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Enveloping Algebras and Geometric Representation Theory

Organised by
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March 4th – March 10th, 2012

Abstract. The workshop brought together experts investigating algebraic Lie theory from the geometric and combinatorial points of view.

Mathematics Subject Classification (2000): 14Lxx, 17Bxx, 20Gxx.

Introduction by the Organisers

The workshop *Enveloping Algebras and Geometric Representation Theory* organized by Shrawan Kumar (Chapel Hill), Peter Littelmann (Köln) and Wolfgang Soergel (Freiburg) was held March 4th—March 10th, 2012. It continues a series of conferences on enveloping algebras, as is indicated by the first part of the title, but the main focus was on geometric and combinatorial methods in algebraic Lie theory.

The meeting was attended by over 50 participants from all over the world, including quite a few younger researchers. We had usually three talks in the morning and two in the afternoon, leaving ample time for discussions, which was amply used by the participants. Wednesday afternoon was reserved for an excursion to Sankt Roman, and on Thursday afternoon we had four shorter talks by younger participants. In addition, we had an 'open problem' session on Thursday evening preceding the social evening.

Particularly interesting seemed to us the work of Leclerc and Hernandez on the Grothendieck rings of certain representations of quantum loop enveloping algebras and its relation to certain twisted derived Hall algebras; The work on geometrically killing the dynamical Weyl group by Ginzburg; the solution to the so-called AGT-conjecture of Alday, Gaiotto and Tachikawa by Vasserot; and the proof of Kostant's

Clifford algebra conjecture by Joseph; he work of Ian Gordon and Ivan Losev on category $\mathcal{O}$ for cyclotomic rational Cherednik algebras, establishing in this case a derived equivalence between different blocks conjectured in general by Rouquier.

Workshop: Enveloping Algebras and Geometric Representation Theory

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Abstracts

The coherent cohomology ring of an algebraic group

Michel Brion

This talk is based on the preprint [3].

To each scheme X over a field k, one associates the graded-commutative k-algebra

$$H^*(X) := \bigoplus_{i\geq 0} H^i(X, \mathcal{O}_X)$$

with multiplication given by the cup product. Any morphism of schemes $f : X \to X'$ induces a pull-back homomorphism of graded algebras

$$f^* : H^*(X') \to H^*(X),$$

and there are Künneth isomorphisms

$$H^*(X) \otimes H^*(Y) \xrightarrow{\cong} H^*(X \times Y).$$

When X is affine, $H^*(X)$ is just the algebra $\mathcal{O}(X)$ of global sections of $\mathcal{O}_X$.

Now consider a k-group scheme G with multiplication map $\mu : G \times G \to G$, neutral element $e_G \in G(k)$, and inverse map $\iota : G \to G$. Then $H^*(G)$ has the structure of a graded Hopf algebra with comultiplication μ^*, counit e_G^* and antipode ι^*. When G is affine, the Hopf algebra $H^*(G) = \mathcal{O}(G)$ uniquely determines the group scheme G; moreover, the category of G-modules is anti-equivalent to that of left comodules over $\mathcal{O}(G)$ (see [8, Exp. I, Prop. 4.7.2]).

On the other hand, when G is an abelian variety, a result of Serre (see [9, Chap. 7, Thm. 10]) asserts that $H^*(G)$ is the exterior algebra $\Lambda^* H^1(G)$; moreover, the vector space $H^1(G)$ has the same dimension as G, and consists of the primitive elements of $H^*(G)$ (recall that $\gamma \in H^*(G)$ is primitive if $\mu^*(\gamma) = \gamma \otimes 1 + 1 \otimes \gamma$).

We generalize this result as follows:

Let G be a group scheme of finite type over k. Then the graded algebra $H^*(G)$ is the exterior algebra of the $\mathcal{O}(G)$-module $H^1(G)$, which is free of finite rank.

If G is connected, then denoting by $P^*(G) \subset H^*(G)$ the graded subspace of primitive elements, we have an isomorphism of graded Hopf algebras

$$H^*(G) \cong \mathcal{O}(G) \otimes \Lambda^* P^1(G).$$

Moreover, $P^i(G) = 0$ for all $i \geq 2$.

As a consequence, the graded Lie algebra $P^*(G) = P^0(G) \oplus P^1(G)$ is abelian. Note that $P^0(G)$ consists of the homomorphisms of group schemes $G \to \mathbb{G}_a$; this k-vector space is finite-dimensional in characteristic 0, but not in prime characteristics (already for $G = \mathbb{G}_a$).

When G is an abelian variety and k is perfect, the structure of $H^*(G)$ follows readily from that of connected graded-commutative Hopf algebras (see [1, Thm. 6.1]) and from the isomorphism of $H^1(G)$ with the Lie algebra of the dual abelian variety (see [6, §13, Cor. 3]). For an arbitrary group scheme G, our main result does not seem to follow from general structure results such as those of Cartier-Gabriel-Kostant (see [10, Thm. 8.1.5]) and Milnor-Moore (see [4, §6]), since $H^*(G)$ is neither connected nor cocommutative. Also, returning to the setting of schemes, the $\mathcal{O}(X)$-module $H^*(X)$ is generally not free. For example, when X is the punctured affine plane, $H^1(X)$ is a torsion module over $\mathcal{O}(X) = k[x, y]$ and is not finitely generated.

The proof of our main result is based on the affinization theorem (see [8, Exp. VIB, Thm. 12.2]). It asserts that G has a smallest normal subgroup scheme H such that the quotient G/H is affine; then $\mathcal{O}(G/H) \cong \mathcal{O}(G)$ via the quotient morphism $G \to G/H$, which is therefore identified with the canonical morphism $G \to \operatorname{Spec} \mathcal{O}(G)$. Moreover, H is smooth, connected and contained in the center of the neutral component G^o; in particular, H is commutative. Also, we have $\mathcal{O}(H) = k$, i.e., H is "anti-affine". In fact, H is the largest anti-affine subgroup scheme of G; we denote it by G_{ant}.

By analyzing the quotient morphism $G \to G/G_{\mathrm{ant}}$, we obtain an isomorphism of $\mathcal{O}(G)$-modules

$$\psi : H^*(G) \xrightarrow{\cong} \mathcal{O}(G) \otimes H^*(G_{\mathrm{ant}})$$

which identifies the pull-back $H^*(G) \to H^*(G_{\mathrm{ant}})$ to $e_G^* \otimes \mathrm{id}$. On the other hand, using the classification of anti-affine groups (see [2, 7]) and additional arguments, we show that the Hopf algebra $H^*(G_{\mathrm{ant}})$ is the exterior algebra of $H^1(G_{\mathrm{ant}})$, a finite-dimensional vector space. This yields the first assertion of our main result.

When G is connected, we show that the above map ψ is an isomorphism of graded Hopf algebras; moreover, $P^1(G) \cong H^1(G_{\mathrm{ant}})$ via pull-back. This yields a description of the primitive elements which takes very different forms in characteristic 0 and in positive characteristics. We mention a rather unexpected consequence:

In characteristic 0*, the group schemes G such that $H^*(G) = k$ are exactly the universal extensions of abelian varieties by vector groups; in positive characteristics, these group schemes are trivial.*

When G acts on a scheme X and $\mathcal{F}$ is a G-linearized quasi-coherent sheaf on X, the cohomology $H^*(X, \mathcal{F})$ is equipped with the structure of a graded left comodule over $H^*(G)$. To describe this structure, it suffices to determine the maps $H^i(X, \mathcal{F}) \to \mathcal{O}(G) \otimes H^i(X, \mathcal{F})$ and $H^i(X, \mathcal{F}) \to H^1(G) \otimes H^{i-1}(X, \mathcal{F})$. The former map just entails the comodule structure of the G-module $H^i(X, \mathcal{F})$; but we do not know any simple description of the latter map, except in very special cases. Another open question is to describe the coherent cohomology ring of group schemes over (say) discrete valuation rings. In this setting, cohomology does not commute with base change, e.g., for degenerations of abelian varieties to tori. Also, while a version of the affinization theorem is known (see [8, Exp. VIB, Prop. 12.10]), the structure of "anti-affine" group schemes is an open problem.

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Orbits on flag varieties

Friedrich Knop

Let k be an algebraically closed field of arbitrary characteristic, G a connected reductive group defined over k, $B \subseteq G$ a Borel subgroup and $X = G/B$ the full flag variety. We are interested in the set of K-orbits on X where K is an arbitrary closed subgroup of G.

If K has an open orbit in X then K is called a spherical subgroup of G. In that case, it is known (Brion [1], Vinberg [5]) that the set $\mathfrak{B}^K = K\backslash X$ of K-orbits on X is finite. In [3], I constructed a natural action of the Weyl group W of G on $\mathfrak{B}$, provided $\operatorname{char} k \neq 2$.

If K is not spherical, then one can extend this result by replacing $\mathfrak{B}$ by the set $\mathfrak{B}_0^K$ of closed K-irreducible K-stable subvarieties Z of X which have of maximal modularity. Here, the modularity of Z is the minimal codimension of an K-orbit in Z. By a theorem of Vinberg, the maximal modularity is attained for $Z = X$.

Theorem[K.-Pezzini] *Let* $\operatorname{char} k \neq 2$. *Then there is a natural action of* W *on* $\mathfrak{B}_0^K$ *generalizing the action of* W *on* $K\backslash X$ *when* K *is spherical.*

For $\operatorname{char} k = 0$, the theorem was proved in [3]. The general case is proved by a reduction to the case $\operatorname{rk} G = 2$ and subsequently by a case-by-case consideration.

If $\operatorname{char} k = 2$, there are counterexamples even for K spherical, the smallest being $G = SL(3)$ and $K = SO(3)$. Instead there is an action of W on the set $\tilde{\mathfrak{B}}^K$ of K-equivariant double covers of K-orbits in X. As an example, we calculated the W-orbit of the open K-orbit in the case $K = O(n) \subseteq G = GL(n)$.

Finally, for $\operatorname{char} k = 0$, the W-orbit of the open K-orbit is linked to the theory of equivariant compactifications of G/K or, more specifically, to its valuation cone $\mathcal{V}(G/K)$. This is the set of G-invariant discrete valuations $v : k(G/K)^* \to$

$\mathbb{Q}$. It is known that $\mathcal{V}(G/K)$ can be embedded into the rational vector space $\mathrm{Hom}(\mathcal{X}(B), \mathbb{Q})$ as a finitely generated convex cone. A fundamental theorem of Brion [2] asserts that in $\mathrm{char}\, k = 0$ this cone is the fundamental domain of a certain finite reflection group attached to G/K. This is conjectured to hold whenever $\mathrm{char}\, k \neq 2$.

But, as it turns out, $\mathcal{V}(G/K)$ exhibits exotic behavior in $\mathrm{char}\, k = 2$. B. Schalke, [4], calculated for example that $\mathcal{V}(SL(3)/SO(3))$ is the union of two Weyl chambers. In particular, it is not the fundamental domain of a finite reflection group. Further calculations showed that also $\mathcal{V}(SL(4)/SO(4))$ is the union of two Weyl chambers but this time with 4 extremal rays. This provides the first example where $\mathcal{V}(G/K)$ is not even simplicial.

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Periodic structures in the affine category $\mathcal{O}$ at positive level

Peter Fiebig
(joint work with Martina Lanini)

0.1. **Affine category $\mathcal{O}$.** Let $\mathfrak{g}$ be a finite dimensional, complex Lie algebra, and $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$ the associated affine Kac-Moody algebra. Note that $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ is the centrally extended loop algebra, and D denotes the outer derivation operator $t\frac{d}{dt}$. Let $\widehat{\mathfrak{h}} \subset \widehat{\mathfrak{b}} \subset \widehat{\mathfrak{g}}$ be the Cartan and the Borel subalgebras. We study the category $\widehat{\mathcal{O}}$, which is the full subcategory of the category of $\widehat{\mathfrak{g}}$-modules that contains all objects that are semisimple for $\widehat{\mathfrak{h}}$ and locally finite for $\widehat{\mathfrak{b}}$.

0.2. **Block decomposition.** For any weight $\lambda \in \widehat{\mathfrak{h}}^\star$ we denote by $L(\lambda)$ the irreducible module with highest weight λ. Let $\sim$ be the equivalence class on $\widehat{\mathfrak{h}}^\star$ generated by $\lambda \sim \mu$ if there exists a non-split extension

$$0 \to L(\lambda) \to X \to L(\mu) \to 0$$

in $\widehat{\mathcal{O}}$. For any $\sim$-equivalence class $\Lambda \subset \widehat{\mathfrak{h}}^\star$ we define $\widehat{\mathcal{O}}_\Lambda$ as the full subcategory of $\widehat{\mathcal{O}}$ that contains all objects M with the property that if $L(\lambda)$ is isomorphic to a subquotient of M, then $\lambda \in \Lambda$. It follows that $\widehat{\mathcal{O}} = \prod_{\Lambda \in \widehat{\mathfrak{h}}^\star/\sim} \widehat{\mathcal{O}}_\Lambda$.

0.3. **The level of a block.** As K acts semisimply on any object of $\widehat{\mathcal{O}}$ and moreover is central, we have that $\lambda \sim \mu$ implies $\lambda(K) = \mu(K)$. Hence, we can associate with any block $\widehat{\mathcal{O}}_\Lambda$ a *level* k_Λ, defined by $k_\Lambda = \lambda(K)$ for some (all) $\lambda \in \Lambda$.

Let $\rho \in \widehat{\mathfrak{h}}^\star$ be a weight that takes the value 1 on each simple coroot. Note that ρ is only defined up to addition of a multiple of δ, the smallest positive imaginary root. Nothing in the following depends on this choice. We call Λ a *critical equivalence class*, if $k_\Lambda = -\rho(K)$. In the usual normalization, this number is $-h^\vee$, where $h^\vee$ is the dual Coxeter number.

0.4. **Equivalence classes made explicit.** Taking together a theorem of Kac-Kazhdan and the BGG-reciprocity we obtain the following description of the equivalence classes in $\widehat{\mathfrak{h}}^\star$ with respect to λ. We denote by $\widehat{\mathcal{W}}$ the affine Weylgroup, and let $(w, \lambda) \mapsto w.\lambda = w(\lambda + \rho) - \rho$ for $w \in \widehat{\mathcal{W}}$ and $\lambda \in \widehat{\mathfrak{h}}^\star$ be the shifted action of $\widehat{\mathcal{W}}$ on $\widehat{\mathfrak{h}}^\star$. As the linear action $\widehat{\mathcal{W}}$ on $\widehat{\mathfrak{h}}^\star$ stabilizes the line $\mathbb{C}\delta$, the shifted action does not depend on the choice of ρ.

Theorem 1 (Linkage principle). *Let $\Lambda \subset \widehat{\mathfrak{h}}^\star$ be an equivalence class.*

(1) *If Λ is non-critical, then Λ is a $\widehat{\mathcal{W}}_\Lambda$-orbit in $\widehat{\mathfrak{h}}^\star$, i.e. $\Lambda = \widehat{\mathcal{W}}_\Lambda.\lambda$ for any $\lambda \in \Lambda$.*
(2) *If Λ is critical, then Λ is a $\widehat{\mathcal{W}}_\Lambda \times \mathbb{Z}\delta$-orbit in $\widehat{\mathfrak{h}}^\star$, i.e. $\Lambda = \widehat{\mathcal{W}}_\Lambda.\lambda + \mathbb{Z}\delta$ for any $\lambda \in \Lambda$.*

From now on we fix a critical equivalence class Λ.

0.5. **Restricted representations at the critical level.** Let $\mathcal{Z}_{\text{crit}}$ be the Feigin-Frenkel center in the critical level and denote by $\mathcal{Z}^-_{\text{crit}}$ and $\mathcal{Z}^+_{\text{crit}}$ its negative and positive graded part, resp. We call an object M of $\widehat{\mathcal{O}}_\Lambda$ *restricted* if z acts on M by the zero homomorphism for any $z \in \mathcal{Z}^-_{\text{crit}}$ and any $z \in \mathcal{Z}^+_{\text{crit}}$. We denote by $\overline{\mathcal{O}}_\Lambda \subset \widehat{\mathcal{O}}_\Lambda$ the full subcategory of restricted objects.

Each simple critical level object $L(\lambda)$ is restricted, and we can, as before, define an equivalence relation $\overline{\sim}$ on Λ by looking at non-split extensions of $L(\lambda)$ and $L(\mu)$ in $\overline{\mathcal{O}}_\Lambda$. Again, we obtain a block decomposition

$$\overline{\mathcal{O}}_\Lambda = \prod_{\overline{\Lambda} \in \Lambda/\overline{\sim}} \overline{\mathcal{O}}_{\overline{\Lambda}}.$$

We then have the following variant of the linkage principle.

Theorem 2 (Restricted linkage principle,[2]). *Let $\overline{\Lambda} \subset \Lambda$ be an equivalence for $\overline{\sim}$. Then $\overline{\Lambda} = \widehat{\mathcal{W}}_\Lambda.\lambda$ for some (all) $\lambda \in \overline{\Lambda}$.*

The Feigin–Frenkel–Lusztig conjecture is a formula for the character $L(\lambda)$ for a critical weight λ in terms of a set of polynomials that are a variant of the well-known Kazhdan–Lusztig polynomials. These are called *periodic polynomials*. For a precise statement, see [2].

0.6. **Blocks and moment graphs.** Suppose that Λ is an equivalence class with the following properties:

(1) Some (equivalently, any) $\lambda \in \Lambda$ is integral.
(2) Λ contains a highest weight λ.
(3) $\mathrm{Stab}_{\widehat{\mathcal{W}}}(\lambda) = \mathcal{W}$, the finite Weyl group.

We denote by $\widehat{\mathcal{G}}_\Lambda$ the moment graph associated to the pair $(\widehat{\mathcal{W}}, \mathcal{W})$. It is constructed as follows. Its set of edges is $\widehat{\mathcal{W}}/\mathcal{W}$, and $\overline{x}$ and $\overline{y}$ are connected by an edge if $\overline{x} \neq \overline{y}$ and if there is a reflection $t \in \widehat{\mathcal{W}}$ with $\overline{tx} = \overline{y}$. This edge is then labelled by the positive coroot associated with t (this is in fact well-defined). Finally, the set of vertices of $\mathcal{G}$ is partially ordered by the induced Bruhat order on $\widehat{\mathcal{W}}/\mathcal{W}$.

0.7. **The stable subgraph of $\mathcal{G}$.** Let $\mathscr{A}$ be the set of alcoves associated with the affinization $\widehat{\mathcal{W}}$ of the finite Weylgroup $\mathcal{W}$ (note that $\widehat{\mathcal{W}}$ is isomorphic to $\mathcal{W} \ltimes \mathbb{Z}R^\vee$, where $R^\vee$ is the set of (finite) coroots. Denote by $\mathscr{A}^-$ the set of alcoves in the antidominant Weyl chamber. We can naturally identify $\widehat{\mathcal{W}}$ with $\mathscr{A}$ in such a way that $\mathscr{A}^-$ corresponds to the longest representatives of $\mathcal{W}$-classes in $\widehat{\mathcal{W}}$. In this way we also obtain an identification of $\mathscr{A}^-$ with $\widehat{\mathcal{W}}/\mathcal{W}$, i.e. with the set of vertices of $\mathcal{G}$.

Now let γ be contained in C^-. Then the map $A \mapsto A - \gamma$ stabilizes $\mathscr{A}^-$. Let $A, B \in \mathscr{A}^-$ be far inside $\mathscr{A}^-$ and suppose that A and B are connected by an edge E. We call this edge *stable* if, for $-\gamma$ far inside $\mathscr{A}^-$, the alcoves $A - \gamma$ and $B - \gamma$ are also connected by an edge that is labelled by the same coroot as E. This yields the *stable* subgraph $\mathcal{G}^{\mathrm{stab}} \subset \mathcal{G}$ (a technical note: We always work with finite subgraphs of $\mathcal{G}$ that live *far inside* $\mathscr{A}^-$ in order to avoid phenomena at the walls).

0.8. **BMP-sheaves on $\mathcal{G}^{\mathrm{stab}}$.** The inclusion $i \colon \mathcal{G}^{\mathrm{stab}} \to \mathcal{G}$ gives rise to a pullback functor $i^* \colon \mathcal{G}\text{-mod}_{\mathbb{C}} \to \mathcal{G}^{\mathrm{stab}}\text{-mod}_{\mathbb{C}}$.

Theorem 3 ([3]). *Let $\mathscr{B}$ be the Braden-MacPherson sheaf on $\mathcal{G}^{\mathcal{J}}$, where $\mathcal{G}^{\mathcal{J}}$ is a piece of $\mathcal{G}$ far inside the antifundamental chamber. Then $i^*\mathscr{B}$ is isomorphic to the Braden-MacPherson sheaf $\mathscr{B}^{stab}$ on $\mathcal{G}^{stab,\mathcal{J}}$.*

In particular, this yields a categorical interpretation of the stabilization phenomenon of affine parabolic Kazhdan–Luztig polynomials. Due to a theorem of Kato, this stabilization phenomenon links the parabolic affine Kazhdan–Lusztig polynomials that are associated with the moment graph $\mathcal{G}$ to the periodic polynomials appearing in the Feigin–Frenkel–Lusztig conjecture. Moreover, the category of sheaves on $\mathcal{G}$ is intimately connected to the category $\widehat{\mathcal{O}}$ at positive level. We hope to be able to employ these results in order to study critical representations.

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Weyl modules and q-Whittaker functions

Michael Finkelberg
(joint work with Alexander Braverman)

0.1. **The q-Whittaker functions.** Let G be a semi-simple, simply connected group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$; we choose a pair of opposite Borel subgroups B, B_- of G with unipotent radicals U, U_-; the intersection $B \cap B_-$ is a maximal torus T of G. It will be convenient for us to denote the weight lattice of T by $\check{\Lambda}$ and the coweight lattice by Λ. In this talk we study certain invariant polynomials $\Psi_{\check{\lambda}}(q,z)$ on T (the invariance is with respect to the Weyl group W of G). Here $z \in T$, $q \in \mathbb{C}^*$ and $\check{\lambda} : T \to \mathbb{C}^*$ is a dominant weight of G. The function $\Psi_{\check{\lambda}}(q,z)$ is a polynomial function of z with coefficients which are rational functions of q (in fact, later were are going to work with a certain modification $\widehat{\Psi}_{\check{\lambda}}(q,z)$ of $\Psi_{\check{\lambda}}(q,z)$ which will be polynomial in q).

The definition of $\Psi_{\check{\lambda}}(q,z)$ is as follows. Let $\check{G}$ denote the Langlands dual group of G with its maximal torus $\check{T}$. In [5] and [14] the authors define (by adapting the so called Kostant-Whittaker reduction to the case of quantum groups) a homomorphism $\mathcal{M} : \mathbb{C}[T]^W \to \mathrm{End}_{\mathbb{C}(q)}\mathbb{C}(q)[\check{T}]$ called the quantum difference Toda integrable system associated with $\check{G}$. For each $f \in \mathbb{C}[T]^W$ the operator $\mathcal{M}_f := \mathcal{M}(f)$ is indeed a difference operator: it is a $\mathbb{C}(q)$-linear combination of shift operators $\mathbf{T}_\beta$ where $\beta \in \Lambda$ and

$$\mathbf{T}_\beta(F(z)) = F(q^\beta z).$$

Remark. In principle the constructions of [5] and [14] depend on a choice of orientation of the Dynkin diagram of $\check{G}$; however one can deduce from the main result of [6] that the resulting homomorphism is independent of this choice.

In particular, the above operators can be restricted to operators acting in the space of functions on the lattice $\check{\Lambda}$ by means of the embedding $\check{\Lambda} \hookrightarrow \check{T}$ sending every $\check{\lambda}$ to $q^{\check{\lambda}}$. For any $f \in \mathbb{C}[\check{T}]^W$ we shall denote the corresponding operator by $\mathcal{M}_f^{\mathrm{lat}}$. The following conjecture should probably be not very difficult; however, at the moment we don't know how to prove it:

Conjecture 0.2. (1) *There exists unique collection of $\mathbb{C}(q)$-valued polynomials $\Psi_{\check{\lambda}}(q,z)$ on T satisfying the following properties:*

a) $\Psi_{\check{\lambda}}(q,z) = 0$ if $\check{\lambda}$ is not dominant.

b) $\Psi_0(q,z) = 1$.

c) Let us consider all the functions $\Psi_{\check{\lambda}}(q,z)$ as one function $\Psi(q,z) : \check{\Lambda} \to \mathbb{C}(q)$ depending on $z \in T$. Then for every $f \in \mathbb{C}[T]^W$ we have

$$\mathcal{M}_f^{\mathrm{lat}}(\Psi(q,z)) = f(z)\Psi(q,z).$$

(2) *The polynomials* $\Psi_{\check{\lambda}}(q,z)$ *are* W*-invariant.*

Of course, the second statement follows from the "uniqueness" part of the first.

Some remarks about the literature are necessary here. First of all, Conjecture 0.2 is easy for $G = SL(N)$. In this case, the functions $\Psi_{\check{\lambda}}(q,z)$ are extensively studied in [8]-[10]. Second, for general G there exists another definition of the q-Toda system using double affine Hecke algebras, studied for example in [4]. Since it is not clear to us how to prove that the definition of q-Toda from [4] and the definition of [5] and [14] are the same, we shall denote the operators from [4] by $\mathcal{M}'_f$. It is easy to see that $\mathcal{M}_f = \mathcal{M}'_f$ for $G = SL(N)$ [1] Similarly we shall denote by $(\mathcal{M}_f^{\mathrm{lat}})'$ their "lattice" version. Then it is shown in [4] that the existence part of Conjecture 0.2 holds for any G if the operators $\mathcal{M}_f^{\mathrm{lat}}$ are replaced by $(\mathcal{M}_f^{\mathrm{lat}})'$. We shall denote the corresponding polynomials by $\Psi'_{\check{\lambda}}(q,z)$.

The main result of this talk will imply the following:

Theorem 0.3. (1) *There exists a collection of* W*-invariant polynomials* $\Psi_{\check{\lambda}}(q,z)$ *on* T *with coefficients in* $\mathbb{C}(q)$ *satisfying a), b) and c) above.*

(2) *Let* $\widehat{\Psi}_{\check{\lambda}}(q,z) = \Psi_{\check{\lambda}}(q,z) \cdot \prod_{i\in I} \prod_{r=1}^{\langle \alpha_i, \check{\lambda}\rangle} (1-q^r)$. *Then* $\widehat{\Psi}_{\check{\lambda}}(q,z)$ *is a polynomial function on* $\mathbb{A}^1 \times T$.

We are going to construct the above polynomials explicitly by algebro-geometric means.

We shall usually refer to the polynomials $\Psi_{\check{\lambda}}$ and $\widehat{\Psi}_{\check{\lambda}}$ as q-Whittaker functions (following [8]-[10]). It is not difficult to see that

$$\lim_{q\to 0} \Psi_{\check{\lambda}} = \lim_{q\to 0} \widehat{\Psi}_{\check{\lambda}} = \chi(L(\check{\lambda}))$$

where $\chi(L(\lambda))$ stands for the character of the irreducible representation $L(\check{\lambda})$ of G with highest weight $\check{\lambda}$.

The main purpose of this talk is to give several (algebro-geometric and representation theoretic) interpretations of the functions $\Psi_{\check{\lambda}}$ and $\widehat{\Psi}_{\check{\lambda}}$; as a byproduct we shall show that $\widehat{\Psi}_{\check{\lambda}}(q,z)$ is *positive*, i.e. it is a linear combination of the functions $\chi(L(\check{\mu}))$ with coefficients in $\mathbb{Z}_{\geq 0}[q]$ (this also implies that $\Psi_{\check{\lambda}}$ is a linear combination of the $\chi(L(\check{\mu}))$'s with coefficients in $\mathbb{Z}_{\geq 0}[[q]]$). All of our results are known for the polynomials $\Psi'_{\check{\lambda}}$ (and thus, in particular, we can show that $\Psi_{\check{\lambda}} = \Psi'_{\check{\lambda}}$) due to [4] and [11] but our proofs are totally different from *loc. cit.*

0.4. **Weyl modules.** Recall the notion of Weyl $\mathfrak{g}[\mathfrak{t}]$-module $\mathcal{W}(\check{\lambda})$ for dominant $\check{\lambda} \in \Lambda_+^\vee$, see e.g. [3]. It is the maximal G-integrable $\mathfrak{g}[\mathfrak{t}]$-quotient module of $\mathrm{Ind}_{\mathfrak{u}[\mathfrak{t}]\oplus\mathfrak{t}}^{\mathfrak{g}[\mathfrak{t}]} \mathbb{C}_{\check{\lambda}}$ where $\mathfrak{u} \subset \mathfrak{g}$ is the nilpotent radical of a Borel subalgebra, containing $\mathfrak{t}$. There is also a natural notion of *dual Weyl module* $\mathcal{W}(\check{\lambda})^\vee$ (one has to replace the induction by coinduction and "quotient module" by "submodule").

[1] In fact, the results of this talk together with the results of [11] imply that $\mathcal{M}_f = \mathcal{M}'_f$ for any G, but we would like to have an independent proof of this fact

Both $\mathcal{W}(\check{\lambda})$ and $\mathcal{W}(\check{\lambda})^\vee$ are endowed with a natural action of $\mathbb{C}^*$ by "loop rotation". When restricted to $G\times\mathbb{C}^*$ the module $\mathcal{W}(\check{\lambda})$ becomes a direct sum of finite-dimensional representations and the character $\chi(\mathcal{W}(\check{\lambda}))$ makes sense; moreover it is a linear combination of $\chi(L(\check{\mu}))$'s with coefficients in $\mathbb{Z}_{\geq 0}[[q]]$. Also we have $\chi(\mathcal{W}(\check{\lambda}))=\chi(\mathcal{W}(\check{\lambda})^\vee)$.

Let $\mathbb{A}^{\check{\lambda}}$ denote the space of all formal linear combinations $\sum\gamma_i x_i$ where $x_i\in\mathbb{A}^1$ and γ_i are dominant weights of G such that $\sum\gamma_i=\check{\lambda}$. The character of $\mathbb{C}[\mathbb{A}^{\check{\lambda}}]$ with respect to the natural action of $\mathbb{C}^*$ is equal to $\prod\limits_{i\in I}\prod\limits_{r=1}^{\langle\alpha_i,\check{\lambda}\rangle}(1-q^r)$. According to [3] there exists an action of $\mathbb{C}[\mathbb{A}^{\check{\lambda}}]$ on $\mathcal{W}(\check{\lambda})$ such that

1) This action commutes with $G[\mathsf{t}]\rtimes\mathbb{C}^*$;

2) $\mathcal{W}(\check{\lambda})$ is finitely generated and free over $\mathbb{C}[\mathbb{A}^{\check{\lambda}}]$.

Let $D(\check{\lambda})$ be the fiber of $\mathcal{W}(\check{\lambda})$ at $\check{\lambda}\cdot 0\in\mathbb{A}^{\check{\lambda}}$. This module is called a Demazure module (for reasons explained in [7]). This is a finite-dimensional $G[\mathsf{t}]\rtimes\mathbb{C}^*$-module (in fact, it is easy to see that the action of $G[\mathsf{t}]$ on $D(\check{\lambda})$ extends to an action of $G[[\mathsf{t}]]$). We are going to prove the following

Theorem 0.5. *Assume that G is simply laced. Then*

(1)

$$\chi(\mathcal{W}(\check{\lambda}))=\Psi_{\check{\lambda}}(q,z) \tag{0.1}$$

(2)

$$\chi(D(\check{\lambda}))=\widehat{\Psi}_{\check{\lambda}}(q,z). \tag{0.2}$$

In particular, $\widehat{\Psi}_{\check{\lambda}}(q,z)$ *is positive in the sense discussed above.*

When G is not simply laced, the above result is still true, if one replaces $G[[\mathsf{t}]]$ by some twisted (in the sense of Kac-Moody groups) version of it.

Theorem 0.5(2) is proved in [13] (for the case $G=SL(N)$) and in [11] for $\widehat{\Psi}'_{\check{\lambda}}$ instead of $\widehat{\Psi}_{\check{\lambda}}$. [2] Thus Theorem 0.5 together with [13], [11] imply that $\widehat{\Psi}'_{\check{\lambda}}=\widehat{\Psi}_{\check{\lambda}}$, but as was mentioned earlier we would like to have an independent proof of this result. We would also like to emphasize that our proof of Theorem 0.5 is geometric (in fact it follows easily from the main result of [2]) and thus it is quite different from the proof in [11].

0.6. **Geometric interpretation and spaces of (quasi-)maps.** To prove Theorem 0.5 it is clearly enough to prove (0.1). This will be done by interpreting both the LHS and the RHS in terms of algebraic geometry.

Let us first do it for the LHS. The quotient $G[[\mathsf{t}]]/T\cdot U_-[[\mathsf{t}]]$ can naturally be regarded as a scheme over $\mathbb{C}$. Any weight $\check{\lambda}$ defines a $G[[\mathsf{t}]]\rtimes\mathbb{C}^*$-equivariant line bundle on this scheme in the standard way. We shall prove

[2] It is important to emphasize that the definition of Demazure modules used in this talk (as fibers of Weyl modules) is not obviously equivalent to the standard definition used in [11]; however, the equivalence of the two definitions is proved in [7]

Theorem 0.7. *There is a natural isomorphism* $\Gamma(G[[\mathsf{t}]]/T\cdot U_-[[\mathsf{t}]],\mathcal{O}(\check\lambda))\simeq\mathcal{W}(\check\lambda)^\vee$. *Similarly,* $\Gamma(G[[\mathsf{t}]]/B_-[[\mathsf{t}]],\mathcal{O}(\check\lambda))\simeq D(\check\lambda)^\vee$.

Remark. Theorem 0.7 is not difficult; it can be thought of as an analog of Borel-Weil-Bott theorem for $G[[\mathsf{t}]]$. Let us also stress, that while the dual Weyl module $\mathcal{W}(\check\lambda)^\vee$ has a natural action of $G[[\mathsf{t}]]$, the Weyl module $\mathcal{W}(\check\lambda)$ itself only has an action of $G[\mathsf{t}]$.

On the other hand, there is a well known connection between the quotient $G[[\mathsf{t}]]/T\cdot U_-[[\mathsf{t}]]$ and the space of based maps $\mathbb{P}^1\to G/B$. Moreover, in [2] we have given a construction of the universal eigen-function of the operators $\mathcal{M}_f$ via the geometry of the above spaces of maps. Using this construction, we can obtain (0.1) from Theorem 0.7 by a (simple) sequence of formal manipulations. Technically, in order to perform this we shall need to consider a compactification of the space of maps by the corresponding space of quasi-maps.

0.8. **Relation to Macdonald polynomials.** For every $\check\lambda$ as above Macdonald defined certain remarkable polynomial $P_{\check\lambda}(q,t,z)$ (we use [12] as our main reference for Macdonald polynomials). This is a W-invariant polynomial on T with coefficients in $\mathbb{C}[[q,t]]$ (which in fact, converge to rational functions in q and t).

Conjecture 0.9. *We have* $P_{\check\lambda}(q,0,z)=\widehat{\Psi}_{\check\lambda}(q,z)$.

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How to convert a quiver variety into a categorical action

Ben Webster

This talk concerns the connections between the theory of categorical actions of Lie algebras, introduced by Chuang and Rouquier [CR08, Roub] and Khovanov and Lauda [KL10] with the theory of quiver varieties as introduced by Nakajima [Nak94] and Lusztig.

Hints at this relation have already appeared in the work of Lauda [Lau10] and Khovanov-Lauda [KL10] connecting the special cases of $\mathfrak{sl}_2$ and $\mathfrak{sl}_n$ to the geometry of Grassmannians and flag varieties, which are special cases of quiver varieties. Also pointing in this direction is the work of Varagnolo-Vasserot [VV] and Rouquier [Roua] connecting the Khovanov-Lauda-Rouquier (or quiver Hecke) algebras to the geometry of perverse sheaves on Lusztig quiver varieties. This accomplishes a geometric categorification of the "upper half" $U_-^+(\mathfrak{g})$ of the quantum group, analogous to the realization of this algebra as a Hall algebra of quivers.

Nakajima has already shown that the way to "double" the Hall algebra construction is to consider "hyperkähler analogues" (in the sense of Proudfoot's thesis) of Lusztig's quiver varieties; these are generally called **Nakajima quiver varieties**. Our work can be seen as a direct geometric categorification of Nakajima's construction, but fits into a larger program of the study of symplectic varieties. Jointly with Braden, Licata and Proudfoot, the author has suggested a scheme for associating certain representation theoretic categories to symplectic singularities. What we aim to describe is the special case of this program for Nakajima quiver varieties.

Theorem ([BK04, BPW])**.** *Associated to every affine symplectic singularity $\mathfrak{M}$ with contracting $\mathbb{C}^*$ action, there is a canonical quantization: a filtered algebra A with center Z given by a polynomial ring, such that for any maximal ideal $\mathfrak{m}$ in Z, the quotient $A/A\mathfrak{m}$ has associated graded $\mathbb{C}[\mathfrak{M}]$ and semi-classical Poisson structure given by that coming from the symplectic structure.*

In the case of the nilcone of a simple Lie algebra $\mathfrak{g}$, the resulting algebra is essentially the universal enveloping algebra $U(\mathfrak{g})$. For Slodowy slices, we arrive at the finite W-algebra. For hypertoric varieties, we arrive at the "hypertoric enveloping algebra." For general quiver varieties, these seem to be new and interesting algebras, which can be explicitly described using non-commutative Hamiltonian reduction.

There is, of course, a beautiful picture relating the representation theory of this algebra to the geometry of the variety T^*G/B, and it is this perspective we intend to generalize.

In particular, if we consider the quotient of $U(\mathfrak{g})$ by the central character of a finite-dimensional representation, this algebra has a natural category of **Harish-Chandra bimodules**, bimodules which are locally finite for the adjoint action.

The derived category $D(\mathsf{HC})$ of bimodules over this quotient with Harish-Chandra cohomology is closed under derived tensor product, and is thus a monoidal category.

Theorem. *The Grothendieck group of $D(\mathsf{HC})$ is canonically isomorphic to the group algebra $\mathbb{Z}[W]$, via the map given by localization of bimodules to twisted D-modules on $G/B \times G/B$ followed by characteristic cycle, and the isomorphism between $\mathbb{Z}[W]$ and the top Borel-Moore homology of the Steinberg variety.*

Thus, Harish-Chandra bimodules can be thought of as a natural categorification of the group algebra of the Weyl group (and by using a geometric grading, this can be upgraded to the Hecke algebra). Thus, it is natural to consider other examples of interesting convolution algebras which appear in symplectic varieties. Perhaps the most interesting is the construction by Nakajima of the universal enveloping algebra in the convolution algebras of quiver varieties. We fix a symmetric Kac-Moody algebra $\mathfrak{g}$ and a highest weight λ; attached to this highest weight is a union of quiver varieties $\mathfrak{Q}^\lambda = \sqcup \mathfrak{Q}^\lambda_\mu$ attached to the different weight spaces of the highest weight rep with weight λ. This space has a natural map to a single (possibly infinite dimensional) affine variety $\mathcal{Q}^\lambda$, which can be thought of as the space of semi-simple representations of the pre-projective algebra up to stabilization.

By analogy with the result above, we expect that the Harish-Chandra bimodules over the quantizations of these quiver varieties to be a natural categorification of $U(\mathfrak{g})$, or at least a quotient of it. On the other hand, we already know such a categorification from the work mentioned above, defined in an essentially combinatorial way.

Theorem ([Web])**.** *There is a functor from the Cautis-Lauda* [CL] *version of the 2-Kac-Moody algebra $\mathcal{U}$ to the 2-category whose objects are weights of μ and whose morphism categories $\mu \to \nu$ are the derived categories of "Harish-Chandra" bimodules between quantizations of $\mathfrak{Q}^\lambda_\mu$ and $\mathfrak{Q}^\lambda_\nu$. Ranging over all λ, this functor is faithful.*

This gives a geometric realization of the totality of the 2-category $\mathcal{U}$ not just its upper half. While of intrinsic interest, this result also has consequences for the structure of the 2-category $\mathcal{U}$. One consequence is:

Theorem ([Web])**.** *The classes of indecomposable 1-morphisms in $\mathcal{U}$ coincide with Lusztig's canonical basis of $\dot{U}$ for all symmetric Lie algebras.*

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Quantum Grothendieck derived and derived Hall algebras

Bernard Leclerc
(joint work with David Hernandez)

The talk was based on the recent preprint [4]. Let $\mathfrak{g}$ be a simple Lie algebra of type A, D, E over $\mathbb{C}$. We denote by $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ a triangular decomposition of $\mathfrak{g}$. Let v be an indeterminate, and let

$$U_v(\mathfrak{g}) = U_v(\mathfrak{n}) \otimes U_v(\mathfrak{h}) \otimes U_v(\mathfrak{n}_-)$$

be the corresponding Drinfeld-Jimbo quantum enveloping algebra over $\mathbb{C}(v)$, defined via a v-analogue of the Chevalley-Serre presentation of $U(\mathfrak{g})$. Using a geometric realization of $U_v(\mathfrak{n})$ in terms of perverse sheaves on varieties of representations of a quiver Q of the same Dynkin type as $\mathfrak{g}$, Lusztig [5] has defined a canonical basis $\mathbf{B}$ of $U_v(\mathfrak{n})$ with favorable positivity properties. This was inspired by a seminal work of Ringel [8], showing that the twisted Hall algebra of the category mod(FQ) of representations of Q over a finite field F, is isomorphic to the specialization of $U_v(\mathfrak{n})$ at $v = \sqrt{|F|}$.

One can associate with $\mathfrak{g}$ another quantum algebra. Let $L\mathfrak{g} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$ be the loop algebra of $\mathfrak{g}$. Let q be a nonzero complex number, which is not a root of unity. Via a q-analogue of the loop presentation of $U(L\mathfrak{g})$, Drinfeld [1] has defined the quantum loop algebra $U_q(L\mathfrak{g})$, an algebra over $\mathbb{C}$. The finite-dimensional representations of $U_q(L\mathfrak{g})$ have attracted a lot of attention, because of their connection with the trigonometric solutions of the quantum Yang-Baxter equation with spectral parameter. In [4] we focus on a tensor subcategory $\mathcal{C}_\mathbb{Z}$ of the category of finite-dimensional $U_q(L\mathfrak{g})$-modules, whose simple objects are parametrized by a discrete set (for the precise definition of $\mathcal{C}_\mathbb{Z}$ see [3, §3.7] or [4, §5.2]). Denote by $\mathcal{R}$ the complexified Grothendieck ring of $\mathcal{C}_\mathbb{Z}$. Let t be another indeterminate. By works of Nakajima [6] and Varagnolo-Vasserot [11], the $\mathbb{C}$-algebra $\mathcal{R}$ has an interesting t-deformation $\mathcal{R}_t$ over $\mathbb{C}(t)$. Hernandez [2] has introduced a slightly different deformation $\mathcal{K}_t$ (defined over $\mathbb{C}(t^{1/2})$). These t-deformations are important because they contain for every simple object L of $\mathcal{C}_\mathbb{Z}$ a "class" $[L]_t$ which can be characterized by axioms similar to those of Lusztig for the canonical basis $\mathbf{B}$.

As a consequence, Nakajima [6] has shown that one can calculate algorithmically the character of L.

Surprisingly, these deformed Grothendieck rings have not been much studied from the ring theoretic point of view, and for instance, to the best of our knowledge, there is no available presentation by generators and relations in the literature. One of our main results is a presentation of $\mathcal{K}_t$, with a similar flavour as the familiar Drinfeld-Jimbo presentation of $U_v(\mathfrak{n})$.

Theorem 0.1. *The algebra $\mathcal{K}_t$ is isomorphic to the $\mathbb{C}(t^{1/2})$-algebra $\mathcal{A}$ presented by generators $y_{i,m}$ $(i \in I, m \in \mathbb{Z})$ subject only to the following relations:*

(R1) *for every $m \in \mathbb{Z}$,*

$$y_{i,m}\, y_{j,m} - y_{j,m}\, y_{i,m} = 0 \qquad \text{if } (\alpha_i, \alpha_j) = 0,$$

$$y_{i,m}^2\, y_{j,m} - (t + t^{-1}) y_{i,m}\, y_{j,m}\, y_{i,m} + y_{j,m}\, y_{i,m}^2 = 0 \qquad \text{if } (\alpha_i, \alpha_j) = -1;$$

(R2) *for every $m \in \mathbb{Z}$ and every $i, j \in I$,*

$$y_{i,m}\, y_{j,\,m+1} = t^{-(\alpha_i,\alpha_j)} y_{j,\,m+1}\, y_{i,m} + \delta_{ij}(1 - t^{-2});$$

(R3) *for every $p > m + 1$ and every $i, j \in I$,*

$$y_{i,m}\, y_{j,p} = t^{(-1)^{p-m}(\alpha_i,\alpha_j)}\, y_{j,p}\, y_{i,m}.$$

This presentation shows that $\mathcal{K}_t$ is obtained by taking an infinite number of copies of $U_t(\mathfrak{n})$ labelled by $m \in \mathbb{Z}$, and then imposing t-boson relations between generators of copies sitting at adjacent integers, and t-commutation relations between generators of non-adjacent copies.

Let $D^b(\mathrm{mod}(FQ))$ be the bounded derived category of $\mathrm{mod}(FQ)$. Toën [10] has attached to this triangulated category an associative algebra called the derived Hall algebra of $D^b(\mathrm{mod}(FQ))$ (see also [12]). Let $DH(Q)$ denote the twisted derived Hall algebra obtained by twisting Toën's multiplication by means of the Ringel form, as in [9]. It follows from our presentation of $\mathcal{K}_t$ that:

Theorem 0.2. (a) *The specialization of $\mathcal{K}_t$ at $t = |F|^{1/2}$ is isomorphic to $DH(Q)$.*

(b) *Under this isomorphism, the classes of fundamental $U_q(L\mathfrak{g})$-modules are mapped to scalar multiples of the classes of indecomposable stalk complexes in $DH(Q)$, and the basis of classes of standard $U_q(L\mathfrak{g})$-modules is mapped to a rescaling of the natural basis of $DH(Q)$ indexed by isoclasses of objects of $D^b(\mathrm{mod}(FQ))$.*

To obtain our presentation of $\mathcal{K}_t$ we first consider a tensor subcategory $\mathcal{C}_Q$ of $\mathcal{C}_\mathbb{Z}$ which "looks like $\mathrm{mod}(FQ)$ inside $D^b(\mathrm{mod}(FQ))$". When Q is a bipartite orientation of the Dynkin diagram and the Coxeter number h of $\mathfrak{g}$ is even, $\mathcal{C}_Q$ is just the subcategory $\mathcal{C}_\ell$ with $\ell = h/2 - 1$, as defined in [3]. The general definition of $\mathcal{C}_Q$ for an arbitrary orientation Q is given in [4, §5.11]. Let $\mathcal{K}_{t,\,Q}$ be the subalgebra of $\mathcal{K}_t$ spanned by the elements $[L]_t$ associated with the simple objects L of $\mathcal{C}_Q$.

The quantum algebra $U_v(\mathfrak{n})$ is endowed with a distinguished scalar product. Let $\mathbf{B}^*$ be the basis of $U_v(\mathfrak{n})$ adjoint to the canonical basis $\mathbf{B}$ with respect to this scalar product. Let $v^{1/2}$ be a square root of v, and set $\mathcal{U}_v(\mathfrak{n}) := \mathbb{C}(v^{1/2}) \otimes U_v(\mathfrak{n})$. The main step for obtaining the presentation of $\mathcal{K}_t$ is:

Theorem 0.3. (a) *There is a* $\mathbb{C}$*-algebra isomorphism* $\Phi\colon \mathcal{K}_{t,Q} \xrightarrow{\sim} \mathcal{U}_v(\mathfrak{n})$ *with* $\Phi(t^{1/2}) = v^{1/2}$.

(b) *For every simple object* L *of* $\mathcal{C}_Q$*, the image* $\Phi([L]_t)$ *belongs to* $\mathbf{B}^*$ *(up to some explicit half-integral power of* v*).*

Thus, it follows from Theorem 0.3 that if N denotes a unipotent group with Lie algebra $\mathfrak{n}$, the tensor category $\mathcal{C}_Q$ is a categorification of the coordinate ring $\mathbb{C}[N]$ together with its dual canonical basis.

Since the bases $\mathbf{B}^*$ and $\{[L]_t\}$ have geometric origin, it is natural to ask for a geometric explanation of Theorem 0.3 (b). In the final part of [4], we show that the quiver representation spaces $E_{\mathbf{d}}$ used by Lusztig to define the canonical basis of $U_v(\mathfrak{n})$ are isomorphic to some particular graded quiver varieties $\mathfrak{M}_0^\bullet(W^{\mathbf{d}})$ used by Nakajima for describing the classes $[L]_t$ of the simple objects L of $\mathcal{C}_Q$. Moreover the intersection cohomology sheaves of closures of $G_{\mathbf{d}}$-orbits in $E_{\mathbf{d}}$ can be identified with the intersection cohomology sheaves of closures of strata in $\mathfrak{M}_0^\bullet(W^{\mathbf{d}})$. This is inspired by a similar result of Nakajima [7] for the category $\mathcal{C}_1$.

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Some examples of parity sheaves

Geordie Williamson

In 2008 I made a bet with Peter Fiebig involving a case of very good wine: I bet that by 2017 there would be "significant progress" on Lusztig's conjecture, and by 2022 a "complete solution". Of course the terms in inverted commas are open to interpretation (there are different versions of Lusztig's conjecture involving different bounds, what about a single counter-example? etc.) However even arrival at a point where we have to debate these questions would indicate significant progress.

In this sense my talk is selfish: I want to convince as many people as possible to start thinking about Lusztig's conjecture, so that I maximise my chances of getting some bottles of good wine. (Not to mention my second selfish motivation: Peter knows a lot more about wine than I do.) The main goal of my talk is to convince you that there are very difficult questions involved, but that things are happening. Recent work shows that Lusztig's conjecture is not the impenetrable fortress that many think it is.

1. Parity sheaves

Let X denote a complex algebraic variety equipped with a Whitney stratification

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$$

by locally closed connected smooth subvarieties. We write d_λ for the complex dimension of X_λ and for $i_\lambda : X_\lambda \hookrightarrow X$ the inclusion. Fix a field k (of characteristic $p \geq 0$). Write $\mathrm{Loc}(X_\lambda)$ for the abelian category of local systems of finite dimensional k-vector spaces on X_λ and $D_\Lambda(X)$ for the full subcategory of the derived category of sheaves of k-vector spaces consisting of Λ-constructible complexes.[1]

A *pariversity* is a function $\dagger : \Lambda \to \mathbb{Z}/2\mathbb{Z}$. We will only ever care about two special pariversities: the *constant* pariveristy $\natural(\Lambda) = \overline{0}$; and the *dimension* pariversity $\diamond(\lambda) = \overline{d_\lambda}$. For a fixed pariveristy $\dagger$ we say that a complex $F \in D_\Lambda(X)$ is $\dagger$*-even* if $\mathcal{H}^m(i_\lambda^? F) = 0$ for $\overline{m} \not\cong \dagger(\lambda)$ modulo 2 and all $\lambda \in \Lambda$ and $? \in \{*, !\}$. Furthermore, F is $\dagger$*-parity* if $F \cong F_0 \oplus F_1$ with F_0 and $F_1[1]$ both $\dagger$-even.

Example: $F \in D_\Lambda(X)$ is $\natural$-even if its stalks and costalks vanish in odd degree.

We make the following (strong) assuption on our stratification:

$$H^{\mathrm{odd}}(X_\lambda, \mathcal{L}) = 0 \qquad \text{for all } \mathcal{L} \in \mathrm{Loc}(X_\lambda). \qquad (P)$$

For example, if all the strata are simply connected this is the assumption that $H^{\mathrm{odd}}(X_\lambda) = 0$. Certainly our assumption (P) forces $\mathrm{Loc}(X_\lambda)$ to be semi-simple.

Fix a pariversity λ. A somewhat surprising consequence of the above assumption is the following theorem, which was discovered in joint work with D. Juteau and C. Mautner: Given any indecomposable (=simple) local system $\mathcal{L}$ on X_λ there exists up to isomorphism at most one indecomposable $\dagger$-parity sheaf $\mathcal{E}^\dagger(\lambda, \mathcal{L})$ extending $\mathcal{L}[d_\lambda]$. Moreover, any indecomposable $\dagger$-parity sheaf is isomorphic to $\mathcal{E}^\dagger(\lambda, \mathcal{L})[m]$ for some λ, $\mathcal{L} \in \mathrm{Loc}(X_\lambda)$ and $m \in \mathbb{Z}$. If it exists we call $\mathcal{E}^\dagger(\lambda, \mathcal{L})$ a *parity sheaf*.

Some remarks:

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[1] Also known as the "bounded derived category of Λ-constructible complexes" although this is slightly misleading.

i) If they exist, the above theorem shows that parity sheaves are classified in the same was as intersection cohomology complexes. However, parity sheaves need not exist and when they do they need not be perverse.
ii) Condition (P) is very restrictive. Sometimes it is useful to replace it with an equivariant version. For example this allows one to discuss parity sheaves on nilpotent cones (under explicit mild restrictions on p) and toric varieties.
iii) Our work on parity sheaves was inspired by work of Soergel who showed the existence and uniqueness of certain complexes on the flag variety obtained as direct summands of direct images from Bott-Samelson resolutions. He obtained his classification by relating the endomorphism algebras of these complexes to the endomorphism algebras of projective objects in "modular category $\mathcal{O}$". Whilst performing calculations on nilpotent cones with sheaf coefficients of characteristics 2 and 3 we noticed a similar phenomenon to that observed by Soergel. This led us to look for a geometric classification.
iv) The proof of the classification result is formally similar to the classification of tilting objects in highest weight categories by Ringel and Donkin.

2. Parity sheaves on flag varieties

For the rest of this talk we will restrict ourselves to the case of $X = G/B$ for a Kac-Moody group G and Borel subgroup B. The reader can certainly think about a finite flag variety of a connected complex reductive algebraic group G, for example $G = GL_n(\mathbb{C})$.

We let W denote the Weyl group of G and consider the stratification with $\Lambda = W$ given by the Bruhat decomposition:

$$X = G/B = \bigsqcup_{\lambda \in \Lambda} X_\lambda = \bigsqcup_{x \in W} BxB/B.$$

Each X_λ is isomorphic to an affine space, and hence our parity assumption (P) is satisfied. One can also show that in this case (using an inductive "Deligne" construction) that parity sheaves exist and are unique for any pariversity $\dagger$.

The following examples hopefully convince the reader of the usefulness of the notion of parity sheaves:

i) if k is a field of characteristic 0 (or of any sufficiently large characteristic) then $\mathcal{E}^\natural(w) \cong \mathrm{IC}(\overline{X_w})$, the intersection cohomology complex of the Schubert variety $\overline{X_w}$.
ii) in any characteristic one has $\mathcal{E}^\diamond(w) = T(w)$, the indecomposable tilting sheaf with support $\overline{X_w}$.

From now on we focus on the case of $\dagger = \natural$. Under this understanding we omit $\dagger$ from all notation below.

3. The p-canonical basis

As above, we assume that $X = G/B$ with its Bruhat stratification. Recall that W is the Weyl group and let S denote the subset of simple reflections. We order W using the Bruhat order.

We consider $\mathcal{H}$ the Hecke algebra of (W, S). This is an associative $\mathbb{Z}[v^{\pm 1}]$-algebra which is free as an $\mathbb{Z}[v^{\pm 1}]$-module with basis $\{H_w \mid w \in W\}$. The multiplication is determined by

$$H_s H_w = \begin{cases} H_{sw} & \text{if } sw > w, \\ (v^{-1} - v)H_w + H_{sw} & \text{if } sw < w. \end{cases}$$

Recall that the Hecke algebra $\mathcal{H}$ possesses a remarkable Kazhdan-Lusztig basis $\{\underline{H}_w \mid w \in W\}$. For example $\underline{H}_s = H_s + vH_{id}$. It has the following positivity properties:

i) $\underline{H}_w = H_w + \sum_{x<w} h_{x,w} H_x$ where $h_{x,w} \in v\mathbb{N}[v]$;
ii) $\underline{H}_x \underline{H}_y = \sum \mu_{xy}^z \underline{H}_z$ with $\mu_{xy}^z \in \mathbb{N}[v^{\pm 1}]$.

We now recall the geometric meaning of this basis. Given a finite dimensional vector space $V = \bigoplus V^i$ let $\mathrm{ch}V = \sum \dim V^{-i} v^i$ denote its Poincaré polynomial. Given $F \in D^b_W(X)$ define

$$\mathrm{ch}F = \sum_{x \in W} \mathrm{ch}H^*(F_x) v^{-\ell(x)} H_x.$$

Then a fundamental theorem of Kazhdan-Lusztig states that $\mathrm{chIC}(x, \mathbb{Q}) = \underline{H}_x$ ("Kazhdan-Lusztig polynomials encode local rational intersection cohomology of Schubert varieties"). This result is the key to understanding the positivity properties stated above.

Because $\mathcal{E}(x, \mathbb{Q}) \cong \mathrm{IC}(x, \mathbb{Q})$ we are tempted to try to understand $\mathrm{ch}\mathcal{E}(x, k)$ in a similar fashion for arbitrary k. Let us set

$${}^p\underline{H}_x := \mathrm{ch}\mathcal{E}(x, k).$$

(Recall that p denotes the characteristic of k.)

Then we have:

i) ${}^p\underline{H} = H_w + \sum_{x<w} {}^p h_{x,w} H_x$ with ${}^p h_{x,w} \in \mathbb{N}[v^{\pm 1}]$. Hence $\{\underline{H}_w \mid w \in W\}$ is a basis, which we call the *p-canonical basis* of $\mathcal{H}$,
ii) ${}^p\underline{H}_w = \sum {}^p m_{x,y} \underline{H}_x$ with ${}^p m_{x,w} \in \mathbb{N}[v^{\pm 1}]$,
iii) ${}^p\underline{H}_x {}^p\underline{H}_y = \sum {}^p \mu_{xy}^z {}^p\underline{H}_z$ with ${}^p\mu_{x,y}^z \in \mathbb{N}[v^{\pm 1}]$.

A note on the proofs of these positivity properties: i) follows easily from the definition of ch and the facts that $i_x \mathcal{E}(x, k) = \underline{k}_{X_x}[\ell(x)]$ and $\mathrm{supp}\mathcal{E}(x, k) \subset \overline{X_x}$. One may show that the characters of indecomposable parity sheaves only depend on the characteristic of k. ii) then follows from the fact that $\mathcal{E}(x, \mathbb{F}_p)$ admit "integral forms" $\mathcal{E}(x, \mathbb{Z}_p)$ and $\mathcal{E}(x, \mathbb{Z}_p) \otimes^L_{\mathbb{Z}_p} \mathbb{Q}_p$ is isomorphic to a direct sum of intersection cohomology complexes. Lastly parity sheaves admit lifts to the equivariant derived category $D_B(X, k)$ where there is a convolution formalism categorifying the multiplication in the Hecke algebra. iii) then follows because the convolution of two parity sheaves is isomorphic to a direct sum of shifts of parity sheaves.

One knows essentially nothing about the basis ${}^p\underline{H}_x$ in general except that, for fixed x, ${}^p\underline{H}_x = \underline{H}_x$ for big enough p, but we can't say how big is enough. To get a feeling for this basis we ask three questions:

Q1) For which p and x is ${}^p\underline{H}_x = \underline{H}_x$?
Q2) For fixed p understand the equivalence classes generated by $x \sim y$ if ${}^p m_{x,y} \neq 0$.
Q3) Describe ${}^p m_{x,y}$ in general.

In the hope of giving some understanding of what these questions involve we consider parity sheaves on the affine Grassmannian which are equivariant with respect to $G[[t]]$-orbits. Equivalently, we consider the elements ${}^p\underline{H}_x$ where x is an element of the affine Weyl group which is maximal in its double coset for the finite Weyl group. Because parity sheaves correspond to tilting modules (if $p > h + 1$), we can translate the above questions as follows:

i) Q1: when is $T(\lambda) = \Delta(\lambda)$ ($\Leftrightarrow \Delta(\lambda) = L(\lambda)$)? This is known (but complicated). For example, it holds if λ belongs to the fundamental alcove.
ii) Q2: which standard modules may occur in composition factors of tilting modules? For regular weights this is the linkage principle.
iii) Q3: determine the multiplicities of standard modules in tilting modules. This is unknown (and is presumably very difficult).

Examples related to Lusztig's conjecture:

i) Soergel has shown that ${}^p\underline{H}_x = \underline{H}_x$ for $p > h$ on a finite flag variety is equivalent to Lusztig's conjecture "around the Steinberg weight".
ii) Fiebig has shown that ${}^p\underline{H}_x = \underline{H}_x$ for certain elements of the affine Weyl group (those indexing weights in the intersection of the principal block and fundamental box) for $p > h$ implies Lusztig's conjecture.

Given i) above it seems sensible to do experiments. Experiments have been made possible by the following result (building on work of Libedinsky and Elias-Khovanov) of Elias and the author: The monoidal category of Soergel bimodules can be described by generators and relations.

We present a summary of our findings: We have $\underline{H}_x = {}^p\underline{H}_x$ for all p in the following table

A_n	B_n	D_n	F_4	G_2	E_6 (partial)
all p for $n < 6$ $p \neq 2$ for $n = 7$	$p \neq 2$ for $n < 6$	$p \neq 2$ for $n < 6$	$p \neq 2, 3$	$p \neq 2, 3$	$p \neq 2, 3$

The entry $p \neq 2$ in A_7 is due to Braden (2002). With his help I have recently been able to determine the full p-canonical basis in A_7 (38 of the 40320 elements of A_7 satisfy ${}^p\underline{H}_x \neq \underline{H}_x$). The exclusions $p \neq 2, 3$ in E_6 are due to Polo and Riche. The entries $p \neq 3$ for F_4 and E_6 give a counterexample to Fiebig's "GKM-conjecture". Thanks are also due to Jean Michel for helping me speed up my programs significantly.

Recently Polo has shown that for all primes p there is an x in a Weyl group of type A_{4p-1} with ${}^p\underline{H}_x \neq \underline{H}_x$!! So to summarise: the next few years will be interesting ones as far as Lusztig's conjecture is concerned!

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Category $\mathcal{O}$ for cyclotomic rational Cherednik algebras

Ian Gordon

(joint work with Ivan Losev)

This is a report on joint work with I.Losev, [3]. Our motivation comes from the set of conjectures made by P.Etingof, [2]. Some of these are confirmed by Shan-Vasserot, [6], by Stroppel-Webster, [7], and by our own work. The conjectures were inspired by developments that connect representation theory with certain symplectic resolutions.

0.1. Let W be a finite complex reflection group acting on its reflection representation $\mathfrak{h}$. Let $H_{\mathbf{p}}(W)$ be the rational Cherednik algebra of W with parameter p. This is an associative noetherian $\mathbb{C}$-algebra which is generated as a subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{h}])$ by the elements of W, multiplication by elements of $\mathbb{C}[\mathfrak{h}]$, and by the Dunkl operators which are deformations depending on p of the partial derivations ∂_y for $y \in \mathfrak{h}$. Let $\mathcal{O}_{\mathbf{p}}(W)$ be the full subcategory of finitely generated $H_p(W)$-modules on which the Dunkl operators act locally nilpotently. This is a highest-weight category, with standard objects $\Delta_p(\lambda)$ labelled by the set $\mathsf{Irrep}(W)$ of irreducible complex representations of W, and with a partial ordering $\prec_{\mathbf{p}}$ depending on the parameter $\mathbf{p}$.

0.2. Under very mild restrictions on W, which are expected to be unnecessary, there is an exact functor $\mathsf{KZ}_{\mathbf{p}} : \mathcal{O}_{\mathbf{p}}(W) \to \mathcal{H}_{q(\mathbf{p})}(W)$-mod where $\mathcal{H}_{q(\mathbf{p})}(W)$ denotes the Hecke algebra of W with parameter $q(\mathbf{p}) = \exp(2\pi\sqrt{-1}\mathbf{p})$. The functor is essentially surjective and fully faithful on projective objects.

Theorem (Rouquier, [5])**.** *Suppose that $\mathbf{p}, \mathbf{p}'$ are parameters attached to W such that $q(\mathbf{p}) = q(\mathbf{p}')$ and such that all the rank one parabolic Hecke subalgebras $\mathcal{H}_{q(\mathbf{p})}(W')$ of $\mathcal{H}_{q(\mathbf{p})}(W)$ are semisimple. If the orderings $\prec_{\mathbf{p}}$ and $\prec_{\mathbf{p}'}$ on $\mathsf{Irrep}(W)$ are equal then there is an equivalence of highest-weight categories $\mathcal{O}_{\mathbf{p}}(W) \sim \mathcal{O}_{\mathbf{p}'}(W)$*

Rouquier conjectures that without compatibility of the orderings, there should be a derived equivalence $D^b(\mathcal{O}_{\mathbf{p}}(W)) \sim D^b(\mathcal{O}_{\mathbf{p}'}(W))$.

0.3. Now fix $\ell \in \mathbb{N}$ and let n be a positive integer. Let $W_n = (\mu_\ell)^n \rtimes \mathfrak{S}_n$, a finite complex reflection group which acts on $\mathfrak{h} = \mathbb{C}^n$ by multiplication by ℓ^{th} roots of unity and by place permutation. The corresponding rational Cherednik algebra $H_{\mathbf{p}}(W_n)$ is sometimes called the cyclotomic rational Cherednik algebra.

Theorem (G.-Losev)**.** *Suppose that* $\mathbf{p}, \mathbf{p}'$ *are parameters attached to* W_n *such that* $q(\mathbf{p}) = q(\mathbf{p}')$. *Then* $D^b(\mathcal{O}_{\mathbf{p}}(W_n)) \sim D^b(\mathcal{O}_{\mathbf{p}'}(W_n))$.

0.4. A parameter $\mathbf{p}$ is called spherical if the idempotent $e = |W_n|^{-1} \sum_{w \in W_n} w \in \mathbb{C}[W_n]$ does not annihilate any irreducible $H_{\mathbf{p}}(W_n)$-representation. The set of spherical parameters is a Zariski-dense open set within the set of all parameters for W_n.

Theorem (G.-Losev)**.** *There is an action of* $\mathfrak{S}_\ell$ *on the set of spherical parameters such that* $\mathcal{O}_p(W_n)$ *is equivalent to* $\mathcal{O}_{\sigma(p)}(W_n)$ *for all* $\sigma \in \mathfrak{S}_\ell$ *and all spherical parameters* p. *Moreover, with a tiny further restriction on the parameter* p, *the equivalence sends* $\Delta_p(\lambda)$ *to* $\Delta_{\sigma(p)}(\sigma(\lambda))$ *where, if we identify* $\mathsf{Irrep}(W_n)$ *with* ℓ-*multipartitions of* n, $\mathfrak{S}_\ell$ *acts by place permutation on* $\mathsf{Irrep}(W_n)$.

0.5. The proofs of these theorems involve the deformation quantizations of a symplectic resolution X of the conical symplectic singularity $(\mathfrak{h} \times \mathfrak{h}^*)/W_n$. For the first, one quantizes the derived McKay correspondence $D^b(\mathrm{Coh}^{W_n}(\mathfrak{h} \times \mathfrak{h}^*)) \sim D^b(\mathrm{Coh}(X))$ to obtain a derived equivalence between $H_{\mathbf{p}}(W_n)$-mod and modules for a deformation quantization of X. Tensoring by line bundles on the quantization of X then relates different values of $\mathbf{p}$, and from there it is possible to pass to $\mathcal{O}$ by using a two-dimensional torus action on X. The second theorem is a consequence of the fact that the space of deformations of X is a covering of a space of deformations of $(\mathfrak{h} \times \mathfrak{h})/W_n$ with deck transformation group $\mathfrak{S}_\ell$. The appropriate deformations of $(\mathfrak{h} \times \mathfrak{h})/W_n$ are the spherical Cherednik algebras $eH_{\mathbf{p}}(W_n)e$, [4]. For spherical parameters, these algebras are Morita equivalent to $H_p(W_n)$, so one can access the symmetry.

0.6. Symplectic resolutions of the spaces $(\mathfrak{h} \times \mathfrak{h})/W$ do not necessarily exist for arbitrary W. The works of several authors, culminating in [1], show such resolutions exist if and only if $W = W_n$ or $W = W(G_4)$, the exceptional complex reflection group labelled G_4 in the list of Shephard-Todd.

0.7. Our second theorem helps a little in trying to describe the structure of $\mathcal{O}_{\mathbf{p}}(W_n)$, and in particular finding character formulae for the irreducible objects or even just describing their Gelfand-Kirillov dimension. For "irrational" parameters we use it as a step in giving an equivalence between $\mathcal{O}_{\mathbf{p}}(W_n)$ and blocks of Lie-theoretic parabolic category $\mathcal{O}$ of type A.

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Hodge Theory and Real Reductive Groups

Kari Vilonen
(joint work with Wilfried Schmid)

Let $G_{\mathbb{R}}$ be a real reductive group which we will assume to be semi-simple to simplify the exposition. The main problem discussed here is to determine all irreducible unitary representations of $G_{\mathbb{R}}$. We write $K_{\mathbb{R}}$ for the maximal compact subgroup, and G, K for the complexifications of $G_{\mathbb{R}}$, $K_{\mathbb{R}}$. Let X be the flag variety of G, i.e. $X = G/B$ where B is any particular Borel subgroup of G. For $\lambda \in H^2(X, \mathbb{C})$ we have, by taking cohomology,

$$\begin{gathered}\{K - \text{equivariant perverse sheaves on } X\}_\lambda \\ \updownarrow \\ \{G_{\mathbb{R}} - \text{equivariant perverse sheaves on } X\}_\lambda \\ \updownarrow \\ \{G_{\mathbb{R}} - \text{representations with infinitesimal character } \chi_\lambda\}\end{gathered}$$

This construction says nothing about unitarity, so far. For the purposes of classifying irreducible unitary representations it suffices to consider the situation where $\lambda \in H^2(X;\mathbb{R})$ is real. In that case we can canonically lift the irreducible representations and the standard representations to the the corresponding category of polarizable mixed Hodge modules $MHM_K(X)$.

Our work can be summarized as follows:

Theorem. *For $\lambda \in H^2(X, \mathbb{R})$ dominant we obtain a faithful exact functor*

$$MHM_K(X)_\lambda \to \{\textit{polarizable mixed Hodge structures}\}$$

This result implies the following

Corollary. *For $\lambda \in H^2(X, \mathbb{R})$ dominant the standard representations carry canonical polarizable mixed Hodge structures.*

Corollary. *For $\lambda \in H^2(X, \mathbb{R})$ dominant the irreducible representations carry a canonical polarized Hodge structure.*

In our arguments we make use of the exact faithful functor

$$gr : MHM_K(X)_\lambda \to Coh_{CM}^{\mathbb{C}^* \times K}(T^*X)$$

The right hand side stands for coherent sheaves on the cotangent bundle which are Cohen-Macauley and $\mathbb{C}^* \times K$-equivariant. We use this functor and an explicit b-function calculation to determine the Hodge structure of tempered representations. Other key ingredients include the identification of the V-filtration and the multiplier ideal filtration, a generalization of a result of Budur-Saito. This result allows us to characterize the V-filtration in terms of an L^2-condition. Finally, we determine the behavior of the Hodge structure under the intertwining operators. This last result gives an algorithm for determining the unitary representations as proposed by Vogan.

Equivariant perverse sheaves on affine Grassmanians

Victor Ginzburg

Let $\mathfrak{g}$ be a complex reductive Lie algebra with Cartan algebra $\mathfrak{t}$. Hotta and Kashiwara defined a holonomic $\mathscr{D}$-module $\mathcal{M}$, on $\mathfrak{g} \times \mathfrak{t}$, called Harish-Chandra module. We relate $\operatorname{gr} \mathcal{M}$, an associated graded module with respect to a canonical *Hodge filtration* on $\mathcal{M}$, to the *isospectral commuting variety*, a subvariety of $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{t} \times \mathfrak{t}$ which is a ramified cover of the variety of pairs of commuting elements of $\mathfrak{g}$. Our main result establishes an isomorphism of $\operatorname{gr} \mathcal{M}$ with the structure sheaf of $\mathfrak{X}_{\mathrm{norm}}$, the normalization of the isospectral commuting variety. We deduce, using Saito's theory of Hodge $\mathscr{D}$-modules, that the scheme $\mathfrak{X}_{\mathrm{norm}}$ is Cohen-Macaulay and Gorenstein. This confirms a conjecture of M. Haiman.

Associated with any *principal nilpotent pair* in $\mathfrak{g}$, there is a finite subscheme of $\mathfrak{X}_{\mathrm{norm}}$. The corresponding coordinate ring is a bigraded finite dimensional Gorenstein algebra that affords the regular representation of the Weyl group. The socle of that algebra is a 1-dimensional space generated by a remarkable W-harmonic polynomial on $\mathfrak{t} \times \mathfrak{t}$. In the special case where $\mathfrak{g} = \mathfrak{gl}_n$ the above algebras are closely related to the $n!$-theorem of Haiman and our W-harmonic polynomial reduces to the Garsia-Haiman polynomial. Furthermore, in the $\mathfrak{gl}_n$ case, the sheaf $\operatorname{gr} \mathcal{M}$ gives rise to a vector bundle on the Hilbert scheme of n points in $\mathbb{C}^2$ that turns out to be isomorphic to the *Procesi bundle*. Our results were used by I. Gordon to obtain a new proof of positivity of the Kostka-Macdonald polynomials established earlier by Haiman.

We will use a special notation $\boldsymbol{\mathcal{D}} := \mathscr{D}_{\mathfrak{g} \times \mathfrak{t}}$ for the sheaf of differential operators on $\mathfrak{g} \times \mathfrak{t}$. We have $\Gamma(\mathfrak{g} \times \mathfrak{t}, \boldsymbol{\mathcal{D}}) = \mathscr{D}(\mathfrak{g} \times \mathfrak{t}) = \mathscr{D}(\mathfrak{g}) \otimes \mathscr{D}(\mathfrak{t})$ where $\mathscr{D}(\mathfrak{g})$, resp. $\mathscr{D}(\mathfrak{t})$, is the algebra of polynomial differential operators on the vector space $\mathfrak{g}$, resp. $\mathfrak{t}$. The subalgebra of $\mathscr{D}(\mathfrak{g})$, resp. $\mathscr{D}(\mathfrak{t})$, formed by the differential operators with constant coefficients may be identified with $\operatorname{Sym} \mathfrak{g}$, resp. with $\operatorname{Sym} \mathfrak{t}$, the corresponding symmetric algebra.

Let the group G act on $\mathfrak{g}$ by the adjoint action. Harish-Chandra [HC] defined a 'radial part' map $\mathsf{rad} : \mathscr{D}(\mathfrak{g})^G \to \mathscr{D}(\mathfrak{t})^W$. This is an algebra homomorphism such that its restriction to the subalgebra of G-invariant polynomials, resp. G-invariant constant coefficient differential operators, reduces to the Chevalley isomorphism $\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{t}]^W$, resp. $(\operatorname{Sym} \mathfrak{g})^G \xrightarrow{\sim} (\operatorname{Sym} \mathfrak{t})^W$.

Given $a \in \mathfrak{g}$, one may view the map $\operatorname{ad} a : \mathfrak{g} \to \mathfrak{g}$, $x \mapsto [a, x]$ as a (linear) vector field on $\mathfrak{g}$, that is, as a first order differential operator on $\mathfrak{g}$. The assignment $a \mapsto \operatorname{ad} a$ gives a linear map $\operatorname{ad} : \mathfrak{g} \to \mathscr{D}(\mathfrak{g})$ with image $\operatorname{ad} \mathfrak{g}$. Thus, one can form a left ideal $\boldsymbol{\mathcal{D}}(\operatorname{ad} \mathfrak{g} \otimes 1) \subset \boldsymbol{\mathcal{D}}$.

Definition 0.1. The Harish-Chandra module is a left $\mathcal{D}$-module defined as follows

$$\mathcal{M} := \mathcal{D}\big/\big(\mathcal{D}\,(\operatorname{ad}\mathfrak{g}\otimes 1) + \mathcal{D}\,\{u\otimes 1 - 1\otimes \mathsf{rad}(u),\ u\in \mathscr{D}(\mathfrak{g})^G\}\big). \tag{0.2}$$

According to an important result of Hotta and Kashiwara [HK1], the Harish-Chandra module is a simple holonomic $\mathcal{D}$-module of '*geometric origin*'. This implies that $\mathcal{M}$ comes equipped with a natural structure of *Hodge module* in the sense of M. Saito [Sa]. In particular, there is a canonical Hodge filtration on $\mathcal{M}$. Taking an associated graded sheaf with respect to the Hodge filtration produces a coherent sheaf $\widetilde{\mathrm{gr}}^{\mathrm{Hodge}}\,\mathcal{M}$ on $T^*(\mathfrak{g}\times\mathfrak{t})$.

The support of the sheaf $\widetilde{\mathrm{gr}}^{\mathrm{Hodge}}\,\mathcal{M}$ turns out to be closely related to the commuting scheme of the Lie algebra $\mathfrak{g}$. Our main idea is to exploit the powerful theory of Hodge modules to deduce new results concerning commuting schemes using information about the sheaf $\widetilde{\mathrm{gr}}^{\mathrm{Hodge}}\,\mathcal{M}$.

Put $\mathfrak{G} = \mathfrak{g}\times\mathfrak{g}$ and let G act diagonally on $\mathfrak{G}$. The *commuting scheme* $\mathfrak{C}$ is defined as the *scheme-theoretic* zero fiber of the commutator map $\kappa: \mathfrak{G}\to\mathfrak{g}$, $(x,y)\mapsto[x,y]$. Thus, $\mathfrak{C}$ is a G-stable closed subscheme of $\mathfrak{G}$; set-theoretically, one has $\mathfrak{C} = \{(x,y)\in\mathfrak{G} \mid [x,y]=0\}$. The scheme $\mathfrak{C}$ is known to be generically reduced and irreducible. It is a long standing open problem whether or not this scheme is reduced.

Let $\mathfrak{T} := \mathfrak{t}\times\mathfrak{t}\subset\mathfrak{G}$. It is clear that $\mathfrak{T}$ is an $N(T)$-stable closed subscheme of $\mathfrak{C}$ and the resulting $N(T)$-action on $\mathfrak{T}$ factors through the diagonal action of the Weyl group $W = N(T)/T$. Therefore, restriction of polynomial functions gives algebra maps

$$\mathsf{res}:\ \mathbb{C}[\mathfrak{G}]^G \twoheadrightarrow \mathbb{C}[\mathfrak{C}]^G \to \mathbb{C}[\mathfrak{T}]^W. \tag{0.3}$$

The *isospectral commuting variety* is defined to be the algebraic set:

$$\mathfrak{X} = \{(x_1,x_2,t_1,t_2)\in\mathfrak{C}\times\mathfrak{T} \mid P(x_1,x_2) = (\mathsf{res}\,P)(t_1,t_2),\ \ \forall P\in\mathbb{C}[\mathfrak{C}]^G\}. \tag{0.4}$$

We view $\mathfrak{X}$ as a *reduced* closed subscheme of $\mathfrak{G}\times\mathfrak{T}$.

To proceed further, we fix an invariant bilinear form $\langle -,-\rangle:\ \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$. This gives an isomorphism $\mathfrak{g}\xrightarrow{\sim}\mathfrak{g}^*$, $x\mapsto\langle x,-\rangle$, resp. $\mathfrak{t}\xrightarrow{\sim}\mathfrak{t}^*$, $t\mapsto -\langle t,-\rangle$. Thus, one gets an identification $T^*(\mathfrak{g}\times\mathfrak{t}) = \mathfrak{G}\times\mathfrak{T}$ so one may view $\widetilde{\mathrm{gr}}^{\mathrm{Hodge}}\,\mathcal{M}$ as a coherent sheaf on $\mathfrak{G}\times\mathfrak{T}$.

Theorem 0.5. *There is a natural $\mathcal{O}_{\mathfrak{G}\times\mathfrak{T}}$-module isomorphism* $\psi_*\mathcal{O}_{\mathfrak{X}_{\mathrm{norm}}} \xrightarrow{\sim} \widetilde{\mathrm{gr}}^{\mathrm{Hodge}}\,\mathcal{M}$.

This theorem combined with some deep results of Saito [Sa] yields the following theorem that confirms a conjecture of M. Haiman, [Ha3, Conjecture 7.2.3].

Theorem 0.6. *$\mathfrak{X}_{\mathrm{norm}}$ is a Cohen-Macaulay and Gorenstein variety with trivial canonical sheaf.*

Corollary 0.7. *The scheme $\mathfrak{C}_{\mathrm{norm}}$ is Cohen-Macaulay.*

The first projection $\mathfrak{G}\times\mathfrak{T}\to\mathfrak{G}$ restricts to a map $p:\ \mathfrak{X}\to\mathfrak{C}$. Therefore, the composite $\mathfrak{X}_{\mathrm{norm}}\to\mathfrak{X}\to\mathfrak{C}$ factors through a morphism $p_{\mathrm{norm}}:\ \mathfrak{X}_{\mathrm{norm}}\to\mathfrak{C}_{\mathrm{norm}}$.

It is immediate to see that p_{norm} is a *finite* $G \times \mathbb{C}^\times \times \mathbb{C}^\times$-equivariant morphism and that the group W acts along the fibers of this morphism.

Let $\mathscr{R} := (p_{\mathrm{norm}})_* \mathcal{O}_{\mathfrak{X}_{\mathrm{norm}}}$. This is a $G \times \mathbb{C}^\times \times \mathbb{C}^\times$-equivariant coherent sheaf of $\mathcal{O}_{\mathfrak{C}_{\mathrm{norm}}}$-algebras. The action of the group W along the fibers of the map p_{norm} gives a W-action on $\mathscr{R}$ by $\mathcal{O}_{\mathfrak{C}_{\mathrm{norm}}}$-algebra automorphisms. Therefore, for any finite dimensional W-representation E, one has a coherent sheaf $\mathscr{R}^E := (E \otimes \mathscr{R})^W$, the E-isotypic component of $\mathscr{R}$. Equivalently, in terms of the contragredient representation E^*, we have $\mathscr{R}^E = \mathrm{Hom}_W(E^*, \mathscr{R})$.

The sheaf $\mathscr{R}$ enjoys the following properties.

Corollary 0.8. (i) *The sheaf $\mathscr{R}$ is Cohen-Macaulay and we have*

$$\mathcal{O}_{\mathfrak{C}_{\mathrm{norm}}} \cong \mathscr{R}^W, \quad \text{resp.} \quad \mathcal{K}_{\mathfrak{C}_{\mathrm{norm}}} \cong \mathscr{R}^{\,sign}.$$

(ii) *There is a $G \times W \times \mathbb{C}^\times \times \mathbb{C}^\times$-equivariant isomorphism $\mathscr{R} \cong \mathscr{H}om_{\mathcal{O}_{\mathfrak{C}_{\mathrm{norm}}}}(\mathscr{R}, \mathcal{K}_{\mathfrak{C}_{\mathrm{norm}}})$. Furthermore, for any finite dimensional W-module E, this gives an isomorphism*

$$\mathscr{R}^{E^* \otimes\, sign} \;\cong\; \mathscr{H}om_{\mathcal{O}_{\mathfrak{C}_{\mathrm{norm}}}}(\mathscr{R}^E, \mathcal{K}_{\mathfrak{C}_{\mathrm{norm}}}).$$

Given $x \in \mathfrak{g}$, let $\mathfrak{g}_x$ denote the centralizer of x in $\mathfrak{g}$. Similarly, write $\mathfrak{g}_{x,y} = \mathfrak{g}_x \cap \mathfrak{g}_y$ for the centralizer of a pair $(x, y) \in \mathfrak{C}$ in $\mathfrak{g}$. We call an element $x \in \mathfrak{g}$, resp. a pair $(x, y) \in \mathfrak{C}$, *regular* if we have $\dim \mathfrak{g}_x = \mathbf{r}$, resp. $\dim \mathfrak{g}_{x,y} = \mathbf{r}$. Let $\mathfrak{g}^r$, resp. $\mathfrak{C}^r$, be the set of regular elements of $\mathfrak{g}$, resp. of $\mathfrak{C}$. One shows that the set $\mathfrak{C}^r$ is a Zariski open and dense subset of $\mathfrak{C}$ which is equal to the smooth locus of the scheme $\mathfrak{C}$.

There is a coherent sheaf $\boldsymbol{g}$ on $\mathfrak{C}_{\mathrm{norm}}$, the "*universal stabilizer sheaf*', such that the geometric fiber of $\boldsymbol{g}$ at each point is the Lie algebra of the isotropy group of that point under the G-action. Any pair $(x, y) \in \mathfrak{C}^r$ may be viewed as a point of $\mathfrak{C}_{\mathrm{norm}}$. The sheaf $\boldsymbol{g}|_{\mathfrak{C}^r}$ is locally free; its fiber at any point $(x, y) \in \mathfrak{C}^r$ equals, by definition, the vector space $\mathfrak{g}_{x,y}$.

Part (i) of the following theorem says that the sheaf $\mathscr{R}$ gives an algebraic vector bundle on $\mathfrak{C}^r$ that has very interesting structures. Part (ii) of the theorem provides a description of the isotypic components $\mathscr{R}^{\wedge^s \mathfrak{t}} = (\wedge^s \mathfrak{t} \otimes \mathscr{R})^W$, coresponding to the wedge powers $\wedge^s \mathfrak{t}$, $s \geq 0$, of the reflection representation of W, in terms of the sheaf $\boldsymbol{g}$.

Theorem 0.9. (i) *The restriction of the sheaf $\mathscr{R}$ to $\mathfrak{C}^r$ is a locally free sheaf. Each fiber of the corresponding algebraic vector bundle is a finite dimensional algebra that affords the regular representation of the group W.*

(ii) *For any $s \geq 0$, there is a natural $G \times \mathbb{C}^\times \times \mathbb{C}^\times$-equivariant isomorphism $\mathscr{R}^{\wedge^s \mathfrak{t}}|_{\mathfrak{C}^r} \;\cong\; \wedge^s \boldsymbol{g}|_{\mathfrak{C}^r}$.*

Given a regular point $\mathbf{x} = (x_1, x_2) \in \mathfrak{C}$, let $\mathscr{R}_{\mathbf{x}}$ be the fiber at $\mathbf{x}$ of (the algebraic vector bundle on $\mathfrak{C}^r$ corresponding to) the locally free sheaf $\mathscr{R}$. By definition, one has $\mathscr{R}_{\mathbf{x}} = \mathbb{C}[p_{\mathrm{norm}}^{-1}(\mathbf{x})]$ where $p_{\mathrm{norm}}^{-1}(\mathbf{x})$, the scheme theoretic fiber of the morphism p_{norm} over $\mathbf{x}$, is a W-stable (not necessarily reduced) finite subscheme of $\mathfrak{X}_{\mathrm{norm}}$. Thus, $\mathscr{R}_{\mathbf{x}}$ is a finite dimensional algebra equipped with a W-action.

The W-module $\mathscr{R}_{\mathbf{x}}$ is isomorphic to the regular representation of W, by Theorem 0.9(i). In particular, one has $\dim \mathscr{R}_{\mathbf{x}}^W = \dim \mathscr{R}_{\mathbf{x}}^{sign} = 1$. The line $\mathscr{R}_{\mathbf{x}}^W$ is clearly spanned by the unit of the algebra $\mathscr{R}_{\mathbf{x}}$. Further, one has a canonical map $\mathscr{R}_{\mathbf{x}} \to \mathscr{R}_{\mathbf{x}}^{sign},\ r \mapsto r^{\,sign}$, the W-equivariant projection to the isotypic component of the sign representation. This map gives, thanks to the isomorphism $\mathcal{K}_{\mathfrak{C}_{\text{norm}}} \cong \mathscr{R}^{\,sign}$ of Corollary 0.8(i), a nondegenerate trace on the algebra $\mathscr{R}_{\mathbf{x}}$. In other words, the assignment $r_1 \times r_2 \mapsto (r_1 \cdot r_2)^{\,sign}$ provides a nondegenerate symmetric bilinear pairing on $\mathscr{R}_{\mathbf{x}}$.

The most interesting fibers of the sheaf $\mathscr{R}$ are, in a sense, the fibers over *principal nilpotent pairs*. Following [Gi], we call a regular pair $\mathbf{e} = (e_1, e_2) \in \mathfrak{C}^r$ a principal nilpotent pair if there exists a rational homorphism $\mathbb{C}^\times \times \mathbb{C}^\times \to G,\ (\tau_1, \tau_2) \mapsto g(\tau_1, \tau_2)$ such that one has

$$\tau_i \cdot e_i \;=\; \operatorname{Ad} g(\tau_1, \tau_2)(e_i) \quad i = 1, 2, \qquad \forall \tau_1, \tau_2 \in \mathbb{C}^\times. \tag{0.10}$$

Associated with the nilpotent pair $\mathbf{e} = (e_1, e_2)$ there is a pair of commuting semisimple elements of $\mathfrak{g}$ defined by $h_s := \frac{\partial g(\tau_1,\tau_2)}{\partial \tau_s}\big|_{\tau_1=\tau_2=1},\ s = 1, 2$. The pair $\mathbf{h} = (h_1, h_2)$ is regular, [Gi, Theorem 1.2]; furthermore, the fiber $p_{\text{norm}}^{-1}(\mathbf{h})$ is a reduced finite subscheme of $\mathfrak{G} \times \mathfrak{T}$. Specifically, writing $W\!\cdot\!\mathbf{h}$ for the W-orbit of the element $\mathbf{h} \in \mathfrak{T}$, one has a bijection $W\!\cdot\!\mathbf{h} \xrightarrow{\sim} p_{\text{norm}}^{-1}(\mathbf{h}),\ w(\mathbf{h}) \mapsto (\mathbf{h}, w(\mathbf{h}))$. Thus, the algebra $\mathscr{R}_{\mathbf{h}} = \mathbb{C}[p_{\text{norm}}^{-1}(\mathbf{h})]$ is a semisimple algebra isomorphic to $\mathbb{C}[W\!\cdot\!\mathbf{h}]$, the coordinate ring of the set $W\!\cdot\!\mathbf{h}$.

One of the central results of the paper is the following theorem motivated, in part, by [Ha3, §4.1]. Part (i) of the theorem describes how $\mathbb{C}[W\!\cdot\!\mathbf{h}] = \mathscr{R}_{\mathbf{h}}$, a semisimple Gorenstein algebra, degenerates to the bigraded Gorenstein algebra $\mathscr{R}_{\mathbf{e}}$.

Theorem 0.11. (i) *There is a W-equivariant $\mathbb{Z}^2$-graded algebra isomorphism* $\mathscr{R}_{\mathbf{e}} \;\cong\; \operatorname{gr}^F \mathbb{C}[W\!\cdot\!\mathbf{h}]$.

(ii) *We have* $\mathscr{R}_{\mathbf{e}}^{i,j} = 0$ *unless* $0 \le i \le \mathbf{d}_1 \;\&\; 0 \le j \le \mathbf{d}_2$, *where* $\mathbf{d}_s := \#R_s^+,\ s = 1, 2$.

From the isomorphism $\mathscr{R}_{\mathbf{e}} = \operatorname{gr}^F \mathbb{C}[W\!\cdot\!\mathbf{h}]$, of Theorem 0.11, we see that $\operatorname{gr}^F \mathbb{C}[W\!\cdot\!\mathbf{h}]$ is a Gorenstein algebra and that the line $(\operatorname{gr}^F \mathbb{C}[W\!\cdot\!\mathbf{h}])^{\,sign}$ is the socle of that algebra. In most cases, one can actually obtain a more explicit description of the socle.

The following result provides a simple description of the socle of the algebra $\operatorname{gr}^F \mathbb{C}[W\!\cdot\!\mathbf{h}]$ in the case of nonexceptional nilpotent pairs.

Theorem 0.12. *For any non-exceptional principal nilpotent pair* $\mathbf{e}$, *we have*

$$(\operatorname{gr}^F \mathbb{C}[W\!\cdot\!\mathbf{h}])^{\,sign} \;=\; \operatorname{gr}_{\mathbf{d}_1,\mathbf{d}_2} \mathbb{C}[W\!\cdot\!\mathbf{h}].$$

Morover, a base vector of the 1-dimensional vector space on the right is provided by the image under the map $\operatorname{gr}^F \mathbb{C}[\mathfrak{T}] \to \operatorname{gr}^F \mathbb{C}[W\!\cdot\!\mathbf{h}]$ *of the class of the following bihomogeneous polynomial:*

$$\Delta_{\mathbf{e}} := \sum\nolimits_{w \in W} \; sign(w) \cdot w(\check{h}_1^{\mathbf{d}_1} \otimes \check{h}_2^{\mathbf{d}_2}) \;\in\; \mathbb{C}[\mathfrak{T}]^{\,sign}. \tag{0.13}$$

One may pull-back the function $\Delta_{\mathbf{e}}$ via the projection $\mathfrak{X}_{\mathrm{norm}} \to \mathfrak{T}$. The resulting W-alternating function on $\mathfrak{X}_{\mathrm{norm}}$ gives a G-invariant section, $\mathbf{s}_{\mathbf{e}}$, of the sheaf $\mathscr{R}^{sign}$. Let $\mathbf{s}_{\mathbf{e}}(\mathbf{e}) \in \mathscr{R}_{\mathbf{e}}^{sign}$ denote the value of the section $\mathbf{s}_{\mathbf{e}}$ at the point $\mathbf{e}$.

Corollary 0.14. *For a non-exceptional principal nilpotent pair* $\mathbf{e}$*, one has:* $\Delta_{\mathbf{e}}(\mathbf{h}) \neq 0$ *and* $\mathbf{s}_{\mathbf{e}}(\mathbf{e}) \neq 0$. *Moreover, we have that* $\mathscr{R}_{\mathbf{e}}^{sign} = \mathscr{R}_{\mathbf{e}}^{\mathbf{d}_1,\mathbf{d}_2} = \mathbb{C} \cdot \mathbf{s}_{\mathbf{e}}(\mathbf{e})$ *is the socle of the algebra* $\mathscr{R}_{\mathbf{e}}$.

Let $\mathrm{Hilb}^n(\mathbb{C}^2)$ be the Hilbert scheme of n points in $\mathbb{C}^2$. In his work on the $n!$-theorem, Haiman introduced a certain *isospectral Hilbert scheme* $\widetilde{\mathrm{Hilb}}^n(\mathbb{C}^2)$, a reduced finite scheme over $\mathrm{Hilb}^n(\mathbb{C}^2)$, see [Ha1]. The main result of *loc cit* says that $\widetilde{\mathrm{Hilb}}^n(\mathbb{C}^2)$ is a normal, Cohen-Macaulay and Gorenstein scheme.

Now, let $G = GL_n$. It turns out that there is a Zariski open dense subset $\mathfrak{C}^\circ \subset \mathfrak{C}^r$ such that the projection $p_{\mathrm{norm}} : p_{\mathrm{norm}}^{-1}(\mathfrak{C}^\circ) \to \mathfrak{C}^\circ$ is closely related to the projection $\widetilde{\mathrm{Hilb}}^n(\mathbb{C}^2) \to \mathrm{Hilb}^n(\mathbb{C}^2)$. Using this relation, we are able to deduce from our Theorem 0.6 that the *normalization* of the isospectral Hilbert scheme is Cohen-Macaulay and Gorenstein.

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Degenerate flag varieties

Evgeny Feigin

Let $\mathfrak{g}$ be a simple Lie algebra with the Cartan decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$ be the Borel subalgebra. Let $\alpha_1, \dots, \alpha_l$ be the set of simple roots and $\omega_1, \dots, \omega_l$ be the fundamental weights. We have $(\omega_i, \alpha_j) = \delta_{i,j}$, where $(\cdot, \cdot)$ is the Killing form on $\mathfrak{h}^*$. A dominant integral weight λ is given by $\sum_{i=1}^l m_i \omega_i$, $m_i \in \mathbb{Z}_{\geq 0}$. For a dominant integral λ let V_λ be the finite-dimensional irreducible highest weight $\mathfrak{g}$-module with highest weight λ and a highest weight vector v_λ such that $\mathfrak{n} v_\lambda = 0$, $h v_\lambda = \lambda(h) v_\alpha$ $(h \in \mathfrak{h})$ and $V_\lambda = \mathrm{U}(\mathfrak{n}^-) v_\lambda$. We denote by G and N^- the Lie groups of the Lie algebras $\mathfrak{g}$ and $\mathfrak{n}^-$.

The (generalized) flag varieties for G are defined as quotient G/P by the parabolic subgroups. These varieties play crucial role in the geometric representation theory. An important feature of the flag varieties is that they can be naturally embedded into the projectivization of the highest weight modules. Namely, let λ be a dominant weight such that the stabilizer of the line $[v_\lambda]$ in G is equal to P. Here and below for a vector v in a vector space V we denote by $[v] \in \mathbb{P}(V)$ the line spanned by v. Then one gets the embedding $G/P \subset \mathbb{P}(V_\lambda)$ as the G-orbit of the highest weight line $[v_\lambda]$. For a dominant weight λ we denote by $F_\lambda \subset \mathbb{P}(V_\lambda)$ the orbit $G[v_\lambda]$ of the highest weight line. These are smooth projective algebraic varieties. It is clear that F_λ depends only on the class of regularity of λ, i.e. $F_\lambda \simeq F_\mu$ if and only if for all i $(\lambda, \omega_i) = 0$ iff $(\mu, \omega_i) = 0$.

Now let $\mathfrak{g}^a$ be the degenerate Lie algebra defined as a direct sum $\mathfrak{b} \oplus \mathfrak{g}/\mathfrak{b}$ of the Borel subalgebra $\mathfrak{b}$ and abelian ideal $\mathfrak{g}/\mathfrak{b}$ (see [2],[3]). The algebra $\mathfrak{b}$ acts on $\mathfrak{g}/\mathfrak{b}$ via the adjoint action. We denote the space $\mathfrak{g}/\mathfrak{b}$ by $(\mathfrak{n}^-)^a$ (a is for abelian). Let $(N^-)^a = \exp(\mathfrak{n}^-)^a$ be the abelian Lie group, which is nothing but the product of $\dim \mathfrak{n}^-$ copies of the group $\mathbb{G}_a$ – the additive group of the field. Let G^a be the semidirect product $B \ltimes (N^-)^a$ of the subgroup B and of the normal abelian subgroup $(N^-)^a$ (the action of B on $(N^-)^a$ is induced by the action of B on $\mathfrak{n}^-$ by conjugation).

In [6], [7] the $\mathfrak{g}^a$-modules V_λ^a were introduced and studied in types A and C. The module V_λ^a is the associated graded module with respect to the PBW filtration on V_λ. For a given dominant integral weight λ the group G^a acts naturally on $\mathbb{P}(V_\lambda^a)$. By definition, the degenerate flag variety F_λ^a is the orbit closure of the highest weight line, i.e.

$$F_\lambda^a = \overline{G^a[v_\lambda]} \subset \mathbb{P}(V_\lambda^a).$$

Note that $F_\lambda^a = \overline{(N^-)^a \cdot [v_\lambda]}$, i.e. the group acts on F_λ^a with an open orbit isomorphic to an affine space. The varieties F_λ^a are hence $\mathbb{G}_a^{\dim \mathfrak{n}}$-varieties. The variety F_λ^a is not a homogeneous G^a-variety in contrast to the classical situation.

We now state several theorems describing geometric properties of flag varieties in type A (the case of $\mathfrak{g} = sp_{2n}$ is worked out in [5]).

Theorem 0.1. *The varieties F_λ^a depend only on the class of regularity of λ.*

For simplicity, in what follows we only consider the case of complete degenerate flag varieties, i.e. λ is a dominant regular weight. We denote the corresponding degenerate flag variety by F_n^a. Let W be an n-dimensional vector space with a basis $w_1, \dots, w_n$. We define the projection operators $pr_k : W \to W$, $k = 1, \dots, n$ by the formula $pr_k(\sum_{i=1}^n c_i w_i) = \sum_{i \neq k} c_i w_i$. The following theorem can be found in [3].

Theorem 0.2. *F_n^a is isomorphic to the variety of collections $(V_i)_{i=1}^k$ of subspaces $V_i \subset W$ satisfying for $i = 1, \dots, n-2$*

$$\dim V_i = i,\ pr_{i+1} V_i \subset V_{i+1}.$$

For any dominant weight μ there exits a map $F_n^a \to \mathbb{P}(V_\mu^a)$. We denote the inverse image of $O(1)$ by $\mathcal{L}_\mu$. The following theorem is proved in [4].

Theorem 0.3.
(1) *The varieties F_n^a are normal locally complete intersections (thus Cohen-Macaulay and Gorenstein).*
(2) *The varieties F_n^a admit crepant resolution of singularities, which is a Bott tower (i.e. can be constructed as a tower of successive $\mathbb{P}^1$ fibrations).*
(3) *The varieties F_n^a have rational singularities and are Frobenius split.*
(4) *For any dominant μ the cohomology groups $H^m(F_n^a, \mathcal{L}_\mu)$ vanish unless $m = 0$, and the zero cohomology is isomorphic to $(V_\mu^a)^*$.*

Finally, we note that the varieties F_n^a can be realized as certain quiver Grassmannians for equioriented type A quivers (see [1]).

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The arc space of horospherical varieties and motivic integration

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(joint work with Victor Batyrev)

Let G be a connected reductive group and $H \subseteq G$ a closed subgroup. The homogeneous space G/H is called *horospherical* if H contains a maximal unipotent subgroup $U \subseteq G$. In this case, the normalizer $N_G(H)$ is a parabolic subgroup $P \subseteq G$ and P/H is an algebraic torus T. The horospherical homogeneous space G/H can be described as an affine torus bundle with the fiber T over the projective homogeneous space G/P. The dimension r of the torus T is called the *rank* of the horospherical homogeneous space G/H. Let M be the lattice of characters of the torus T, and $N = \mathrm{Hom}(M, \mathbb{Z})$ the dual lattice. According to the Luna-Vust theory [LV83], any G-equivariant embedding $G/H \hookrightarrow X$ of a horospherical homogeneous space G/H can be described combinatorially by a colored fan Σ in the r-dimensional vector space $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. In the case $H = U$, G-equivariant embeddings of G/U have been considered independently by Pauer [Pau81, Pau83]. Equivariant embeddings of horospherical homogeneous spaces are generalizations of the well-known toric varieties which are torus embeddings $T \hookrightarrow X$ ($G = T$, $H = \{e\}$).

Our work is motivated by some known formulas for stringy invariants of toric varieties. Let X be a $\mathbb{Q}$-Gorenstein toric variety defined by a fan $\Sigma \subset N_{\mathbb{R}}$ and denote by $|\Sigma| \subset N_{\mathbb{R}}$ its support. Then there is a piecewise linear function $\omega_X : |\Sigma| \to \mathbb{R}$ such that its restriction to every cone $\sigma \in \Sigma$ is linear and ω_X has value -1 on all primitive lattice generators of 1-dimensional faces of σ. It was shown in [Ba98] that the stringy E-function of the toric variety X can be computed by the formula

$$E_{\mathrm{st}}(X; u, v) := \big(uv - 1\big)^r \sum_{n \in |\Sigma| \cap N} (uv)^{\omega_X(n)}. \tag{0.1}$$

We prove a similar to (0.1) formula for any $\mathbb{Q}$-Gorenstein horospherical variety X defined by a colored fan Σ:

$$E_{\mathrm{st}}(X; u, v) := E(G/H; u, v) \sum_{n \in |\Sigma| \cap N} (uv)^{\omega_X(n)}, \tag{0.2}$$

where $\omega_X : |\Sigma| \to \mathbb{R}$ is a certain Σ-piecewise linear function.

In contrast to toric varieties, the stringy E-function of a locally factorial horospherical variety X needs not to be a polynomial. If X is smooth, then $E_{\mathrm{st}}(X; u, v) = E(X; u, v)$ is polynomial and in particular the *stringy Euler number*, $e_{\mathrm{st}}(X) := E_{\mathrm{st}}(X; 1, 1)$, is equal to the usual Euler number $e(X) := E(X; 1, 1)$. If X is a locally factorial horospherical variety whose minimal orbits are projective, then we show that $e_{\rm st}(X) \geq e(X)$ and that the equality holds if and only if X is smooth. We conjecture that the equality

$$e_{\rm st}(X) = e(X)$$

can be used as a smoothness criterion for arbitrary locally factorial spherical varieties.

The key idea behind the formula (0.2) for toric varieties is the isomorphism

$$T(\mathcal{K})/T(\mathcal{O}) \simeq N,$$

where $\mathcal{O} := \mathbb{C}[[t]]$, $\mathcal{K} := \mathbb{C}((t))$ and $T(\mathcal{O})$ (resp. $T(\mathcal{K})$) denotes the set of $\mathcal{O}$-valued (resp. $\mathcal{K}$-valued) points in T. We remark that the *stringy motivic integral* over the arc space $X(\mathcal{O})$ of a toric variety X is equal to its restriction to the arc space $T(\mathcal{K})$. The latter contains countably many orbits of the maximal compact subgroup $T(\mathcal{O}) \subset T(\mathcal{K})$ that are parametrized by elements n of the lattice N. The stringy motivic integral over a $T(\mathcal{O})$-orbit corresponding to an element $n \in N$ is equal to $(\mathbb{L}-1)^r \mathbb{L}^{\omega_X(n)}$ where $(\mathbb{L}-1)^r$ is the stringy motivic volume of the torus T and $\mathbb{L}$ is the class of the affine line in the Grothendieck ring $\mathrm{K}_0(\mathrm{Var}_{\mathbb{C}})$ of algebraic varieties. Our approach in the proof of the formula (0.2) is to use a more general bijection

$$G(\mathcal{O}) \setminus (G/H)(\mathcal{K}) \simeq N$$

which holds for any horospherical homogeneous space G/H, see [LV83] and [GN10].

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Weyl modules for twisted loop algebras and beyond

Ghislain Fourier

(joint work with Nathan Manning, Prasad Senesi)

Let $\mathfrak{g}$ be a simple complex Lie algebra and $\mathcal{L}(\mathfrak{g}) := \mathfrak{g} \otimes \mathbb{C}[t^{\pm}]$ be the associated loop algebra with the bracket given by

$$[x \otimes t^n, y \otimes t^m] := [x, y] \otimes t^{n+m}.$$

In this talk we want to relate the representation theory of $\mathcal{L}(\mathfrak{g})$ to the representation theory of the twisted loop algebra $\mathcal{L}^\sigma(\mathfrak{g})$ defined as follows:
Let σ be Dynkin diagram automorphism of $\mathfrak{g}$ of order m, then σ acts on $\mathbb{C}[t^{\pm}]$ by

$\sigma(t^n) = \zeta_m^{-n} t^n$, where ζ is a m-th root of unity. This induces an automorphism of $\mathcal{L}(\mathfrak{g})$ and we denote by $\mathcal{L}^\sigma(\mathfrak{g})$ the set of fix points. In detail

$$\mathcal{L}^\sigma(\mathfrak{g}) = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_{\bar{n}} \otimes t^n$$

where $\bar{n}$ denotes the residue of n devided by m and $\mathfrak{g}_\epsilon$ the eigenvectors of eigenvalue ζ_m^ϵ.
This algebra appears naturally in the context of twisted affine Kac-Moody algebras.
Denote $\mathcal{F}$ the category of finite–dimensional $\mathcal{L}(\mathfrak{g})$-modules and $\mathcal{F}^\sigma$ the category of finite–dimensional $\mathcal{L}^\sigma(\mathfrak{g})$-modules. Since $\mathcal{L}^\sigma(\mathfrak{g}) \subset \mathcal{L}(\mathfrak{g})$, we have a natural restriction functor

$$\mathcal{R} : \mathcal{F} \longrightarrow \mathcal{F}^\sigma.$$

In this talk I want to explain how to construct a left inverse to this functor. In fact, it turns out that one has to decompose $\mathcal{F}^\sigma$ into a sum of smaller categories. For each of these categories one can define such a functor. Furthermore, I want to explain the implications to global Weyl modules.
For $\lambda \in P^+$, the dominant integral weights for $\mathfrak{g}$, denote by $L(\lambda)$ the corresponding simple finite–dimensional highest weight module. For each $a \in \mathbb{C}^*$, we have an evaluation map

$$ev_a : \mathcal{L}(\mathfrak{g}) \longrightarrow \mathfrak{g} \, : \, x \otimes p(t) \mapsto p(a)x.$$

By pullback we obtain for each $L(\lambda)$ and a a simple $\mathcal{L}(\mathfrak{g})$-module

$$L(\lambda)_a := ev_a^*(L(\lambda)).$$

Even more, for $\lambda_1, \ldots, \lambda_K$ and $a_1, \ldots a_K$ pairwise distinct non–zero complex numbers

$$L(\lambda_1)_{a_1} \otimes \ldots \otimes L(\lambda_K)_{a_K}$$

is a simple $\mathcal{L}(\mathfrak{g})$-module.

Theorem 0.1. [4] *Every finite–dimensional $\mathcal{L}(\mathfrak{g})$-module is such a tensor product of evaluation modules*

To each such a module we can associate a function

$$f : \mathbb{C}^* \longrightarrow P^+ \; : \; f(a_i) := \lambda_i \; ; \; f(b) = 0 \text{ else }.$$

This function is finitely supported. Denote by $\mathcal{E}$ the set of finitely supported functions $\mathbb{C}^* \longrightarrow P^+$, and to each such a function f we attach

$$L(f) := \bigotimes_{a \in \mathbb{C}^*} L(f(a))_a.$$

This construction gives the parametrization of the simple modules by the set $\mathcal{E}$.
Let σ be the automorphism of $\mathcal{L}^\sigma(\mathfrak{g})$, then σ acts on $\mathbb{C}^*$ by multiplication by ζ_m and on P^+ by the induced Dynkin diagram automorphism. Let $\mathcal{E}^\sigma$ be the set of equivariant finitely supported functions, in detail $\sigma(f(a)) = f(\sigma(a))$ for all $a \in \mathbb{C}^*$.

Theorem 0.2. [8] *$\mathcal{E}^\sigma$ parametrizes the simple finite–dimensional modules of $\mathcal{L}^\sigma(\mathfrak{g})$, for $f \in \mathcal{E}^\sigma$ denote by $L^\sigma(f)$ the corresponding simple module.*

Denote by $\mathrm{supp}(f) := \{a \in \mathbb{C}^* \mid f(a) \neq 0\}$. In the following X will always denote a finite subset of $\mathbb{C}^*$, such that $\langle\sigma\rangle.a \cap \langle\sigma\rangle.b = \emptyset$ for all $a \neq b \in X$.
Let f be equivariant and $X \subset \mathrm{supp}(f)$ such that $\langle\sigma\rangle.X = \mathrm{supp}(f)$, then

$$\mathcal{R}(V(f_{|X})) \cong L^\sigma(f).$$

Even more:

Theorem 0.3. [2] *Suppose M is a finite–dimensional $\mathcal{L}^\sigma(\mathfrak{g})$ module, then there exists an ideal $I_M^\sigma \subset \mathcal{L}^\sigma(\mathfrak{g})$ of finite codimension and an ideal $I_M \in \mathcal{L}(\mathfrak{g})$ of finite codimension such that*

- *$I_M^\sigma.M = 0$,*
- *$\mathcal{L}^\sigma(\mathfrak{g})/I_M^\sigma \cong \mathcal{L}(\mathfrak{g})/I_M$ as Lie algebras.*

The isomorphism depends on the choice of representatives, so depends on choosing a subset $X \subset \mathrm{supp}(M)$ (where the support of M is the union of the supports of all simple subquotients of M). This gives for each X a functor $\mathcal{U}_X$ from the subcategory $\mathcal{F}_X^\sigma$ of $\mathcal{L}^\sigma(\mathfrak{g})$-modules supported on $\langle\sigma\rangle.X$ to the category $\mathcal{F}_X$ of $\mathcal{L}(\mathfrak{g})$-modules supported on X.

Theorem 0.4. [5]

$$\mathcal{R}_X \circ \mathcal{U}_X = id_{\mathcal{F}_X^\sigma} \text{ and } \mathcal{U}_X \circ \mathcal{R}_X = id_{\mathcal{F}_X}.$$

The Global Weyl module $W(\lambda)$ is defined ([1]) to be projective object in the category of integrable $\mathcal{L}(\mathfrak{g})$-modules with weights bounded above by λ. It has a weight space decomposition with respect to $\mathfrak{h} \otimes 1$, and the highest weight space of weight λ is therefore a quotient of $\mathbf{U}(\mathfrak{h} \otimes \mathbb{C}[t^\pm])$. Denote this highest weight space by $\mathbf{A}_\lambda$, then:

Theorem 0.5. [4] *For $\lambda \in P^+$*

- *$\mathbf{A}_\lambda$ is a polynomial ring in finitely many variables.*
- *$W(\lambda)$ is a right module for $\mathbf{A}_\lambda$.*

In was shown in a series of papers ([4], [3], [6], [6], [9]) that $W(\lambda)$ is a free $\mathbf{A}_\lambda$-module of finite rank.
$\mathfrak{g}_0$ is a simple Lie algebra, denote the dominant integral weights by P_0^+, we have a canonical map $\pi : P^+ \longrightarrow P_0^+$. Then by the same construction one obtains the twisted global Weyl module and one can show

Theorem 0.6. [7] *For $\mu \in P_0^+$*

- *$\mathbf{A}_\mu^\sigma$ is polynomial ring.*
- *$W^\sigma(\mu)$ is a free $\mathbf{A}_\mu^\sigma$-module of finite rank.*
- *The rank of $W^\sigma(\mu)$ as a $\mathbf{A}_\mu^\sigma$-module is the same as the rank of $W(\lambda)$ as a $\mathbf{A}_\lambda$-module for any $\lambda \in \pi^{-1}(\mu)$.*
- *$W^\sigma(\mu) \subset \bigoplus_{\lambda \in \pi^{-1}(\mu)} W(\lambda)$, where the generator of the LHS is mapped to the sum of the generators on the RHS.*

This generalizes the finite–dimensional pictures, there is no inverse functor for twisted global Weyl modules, but one can realize them in a direct sum of untwisted global Weyl modules.

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Prime representations from a homological perspective

Vyjayanthi Chari

(joint work with Adriano Moura, Charles Young)

The study of finite-dimensional representations of quantum affine algebras has been an active field of research for at least two decades. The abstract classification of the simple representations was given in [3], [4], and much of the subsequent work has focused on understanding the structure of these representations. This has proved to be a difficult task and a complete understanding outside the case of $\mathfrak{sl}_2$ is still some distance away. A number of important methods have been developed: for instance, the work of [5], [6] on q–characters has resulted in a deeper combinatorial understanding of these representations. The geometric approach of H. Nakajima and the theory of crystal bases of M. Kashiwara have also been very fruitful. Another powerful tool is the T–system [7, 11, 13], which was recently shown [12] to extend beyond Kirillov-Reshetikhin modules to wider classes of representations. A connection with the theory of cluster algebras has been established recently in [9, 14].

The study of the structure of the irreducible representations can be reduced to the so called prime ones, namely those simple representations which cannot be written as a tensor product of two non-trivial simple representations. Clearly any finite–dimensional simple representation can be written as a tensor product of simple prime representations and one could then focus on understanding the prime representations. This was the approach used in [3] for the $\mathfrak{sl}_2$-case, but generalizing this approach is very difficult. Many examples of prime representations are known in general, for example the Kirillov–Reshethikhin modules are prime and, more generally, the minimal affinizations are also prime and other examples may be found for instance in [9]. However, except in the $\mathfrak{sl}_2$-case where the simple prime representations are precisely the Kirillov-Reshetikhin modules (which are also the evaluation modules), the classification of the prime representations is not known.

Our work is motivated by an effort to understand the simple prime representations via homological properties. Thus, let $\hat{\mathcal{F}}$ be the category of finite-dimensional representations of the quantum affine algebra and denote by $V(\boldsymbol{\pi})$ the irreducible representation associated to the Drinfeld polynomial $\boldsymbol{\pi}$. We construct in a natural way a non-trivial self-extension of any object V of $\hat{\mathcal{F}}$ which motivates the natural question of characterizing the simple objects which satisfy

$$\dim \operatorname{Ext}^1_{\hat{\mathcal{F}}}(V, V) = 1. \tag{0.1}$$

Our first result shows that any simple V satisfying (0.1) is of the form $V(\boldsymbol{\pi}_0^s)$ for some $s \geq 1$ where $\boldsymbol{\pi}_0$ is such that $V(\boldsymbol{\pi}_0)$ is prime. Hence, if $V(\boldsymbol{\pi}_0)$ is a real prime in the sense of [9], then using [8] we see that V is a tensor power of $V(\boldsymbol{\pi}_0)$.

In the case of $\mathfrak{sl}_2$ we prove the stronger result that a simple object V satisfies (0.1) if and only if V is prime. It is natural and now obviously interesting to ask if such a result remains true for general $\mathfrak{g}$. Our next result provides partial evidence for this to be true. Namely, we prove for a large family of simple prime representations including the minimal affinizations that the space of self-extensions is one-dimensional. Our results go beyond minimal affinizations and we prove that the representations $S(\beta)$ defined in [9] have a one–dimensional space of extensions as long as β is a positive root in which every simple root occurs with multiplicity one. It is worth comparing our results in the quantum case with their non-quantum counter parts. One can define in a similar way the notion of prime representations for the category of finite-dimensional representations of an affine Lie algebra $\hat{\mathfrak{g}}$. It is known through the work of [1], [10] that if V, V' are irreducible finite-dimensional representations of $\hat{\mathfrak{g}}$, then

$$\operatorname{Ext}^1_{\hat{\mathfrak{g}}}(V, V') \cong \operatorname{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V, V').$$

It is now easily seen that there exist examples of simple prime representations V of the affine Lie algebra such that $\operatorname{Ext}^1_{\hat{\mathfrak{g}}}(V, V)$ has dimension at least two. We give an example of a simple representation of the quantum affine algebra which has a one–dimensional space of self–extensions but whose classical limit, although also prime and simple, has a two dimensional space of self–extensions.

We identify a certain class of simple modules satisfy (0.1). Our results show that this implies that these modules are prime. The latter fact that the modules are prime can also be proved by other methods as well, see [9] for the modules of type $S(\beta)$ and [2] for remarks on minimal affinizations. Our goal here is really to provide evidence towards the conjecture for general $\mathfrak{g}$, that V satisfies (0.1) iff V is prime.

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One-parameter contractions of Lie-Poisson brackets

Oksana Yakimova

Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{A}^n = \mathbb{A}^n_{\mathbb{K}}$ the n-dimensional affine space with the algebra of regular functions $\mathcal{A} = \mathbb{K}[x_1, \dots, x_n]$. Set $\Omega = \Lambda^{\bullet}_{\mathcal{A}}(dx_i)$ and $W = \Lambda^{\bullet}_{\mathcal{A}}(\partial_{x_i})$, both are graded skew-symmetric algebras and free $\mathcal{A}$-modules with bases consisting of skew-monomials in dx_i and $\partial_i = \partial_{x_i}$, respectively. In particular, $\Omega^0 = W^0 = \mathcal{A}$ and $\Omega^1 = \langle dx_i \mid 1 \leqslant i \leqslant n\rangle_{\mathcal{A}}$, $W^1 = \langle \partial_i \mid 1 \leqslant i \leqslant n\rangle_{\mathcal{A}}$. Extending a non-degenerate $\mathcal{A}$-pairing $dx_i(\partial_j) = \delta_{ij}$, we view $\Omega^k = \Lambda^k_{\mathcal{A}}\Omega^1$ and $W^k := \Lambda^k_{\mathcal{A}} W^1$ as dual $\mathcal{A}$-modules.

Let $\omega = dx_1 \wedge \ldots \wedge dx_n$ be the volume form. There is an $\mathcal{A}$-linear map

$$\frac{1}{\omega} : \Omega^k \to (\Omega^{n-k})^* \cong W^{n-k}$$

such that for $f \in \Omega^k$ and $g \in \Omega^{n-k}$, $(f/\omega)(g) = a$, where $f \wedge g = a\omega$ (with $a \in \mathcal{A}$).

Suppose that $\mathcal{A}$ possesses a Poisson structure $\{\ ,\ \}\colon\ \mathcal{A}\times\mathcal{A} \to \mathcal{A}$ and let π denote the corresponding *Poisson tensor*, an element of $\mathrm{Hom}_{\mathcal{A}}(\Omega^2, \mathcal{A}) \cong W^2$ satisfying $\pi(df \wedge dg) = \{f, g\}$ for all $f, g \in \mathcal{A}$. For $\xi \in \mathbb{A}^n$, π_ξ is a skew-symmetric matrix with entries $\{x_i, x_j\}(\xi)$. The *index* of the Poisson algebra $\mathcal{A}$, denoted $\mathrm{ind}\,\mathcal{A}$, is defined as

$$\operatorname{ind}\mathcal{A} := n - \operatorname{rk}\pi, \text{ where } \operatorname{rk}\pi = \max_{\xi\in\mathbb{A}^n} \operatorname{rk}\pi_\xi.$$

Set $\operatorname{Sing}\pi := \{\xi \in \mathbb{A}^n \mid \operatorname{rk}\pi_\xi < \operatorname{rk}\pi\}$. Given $k \in \mathbb{N}$, we let

$$\Lambda^k\pi := \underbrace{\pi\wedge\pi\wedge\ldots\wedge\pi}_{k \text{ factors}},$$

be an element of W^{2k}. Note that $\Lambda^k\pi = 0$ for $k > (\operatorname{rk}\pi)/2$ and $\Lambda^k\pi \neq 0$ for $k \leqslant (\operatorname{rk}\pi)/2$. Besides, $(\Lambda^{(\operatorname{rk}\pi)/2}\pi)_\xi = 0$ if and only if $\xi \in \operatorname{Sing}\pi$.

An element $a \in \mathcal{A}$ is *central*, if $\{a, \mathcal{A}\} = 0$, the set $Z\mathcal{A} = Z(\mathcal{A}, \pi)$ of all central elements is called the *Poisson centre* of $\mathcal{A}$. As is well-known, $\operatorname{tr.deg}_{\mathbb{K}} Z\mathcal{A} \leqslant \operatorname{ind}\mathcal{A}$.

Example 1. Let $\mathfrak{q}$ be a finite-dimensional Lie algebra over $\mathbb{K}$ with a basis $x_1, \ldots, x_n$. Then the Poisson tensor on $\mathfrak{q}^*$ is given by $\pi = \sum_{i<j}[x_i, x_j]\partial_i \wedge \partial_j$. Here $\mathcal{A} = \mathcal{S}(\mathfrak{q}) = \mathbb{K}[\mathfrak{q}^*]$, and $Z\mathcal{S}(\mathfrak{q}) = \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$. Note also that $\operatorname{ind}\mathcal{S}(\mathfrak{q}) = \operatorname{ind}\mathfrak{q} = \min_{\gamma\in\mathfrak{q}^*} \dim\mathfrak{q}_\gamma$.

For $g_1, \ldots, g_m \in \mathcal{A}$, the *Jacobian locus* $\mathcal{J}(g_1, \ldots, g_m)$ consists of all $\xi \in \mathbb{A}^n$ such that the differentials $d_\xi g_1, \ldots, d_\xi g_m$ are linearly dependent. In other words, $\xi \in \mathcal{J}(g_1, \ldots, g_m)$ if and only if $(dg_1 \wedge \ldots \wedge dg_m)_\xi = 0$. The set $\mathcal{J}(g_1, \ldots, g_m)$ is Zariski closed in $\mathbb{A}^n$ and it coincides with $\mathbb{A}^n$ if and only if $g_1, \ldots, g_m$ are algebraically dependent.

The following statement can be extracted from the proofs of [6, Theorem 3.1], [8, Theorem 1.2], [7, Theorem 1.2].

Lemma 1. *Let $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]$ be a Poisson algebra of index ℓ and let $\{F_1, \ldots, F_\ell\} \subset Z\mathcal{A}$ be a set of algebraically independent polynomials. Then there are coprime $q_1, q_2 \in \mathcal{A} \setminus \{0\}$ such that*

$$q_1 \frac{dF_1 \wedge \ldots \wedge dF_\ell}{\omega} = q_2 \Lambda^{(n-\ell)/2}\pi\,.$$

Of particular interest are situations where $q_1, q_2 \in \mathbb{K}$ for q_1, q_2 as above. This can be guarantied by "codim-2" conditions, see e.g. [8, Theorem 1.2]. If $\dim\operatorname{Sing}\pi \leqslant n-2$, then q_1 must be a scalar. If $\dim\mathcal{J}(F_1, \ldots, F_\ell) \leqslant n-2$, then q_2 must be a scalar.

Definition 1. We will say that $F_1, \ldots, F_\ell$ satisfy the *Kostant equality*, if $(dF_1 \wedge \ldots \wedge F_\ell)/\omega = \Lambda^{(n-\ell)/2}\pi$ and that a Poisson algebra $\mathcal{A}$ (or a Lie algebra $\mathfrak{q}$) is of *Kostant type*, if $Z\mathcal{A}$ is generated by ℓ polynomials satisfying the Kostant equality.

By the Kostant regularity criterion [5, Theorem 9], a reductive Lie algebra is of Kostant type. Other examples of such algebras are the centralisers $\mathfrak{g}_e$ of nilpotent elements in $\mathfrak{sl}_m$ and $\mathfrak{sp}_{2m}$ [8], and truncated seaweed (biparabolic) subalgebras of $\mathfrak{sl}_m$ and $\mathfrak{sp}_{2m}$ [4]. In favourable circumstances, *contractions* lead to new Lie algebras of Kostant type, for instance, semi-direct products related to symmetric decompositions $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, [7], [10].

1. Contractions and the Kostant equality

Suppose that there is a family of automorphisms φ_t of $\mathbb{K}^n = \langle x_1, \ldots, x_n\rangle_{\mathbb{K}}$ given by a regular map $\mathbb{K}^{\times} \to \mathrm{GL}(\mathbb{K}^n)$. We extend φ_t to $\mathcal{A}$, Ω, and W; and set $\pi_t = \varphi_t^{-1}(\pi)$. This is a new Poisson tensor on $\mathbb{A}^n$. Here $Z(\mathcal{A}, \pi_t) = \varphi_t^{-1}(Z\mathcal{A})$. We say that $\tilde{\pi} = \lim_{t\to 0} \pi_t$ is a contraction of π, if this limit exists. For each $H \in \mathcal{A}$, define its *highest (t-) component* as a non-zero polynomial $H^\bullet$ such that $H^\bullet = \lim_{t\to 0} t^d \varphi_t^{-1}(H)$ for some (unique) $d =: \deg_t H$. It is not difficult to show that $H^\bullet \in Z(\mathcal{A}, \tilde{\pi})$ for any $H \in Z\mathcal{A}$.

Theorem 1. *Suppose that* $\operatorname{ind} \mathcal{A}$ *stays the same under a contraction* $\pi_t \rightsquigarrow \tilde{\pi}$ *and* $\varphi_t(\omega) = t^{D_t}$. *If the Kostant equality holds for a set of polynomials* $F_1, \ldots, F_\ell \in Z(\mathcal{A}, \pi)$, *then*

- (i) $\sum \deg_t F_i \geqslant D_t$, *and* $F_i^\bullet$ *are algebraically independent if and only if* $\sum \deg_t F_i = D_t$;
- (ii) *if* $\sum \deg_t F_i = D_t$, *then* $F_i^\bullet$ *satisfy the Kostant equality with* $\tilde{\pi}$;
- (iii) *if we have an equality in* (i), $\dim \operatorname{Sing} \tilde{\pi} \leqslant n - 2$, *and each* $F_i^\bullet$ *is a homogeneous polynomial, then* $F_i^\bullet$ *generate* $Z(\mathcal{A}, \tilde{\pi})$.

The idea of a proof is that one contracts both sides of the Kostant equality. To show part (iii), one needs a characteristic zero version of Skryabin's result, see [8, Theorem 1.1], which states that here the subalgebra generated by $F_i^\bullet$ is algebraically closed in $\mathcal{A}$.

2. Applications to E. Feigin's contraction

Suppose that $\mathbb{K}$ is algebraically closed. Let $\mathfrak{g} = \operatorname{Lie} G$ be a simple Lie algebra of rank ℓ, $B \subset G$ a Borel subgroup, and $\mathfrak{b} = \operatorname{Lie} B$ a Borel subalgebra. Fix a decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$, where $\mathfrak{n}^-$ is the nilpotent radical of an opposite Borel, and consider a one-parameter contraction of $\mathfrak{g}$ given by $\varphi_t|_{\mathfrak{b}} = \mathrm{id}$, $\varphi_t|_{\mathfrak{n}^-} = t\mathrm{id}$. For the resulting Lie algebra $\tilde{\mathfrak{g}}$, we have $\tilde{\mathfrak{g}} = \mathfrak{b} \ltimes \mathfrak{n}^-$, where $\mathfrak{n}^-$ is an Abelian ideal. This contraction was recently introduces by E. Feigin in [1] and was further studied in e.g. [2] and [3].

Here $\operatorname{ind} \tilde{\mathfrak{g}} = \ell$ [9] and $\deg_t F$ is the highest degree of F with respect to $\mathfrak{n}^-$. Let $F_i \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ with $1 \leqslant i \leqslant \ell$ be homogeneous generators such that $\deg F_i \leqslant \deg F_{i+1}$. Again by [9], $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is freely generated by $F_i^\bullet$.

Lemma 2. *Assume that* F_i *are normalised to satisfy the Kostant equality. Then* $F_i^\bullet$ *satisfy the Kostant equality with* $\tilde{\pi}$ *and therefore* $\tilde{\mathfrak{g}}$ *is a Lie algebra of Kostant type. Besides,* $\deg_t F_i = \deg F_i - 1$ *for all* i.

Proof. If $\deg_t F_i = \deg F_i$, i.e., $F_i^\bullet \in \mathcal{S}(\mathfrak{n}^-)$, then $F_i^\bullet \in \mathcal{S}(\mathfrak{n}^-)^{\mathfrak{b}} = \mathbb{K}$. Hence $\deg_t F_i \leqslant \deg F_i - 1$ and $\sum \deg_t F_i \leqslant \dim \mathfrak{b} - \ell = \dim \mathfrak{n} = D_t$. By Theorem 1, $\deg_t F_i = \deg F_i - 1$, the polynomials $F_i^\bullet$ are algebraically independent and satisfy the Kostant equality with $\tilde{\pi}$. □

Let $\alpha_1, \ldots, \alpha_\ell$ be a set of the simple roots, δ the highest root, e_δ a highest root vector, and $r_i = [\delta : \alpha_i]$ the i-th coefficient in the decomposition of δ, i.e.,

$\delta = \sum r_i\alpha_i$. Choose also non-zero $f_i \in \mathfrak{g}_{-\alpha_i}$. Then up to a non-zero scalar, $F_\ell^\bullet = e_\delta \prod f_i^{r_i}$. Using the Kostant equality we can show that $\Lambda^{(n-\ell)/2}\tilde{\pi} = pR$, where $p = \prod f_i^{r_i-1}$, $R \in W^{n-\ell}$, and the zero set of R in $\tilde{\mathfrak{g}}^*$ has codimension grater than or equal to 2. In particular, outside of type A we get curious examples of Lie algebras of Kostant type that does not have the "codim-2" property. However, the quotient map $\mathbb{K}[\tilde{\mathfrak{g}}^*] \to \mathbb{K}[\tilde{\mathfrak{g}}^*]^{\tilde{\mathfrak{g}}}$ is equidimensional and $\mathbf{U}(\tilde{\mathfrak{g}})$ is a free $Z\mathbf{U}(\tilde{\mathfrak{g}})$-module for all simple $\mathfrak{g}$, [9].

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Affine Mirković-Vilonen polytopes

Pierre Baumann

(joint work with Joel Kamnitzer and Peter Tingley)

Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be a semisimple Lie algebra over $\mathbb{C}$. We adopt the usual notation: $\mathfrak{h}_\mathbb{R}^*$ is the $\mathbb{R}$-vector subspace of $\mathfrak{h}^*$ spanned by the roots, W is the Weyl group, and w_0 is the longest element in W. Let C_+ be the dominant chamber in $\mathfrak{h}_\mathbb{R}$ and let r be the number of positive roots. Let $B(\infty)$ be the Kashiwara crystal of $U_q(\mathfrak{n}_+)$.

In his PhD thesis, Kamnitzer constructed an injective family $(\mathrm{Pol}(b))_{b\in B(\infty)}$ of lattice convex polytopes in $\mathfrak{h}_\mathbb{R}^*$. These polytopes are called MV polytopes. The normal fan of each MV polytope is coarser than the Weyl fan in $\mathfrak{h}_\mathbb{R}$. Among the polytopes having this property, MV polytopes are characterized by constraints on the shape of their 2-faces (these contraints are called the tropical Plücker relations).

The aim of the talk is to explain a method that generalizes this construction to the case of an affine Kac-Moody algebra $\mathfrak{g}$. Before giving the rough idea, let us draw attention to two features of the theory:

(1) The crystal $B(\infty)$ canonically parametrizes several interesting bases of $U(\mathfrak{n}_+)$: the canonical basis of Lusztig [11] (global crystal basis of Kashiwara [9]), the semicanonical basis of Lusztig [13], and the basis given by MV cycles through the geometric Satake equivalence [14]. It can be shown that the transition matrices between these bases is lower unitriangular w.r.t. the order on $B(\infty)$ given by the containment of the MV polytopes.
(2) MV polytopes encode the Lusztig parametrizations of the crystal $B(\infty)$, which arise when one compares the canonical basis with PBW bases in the quantum group $U_q(\mathfrak{n}_+)$ (see for instance [6]). Specifically, take $b \in B(\infty)$. Any reduced word $\mathbf{i} = (i_1, \ldots, i_r)$ for w_0 defines a gallery (a sequence of contiguous chambers) $C_+,\ s_{i_1}C_+,\ s_{i_1}s_{i_2}C_+,\ \ldots,\ -C_+$ in $\mathfrak{h}_\mathbb{R}$, whence a sequence of vertices of $\mathrm{Pol}(b)$, which pictures a path in the 1-skeleton of the polytope. In this context, the lengths of the edges in this path form the Lusztig datum of b in direction $\mathbf{i}$.

Originally, MV poytopes were defined as the moment polytopes of the MV cycles, in the context of the geometric Satake equivalence. This approach is related to the third basis mentioned in item (1) above, and has not yet been extended to the affine setup. Our method relies on the setup used to defined the semicanonical basis; it has the drawback of being restricted to the case of a symmetric Cartan matrix.

The main ingredients of the construction are:

- The notion of the Harder-Narasimhan polytope $P(T)$ of an object T of an abelian finite length category $\mathcal{A}$. This polytope sits in the realified Grothendieck group $K(\mathcal{A}) \otimes_\mathbb{Z} \mathbb{R}$ of the category; it is defined as the convex hull of the classes $[X]$ of all subobjects $X \subseteq T$.
- The completed preprojective algebra Λ on the Dynkin diagram of $\mathfrak{g}$, and Buan, Iyama, Reiten and Scott's tilting theory for Λ-mod [7].
- The fact, due to Kashiwara and Saito [10] (see also [12]), that to each $b \in B(\infty)$ of weight ν corresponds canonically an irreducible component Λ_b of the affine variety of Λ-module structures in dimension-vector ν.

One can then show that in finite type, $\mathrm{Pol}(b)$ is the HN polytope $P(T)$ for T a general point in Λ_b.

Alone, this construction cannot be used to define the MV polytope $\mathrm{Pol}(b)$ in the affine symmetric type, because the HN polytope $P(T)$ does not carry enough information to recover Λ_b. Somehow, one needs to take into account the multitude and the multiplicity of the imaginary roots. The solution is to decorate these polytopes with a family of partitions, indexed by the set of all chamber coweights for the finite (classical) root system.

When the definitions are correctly set, all the properties mentioned above in the finite type case also hold in the affine type case: MV polytopes are in canonical bijection with $B(\infty)$; they are characterized by the fact that their normal fan is coarser than the Weyl fan and by conditions on their 2-faces (see Figure 1).

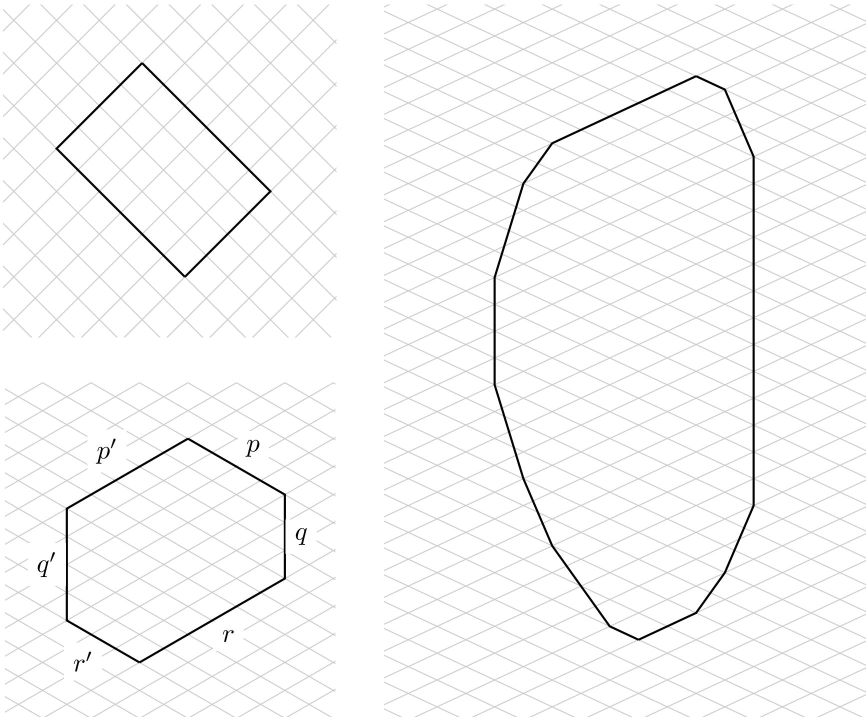

FIGURE 1. Examples of 2-faces of affine MV polytopes. These faces are of type $A_1 \times A_1$ (top left), A_2 (bottom left), and $A_1^{(1)}$ (right). In type A_2, the tropical Plücker relation is $q' = \min(p, r)$. The analogue for type $A_1^{(1)}$ of this condition can be found in [3].

In the affine type case, similarly to the situation for the finite type, PBW bases can be used as an approximation to the canonical basis [4, 5, 8], whence a notion of Lusztig parametrization for $B(\infty)$. Our MV polytopes encode all the existing Lusztig parametrizations. From this point of view, the conditions on the 2-faces can be seen as a replacement for Lusztig's piecewise linear bijections $R_{\mathbf{i}}^{\mathbf{i}'}$ [11].

For the seek of completeness, let us mention the papers [15, 16], whose relation to the work reported here is however not clear.

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Sheaf-theoretic Koszul duality for Kac-Moody groups

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(joint work with Roman Bezrukavnikov)

The formalism of Koszul duality in representation theory goes back to the work of Beilinson, Ginzburg, Schechtman [BGS88] and Soergel [So90] from 1980's, and was developed later by these and other authors in [BGS96], [BG99] etc. The formalism uncovers some intriguing phenomena. On the one hand, it shows that some categories of representations (such as Bernstein-Gel'fand-Gel'fand category $\mathcal{O}$) are "controlled" by Koszul quadratic algebras; this fact, closely related to Kazhdan-Lusztig conjectures, is proven using purity theorem about Frobenius (or Hodge) weights on Ext's between irreducible perverse sheaves. On the other hand, the duality (or rather equivalence) between derived categories of representations has some interesting geometric properties. In particular, it interchanges the Lefschetz $\mathfrak{sl}(2)$ (i.e. the $\mathfrak{sl}(2)$ containing multiplication by the first Chern class of an ample line bundle acting on cohomology of a smooth projective variety) with the Picard-Lefschetz $\mathfrak{sl}(2)$ (i.e. $\mathfrak{sl}(2)$ containing the logarithm of monodromy acting on cohomology of nearby cycles), which is formally similar to a key property of mirror symmetry.

In this paper we extend the result of [So90] and [BGS96] to a much more general setting: we replace a semi-simple algebraic group considered in *loc. cit.* by an arbitrary Kac-Moody group. We work with mixed ℓ-adic sheaves on (ind-)varieties over a fixed finite field $k = \mathbb{F}_q$. For such a variety with an action of an algebraic group A, let $D^b_m(A\backslash X)$ denote the derived category of A-equivariant mixed $\overline{\mathbb{Q}_\ell}$ sheaves on X.

Let G be a Kac-Moody group defined over k. Let $B = UH$ be a Borel subgroup of G with unipotent radical U and Cartan subgroup H. The ind-variety G/B is called the flag variety of G and G/U is called the enhanced flag variety of G.

Let $G^\vee$ be the Langlands dual Kac-Moody group of G. This is a Kac-Moody group with root system dual to that of G, with Borel subgroup $B^\vee = U^\vee H^\vee$. Let W be the Weyl group of G and $G^\vee$, which is a Coxeter group with simple reflections Σ (in bijection with simple roots of G). Let $\Theta \subset \Sigma$ be such that the subgroup W_Θ generated by Θ is finite, hence determining a parabolic subgroup P_Θ of G. The main results consist of four equivalences of derived categories in the spirit of Koszul duality:

Theorem. *There are equivalences of triangulated categories:*

- *Equivariant-monodromic duality, which is a monoidal equivalence:*
$$\Phi : D^b_m(B\backslash G/B) \xrightarrow{\sim} \widehat{D}^b_m(B^\vee \vdots G^\vee \vdots B^\vee);$$
- *"Self-duality":*
$$\Psi : D^b_m(B^\vee\backslash G^\vee/U^\vee) \xrightarrow{\sim} D^b_m(U\backslash G/B);$$
- *Parabolic-Whittaker duality:*
$$\Phi_\Theta : D^b_m(P_\Theta\backslash G/B) \xrightarrow{\sim} \widehat{D}^b_m((U^{\vee,\Theta}U^{\vee,-}_\Theta, \chi)\backslash G^\vee \vdots B^\vee);$$
- *"Paradromic-Whittavariant" duality:*
$$\Psi_\Theta : D^b_m(P^\vee_\Theta\backslash G^\vee/U^\vee) \xrightarrow{\sim} D^b_m((U^\Theta U^-_\Theta, \chi)\backslash G/B);$$

We need to explain the notations. The category $\widehat{D}^b_m(B^\vee \vdots G^\vee \vdots B^\vee)$ is a certain completion of the category $D^b_m(B^\vee \vdots G^\vee \vdots B^\vee)$, the latter being the derived category of left $U^\vee$-equivariant mixed complexes on the enhanced flag variety $G^\vee/U^\vee$, which, along the $H^\vee$-orbits (under the action given by either left or right multiplication) have unipotent monodromy. The completion procedure adds objects with free unipotent monodromy to the monodromic category.

In the target of the last equivalence Ψ_Θ, U^Θ is the unipotent radical of P_Θ, and U^-_Θ is the unipotent radical of a Borel subgroup of L_Θ (the Levi subgroup of P_Θ), which is opposite to the standard Borel. The left quotient by $(U^\Theta U^-_\Theta, \chi)$ means taking mixed complexes which are left equivariant under $U^\Theta U^-_\Theta$ against a generic character $\chi : U^-_\Theta \to \mathbb{G}_a$. Such a construction is a geometric analogue of Whittaker models. The meaning of $(U^{\vee,\Theta}U^{\vee,-}_\Theta, \chi)$ in the target of Φ_Θ is similar, with G replaced by $G^\vee$.

The equivalences in the theorem enjoy the following properties:

- They respect the relevant monoidal structures. For example, both sides of the equivariant-monodromic duality carry monoidal structures given by convolution of sheaves, and Φ is a monoidal functor. Similarly, both sides of the parabolic-Whittaker duality are module categories under the respective monoidal categories in the equivariant-monodromic duality (given by convolution on the right), and Φ_Θ respects these module category structures.
- They send standard (resp. costandard) sheaves to standard (resp. costandard) sheaves. The spaces in question have Schubert stratifications indexed by (cosets of) the Weyl group. The standard and costandard sheaves are ! and $*$-extensions of constant sheaves (or free-monodromic sheaves) on the strata.
- They send intersection cohomology sheaves to indecomposable (free monodromic) tilting sheaves. For example, under the equivalence Φ, the intersection cohomology sheaf of the closure of the Schubert stratum $BwB/B \subset G/B$ is sent to the free-monodromic tilting sheaf supported on the closure of $B^\vee wB^\vee/U^\vee \subset G^\vee/U^\vee$. In the case of Ψ and Ψ_Θ, they also send indecomposable tilting sheaves to intersection cohomology sheaves.
- They are exact functors between triangulated categories, but *not* t-exact with respect to the perverse t-structures. Under all these equivalences, the Tate twist (1) becomes the functor $[-2](-1)$.

We comment on the precise relation between our work and other's. First, [So90] works with a regular integral block in category $\mathcal{O}$ of highest weight modules over the semi-simple Lie algebra. By Beilinson-Bernstein Localization Theorem this category is identified with a category of perverse sheaves on the flag variety. In this paper we work directly with the geometric category of sheaves and its generalizations. (A generalization of Localization Theorem to a general Kac-Moody group is not known, so one can not restate our result in terms of modules in this more general setting). The parabolic-singular variant of Koszul duality developed in [BGS96] involves singular category $\mathcal{O}$. By [MS97] the latter is equivalent to the category of "generalized Whittaker" perverse sheaves on the flag variety; hence the appearance of Whittaker sheaves in the present paper.

We would also like to point out the equivalences in the above theorem generalize the variant of Koszul duality equivalence suggested in [BG99] rather than the original equivalences of [So90] and [BGS96]. While the latter send irreducible objects to projective ones, the former sends irreducible objects to tilting ones. The advantage of the "tilting" version of the equivalence is that it turns out to be a monoidal functor (in the cases when the categories in question are monoidal); in the finite dimensional group case this verifies a conjecture in [BG99, Conjecture 5.18]. For a finite dimensional semi-simple group, the two functors differ by a long intertwining functor (Radon transform). In the Kac-Moody setting there is a more essential difference between the two formulations; in fact, the categories we consider do not have enough projectives, so the requirement for the functor to send irreducibles to projectives does not apply here. So we work out a generalization of the "tilting" version of the formalism, and show that the resulting equivalences are monoidal (when applicable). The price to pay for including monoidal categories into consideration is additional technical difficulties of foundational nature (appearing already in the finite dimensional semi-simple group case), such as the formalism of completions of derived categories.

The proof of the theorem uses the strategy in [So90]: two functors $\mathbb{H}$ and $\mathbb{V}$ are defined on the two sides of the equivariant-monodromic equivalence and we use $\mathbb{H}$ (resp. $\mathbb{V}$) to calculate the Ext algebra of the intersection cohomology sheaves (resp. the endomorphism algebra of the free-monodromic tilting sheaves). This way we show that both sides of the equivalence are governed by the same differential graded algebra. The other equivalences are more or less formal consequences of the first one.

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Projective Normality of Model Wonderful Varieties

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(joint work with Paolo Bravi, Andrea Maffei)

Let G be a semisimple and connected complex algebraic group.

Definition. A G-variety M is called *wonderful* (of rank n) if it is smooth and projective and it satisfies the following conditions:

- M possesses an open orbit whose complement is a union of n smooth prime divisors (the *boundary divisors*) with non-empty transversal intersections;
- Any orbit closure in M equals the intersection of the boundary divisors which contain it.

Let M be a wonderful variety and let $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}(M)$ be globally generated line bundles: is the multiplication of sections

$$m_{\mathcal{L},\mathcal{L}'} : \Gamma(M,\mathcal{L}) \times \Gamma(M,\mathcal{L}') \longrightarrow \Gamma(M,\mathcal{L}\otimes\mathcal{L}')$$

surjective? In particular, if these multiplications are all surjective, it follows that the complete linear system of any ample line bundle embeds M as a projectively normal variety. A trivial case is that of a flag variety: these are the wonderful varieties of rank zero and in this case the surjectivity of the multipication is an easy consequence of the irreducibility of the modules of sections.

An important class of wonderful varieties was introduced by De Concini and Procesi in the context of symmetric varieties [4]. In this case, it was shown by Chirivì and Maffei [3] that $m_{\mathcal{L},\mathcal{L}'}$ is surjective for every globally generated line bundles, giving in this way a positive answer to a question raised by Faltings [5].

Wonderful varieties were then considered in full generality by Luna, who proposed a general approach to attack the problem of their classification via combinatorial invariants [7]. Another remarkable class of wonderful varieties arises in the context of model varieties, which have been classified by Luna in [8], where it is shown that there exists a wonderful variety M_G^{mod} whose orbits parametrize the model varieties for G.

Let M be a wonderful G-variety and fix a maximal torus and a Borel subgroup $T \subset B \subset G$. Denote B^- the opposite Borel subgroup of B, let $z \in M$ be the unique B^- fixed point and denote $Y = Gz$ the unique closed orbit of M. Define Σ as the set of T-weights occurring in the T-module T_zM/T_zY: its elements are called the *spherical roots* of M and they naturally correspond to the local equations of the boundary divisors, which are G-stable. If $\sigma \in \Sigma$, we denote by M^σ the associated boundary divisor.

By the work of Brion [1], the Picard group of M is freely generated by the classes of the B-stable prime divisors which are not G-stable: such prime divisors are called the *colors* of M. Moreover, the semigroup of globally generated line bundles correspond to the free semigroup generated by the colors. If Δ denotes the set of colors of M, we get then a natural pairing $c : \Sigma \times \Delta \longrightarrow \mathbb{Z}$ (called the *Cartan pairing* of M), defined by the identity $[M^\sigma] = \sum_\Delta c(\sigma, D)[D]$ and which induces an embedding of $\mathbb{Z}[\Sigma]$ inside $\mathbb{Z}[\Delta]$.

The triple (Σ, Δ, c) is a main part of the combinatorial datum that Luna attached to a wonderful variety (the *spherical system* of M). If M is a wonderful symmetric variety (of non-exceptional type), then the situation is very nice: Σ is the set of simple roots of a root system Φ_Σ (the *restricted root system*), Δ is identified with the set of fundamental weights of Φ_Σ and c is the Cartan pairing of Φ_Σ. In the general case the situation can be more complicated, however by the work of Brion [2] and Knop [6] there always exists a root system Φ_Σ with basis Σ and we may think the triple (Σ, Δ, c) as a generalization of a root system.

As in the case of a root system, the semigroup $\mathbb{N}[\Delta]$ is naturally equipped with a partial order $\leqslant_\Sigma$, defined as follows: if $E, F \in \mathbb{N}[\Delta]$, then $E \leqslant_\Sigma F$ if and only if $F - E \in \mathbb{N}[\Sigma]$. In the case of a root system, this is the partial order on the semigroup of dominant weights studied by Stembridge [9]. The partial order $\leqslant_\Sigma$ is tightly related to the description of the sections of a line bundle on M. If $E \in \mathbb{N}[\Delta]$, denote $\Delta(E) = \{D \in \mathbb{N}[\Delta] \,:\, D \leqslant_\Sigma E\}$.

If $E \in \mathbb{N}[\Delta]$, denote $\mathcal{L}_E \in \mathrm{Pic}(M)$ the associated line bundle and $s_E \in \Gamma(M, \mathcal{L}_E)$ the canonical section, which is B semi-invariant, denote moreover $V_E = \langle G s_E \rangle \subset \Gamma(M, \mathcal{L}_E)$ the generated submodule. Similarly, if $\gamma = \sum a_\sigma \sigma \in \mathbb{N}[\Sigma]$, denote $\mathcal{L}_\gamma \in \mathrm{Pic}(M)$ the line bundle associated to $M^\gamma = \sum a_\sigma M^\sigma$ and denote $s^\gamma \in$

$\Gamma(M, \mathcal{L}_\gamma)$ the canonical section, which is G invariant. Then it holds the following decomposition:

$$\Gamma(M, \mathcal{L}_E) = \bigoplus_{D \in \Delta(E)} s^{E-D} V_D.$$

Let $E, F \in \mathbb{N}[\Delta]$ and consider the multiplication of sections $m_{E,F} : \Gamma(M, \mathcal{L}_E) \times \Gamma(M, \mathcal{L}_F) \longrightarrow \Gamma(M, \mathcal{L}_{E+F})$. An easy inductive argument reduces the study of the surjectivity of $m_{E,F}$ to a particular set of triples.

Definition. Let $E, F \in \mathbb{N}[\Delta]$ and let $D \in \Delta(E+F)$. The triple (E, F, D) is called a *low triple* if the following condition is satisfied: if $E' \in \Delta(E)$ and $F' \in \Delta(F)$ are such that $D \in \Delta(E' + F')$, then $E' = E$ and $F' = F$.

The notion of low triple was introduced by Chirivì and Maffei in [3] in order to study the surjectivity of the multiplication map in the case of a symmetric wonderful variety however our definition is slightly more general than the original one. Notice that if (E, F, D) is a low triple and if $\gamma = E + F - D$, then $s^\gamma V_D \subset \Gamma(M, \mathcal{L}_E)\Gamma(M, \mathcal{L}_F)$ if and only if $s^\gamma V_D \subset V_E V_F$.

Suppose that $E, F \in \mathbb{N}[\Delta]$ are such that $F <_\Sigma E$ and suppose that F is maximal with this property: then we say that E *covers* F and we say that $E - F$ is a *covering relation* for (Σ, Δ, c). If $\gamma = \sum_\Delta n_D D$, define its *positive height* $\mathrm{ht}^+(\gamma) = \sum_{n_D > 0} n_D$. If (Σ, Δ, c) is identified with the triple of the root system Φ_Σ as in the case of a non-exceptional symmetric variety, then it is very easy to show that $\mathrm{ht}^+(\gamma) \leq 2$ for every covering relation $\gamma \in \mathbb{N}[\Sigma]$.

Theorem 1. *Let M be a wonderful variety with triple (Σ, Δ, c) and suppose that the following conditions are fulfilled:*

- *If (E, F, D) is a low triple with $E, F \in \Delta$, then $s^{E+F-D} V_D \subset V_E V_F$.*
- *If $\gamma \in \mathbb{N}[\Sigma]$ is a covering relation, then $\mathrm{ht}^+(\gamma) \leqslant 2$.*

Then the multiplication $m_{E,F}$ is surjective for every $E, F \in \mathbb{N}[\Delta]$.

We conjecture that the second condition of previous theorem is always satisfied. In particular, this would imply that the multiplication $m_{E,F}$ is surjective for every $E, F \in \mathbb{N}[\Delta]$ if and only if it is surjective for every $E, F \in \Delta$. On the other hand, following example shows that the first condition of previous theorem may not be fulfilled: in particular the multiplication $m_{E,F}$ may not be surjective.

Example. Let $G = \mathrm{SO}(9)$ and consider the model wonderful variety M_G^{mod}. Denote $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ the simple roots of G and $\omega_1, \omega_2, \omega_3, \omega_4$ the fundamental weights of G, enumerated as usual. Then the restriction of line bundles to the closed orbit is injective and we may describe spherical roots and colors of M_G^{mod} as follows:

$$\Sigma = \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, 2\alpha_4\}, \qquad \Delta = \{\omega_1, \omega_2, \omega_3, 2\omega_4\}.$$

Consider the low triple $(\omega_2, \omega_2, \omega_1)$: then $V(\omega_1) \not\subset V(\omega_2)^{\otimes 2}$, hence $s^{2\omega_2 - \omega_1} V_{\omega_1} \not\subset V_{\omega_2}^2$ and the multiplication m_{ω_2, ω_2} is not surjective.

In the case of a model wonderful variety, by classifying the covering relations and using the reduction of previous theorem, we proved the following theorem.

Theorem 2. *Let G be a semisimple connected group of classical type and consider the associated model wonderful variety M_G^{mod}. Then the multiplication of sections is surjective for any couple of globally generated line bundles on M_G^{mod} if and only if G has no adjoint factors of type B_r with $r \geq 4$.*

Actually, the counterexample given for the model wonderful variety of SO(9) does not express a lack of the multiplication, but rather a lack of the tensor product. Indeed $V(\omega_1) \not\subset V(\omega_2)^{\otimes 2}$ but $V(2\omega_1) \subset V(2\omega_2)^{\otimes 2}$: this expresses the fact that the saturation property does not hold for SO(9). Notice that if we assume that the multiplication of M is generic as much as possible, then the saturation property for the tensor product of G would imply an analogous saturation property for the multiplication of M.

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Analogies between smooth representation of p-adic groups and affine Kac-Moody algebras

Masoud Kamgarpour

Let $\mathfrak{g}$ be a reductive Lie algebra and let G denote the corresponding connected reductive group. For instance, one can take G to be the general linear group GL_N. Let F be a local non-Archimedean field. Let $\mathcal{O}$ denote the ring of integers of F and let t denote a uniformizer for $\mathcal{O}$. Let $K_n \subseteq G(\mathcal{O})$ denote the preimage of the identity element under the natural morphism $G(\mathcal{O}) \to G(\mathcal{O}/t^n)$. A representation of (π, V) of $G(F)$ is *smooth* if, for every $v \in V$, there exists a positive integer N such that

$$\pi(g).v = v, \qquad \forall v \in K_N.$$

Let $\mathscr{R}(G)$ denote the category of smooth representations of G. This category plays a prominent role in Langlands program [L70], [V93].

Let $\widehat{\mathfrak{g}} = \mathfrak{g}((t)) \oplus \mathbb{C}$ denote the affine Kac-Moody algebras associated to $\mathfrak{g}$. The representations of $\widehat{\mathfrak{g}}$ have a parameter, known as the *level*, which captures the character by which $\mathbb{C} \subset \widehat{\mathfrak{g}}$ acts. In what follows, we work with a special value of this parameter called the *critical level*; see, for instance, [F07]. A representation V of $\widehat{\mathfrak{g}}$ at the critical level is *smooth* if, for every vector $v \in V$, there exists a positive integer N such that

$$\mathfrak{g} \otimes t^N \mathbb{C}[[t]].v = 0.$$

Let $\widehat{\mathfrak{g}}_{\mathrm{crit}} - \mathrm{mod}$ denote the category of smooth representations of $\widehat{\mathfrak{g}}$ at the critical level. Note that $\widehat{\mathfrak{g}}_{\mathrm{crit}} - \mathrm{mod}$ is equivalent to the category of modules over a certain algebra $U^{\mathrm{sm}}_{\mathrm{crit}}(\widehat{\mathfrak{g}})$ which, in turn, is related to the universal enveloping algebra of $\widehat{\mathfrak{g}}$. The category $\widehat{\mathfrak{g}}_{\mathrm{crit}} - \mathrm{mod}$ plays a central role in the geometric Langlands program [BD], [FG06], [F07]. In contrast to $\mathscr{R}(G)$, however, our knowledge of $\widehat{\mathfrak{g}}_{\mathrm{crit}} - \mathrm{mod}$ is limited. Roughly speaking, the part of $\widehat{\mathfrak{g}}_{\mathrm{crit}} - \mathrm{mod}$ which is well-understood corresponds to the category $\widehat{\mathcal{O}}_{\mathrm{crit}}$. It is natural to try to transport the methods available for studying $\mathscr{R}(G)$ to the setting of $\widehat{\mathfrak{g}}_{\mathrm{crit}} - \mathrm{mod}$.

Parabolic induction. One of the basic ways to produce representations is via induction. When dealing with reductive group, it is natural to consider induction of representations from parabolic subgroups. Choose a Borel subgroup $B \subset G$ and a maximal split torus $T \subset B$. If $G = \mathrm{GL}_N$, then we can take B (resp. T) to be the group of upper triangular (resp. diagonal) matrices. Let $\chi : T(F) \to \mathbb{C}^\times$ be a smooth character; i.e., χ is trivial on $T(t^n\mathcal{O})$ for some positive integer n. Inflating χ to $B(F)$ and inducing to $G(F)$, we obtain a representation

$$\mathscr{B}(\chi) := \mathrm{ind}_{B(F)}^{G(F)} \chi$$

known as the principal series representation associated to χ. Using the fact that $G(F)/B(F)$ is compact, one can show that $\mathscr{B}(\chi)$ is a smooth representation; see, for instance, [C75].

Let us consider the analogous construction in $\widehat{\mathfrak{g}}_{\mathrm{crit}} - \mathrm{mod}$. Let $\widehat{\mathfrak{h}}$ and $\widehat{\mathfrak{b}}$ denote the corresponding subalgebras of $\widehat{\mathfrak{g}}$. Given a (smooth) module $V \in \widehat{\mathfrak{h}}_{\mathrm{crit}} - \mathrm{mod}$, we can inflate it to a smooth module in $\widehat{\mathfrak{b}}_{\mathrm{crit}} - \mathrm{mod}$ and then form the induced module

$$U^{\mathrm{sm}}_{\mathrm{crit}}(\widehat{\mathfrak{g}}) \otimes_{U^{\mathrm{sm}}_{\mathrm{crit}}(\widehat{\mathfrak{b}})} V.$$

This is a free module over $U^{\mathrm{sm}}_{\mathrm{crit}}(\widehat{\mathfrak{n}}^-)$; therefore, it is *not* a smooth module. To remedy this shortcoming, Feigin and Frenkel [FF88], [FF90] (following Wakimoto) proposed to do "semi-infinite" induction rather than the naive induction. In this approach, one starts with a representation χ of $\widehat{\mathfrak{h}}$ at *level zero* (or, equivalently, a smooth character of $\mathfrak{h}((t))$) and produces a representation $\mathbb{W}(\chi) \in \widehat{\mathfrak{g}}_{\mathrm{crit}} - \mathrm{mod}$, known as the *Wakimoto module* associated to χ. We will not reproduce the construction of $\mathbb{W}(\chi)$, referring the reader to [F07], [FG06]. We point out, however, that the Wakimoto modues produced in *loc. cit* are *Iwahori integrable*, even if the original character is not Iwahori integrable.[1] It is natural to try to construct Wakimoto modules that are not Iwahori integrable (see Problem 5).

[1] I thank Dennis Gaitsgory for pointing this out to me; see also [FG06, §11]. Recall that a module in $\widehat{\mathfrak{g}}_{\mathrm{crit}} - \mathrm{mod}$ is Iwahori integrable if $\mathrm{Lie}[I, I]$ acts locally nilpotently, and $\mathfrak{h}$ acts semisimply with eigenvalues corresponding to integral weights [FG06, §5.2].

Categorical decomposition. Recall that the center of an abelian category is defined to be the algebra of endomorphisms of the identity functor. Using the center of the abelian category $\mathscr{R}(G)$, Bernstein obtained a canonical direct product decomposition

$$\mathscr{R}(G) = \prod_{\mathfrak{s}\in\mathscr{B}(G)} \mathscr{R}_{\mathfrak{s}}(G).$$

The objects of $\mathscr{R}_{\mathfrak{s}}(G)$ are all representations (π, V) such that all irreducible subquotients of π have fixed supercuspidal support modulo unramified twist. Thus the indexing set $\mathscr{B}(G)$ consists of irreducible supercuspidal representations of Levi subgroups of G modulo G-conjugation and twisting by unramified characters. Each category $\mathscr{R}_{\mathfrak{s}}(G)$ is a module category over an algebra $\mathcal{A}_{\mathfrak{s}}$ which can be described explicitly. The Morita equivalence class of $\mathcal{A}_{\mathfrak{s}}$ is unique. Moreover, every $\mathscr{R}_{\mathfrak{s}}(G)$ is indecomposible. For these facts and more details, see [Ber84] and [R09]. It is natural to ask whether a similar direct product decomposition of $\widehat{\mathfrak{g}}_{\mathrm{crit}}-$mod exists.

Problem 4. Using the center $\mathscr{Z}$ of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ − mod, express this abelian category as a direct product.

Induction via compact open subgroups. Ever since papers of Howe [H73], [H77], inducing representations from compact open subgroups of $G(F)$ has become one of the main tools for constructing smooth representations of $G(F)$. Bushnell and Kutzko [BK98] have organized this in the theory of types. They proposed that for every $\mathfrak{s} \in \mathscr{B}(G)$, there should exist a compact open subgroup K and a representation ρ of K, such that the induced representation $\mathrm{ind}_K^{G(F)}\,\rho$ is a progenerator for $\mathscr{R}_{\mathfrak{s}}(G)$. (This implies that the endomorphism ring of this representation is Morita equivalent to $\mathcal{A}_{\mathfrak{s}}$). The pair (K, ρ) is known as an $\mathfrak{s}$-type.

It is known that every smooth character $\bar{\mu} : T(\mathcal{O}) \to \mathbb{C}$ defines a canonical element in $\mathscr{B}(G)$, and therefore a block which we denote by $\mathscr{R}_{\bar{\mu}}$. Suppose the character $\bar{\mu}$ factors through a character $\mu : T(\mathcal{O}/t^{n+1}) \to \mathbb{C}^\times$. Note $T(\mathcal{O}/t^{n+1})$ identifies canonically with $T_n(\mathbb{F}_q)$, where T_n is the scheme of n-jets of T. Similarly we have the jet groups G_n and B_n. Now we can form the induced representation

$$\mathscr{W}(\bar{\mu}) := \mathrm{ind}_{G(\mathcal{O})}^{G(F)}\,\mathrm{ind}_{B_n(\mathbb{F}_q)}^{G_n(\mathbb{F}_q)}\,\mu.$$

Alternatively, we can describe $\mathscr{W}(\bar{\mu})$ as follows. Let I_n denote the inverse image of $B(\mathcal{O}/t^{n+1})$ under the natural morphism $G(\mathcal{O}) \to G(\mathcal{O}/t^{n+1})$. Then $\bar{\mu}$ defines a character $\mu : I_n \to \mathbb{C}^\times$. It is clear that

$$\mathscr{W}(\bar{\mu}) \simeq \mathrm{ind}_{I_n}^{G(F)}\,\mu.$$

In some cases, one can show that (I_n, μ) is a type for the block $\mathscr{R}_{\bar{\mu}}$; that is, $\mathscr{W}(\bar{\mu})$ is a progenerator of $\mathscr{R}_{\bar{\mu}}$ [R98], [BK99].

Verma modules and Wakimoto modules. Let us consider the analogue of $\mathscr{W}(\bar{\mu})$ in $\widehat{\mathfrak{g}}_{\mathrm{crit}}-$mod. Let $\mathfrak{h}_n$, $\mathfrak{b}_n$, and $\mathfrak{g}_n$ denote the n-jets of $\mathfrak{h}$, $\mathfrak{b}$, and $\mathfrak{g}$, respectively. These are Lie algebras over $\mathbb{C}$[2] Let $\theta : \mathfrak{h}_n \to \mathbb{C}$ be a character, and let $V(\theta) :=$

[2]Note, however, that $\mathfrak{g}_n$ is not reductive for $n > 0$.

$U(\mathfrak{g}_n) \otimes_{(\mathfrak{b}_n)} \theta$ denote the associated $\mathfrak{g}_n$-module. Let

$$\mathbb{V}(\theta) := \mathrm{Ind}_{\mathfrak{g}[[t]]\oplus\mathbb{C}}^{\widehat{\mathfrak{g}}_{\mathrm{crit}}} V(\theta) \in \widehat{\mathfrak{g}}_{\mathrm{crit}} - \mathrm{mod}$$

denote the associated module at the critical level. It is easy to show that this module is smooth. In analogy with above, one can give a realization of $\mathbb{V}(\theta)$ as induced from a character of $\mathrm{Lie}(I_n)$. In particular, under some integrality conditions on θ, one expects that $\mathbb{V}(\theta)$ is I_n-integrable. When $n = 0$ (so $\mathfrak{h}_0 = \mathfrak{h}$) the representations $V(\theta)$ and $\mathbb{V}(\theta)$ are known as the Verma modules of $\mathfrak{g}$ and $\widehat{\mathfrak{g}}_{\mathrm{crit}}$, respectively, and have been studied intensively. For $n > 0$, very little is known about these representations.

One way to obtain some information about $\mathbb{V}(\theta)$ is to relate it to Wakimoto modules. Note that we have a natural morphism

$$\mathrm{Res}_n : \mathfrak{h}((t)) \to \mathfrak{h}_n \qquad \sum a_i t^i \mapsto (a_{-1}, \cdots, a_{-n-1}).$$

Therefore, given a character $\theta \in \mathfrak{h}_n \to \mathbb{C}$, one can define a character $\chi_\theta = \theta \circ \mathrm{Res}_n : \mathfrak{h}((t)) \to \mathbb{C}$.

Problem 5. (i) For every $\theta \in \mathfrak{h}_n^*$, define a Wakimoto module $\mathbb{W}(\chi_\theta)$, so that we have a natural nontrivial morphism $\mathbb{V}(\theta) \to \mathbb{W}(\chi_\theta)$.
(ii) Prove that, under some additional constraints on θ, the natural morphism is an isomorphism.

In the case $n = 0$, Feigin and Frenkel solve the above problem. This is the key step in Frenkel's proof of the Kac-Kazhdan conjecture [F07, §6.4].

Geometrization of principal series types. One can relate representations of $G(F)$ with those of $\widehat{\mathfrak{g}}_{\mathrm{crit}}$ via geometrization. The representation $\mathscr{W}(\bar{\mu})$ is realized on the space of (I_n, μ)-invariant functions on $G(F)$. Let $\mathscr{H}(\bar{\mu})$ denote the algebra of (I_n, μ)-biinvariant functions on $G(F)$. For simplicity, assume that the character $\bar{\mu}$ is regular; i.e., its stabilizer in the Weyl group is trivial. Then one has a canonical isomorphism

$$\mathscr{H} = \mathrm{End}(\mathscr{W}(\bar{\mu})) \simeq \mathrm{K}_0(\mathrm{Rep}(\check{T}))).$$

where $\check{T}$ denotes the dual complex torus. In [MS11], we study the category $\mathscr{W}_{\mathrm{geom}}$ (resp. $\mathscr{H}_{\mathrm{geom}}$) consisting of "$(I_n, \mu)$-equivariant" (resp. (I_n, μ)-biequivariant) appropriately twisted D-modules on $G((t))$. We prove that the category $\mathscr{H}_{\mathrm{geom}}$ is equivalent to $\mathrm{Rep}(\check{T})$. Moreover, we show that $\mathscr{H}_{\mathrm{geom}}$ acts on $\mathscr{W}_{\mathrm{geom}}$ by convolution.[3]

Taking global sections (as $\mathcal{O}$-modules), one defines a functor

$$\Gamma : \mathscr{W}_{\mathrm{geom}} \to \widehat{\mathfrak{g}}_{\mathrm{crit}} - \mathrm{mod}.$$

It is natural to ask how the center $\mathscr{Z}$ acts on the image of this functor. One can show that the global sections of the easiest D-module on $\mathscr{W}_{\mathrm{geom}}$ (the so called δ D-module) is equal to $\mathbb{V}(\mu)$. Moreover, if one can determine how $\mathscr{Z}$ acts on this

[3] In fact, in [MS11], we work with perverse sheaves. One can, however, translate our paper into the language of D-modules.

D-module, that would allow one to determine how it acts on the entire image. We now give a conjectural description of the action of the center on $\mathbb{V}(\mu)$.

A conjecture about the action of $\mathscr{Z}$. According to a theorem of Feigin-Frenkel [F07], $\mathscr{Z}$ is isomorphic to the algebra of functions on the space of *$\check{G}$-opers*. Here, $\check{G}$ is the Langlands dual group, obtained from G by switching the roots and coroots and characters and cocharacters. Similarly, we have the dual algebras $\check{\mathfrak{g}}$ and $\check{\mathfrak{b}}$. A $\check{G}$-oper on the disk with singularity of order less than or equal to $n+1$ is the space of $\check{N}[[t]]$-gauge equivalence classes of operators

$$\nabla = \partial_t + \frac{1}{t^{n+1}}(p_{-1} + \mathbf{v}(t)), \qquad \mathbf{v}(t) \in \check{\mathfrak{b}}[[t]].$$

The group $\check{N}$ denotes the unipotent radical of the Borel $\check{B} \subset \check{G}$. The gauge quivalence relation is defined by the conjugation action of $\check{N}[[t]]$ on $\mathbf{v}(t) \in \check{\mathfrak{b}}[[t]]$. Finally, $p_{-1} := f_1 + \cdots f_l \in \check{\mathfrak{g}}$, where the f_i's are generators of the (one-dimensional) root subspace of $\check{\mathfrak{n}}^-$; see [F07, §4.2.4]. To an oper, we associate its residue

$$\mathrm{Res}_n(\nabla) := p_{-1} + \mathrm{ev}_n(\mathbf{v}(t)),$$

where $\mathrm{ev}_n : \check{\mathfrak{b}}[[t]] \to \check{\mathfrak{b}}[[t]]/t^{n+1} = \check{\mathfrak{b}}_n$ is the canonical morphism. Under gauge transformation by an element $x(t) \in \check{N}[[t]]$, the residue gets conjugated by $\mathrm{ev}_n(x) \in \check{N}_n$. Therefore, the projection of $\mathrm{Res}_n(\nabla)$ onto

$$\check{\mathfrak{g}}_n/\check{G}_n = \mathrm{Spec}(k[\check{\mathfrak{g}}_n]^{\check{G}_n}) \simeq \mathrm{Spec}(k[(\mathfrak{h}^*/W)_n]) = (\mathfrak{h}^*/W)_n.$$

is well-defined. The middle isomorphism in the above line is due to Beilinson and Drinfeld [BD] (see also [M01, Appendix]). Thus, we obtain a morphism

$$\mathrm{Res}_n : \mathrm{Op}_{\check{G}}^{\mathrm{ord}_n}(\mathcal{D}) \to (\mathfrak{h}^*/W)_n$$

For $\chi \in (\mathfrak{h}^*/W)_n$, we denote by $\mathrm{Op}_{\check{G}}^{\mathrm{ord}_n}(\mathcal{D})_\chi$ the subscheme of opers with singularity at most n and residue χ.

Conjecture 1. If the center $Z(\mathfrak{g}_n)$ acts on $V \in \mathfrak{g}_n-\mathrm{mod}$ through the character $\Lambda \in (\mathfrak{h}^*/W)_n$ then $\mathrm{Ind}_{\mathfrak{g}[[t]]\oplus\mathbb{C}}^{\widehat{\mathfrak{g}}_{\mathrm{crit}}}(V)$ is scheme theoretically supported on the subscheme $\mathrm{Op}_{\check{G}}^{\mathrm{ord}_n}(\mathcal{D})_{-\Lambda}$.

One can show that $Z(\mathfrak{g}_n) \simeq \mathbb{C}(\mathfrak{h}^*/W)_n$; therefore, the first statement of the above conjecture makes sense. We note that for $\theta \in \mathfrak{h}_n^*$, one can explicitly determine the action of $Z(\mathfrak{g}_n)$ on $V(\theta)$. Therefore, the above conjecture will, in particular, give us information about the action of $\mathscr{Z}$ on $\mathbb{V}(\theta)$. In the case $n=0$, this conjecture is a theorem of Frenkel [F07, §9].

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Koszul Categories and Mixed Hodge Modules

Sarah Kitchen

(joint work with Pramod Achar)

In this talk, I report on joint work with Pramod Achar. We considered the following problem: Let X be a smooth complex algebraic variety, and $\mathscr{S}$ a stratification of X by affine spaces.

Question: Can we generate a Koszul category from the category of mixed Hodge modules on X (constructible along $\mathscr{S}$) by a general procedure, which gives a grading on the category of $\mathscr{S}$-constructible $\mathbb{Q}$-perverse sheaves on X?

The motivation for this problem comes from [BGS]. In [BGS], the authors produce gradings for flag varieties stratified by Bruhat cells in ℓ-adic mixed perverse sheaves and in mixed Hodge modules by completely different methods. While the ℓ-adic construction was quite general, their construction for mixed Hodge modules relied heavily on the specific geometry of the Bruhat stratification. We wanted to understand why these cases had to be handled so differently.

The first part of the talk is used to define/recall the definitions of mixed and Koszul categories, and to introduce gradings as "mixed version" of a non-mixed artinian category. The second part of the talk is used to partially define mixed Hodge modules, but more importantly to isolate features of the category of mixed Hodge modules that are important for the main theorem. Two of those features are as follows. For X and $\mathscr{S}$ as above (in fact we need stricter conditions on $\mathscr{S}$ than just affineness, but I supress the precise details in the talk for the sake of time), and let X_s denote a stratum in $\mathscr{S}$. Let $\mathrm{MHM}_{\mathscr{S}}(X)$ be the category of $\mathscr{S}$-constructible mixed Hodge modules on X. We have:

- The irreducible objects of $\mathrm{MHM}_{\mathscr{S}}(X)$ are all Tate twists $\mathscr{L}_s(n)$ of the unique (up to isomorphism) weight $\dim X_s$ Hodge module $\mathscr{L}_s$ with the IC-sheaf for X_s as its underlying perverse sheaf.
- For M a mixed Hodge module of weight $\leq m$ and N an object of weight $\geq n$, we have $\mathrm{Ext}^i(M,N) = 0$ for all $m < n+i$.

For Koszulity, we would also need $\mathrm{Ext}^1(M,N) = 0$ for $m > n+1$ when M and N are simple.

The authors of [AR] introduce in their paper a general procedure which in our case indeed produces a Koszul category. If $\mathcal{M}$ is a mixed category and $\mathcal{M}_i$ denote the full subcategory of pure objects of weight i, then let

$$\mathcal{A} = \{X \in \mathrm{D^b}(\mathcal{M}) \mid \mathrm{H}^i(X) \in \mathcal{M}_i \ \forall i\}$$

be the category of pure complexes in $\mathrm{D^b}(\mathcal{M})$. From $\mathcal{A}$ we can build its homotopy category $\mathrm{K^b}(\mathcal{A})$ and a result of [AR] is that $\mathrm{K^b}(\mathcal{A})$ admits a t-structure with heart $\mathcal{M}^\diamond$, which we call the *winnowing* of $\mathcal{M}$, that is again a mixed abelian category. They also show that $\mathcal{M}^\diamond$ is Koszul when the realization functor $\mathrm{D^b}(\mathcal{M}^\diamond) \to \mathrm{K^b}(\mathcal{A})$ is an equivalence. In the category of pure complexes $\mathcal{A}$, cohomological degree and weight have essentially been identified. So, while the objects in $\mathcal{M}^\diamond$ are double complexes of objects in $\mathcal{M}$ that are supported along the diagonal, the point is that by winnowing, we have effectively spread the weights out with the cohomological degrees. This produces the desired ext-vanishing.

Applying this to $\mathcal{M} = \mathrm{MHM}_{\mathscr{S}}(X)$, we can state the main theorem of [AK]:

Theorem 0.1. (1) *There is a grading*

$$\mathrm{MHM}_{\mathscr{S}}(X)^\diamond \to \mathrm{Perv}_{\mathscr{S}}(X)$$

(2) *There exists a canonical exact faithful functor*

$$\beta : \mathrm{MHM}_{\mathscr{S}}(X) \to \mathrm{MHM}_{\mathscr{S}}(X)^\diamond$$

(3) *If X is a flag variety, $\mathscr{S}$ is the stratification by Bruhat cells, and $\mathrm{MHM}_{\mathscr{S}}(X)'$ is the category obtained in* [BGS], *the restriction of β to $\mathrm{MHM}_{\mathscr{S}}(X)'$ is an equivalence of categories.*

The construction of $\mathrm{MHM}_{\mathscr{S}}(X)'$ in [BGS] depended on non-canonical choices. That this could not be avoided in constructing a Koszul subcategory of $\mathrm{MHM}_{\mathscr{S}}(X)$ is reflected in this theorem by the fact that we can not find a canonical embedding from $\mathrm{MHM}_{\mathscr{S}}(X)^{\diamond}$ to $\mathrm{MHM}_{\mathscr{S}}(X)$.

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On the Geometric Hecke Algebra

Rahbar Virk

This is a report on work in progress [5]. We explain how several *a priori* disparate braid group actions on derived categories of sheaves are representations of the geometric Hecke algebra.

Let G be a connected reductive group over the complex numbers. Let $B \subseteq G$ be a Borel subgroup. The *geometric Hecke algebra*, denoted $\mathscr{H}$, is the B-equivariant derived category of mixed Hodge modules on the flag variety G/B. The *convolution* bifunctor $- \cdot -\colon \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ is defined by the formula

$$M \cdot N = m_!(M \mathbin{\widetilde{\boxtimes}} N),$$

where $M \mathbin{\widetilde{\boxtimes}} N$ denotes the equivariant descent of $M \boxtimes N$ to $G \overset{B}{\times} G/B$, and m is the map induced by the multiplication on G. This is an associative operation and endows $\mathscr{H}$ with a monoidal structure.

The B-orbits in G/B (Schubert cells) are parameterized by the Weyl group W. For each $w \in W$, let $i_w\colon X_w \hookrightarrow G/B$ denote the inclusion of the orbit corresponding to w. Let $\ell\colon W \to \mathbf{Z}_{\geq 0}$ denote the length function. Set $\mathbf{T}_w = i_{w!}\underline{X_w}[\ell(w)]$, where $\underline{X_w}$ is the constant sheaf on X_w.

Convolution endows the Grothendieck group of $\mathscr{H}$ with a ring structure. It is folklore that this ring is isomorphic to the Hecke algebra of W. In fact,

$$\text{if } \ell(ww') = \ell(w) + \ell(w'), \text{ then } \mathbf{T}_w \cdot \mathbf{T}_{w'} = \mathbf{T}_{\mathbf{ww'}}.$$

In other words, the $\mathbf{T}_w$, $w \in W$, satisfy the braid relations (at the categorical as well as Grothendieck group levels). We go a bit further and show that the $\mathbf{T}_w$ are invertible objects under convolution (the unit is given by $\mathbf{T}_e$). The proof of invertibility is essentially an SL_2 computation combined with the Artin-Grothendieck vanishing theorem for affine morphisms.

An $\mathscr{H}$*-action* or $\mathscr{H}$*-representation* on a category $\mathcal{C}$ is a monoidal functor from $\mathscr{H}$ to the category of endofunctors of $\mathcal{C}$. As the $\mathbf{T}_w$ are invertible, $\mathscr{H}$-actions give actions of the braid group of W, via auto-equivalences, on $\mathcal{C}$. We now outline a menagerie of such $\mathscr{H}$-actions.

Let X be a variety with G-action. A small variation of the formula for convolution defines an $\mathscr{H}$-action on the B-equivariant derived category of X. The case when X is a spherical variety is of particular interest in representation theory.

The G-equivariant derived category of $G/B \times G/B$ is equivalent to $\mathscr{H}$. In terms of the former category, convolution can be described by the usual formalism of convolution of kernels. That is, by the formula

$$M \cdot N = p_{13*}(p_{12}^* M \otimes p_{23}^* N)[-\dim G/B],$$

where $p_{??}\colon G/B \times G/B \times G/B \to G/B \times G/B$ denotes projection on the named factors. Let X be a variety. A variation of the above formula gives an $\mathscr{H}$-action on the derived category of mixed Hodge modules on $G/B \times X$.

Forgetting mixed structures, we also obtain $\mathscr{H}$-actions on ordinary (equivariant and non-equivariant) derived categories of sheaves. Via the Riemann-Hilbert correspondence, these transfer to the setting of D-modules. In this way, we recover the intertwining functors of [1].

We now sketch how to obtain $\mathscr{H}$-actions on categories of *coherent* sheaves. The preliminary nature of some of these results forces us to be a bit fuzzy. Consequently, until a published version of [5] appears, the statements that follow should be treated with a dose of skepticism.

Part of the data underlying a mixed Hodge module on a smooth variety X is a D-module on X endowed with a good filtration (the Hodge filtration). Taking the associated graded of this filtered D-module defines a functor from mixed Hodge modules to $\mathbf{C}^*$-equivariant coherent sheaves on the cotangent bundle T^*X. Specializing to $X = G/B$, in which case $T^*(G/B) = \tilde{\mathcal{N}}$ (the enhanced nilpotent cone, à la the Springer resolution), this functor was exploited, at the level of Grothendieck groups, by T. Tanisaki [4]. We extend the results of [4] to the categorical level, and obtain an $\mathscr{H}$-action on $G \times \mathbf{C}^*$-equivariant coherent sheaves on $\tilde{\mathcal{N}}$. This extends the braid group actions of M. Khovanov and R. Thomas [2] to arbitrary type. Further, this action coincides (modulo minor 'twists') with the braid group actions constructed by R. Bezrukavnikov and S. Riche [3]. However, the results of [3] hold over arbitrary fields, ours are only over the complex numbers. This is perhaps indicative that our definition of $\mathscr{H}$ is itself just the shadow of a more fundamental motivic construct.

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Generalization of the Macdonald formula for Hall-Littlewood polynomials

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The symmetric Hall-Littlewood polynomials $P_\lambda(x,q)$ have an intrinsic meaning in combinatorial representation theory generalizing other important families of symmetric functions i.e. the monomial symmetric functions and the Schur functions. Originally P. Hall defined the Hall-Littlewood polynomials for type A_n as a family of symmetric functions associated to certain elements in the Hall algebra. Later, Littlewood defined them explicitly in terms of the Weyl group W and a coweight lattice $X^\vee$ for type A_n [Li]. This formula led to defining Hall-Littlewood polynomials of arbitrary type by replacing W and $X^\vee$ in Littlewood's definition by a Weyl group and a coweight lattice of arbitrary type. These polynomials coincide with the so-called Macdonald spherical functions [Mac2], thus both names appear in the literature denoting the same objects.
There are various explicit combinatorial formulas for the Hall-Littlewood polynomials proven by Gaussent-Littelmann, Macdonald, Lenart, Schwer, Haiman-Haglund-Loehr [GL1] [Mac1],[L1], [L2], [S], [HHL] to name only a few. The first and probably most famous combinatorial formula, the Macdonald formula, is exclusively for type A_n. This formula is in terms of Young tableaux of type A_n. Most recently, Gaussent-Littelmann developed a formula for Hall-Littlewood polynomials for arbitrary type as sum over positively folded one-skeleton galleries in the standard apartment of the affine building. This formula has a geometric background which relates it closely to the Schwer formula which is a sum over positively folded alcove galleries in the standard apartment of the affine building. Let us explain the geometric background and their connection more precisely:
Express a given Hall-Littlewood polynomial $P_\lambda(x,q)$ of arbitrary type in the monomial basis $\{m_\mu(x)\}_{\mu\in X^\vee_+}$:

$$P_\lambda(x,q) = \sum_{\mu\in X^\vee_+} q^{-\langle\lambda+\mu,\rho\rangle} L_{\lambda,\mu}(q) m_\mu(x)$$

with $L_{\lambda,\mu}(q) \in \mathbb{Z}[q]$.
The Satake isomorphism yields that the Laurent polynomial $L_{\lambda,\mu}(q)$ can be calculated by counting points in a certain intersection of orbits in an affine Grassmannian depending on the coweights λ and μ over a finite field $\mathbb{F}_q$. Both, Schwer and Gaussent-Littelmann use this approach by describing the elements in this intersection with galleries in the standard apartment of the affine building, namely Gaussent-Littelmann use one-skeleton galleries whereas Schwer uses alcove galleries. In geometric terms using different types of galleries results from choosing different Bott-Samelson type varieties. Gaussent-Littelmann refer to this connection between the formulas as "geometric compression". One major advantage of using one-skeleton galleries instead of alcove galleries is that there is a one-to-one correspondence between the positively folded one-skeleton galleries of type λ and target μ for some dominant coweights λ and μ and the semistandard Young tableaux of shape λ and content μ, for classical types. This correspondence leads to the question whether it is possible to calculate the contribution to the Gaussent-Littelmann formula of a positively folded one-skeleton gallery δ directly from the associated semistandard Young tableau T_δ. We give a positive answer to this question by developing the so-called combinatorial Gaussent-Littelmann formula. The key ingredient for the proof of this formula is a recurrence for a certain set of positively folded galleries of chambers in the standard apartment of the residue building that appears in the Gaussent-Littelmann formula.
It turns out that the Macdonald formula and the combinatorial Gaussent-Littelmann formula coincide for type A_n. In fact, the Macdonald formula is a closed formula for the recursively defined combinatorial Gaussent-Littelmann formula. Apparently, the first indicator for the equality of the two formulas is that they are both sums over semistandard Young tableaux. Further, in the combinatorial Gaussent-Littelmann formula the contribution of a semistandard Young tableau is a product of contributions of the columns of the tableau. These contributions only depend on the column itself and, if existing, on the column to the right. Reformulating the Macdonald formula reveals this property in the formula, too, although it is not at all obvious at first glance. We prove the equality of the two formulas by showing that the contribution of every column is the same.
Since the Macdonald formula is valid only for type A_n the formula of Gaussent-Littelmann generalizes the Macdonald formula and provides it with a geometric background.

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Spherical Hecke algebras for Kac-Moody groups

Stephane Gaussent
(joint work with Guy Rousseau)

Let G be a connected reductive group over a local non-archimedean field $\mathcal{K}$ and let $K \subset G$ be an open compact subgroup. The space $\mathcal{H}$ of complex functions on G, bi-invariant by K and with compact support is an algebra for the natural convolution product. Ichiro Satake [Sa63] studied this algebra $\mathcal{H}$ to define the spherical functions and proved, in particular, that $\mathcal{H}$ is commutative for good choices of K. We know now that one of the good choices for K is the fixator of some special vertex for the action of G on its Bruhat-Tits building $\mathscr{I}$, whose structure is explained in [BrT72]. Moreover $\mathcal{H}$, now called the spherical Hecke algebra, may be entirely defined with $\mathscr{I}$, see [P06].

Kac-Moody groups are interesting generalizations of reductive groups and it is natural to try to generalize the spherical Hecke algebra to the case of a Kac-Moody group. But there is now no good topology on G and no good compact subgroup, so the "convolution product" has to be defined only with algebraic means. Alexander Braverman and David Kazhdan [BrK10] succeeded in defining such a spherical Hecke algebra, when G is split and untwisted affine. For a well chosen subgroup K, they define $\mathcal{H}$ as an algebra of $K-$bi-invariant complex functions with "almost finite" support. There are two new features: the support has to be in a subsemigroup G^+ of G and it is an infinite union of double classes. Hence, $\mathcal{H}$ is naturally a module over the ring of complex Laurent polynomials.

1. Definition

So, let G be an almost split Kac-Moody group over a local non-archimedean field $\mathcal{K}$. Our idea is to build this spherical Hecke algebra using the hovel associated to G that we built in [GR08], [Ro12] and [Ro13].

The hovel $\mathscr{I}$ is a set with an action of G and a covering by subsets called apartments, in one-to-one correspondence with the maximal split subtori, hence permuted transitively by G.

Each apartment A is a finite dimensional real affine space and its stabilizer N in G acts on it via a generalized affine Weyl group $W = W^v \ltimes Y$, where $Y \subset \overrightarrow{\mathbb{A}}$ is a discrete subgroup of translations in a chosen apartment. This group stabilizes a set $\mathcal{M}$ of affine hyperplanes called walls.

Even though $\mathscr{I}$ looks much like the Bruhat-Tits building of a reductive group, there are two big differences. The set $\mathcal{M}$ is not a locally finite system of hyperplanes anymore (as the root system Φ is infinite) and two points in $\mathscr{I}$ are not always in a same apartment, the Cartan decomposition fails $G \neq KNK$ (this is why $\mathscr{I}$ is called a hovel). But on $\mathscr{I}$, we can show that there is a $G-$invariant preorder $\leq$ which induces on each apartment A the preorder given by the Tits cone $\mathcal{T} \subset \overrightarrow{A}$. Two points x and y such that $x \leq y$ are in a same apartment $A = g \cdot \mathbb{A}$, for $g \in G$. There also exists a vectorial distance $d^v : \mathscr{I} \times_{\leq} \mathscr{I} \to C_f^v$ taking values in the fundamental Weyl chamber and defined by $d^v(x,y) = (g^{-1}y - g^{-1}x)^{++}$, where $(z)^{++}$ means the projection of $z \in \mathcal{T}$ onto $\overline{C_f^v}$.

Now, we consider the fixator K in G of a special point 0 in $\mathbb{A}$. Then the set

$$G^+ = \{g \in G \mid 0 \leq g.0\}$$

is a semigroup, and $K \backslash G^+/K$ is in bijection with the subsemigroup $Y^{++} = Y \cap C_f^v$ of Y. Given a ring R, a function $\phi : G^+/K \to R$ is said to have an almost finite support if $supp(\phi) \subset \bigcup_{i=1}^n (\lambda_i - Q_+^\vee) \cap Y^{++}$, where $\lambda_i \in Y^{++}$ and $Q_+^\vee$ is the subsemigroup of Y generated by the fundamental coroots $(\alpha_i^\vee)$. Note that $(\lambda - Q_+^\vee) \cap Y^{++}$ is infinite except when G is reductive.

The spherical Hecke algebra $\mathcal{H}_R$ is the space of $K-$invariant functions on G^+/K with values in a ring R and almost finite support. The convolution product is defined by

$$(\phi * \psi)(z) = \sum_{0 \leq x \leq z} \phi(x)\psi(d^v(x,z)).$$

2. Results

Theorem 6. *Assume that $(\alpha_i^\vee)$ is free in $Y \otimes \mathbb{R}$. Then for any ϕ and ψ in $\mathcal{H}_R$, $\phi * \psi$ is well defined and belongs to $\mathcal{H}_R$. Hence $\mathcal{H}_R$ is an algebra.*

The structure constants of $\mathcal{H}_R$ are the non-negative integers $m_{\lambda,\mu}(\nu)$ (for $\lambda, \mu, \nu \in Y^{++}$) such that

$$c_\lambda * c_\mu = \sum_{\nu \in Y^{++}} m_{\lambda,\mu}(\nu) c_\nu,$$

where c_λ is the characteristic function of $K\lambda K$. Each chamber (= alcove) in $\mathscr{I}$ has only a finite number of adjacent chambers along a given panel. These numbers are called parameters of $\mathscr{I}$ and their set $\mathcal{Q}$ is finite; in the split case, there is only one parameter q: the number of elements of the residue field κ of $\mathcal{K}$.

Theorem 7. *The structure constants $m_{\lambda,\mu}(\nu)$ are polynomials in these parameters with integral coefficients depending only on the geometry of an apartment.*

Suppose now that the group G is split and still that $(\alpha_i^\vee)$ is free in $Y \otimes \mathbb{R}$, then we can show:

Theorem 8. *There exists an involution θ on G such that*

$$\theta(t) = t^{-1}, \forall t \in \mathcal{T}, \quad \theta(K) \subset K.$$

The spherical Hecka algebra $\mathcal{H}_R$ is commutative.

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Categorification of finite dimensional modules over orthosymplectic Lie superalgebras

Caroline Gruson

(joint work with Vera Serganova)

This is a joint work with Vera Serganova (UC Berkeley), see C. Gruson, V. Serganova, *Bernstein-Gel'fand-Gel'fand reciprocity and indecomposable projective modules for classical algebraic supergroups*, available on Arxiv 0370570, submitted.

Consider an orthosymplectic Lie superalgebra $\mathfrak{g} = \mathfrak{osp}(2m+1, 2n)$.

It is well known that the category $\mathcal{F}$ of finite dimensional $\mathfrak{g}$-modules is not semi-simple. This category splits into blocks, and there is a family of *translation functors* which allow to go from one block to another one.

In this work, we describe a way to re-interpret certain translation functors as linear operators (matrices in $\mathfrak{gl}(\infty)$) acting on a specific representation of $\mathfrak{gl}(\infty)$, namely if V is the standard module of $\mathfrak{gl}(\infty)$ and V^* is the costandard, we make $\mathfrak{gl}(\infty)$ act on $\Lambda^m(V^*) \otimes \Lambda^n(V)$.

We use this description to express indecomposable projective modules of $\mathcal{F}$ in terms of the *standard modules*. Actually, the standard modules in this case are obtained as virtual modules in the Grothendieck group of $\mathcal{F}$. Let me explain who they are: simple modules in $\mathcal{F}$ have a highest weight (with respect to a subalgebra $\mathfrak{b}$ of $\mathfrak{g}$.

Consider the flag supervariety G/B, where $G = SOSP(2m+1, 2n)$ and B is the sub-supergroup of G with Lie superalgebra $\mathfrak{b}$, and construct a line bundle $\mathcal{L}_\lambda$ on it coming from the character of B which corresponds to the highest weight.

The cohomology $H^*(G/B, \mathcal{L}_\lambda^*)^*$ is equipped with a structure of $\mathfrak{g}$-module and we define the *Euler characteristic* to be the alternate sum of those cohomology groups in the Grothendieck group.

Those virtual modules generate a proper subgroup of the Grothendieck group of $\mathcal{F}$, and the Euler characteristics are in one-to-one correspondence with a well-chosen basis of the $\mathfrak{gl}(\infty)$-module $\Lambda^m(V^*) \otimes \Lambda^n(V)$.

We describe an algorithm expressing an indecomposable projective modules of $\mathcal{F}$ as a linear combination with (non necessarily positive) integral coefficients of Euler characteristics.

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Affine W-algebras and quiver varieties

Eric Vasserot

Given a connected reductive group G over $\mathbb{C}$, Kazhdan-Lusztig and Ginzburg have proved that the affine Hecke algebra of G is isomorphism to the Grothendieck group of the category of $G \times \mathbb{C}^\times$-equivariant coherent sheaves over the Steinberg variety of G. Similarly, one can associate to a quiver Γ with a (finite) set of vertices I and a (finite) set of arrows H a symmetrizable Kac-Moody algebra $\mathfrak{g}_\Gamma$ and a quiver variety. The quiver variety depends of the choice of a dimension vector $w \in \mathbb{N}I$. It is a quasi-projective variety M_w which is equipped with an algebraic action of the group $G_w = \prod_{i \in I} GL(w_i)$. Nakajima has proved that there is an algebra homomorphism ϕ_w from the quantized enveloping algebra $U_q(L\mathfrak{g}_\Gamma)$ of the loop algebra of $\mathfrak{g}_\Gamma$ to the Grothendieck group $K^{G_w \times \mathbb{C}^\times}(Z_w)$ of the category of $G_w \times \mathbb{C}^\times$-equivariant coherent sheaves over a Steinberg-type variety Z_w associated with M_w. In general, the map ϕ_w is neither injective nor surjective. It is an important question to understand which algebra maps indeed to $K^{G_w \times \mathbb{C}^\times}(Z_w)$. When the quiver is not of finite type some new algebras are involved. In this talk we consider the particular where Γ consists of one vertex an one loops.

In this case w is simply an integer $r \geqslant 0$ and the corresponding quiver variety M_r is the moduli space of torsion free sheaves of rank r over $\mathbb{P}^2$ with a trivialisation over a fixed line $\mathbb{P}^1 \subset \mathbb{P}^2$. Further, instead of the equivariant K-theory of M_r we'll consider the equivariant cohomology group $H^D(M_r)$ where D is the torus $(\mathbb{C}^\times)^{r+2}$. The D-action on M_r comes from the $(\mathbb{C}^\times)^2$-action $\mathbb{P}^2$ and the $(\mathbb{C}^\times)^r$-action on the trivialization. Recall that the space $H^D(M_r)$ is a module over the algebra

$$H^D(point) = R_r = \mathbb{C}[x, y, e_1, \dots, e_r],$$

where x, y come from the characters of the $(\mathbb{C}^\times)^2$-action and $e_1, \dots, e_r$ come from the characters of the $(\mathbb{C}^\times)^r$-action. Set

$$\vec{e} = (e_1, \dots, e_r), \qquad \kappa = -y/x, \qquad \mathbf{L}_K^r = H^D(M_r) \otimes_{R_r} K_r,$$

where K_r is the fraction field of R_r. Our aim is to prove the following theorem, known as (a version of the) AGT conjecture.

Theorem *(a) There is a representation of the affine W-algebra $W_k(\mathfrak{gl}_r)$ of level $k = \kappa - r$ on $\mathbf{L}_K^r$, identifying it with the Verma module of highest weight*

$$\beta = \vec{e}/y - \xi\rho/\kappa.$$

(b) This action is quasi-unitary with respect to the intersection pairing on $\mathbf{L}_K^r$.

(c) The Gaiotto state $G = \sum_{n \geqslant 0} G_n$*, with $G_n = [M_{r,n}]$, is a Whittaker vector of this Verma module.*

The proof consists of three steps. First we define a new algebra SH. It should be regared as the Yangian of an Heisenberg algebra of rank one, i.e., of the algebra generated by elements c, b_n, $n \in \mathbb{Z}$, satisfying the relations

$$[b_n, b_m] = n\delta_{n+m,0}\, c.$$

We do not give an explicit presentation of SH. We define it as a central extension of the limit of a projective system of algebras built using Cherednik's degenerate double affine Hecke algebras of $GL(N)$, as N goes to ∞. Next we prove that SH acts on $\mathbf{L}_K^r$. This part of the argument is rather standard. It uses some correspondences on the quiver variety. The algebra SH is graded. We define its degreewise completion $U(SH)$ to be the projective limit of algebras

$$U(SH) = \bigoplus_{s \in \mathbb{Z}} U(SH)[s], \qquad U(SH)[s] = \varprojlim_N SH[s]/\mathscr{I}_N[s]$$

$$\mathscr{I}_N[s] = \sum_{t \geqslant N} SH[t-s]\, SH[-t].$$

The final step consists to prove that $U(SH)$ is isomorphic to the current algebra $U(W_k(\mathfrak{gl}_r))$ of $W_k(\mathfrak{gl}_r)$. In order to prove this an important step is to prove that SH is a topological Hopf algebra. This means that there is a comultiplication

$$SH \to SH \widehat{\otimes} SH,$$

where $\widehat{\otimes}$ is a completion of the tensor product. We do not know how to construct this coproduct by elementary methods. Our argument uses our previous works. There we studied an algebra **SH** similar algebra to SH, which acts on the equivariant K-theory group of M_r. Further, we proved that **SH** is closely related to the Hall algebra of the category of coherent sheaves over an elliptic curve. This automatically implies that **SH** is equipped with a topological comultiplication Δ. Then, by a degeneration argument, we construct the comultiplication of SH using this Δ. Once this comultiplication is given, it is possible to compare $U(SH)$ with $U(W_k(\mathfrak{gl}_r))$ using Feigin-Frenkel's theorem, which characterizes $W_k(\mathfrak{gl}_r)$ as the kernel of some *screening* operators in the Fock space of a rank r Heisenberg algebra. Okounkov and Maulik have obtain another proof of the theorem above. They use technics from symplectic geometry.

Zhelobenko Invariants and the Kostant Clifford Algebra Conjecture

Anthony Joseph

We summarize results in [4], [5] as presented in Oberwolfach on Friday 9 March 2012.

1. The Conjectures

1.1. **The Clifford Algebra Conjecture.** Let $\mathfrak{g}$ be a complex simple Lie algebra and $C(\mathfrak{g})$ its Clifford algebra. Through triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, the latter admits a Harish-Chandra map ϕ onto the Clifford algebra $C(\mathfrak{h})$ of $\mathfrak{h}$. Set $\ell = \dim \mathfrak{h}$. Let $m_i : i = 1, 2, \ldots, \ell$, be the exponents of $\mathfrak{g}$ taken in increasing order. Let $\pi = \{\alpha_i\}_{i=1}^{\ell}$ be the choice of simple roots relative to the above triangular decomposition.

In about 1997, Kostant suggested to Y. Bazlov a rather precise conjecture as to the form ϕ should take when restricted to a space of P of primitive invariants of $C(\mathfrak{g})$. Let $p_i : i = 1, 2, \ldots, \ell$ be an orthogonal basis of P with p_i of degree $2m_i + 1$, specifically as described in [7, Thm. 85]. Identify $\mathfrak{h}$ with $\mathfrak{h}^*$ through the Killing form and then with the Cartan subalgebra $\mathfrak{h}^\vee$ of the Langlands dual $\mathfrak{g}^\vee$. Let $e^\vee, h^\vee, f^\vee$ be a principal s-triple for $\mathfrak{g}^\vee$ with $h^\vee \in \mathfrak{h}^\vee$ and $e^\vee$ the sum of simple coroot vectors. One may remark that $h^\vee$ identifies with the half-sum ρ of the positive roots.

The Kostant Clifford algebra conjecture states that $\phi(p_i)$ is a zero weight vector in a simple module for the above s-triple of dimension $2m_i + 1$. It was settled by Bazlov for $\mathfrak{g}$ of type A in his thesis [1].

1.2. **Reduction to an Enveloping Algebra Conjecture.** Let $U(\mathfrak{g})$ its enveloping algebra of $\mathfrak{g}$. Recall that the Harish-Chandra map φ is the projection of $U(\mathfrak{g})$ onto $U(\mathfrak{h})$ defined by the direct sum decomposition $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+)$.

Let $\mathscr{F}$ denote the canonical filtration on $U(\mathfrak{g})$. Let φ_λ denote the composition of φ with evaluation at $\lambda \in \mathfrak{h}^*$. Let Φ^- (resp Φ^-_λ) be the map of $\mathfrak{g} \otimes U(\mathfrak{g})$ onto $\mathfrak{g} \otimes S(\mathfrak{h})$ (resp. $\mathfrak{g}$) defined by applying the identity map to the first factor and φ (resp. φ_λ) to the second factor. Using [7, Thm. 89] the Kostant Clifford algebra conjecture easily reduces to showing that

$$e^{\vee(m+1)} \Phi^-_\rho (\mathfrak{g} \otimes \mathscr{F}^m U(\mathfrak{g}))^{\mathfrak{g}} = 0. \qquad (1)$$

1.3. **A Symmetric Algebra Result.** There is a symmetric algebra version of (1) noted in the thesis work of R. Rohr [9]. In this φ is replaced by the Chevalley restriction map $\varphi^0 : S(\mathfrak{g}) \to S(\mathfrak{h})$. Then Φ^0 and Φ^0_ρ are defined by replacing φ by φ^0. Rohr's result states that

$$e^{\vee(m+1)} \Phi^0_\rho (\mathfrak{g} \otimes \mathscr{F}^m S(\mathfrak{g}))^{\mathfrak{g}} = 0. \qquad (2)$$

The proof is rather easy at least for $\mathfrak{g}$ simply-laced. Moreover in the latter case it immediately extends to the adjoint module $\mathfrak{g}$ being replaced by any finite dimensional $\mathfrak{g}$ module V. By contrast the corresponding assertion fails for (1) when V is arbitrary. Thus the proof of (1) cannot be expected to follow by formal arguments and to be too easy.

1.4. **An Analogue Enveloping Algebra Conjecture.** Let V be a finite dimensional $\mathfrak{g}$ module and V_0 its zero weight subspace. Consider $V \otimes U(\mathfrak{g})$ as a $U(\mathfrak{g})$ bimodule through the rule

$$x(v \otimes a)y = xv \otimes ay + v \otimes xay, \forall x, y \in \mathfrak{g}, v \in V, a \in U(\mathfrak{g}).$$

Let Φ^+ denote the projection onto V defined by the direct sum decomposition $V \otimes U(\mathfrak{g}) = V \otimes U(\mathfrak{h}) \oplus (\mathfrak{n}^-(V \otimes U(\mathfrak{g})) + (V \otimes U(\mathfrak{g}))\mathfrak{n}^+)$. One may remark that Φ^- obtains from Φ^+ by treating V as the trivial $\mathfrak{g}$ module in the above. Analogous to (1) one can ask if

$$e^{\vee(m+1)}\Phi^+_\rho(\mathfrak{g} \otimes \mathscr{F}^m U(\mathfrak{g}))^{\mathfrak{g}} = 0. \tag{3}$$

2. Method of Proof

2.1. **A Slight Generalization.** The argument in the proof of (2) is based on the vanishing of the double commutator $[e^\vee, [e^\vee, \rho]]$. Thus it is immediate that it holds for all multiples of ρ. We shall use this fact to show that (1) and (3) hold for all multiples of ρ.

2.2. **The Zhelobenko Operators.** Zhelobenko [10] introduced a family of Ξ^+ of operators which act like the identity on invariants and factor through the projection onto $V \otimes U(\mathfrak{h})$ defined by Φ^+. Thus $\Phi^+(V \otimes U(\mathfrak{g}))^{\mathfrak{g}} \subset (V_0 \otimes S(\mathfrak{h}))^{\Xi^+}$. Rather recently Khoroshkin, Nazarov and Vinberg [6] showed that equality holds. This is a key ingredient in our proof of (3).

2.3. **The Analogue Zhelobenko Operators.** Seemingly it is not possible to define operators which similarly factor through the projection defined by Φ^-, yet one may define directly on its image, a set of operators Ξ^- with the property that $\Phi^-(V \otimes U(\mathfrak{g}))^{\mathfrak{g}} \subset (V_0 \otimes S(\mathfrak{h}))^{\Xi^-}$, and furthermore that equality holds. Even more surprisingly the two sets of operators defined here and in 2.2 are given by different yet almost identical formulae. This has the consequence that the proof of (1) follows closely the proof of (3).

2.4. **A Key Lemma.** The formulae for the Zhelobenko operators and their analogues, only take a pleasant form with respect to a basis of V_0 specific to the particular element of Ξ^+ (or of Ξ^-) in question. However in the case when V is the adjoint module, these formulae are sufficiently simple to make calculations possible. In particular there is a natural basis of $V_0 = \mathfrak{h}$ in which to express these invariants, namely that given by the fundamental weights $\varpi_i : i = 1, 2, \dots, \ell$. Indeed taking $\varepsilon = +1, 0, -1$ according to the three images occurring in equations

(1) − (3) we may write an invariant in the form

$$\sum_{i=1}^{\ell}(\varpi_i \otimes q_i^\varepsilon).$$

We remark that for $\varepsilon = 0$, the expression is an invariant under the Weyl group W and can be viewed as the differential of a homogeneous element of $S(\mathfrak{h})^W$. This implies that q_i^0 is antisymmetric with respect to the simple reflection s_i, hence divisible by the simple coroot $\alpha_i^\vee$ with the quotient $P_i^0 = q_i^0/\alpha_i^\vee$ being s_i invariant.

A similar reduction is possible when $\varepsilon = \pm 1$ except that there is a translated action of the Weyl group. To remove the effect of this translation one introduces the automorphism θ of $S(\mathfrak{h})$ defined by $\theta(q)(\lambda) = q(\lambda + \rho), \forall \lambda \in \mathfrak{h}^*$. Then setting $Q_i^\varepsilon = \theta^{-1}(q_i^\varepsilon)$ one obtains that Q_i^ε is divisible by $\varepsilon + \alpha_i^\vee$ and that the quotient $P_i^\varepsilon = Q_i^\varepsilon/(\varepsilon + \alpha_i^\vee)$ is s_i invariant.

Define the linear operator A_i on $S(\mathfrak{h})$ by $A_i f = \frac{f - s_i f}{\alpha_i^\vee}$. Remarkably

Lemma 9. *For all $i, j = 1, 2, \ldots, \ell$, $\varepsilon = +1, 0, -1$, one has*

$$A_i P_j^\varepsilon = \alpha_j^\vee(\alpha_i)\frac{P_i^\varepsilon - P_j^\varepsilon}{\varepsilon + s_i(\alpha_j^\vee)}.$$

3. The BGG and Zhelobenko Monoids

3.1. **The Monoids.** The above lemma allows one to compare the families $\{P_i^\varepsilon\}_{i=1}^\ell$ and then to deduce (1) and (3) from (2). The operators $A_i : i = 1, 2, \ldots, \ell$, were introduced by Bernstein, Gelfand and Gelfand [2]. They have square zero and satisfy the braid relations, hence generate a monoid $\mathbf{A}$ which naturally identifies with W as a set and which we call the BGG monoid. The action of $\mathbf{A}$ on any family $\{P_i^\varepsilon\}_{i=1}^\ell$ generates a module $\mathbf{P}^\varepsilon$ for $\mathbf{A}$ which we call the Zhelobenko monoid. (Its structure is independent of ε.)

3.2. **The Structure Constants for the Langlands dual.** Let Δ (resp. $\Delta^\vee$) denote the set of roots (resp. coroots) defined with respect to the pair $(\mathfrak{g}, \mathfrak{h})$. Set $\Delta^+ = \Delta \cap \mathbb{N}\pi$, Let $x_{\gamma^\vee} : \gamma^\vee \in \Delta^\vee$ denote (part of) a Chevalley basis for the Langlands dual $\mathfrak{g}^\vee$. Define (as usual) the structure constants by $[x_{\alpha^\vee}, x_{\beta^\vee}] = N_{\alpha^\vee,\beta^\vee} x_{\alpha^\vee+\beta^\vee}$, whenever $\alpha^\vee, \beta^\vee, \alpha^\vee + \beta^\vee$ are all non-zero coroots, setting $N_{\alpha^\vee,\beta^\vee} = 0$, when $\alpha^\vee + \beta^\vee$ is not a coroot.

3.3. **A Key Proposition.** If Δ is simply-laced, then it admits a natural identification with $\mathbf{P}^\varepsilon$. Otherwise we construct an injection $\gamma^\vee \mapsto P_{\gamma^\vee}^\varepsilon$ of $\Delta^\vee$ into $\mathbf{P}^\varepsilon$ satisfying in particular $P_i^\varepsilon = P_{\alpha_i^\vee}^\varepsilon : i = 1, 2, \ldots, \ell$. Under appropriate further choices one obtains the following

Proposition 3.1. *For all $\gamma \in \Delta^+ \setminus \pi$ one has* $P_{\gamma^\vee}^\varepsilon = \frac{\sum_{\alpha \in \pi} N_{-\alpha^\vee,\gamma^\vee} P_{\gamma^\vee - \alpha^\vee}^\varepsilon}{\varepsilon + \gamma^\vee}$.

4. Completion of Proof

4.1. **The Principal Filtration on the Cartan subalgebra.** Identify $\mathfrak{h}$ with $\mathfrak{h}^\vee$ as in 1.1 and set

$$\mathscr{F}^m(\mathfrak{h}) = \{h \in \mathfrak{h} | e^{\vee(m+1)}h = 0\}.$$

4.2. **Main Result.** To deduce (1) and (3) from (2) we need to show that the ℓ-tuples $\{P_i^\varepsilon(\rho)\}_{i=1}^\ell$ are proportional for $\varepsilon = +1, 0, -1$. We generalize this according to 2.1. Fix $s \in \mathbb{C}$. A key point is that the ratios of the $\varepsilon + \gamma^\vee(s\rho)$ are constant for coroots of fixed height. Then the desired conclusion obtains by reverse induction on coroot height using Proposition 3.3. However here one has to take account of the fact that the matrix changing root height by one involving the structure constants is not invertible. Yet this is exactly the same matrix defined by the action of $f^\vee$ on the space $\bigoplus_{\gamma \in \Delta^{\vee+}} \mathbb{C}x_{\gamma^\vee}$. Now the cokernel of the latter determines the exponents of $\mathfrak{g}^\vee$ (equivalently of $\mathfrak{g}$) and so at each step one can introduce lower order terms coming from lower order invariants to ensure that the reverse induction proceeds. In this argument one must also avoid values of s for which the $\varepsilon + \gamma^\vee(s\rho)$ vanish. However polynomiality in s implies that vanishing is a closed condition and hence holds for all $s \in \mathbb{C}$.

On the other hand to ensure that the exponents in the left hand sides of (1) and (3) are the smallest possible for vanishing we must exclude the zeros of the corresponding determinants [8] and [3, 3.3, 3.6]. These are given by the sets $F_0^\pm = \{s \in \mathbb{C} | \pm 1 + (s+1)\gamma^\vee(\rho) = 0 : \gamma^\vee \in \Delta^{\vee+}\}$. Combining the above we obtain the

Theorem 4.1. *For all $s \in \mathbb{C} \setminus F_0^\pm$, one has*

$$\Phi_{s\rho}^\pm((\mathfrak{g} \otimes \mathscr{F}^m(U(\mathfrak{g}))^\mathfrak{g}) = \mathscr{F}^m\mathfrak{h}.$$

This settles the Kostant Clifford Algebra conjecture as well as (1) and (3) in a rather precise form.

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Problem Session

Question by Shrawan Kumar
Let $V_m := \mathbb{C}^m$. For $j \leq m$ $L_j := S^j(S^m(V_m))$ is acted upon by $SL_m(\mathbb{C})$ in the obvious way. Let $e_1, \dots, e_m$ be a basis of V_m and

$$v_0 = e_1 \otimes \cdots \otimes e_n \in S^m(V_m)$$

does $v^j = v_0 \otimes \cdots \otimes v_0$ generate L^j as a $SL_m(\mathbb{C})$-module?
Question by Ben Webster
Let G be a reductive (semi-simple or simple) complex group. As usual define $G[[t]] = G(\mathbb{C}[[t]])$ and $G((t)) = G(\mathbb{C}[[t]])$ One defines

$$G_1[t^{-1}] := \ker(ev(t^{-1} = 0) : G[[t]] \to G)$$

where t^{-1} is regarded as a formal variable. Clearly $G_1[t^{-1}]$ acts on the affine Grassmannian $Gr = G((t))/G[[t]]$ from the left. Set $e = t^0$,

$$W^\lambda = G_1[t^{-1}][e] \cap \overline{G[[t]][t^\lambda]}$$

How to describe

$$I(W^\lambda) \subset \mathbb{C}[G_1[t^{-1}][e]]$$

Question by Michael Duflo
Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a super Lie algebra over $\mathbb{C}$ with universal enveloping algebra $U(\mathfrak{g})$. *When is $U(\mathfrak{g})$ prime?*
Question by Daniel Juteau
Let G a complex reductive group with maximal torus T and Weyl group W. Then W operates on G/T without fixed points. The induced representation

$$W \looparrowright H^*(G/T; \mathbb{C})$$

is a graded version of the regular representation. Now let $\Bbbk$ be any field, then $R\Gamma(G/T, \Bbbk)$ is a perfect complex of $\Bbbk[W]$-modules. *Is it possible to give a desciption (as above) of this complex in general?*
Question by Ghislain Fourier
Let A be a finitely generated associative algebra over $\mathbb{C}$ and S_m the m-th symmetric group. Let S_m act on $A^{\otimes m}$ in the obvious way. *When does*

$$(A^{\otimes m})^{S_m} \cong \mathbb{C}[x_1, \dots, x_m]$$

hold?
Question by Carl Mautner
Let G be a reductive group and Gr the corresponding affine Grassmannian stratified by $G[[t]]$-orbits and $\mathscr{E}(\lambda, \Bbbk)$ the parity sheaf corresponding to $G[[t]]$-orbit Gr^λ. It is known that $\mathscr{E}(\lambda, \Bbbk)$ is a tilting sheaf if *char* $\Bbbk > n + 1$, where n is the Coxeter number. *For which characteristics is ${}^pH^*(\mathscr{E}(\lambda, \Bbbk))$ a tilting sheaf?*
Question by Claudio Procesi

Let A be an associative algebra of a field of characteristic zero. Assume there exists an $n \in \mathbb{N}$ such that

$$a^n = 0 \text{ for all } a \in A$$

Is it possible to show that

$$A^{\binom{n+1}{2}} = 0$$

Reporter: Oliver Straser

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