Mathematisches Forschungsinstitut Oberwolfach

Report No. 05/2012

DOI: 10.4171/OWR/2012/05

Recent Developments in the Numerics of Nonlinear Hyperbolic Conservation Laws and their Use in Science and Engineering

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January 15th – January 21st, 2012

Abstract. Modern numerical methods for hyperbolic conservation laws rely on polynomials of high degree, mostly orthogonal polynomials, on triangular or quadrilateral meshes. Due to shocks stability is an issue and modern means of filtering like spectral viscosity is required. Additional TV-filters are needed in most cases as postprocessors and the choice of the solver for the differential equations to integrate in time is crucial. The workshop was organised to bring together researchers from different areas of mathematics in order to fuel the research on high-order efficient and robust numerical methods.

Mathematics Subject Classification (2000): 34-99, 35F25, 35L65, 41A10, 76-99.

Introduction by the Organisers

The workshop *Recent Developments in the Numerics of Nonlinear Hyperbolic Conservation Laws and their Use in Science and Engineering*, organised by Rainer Ansorge (Hamburg), Hester Bijl (Delft), Andreas Meister (Kassel) and Thomas Sonar (Braunschweig) was held January 15th–January 21st, 2012. This meeting was well attended with 45 participants with broad geographic representation from many continents. This workshop was in a sense an experiment. Since modern numerical methods like Discontinuous Galerkin or Spectral Element Finite Difference methods are based on orthogonal polynomials on simplizes and use modal filters and many more mathematical devices from different areas of research we decided to invite renowned researchers from numerical methods for ODEs, image processing,

approximation theory, and numerical methods for hyperbolic conservation laws. Although there was some confusion in the beginning since the image processing people did not know in advance whether they were invited to the right conference or not these confusions could be washed away. At the end of the workshop we heard from several researchers that this was indeed an extraordinary successful workshop in which specialists from so different areas talked with each other for the first time.

The talks ranged from new Runge-Kutta solvers, new filters in image processing, Discontinuous Galerkin methods, Spectral Difference methods, Finite Difference operators, implicit solvers, and finite volume methods to the modeling of shocks, salt distribution in the baltic sea, a new model of atmospheric flow and its numerics, and many more. Discussions were lively and many different research areas met for the first time resulting in interesting talks and contacts.

The workshop was a tremendous success and we are looking forward to repeat this kind of conference in Oberwolfach again in a few years.

Workshop: Recent Developments in the Numerics of Nonlinear Hyperbolic Conservation Laws and their Use in Science and Engineering

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Abstracts

Recent developments in very high order Residual Distribution Schemes for inviscid and viscous problems.

Rémi Abgrall
(joint work with Mario Ricchiuto, Dante de Santis)

The numerical simulation of compressible flow problems, or more generally speaking, of partial differential equations (PDEs) of hyperbolic nature, has been the topic of a huge literature since the seminal work of von Neuman in the 40's. Among the "hot" topics of the field has been, since the works of Lax, Wendroff, Godunov, Mc Cormack, van Leer, Roe, Harten, Yee and Osher, to give a few names, the development of robust, parameter free and accurate schemes. Among the most successful methods one may quote the van Leer's MUSCL method [23] and modified flux approach of Roe. These techniques are only second order accurate. The accuracy can be improved via the ENO/WENO methods by Harten, Shu and others.

The emergence of modern parallels computers, another concern has emerged: what about accuracy and efficiency ? Indeed, it is now important to develop robust algorithms that scale correctly on parallel architecture. This can be achieved more or less easily if the stencil of the numerical scheme is as compact as possible. Good candidates are the schemes relying on finite element technology, such as the Discontinuous Galerkin methods [10] or the stabilized continuous finite element methods [14]. In these methods, the numerical stencil is the most possible compact one.

In these notes, we discuss in some details of another class of numerical schemes, the so-called Residual Distribution schemes , also denoted by Fluctuation Splitting schemes. The history of these schemes can be traced back to the work of P.L. Roe [20] and even his famous 1981 paper [21] where he does not define a finite volume scheme but a true residual distribution scheme. Indeed, the first RD scheme ever was probably presented by Ni [15]. The idea was to construct a scheme with the most compact computational stencil that can ensure second order accuracy. This scheme had some similarities with the Lax Wendroff one.

If these RD share many similarities with more established schemes such as the SUPG scheme by Hughes and coworkers [13], the driving idea is (i) to introduce the upwinding concept, (ii) to manage such that a provable or a practical maximum principle is achieved without any parameter to tune. In our opinion, (ii) is the most important feature.

In Roe's paper and the first RD papers, the main idea was to introduce upwinding into the numerical formulation of the problems, coupled in a very clever way, with a technique to reach second order accuracy for steady problem. This has been presented in a series of papers and VKI reports, see e.g. [11, 22]. Two schemes had emerged at the time : the N scheme by Roe and the PSI scheme by R. Struijs, see [22]. The first one is probably the optimal first order strategy for scalar

problems using triangular meshes, the second one the best second order scheme on these type of meshes, for steady problems again. When dealing with systems or non triangular meshes, the situation became more complex, and it appeared that the upwinding concept had to be relaxed a bit.

Since the early days, many contributions have been given. Among the issues, two are more difficult because they do not cast a priori naturally in the original RD framework. An interesting contribution on the approximation of viscous problem is the work of H. Nishikawa [16]. Last, and up to our knowledge the first contribution on higher than second order accurate RD scheme is due to Caraeni [9], as well as early work on unsteady and viscous problem. One of the main differences with the approach emphasized in this paper is that Caraeni's schemes are not as compact as here

This paper presents the authors' view of what is the current status of RD scheme for *steady* problems. In a first part, we present a reinterpretation of standard finite volume schemes and show on a simple example how maximal accuracy can be reached with a minimal stencil. In the second part, we present a general framework to describe what a RD scheme is and provide some examples. Then we give a very formal variational formulation and some connection with more established schemes such as the Discontinuous Galerkin schemes or the stabilized continuous finite element methods. In a second part, we discuss a systematic way of getting a non oscillatory scheme, without tunable parameter, even in the system case. Numerical examples are given for illustration. The last section is devoted to some extensions, in particular the unsteady case and the viscous case. This sumamrizes [1, 2, 3, 4, 5, 6, 7, 8].

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Fast implicit solvers for unsteady Navier-Stokes: 3D-DG

Philipp Birken
(joint work with Mark Haas, Gregor Gassner, Claus-Dieter Munz)

We consider the computation of unsteady compressible viscous flows. These problems are particularly stiff, making implicit time integration schemes very interesting. To make implicit methods truly attractive, fast solvers for the appearing linear or nonlinear equation systems are needed.

One focus in the CFD community for decades has been the design of fast solvers for steady flows with the result being superfast nonlinear multigrid methods for the steady Euler equations [7]. Their use is ubiquitious in industry codes and widespread in academia. By contrast, Newton methods are shunned because they are supposedly difficult to implement, need a lot of storage and are slow to converge since the canonical initial guess, namely freestream values, is typically far away from the steady state.

To transfer an existing multigrid code to unsteady codes, the nonlinear system is solved using the dual time stepping approach of Jameson [10] that just reuses the

existing multigrid code. While this comes at almost no additional implementation cost, it also leads to a significant decrease in convergence speed. The reason for this is essentially, that unsteady flows are different from steady flows and that a method for the steady Euler equations is not necessarily a good method for the unsteady Navier-Stokes equations.

Regarding time integration, typically BDF methods are used for higher order in time. This is due to the fallacy that they need just one nonlinear equation system, as opposed to implicit Runge-Kutta methods, which need several per time step and therefore must be less efficient.

All in all, it must be said that the standard schemes in use do not provide efficient computations of unsteady viscous compressible flows. We argue that there are two ways forward:

- First, the redesign of the multigrid methods, since these strongly depend on the equations and the convergence decrease can be attributed to that.
- Second, the use of preconditioned Jacobian-Free Newton-Krylov (JFNK) methods [11] inside time adaptive implicit Runge-Kutta methods with an intelligent choice of all solver tolerances.

For the first approach, consider the linear advection equation with periodic boundary conditions, discretized in space using a first order upwind finite volume scheme and using the implicit Euler method in time. For a steady state computation, the eigenvalues would then be $\lambda(\Theta) = -a/\Delta x(1 - e^{-i\Theta})$, whereas for an unsteady computation, we would obtain $\lambda(\Theta) = -1 - a\Delta t/\Delta x(1 - e^{-i\Theta})$ with $\Theta \in [\pi, \pi]$. This suggests that a multigrid smoother optimized for the eigenvalues of the first case will not behave very well for the second case. In [2], this point is elaborated for the case of explicit Runge-Kutta smoothers. An improvement of up to a factor of two is seen just by reoptimizing the parameters of the smoother.

Regarding the second approach, it has already been shown by Bijl et. al. that DIRK methods are competitive with BDF schemes at engineering tolerances [1]. Time adaptivity is trivial due to embedded schemes of lower order, leading again to significant improvements in CPU time, as demonstrated for example in [6] for a thermal Fluid Structure Interaction problem. To make these schemes truly efficient, inexact Newton schemes should be used, combined with a JFNK scheme, resulting in second order convergence for the nonlinear systems [8]. The linear systems are solved using preconditioned GMRES, where the preconditioner is a crucial ingredient for efficiency. Note that with the exception of the preconditioner, this type of schemes is reasonably easy to implement and flexible regarding a change of discretization or the underlying equation.

Of particular use are preconditioners that work well in parallel. Prime examples are multigrid schemes, but it turns out that using the FAS multigrid as a nonlinear preconditioner does not work well [5, 4]. Nevertheless, linear multigrid schemes as discussed above are good candidates. Another idea is to exploit that we have a sequence of linear systems. This is done in [3] using the ideas of triangular preconditioner updates.

In the context of finite volume schemes, reasonable preconditioners exist and implicit schemes are used for a large number of applications of unsteady flows. However, when considering high order methods, the situation becomes different and explicit schemes become hard to beat. For DG schemes, the major reason for this is the significantly increased number of unknowns per cell, leading to a Jacobian consisting of large blocks, which makes the setup and application of preconditioners extremely costly. For a modal DG scheme, we suggest the ROBO-SGS preconditioner (reduced offblock order symmetric Gauss-Seidel), a variant of SGS where the off diagonal blocks are computed only to a reduced order. Using a polymorphic modal-nodal DG scheme [9], we are able to gain a factor of 10 compared to an explicit scheme.

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Discrete Flux-Corrected Transport and Applications in Image Processing

Michael Breuss

In the focus of this contribution is a completely discrete approach to construct flux-corrected transport (FCT) schemes. We show how to use the viscosity form of a numerical method to derive in a discrete framework such schemes. They differ from classic FCT of Boris and Book by a systematic compression. Exactly this makes

them ideal for some applications in image processing like e.g. in mathematical morphology.

The first work in which this framework has been described in detail is given by [3]. There, it has been applied to deal with the hyperbolic initial value problems

$$u_t = \pm\|\nabla u\|, \quad u(\mathbf{x}, t = 0) = f(\mathbf{x}) \tag{1}$$

These are the constituting (Eikonal-type) equations of *dilation* and *erosion* (corresponding to '+' and '−' in (1), respectively) that are the building blocks of mathematical morphology. The given input image f is usually processed by a well-engineered chain of dilation/erosion processes. The method has been extended and analysed in more detail in the context of scalar (grey-value) data [4], and it has been extended to work with tensor-valued data (for symmetric positive definite matrices) as well [1, 2].

We also show how to use Markov chain models to construct discrete FCT schemes for solving hyperbolic conservation laws. The investigation of this approach to construct such numerical schemes has started with the recent work [5].

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Analysing numerical mixing and dissipation for discretisations of the advection equation, with applications to ocean modelling

Hans Burchard

1. Introduction

Large and regional scale ocean models based on the primitive equations (hydrostatic Reynolds-averaged Navier-Stokes equations) are typically of relatively coarse spatial resolution, given the often complex bathymetries and coast lines. Therefore, non-linear terms such as the advection terms in scalar and momentum equations are often approximated with low accuracy. The purpose of this presentation is therefore to derive the tracer variance decay method as analysis method for numerical mixing by [1] for a layered model in one spatial dimension including

varying layer thickness. In a second step, the tracer variance method is extended towards a velocity variance method, i.e., it is applied to the momentum equation.

2. Dynamic equations

In this simple one-dimensional setting, the following dynamic equations are to be discretised: An equation for the horizontal momentum u (not shown) an equation for the scalar tracer, s,

$$\partial_t (sh) + \partial_x (hus) - \partial_x (K_h h \partial_x s) = 0, \tag{1}$$

with horizontal diffusivity, K_h, and a dynamic equation for the depth h (not shown). Integration of (1) with respect to x and application on of periodic boundary conditions shows that the total tracer is conserved: $\partial_t \int_0^L sh \, \mathrm{d}x = 0$. Multiplying (1) with $2s$ results in

$$\partial_t \left(s^2 h\right) + \partial_x \left(hus^2\right) - \partial_x \left(K_h h \partial_x \left(s^2\right)\right) = -2K_h h \left(\partial_x s\right)^2. \tag{2}$$

3. Discretisation

For demonstration purposes, an explicit first-order upwind discretisation is investigated for the tracer concentration equation for s, (1), shown here for the example of non-negative velocity and $K_h = 0$:

$$\frac{s_i^{n+1} h_i^{n+1} - s_i^n h_i^n}{\Delta t} + \frac{u_{i+1/2}^n h_{i+1/2}^n s_i^n - u_{i-1/2}^n h_{i-1/2}^n s_{i-1}^n}{\Delta x} = 0. \tag{3}$$

For the velocity equation, a similar method is applied.

4. Numerical mixing

Multiplication of (3) by $(s_i^{n+1} + s_i^n)$ results after a number of reformulations in

$$\begin{aligned} &\frac{\left(s_i^{n+1}\right)^2 h_i^{n+1} - \left(s_i^n\right)^2 h_i^n}{\Delta t} + \frac{u_{i+1/2}^n h_{i+1/2}^n \left(s_i^n\right)^2 - u_{i-1/2}^n h_{i-1/2}^n \left(s_{i-1}^n\right)^2}{\Delta x} = \\ &-2\frac{\Delta x}{2} u_{i-i/2}^n h_{i-1/2}^n \left(1 - u_{i-1/2}^n \frac{\Delta t}{\Delta x} \frac{h_{i-1/2}^n}{h_i^{n+1}}\right) \left(\frac{s_i^n - s_{i-1}^n}{\Delta x}\right)^2. \end{aligned} \tag{4}$$

The left hand side of (4) is the explicit first order upstream scheme applied to the square of the tracer. The right hand side of (4) is the numerical violation of the variance conservation which has been shown in (2) to hold for $K_h = 0$, and can be interpreted as the discretisation of the right hand side of (2), with the difference that here the numerical diffusivity instead of the physical diffusivity is used. The right hand side of (4) is the numerical variance decay for the first order upwind scheme and will be denoted as numerical mixing. Since for more complicated, non-linear advection schemes a closed formulation for the numerical mixing can in general not be obtained, a modified approach is applied. After the scalar advection step, an additional advection is carried out for the square of the tracer, using the identical advection scheme. Since that method conserved the square of the scalar (because the advection scheme is conservative), the numerical

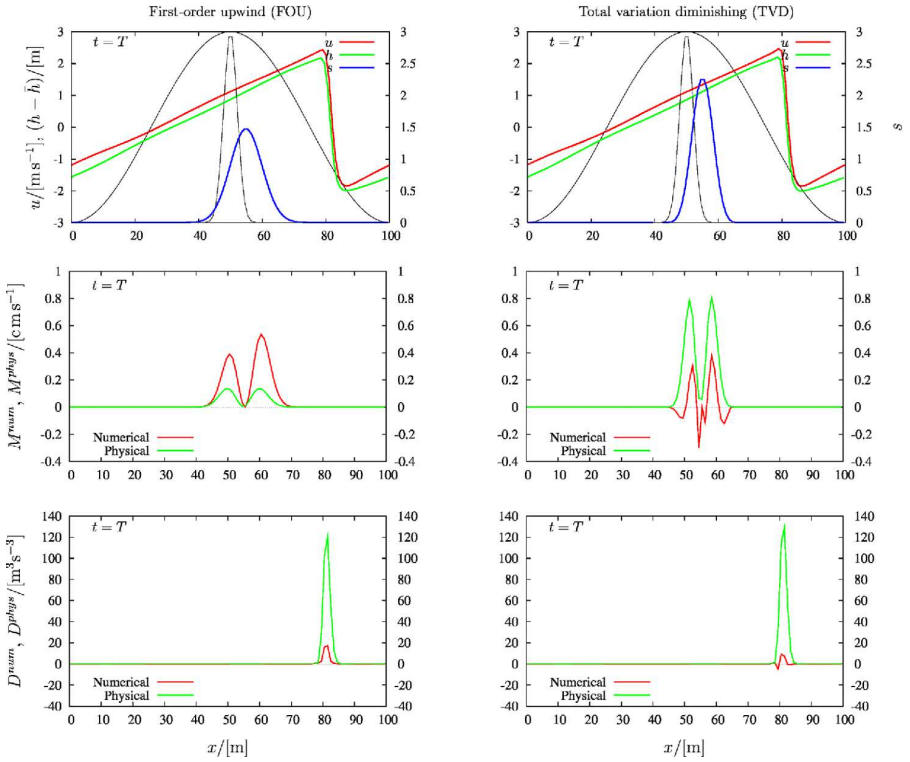

FIGURE 1. Low-resolution reference solution for u, $h-\bar{h}$ and s at $t = T$ for the FOU scheme and for the TVD scheme. The initial conditions for u, $h-\bar{h}$ and s are indicated as thin black curves. Middle panels: Numerical and physical mixing for both schemes at $t = T$. Lower panels: Numerical and physical dissipation for both schemes at $t = T$.

dissipation can be calculated as the difference between the advected square of the scalar and the square of the advected scalar, divided by the time step, see [1] for details. Numerical dissipation, i.e., the kinetic energy loss due to momentum advection, can be calculated in an equivalant way and it not shown here.

5. Example calculations

In the following, results for numerical mixing and dissipation are shown for the discretisations of the dynamic equations (1) and for u and h under consideration of the periodic boundary conditions and harmonic initial conditions. The following parameters are used: $L = 100\mathrm{m}$, $T = 10\mathrm{s}$, $g = 10\mathrm{m\,s^{-2}}$, $\bar{h} = 10\mathrm{m}$ (average depth), $\eta_0 = 3\mathrm{m}$ (initial amplitude of layer thickness), $A_h = 4\mathrm{m^2 s^{-1}}$ (horizontal viscosity) and $K_h = 0.2\mathrm{m^2 s^{-1}}$. Numerical results for the FOU and TVD (Total Variation

Diminishing) schemes at $t = T$ are shown in figure 1. As spatial resolution $\Delta x = L/100 = 1.0$m was chosen and the time step was $\Delta t = T/500 = 2 \cdot 10^{-2}$s. As expected from (4), M^{num} is non-negative for the FOU scheme. In contrast to the FOU scheme, the results for the TV scheme show much smaller numerical than physical mixing. The numerical mixing is partially negative which is due to the non-monotone part of the TVD scheme. Physical mixing is much larger for the TVD scheme than physical mixing for the FOU scheme, because larger gradients and thus larger curvature (second derivative of the tracer) is left. The results for the velocity advection confirm these results (lower panels in figure 1), although less drastically: numerical dissipation is higher (and non-negative) for the FOU scheme, and has some negative values for the TVD scheme. And physical dissipation is slightly higher for the TVD scheme than for the FOU scheme.

6. Conclusions

The implementation of the numerical mixing and dissipation analysis into three-dimensional ocean models is straight-forward and has been demonstrated for the numerical mixing analysis by [2]. As argued above, the method is locally not exact for multi-step or implicit schemes due to the dependence of the discrete advection term on intermediate results. However, since generally the effect of advection schemes is local and model dynamics are highly variable, some temporal or spatial averaging should result in a sufficiently accurate estimate of the numerical mixing. The same should apply for the numerical dissipation analysis which has not yet been performed for three-dimensional ocean models.

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Dealing with parasitic behaviour in G-symplectic integrators

John Butcher
(joint work with Yousaf Habib, Adrian Hill)

The aim of this work is to understand the possible role in the long-term integration of conservative systems by "G-symplectic" methods. Although symplectic behaviour, or the exact conservation of quadratic invariants, for irreducible methods of this type, is not possible [2], there is a G-generalization, similar to the generalization introduced by Dahlquist [3] in the study of non-linear dissipative methods. Specific issues in this research include the role of time-reversal symmetry in conservative integration, the elimination of parasitic effects and the construction and implementation of specific methods of increasingly high orders.

A Runge-Kutta method (A, b^T, c) with the property that $M = \mathrm{diag}(b)A + A^T\mathrm{diag}(b) - bb^T = 0$ is said to be symplectic or canonical. Such methods conserve quadratic invariants and symplectic behaviour. Although genuine multivalue methods cannot possess an equivalent property, G-symplectic general linear methods can give excellent results.

We will be considering differential equations of the form $y'(x) = f(y(x)), y(x_0) = y_0$,. where $f : X \to X$ and X is an inner-product space. We will often be talking about a quadratic form Q on X and we will want to ask questions about how $\langle y_n, Qy_n\rangle$ behaves for $n = 0, 1, 2, \ldots$, where y_n is a numerical approximation to the solution after n time steps..

The following identity relating the input y_{n-1} to step number n, the output y_n and the stage derivatives F_i can be verified:

$$\langle y_n, Qy_n\rangle = \langle y_{n-1}, Qy_{n-1}\rangle + 2h\sum_i b_i\langle F_i, QY_i\rangle - h^2\sum_{ij} m_{ij}\langle F_i, QF_j\rangle,$$

where Q is a quadratic form. If $\langle y(x), Qy(x)\rangle$ is conserved, then $\langle y'(x), Qy(x)\rangle = 0$ and for a Runge-Kutta method $\langle F_i, QY_i\rangle = 0$. Hence, if $M = 0$, $\langle y_n, Qy_n\rangle = \langle y_{n-1}, Qy_{n-1}\rangle$. This means that all quadratic invariants of a problem are conserved and, furthermore, the method observes symplectic behaviour.

We now generalize from Runge–Kutta methods, with a single input and output, to general linear methods with multiple inputs and outputs. Let r denote the number of inputs (and outputs) and s the number of stages. The inputs and outputs to step n are respectively $y_i^{[n-1]}, y_i^{[n]}, i = 1, 2, \ldots, r$. Write the stage values and stage derivatives as $Y_i, F_i, i = 1, 2, \ldots, s$. A particular method in this family is characterized by a partitioned $(s + r) \times (s + r)$ matrix

$$\begin{bmatrix} A & U \\ B & V \end{bmatrix}$$

and the quantities computed in a step satisfy

$$Y_i = h\sum_{j=1}^{s} a_{ij}F_j + \sum_{j=1}^{r} u_{ij}y_j^{[n-1]}, \qquad i = 1, 2, \ldots, s,$$

$$y_i^{[n]} = h\sum_{j=1}^{s} b_{ij}F_j + \sum_{j=1}^{r} v_{ij}y_j^{[n-1]}, \qquad i = 1, 2, \ldots, r.$$

The matrix M, in the theory of Runge–Kutta methods, generalizes, in the case of general linear methods, to

$$M = \begin{bmatrix} DA + A^TD - B^TGB & DU - B^TGV \\ U^TD - V^TGB & G - V^TGV \end{bmatrix},$$

where D is a positive diagonal matrix. If $M = 0$, it is found that $\|y^{[n]}\|_G = \|y^{[n-1]}\|_G$, with a similar result for any quadratic invariant, including the wedge product occurring in the definition of symplectic methods. We will refer to methods with this property as being "G-symplectic". It is natural to ask the question "Is this a useful property?"

We will consider how parasitic solutions can destroy computed results, for the single case $V = \mathrm{diag}(1,-1)$. Let $-\mu$ denote the $(2,2)$ element of BU. At the beginning of step n, suppose the second component of $y^{[n-1]}$, has been corrupted by $(-1)^{n-1}z_{n-1}$. This will produce a perturbation in stage number i of approximately $(-1)^{n-1}u_{i2}z_{n-1}$. The corresponding perturbation in stage derivative number i is approximately $(-1)^{n-1}\frac{\partial f}{\partial y}u_{i2}z_{n-1}$. Substitute into the formula for the second output value and we find, approximately,

$$(-1)^n z_n = (-1)^{n-1}\Big(- z_{n-1} + \sum_{i=1}^{s} b_{2i}u_{i2}\frac{\partial f}{\partial y}z_{n-1}\Big) = (-1)^n\Big(1 + h\mu\frac{\partial f}{\partial y}\Big)z_{n-1}.$$

Hence, z_n approximately satisfies the differential equation $z' = \mu\frac{\partial f}{\partial y}z$. It looks as though we will be better off if $\mu = 0$.

Method **P** below was introduced in [1] and has order 4. Alongside it we consider method **N** which also has order 4.

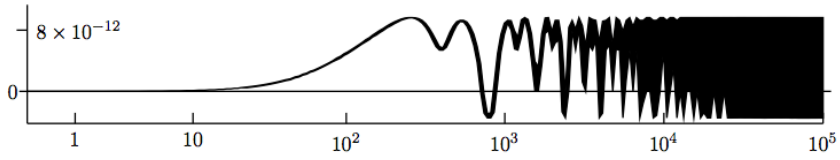
Method **P:**
$$\left[\begin{array}{cc} A & U \\ B & V \end{array}\right] = \left[\begin{array}{cc|cc} \frac{3+\sqrt{3}}{6} & 0 & 1 & \frac{3+2\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} & \frac{3+\sqrt{3}}{6} & 1 & -\frac{3+2\sqrt{3}}{3} \\ \hline \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & -1 \end{array}\right].$$

Method **N:**
$$\left[\begin{array}{cc} A & U \\ B & V \end{array}\right] = \left[\begin{array}{cc|cc} \frac{3-\sqrt{3}}{6} & 0 & 1 & -\frac{3-2\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} & \frac{3-\sqrt{3}}{6} & 1 & \frac{3-2\sqrt{3}}{3} \\ \hline \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & -1 \end{array}\right].$$

For **P**, $\mu = 1 + \frac{2\sqrt{3}}{3}$ and for **N**, $\mu = 1 - \frac{2\sqrt{3}}{3}$.

Numerical tests with the simple pendulum with a large amplitude show a breakdown of the performance afetr a few thousand time step, for both **P** and **N**. However, it is known that the growth factors accumulate over successive steps. Hence, it should be possible to combine sequences of **P** and **N** so that the total of all μ values is bounded.

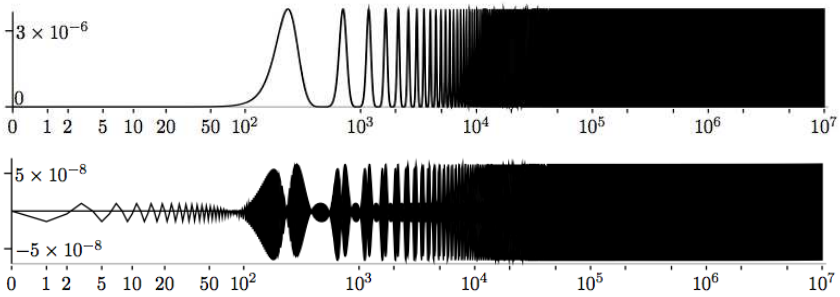
The following figure shows the deviation of the Hamiltonian from its initial value in a calculation based on this cancellation approach over 10^5 time steps.



It is also possible to construct methods for which $\mu = 0$. We will present two methods, of orders 2 and 4, respectively.

$$p=2: \left[\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 1 & 1 \\ -\frac{1}{12} & \frac{1}{4} & 0 & 0 & 1 & \frac{1}{2} \\ -\frac{1}{4} & \frac{5}{6} & \frac{1}{4} & 0 & 1 & -\frac{1}{2} \\ -\frac{1}{3} & 1 & \frac{1}{3} & 0 & 1 & -1 \\ \hline -\frac{1}{6} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{6} & 1 & 0 \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{6} & 0 & -1 \end{array}\right] \qquad p=4: \left[\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 1 & -\frac{\sqrt{5}}{12} & -\frac{1}{12} \\ -\frac{1}{12} & \frac{1}{4} & 0 & 0 & 1 & 0 & -\frac{1}{12} \\ -\frac{7}{60} & \frac{7}{10} & \frac{1}{4} & 0 & 1 & 0 & \frac{1}{12} \\ -\frac{1}{5} & \frac{7}{10} & \frac{1}{2} & 0 & 1 & \frac{\sqrt{5}}{12} & \frac{1}{12} \\ \hline -\frac{1}{10} & \frac{3}{5} & \frac{3}{5} & -\frac{1}{10} & 1 & 0 & 0 \\ -\frac{1}{5} & \frac{6}{5} & -\frac{6}{5} & \frac{1}{5} & 0 & 0 & 1 \\ \frac{\sqrt{5}}{5} & 0 & 0 & -\frac{\sqrt{5}}{5} & 0 & -1 & 0 \end{array}\right]$$

The deviations of the Hamiltonian from its initial value, for the simple pendulum, is shown for each of these methods for 10^7 time steps.



These numerical simulations are encouraging and are reason enough to try to understand these methods better, both theoretically and in practice.

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Energy-Stable Weighted Essentially Non-Oscillatory Finite-Difference Schemes

Mark H. Carpenter

Conventional WENO formulations encounter serious challenges at nodes that are near domain boundaries. To illustrate this, first recall that all high-order finite-difference formulations use "inward biased" stencils near boundaries that maintain the accuracy and stability of the interior scheme. Next, note that WENO schemes invoke stencil biasing mechanics throughout the domain, including nodes that adjoin the boundaries (e.g., fourth-order WENO schemes test three candidate stencils at each node). Thus, not only must the boundary closures that are used

for WENO schemes be 1) inward biased, 2) stable, and 3) accurate, they also must be stable for any possible combination of candidate stencils that *occur anywhere in the domain*. Simultaneously satisfying these constraints is a remote possibility if left strictly to chance.

Fourth-order boundary closures for the "finite-domain" operator are described in a recent article by Fisher et al. [2]. These closures and near-wall biasing mechanics complement the *periodic domain* ESWENO finite-difference methodology reported in Refs. [5] and [1]. The new interior/boundary ESWENO schemes retain all of the salient features of the original periodic schemes, including: 1) conservation and L_2-energy stability for constant coefficient (linear) hyperbolic systems, 2) design-order accuracy throughout the domain, including regions near the boundaries or near smooth extrema, and 3) full WENO stencil biasing mechanics at all possible points. The finite-domain ESWENO schemes are constructed by adding a "special" nonlinear artificial dissipation term to a conventional WENO scheme. The additional term is design-order accurate for smooth solutions including extrema, and is constructed such that the resulting ESWENO scheme is stable in the L_2-energy norm.

Herein, sixth-order ESWENO schemes are described for the "finite-domain" problem. The design strategy for the for these schemes is equivalent to that used to build the periodic domain ESWENO schemes:

- Develop a sixth-order finite-domain target scheme that is stable, conservative, and accurate for smooth flows. This task is accomplished using the summation-by-parts (SBP) [4, 5] matrix operator framework.
- Recast the target scheme in the "dual grid" framework of the conventional WENO approach, whereby the solution is stored and advanced at the grid points, while the interface fluxes that are constructed at the "half points" ensure conservation. A special set of flux points and interpolants are required near the boundaries to accomplish this task.
- Develop a finite-domain WENO biasing strategy that allows all stencil weights to deviate from their target values. Precise control of the the biasing mechanics ensures design-order accuracy for smooth solutions and essentially non-oscillatory properties at discontinuities.
- Test the stability of the finite-domain WENO scheme.If it is unstable, add a design-order artificial dissipation term that ensures an L_2-energy estimate for the combined operator.

1. Numerical Tests

The accuracy and stability characteristics of the new finite-domain ESWENO framework were tested by using the one-dimensional advection equation and the quasi-one-dimensional and two-dimensional Euler equations. As a more practical test, the framework was implemented in a two-dimensional reacting Navier-Stokes solver and was used to simulate supersonic nonreacting and reacting shear layers. All simulations were integrated in time or pseudo-time with the same five-step, fourth-order Runge-Kutta solver.

The propagation of the isentropic Euler vortex is used to show the efficacy of the new ESWENO schemes. (An exact solution to the convected vortex can be found in ref. [2].) The physical domain is a rectangular box containing a cylindrical hole, and is discretized using a multi-domain ESWENO techniques on five individual domains. Gridlines are continuous at all domain interfaces, but their derivatives are not (i.e. C_0 connectivity). The exact solution is used on all boundaries for the physical boundary conditions, imposed weakly using the SAT penalty method. Reflections from the boundaries are design-order accurate and are weakly influenced by the choice of the SAT penalty term.

The L_2 convergence rates for the ESWENO$_{5\text{-}6\text{-}5}$ operator are tabulated in table 1. The results show that the linear operators recover the design-order accuracy. The ESWENO operator asymptotically approach the linear target values as the resolution improves.

TABLE 1. L_2 error and convergence rates for the ESWENO$_{5\text{-}6\text{-}5}$ and linear block-norm central difference operators for the two-dimensional inviscid vortex.

	Linear Block Norm		ESWENO$_{5\text{-}6\text{-}5}$	
Number of Cells	**L_2 Error**	**L_2 Rate**	**L_2 Error**	**L_2 Rate**
Resolution	L_2 Error	L_2 Rate	L_∞ Error	L_∞ Rate
32×32	3.28-05	-	1.28e-04	-
64×64	5.89e-07	5.80	2.68e-06	5.58
128×128	9.81e-09	5.91	4.52e-08	5.89

2. Conclusions

A general strategy is presented in Refs. [1] and [5] for constructing Energy-Stable Weighted Essentially Non-Oscillatory (ESWENO) finite-difference schemes on *periodic* domains. ESWENO schemes up to eighth-order are developed and proved to be stable in the energy norm for both continuous and discontinuous solutions of systems of linear hyperbolic equations. Herein, boundary closures are presented for the sixth-order ESWENO scheme that maintain wherever possible the WENO stencil biasing properties, while satisfying the summation-by-parts (SBP) operator convention, thereby ensuring stability in an L_2 norm. A novel set of non-uniform flux interpolation points is necessary near the boundaries to simultaneously achieve 1) accuracy, 2) the SBP convention, and 3) WENO stencil biasing mechanics. The novelty lies in the recognition that the discrete set of flux points $\bar{\mathbf{x}}$ must be consistent with (i.e., derived from) the stability norm $\mathcal{P}$ that is used in the SBP formulation. Using the new flux points $\bar{\mathbf{x}}$, third-order, and fifth-order boundary closures are developed that achieve stability in a diagonal and block norm, respectively.

Extensive numerical validation is presented to assess the efficacy of the new boundary closures. The test problems included 1) the unsteady one-dimensional

linear wave equation (hyperbolic), 2) steady quasi-one-dimensional nozzle flow that is simulated by solving the nonlinear Euler equations, 3) unsteady propagation of a two-dimensional Euler vortex, 4) unsteady convection-diffusion-reaction of a supersonic hydrogen-air mixing layer that is simulated by solving the two-dimensional Navier-Stokes equations, and 5) laminar flow around a multi-element airfoil The global accuracy of the third-order diagonal-norm operators and fifth-order block-norm operators is shown to be design order; that is, four and six, respectively, for all test cases. Furthermore, the accuracy of the new ESWENO operators are shown to be quite close to the linear target operator.

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A Simple Eulerian Finite-Volume Method for Compressible Fluids in Domains with Moving Boundaries

Alina Chertock

(joint work with Alexander Kurganov)

We are interested in developing a simple, accurate, and robust numerical method for computing compressible fluids in the domains with moving boundaries. In the two-dimensional (2-D) case, the governing equations are the compressible Euler equations:

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(E+p) \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}_y = 0,$$

where ρ is the fluid density, u and v are the velocities, E is the total energy, and p is the pressure. The system is closed using the equation of state (EOS), which, for ideal gases, reads:

$$E = \frac{p}{\gamma - 1} + \frac{\rho}{2}(u^2 + v^2), \quad \gamma = \text{const}.$$

We also introduce the notation $c := \sqrt{\gamma p/\rho}$ for the speed of sound.

Our goal is to apply Eulerian finite-volume (FV) methods to the problems with changing geometries. To this end, we place the fluid domain into the computational domain of a fixed size, which is divided into Cartesian cells. At every time moment, each cell is marked as either internal, external, or boundary one. The internal cells are fully occupied by the gas, the external cells are located outside of the fluid domain and play the role of the so-called ghost cells, while the boundary cells form a thin layer between the internal and external ones. The boundary cells have to be introduced since in the case of moving boundaries, the fluid domain boundary cannot, in general, be forced to coincide with the cell edges. As a result, the boundary cells are only partially filled with the gas, which is very inconvenient since within the FV computational framework, numerical solutions are represented in terms of the cell averages. One of the possible ways to treat the boundary cells is to split each of them into two smaller cells: the internal and the external ones. However, this would significantly increase the complexity of the entire solution algorithm and may lead to very small time steps (see, e.g., [1, 3, 4]. We prefer an alternative, simpler approach, in which the averages are computed over the internal cells only and the data contained in the boundary cells are not used for the computation of numerical fluxes. We treat the boundary cells similarly to the way the so-called "mixed" cells have been treated in [2]. Namely, we only approximate the point values at the edges of these cells, required in the numerical flux computations. These point values are obtained using the solid wall extrapolation followed by the interpolation in the phase space (by solving the Riemann problem between the internal cell averages and the extrapolated ones). The numerical solution is then evolved in internal cells only using a FV method.

The proposed computational framework is general and may be used in conjunction with one's favorite FV method. In this talk, we present the semi-discrete second-order central-upwind scheme developed in [5, 6, 7]. This Godunov-type scheme enjoys all major advantages of Riemann-problem-solver-free, non-oscillatory central schemes and, at the same time, have a certain "built-in" upwind nature. The robustness of the new approach is illustrated on a number of one- and two-dimensional numerical examples.

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Discontinuous Galerkin method - a robust solver for compressible flow

Miloslav Feistauer
(joint work with Jan Česenek and Václav Kučera)

In the numerical solution of compressible flow, it is necessary to overcome several obstacles. Let us mention the necessity to resolve accurately shock waves, contact discontinuities and (in viscous flow) boundary layers, wakes and their interaction. All these phenomena are connected with the simulation of high speed flow with high Mach numbers. However, it appears that the solution of low Mach number flow is also rather difficult. This is caused by the stiff behaviour of numerical schemes and acoustic phenomena appearing in low Mach number flows at incompressible limit. In this case, standard finite volume schemes fail. This led to the development of special finite volume techniques allowing the simulation of compressible flow at incompressible limit, which are based on modifications of the Euler or Navier-Stokes equations. However, these techniques could not be applied simultaneously to the solution of high speed flow. Therefore, further attempts were concentrated on the extension of these methods to the solution of flows at all speeds. A success in this direction was achieved by several authors. Main ingredients of these techniques are finite volume schemes applied on staggered grids, combined with multigrid, the use of the pressure-correction, multiple pressure variables and flux preconditioning.

At the Faculty of Mathematics and Physics of the Charles University in Prague our group (M. Feistauer, V. Dolejší, V. Kučera, M. Vlasák together with several PhD and Master students) is concerned with numerical solution of nonlinear convection-diffusion problems and compressible flow by the discontinuous Galerkin method (DGFEM), which employs piecewise polynomial approximations without any requirement on the continuity on interfaces between neighbouring elements. We found that this technique, equipped with several important ingredients, becomes an accurate, efficient and robust method allowing the solution of compressible flow with a wide range of the Mach number and Reynolds number.

Our main goal is to develop an efficient and robust DGFEM for the numerical simulation of compressible flow in time dependent domains with applications to fluid-structure interaction (FSI). The interaction of fluid flow with vibrating bodies plays a significant role in many areas of science and technology. We mention, for example, development of airplanes (vibrations of wings) or turbines (blade vibrations), some problems from civil engineering (interaction of wind with constructions as bridges, TV towers or cooling towers of power stations), car industry (vibrations of various elements of a carosery), but also in medicine (hemodynamics or flow in the glottis with vibrating vocal folds). In a number of these examples the

moving medium is a gas, i.e. compressible flow. For low Mach number flows incompressible models are used (as e.g. in [1], [7]), but in some cases compressibility plays an important role.

The solution of fluid-structure interaction requires the coupling of the solution of equations describing the fluid flow with equations describing the structural behaviour. Due to the deformation and/or vibrations of structures, the computational domain is time dependent. In the numerical simulation of compressible flow in time dependent domains it is necessary to overcome difficulties caused by the change of the domain and computaional mesh geometry in time, nonlinear convection dominating over diffusion, which leads to boundary layers and wakes for large Reynolds numbers and instabilities caused by acoustic effects for low Mach numbers.

The goal of our research is the numerical simulation of interaction of compressible flow with structures, particularly the flow induced airfoil vibrations or the flow past an elastic wall vibrating due to the influence of an airflow. We are concerned with the generalization of the methods from [5], [4] and [3] to the solution of compressible inviscid and viscous flow in time dependent domains. The main ingredients of the method is the discontinuous Galerkin space semidiscretization of the Euler and Navier-Stokes equations written in the ALE (arbitrary Lagrangian-Eulerian) form, semi-implicit time discretization using backward difference formula or space-time DGFEM, suitable treatment of boundary conditions and the shock capturing avoiding Gibbs phenomenon manifested by spurious overshoots and undershoots near internal and boundary layers.

The numerical solution of compressible flow is coupled with the solution structure problem. In the case of the simulation of flow-induced vibrations of an elastically supported airfoil, the airfoil motion is described by a system of second-order ordinary differential equations discretized by the Runge-Kutta method. For the simulation of the interaction of flow with an elastic body, a dynamic elasticity system is used, which is solved by the conforming finite element method. Special attention is paid to the construction of the ALE mapping and the realization of the coupling of the flow and structure problems. Some details of this technique are described in [5], [6], [2].

Numerical experiments carried out for flow past an oscillating airfoil and in a channel with moving elastical walls demonstrate the applicability of the developed method to technically relevant problems and show the accuracy and robustness of the worked out technique.

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Numerics of the shallow water equations

Jiří Felcman
(joint work with Oto Havle, Libor Kadrnka)

1. Introduction

The shallow water equations (SWE) can be formulated as a two dimensional hyperbolic system for an unknown vector valued function $\boldsymbol{w}$

$$\frac{\partial \boldsymbol{w}}{\partial t} + \sum_{s=1}^{2} \frac{\partial}{\partial x_s} \boldsymbol{f}_s(\boldsymbol{w}) = \boldsymbol{s}(\boldsymbol{w}), \tag{1}$$

with fluxes

$$\boldsymbol{f}_s(\boldsymbol{w}) = (hv_s, hv_1v_s + \delta_{s1}\frac{1}{2}gh^2, hv_2v_s + \delta_{s2}\frac{1}{2}gh^2)^{\mathrm{T}}$$

and a source term $\boldsymbol{s}(\boldsymbol{w}) = (0, -gh\nabla z)^{\mathrm{T}}$. The unknown variables $\boldsymbol{w} = \boldsymbol{w}(x,t)$, $x = (x_1, x_2)$, form a vector $\boldsymbol{w} \equiv (h, h\boldsymbol{v})^{\mathrm{T}} \equiv (h, hv_1, hv_2)^{\mathrm{T}}$, where h and $\boldsymbol{v}$ denote the water height and velocity, respectively. The gravity constant $g > 0$ and a topography function $z = z(x)$ are given.

A standard finite volume method (FVM) [1], [2] can be applied for finding the value of the approximate solution $\boldsymbol{w}_i^k$ on the finite volume D_i at time t_k, $\boldsymbol{w}(\cdot, t_k)|_{D_i} \approx \boldsymbol{w}_i^k = const.$:

$$\boldsymbol{w}_i^{k+1} = \boldsymbol{w}_i^k - \frac{\tau_k}{|D_i|} \sum_{j \in S(i)} |\Gamma_{ij}| \boldsymbol{H}\left(\boldsymbol{w}_i^k, \boldsymbol{w}_j^k, \boldsymbol{n}_{ij}\right). \tag{2}$$

Here we adopt the notation and its description from [1, 3] where we dealt with the finite volume method for the Euler equations written as the hyperbolic system.

Remark The approximation of the source term $\boldsymbol{s}(\boldsymbol{w})$ is included into the definition of the numerical flux in relation (2).

In [4] a time marching finite volume method for SWE was proposed and numerically tested for a one-dimensional case. The novelty of that approach lies in the design of the new numerical flux of Vijayasundaram type, its analysis and numerical examples. We further extend the proposed approach by enhancing the

method by mesh adaptation. We consider the finite volume solution of system (1) as piecewise constant vector-valued functions $\mathbf{w}^k_{\mathcal{D}^k}, k = 0, 1, \ldots$, defined almost everywhere in the computational domain Ω so that $\mathbf{w}^k_{\mathcal{D}^k}\big|_{\overset{\circ}{D}_i} = \boldsymbol{w}^k_i$ for all $i \in J$, where $\overset{\circ}{D}_i$ is the interior of D_i and $\boldsymbol{w}^k_i$ are obtained from the finite volume formula (2).

Once a numerical solution $\mathbf{w}^k_{\mathcal{D}^k}$ is obtained at the given time level t_k on the finite volume mesh $\mathcal{D}^k$, the vertices of the mesh $\mathcal{D}^k$ will be redistributed using an iteration procedure. At each iteration, the vertices are moved according to a variational principle. After this iteration loop the new mesh $\mathcal{D}^{k+1}$ is available and the finite volume solution $\mathbf{w}^{k+1}_{\mathcal{D}^{k+1}}$ on the adapted mesh $\mathcal{D}^{k+1}$ at the time step t_{k+1} is computed.

2. Adaptive Finite Volume Method

The basic FVM (2) yields for the given mesh $\mathcal{D}$ and the piecewise constant solution $\mathbf{w}^k_{\mathcal{D}}$ on it the passage to the update $\mathbf{w}^{k+1}_{\mathcal{D}}$ without the change of the mesh:

$$\mathbf{w}^{k+1}_{\mathcal{D}} := \mathrm{FVsol}\left(\mathbf{w}^k_{\mathcal{D}}, \mathcal{D}\right). \tag{3}$$

The adaptive FVM changes adaptively the computational mesh from $\mathcal{D}^k$ to $\mathcal{D}^{k+1}$ during the time evolution and computes the new approximation $\mathbf{w}^{k+1}_{\mathcal{D}^{k+1}}$ on the adapted mesh $\mathcal{D}^{k+1}$:

$$\mathbf{w}^{k+1}_{\mathcal{D}^{k+1}} := \mathrm{AFVsol}\left(\mathbf{w}^k_{\mathcal{D}^k}, \mathcal{D}^k\right). \tag{4}$$

Assignment (4) consists of three steps: 1. Mesh adaptation, 2. Solution recovery and 3. Finite volume update:

$$\mathcal{D}^{k+1} := \mathrm{MeshAdapt}\left(\mathbf{w}^k_{\mathcal{D}^k}, \mathcal{D}^k\right), \tag{5}$$

$$\widetilde{\mathbf{w}}^k_{\mathcal{D}^{k+1}} := \mathrm{SolRecovery}\left(\mathbf{w}^k_{\mathcal{D}^k}, \mathcal{D}^k, \mathcal{D}^{k+1}\right), \tag{6}$$

$$\mathbf{w}^{k+1}_{\mathcal{D}^{k+1}} := \mathrm{FV}\left(\widetilde{\mathbf{w}}^k_{\mathcal{D}^{k+1}}, \mathcal{D}^{k+1}\right). \tag{7}$$

In what follows we shall describe these three steps (5) – (7) and explain the used symbolic notation.

2.1. Mesh Adaptation.

In [5] the mesh generation based on the variational principle was proposed. Here we present its application for the shallow water equations. The vertices of the adapted finite volume mesh in the computational domain $\Omega \subset I\!R^2$ are obtained as images of vertices of a uniform mesh in a rectangular domain Ω_ξ using a one-to-one coordinate transformation $x : \Omega_\xi \ni \xi \longmapsto x(\xi) \in \Omega$. Since we are interested in problems with strong gradients and shocks, it is desirable to refine the mesh in the vicinity of such phenomena. The Winslow's monitor function can be adapted for the SWE. The gradient of the depth of water $h = h(x, t_k)$ fixed at time t_k is chosen as a significant scalar quantity on which the mesh redistribution at time level t_k of the time marching procedure (2) is based. The details can be found in [6].

2.2. **Solution Recovery.** After the adaptive mesh is constructed it is necessary to recompute the solution $\mathbf{w}^k_{\mathcal{D}^k}$ on the old mesh $\mathcal{D}^k$ to its recovery $\widetilde{\mathbf{w}}^k_{\mathcal{D}^{k+1}}$ on the newly adapted mesh $\mathcal{D}^{k+1}$. According to [5] the geometric mass conservation law has to be satisfied in this computational step. It reads

$$\sum_i |D_i^k| \boldsymbol{w}_i^k = \sum_i |D_i^{k+1}| \widetilde{\boldsymbol{w}}_i^k, \tag{8}$$

where $|D_i|$ denotes the two-dimensional measure (area) of the finite volume D_i, $\widetilde{\boldsymbol{w}}_i^k = \widetilde{\mathbf{w}}^k_{\mathcal{D}^{k+1}}\big|_{\overset{\circ}{D}_i^{k+1}}$ and the symbol $\circ$ denotes the interior of D_i^{k+1}. The perturbation method from [5] is applied.

2.3. **Finite Volume Method.** We seek a finite volume scheme, which preserves a class of stationary solutions:

$$h(x,t) = H_0 - z(x), \qquad \boldsymbol{v}(x,t) = 0. \tag{9}$$

The numerical scheme can be written in a form similar to the standard conservative finite volume scheme:

$$\boldsymbol{w}_i^{k+1} = \boldsymbol{w}_i^k - \frac{\tau^k}{|D_i|} \sum_{j \in S(i)} |\Gamma_{ij}| \boldsymbol{H}_{total}\left(\boldsymbol{w}_i^k, \boldsymbol{w}_j^k, z_i, z_j, \boldsymbol{n}_{ij}\right) \tag{10}$$

The numerical flux $\boldsymbol{H}_{total}\left(\boldsymbol{w}_i^k, \boldsymbol{w}_j^k, z_i, z_j, \boldsymbol{n}_{ij}\right)$ is based on the Vijayasundaram type numerical flux. However, we add a correction term, which is necessary to ensure the discrete version of (9). We use the piecewise-constant approximation of the topography function z. The details can be found in [4].

Recently the higher order version of the proposed algorithm is in progress. It is based on the higher order reconstruction of the piecewise constant approximate solution followed by a new slope limiting procedure introduced in [7].

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On one dimensional low Mach number applications

Ingenuin Gasser
(joint work with Maria Bauer, Elisabetta Felaco)

The fluid dynamic applications we have in mind are

- chimneys [FG]
- exhaust tubes [GR]
- solar updraft towers [SS, G]
- energy towers [Z, BG]
- gas pipelines [BGH]
- receivers in parabolic trough power station
- road and railway tunnel fires [GS, G1].

The fluids under consideration are mainly gases or gas mixtures. All the mentioned applications have the following properties in common:

- The main features of the flows are one dimensional or the application can be roughly described by one dimensional models.
- The flows are characterised by a low Mach number.
- The temperature variations in the flows cannot be negleted. Some of the mentioned flows are even buoyancy driven.

The underlying equations are obtained by balancing mass, momentum and energy. Typically we have inviscid compressible gas dynamic equations. After an appropriate scaling small parameters can be identified. Here the Mach number - the relation between fluid and sound velocity - is always small. Typically this requires increased numerical effort when solving the fully compressible equations.

However we use the smallness of the Mach number to perform an asymptotics and to further simplify the models. In combination with a possible small Froud number various asymptotic regimes are possible: small Mach number, Boussinesq, quasistatic approximations etc. We analyse the resulting asymptotic equations. Finally we perform numerical simulations in order to validate the models. The asymptotic models result to recover the main features of the flows and to be numerically convenient.

In fact, the aim is to obtain simple models which allow fast robust numerical simulation tools. This is important when applying control or optimisation procedures or when considering applications on networks.

In the talk we focus on two applications: the chimney and the Energy tower. For both we show the modelling approach and numerical simulations.

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On the Accuracy of High Order Methods for Underresolved Multi-Scale Problem Simulations

Gregor J. Gassner
(joint work with Andrea D. Beck, David Kopriva, Claus-Dieter Munz)

The term *order* or *order of convergence* of a discretization method describes the behavior of the discretization error with respect to the discretization parameter h when refining the discretization, i.e. for $h \to 0$. For 'small' h (well resolved approximations) the error behaves like $\mathcal{O}(h^r)$, where r is the order of convergence. It is common rational that for well resolved approximations of sufficient smooth problems, discretization methods with larger values of r, i.e. *high order methods*, yield better efficiency with respect to computational effort for a given goal accuracy compared to low order methods. This actual trend towards high order methods shows itself in recent European project, such as ADIGMA and its follow-on IDIHOM, where the applicability and performance of high order methods for industrial computational fluid dynamics (CFD) is investigated and evaluated.

A silent assumption, often not clearly stated in the high order CFD community is that one needs *sufficient smoothness* of the underlying problem for high order discretizations to be efficient. For a multi-scale problem, like most relevant fluid flow problems (turbulence!), the solution is smooth, but can often only be coarsly resolved due to the large range of occuring spatial and temporal scales and limited computer resources. This has the consequence of *insufficient grid resolution induced artificial roughness* of the underlying computational problem.

With respect to the discretization parameter, this means that h is 'large' and thus the theoretical error behavior considerations for $h \to 0$ are not valid and not usefull to judge the accuracy of high order methods. Statements about the superiority of high order methods thus cannot be simpley translated to the underresolved case, which is the common case in partical fluid flow simulations and furthermore, common in all multi-scale simulations.

As error convergence behavior of a discretization for 'small' h is not appropriate, other quality considerations of a discretization method have to be investigated, such as dispersion and dissipation properties for a large range of scales. In case of Discontinuous Galerkin (DG) discretizations, it can be shown that the high order

variants yield superior dispersion and dissipation behavior over a broader range of scales [1].

Motivated by this observation and the general question of accuracy for high order methods in coarse resolved multi-scale simulations, the talk presents recent results [2], where the accuracy of high order DG methods for coarse resolved turbulence simlation is investigated. It was found, that even in the case of underresolution, high order approximations yield superior efficiency compared to their lower order variants due to the better dispersion and dissipation behavior. It was furthermore found, that a very high order accurate DG discretization ($r = 16$ in this case) yields even better accuracy than state of the art large eddy simulation (LES) methods for the same number of degrees of freedom (DOF) for the considered example. The suprise stems from the fact that those methods are particularly tuned to capture coarse resolved turbulence, whereas the considered high order method can be directly applied to all sorts of other multi-scale problems.

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IMEX Methods for Hyperbolic Systems with Stiff Relaxation Terms

Willem Hundsdorfer

The numerical solution of hyperbolic systems with relaxation terms

$$u_t = \nabla f(u) + \frac{1}{\epsilon} g(u) \tag{1}$$

leads to different and conflicting demands on standard implicit or explicit time integration methods. On the one hand, if $\epsilon > 0$ is small, the relaxation term becomes very stiff, which calls for an implicit method. On the other, hand, if the convection term is discretized in space with a scheme based on flux-limiting or with a WENO scheme, for example, then explicit time stepping becomes more attractive for that part. This is due to the fact that the nonlinearities introduced by such spatial discretzations are rather complicated, but also implicit methods do not offer substantially better monotonicity properties than explicit methods.

These conflicting demands can be met by so-called IMEX time stepping methods, where explicit and implicit methods are combined. In this talk we will mainly focus on linear multistep methods. First we will briefly describe the monotonicity theory for implicit and explicit multistep methods, and then discuss the construction of suitable IMEX multistep methods.

This review is based on work with Steve Ruuth [4, 5], on the selection of multistep methods and the construction of IMEX methods, and on more recent work

with Anna Mozartova and Marc Spijker [2, 3], on monotonicity theory for linear multistep methods.

Monotonicity and boundedness for linear multistep methods.
Spatial discretization of a conservation law $u_t = \nabla f(u)$ by finite differences or finite volumes leads to an ODE system in a vectorspace $\mathbb{V}$

$$u'(t) = F(u(t)) \tag{2}$$

with initial value $u(0) = u_0$. In the usual applications $\mathbb{V} = \mathbb{R}^m$, with m proportional to the number of cells. To solve these resulting ODE systems we consider numerical approximations $u_n \approx u(t_n)$, $t_n = n\Delta t$, by linear multistep methods

$$u_n = \sum_{j=1}^{k} a_j u_{n-j} + \Delta t \sum_{j=0}^{k} b_j F(u_{n-j}) \tag{3}$$

for $n \geq k$. The starting values for this multistep recursion, $u_0, u_1, \ldots, u_{k-1} \in \mathbb{V}$, are supposed to be given, or computed by a Runge-Kutta method.

For suitable spatial discretizations of the PDE, the resulting ODE system (2) will satisfy

$$\|v + \tau_0 F(v)\| \ \leq \ \|v\| \qquad \text{for all } v \in \mathbb{V} \tag{4}$$

in some suitable norm (e.g. the maximum norm) or semi-norm (e.g. the discrete total variation semi-norm); see [1], for example. Here, τ_0 is determined by the problem and spatial discretization. ¿From (4) it follows that approximations found with the forward Euler method satisfy $\|u_n\| \leq \|u_0\|$ for $n \geq 1$ whenever $\Delta t \leq \tau_0$.

For multistep method we can consider the following property under the basic assumption(4):

$$\|u_n\| \leq \mu \max_{1 \leq j < k} \|u_j\| \qquad \text{for all } \Delta t \leq \gamma\tau_0 \text{ and } n \geq 1\,. \tag{5}$$

If this holds with $\mu = 1$, the method is said to be *monotone* or *strong stability preserving* (SSP) with stepsize coefficient $\gamma > 0$. If it holds with some $\mu \geq 1$ the method is called *bounded*. With the total variation semi-norm, these properties are known as *total variation diminishing* (TVD) if $\mu = 1$, and *total variation bounded* (TVB) if $\mu \geq 1$.

It is easily verified that a linear multistep method is monotone with stepsize coefficient $\gamma > 0$ if $b_0 \geq 0$ and $a_j \geq \gamma b_j \geq 0$ for $1 \leq j \leq m$. The monotonicity property is therefore very easy to verify, but it turns out that most linear multistep methods of practical importance do not satisfy it, due to the requirement that all coefficients are nonnegative. It is therefore interesting to look at more relaxed properties, such as boundedness.

¿From [3] we have the following results. Let $\rho_n = 0$ for $n < 0$, $\rho_0 = 1$, and

$$\rho_n = \sum_{j=1}^{k} a_j \rho_{n-j} + \gamma \sum_{j=0}^{k} b_j \rho_{n-j} \qquad \text{for } n \geq 1\,.$$

for $n \geq 1$. Then, under some technical assumptions, it can be shown that the essential condition for the boundedness property (5), with $\mu \geq 1$, is

$$\sum_{j=0}^{k} b_j \rho_{n-j} \geq 0 \qquad \text{for all } n \geq 0\,. \tag{6}$$

Moreover, if this holds then the combination of the linear multistep method and a Runge-Kutta starting procedure to compute $u_1, \ldots, u_{k-1}$ can still be monotone: $\|u_n\| \leq \|u_0\|$ $(n \geq 1)$ whenever $\Delta t \leq \gamma \tau_0$.

As an example, the implicit BDF2 method has stepsize coefficient $\gamma = \frac{1}{2}$ for boundedness and also for monotonicity with implicit Euler start. This γ is of course not very large; to justify the use of an implicit method a much larger γ would be required. In fact, for the extrapolated BDF2 method, which is the explicit counterpart of the implicit method, we have $\gamma = \frac{5}{8}$ for boundedness and for monotonicity with explicit Euler start.

Also for other implicit methods, the allowable step sizes are not large compared to explicit methods. Since properties like TVD/TVB, or generalizations to other semi-norms, are crucial for convergence towards the proper solutions of conservation laws with shocks, it can be concluded that explicit methods are better suited than implicit ones for such equations.

IMEX linear multistep methods
Spatial discretization of (1) leads to ODE systems of the form

$$u'(t) = F(u(t)) + G(u(t))\,, \tag{7}$$

where F is the discretized convection term and G stand for the stiff relaxation term. An IMEX linear multistep method for such a system is given by

$$u_n = \sum_{j=1}^{k} a_j u_{n-j} + \Delta t \sum_{j=1}^{k} \hat{b}_j F(u_{n-j}) + \Delta t \sum_{j=0}^{k} \hat{b}_j G(u_{n-j}) \tag{8}$$

for $n \geq k$. Here the $\hat{b}_j$ $(1 \leq j \leq k)$ are the weights of an explicit method and the b_j $(0 \leq j \leq k)$ of an implicit method.

Such combinations of implicit and explicit methods were introduced in 1980 by Crouzeix and by Varah. Runge-Kutta combinations of implicit and explicit methods are also known but for those methods the structure of the local errors are complicated. For (8) it is known that having order p for both the implicit and the explicit method, guarantees an order p for the combined IMEX scheme. Moreover, in contrast to Runge-Kutta combinations, the local errors of the IMEX linear multistep methods are not affected by the stiffness of the system. Therefore, the phenomenon of *order reduction* is avoided.

To construct IMEX linear multistep methods, one can start with a good implicit method and then find a suitable explicit method. The implicit BDFk methods can thus be combined with the extrapolated, explicit counterparts, and this yields very good IMEX schemes.

An other possibility is to start with an explicit method and combined it with a suitable implicit method, which should have good stability and damping properties. Starting from the well-known explicit Adams-Bashforth methods, one can thus obtain IMEX methods of order k. However, these schemes turn out to be not very suitable for equations of the type (1), mainly because the monotonicity and boundedness properties of the Adams-Bashforth methods are not very good.

Much better methods were obtained in [5] by using explicit methods of order k that have the TVB property with large stepsize coefficients $\gamma > 0$. The resulting IMEX-TVB methods did give good results in the numerical tests in [5].

Compared to the IMEX Runge-Kutta methods, the linear multistep methods (8) are promising because of the simple structure of the local errors and the favourable TVD/TVB properties, taking into account the amount of work per step. On the other hand, the Runge-Kutta methods are self-starting, and good IMEX combinations were obtained in [6].

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Adaptive ADER Methods using Kernel-based Polyharmonic Spline WENO Reconstruction

Armin Iske

(joint work with Terhemen Aboiyar and Emmanuil H. Georgoulis)

During the last 30 years, *finite volume methods* (FVM) have gained great popularity as numerical schemes for hyperbolic problems. Classical FVM are typically of low order (cf. e.g., [9, 10]). High order FVM include *essentially non-oscillatory* (ENO) and the more sophisticated *weighted essentially non-oscillatory* (WENO) schemes (see, e.g., [1, 5, 7, 10, 15, 16, 24] and references therein) for the discretisation of the space variables. Available high order FVM often combine high order space discretisations with one-step or multistep time stepping methods, such as *total variation diminishing* Runge-Kutta (TVD-RK) methods. However, high order TVD-RK methods have a reduced region of absolute stability [6]. This essentially limits the order of TVD-RK time discretisations [11].

In the WENO framework, a collection of stencils is first selected in the neighbourhood of each cell (control volume). For each stencil, a high order recovery function is computed from the stencil's cell average values. The WENO reconstruction for the cell is then given by a weighted sum of the stencils' recovery functions. The weights are chosen so that unphysical oscillations in regions of low regularity of the solution (e.g. near sharp fronts and shocks) are avoided. In available (multivariate) WENO methods, local (multivariate) polynomial interpolation is used in the reconstruction step. Using multivariate polynomial interpolation in WENO reconstructions, however, has restrictions. Indeed, when using polynomial interpolation, the size of each stencil is required to match the dimension of the polynomial ansatz space; this reduces the flexibility in the stencil selection. This restriction is particularly severe for unstructured meshes, where for the sake of numerical stability enhanced flexibility in the stencil selection is of vital importance [1]. A standard remedy to circumvent this restriction is the use of least-squares approximation in the reconstruction step. Standard least-squares approximation, however, does usually *not* minimize the relevant quantities which are used for the definition of the WENO weights. This is regarded as a major drawback of least-squares reconstructions, especially when it comes to approximate the solution in regions of low regularity. Therefore, the development of stable high order FVM for hyperbolic problems is still an ongoing challenge.

Recently, ADER methods were introduced in [20, 21], and further developed in [17, 19, 22], to obtain finite volume schemes of arbitrary high order. In the concept of ADER, **A**rbitrary high order **DER**ivatives are used to construct high order flux evaluations, using generalised Godunov methods. The high order flux evaluation of the ADER method can be combined with high order finite volume space discretisations (e.g., WENO reconstructions), leading to an ADER-FVM of arbitrary high order. ADER schemes have very recently gained considerable popularity in a wide range of applications from engineering and physics, see e.g., [3, 12, 13, 14, 23].

The application of ADER schemes to nonlinear hyperbolic problems is a subject of active research. For the multi-dimensional case, ADER schemes on Cartesian grids can be found in [13, 14, 18]. ADER methods in combination with the discontinuous Galerkin method are introduced in [4]. *Adaptive* ADER methods on unstructured triangular meshes have recently been developed in [8], giving more flexible efficient high order ADER schemes. In [8], adaptive ADER methods are combined with high order polynomial WENO reconstructions.

Mesh adaptivity is particularly important for the computational efficiency of FVM. Indeed, the presence of lower dimensional locally singular behaviour of the solutions to hyperbolic problems (e.g., sharp gradients and discontinuities) require different resolutions in different regions of the computational domain. This can be accomplished using adaptive methods.

This talk is concerning the development of new stable adaptive FVM on unstructured meshes of arbitrary high order, both in space and in time, using the ADER methodology. More specifically, a novel framework for WENO reconstruction is

proposed, whereby *kernel-based* interpolation (rather than polynomial interpolation) is utilised in the WENO reconstruction step. Kernel-based interpolation leads to ansatz spaces of variable dimensions, thereby can be applied to WENO reconstructions based on stencils of variable sizes, hence enhancing the flexibility of WENO reconstruction schemes. Our preferred choice of kernels are the family of the radial *polyharmonic splines*. Apart from the enhanced flexibility in the stencil selection, this particular choice has specific advantages concerning the numerical stability of local interpolation. Moreover, they are easily implemented in any space dimension. Furthermore, polyharmonic spline reconstruction leads to natural choices of *oscillation indicators*, as per required in the WENO reconstruction step. More specifically, the oscillation indicators are defined through the natural Sobolev semi-norms associated with the polyharmonic spline interpolation problem. For each stencil, the polyharmonic spline interpolant is the unique minimiser of the Sobolev semi-norm among all interpolants in the corresponding Sobolev space. Thus, the polyharmonic spline interpolant is – in that sense – the least oscillatory interpolant in the Sobolev space, which in turn improves the stability of the employed WENO reconstruction. This is supported by the numerical experiments presented in this talk. Moreover, the performance of the proposed method is illustrated in our previous work [2] by a series of numerical experiments, including linear advection, Burgers' equation, Smolarkiewicz's deformational flow test, and the five-spot problem.

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Some Reasons this Analyst Wants Better (High Order) Numerical Solutions

Barbara Lee Keyfitz

In this talk, I give some indication of how mathematicians working on the analysis of hyperbolic conservation laws can benefit from the results of numerical simulations of conservation laws.

One current direction of research in conservation laws is the study of multidimensional problems. Here the question is well-posedness. For systems in one space variable and time, there is now a satisfactory theory in the space BV of functions of bounded variation [2]. However, a result of Rauch [5], based on a linear theory due to Brenner [1], indicates that for multidimensional problems there are obstacles to well-posedness in BV. The lack of a good basis for a theory has led several groups of mathematicians to look at self-similar problems in two space variables and time. That is, we reduce a system $U_t + F(U)_x + G(U)_y = 0$, $U \in \mathbb{R}^n$, to a system $\widetilde{F}_\xi + \widetilde{G}_\eta = S$, where $\xi = x/t$, $\eta = y/t$, $\widetilde{F} = F - \xi U$, $\widetilde{G} = G - \eta U$

and $S(U) = -2U$ appears as a source term when the reduced system is written in conservation form.

Using some simplified model equations, such as the unsteady transonic small disturbance equation (UTSDE), and also the equations of isentropic compressible inviscid gas dynamics, we have established some analytic results, such as the existence of a subsonic flow in the neighborhood of the shock reflection point in transonic regular reflection. However, as is well-known, there is a range of wedge angles (small) and Mach numbers (close to unity) for which neither regular nor Mach reflection is mathematically possible. Seeking some insight into this situation, in 2002 Allen Tesdall and John Hunter, [9], performed careful numerical simulations on the UTSDE, and discovered a new pattern, which they named Guderley Mach Reflection after the German aerodynamicist who, fifty years ago, predicted that a rarefection wave might appear in cases where the amount of vorticity present was not sufficient to resolve a discontinuity. The pattern involves a series of reflections, each involving a rarefaction and a small supersonic patch. These results, later replicated in other systems, [11, 12], were a direct motivation for experiments which confirmed the phenomenon, [6, 7].

Our attempts to analyze this are only beginning. Besides the complication of a possibly infinite cascade of reflections, we do not yet have a clear understanding of the interaction of the rarefaction with the sonic line. We have a steady model that includes some aspects of this interaction, which is distinguished by having a two-way interaction across the sonic line between the supersonic and subsonic regions, [4]; and we have done some preliminary analysis on a self-similar model, [10].

A second interesting question arising in this and other multidimensional steady or self-similar problems, concerns the formation point of a transonic shock. While reasonably careful simulations appear to show that the shock forms on the sonic line itself, refining the calculations often demonstrates that the shock formation point is inside the supersonic region, [8]. In this talk, I present some examples that show a shock can actually form on the sonic line, as well as some evidence that such behavior is non-generic. In addition, one recent example, a shock diffracting around a screen, [3], appears to exhibit a shock with an exactly sonic formation point. Like Guderley Mach Reflection, this phenomenon was discovered numerically, and presents an interesting target for analysis.

The reseach described in this talk was supported in part by the National Science Foundation and the Department of Energy.

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Sound-proof model equations for atmospheric flows and associated multiscale time integrators

Rupert Klein

The numerical integration of the compressible flow equations for meteorological applications is challenging for at least two reasons. First, atmospheric flows are of low Mach number implying the usual obstacles for numerical integration and, secondly, they belong to a three-time-scale asymptotic limit regime for advection, internal gravity waves, and sound, making matters even more delicate. Meteorologists are interested in following accurately the dynamics of advection and internal waves while disregarding most of the effects of compressibility. There are very long wave compressible modes, however, namely the Lamb-waves, which are considered to be of some relevance and are therefore not to be disregarded entirely in numerical integrations.

Validity of sound-proof models

In my first presentation I have discussed theoretical arguments based on multiple scales asymptotics that justify the use of well-known "sound-proof approximations" for atmospheric motions in this three-time-scale regime, see [1]. These reduced models still involve two asymptotically separated time scales, namely those of advection and internal waves, but suppress sound propagation by nonhomogeneous algebraic divergence constraints for the flow velocity. By formal asymptotics, I have shown that the solutions of these sound-proof models will stay close to those of the full compressible flow equations over time scales comparable to the slow advection time, given properly prepared initial data.

The main technique in this analysis is spectral expansion of the solution in horizontally propagating harmonic waves with wavelength-dependent vertical structure. The determination of the vertical structure functions can be reduced to solving a single effective regular Sturm-Liouville problem in the case of the sound-proof equations, and to a perturbed Sturm-Liouville problem that is weakly nonlinear in the eigenvalue for the compressible case. Showing that the structure functions and eigenvalues of the sound-proof and the weakly compressible cases are asymptotically close, we find a criterion for the intermediate internal wave time scale that guarantees asymptotic validity of the sound-proof approximation over advective times.

The spectral expansion ansatz also gives hints at how to attempt a rigorous proof of validity of the sound-proof models using the technique of energy estimates as pioneered by Klainerman and Majda in their seminal papers on the zero Mach number approximation, [2].

A multiscale time integrator

In my second presentation I have discussed a novel numerical semi-implicit time integration scheme that relies on multigrid technology to explicitly distinguish the principally different behavior of long and short wave modes in the atmosphere [3]. While long wave modes may well be compressible (the Lamb waves) and should thus be transported accurately as acoustic modes, short wave flow components are effectively incompressible and should therefore follow the abovementioned sound-proof limiting dynamics. By applying a different time integration scheme to each level of the multigrid hierarchy, one can construct suitable scale-adaptive integration schemes that observes these limits automatically.

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A robust numerical method for compressible MHD applied to astrophysical flow simulations

Christian Klingenberg

(joint work with François Bouchut, Knut Waagan)

The ideal MHD equations are a central model in astrophysics, and their solution relies upon stable numerical schemes. We present an implementation of a new method, which possesses excellent stability properties. We have developed approximate Riemann solvers for ideal MHD equations based on a relaxation approach in [1, 2]. These lead to entropy consistent solutions with good properties like guaranteed positive density. We describe the extension to higher order and

multiple space dimensions. Numerical tests demonstrate that the theoretical stability properties are valid in practice with negligible compromises to accuracy. The result is a highly robust scheme with state-of-the-art efficiency. The scheme's robustness is due to entropy stability, positivity and properly discretised Powell terms. The implementation takes the form of a modification of the MHD module in the FLASH code, an adaptive mesh refinement code. We compare the new scheme with the standard FLASH implementation for MHD [3, 4]. Results show comparable accuracy to standard FLASH with the Roe solver, but highly improved efficiency and stability, particularly for high Mach number flows and low plasma b. The tests include 1D shock tubes, 2D instabilities and highly supersonic, 3D turbulence. We consider turbulent flows with RMS sonic Mach numbers up to 10, typical of gas flows in the interstellar medium. We investigate both strong initial magnetic fields and magnetic field amplification by the turbulent dynamo from extremely high plasma b [5].

Usung the FLASH code we present new 3D magnetohydrodynamic simulations of a supernova-driven, stratified interstellar medium [6, 7]. We examine whether inclusion of magnetic fields and a greater vertical extent to the simulation domain produce a gas distribution that better matches the observations. We study the change of magnetic energy over time in our models, showing that it appears to reach a steady state after a few hundred megayears, presumably supported by a turbulent dynamo driven by the supernova explosions.

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Orthogonal polynomials in several variables potentially useful in pde

Tom H. Koornwinder

A system of orthogonal polynomials (OP's) $\{p_n\}_{n=0}^{\infty}$ on $\mathbb{R}$ with respect to a positive measure μ on $\mathbb{R}$ is called *classical* if there is a second order differential operator L such that $Lp_n = \lambda_n p_n$ $(n = 0, 1, 2, \ldots)$ for certain eigenvalues λ_n. By a theorem of Bochner [1] there are three families of classical OP's (up to an affine transformation of the argument of the OP):

1. Hermite: $p_n = H_n$, $d\mu(x) = e^{-x^2}\,dx$ on $\mathbb{R}$,
   $(Lf)(x) = \frac{1}{2}f''(x) - xf'(x)$, $\lambda_n = -n$.
2. Laguerre: $p_n = L_n^{\alpha}$, $d\mu(x) = x^{\alpha}e^{-x}\,dx$ on $[0,\infty)$, $\alpha > -1$,
   $(Lf)(x) = xf''(x) + (\alpha + 1 - x)f'(x)$, $\lambda_n = -n$.
3. Jacobi: $p_n = P_n^{(\alpha,\beta)}$, $d\mu(x) = (1-x)^{\alpha}(1+x)^{\beta}\,dx$ on $[-1,1]$, $\alpha, \beta > -1$,
   $(Lf)(x) = (1-x^2)f''(x) + (\beta - \alpha - (\alpha+\beta+2)x)f'(x)$, $\lambda_n = -n(n+\alpha+\beta+1)$.

Let μ be a positive measure on $\mathbb{R}^d$ such that $\int_{\mathbb{R}} |x^{\alpha}|\,d\mu(x) < \infty$ $(\alpha \in (\mathbb{Z}_{\geq 0})^d)$ and the support of μ has nonempty interior. Let $\mathcal{P}_n$ consist of all polynomials p of degree $\leq n$ such that $\int_{\mathbb{R}^d} pq\,d\mu = 0$ for all polynomials q of degree $< n$. Then $\mathcal{P}_n$ has the same dimension $\binom{n+d-1}{n}$ as the space of homogeneous polynomials of degree n in d variables. Furthermore, the spaces $\mathcal{P}_n$ $(n = 0, 1, 2, \ldots)$ are mutually orthogonal in $L^2(\mu)$. We call $\{\mathcal{P}_n\}_{n=0}^{\infty}$ a *system of orthogonal polynomials* with respect to the measure μ.

As a refinement of this notion we may choose an orthogonal basis $\{p_{\alpha}\}_{\alpha_1+\cdots+\alpha_d=n}$ for each space $\mathcal{P}_n$, and call the polynomials p_{α} orthogonal polynomials. Of course, there are many ways to choose such orthogonal bases.

A system $\{\mathcal{P}_n\}$ of orthogonal polynomials in d variables is called *classical* if there is a second order pdo L acting on the space of polynomials such that $\mathcal{P}_n$ is an eigenspace of L for a certain eigenvalue λ_n $(n = 0, 1, 2, \ldots)$. As a refinement there may be, apart from $L = L_1$, $d-1$ further pdo's $L_2, \ldots, L_d$ such that $L_1, L_2, \ldots, L_d$ commute, are self-adjoint with respect to μ, and have one-dimensional joint eigenspaces. Then we have OP's p_{α} with $L_j p_{\alpha} = \lambda_{\alpha}^{(j)} p_{\alpha}$.

It was shown by Krall & Sheffer [8] and Kwon, Lee & Littlejohn [9] that there are five families of classical orthogonal polynomials in 2 variables, as follows:

1. $d\mu(x,y) = e^{-x^2-y^2}\,dx\,dy$ on $\mathbb{R}^2$, $L = \frac{1}{2}(\partial_{xx} + \partial_{yy}) - x\partial_x - y\partial_y$, $\lambda_n = -n$.
2. $d\mu(x,y) = x^{\alpha}y^{\beta}e^{-x-y}\,dx\,dy$ on $[0,\infty)\times[0,\infty)$, $\alpha, \beta > -1$,
   $L = x\partial_{xx} + y\partial_{yy} + (1+\alpha-x)\partial_x + (1+\alpha-y)\partial_y$, $\lambda_n = -n$.
3. $d\mu(x,y) = y^{\beta}e^{-x^2-y}\,dx\,dy$ on $\mathbb{R}\times[0,\infty)$, $\beta > -1$,
   $L = \frac{1}{2}\partial_{xx} + y\partial_{yy} - x\partial_x + (1+\beta-y)\partial_y$, $\lambda_n = -n$.
4. $d\mu(x,y) = x^{\alpha}y^{\beta}(1-x-y)^{\gamma}\,dx\,dy$ on $\{(x,y) \in \mathbb{R}^2 \mid x, y \geq 0,\ x+y \leq 1\}$,
   $\alpha, \beta, \gamma > -1$, $L = x(1-x)\partial_{xx} + y(1-y)\partial_{yy} - 2xy\partial_{xy} + \big(\alpha + 1 - (\alpha+\beta+\gamma+3)x\big)\partial_x + \big(\beta + 1 - (\alpha+\beta+\gamma+3)y\big)\partial_y$, $\lambda_n = -n(n+\alpha+\beta+\gamma+2)$.

5. $d\mu(x,y) = (1-x^2-y^2)^\alpha\,dx\,dy$ on $\{(x,y)\in\mathbb{R}^2 \mid (x^2+y^2\le 1\}$, $\alpha > -1$,
   $L = (1-x^2)\partial_{xx} + (1-y^2)\partial_{yy} - 2xy\partial_{xy} - (2\alpha+3)(x\partial_x + y\partial_y)$,
   $\lambda_n = -n(n+2\alpha+2)$.

Orthogonal bases $\{p_{n,k}\}_{k=0,1,\dots,n}$ for $\mathcal{P}_n$ $(n = 0,1,2,\dots)$ in these five cases can be obtained by Gram-Schmidt orthogonalization of the monomials $1, x, y, x^2, xy, y^2, \dots, x^n, x^{n-1}y, \dots, x^{n-k}y^k, \dots$. The resulting polynomials are as follows.

1. $p_{n,k}(x,y) = H_{n-k}(x)H_k(y)$.
2. $p_{n,k}(x,y) = L^\alpha_{n-k}(x)L^\beta_k(y)$.
3. $p_{n,k}(x,y) = H_{n-k}(x)L^\beta_k(y)$.
4. $p_{n,k}(x,y) = P^{(\alpha,\beta+\gamma+2k+1)}_{n-k}(1-2x)\,(1-x)^k P^{(\beta,\gamma)}_k\big(1-2y/(1-x)\big)$.
5. $p_{n,k}(x,y) = P^{(\alpha+k+\frac12,\alpha+k+\frac12)}_{n-k}(x)\,(1-x^2)^{k/2}P^{(\alpha,\alpha)}_k\big(y/\sqrt{1-x^2}\,\big)$.

The expansions in monomials of these polynomials $p_{n,k}$ do not involve all monomials $x^{m-j}y^j$ with (m,j) equal or less than (n,k) in the lexicographic ordering. For classes 1, 2 and 3 $p_{n,k}(x,y)$ only contains monomials $x^{m-j}y^j$ with $m-j \le n-k$ and $j \le k$. For classes 4 and 5 $p_{n,k}(x,y)$ only contains monomials $x^{m-j}y^j$ with $m \le n$ and $j \le k$. Furthermore, in these five cases there is a second order differental operator L_2 commuting with L which has the $p_{n,k}$ as eigenfunctions with eigenvalue only depending on k.

The OP's $p_{n,k}$ for case 4 (on the triangular region), as explicitly given above, were introduced by Proriol [10] in 1967. They were mentioned in the survey paper by Koornwinder [7] in 1975. Their special case $\alpha = \beta = \gamma = 0$ (constant weight function) was rediscovered by Dubiner [2] in 1991, who was motivated by applications to finite elements. Dubiner's paper was much quoted in this context. For a while, the special functions and finite elements communities were not aware that they had a joint interest. But in 2000 Hesthaven & Teng [4] referred to Proriol's paper, while later Karniadakis & Sherwin in their book [6] had ample references to papers on special functions. Conversely, in 2001 Dunkl & Xu referred in their book [3] to Dubiner's paper.

Another important orthogonal system for case 5 on the disk is as follows.

$$R^\alpha_{m,n}(z) := \text{const.}\begin{cases} P^{(\alpha,m-n)}_n(2|z|^2-1)z^{m-n}, & m \ge n,\\ P^{(\alpha,n-m)}_m(2|z|^2-1)\overline{z}^{n-m}, & n \ge m\end{cases}$$
$$((m,n)\in(\mathbb{Z}_{\ge0})^2,\ z\in\mathbb{C},\ \alpha>-1).$$

Then $R^\alpha_{m,n}(z) = \text{const.}\, z^m\overline{z}^n + \text{polynomial in } z, \overline{z} \text{ of lower degree.}$ and

$$\int_{x^2+y^2<1} R^\alpha_{m,n}(x+iy)\,\overline{R^\alpha_{k,l}(x+iy)}\,(1-x^2-y^2)^\alpha\,dx\,dy = 0 \quad ((m,n)\ne(k,l)).$$

For $\alpha = 0$ these polynomials are called *Zernike polynomials*. They were introduced by Zernike [11] in 1934 for applications in optics and are still much used there. The polynomials $R^{\alpha}_{m,n}$ for general α first occurred in Zernike & Brinkman [12].

For numerical applications it is important that Jacobi polynomials can be approximated by polynomials which are orthogonal on finitely many equidistant points. These are the *Hahn polynomials* $Q_n(x;\alpha,\beta,N)$ $(n = 0, 1, \ldots, N)$ satisfying

$$\sum_{x=0}^{N} (Q_n Q_m)(x;\alpha,\beta,N) \binom{\alpha+x}{x} \binom{\beta+N-x}{N-x} = 0 \quad (n \neq m).$$

The approximation is: $\lim_{N\to\infty} Q_n(Nx;\alpha,\beta,N) = \text{const.}\, P_n^{(\alpha,\beta)}(1-2x)$.

From the Hahn polynomials we can build polynomials (Karlin & McGregor [5])

$$Q_{n,k}(x,y;\alpha,\beta,\gamma,N) := Q_{n-k}(x;\alpha,\beta+\gamma+2k+1,N-k)\binom{N-x}{k} Q_k(y;\beta,\gamma,N-x)$$

which are orthogonal on the set $\{(x,y) \in \mathbb{Z}^2 \mid x, y \geq 0,\ x+y \leq N\}$ with respect to the weights

$$w(x,y;\alpha,\beta,\gamma,N) := \binom{\alpha+x}{x} \binom{\beta+y}{y} \binom{\gamma+N-x-y}{N-x-y}.$$

They approximate the polynomials of class 4 on the triangle:

$$\lim_{N\to\infty} Q_{n,k}(Nx,Ny;\alpha,\beta,\gamma,N) = \text{const.}\, p_{n,k}^{\alpha,\beta,\gamma}(x,y),$$

which looks promising for applications.

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Implementation and Efficiency of Discontinuous Galerkin Spectral Element Methods for Fluid Flow Problems

David A. Kopriva

Discontinuous Galerkin spectral element methods (DGSEMs) are high order methods with many features to make them attractive for use to compute highly accurate solutions to fluid flow and wave propagation problems. They are geometrically flexible, like finite element methods, and can be used in arbitrarily complex geometries. They are designed so that one increases the number of degrees of freedom either by increasing the order of approximation or by increasing the number of elements. The result is an approximation that can be both exponentially convergent in the polynomial order of the approximation and high order in the element size. The methods have been shown to have exponentially small dissipation and dispersion errors, which makes them ideal for wave propagation problems. The approximations are highly localized, making boundary conditions and parallelization easy to implement. Finally, the DGSEMs are robust, at least when compared to strong form spectral methods.

Conventional wisdom, however, states that DG spectral element methods are: (i) Too hard to implement and (ii) Less efficient than other methods, especially compact finite difference methods. We can show that, as usual, conventional wisdom is not necessarily correct. We will describe an efficient and simple to implement form of the spectral element method, and examine strategies for reducing its issues with stiffness. We will also compare the approximation with an optimized compact finite difference method to discuss the issue of relative efficiency.

We solve problems of compressible flow, approximating flows modeled by a system of conservation laws

$$\vec{q}_t + \nabla \cdot \vec{f} = 0, \tag{1}$$

with fluxes

$$\vec{f} = \vec{f}^i + \vec{f}^v \tag{2}$$

for either inviscid problems modeled by the Euler equations of gas-dynamics or viscous problems modeled by the compressible Navier-Stokes equations

The development of a DGSEM approximation has the following steps: The domain of interest is decomposed into multiple elements, which can be arbitrarily complex. Each element is mapped onto a reference element, on which a strong form of the equations still applies, namely

$$\tilde{q}_t + \nabla \cdot \tilde{f} = 0,$$

where

$$J\vec{a}^i = J\nabla\xi^i = \vec{a}_j \times \vec{a}_k = \frac{\partial \vec{X}}{\partial \xi^j} \times \frac{\partial \vec{X}}{\partial \xi^k} \quad (i,j,k) \text{ cyclic}$$

$$J = \vec{a}_i \cdot (\vec{a}_j \times \vec{a}_k) \quad (i,j,k) \ cyclic$$

The approximation imposes three constraints: One approximates the solutions and fluxes by polynomials in each element $\tilde{q} \approx \tilde{Q} \in \mathbb{P}^N$, $\tilde{f} \approx \tilde{F} \in \mathbb{P}^M$, uses a weak form of the equations,

$$\int_E \left(\tilde{Q}_t + \nabla \cdot \tilde{f} \right) \phi = 0,$$

and assumes no continuity on $\phi \in \mathbb{P}^N$ between elements. With these approximations one gets the (weak form) of the DG approximation on the reference element

$$\int_E \tilde{Q}_t \phi d\xi + \int_{\partial E} \tilde{F}^* \cdot \hat{n}_\xi \phi dS - \int_E \tilde{F} \cdot \nabla \phi d\xi = 0. \tag{3}$$

The result is a *framework* for approximations. One can use Quad/Hex or Tri/Tet elements with nodal or modal bases. Different polynomials can be used for that basis. Boundary and interior unknowns can be approximated with different polynomial orders. The solution and fluxes can be approximated with different orders. Integrals can be computed exactly or with quadrature. The quadrature can be exact or inexact. One can apply the approximation to (3) or integrate by parts again to get a penalty-like approximation.

The "Classical" spectral element approach gives an easy to implement and effective approximation. We use quadrilateral/ hexahedral elements, which allows the use of an efficient nodal tensor product basis. The nodal basis allows for simple application to nonlinear/variable coefficient/general complex geometry problems. All approximations use the same polynomial order, which simplifies coding. Finally, we use a Legendre basis with Gauss-type quadrature to approximate the integrals. The result is an element-wise approximation at each mesh point of the form

$$\frac{dQ_{i,j,k}}{dt} + \sum_{n=1}^{3} D_{\xi^n} \tilde{F}^n_{i,j,k} = 0, \tag{4}$$

where

$$D_{\xi^1} \tilde{F}^1_{i,j,k} = \left[\widetilde{F}^*(1,\eta_j,\zeta_k) \frac{\ell_i(1)}{w_i^{(\xi)}} - \widetilde{F}^*(-1,\eta_j,\zeta_k) \frac{\ell_i(-1)}{w_i^{(\xi)}} \right] + \sum_{m=0}^{N} \widetilde{F}_{m,j,k} \hat{D}^{(\xi)}_{im}.$$

The primary work is therefore to compute the fluxes $\tilde{F}^n_{i,j,k}$ from the solution, evaluation of the Riemann solver, e.g. $\widetilde{F}^*(-1,\eta_j,\zeta_k)$, a series of dot products, and a series of matrix-vector products. We see, at least in terms of implementation, the approximation is an extension (through the matrix-vector product term) of a standard finite volume scheme.

Finite difference approximations are assumed to be more efficient than (4) for two reasons. First, because of the matrix-vector product, the work to compute the spectral approximation is $O\left(N^2\right)$, whereas it is only $O\left(N\right)$ for a finite difference approximation. The second is that the time step restriction on the spectral approximation is $O\left(N^{-2}\right)$ vs $O\left(N^{-1}\right)$ for finite differences. Formal analysis does

not describe the whole situation, however. First, computer architectures today are very efficient at computing matrix-vector multiplication. Second, boundary approximations for the finite difference approximation are often lower order than in the interior.

Compact finite difference methods require the solution of a tri-diagonal system - twice. The first is to compute the derivative. The second is to add a filter. The cost of the filter must be added because the (centered) schemes cannot be run without one. As a result, the cost to compute a derivative is not necessarily lower for the finite difference scheme vs the spectral. The cost per grid point is not necessarily more expensive, either. Comparing Euler and Navier-Stokes codes, we see that for the compressible Navier Stokes equations, a spectral element approximation of the same order can cost only 1/3 that of the compact finite difference scheme. Finally, because the order of approximation is usually lower at the boundaries, users of finite difference codes refine the mesh near boundaries, making the problems more stiff, and negating the time step advantage. The result is that in practice a finite difference approximation is not necessarily more stiff than the spectral.

High order methods are stiff, nonetheless, and efficient strategies are needed to speed convergence to steady-state or to take larger time steps when accuracy is limited by stability, not accuracy, considerations. To start, we compare implicit vs explicit time integrators. Explicit time integrators are relatively inexpensive, but one must take large numbers of time steps. Used with a DG spectral approximation, however, they are embarrassingly parallel. We can also use explicit local time stepping to work hard only on elements where the time steps are severely limited. On the other hand, implicit schemes for the spectral element methods are very expensive per step. They are more complicated to program, are much more memory intensive, and the performance is critically affected by the availability of good preconditioners.

We compare strategies for the efficient time integration of spectral element approximations. For steady-state serial computations, at least in two space dimensions, implicit approximations are very effective. However the tradeoff between memory requirements for the preconditioner and convergence rate is severe. In three dimensions, memory restricts approximations to relatively low order, and parallel strategies are the most efficient. The same is true for time dependent problems. Whereas an explicit computation may take 11K time steps per period in a wave propagation problem vs 64 with an implicit computation, the expense of the implicit scheme per time step can result in less than a factor of three in overall speed up. This difference can be easily erased by parallel and/or local time step strategies. The result is that it is not given that implicit approximations for high order spectral approximations of compressible flow problems are the natural way to make the methods more efficient.

New Adaptive Artificial Viscosity Method for Hyperbolic Systems of Conservation Laws

Alexander Kurganov
(joint work with Yu Liu)

We study finite volume Godunov-type schemes for 1-D,

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{0}, \tag{1}$$

and 2-D,

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x + \mathbf{g}(\mathbf{u})_y = \mathbf{0}, \tag{2}$$

hyperbolic systems of conservation laws. In the above formulae, $\mathbf{u} \in \mathbb{R}^N$ is a vector of conserved quantities and $\mathbf{f}(\mathbf{u})$ and $\mathbf{g}(\mathbf{u})$ are fluxes.

Godunov-type schemes form a class of projection-evolution methods, in which at each time step the computed solution is approximated by a global conservative, sufficiently accurate and non-oscillatory piecewise polynomial reconstruction, which is evolved in time (from a current time level to the next one) according to the integral form of (1) (or (2), respectively). The latter is obtained by integrating the studied system of conservation laws over the space-time control volume of size $\Delta t \times \Delta x$ (or $\Delta t \times \Delta x \times \Delta y$, respectively), where Δt is a small temporal scale and Δx and Δy are small spatial scales. First-order Godunov-type schemes are obtained using the first-order piecewise constant reconstruction, while their higher-order extensions hinge on replacement of the first-order reconstruction with a higher-order one, which consists of either linear, parabolic, cubic or even higher degree polynomial pieces.

Many of the Godunov-type schemes may be written in a particularly simple semi-discrete form (method of lines), obtained by integrating (1) ((2)) over grid cells of size Δx ($\Delta x \times \Delta y$) and replacing the fluxes at the cell interfaces (flux integrals along the cell boundaries) by appropriate numerical fluxes. In the 1-D case, a semi-discrete scheme on a uniform grid with $x_j := j\Delta x$ can be written as follows. Let $x_{j\pm\frac{1}{2}} := x_j \pm \Delta x/2$ and the computational domain consists of the cells $I_j := [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$. Then, the cell averages of $\mathbf{u}(x,t)$ over I_j at time t, denoted by

$$\bar{\mathbf{u}}_j(t) \approx \frac{1}{\Delta x} \int\limits_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \mathbf{u}(x,t)\,dx,$$

are evolved in time by numerically solving the following system of ODEs:

$$\frac{d}{dt}\bar{\mathbf{u}}_j(t) = -\frac{\mathbf{H}_{j+\frac{1}{2}}(t) - \mathbf{H}_{j-\frac{1}{2}}(t)}{\Delta x}, \tag{3}$$

where $\mathbf{H}_{j+\frac{1}{2}}$ is a numerical flux, obtained using the piecewise polynomial approximation, reconstructed at time t using the available cell averages $\{\bar{\mathbf{u}}_j(t)\}$. In our numerical experiments, we have used the central-upwind fluxes developed in [5, 6] (for completeness, both the 1-D and 2-D central-upwind fluxes are briefly reviewed

in Appendixes A and B). However, we would like to stress that the proposed adaptive artificial viscosity method is not tied to any specific numerical flux and can be implemented with one's favorite flux, for which the semi-discrete scheme (3) is linearly stable.

It is well-known that the systems (1) and (2) admit nonsmooth solutions that may contain shocks, contact discontinuities and rarefaction waves. Therefore, when such solutions are to be captured, linearly stable methods may develop large spurious oscillations and even blow up. Thus, a good numerical method must be nonlinearly stable. Nonlinear stability of Godunov-type schemes is typically guaranteed by enforcing non-oscillatory nature of the piecewise polynomial reconstruction with the help of nonlinear limiters. However, such limiters may be very complicated and computationally expensive, especially when a high-order multidimensional scheme is to be designed (see, e.g., [7] and references therein). Alternatively, one may use less computationally expensive unlimited reconstructions, while enforcing nonlinear stability by adding artificial viscosity to the PDE system in the regions of discontinuities. Obviously, to ensure consistency of the numerical approximation, this artificial viscosity must vanish as $\Delta \to 0$, where in the 1-D case, $\Delta := \max(\Delta t, \Delta x)$ and in the 2-D case, $\Delta := \max(\Delta t, \Delta x, \Delta y)$.

The idea of adding artificial viscosity was first proposed in [8] and since then it was notably adopted in many works. The major difficulty in designing a highly accurate and robust artificial viscosity method is to make sure that a sufficient amount of stabilizing diffusion is added wherever it is needed to ensure stability, while in the rest of the computational domain the diffusion must be either switched off or small enough not to affect the high accuracy of the scheme there. At the same time, if the viscosity coefficient is too large in the areas of discontinuity, the solution will be overly smeared there. Therefore, to achieve overall high resolution, the viscosity should be added in an adaptive way using a certain indicator, which should automatically pick rough parts of the computed solution and determine the (optimal) amount of viscosity needed to be added there.

We propose a new adaptive artificial viscosity method. In our method, the viscosity coefficients are chosen to be proportional to the size of the weak local residual, which was originally developed in [3] and then used in [1, 4] as a smoothness indicator for several adaption algorithms. The key point we use here is that for a convergent numerical method of formal order r, the weak local residual is proportional to Δ near (nonlinear) shocks, while it is much smaller ($\sim \Delta^\alpha$, α is close to 2) at (linear) contact waves, and tiny in the smooth parts of the solution ($\sim \Delta^4$). Therefore, the artificial viscosity vanishes as one refines the grid, so that it is consistent with the original hyperbolic equations. Moreover, the rate at which the viscosity coefficients decay, allows us to achieve the main goal—to stabilize the solution at shock regions without oversmearing contact discontinuities or affecting the high resolution of smooth parts of the computed solution.

The key question one has to address to make the adaptive artificial viscosity method robust is how to select the coefficients of proportionality in such a way that the computed solution is non-oscillatory, but its discontinuous parts are well

resolved, that is, they are not overly smeared. To tune the viscosity coefficients, we adopt the strategy proposed in [2]: for the problem at hand, the coefficients are first adjusted on a very coarse mesh and then used for the high-resolution computation on finer meshes.

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Numerical modeling of some geophysical flows

Mária Lukáčová-Medviďová

(joint work with Michael Dudzinski, Sebastian Noelle, Anna Hundertmark)

Typical difficulties appearing in oceanographical or atmospherical flows applications are:

- nonlinear interaction of genuinely multidimensional waves
- multiscale phenomena (wave speeds differing by orders of magnitude)
- preservation of important equilibria exactly or at least up to high order of accuracy
- satisfying entropy inequality.

In general the multidimensional hyperbolic balance laws can be written in the following way

$$\mathbf{u}_t + \sum_{i=1}^{d} \mathbf{f}_i(\mathbf{u})_{x_i} = \mathbf{b}(\mathbf{u}, x, y),$$

where $\mathbf{u}$ is the vector of conservative variables, $\mathbf{f}_i$, $i = 1, 2, \ldots, d$ are flux functions, d denotes the dimension and $\mathbf{b}(\mathbf{u}, x, y)$ is the source term. Both the solution as well as the source term might be discontinuous, the later consists usually of nonconservative products. Typical examples are the shallow water equations with

the source terms modelling bottom topography and the Coriolis forces, or the Euler equations with the source term modelling gravitational forces.

Most geophysical phenomena show a very localized behaviour, i.e. we have small regions with strong interactions in a larger surrounding area with almost steady solutions. To resolve the interesting structures correctly and efficiently *grid adaptation* is an inevitable tool. In [1] Bollermann, Lukáčová-Medvid'ová and Noelle extended the multidimensional finite volume evolution Galerkin (FVEG) scheme to non-uniform, adaptive grids, allowing fine resolutions in the area of interest and a minimal cell number in steady regions. Another crucial point in solving balance laws is the treatment of source terms. For correct solutions, it is necessary to evaluate the source term in such a way that certain steady states are kept numerically. In fact, we have to make sure that the numerical flux and the numerical source term cancel each other exactly, so that no spurious oscillations occur. This property is called *well-balancing*. The adaptive FVEG scheme developed in [1] preserves important equilibrium states.

Many important properties of the shallow water model rely on the fact that the water height is positive. Typical problems including the occurrence of dry areas are the run-up waves at a coast or tsunamis. In order to simulate these types of problems we have developed in [2] numerical schemes that can handle the moving shoreline in a stable and efficient way. In fact, we have developed a relatively simple general approach to ensure the *positivity preservation* for arbitrary finite volume schemes.

For modeling of stratified flows at least two (non-mixing) layers of fluids have to be considered. Such flows arise typically in atmoshperical flows or in oceanography. For example at the Strait of Gibraltar the upper layer corresponds to the lighter Atlantic whereas the lower one to the denser Mediterranean water. Mathematically, the *multi-layer shallow water system* belongs to a class of balance laws in non-conservative form. The *non-conservativity* of the individual layers make it nontrivial to give a sense for a weak solution, cf. [4]. Pares et al. developed the concept of the so-called path-conservative schemes, see, e.g. [10, 11, 3] and the references therein. In [7] Karni and Abgrall illustrated that the path-conservative schemes are in general not able to compute a correct solution to the non-conservative hyperbolic problems, even if the correct path (in phase space) is known. In [5] we have generalized the FVEG scheme for the non-conservative hyperbolic systems and applied it for the two-layer shallow water equations with dry/wet fronts. Our discretization of the source terms is based on the idea that the total momentum is conservative. This and the well-balancy of the scheme then determine a suitable numerical quadrature for the source term approximation.

Another characteristic feature of many geophysical flows is their *multiscale behaviour* with wave speeds differing by orders of magnitude. If explicit time discretization is used for numerical approximation to a governing system that supports multiscale waves, the maximum stable time step will be limited by wave speed of the most rapidly propagating waves. For geophysical flows this means that the waves that carry the least energy and are of little physical significance enforce a severe stability constraint. In order to obtain a reasonably efficient numerical model for simulation of geophysical flows (e.g. atmospheric circulation), it is necessary to circumvent the stability constraint associated with acoustic waves and put the stability limit into closer agreement with the time step limitations arising from accuracy considerations. In [6] Hundertmark, Lukáčová-Medvid'ová and Prill have derived two types of the large time step FVEG scheme; the explicit as well as fully implicit scheme. The first numerical simulations confirm the efficiency of the explicit large time step scheme, while the fully implicit approach yields in some cases more robust scheme. In our recent work we are working in collaboration with colleagues from meteorology, cf. [9], on the development of linear semi-implicit large time step FVEG scheme. The fully nonlinear flux is splitted into a linear part governing the fast waves and the rest nonlinear part governing the convective waves. In order to omit the strict stability condition dictated by fast waves, the linear operator is approximated in time in the implicit way, while the nonlinear one is approximated in an explicit way. This yields a desired CFL stability condition depending just on the slow waves.

Finally, we should point out that our numerical solution should satisfy the discrete entropy inequality. On the other hand, it is a well-known fact that numerical schemes based on a linearization strategy can produce solutions violating the entropy condition. In [8] Lukáčová-Medvid'ová and Tadmor studied theoretically the *entropy stability* of a class of finite volume methods (including the FVEG schemes) for one-dimensional systems of hyperbolic conservation laws. Following Tadmor [12] we have derived the second-order numerical viscosity which guarantees the entropy stability of the Roe-type FV schemes. Numerical experiments confirm that the resulting schemes have just the right amount of numerical viscosity: small enough to retain sharp shock profiles, yet large enough to enforce a correct resolution of sonic rarefactions. In [2] we have also derived a two-dimensional entropy fix that can be viewed as a multi-dimensional generalization of the Harten and Hyman entropy correction.

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Discontinuous Galerkin schemes based on reconstruction and defect corrections

Claus-Dieter Munz
(joint work with Alexander Filimon, Michael Dumbser, Gregor Gassner)

Discontinuous Galerkin (DG) methods are powerful computational tool for the solution of systems of conservation laws. A comprehensive description of the development of discontinuous Galerkin schemes and their applications can be found, e.g., in the recent text book by Hesthaven and Warburton [1]. The idea to apply a reconstruction to the solution of a discontinuous Galerkin scheme was introduced by Cockburn et al. in [2]. At the final time of the simulation they enhanced the accuracy by a post-processing of the data. However, information which is lost due to numerical dissipation during time evolution can not be restored by this approach. In the time dependent case, Dumbser et al. [3] combined the DG finite element variational formulation and reconstruction in every time step. This approach is based on an explicit time integration and has since been applied to a wide range of applications. In this paper, we combine reconstruction the variational formulation for steady state simulations. The pure post-processing according to [2] can not guarantee that the improved solution is a steady state. We use in our approach the improved approximate solution and a corresponding high-order approximation to estimate the defect or local discretization error of the obtained solution only. By including the estimated error as an additional term on the right hand side,

this scheme commutes a corrected solution to higher-order accuracy. The advantage within this approach is that the improved higher order solution is a steady state. For the correction one only needs the inversion of the basic lower-order DG scheme, however, it has to be solved several times within an iteration loop which is often called iterated defect correction, see, e.g., [4].

The main idea of the considered defect correction approach in an abstract form is the following. We write the problem to be numerically solved as

$$Lu = 0, \tag{1}$$

where u denotes the desired solution and L denotes some differential operator, for example the compressible Navier-Stokes equations. The numerical approximation, in our case a discontinuous Galerkin scheme based on piecewise polynomials of degree N, is written as

$$L_h u_h = 0, \tag{2}$$

where L_h is the approximation of the operator L, u_h is the approximate solution and h denotes a discretization parameter such as the mesh width. L_h is imposed to be stable and can hopefully be inverted in an efficient way. This method is called the *basic scheme*. Beside the basic scheme, a higher-order scheme of order $M+1$, with $M > N$ is available, called the *correction scheme* and abbreviated as

$$S_h w_h = 0, \tag{3}$$

with the approximate solution w_h. We note that this correction scheme will not be solved for the solution w_h, but serves as an error approximation only.

The iterated defect correction approach then has the following structure. The initial approximate solution is obtained by solving the basic scheme (1):

$$u_h^{(0)} = L_h^{-1} r_h^{(0)} \qquad \text{with} \qquad r_h^{(0)} := 0. \tag{4}$$

Next the iteration cycle from level k to $k+1$ is described, starting from the steady state solution $u_h^{(k)}$ and the residual $r_h^{(k)}$ at the old iteration level k. It consists of the following steps:

(1) **Reconstruction** : To estimate the local discretization error, a higher-order solution $w_h^{(k)}$ of degree $M > N$ is reconstructed from the lower-order solution $u_h^{(k)}$ of degree N. As usual, the reconstruction operator is based on the solution in a certain neighborhood of the respective control volume.

(2) **Defect Estimation**: This reconstruction allows to estimate the local defect by the higher-order correction scheme (3) as

$$d_h^{(k+1)} := S_h w_h^{(k)}. \tag{5}$$

This defect is used to calculate a new right hand side for the basic scheme:

$$r_h^{(k+1)} := r_h^{(k)} - d_h^{(k+1)}. \tag{6}$$

(3) **Correction** : Next, the basic solution method is applied to this modified equation to yield the improved steady state approximate solution

$$u_h^{(k+1)} = L_h^{-1} r_h^{k+1} = L_h^{-1} \left(L_h u_h^{(k)} - \tilde{d}_h^{(k+1)} \right). \tag{7}$$

This procedure is iteratively applied until some stopping criterion is achieved. Here we demand that $\tilde{d}_h^{(k+1)}$ becomes small.

The discontinuous Galerkin scheme allows an approximate solution u_h which is the union of local polynomials defined in each grid cell and may be discontinuous across grid cell interfaces:

$$u_h(\vec{x}, t) := u_i(\vec{x}, t) = \sum_{l=1}^{\mathcal{N}} \hat{u}_{i,l}(t) \phi_{i,l}(\vec{x}) \quad \text{for} \quad \vec{x} \in Q_i. \tag{8}$$

Here, Q_i is an auxiliary grid cell, the functions $\phi_{i,l} = \phi_{i,l}(\vec{x})$ are the basis functions spanning the space of polynomials of degree N restricted to the cell Q_i, whereas the polynomial coefficients $\hat{u}_{i,l}(t)$ are the time dependent degrees of freedom. The discontinuous Galerkin method is applied to advection diffusion reaction equations. After the spatial discretization a system of ordinary differential equations in time has to be solved which consists in every grid cell of $\mathcal{N}$ equations being coupled to the neighbors by the interface fluxes:

$$\frac{d}{dt} \hat{u}_i = T_V \left(\hat{u}_i, \vec{\phi} \right) + T_S \left(\hat{u}_i, \hat{u}_i^+, \vec{\phi} \right) + T_R \left(\hat{u}_i, \vec{\phi} \right) \tag{9}$$

where T_V, T_S, and T_R contain the cell volume integrals of the fluxes, the cell interface integrals of the fluxes and the source terms, respectively. The coupling to the adjacent grid cells is expressed in T_S by $\hat{u}_i^+$ which contains the values from the neighboring cells. The vector $\vec{\phi}$ contains the components of all the basis functions. The correction scheme is now again a DG variational formulation and reads as

$$\frac{d}{dt} \hat{u}_i = T_V \left(\hat{w}_i, \vec{\phi} \right) + T_S \left(\hat{w}_i, \hat{w}_i^+, \vec{\phi} \right) + T_R \left(\hat{w}_i, \vec{\phi} \right) \tag{10}$$

where $\hat{w}_i$ denotes the vector of the coefficients of the reconstructed polynomial w_i for the grid cell Q_i. The approximate solution w_h has the same form as (8), but is a polynomial of degree $M > N$. The first $\mathcal{N}$ coefficients are the same as for u_h, while the coefficients for the enrichment of the basis are obtained by reconstruction based on [3]. The cell volume and cell interface integrals as well as the source terms in (10) are calculated from the reconstructed higher-order accurate solution.

The building blocks for the interated defect corrections are then given by starting vector $u_h^{(0)}$ which is the steady state solution of the basic scheme with order of accuracy $N + 1$, the reconstruction and the defect estimation. The reconstruction enhances the piecewise polynomial approximation $u_h^{(k)}$ of degree N to $w_h^{(k)}$ of degree M. The defect of this solution is estimated by (10). The estimated local discretization error is subtracted from the right hand side of the basic scheme, resulting in the modified equation. Applying the solution procedure of the basic scheme to this modified equation, the approximate solution $u_h^{(k+1)}$ is obtained.

According to a convergence criterion the whole procedure is stopped or repeated by returning to the first step within the interation procedure.

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Nonlinear and Linear Boundary Conditions for Wave Propagation Problems

Jan Nordström

We discuss linear and nonlinear boundary conditions for wave propagation problems. The concepts of well-posedness and stability are discussed by considering a specific example of boundary treatment occurring in the modelling of earthquakes. The implications and difficulties are related to events occuring in fully nonlinear equations. Numerical simulations illustrate the theoretical discussion.

The principles for construction stable and convergent high order finite difference schemes for linear and nonlinear boundary conditions are discussed in the context of wave propagation problems in earthquake simulations. The first requirement for obtaining reliable solutions is well-posedness, see [1],[2]. A typical recipie is:

- Energy method and *choice of boundary conditions*
- Uniqueness and existence

Once we have a well-posed problem, it is meaningful to construct a numerical approximation. A recipie for constructing a stable and convergent scheme is:

- Energy method and *choice of penalty parameters*
- Convergence

For linear problems, the recipie outlined above guarantees that the scheme converges to a reliable solution. However, as we will see, this is not always the case for nonlinear boundary conditions.

The material is modeled as linear elastic with frictional sliding occuring on thin internal interfaces. The internal interfaces, or faults, are governed by highly nonlinear friction laws. The friction laws relate the slip velocities to the tractions acting on the fault. The elastic wave equations govern the wave propagation between the faults. Theis will result in a linear problem with nonlinear boundary conditions (the friction laws). For a readup on these problems see [3],[4]. Finally, the relation to fully nonlinear problems are shortly discussed and a few examples are given.

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Efficient Filtering Techniques for Stabilization and Postprocessing of discontinuous Galerkin solutions to hyperbolic conservation laws

Sigrun Ortleb

(joint work with Andreas Meister, Thomas Sonar)

This talk is concerned with the development of efficient shock capturing strategies for DG methods on triangular grids solving hyperbolic conservation laws such as the Euler equations of gas dynamics or the shallow water equations. In this context, in collaboration with Andreas Meister and Thomas Sonar, a low-cost damping mechanism based on adaptive modal filtering applied to the Proriol-Koornwinder-Dubiner basis was invented in [15, 16].

In order to deal with the possible instability of high order methods due to Gibbs oscillations, different damping strategies have been proposed for the DG scheme. Often, limiters are applied as in the case of the Runge-Kutta discontinuous Galerkin (RKDG) method developed by Cockburn and Shu in a series of papers, see the review [3]. There, the polynomial degree N of the limited solution in a given "troubled" cell is reduced to $N = 1$ and a modified minmod function is employed to redefine the slope. Modifying the minmod function avoids an approximation of only first order at smooth extrema. Nevertheless, once the minmod limiter is enforced in a given cell, the information inherent in the higher order coefficients is lost. Consequently, so-called moment limiters starting from the higher order coefficients have been suggested in [1, 12, 22]. However, at discontinuities, also these more sophisticated limiters generally reduce the polynomial degree to $N = 1$. A different approach is to use WENO or HWENO reconstructions in regions marked by a shock sensor, see [18, 23], but these techniques are computationally expensive. Directly introducing artificial viscosity terms in the semidiscrete formulation has been considered e.g. in [6, 10, 17]. These terms additionally have to be implemented within the DG scheme and in case of explicit time stepping, the bounds on the admissible time step size are more restrictive due to these additional stabilizing terms.

Our new damping mechanism is therefore related to ideas originating from the class of spectral methods. Spectral methods may apply spectral viscosity [21] in order to stabilize the calculation, introducing diffusion-like terms that depend

on the spatial resolution and distinguish different frequencies of the numerical solution. This concept has successfully been applied to large-eddy simulations in [11]. As suggested in [7], the introduction of spectral viscosity can be carried out within the spectral filtering framework as a so-called modal filter directly applied to the expansion coefficients – if the viscosity is formulated by means of the Sturm-Liouville operator corresponding to the chosen basis. Spectral filtering has broad applications as a stabilization technique, such as propagation of shock waves [4, 5], large-eddy simulation [13, 20] and atmospheric modelling [14]. Spectral filtering on triangular grids has also been considered in [9], where it was used to prevent aliasing instabilities.

We hence derive a relationship between modal filtering for discontinuous Galerkin methods on unstructured triangular grids and the introduction of spectral viscosity to the scheme where the corresponding high order viscosity term is based on the Sturm-Liouville operator associated to the Proriol-Koornwinder-Dubiner(PKD) polynomials. Consequently, the viscosity can be efficiently implemented as a modal filter which is applied to the numerical solution after each time step of the basic DG scheme. As for each time step only the multiplication of the coefficients of the approximate solution with a precomputed factor is necessary, this filtering approach has the advantage of a reduced computational cost compared to moment limiters, (H)WENO reconstruction or the introduction of regular diffusion. With respect to an increasing polynomial degree N, high order accuracy of modal filtering applied to the PKD expansions of sufficiently smooth functions has been proven in [15]. However, with respect to grid refinement, an adaptive application is necessary.

The aim of modal filtering in general is to stabilize the numerical scheme, but by construction no effort is made to obtain an oscillation-free approximation. Indead, there will still be spurious Gibbs oscillations close to a discontinuity of the exact entropy solution, that may nevertheless have the potential to contain high order information. In general, spectral filtering is hence used in combination with a second protagonist: a postprocessing technique used to reconstruct from the available information a more accurate and less oscillatory pointwise approximation. The computational cost of a postprocessing technique has only a small effect on the overall cost of the scheme, as the postprocessing procedure is only carried out at output times to visualize an essentially non-oscillatory numerical solution. Hence, cost optimization is of less importance here.

Reprojection techniques such as the Gegenbauer reconstruction method [8] are a powerful approach especially in one space dimension and provide sophisticated means to reconstruct a high order oscillation-free approximation even close to discontinuities, but are difficult to apply in higher space dimensions. As an alternative to these reprojection methods, Sarra [19] studied the application of the so-called digital total variation (DTV) filter as a possible "black box" postprocessing tool for spectral methods. The DTV filter was developed by Chan, Osher and Shen [2] as an image processing technique and is based on the minimization of a penalized TV energy. It is therefore well-suited for the purpose of removing oscillations –

which are regarded as noise – while preserving edges, i.e. discontinuities of the exact solution. In particular, no a-priori edge detection is necessary. In the case of DG solutions on triangular grids, our results in [15, 16] show that the application of DTV postprocessing is a very promising approach to reconstruct important information from the slightly oscillatory approximations. This is especially the case for a novel adaptive variant of the DTV filter and for DTV graphs based on the underlying DG triangulation.

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Time discretisation methods for the incompressible Navier-Stokes equations

Joachim Rang

This paper considers the simulation of time–dependent laminar flows which are described by the incompressible Navier–Stokes equations. One example is the flow around a cylinder [15]. Discretisations in space and time as well as a linearisation for solving the nonlinear problem in each discrete time are required. There are many possible approaches, see [4] for a comprehensive presentation, and the quest for optimal methods is still an active field of research. With respect to the spatial discretisation an inf–sup stable finite element method [3] is used. It has been demonstrated in a number of numerical studies, e.g. in [15, 7, 6], that the pair of second order velocity Q_2 and first order discontinuous pressure P_1^{disc} on quadrilateral and hexahedral meshes is among the best performing finite element methods.

The topic of the paper is the temporal discretisation of the incompressible Navier–Stokes equations. By far the most simulations of incompressible flows use explicit schemes or simple implicit schemes, like the backward Euler scheme, the Crank–Nicolson scheme or the fractional–step θ–scheme. In this paper we concentrate on implicit schemes, which are appropriate for laminar flow simulations and which avoid the nasty CFL condition. Studies [8] and [9] showed that for obtaining accurate results at least a second order time–stepping scheme is necessary. For this reason, only schemes with at least this accuracy will be considered.

The main focus of this paper is on an adaptive time step control. An adaptive time step control needs some error indicator or estimator on which the determination of the next time step is based. This error estimator suggests a new time step size to reach a given accuracy. If the time step size is too small a lot of unnecessary computational work has to be done. Otherwise, if the time step size is too large, the results may become too inaccurate.

An adaptive time step control may undoubtedly improve the accuracy and efficiency of incompressible flow simulations substantially. However, with the simple implicit schemes commonly used, an efficient time step control is hard to achieve. For this reason simulations with implicit temporal discretisations and adaptive time step control are rather rare in the literature. In [18] it is proposed to compare the results of a Crank–Nicolson step and a step of the fractional–step θ–scheme, which are both of second order. The quality of the time step is estimated on the

basis of different constants in the error estimates. This approach roughly doubles the computational efforts compared to using just one of these schemes. A different approach is considered in [1] where space–time adaptive finite elements are used and the numerical error is approximated with two numerical solutions obtained at different times. The predictor–corrector schemes offer another possibility, for example the Adams–Bashford method combined with the Crank-Nicolson scheme, see [4].

In [17] space–time adaptivity is used with embedding techniques for the control of the time step length. This approach is more efficient but it requires the use of more sophisticated time stepping schemes. This will be performed in this paper: two classes of diagonally implicit Runge–Kutta methods, for example the pressure–corrected fractional–step θ–scheme, and linearly implicit Runge–Kutta methods (Rosenbrock–Wanner methods (ROW-methods)) will be studied. Both classes of methods allow the computation of a second numerical solution with almost the same Runge–Kutta coefficients such that an effective time step control can be achieved [10, 5]. In the case of DIRK-methods schemes like ESDIRK3 and ESDIRK4 [2] or DIRK34 [14] can be used. Common ROW-methods may be ROS3P [11], RODASP [16], ROWDAIND2 [12], and ROS34PW2 [13].

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Towards Numerical Simulation of Fluid-Structure-Acoustics Interaction

Michael Schäfer
(joint work with D. Sternel, M. Kornhaas, F. Flitz, S. Nowak)

Noise generation in urban environments often is caused by turbulent flows or vibrating structures. Various possible interactions between flows, structures and acoustics can result in rather complex multi-physical processes. Thus, the modeling and simulation of corresponding effects is quite challenging. Such a simulation tool should be able to adequately handle flow-acoustics, structure-acoustics and fluid-structure couplings together with all mutual interactions. In this contribution we report on an approach towards the fully coupled numerical simulation of such kind of problems.

Fluid-structure interaction is accomplished by an implicit partitioned approach based on the in-house flow solver FASTEST, the structural solver FEAP and the MpCCI coupling interface. A high numerical efficiency is ensured by the proper involvement of geometrical multigrid methods. It turns out that a global multigrid approach in which the fluid-structure iteration acts as a smoother shows favorable convergence properties.

For flow-acoustics coupling we employ a numerical scheme for the simulation of aerodynamic noise caused by flows at low Mach numbers. To account for sound propagation the flow solver FASTEST is extended by a fully parallelized high resolution (HR) finite-volume scheme that solves the linearized Euler equations (LEE) on boundary fitted, block-structured hexahedral meshes. Aeroacoustic sources are obtained from the unsteady calculated flow field following the basic ideas of Hardin's and Pope's acoustic/viscous splitting technique. In order to speed up the coupled simulation and to account for the very different length scales of small turbulence structures and the long-wave acoustics, the acoustic field is computed on hierarchically coarsened flow grids. Further, different time scales are employed so that multiple acoustic time steps can be performed within one flow time step.

One key issue for handling the structure-acoustics is the proper treatment of moving grids. This is realized via an arbitrary Lagrangian-Eulerian (ALE) formulation.

Results for various test cases are given for verification and validation of the different approaches. Comparative studies illustrate the accuracy and efficiency of the methods. Details about the approaches and the results can be found in Refs. [1, 2, 3, 4].

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ENO reconstruction and ENO interpolation are stable

Eitan Tadmor

(joint work with Ulrik S. Fjordholm, Siddhartha Mishra)

The ENO reconstruction procedure was introduced in 1987 by Harten et. al. [9] in the context of accurate simulations for piecewise smooth solutions of nonlinear conservation laws. Since then, the ENO procedure and its extensions, [19, 10, 20, 11, 12, 13], have been used with a considerable success in Computational Fluid Dynamics; we refer to the review article of Shu [21] and the references therein. Moreover, ENO and its various extensions, in particular, with subcell resolution scheme (ENO-SR), [10], have been applied to problems in data compression and image processing in [14, 1, 5, 17, 4, 6, 2, 3] and references therein.

There are only a few rigorous results about the global accuracy of the ENO procedure. In [2], the authors proved the second-order accuracy of ENO-SR reconstruction of piecewise-smooth C^2 data. Multi-dimensional global accuracy results for the so-called ENO-EA method were obtained in [3]. Despite the extensive literature on the construction and implementation of ENO method and its variants for the last 25 years, we are not aware of any global, mesh independent, *stability results*. This brings us to the main result of this paper, stating the stability of the ENO reconstruction procedure in terms of the following sign property.

Theorem [The sign property [7, 8]]. *At each cell interface, the jump of the reconstructed ENO pointvalues cannot have an opposite sign to the jump in the underlying cell averages. Thus, even though the reconstructed polynomial may have large variations within each cell, its jumps at cell interfaces always have the same sign as the jumps of the cell averages. Moreover, the relative size of these jumps – after and before reconstruction, is uniformly bounded.*

Remark 1.1. We emphasize that the Theorem is valid for any order of ENO reconstruction and for any mesh size. It is valid for non-uniform meshes and makes no assumptions on the underlying function, other than that the cell averages must be well-defined, which is guaranteed if e.g. for L^1 data. This is a remarkable rigidity property of the piecewise-polynomial interpolation.

Remark 1.2. The stability asserted in the Theorem is realized in terms of the reconstructed pointvalues at cell interfaces. These are precisely the input for the construction of high-order accurate finite volume schemes for nonlinear conservation laws (see Shu [21]), and the relation between these values and the cell averages will be the main point of study in this paper. This approach was taken in [10], where we use the sign property to construct arbitrarily high-order accurate entropy stable ENO schemes for systems of conservation laws.

Remark 1.3. The proof of both the sign property depends on the judicious choice of stencils in the ENO Algorithm, and it may fail for other choices of ENO-based algorithms. In particular, the popular WENO methods, which are based on upwind or central weighted ENO stencils, [15, 16, 18], fail to satisfy the sign property, as can be easily confirmed numerically.

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Spectral Difference Method using PKD Polynomials

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(joint work with Thomas Sonar)

Our main idea is to achieve an high order Spectral Difference Method (SDM) on triangulations as proposed in Wang et. al. [4] based on various basis polynomials, espacially PKD polynomials [3] which form an orthogonal basis of the triangle $\mathbb{T}_2 = \{(r,s)| -1 \leq r,s \leq 1, r+s \leq 0\}$ and are defined as

$$\phi_k(r,s) := g_{\ell m}(r,s) := P_\ell^{0,0}\left(\frac{2(1+r)}{(1-s)} - 1\right)(1-s)^\ell P_m^{2\ell+1,0}(s),$$

where $P_n^{\alpha,\beta}$ are Jacobi polynomials defined on $[-1,1]$. Furthermore, additional filtering and enhanced edge detection shall be used to deal with discontinuous solutions.
The SDM can be seen as a kind of flux reconstruction method where the flux $\mathcal{F}$ is expanded, using certain basis polynomials Φ_k, as

$$\mathcal{F}(u(\mathbf{x},t)) = \sum_{k=1}^{N} \hat{F}_k(t)\Phi_k(\mathbf{x})$$

(where in its original form Lagrange polynomials $\Phi_k := L_k$ were chosen) and inserted in the underlying conservation law

$$u_t(\mathbf{x},t) + \nabla \cdot \mathcal{F}(u(\mathbf{x},t)) = 0.$$

Application of a coordinate transform ψ to the standard element $\mathbb{T}_2$ yields the discrete update scheme

$$u_t(\mathbf{x}) = -\sum_{k=1}^{N} \hat{F}_k(t) \cdot \mathcal{J}\psi \nabla_{r,s} \Phi_k(\psi^{-1}(\mathbf{x})). \tag{1}$$

Here, $\nabla_{r,s}\Phi_k(\psi^{-1}(\mathbf{x}))$ are universal coefficients, and only the Jacobian $\mathcal{J}\psi$ has to be stored for each triangle. To be able to deal with general (non-linear) conservation equations or systems, high order modal filters based on the choosen polynomials are applied to avoid spurious oscillations and yield similar results as for the Discontinuous Galerkin Method (DGM) used by Meister et al. [1].
Extending the SDM to arbitrary basis polynomials gives rise to two different approaches to compute the resulting coefficients $\hat{F}_k(t)$ in each timestep t, quadrature on the one hand and interpolation on the other. Since many quadrature point are needed for the quadrature approach, we use two dimensional Lobatto interpolation points based on [2] to compute $\hat{F}_k(t)$. For both single equations (Advection, Burgers) as well as Euler systems numerical tests and convergence studies are carried out; an example is given in table 2 where the slight instability of the SDM for higher orders on triangles is visible. To capture discontinuities using additional filtering, we apply edge detectors based on the conjugated fourier partial sum and extend the idea of concentration kernels proposed by Gelb et al. [5], [6], which accelerates the slow convergence of the partial sums, to two dimensions using Móricz [7] results. Furthermore, we give a first approach how to compute the fourier coefficients directly and exact from some given PKD coefficients $\hat{u}_k(t)$ in a triangle. In future work, numerical tests with these extensions will be carried out and some stabilization approaches for the SDM will be presented.

The authors thankfully acknowledge financial support by Deutsche Forschungsgemeinschaft (DFG) through grant SO 363/11-1.

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Reporter: Andreas Meister

p	# triangles	L_1-error	EOC
1	68	7.969320e-01	
1	272	2.386226e-01	1,73972594082076
1	1088	6.354128e-02	1,90896462981423
1	4352	1.621991e-02	1,96992833810494
2	68	1.612112e-01	
2	272	2.541214e-02	2,66536219981066
2	1088	3.773871e-03	2,75140085823421
2	4352	5.832253e-04	2,69391989951972
3	68	2.320903e-02	
3	272	1.863603e-03	3,6385197647317
3	1088	1.556608e-04	3,58161697606403
3	4352	1.392347e-05	3,48281496799031
4	68	2.327890e-03	
4	272	1.025595e-04	4,5044898493104
4	1088	5.428514e-06	4,23975999429687
4	4352	2.869398e-06	0,919809238033143
5	68	2.189395e-04	
5	272	5.198862e-06	5,39619259136033
5	1088	1.920169e-07	4,75889066274948
5	4352	6.509167e-07	-
6	68	1.641784e-05	
6	272	2.253653e-07	6,18685512473208
6	1088	2.519384e-08	3,1611224606827
6	4352	4.551174e-05	-

TABLE 2. EOC and L_1-error for the advection equation, sine wave with periodic boundary conditions.

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