2\delta-1}.$$

Note that in here the second assertion represents a significant strengthening of the result that $\limsup_{t\to\infty} d(\xi_t, G(i))/\log t = (2\delta-1)^{-1}$ for m_δ almost everywhere ξ, which was obtained in [11] for arbitrary geometrically finite Kleinian groups with parabolic elements, generalising work of Sullivan for cofinite Kleinian groups [10]. Let us point out that in this result the *Limes superior* cannot be replaced by a *Limes inferior.* Also, let us remark that the first statement in the above corollary is closely related to the well-known result by Khintchine for continued fractions ([7]), which asserts that Lebesgue almost everywhere, we have that $\limsup_{n\to\infty} \log a_n/\log n = 1$. In fact, by an elemetary observation, it immediately follows from our Corollary that for essentially free, finitely generated Fuchsian groups with parabolic elements we have m_δ almost everywhere that $\limsup_{n\to\infty} \log X_n/\log n = (2\delta-1)^{-1}$. Again, let us remark that in here the *Limes superior* can not be replaced by a *Limes inferior.*

Further, we point out that the above mentioned results by Galambos and Philipp exclusively concern the Gauss system, for which the limit set is the whole unit interval $[0,1]$. In contrast to this, the conformal dynamical systems which we consider in this paper have limit sets which are of fractal nature. Hence, one of the novelties of our analysis is that we obtain strong distributional convergence and asymptotic estimates for processes which are defined on conformal attractors with parabolic elements and of Hausdorff dimension strictly less than 1.

References

[1] A. F. Beardon, *Inequalities for certain Fuchsian groups*, Acta Math. **127** (1971), 221–258. MR 0286996 (44 #4203)
[2] D. Dolgopyat, On Statistical Properties of Geodesic Flows on Negatively Curved Surfaces. PhD thesis (Princeton University), 1997.
[3] M. Fréchet, *Sur la loi de probabilité de l'écart maximum.*, Annales Soc. Polonaise **6** (1928), 93–122.

[4] J. Galambos, *The distribution of the largest coefficient in continued fraction expansions*, Quart. J. Math. Oxford Ser. (2) **23** (1972), 147–151. MR 0299576 (45 #8624)
[5] ———, *The largest coefficient in continued fractions and related problems*, Diophantine approximation and its applications (Proc. Conf., Washington, D.C., 1972), Academic Press, New York, 1973, pp. 101–109. MR 0352028 (50 #4516)
[6] J. Jaerisch, M. Kesseböhmer B. O. Stratmann, A Fréchet law and an Erdős-Philipp law for maximal cuspidal windings, preprint in *arXiv:1109.3583*, submitted.
[7] A. Y. Khinchin, *Continued fractions*, The University of Chicago Press, Chicago, Ill.-London, 1964. MR MR0161833 (28 #5037)
[8] W. Philipp, *A conjecture of Erdős on continued fractions*, Acta Arith. **28** (1975/76), no. 4, 379–386. MR 0387226 (52 #8069)
[9] M. Pollicott, *Limiting distributions for geodesics excursions on the modular surface*, Spectral analysis in geometry and number theory, Contemp. Math., vol. 484, Amer. Math. Soc., Providence, RI, 2009, pp. 177–185. MR 1500147 (2010c:37074)
[10] ———, *Disjoint spheres, approximation by imaginary quadratic numbers, and the logarithm law for geodesics*, Acta Math. **149** (1982), no. 3-4, 215–237. MR 688349 (84j:58097)
[11] B. Stratmann and S. L. Velani, *The Patterson measure for geometrically finite groups with parabolic elements, new and old*, Proc. London Math. Soc. (3) **71** (1995), no. 1, 197–220. MR 1327939 (97f:58023)

Random iterations of rational functions

David Simmons

It is a theorem of Denker and Urbański that if T is a rational map of degree at least two and if $\phi : \widehat{\mathbb{C}} \to \mathbb{R}$ is Hölder continuous and satisfies the "thermodynamic expanding" condition $P(T, \phi) > \sup(\phi)$, then there exists exactly one equilibrium state μ for T and ϕ, and furthermore $(\hat{\mathbb{C}}, T, \mu)$ is metrically exact. In [5] I extended these results to a random setting, considering what I called a random holomorphic action on $\widehat{\mathbb{C}}$. Pressure and entropy are as defined by T. Bogenschütz in his paper [2] which proves a variational principle for random dynamical systems.

Definition 1. A *(measurable) random dynamical system* consists of

- A probability space $(\Omega, \mathbf{P})$ [i.e. $\mathbf{P}$ is a probability measure on the measurable space Ω]
- An ergodic invertible measure-preserving transformation $\theta : \Omega \to \Omega$
- A measurable space $\mathbb{X}$
- A measurable transformation $\mathbb{T} : \mathbb{X} \to \mathbb{X}$
- A measurable map $\pi : \mathbb{X} \to \Omega$ such that the diagram commutes, i.e. $\pi \circ \mathbb{T} = \theta \circ \pi$.

Definition 2. A *random holomorphic action on the Riemann sphere* $\widehat{\mathbb{C}}$ consists of

- A probability space $(\Omega, \mathbf{P})$
- An ergodic invertible measure-preserving transformation $\theta : \Omega \to \Omega$
- A Borel measurable map $T : \Omega \to \mathcal{R}$, where $\mathcal{R}$ is the set of all rational functions (holomorphic endomorphisms of $\widehat{\mathbb{C}}$), endowed with the compact-open topology [This map will usually be denoted in subscript i.e. $T_\omega := T(\omega)$]

The triple $(\Omega, \mathbf{P}, \theta)$ is called the *base system* and the map T is called the *action on* X.

If $(\Omega, \mathbf{P}, \theta, T)$ is a random holomorphic action on X, we construct a random dynamical system in a natural way as a skew-product:

$$\begin{aligned} \mathbb{X} &:= \Omega \times X \\ \mathbb{T}(\omega, x) &:= (\theta(\omega), T_\omega(x)) \\ \pi(\omega, x) &:= \pi_1(\omega, x) = \omega \end{aligned}$$

The sextuple $(\Omega, \mathbf{P}, \theta, \mathbb{X}, \mathbb{T}, \pi)$ is a random dynamical system since $\pi \circ \mathbb{T} = \theta \circ \pi$. It is called the *random dynamical system associated with* $(\Omega, \mathbf{P}, \theta, T)$.

For shorthand we write

$$T^n_\omega(x) := \pi_2(\mathbb{T}^n(\omega, x)) = T_{\theta^{n-1}(\omega)} \circ \ldots \circ T_\omega(x)$$

so that $T^n_\omega \in \mathcal{R}$. (The map T^n_ω is called a *pseudo-iterate.*)

The concepts of metric entropy and topological pressure can be defined for a random dynamical system associated with a random holomorphic action $(\Omega, \mathbf{P}, \theta, T)$. A variational principle holds in this contexts, yielding a concept of an equilibrium state. Such notions are discussed in [2].

In studying random holomorphic actions, my main result was a generalization of the following theorem:

Theorem 1 ([3])**.** *Suppose that T is a complex rational map of degree at least two and suppose that $\phi : \widehat{\mathbb{C}} \to \mathbb{R}$ is Hölder continuous and satisfies*

$$P(\phi) > \sup(\phi). \tag{1}$$

Then there is a unique equilibrium state for (T, ϕ).

Gromov [4] proved that $h_{\text{top}}(T) = \ln(\deg(T))$ for rational functions. Thus (1) follows from the easy to check condition

$$\sup(\phi) - \inf(\phi) < \ln(\deg(T)).$$

For the remainder of this abstract, fix a random holomorphic action $(\Omega, \mathbf{P}, \theta, T)$ on $\widehat{\mathbb{C}}$ and a random potential function $\phi : \Omega \to \mathcal{C}(\widehat{\mathbb{C}}, \mathbb{R})$. Assume that the set

$$\{\deg(T_\omega) : \omega \in \Omega\}$$

is bounded and does not contain 0 or 1. Also assume that the integrability condition

$$\int \ln \sup_{x \in \widehat{\mathbb{C}}} (\|T'_\omega(x)\|_s) \mathrm{d}\mathbf{P}(\omega) < \infty$$

is satisfied. (Here $\|T'_\omega(x)\|_s$ is the derivative of T_ω at x with respect to the spherical metric.) In particular, this assumption is satisfied if $T(\Omega)$ is relatively compact.

For each $\omega \in \Omega$ and $n \in \mathbb{N}$, we define the *Perron-Frobenius operator* $L_\omega^n : \mathcal{C}(\widehat{\mathbb{C}}, \mathbb{R}) \to \mathcal{C}(\widehat{\mathbb{C}}, \mathbb{R})$ via the equation

$$L_\omega^n[f](p) := \sum_{x \in (T_\omega^n)^{-1}(p)} \exp\left(\sum_{j=0}^{n-1} \phi_{\theta^j(\omega)}(T_\omega^j(x))\right) f(x).$$

(The sum is counted with multiplicity.)

Theorem 2 ([5])**.** *Fix $\alpha > 0$. Suppose that the integrability condition*

$$\int \|\phi_\omega\|_\alpha \mathrm{d}\mathbf{P}(\omega) < \infty$$

holds, and suppose that for each $\omega \in \Omega$, there exists $\lambda_\omega > 0$ so that $L_\omega[\mathbf{1}] = \lambda_\omega \mathbf{1}$. Then there exists a unique equilibrium state of $(\mathbb{X}, \mathbb{T}, \phi)$ over $(\Omega, \mathbf{P}, \theta)$. Also,

$$P_{\phi,\mathbf{P}}(\mathbb{T}|\theta) = \int \ln(\lambda_\omega) \mathrm{d}\mathbf{P}(\omega).$$

Corollary 3. *There exists a unique measure of maximal relative entropy of $(\mathbb{X}, \mathbb{T})$ over $(\Omega, \mathbf{P}, \theta)$. Futhermore*

$$h_{\mathrm{top},\mathbf{P}}(\mathbb{T}|\theta) := P_{0,\mathbf{P}}(\mathbb{T}|\theta) = \int \ln(\deg(T_\omega)) \mathrm{d}\mathbf{P}(\omega),$$

generalizing the deterministic equation $h_{\mathrm{top}}(T) = \ln(\deg(T))$ [4].

Proof. If $\phi = 0$, then $L_\omega[\mathbf{1}] = \deg(T_\omega)\mathbf{1}$. □

Theorem 4 ([5])**.** *Fix $\alpha > 0$ and $0 \le \tau < 1$. For every rational function T_0 of degree at least two, there exists a neighborhood $\mathcal{B}$ of T_0 in the compact-open topology such that the following holds:*

If $(\Omega, \mathbf{P}, \theta, T)$ is a random holomorphic action on $\widehat{\mathbb{C}}$ with $T(\Omega) \subseteq \mathcal{B}$, if $\phi : \Omega \to \mathcal{C}(\widehat{\mathbb{C}}, \mathbb{R})$ is a random potential function, and if:

$$\sup_{\omega \in \Omega} \|\phi_\omega\|_\alpha < \infty$$
$$\sup(e^{\phi_\omega}) \le \tau \inf(L_\omega[\mathbf{1}]) \;\; \forall \omega \in \Omega,$$

then there exists a unique equilibrium state of $(\mathbb{X}, \mathbb{T}, \phi)$ over $(\Omega, \mathbf{P}, \theta)$.

Note that the assumption

$$\sup(e^{\phi_\omega}) \le \tau \inf(L_\omega[\mathbf{1}]) \;\; \forall \omega \in \Omega \tag{2}$$

follows from the stronger and easily verifiable bound

$$\sup(\phi_\omega) - \inf(\phi_\omega) \le \ln(\tau) + \ln\deg(T_\omega) \;\; \forall \omega \in \Omega. \tag{3}$$

Recall that the Denker-Urbański theorem assumed

$$P(\phi) > \sup(\phi) \tag{4}$$

which also follows from (3). This in a sense justifies substituting (2) for (4).

References

[1] J. Bahnmüller and T. Bogenschütz, *A Margulis-Ruelle inequality for random dynamical systems*, Arch. Math. (Basel) 64 (1995), no. 3, 246–253.
[2] T. Bogenschütz, *Entropy, pressure, and a variational principle for random dynamical systems*, Random and Computational Dynamics 1 (1992/93), no. 1, 99–116.
[3] M. Denker and M. Urbański, *Ergodic theory of equilibrium states for rational maps*, Nonlinearity 4 (1991), no. 1, 103–134
[4] M. Gromov, *On the entropy of holomorphic maps*, Enseign. Math. (2) 49 (2003), no. 3-4, 217–235
[5] D. Simmons, *Random iteration of rational functions*, submitted

Hausdorff Dimension of radial Julia sets of meromorphic functions

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(joint work with M. Urbański)

Complex dynamics is a field originated in the works of Pierre Fatou and Gaston Julia. Of course, the problem of linearization for a fixed point was studied before (Böttcher, Koenings and others) and definitely it was an inspiration for the idea of creating this separate branch of mathematics, but numerous and extensive works of Fatou and Julia were the place where complex dynamics was born and maturated. The field became wildly known and popular when about three decades ago first computer images of Mandelbrot set and Julia sets appeared. Complex dynamics got an interest of many researches who started to investigate a variety of interesting and exiting topics in this field. One of them is the geometry of Julia sets and one of the ways to describe and analyze the complex nature of this object is its Hausdorff dimension. Here we study the behavior of this dimension under analytic perturbations.

Probably the first result indicating how the Hausdorff dimension of Julia sets changes under analytic perturbations is the result of Ruelle in [11]. The main technique Ruelle used was thermodynamic formalism. We refer the reader to the books of Zinsmister [15] and Przytycki & Urbański [9] for a modern exposition of thermodynamic formalism and contemporary approach to the problem of real analyticity of Hausdorff dimension.

The problem of real analyticity of the Hausdorff dimension was further studied for many families of rational and meromorphic functions (see e.g. [14], [13], [12], [6], [1] and [7]). Here we continue this line of investigation.

Therefore, let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a meromorphic function. The Fatou set of f consists of all points $z \in \mathbb{C}$ that admit an open neighborhood U_z such that all the forward iterates f^n, $n \geq 0$, of f are well-defined on U_z and the family of maps $\{f^n|_{U_z} : U_z \to \mathbb{C}\}_{n=0}^{\infty}$ is normal. The Julia set $\mathcal{J}(f)$ is then defined as the complement of the Fatou set of f in $\mathbb{C}$. By $\operatorname{sing}(f^{-1})$ we denote the set of singularities of f^{-1}. We define the *postsingular set* of $f : \mathbb{C} \to \hat{\mathbb{C}}$ as $\mathcal{P}(f) = \overline{\bigcup_{n=0}^{\infty} f^n(\operatorname{sing} f^{-1})}$.

The primary object of our study in this paper, the *radial Julia set* $\mathcal{J}_r(f)$ of f is defined as $\mathcal{J}_r(f) := \{z \in \mathcal{J}(f) : \omega(z) \setminus \mathcal{P}(f) \neq \emptyset\}$.

Our two main theorems are the following Theorem 1 and Theorem 2. In these theorems we establish real–analyticity of Hausdorff dimension of radial Julia sets under weakest, up to our knowledge, conditions.

Theorem 1. *Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a nicely strongly regular tame meromorphic function. Let $\Lambda \subset \mathbb{C}^d$ be an open set and let $\{f_\lambda\}_{\lambda\in\Lambda}$ be an analytic family of meromorphic functions such that*

(1) *$f_{\lambda_0} = f$ for some $\lambda_0 \in \Lambda$,*
(2) *there exists an holomorphic motion $H : \Lambda \times \overline{\mathcal{J}_{\lambda_0}} \to \mathbb{C}$ such that each H_λ is a topological conjugacy between f_{λ_0} and f_λ on $\mathcal{J}_{\lambda_0}$.*

Then the map $\lambda \mapsto \mathrm{HD}(\mathcal{J}_\lambda)$ is real–analytic on some neighborhood of λ_0.

Theorem 2. *Suppose that $f : \mathbb{C} \to \hat{\mathbb{C}}$ is a dynamically regular meromorphic function of divergence type which belongs to class $\mathcal{S}$. If $\Lambda \subseteq \mathbb{C}$ is an open set, $\{f_\lambda\}_{\lambda\in\Lambda}$ is an analytic family mermorphic functions and $f_{\lambda_0} = f$ for some $\lambda_0 \in \Lambda$, then the function $\Lambda \ni \lambda \mapsto \mathrm{HD}(\mathcal{J}_r(f_\lambda))$ is real–analytic in some open neighborhood of λ_0 contained in Λ.*

One of our two main techniques employed in the proofs of these theorems is the, recently emerging, concept of nice sets. These sets were introduced and extensively studied by Przytycki and Rivera-Letelier ([10], [8]) in the context of Collet-Eckmann rational mappings. A general construction of nice sets for transcendental functions can be found in [2]. Here we use them to construct appropriate conformal iterated function systems and then to apply the developed machinery of graph directed Markov systems from [4] and [5]. While doing this, as an actually auxiliary step, we obtain new, up to our knowledge, results about real analyticity of the Hausdorff dimension of limits sets of (infinite) conformal graph directed Markov systems. The following Theorem 3 and Theorem 4 in particular extend those from [12] and [1]. The number $b(S_\lambda)$ refers here to the Bowen's parameter of the system S_λ.

Theorem 3. *If $\{S_\lambda\}_{\lambda\in\Lambda}$ is a weakly regularly analytic family of finitely primitive conformal graph directed Markov systems, then the function $\Lambda \ni \lambda \mapsto b(S_\lambda) \in \mathbb{R}$ is real–analytic on some neighborhood of every strongly regular parameter $\lambda_0 \in \Lambda$. In addition, if the Bowen's parameter is equal to the Hausdorff dimension of the limit set, we thus automatically get real analyticity of Hausdorff dimension.*

Theorem 4. *If $\Lambda \subseteq \mathbb{C}^d$ is an open set and $\{S_\lambda\}_{\lambda\in\Lambda}$ is an analytic family of finitely primitive conformal graph directed Markov systems such that S_{λ_0} is strongly regular for some $\lambda_0 \in \Lambda$ and there exists a holomorphic motion $H : \Lambda \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that*

$$\varphi_e^\lambda(H(\lambda, z)) = H(\lambda, \varphi_e^{\lambda_0}(z))$$

for all $\lambda \in \Lambda$ and all $z \in \mathcal{J}_{\lambda_0}$, then the Bowen's parameter function $\Lambda \ni \lambda \to b(S_\lambda)$ is real–analytic on some sufficiently small neighborhood of λ_0.

Note that although we assume in the latter theorem seemingly more, namely the existence of an appropriate holomorphic motion, however, on the other hand, we merely assume here analyticity of the family of graph directed Markov systems, which is much weaker than weakly regular analyticity required in Theorem 3. Staying in the realm of abstract Conformal Graph Directed Markov Systems we are able to provide a very mild sufficient condition, called periodical separation, which entails the existence of a suitable holomorphic motion. Theorem 4 gets then very weak hypotheses indeed. This is however not quite the end of the story about directed Markov systems. The point is that those systems constructed in the proof of Theorem 1 are not known to satisfy the Open Set Condition. To remedy this we invoke the theory of conformal Walters expanding maps developed in in [3].

Having Conformal Iterated Function Systems produced with the help of nice sets, we were also able to show, as a straightforward consequence of the theory of Conformal Graph Directed Markov Systems, that the canonical Hausdorff measure restricted to the radial Julia set of a tame meromorphic function is σ-finite.

References

[1] Hasina Akter and Mariusz Urbański. Real analyticity of Hausdorff dimension of Julia sets of parabolic polynomials $f_l(z) = z(1 - z - lz^2)$. *Illinois J. Math.*, Preprint, 2011.

[2] Neil Dobbs. Nice sets and invariant densities in complex dynamics. *Math. Proc. Cambridge Philos. Soc.*, 150(1):157–165, 2011.

[3] Janina Kotus and Mariusz Urbański. Conformal, geometric and invariant measures for transcendental expanding functions. *Math. Ann.*, 324(3):619–656, 2002.

[4] R. Daniel Mauldin and Mariusz Urbański. Dimensions and measures in infinite iterated function systems. *Proc. London Math. Soc*, 73(3):105–154, 1996.

[5] R. Daniel Mauldin and Mariusz Urbański. *Graph directed Markov systems*, volume 148 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2003. Geometry and dynamics of limit sets.

[6] Volker Mayer and Mariusz Urbański. Geometric thermodynamical formalism and real analyticity for meromorphic functions of finite order. *Ergod. Th. & Dynam. Sys.*, 28:915–946, 2008.

[7] Volker Mayer and Mariusz Urbański. Thermodynamical formalism and multifractal analysis for meromorphic functions of finite order. *Mem. Amer. Math. Soc.*, 203(954):vi+107, 2010.

[8] Feliks Przytycki and Juan Rivera-Letelier. Statistical properties of topological Collet-Eckmann maps. *Ann. Sci. École Norm. Sup. (4)*, 40(1):135–178, 2007.

[9] Feliks Przytycki and Mariusz Urbański. *Conformal fractals: ergodic theory methods*, volume 371 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2010.

[10] Juan Rivera-Letelier. A connecting lemma for rational maps satisfying a no-growth condition. *Ergodic Theory Dynam. Systems*, 27(2):595–636, 2007.

[11] David Ruelle. Repellers for real analytic maps. *Ergodic Theory Dynamical Systems*, 2(1):99–107, 1982.

[12] Mariusz Urbański. Analytic families of semihyperbolic generalized polynomial-like mappings. *Monatsh. Math.*, 159(1-2):133–162, 2010.

[13] Mariusz Urbański and Anna Zdunik. Real analyticity of Hausdorff dimension of finer Julia sets of exponential family. *Ergodic Theory Dynam. Systems*, 24(1):279–315, 2004.

[14] Mariusz Urbański and Michel Zinsmeister. Geometry of hyperbolic Julia-Lavaurs sets. *Indagationes Math.*, 12, 2001.

[15] Michel Zinsmeister. *Thermodynamic formalism and holomorphic dynamical systems*, volume 2 of *SMF/AMS Texts and Monographs*. American Mathematical Society, Providence, RI, 2000. Translated from the 1996 French original by C. Greg Anderson.

A Central Limit Theorem for certain infinite measure preserving transformations

Roland Zweimüller
(joint work with D. Kocheim)

The study of ergodic and probabilistic properties of dynamical systems with an infinite invariant measure has recently led to a number of interesting results which generalize classical theorems for null-recurrent Markov chains to the weakly dependent processes generated by certain types of infinite measure preserving transformations. In the present talk I report on some ongoing joint work with David Kocheim. Very similar results have been obtained (independently and by different methods) under the supervision of Sebastien Gouezel. As the results have not yet reached a final optimized form, I refrain from formulating a general statement, and focus on a specific example instead.

The best-known conservative (i.e. recurrent) infinite measure preserving system is the (shift-space representation of) the simple symmetric random walk $(\xi_n)_{n\geq 0}$ on $\mathbb{Z}$, where $\xi_0 := 0$ and $\xi_n := \sum_{k=1}^{n} \eta_k$ with $(\eta_k)_{k\geq 1}$ i.i.d. and $\Pr[\eta_k = \pm 1] = 1/2$. Letting $S_n(A) := \sum_{k=0}^{n-1} 1_A(\xi_k)$ denote the occupation times of some finite set $A \subseteq \mathbb{Z}$ of states, the following distributional limit theorem is well known,

$$n^{-\frac{1}{2}} S_n(A) \Longrightarrow |\mathcal{N}| \,, \tag{1}$$

where $\mathcal{N}$ is a Gaussian random variable. A refinement has been given by Dobrushin [5], who showed that for sets A, B of the same cardinality,

$$n^{-\frac{1}{4}} (S_n(A) - S_n(B)) \Longrightarrow \mathcal{N}_1 \cdot |\mathcal{N}_2|^{\frac{1}{2}} \tag{2}$$

with independent Gaussian variables $\mathcal{N}_1$ and $\mathcal{N}_2$. For more general random-walk results of this type we refer to [3].

Limit theorems for occupation times of null-recurrent Markov processes generalizing (1) are often referred to as Darling-Kac type results, cf. [4]. Corresponding statements have been given for certain classes of infinite-measure preserving dynamical systems, see [1], [2] or [6] for two different approaches. For a simple explicit example, consider Boole's transformation $T : \mathbb{R} \to \mathbb{R}$ given by $Tx := x - \frac{1}{x}$, which preserves Lebesgue measure λ and is conservative ergodic. Here the occupation times $S_n(A) := \sum_{j=0}^{n-1} 1_A \circ T^j$ of any Borel set A of finite measure again satisfy (1), where the $S_n(A)$ are regarded as random variables on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, P)$ where P is any probability measure with $P \ll \lambda$.

While Darling-Kac results turn out to hold under fairly weak conditions, generalizing (2) to dynamical systems requires stronger assumptions. We were able to do so for conservative ergodic measure preserving maps T on an infinite measure space $(X, \mathcal{A}, \mu)$ which possess reference sets Y of finite measure for which the return map T_Y is Gibbs-Markov, and such that the return-time function φ has a regularly varying tail of index $\frac{1}{2}$. Moreover, the Markovian subsystem obtained by restricting T_Y to $\bigcap_{m\geq 0} T_Y^{-m}\{\varphi \leq L\}$ should again be irreducible (for L large enough). Then (2) holds for $A, B \subseteq Y$ of equal measure. More generally,

$f = 1_A - 1_B$ may be replaced by any sufficiently regular (Lipschitz) function f with $\int f d\mu = 0$ which is supported on Y. Specifically, for Boole's transformation (2) holds whenever A, B are bounded Borel sets with $0 < \lambda(A) = \lambda(B) < \infty$. Analogous results apply when the return-time tail is regularly varying of some different order $\alpha \in (0, 1)$, in which case the conclusion becomes

$$n^{-\frac{\alpha}{2}}(S_n(A) - S_n(B)) \Longrightarrow \mathcal{N} \cdot \mathcal{M}_\alpha^{1/2} \tag{3}$$

with independent variables $\mathcal{N}$ and $\mathcal{M}_\alpha$, where $\mathcal{M}_\alpha$ has a Mittag-Leffler law of index α (that is, $\mathcal{M}_\alpha = \mathcal{G}_\alpha^{-\alpha}$ in distribution, where $\mathcal{G}_\alpha$ is a positive α-stable variable).

The main point of our proof is to establish some asymptotic independence of the processes $(\sum_{j=0}^{m-1} \varphi \circ T_Y^j)_{m\geq 0}$ and $(\sum_{j=0}^{m-1} f \circ T_Y^j)_{m\geq 0}$ generated by the induced system. Here, a key observation is that the return-time sums $\sum_{j=0}^{m-1} \varphi \circ T_Y^j$ can be well approximated (in measure) by finitely many order statistics. We then establish a conditional CLT for $(\sum_{j=0}^{m-1} f \circ T_Y^j)_{m\geq 0}$ with the conditioning events given in terms of order statistics. This is carried out by analyzing Fourier perturbations of the transfer operator of T_Y.

References

[1] J. Aaronson: *An Introduction to Infinite Ergodic Theory.* AMS 1997.

[2] J. Aaronson: *The asymptotic distributional behaviour of transformations preserving infinite measures.* Journal d'Analyse Mathématique **39** (1981), 203-234.

[3] A.N. Borodin, I.A. Ibragimov: *Limit theorems for functionals of random walks.* Proc. Steklov Inst. Math. **195** (1994).

[4] D.A. Darling, M. Kac: *On occupation times for Markoff processes.* Trans. Amer. Math. Soc. **84** (1957), 444-458.

[5] R. Dobrushin: *Two limit theorems for the simplest random walk on a line.* Uspehi Mat. Nauk (N.S.) **10** (1955), no. 3 (65), 139–146.

[6] R. Zweimüller: *Infinite measure preserving transformations with compact first regeneration.* Journal d'Analyse Mathématique **103** (2007), 93-131.

Measures on the Julia set

Christophe Dupont

We deal with the dynamics of holomorphic maps on projective spaces. Let us focus on a result of Zdunik [13] concerning rational maps acting on the Riemann sphere.

Theorem 1. *Let f be a rational map acting on the Riemann sphere $\mathbb{CP}^1$ and let μ be its maximal entropy measure. Then $\dim_H J = \dim_H \mu$ if and only if f belongs to the family of maps $\{z^{\pm d}, Tchebichev, Lattès\}$.*

In the statement, J denotes the support of μ (which coincides with the Julia set of f) and $\dim_H \mu$ stands for the infimum of the Hausdorff dimension of Borel sets of positive μ-measure. Zdunik's theorem then asserts that, for a generic rational map, the *topological* support of μ is strictly larger than the *Borel* support of μ. A related result was proved by Mayer [11], who characterized the same set of maps in terms of the absolute continuity of μ with respect to conformal measures.

The proof of the above theorem relies on the *conformal* property of rational maps. Precisely, the idea is to construct for any generic map an ergodic measure ν which lies between μ and J. The arguments use Mañé's formula [10] about the dimension of ergodic measures with positive entropy and involve the pressure of the derivative of f. An important tool is also the Central Limit Theorem for the singular observable $\log|f'|$ which was established for rational maps by Przytycki, Urbański and Zdunik in the previous article [12].

The aim of the talk is to introduce recent results concerning the higher dimensional case, i.e. for holomorphic self maps of projective space $\mathbb{CP}^k$. We refer to the book of Dinh and Sibony [7] for an introduction to the dynamical properties of such mappings. The following result provides the Lattès part of Zdunik's theorem.

Theorem 2. *Let f be a holomorphic self map of $\mathbb{CP}^k$ and let μ be its maximal entropy measure. Then $\dim_H \mu = 2k$ if and only if f is Lattès.*

That follows from the combination of Berteloot-Dupont [2], Berteloot-Loeb [3], Dinh-Dupont [5] and Dupont [8]. Lattès maps also appear in the problem of commuting endomorphisms, solved by Dinh-Sibony [6]. One of the difficulties is the lack of conformality: the k Lyapunov exponents of μ, which are positive by Briend-Duval [4], are central in the arguments. In particular we had to develop a theory of normal form for inverse branches which, in some sense, plays the role of Koebe distortion theorem.

We shall also discuss the following result of [9].

Theorem 3. *Let f be a holomorphic self map of $\mathbb{CP}^2$ and let μ be its maximal entropy measure. Then*

$$\dim_H \mu \geq \frac{\log d}{\lambda_1} + \frac{\log d}{\lambda_2},$$

where d is the algebraic degree of f and λ_1, λ_2 are the Lyapunov exponents of μ.

That provides half of the expected Mañé's formula in higher dimension, which was conjectured by Binder and de Marco in [1]. The proof consists in studying the distribution of inverse branches in the projective plane: the main tools are normal forms for inverse branches and elements of complex geometry. That bound is actually true for ergodic measures with large entropy (between $\log d$ and $\log d^2$), hence it may be of interest for a possible extension of Zdunik's approach to $\mathbb{CP}^2$.

References

[1] I. Binder, L. DeMarco, *Dimension of pluriharmonic measure and polynomial endomorphisms of $\mathbf{C}^n$,* Int. Math. Res. Not., **11** (2003), 613-625.

[2] F. Berteloot, C. Dupont, *Une caractérisation des exemples de Lattès par leur mesure de Green,* Comment. Math. Helv., **80** (2005).

[3] F. Berteloot, J.-J. Loeb, *Une caractérisation géométrique des exemples de Lattès de $\mathbb{CP}^k$,* Bull. Soc. Math. Fr., **129** (2001), no. 2, 175-188.

[4] J.Y. Briend, J. Duval, *Exposants de Liapounoff et distribution des points périodiques d'un endomorphisme de $\mathbb{CP}^k$,* Acta Math., **182** (1999), no. 2, 143-157.

[5] T.-C. Dinh, C. Dupont, *Dimension de la mesure d'equilibre d'applications meromorphes,* J. Geom. Anal., **14** (2004), no. 4, 613-627.

[6] T.-C. Dinh, N. Sibony, *Sur les endomorphismes holomorphes permutables de* $\mathbb{P}^k$*,* Math. Ann., **324** (2002), no. 1, 33-70.

[7] T.-C. Dinh, N. Sibony, *Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings,* Lecture Notes in Math., **1998**, (2010).

[8] C. Dupont, *Bernoulli coding map and almost sure invariance principle for endomorphisms of* $\mathbb{CP}^k$, Probab. Theory Relat. Fields, **146** (2010), no. 1, 337-359.

[9] C. Dupont, *On the dimension of invariant measures of endomorphisms of* $\mathbb{CP}^k$, Math. Ann., **349** (2011), no. 3, 509-528.

[10] R. Mañé, *The Hausdorff dimension of invariant probabilities of rational maps,* Lecture Notes in Math., **1331**, (1988).

[11] V. Mayer, *Comparing measures and invariant line fields*, Ergodic Theory Dynam. Systems, **22** (2002), no. 2, 555-570.

[12] F. Przytycki, M. Urbański, A. Zdunik, *Harmonic, Gibbs and Hausdorff measures on repellers for holomorphic maps. I,* Ann. of Math. (2) **130** (1989), no. 1, 1-40.

[13] A. Zdunik, *Parabolic orbifolds and the dimension of the maximal measure for rational maps,* Invent. Math. **99** (1990), 627-649.

From Farey and Gauss to modern times, and variations of the theme

Bernd Otto Stratmann

(joint work with M. Kesseböhmer)

The talk consisted of two parts. In the first part we discussed recent results for the Farey map, which are analogs of the famous Gauss problem for the Gauss map. In the second part we reported on recent results on the differentiability of Minkowski's question mark function, which generalises a classical result by Salem and which make use of our previously obtained thermodynamical analysis of the Farey map.

In the **first part of the talk** we first recalled that for the Gauss map $G : [0,1] \to [0,1]$, given by $G : x \mapsto 1/x \mod (1)$, Gauss pointed out in a letter to Laplace in 1812 (see [1]) that for the Lebesgue measure λ on $[0,1]$ one has, for each $x \in (0,1]$ and for n tending to infinity,

$$\lambda(G^{-n}((x,1))) \sim \nu((x,1)) \, (= -\log_2((1+x)/2)) \, ,$$

where ν refers to the Gauss measure (which is invariant under G, that is, we have that $\nu(G^{-1}(A)) = \nu(A)$, for all $A \subset (0,1]$ Borel measurable) and where $c_n \sim d_n$ means that $\lim_{n\to\infty} c_n/d_n = 1$.

Recently, we addressed and answered an analogous question for the Farey map $F : [0,1] \to [0,1]$, which is given by

$$F(x) := \begin{cases} x/(1-x) & \text{for} \quad x \in [0,1/2] \\ (1-x)/x & \text{for} \quad x \in (1/2,1] \, , \end{cases}$$

More precisely, for arbitrary $x \in (0,1]$, we investigated the asymptotic behaviour of $\lambda(F^{-n}((x,1)))$. The answer to this question is given in the following theorem, where μ refers to the (infinite) F-invariant measure absolutely continuous with respect to λ.

Theorem 1. (*see* [4]) *For n tending to infinity, we have*

$$\lambda(F^{-n}((x,1))) \sim \frac{\mu((x,1))}{\log n} \left(= -\frac{\log x}{\log n} \right).$$

The talk gave a sketch of the proof in the special situation $x = 1/2$, which turns out to be of independent number theoretical interest. In this special situation one immediately verifies that, for each $n \in \mathbb{N}$,

$$F^{-(n-1)}((1/2,1)) = \{[a_1, a_2, \ldots] \in [0,1] : \sum_{i=1}^{k} a_i = n \text{ for some } k \in \mathbb{N}\} \, (=: \mathcal{C}_n),$$

where $[a_1, a_2, \ldots]$ refers to the regular continued fraction expansion of an element from $(0,1]$. Using this observation, we then used and extended infinite ergodic theory and obtained the following theorem.

Theorem 2. (*see* [4]) *For n tending to infinity, we have*

$$\lambda(\mathcal{C}_n) \sim \frac{1}{\log_2 n}.$$

Hence, we in particular have that

(a) $\sum_{k=1}^{n} \lambda(\mathcal{C}_k) \sim n/\log_2 n$;
(b) $\lim_{n\to\infty} \lambda(\mathcal{C}_n) = 0$.

Also, more recently, we additionally obtained the following generalisation of these results. Here, δ_x denotes the Dirac distribution at x and $*\lim$ the weak limit of measures. Also, all appearing fractions are assumed to be reduced.

Theorem 3. (*see* [5]) *For each rational number $v/w \in (0,1]$ we have that*

$$*\lim_{n\to\infty} \left(\log(n^{vw}) \sum_{p/q \in F^{-n}\{v/w\}} q^{-2}\, \delta_{p/q} \right) = \lambda.$$

Moreover, most recently, variations of the techniques which led to these results gave rise to the following estimate on the growth rate of the Poincaré series associated with a Kleinian group G with parabolic elements and with exponent of convergence δ. Here, d denotes hyperbolic metric of the hyperbolic space $\mathbb{H}$, $r_{\max}$ to the maximal possible rank of the parabolic elements of G and $|g|$ to the word length of an element $g \in G$.

Theorem 4. (*see* [6]) *For a finitely generated, essentially free Kleinian group G with parabolic element we have, for each $z, w \in \mathbb{H}$,*

$$\sum_{g \in G, |g| \le n} e^{-\delta d(z, g(w))} \asymp \begin{cases} n^{2\delta - r_{\max}} & \text{for} \quad \delta < (r_{\max}+1)/2 \\ n/\log n & \text{for} \quad \delta = (r_{\max}+1)/2 \\ n & \text{for} \quad \delta > (r_{\max}+1)/2. \end{cases}$$

In the **second part of the talk** we first recalled that Minkowski's question mark function $Q : [0,1] \to [0,1]$ is given for all $x = [a_1, a_2, \ldots]$ by

$$Q(x) := \sum_{k\in\mathbb{N}} (-1)^{k+1}\ 2^{1-\sum_{i=1}^{k} a_i}.$$

It is well known that Q is equal to the distribution function of the measure of maximal entropy μ_{top} for the Farey system $([0,1], F)$, and that, by a result of Salem (see [7]), the derivative Q' of Q is equal to zero λ-almost everywhere. In order to sketch the proof of our generalisation of this result of Salem, we first recalled the outcome of our previously obtained thermodynamical analysis of the Farey map from [2]. We then proceeded by explaining how this analysis can be used to obtain the following complete picture for the derivative of Q.

Theorem 5. (*see* [3]) *The unit interval can be written as the disjoint union of the three sets*

$$\Lambda_0 := \{x : Q'(x) = 0\}, \ \ \Lambda_\infty := \{x : Q'(x) = \infty\},$$

and

$$\Lambda_\sim := \{x : Q'(x) \ \text{does not exist and} \ Q'(x) \neq \infty\}.$$

Moreover, for the Hausdorff dimensions $\dim_H$ *of these sets we have that*

$$7/8 \approx \dim_H(\mu_{top}) < \dim_H(\Lambda_\sim) = \dim_H(\Lambda_\infty) = \dim_H(\mathcal{L}(h_{\text{top}})) < \dim_H(\Lambda_0) = 1,$$

where $\mathcal{L}(h_{\text{top}})$ *refers to the level set of the multifractal decomposition of the Farey map* F *at the topological entropy* $h_{\text{top}} = \log 2$*, and* $\dim_H(\mu_{top})$ *denotes the Hausdorff dimension of the measure of maximal entropy of the Farey system* $([0,1], F)$*.*

References

[1] C. F. Gauss, Letter to Gauss, in *Gauss Werke*, Vol. 10, p. 371 - 374 (1917; Letter dated 30 January 1812, French).

[2] M. Kesseböhmer, B. O. Stratmann, A multifractal analysis for Stern-Brocot internals, continued fractions and Diophantine growth rates, *Journal für die reine und angewandte Mathematik (Crelle's Journal)* 605 (2007) 133-163.

[3] M. Kesseböhmer, B. O. Stratmann, Fractal analysis for sets of non-differentiability of Minkowski's question mark function, (preprint in *arXiv:0706.0453v1*; *Journal of Number Theory* 128 (2008) 2663-2686.

[4] M. Kesseböhmer, B. O. Stratmann, On the Lebesgue measure of sum-level sets for continued fractions, (preprint in *arXiv:0901.1787*); *Discrete and Continuous Dynamical Systems* 14 (2011).

[5] M. Kesseböhmer, B. O. Stratmann, A dichotomy between uniform distributions of the Stern–Brocot and the Farey sequence, (preprint in *arXiv:1009.1823*); *Uniform Distribution Theory* 7 no. 2 (2012).

[6] M. Kesseböhmer, B. O. Stratmann, A note on the algebraic growth rate of Poincaré series for Kleinian groups, (preprint in *arXiv:org:0910.5560*); to appear in: *Contributions to Analytic and Algebraic Number Theory - Festschrift for S. J. Patterson* (2012), Spinger Verlag.

[7] R. Salem, On some singular monotonic functions which are strictly increasing, *Trans. Amer. Math. Soc.*, 53 (1943) 427–439.

Open Problems

During the workshop there has been a round table discussion about thermodynamic formalism which all participants attended as well as an *Open Problem Session* to which almost every participant contributed. In the following we state some of the problems which were given by the participants during this session.

Problem 1. (by *Eugen Mihailescu*)
In *Entropy production and folding of the phase space in chaotic dynamics* (arXiv:1104.2342) the author obtained the following result. Consider the hyperbolic toral endomorphism on $\mathbb{T}^2$ given by $f(x,y) = (2x+2y, 2x+3y)$ (mod 1) and its smooth perturbation $g(x,y) = (2x+2y+\epsilon sin 2\pi y,\ 2x+3y+2\epsilon sin 2\pi y)$ (mod 1). Then the inverse SRB measure of g has negative entropy production, while the SRB measure of g has positive entropy production, that is, $e_g(\mu_g^-) < 0$ and $e_g(\mu_g^+) > 0$.

This shows that there exist examples for which the inverse SRB measure has negative entropy production. The problem is to find other folded fractals Λ, which are hyperbolic for non-invertible maps f, and families of f-invariant measures μ on Λ such that the entropy production of μ is negative (or positive).

Problem 2. (by *Mariusz Urbański*)
Let E_{even} be the set of all irrational numbers in the interval $[0;1]$ whose continued fraction expansion entries are all even and let h denote its Hausdorff dimension. It was proven in D. Mauldin and the author in *Conformal iterated function systems with applications to the geometry of continued fractions* (Transactions of A.M.S. 351 (1999), 4995-5025) that the h-dimensional Hausdorff measure of E_{even} vanishes while its h-dimensional packing measure P_h is positive and finite. It was proven by D. Mauldin and the author in *Graph Directed Markov Systems: Geometry and Dynamics of Limit Sets* (Cambridge University Press, 2003) that the Gauss map G restricted to E_{even} admits a unique probability G-invariant measure μ equivalent to P_h. It was also shown there, that the Radon-Nikodym derivative $d\mu/dP_h$ has a (unique) real-analytic extension to the whole interval $[0;1]$. The problem is to find a closed explicit formula for this derivative, or else to show that such a formula does not exist.

Problem 3. (by *Hiroki Sumi*)
Phenomena which are caused by noise or randomness are called *noise-induced phenomena*. In fact, by (numerical) experiments, many physicists have investigated noise-induced phenomena. As explained in the author's talk, the author has shown that regarding the i.i.d. random dynamics of complex polynomials on the Riemann sphere, generically, "noise-induced order" really occurs. However, there are many noise-induced phenomena which have been found by numerical observations but for which no rigorous proofs have been given so far. Therefore, the following is a very important problem:

Give mathematically complete proofs for various noise-induced phenomena which are observed by (numerical) experiments.

Note that nature has many random terms and many physicists are interested in describing nature by using random dynamical systems. These noise-induced phenomena are very important in physics and, quite certainly, they also give rise to many interesting problems in mathematics.

Problem 4. (by *Manuel Stadlbauer*)
Recall that the quadratic family is defined by

$$T_\lambda : [0,1] \to [0,1],\ x \mapsto \lambda x(1-x),\ \text{ for } \lambda \in [1,4].$$

By the celebrated results of Benedicks and Carleson in *On iterations of* $1 - ax^2$ *on* $(-1,1)$ *(Ann. of Math. (2)*, 122 (1), 1985, 1–25), T_λ admits an invariant probability measure for almost all $\lambda \in (1,4]$. Hence, it seems to be plausible that the following question has a positive answer. Assume that P is a probability measure on $[1,4]$ which is absolutely continuous to Lebesgue measure, and that (Λ_i) is a sequence of independent random variables such that Λ_1 is distributed according to P. Is it then true that the random dynamical system given by

$$(x,(\lambda_1,\lambda_2,\ldots)) \mapsto (T_{\lambda_1}(x),(\lambda_2,\lambda_3,\ldots))$$

admits a random invariant measure which is absolutely continuous with respect to Lebesgue? Moreover, is there a random decay of correlation, that is, is the system relatively exact? So far, it is known only that there are positive answers to these questions if the support of the distribution is contained in small neighbourhoods of hyperbolic parameters (see, e.g. V. Baladi and M. Viana, *Strong stochastic stability and rate of mixing for unimodal maps*, Ann. Sci. Éc. Norm. Supér., 1996).

Problem 5. (by *Stephen Muir*)
Consider a classical lattice gas space with countably infinitely many states, that is, a multidimensional shift $\mathbb{N}^{\mathbb{Z}^d}$. An *interaction potential* is a family Φ of bounded functions $\Phi_\Lambda : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$, indexed by finite subsets $\Lambda \subset \mathbb{Z}^d$, where each Φ_Λ depends only on the coordinates within Λ. We assume them to be translation invariant. Letting T^λ be the translation of the site $\lambda \in \mathbb{Z}^d$ to the origin $0 \in \mathbb{Z}^d$, this means we assume that $\Phi_\Lambda \circ T^\lambda = \Phi_{\Lambda+\lambda}$ for every $\lambda \in \mathbb{Z}^d$ and every finite subset $\Lambda \subset \mathbb{Z}^d$. Let us define the following two Banach spaces

$$\mathcal{B} := \left\{ \Phi \ :\ \sum_{\substack{\Lambda \ni 0 \\ |\Lambda|<\infty}} |\Lambda|^{-1} \|\Phi_\Lambda\|_\infty < \infty \right\}, \mathcal{S} := \left\{ \Phi \ :\ \sum_{\substack{\Lambda \ni 0 \\ |\Lambda|<\infty}} \|\Phi_\Lambda\|_\infty < \infty \right\}.$$

The *local energy function* associated to an interaction Φ is given by

$$A_\Phi := \sum_{\substack{\Lambda \ni 0, |\Lambda|<\infty, \text{and} \\ \text{lex.ord.}(0\in\Lambda) = \left\lceil \frac{|\Lambda|}{2} \right\rceil}} \Phi_\Lambda,$$

where "lex.ord." refers to the lexicographic order on $\mathbb{Z}^d$. Using an idea sketched by Ruelle in his book *Thermodynamic Formalism*, it is easy to show that $A_{\mathcal{B}} := \{A_\Phi : \Phi \in \mathcal{B}\} = UC\left(\mathbb{N}^{\mathbb{Z}^d}, \mathbb{R}\right)$. On the other hand, an exact description of $A_{\mathcal{S}} := \{A_\Phi : \Phi \in \mathcal{S}\}$ appears to be unknown. The following partial result is proved in the author's thesis of (University of North Texas, 2011):
For $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ let $\delta_n(f) := \sup\{|f(x) - f(y)| : \pi_{\Lambda_n}(x) = \pi_{\Lambda_n}(y)\}$, where Λ_n is the d-dimensional cube of sites $\lambda \in \mathbb{Z}^d$ for which $\max\{|\lambda_i| : 1 \leq i \leq d\} < n$. Then let

$$R_d := \left\{ f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R} \mid \sum_{n \geq 1} n^d \delta_n(f) < \infty \right\}.$$

In Chapter 7 of the author's thesis it was shown that $R_d \subseteq A_{\mathcal{S}} \subseteq R_{d-1}$. However, it is unknown whether there is an equality in this sequence of containments or whether both inclusions are strict. If both inclusions are strict, one should try to characterise the image of the small space of interactions in terms of the modulus of continuity of the local energy function.

Note that an immediate motivation to solve this problem is that if the upper inclusion were shown to be strict, then the variational principle of the author's thesis would apply to a larger class of models (those with local energies in R_{d-1}) than any existing treatments by interactions, (see e.g. H.O.Georgii, *Gibbs Measures and Phase Transitions*) which require $\Phi \in \mathcal{S}$ for the construction of Gibbs measures.

Problem 6. (by *Anna Zdunik*)
Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f_\lambda(z) := \lambda \exp(z)$. It was proved in my joint work with Mariusz Urbański, that if f_λ is hyperbolic (or equivalently, if the singular value 0 is attracted by an attractive periodic orbit), then we have for the Hausdorff dimension $\dim_H$ of the radial Julia set $J_r(f_\lambda)$ that

$$1 < \dim_H(J_r(f_\lambda)) < 2.$$

This raises the following two questions.

Question 1: What is the supremum of the values $\dim_H(J_r(f_\lambda))$, *where* λ *varies over all possible hyperbolic parameters?*

Question 2: Is the function $(0, 1/e) \ni \lambda \mapsto \dim_H(J_r(f_\lambda))$ *increasing?*

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