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Mini-Workshop: Hypergraph Turán Problem

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Abstract. This mini-workshop focused on the hypergraph Turán problem. The interest in this difficult and old area was recently re-invigorated by many important developments such as the hypergraph regularity lemmas, flag algebras, and stability. The purpose of this meeting was to bring together experts in this field as well as promising young mathematicians to share expertise and initiate new collaborative projects.

Mathematics Subject Classification (2000): 05D05, 05C35.

Introduction by the Organisers

The mini-workshop *Hypergraph Turán Problem*, organised by Penny Haxell (Waterloo), Dhruv Mubayi (Chicago), Oleg Pikhurko (Coventry), and Tibor Szabó (Berlin) was held 8–14 April 2012. This meeting was attended by 17 participants from 6 different countries. The purpose of the mini-workshop was to bring together researchers of different backgrounds and seniority so that they can communicate about recent developments, share their expertise, and continue or initiate collaborative projects. In particular, the organizers invited quite a few researchers who are early in their careers; for example, for 4 participants it was the first time that they were at the MFO.

The schedule was designed to give the participants ample free time for collaboration and discussions. There were 15 talks in total (in mornings) and a problem session (on Monday afternoon). The format of talks varied from a general introduction to some important aspect (such as the 2-part lecture by John Talbot on flag algebras) to a short communication of a recent result.

The hypergraph Turán problem is about 70 years old. The basic question here is to estimate the *Turán function* $\mathrm{ex}(n, F)$ which is the maximum number of edges in a hypergraph G on n vertices that does not contain the given forbidden k-graph F. This fundamental problem, relating global and local parameters, is notoriously difficult and wide open in general. For example, the famous conjecture of Turán that $\mathrm{ex}(n, K_4^3) = (\frac{5}{9} + o(1))\binom{n}{3}$, where K_4^3 denotes the complete 3-graph on 4 vertices, is still open despite the $1000 prize of Erdős. Nonetheless this area is a great success if one judges by the wealth of ideas, methods, and connections that were discovered during the decades of active attempts. The talks presented at the mini-workshop reflected this variety very well, relating Turán-type questions to Ramsey theory, quasi-randomness, extremal problems on hypercubes, decomposition theorems, matchings, H-factors, and counting independent sets in hypergraphs.

One important general development that was motivated by the hypergraph Turán problem was the semidefinite method of Razborov built upon his flag algebras framework. A number of new results obtained by this method were presented at the mini-workhop; also, some participants investigated whether the semidefinite method may apply to extremal problems for other structures such as permutations or monochromatic arithmetic progressions.

The mini-workshop was quite active in terms of ongoing and new collaboration. The joint research projects that were carried out during the workshop included applying flag algebras to permutation densities (Dan Král' and Oleg Pikhurko), co-degree Turán densities (Oleg Pikhurko and Emil Vaughan), hypergraph Ramsey problems for loose cycles (Alexandr Kostochka and Dhruv Mubayi), tight cycles (Dhruv Mubayi and Vojtěch Rödl) and extremal hypergraphs for packing and covering (Penny Haxell and Tibor Szabó).

The mini-workshop was a great success. We are very grateful to the Oberwolfach Mathematical Institute for providing such a stimulating and productive environment.

Mini-Workshop: Hypergraph Turán Problem

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Abstracts

Turán densities and stability via Razborov's flag algebra method

John Talbot

(joint work with Rahil Baber)

If $\mathcal{F}$ is a family of r-graphs then we say that an r-graph G is $\mathcal{F}$*-free* if G contains no subgraph isomorphic to any member of $\mathcal{F}$. The maximum number of edges in an $\mathcal{F}$-free r-graph on n vertices is denoted by $\mathrm{ex}(n, \mathcal{F})$. Since this is often very difficult to calculate we introduce the *Turán density* of $\mathcal{F}$ to be $\pi(\mathcal{F}) = \lim_{n\to\infty} \mathrm{ex}(n, \mathcal{F})/\binom{n}{r}$.

Given a family of r-graphs $\mathcal{F}$ we would like to be able to compute $\mathrm{ex}(n, \mathcal{F})$, or failing this find $\pi(\mathcal{F})$. Another related question one can ask is: what structure do $\mathcal{F}$-free r-graphs with almost the maximum number of edges have? We say that a sequence of $\mathcal{F}$-free r-graphs $\{G_n\}_{n=1}^{\infty}$ is *almost extremal for* $\mathcal{F}$ if each G_n is an $\mathcal{F}$-free r-graph on n vertices and $d(G_n) = \pi(F) + o(1)$.

Razborov [4] recently introduced a powerful new tool to aid in the computation of Turán densities. Using his flag algebra "semidefinite" method he gave the following induced Turán density result. (We say G is $\mathcal{F}$*-induced-free* if G has no *induced* subgraph isomorphic to a member of $\mathcal{F}$ and define $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F})$ and $\pi_{\mathrm{ind}}(\mathcal{F})$ in the obvious way.)

Theorem 1 (Razborov [4])**.** *Let $K_4^{(3)}$ be the complete 3-graph with 4 vertices and let E_1 be the 3-graph with 4 vertices and a single edge. If $\mathcal{F}_1 = \{K_4^{(3)}, E_1\}$ then $\pi_{ind}(\mathcal{F}_1) = 5/9$.*

A sequence of 3-graphs that are $\mathcal{F}_1$-induced-free and asymptotically have density 5/9 is given by Turán's construction T_n. (T_n is the 3-graph with vertex set $[n]$ partitioned into three classes V_0, V_1, V_2 as equally as possible with edges consisting of all triples meeting each class once or meeting V_i once and V_{i+1} twice for some $0 \leq i \leq 2$.)

Building on Razborov's result Pikhurko [2] gave the following stability result. (He then used stability to determine the exact value of $\mathrm{ex}_{\mathrm{ind}}(n, \mathcal{F}_1)$ for sufficiently large n but our focus here is only on Turán density and stability results.)

Theorem 2 (Pikhurko [2])**.** *If $\{G_n\}_{n=1}^{\infty}$ is a sequence of $\mathcal{F}_1$-induced-free graphs, $|V(G_n)| = n$, and $d(G_n) = 5/9 + o(1)$ then we can make G_n isomorphic to T_n by changing at most $o(n^3)$ edges.*

Following Pikhurko we now outline how one can often construct a stability result for an almost extremal $\mathcal{F}$-free sequence of 3-graphs $\{G_n\}_{n=1}^{\infty}$ given a "flag algebra proof" determining the Turán density $\pi(\mathcal{F})$.

Let $\mathcal{F}$ be a family of 3-graphs. For any $k \geq 1$ define $\mathcal{H}_k$ to be the family of all $\mathcal{F}$-free 3-graphs on k vertices up to isomorphism. If G is a 3-graph of order at least k let $p(H; G)$ be the *induced density* of H in G: this is the probability that if $A \subseteq V(G)$ is a set of $|V(H)|$ vertices chosen uniformly at random then the subgraph of G induced by A is isomorphic to H.

We say that an $\mathcal{F}$-free 3-graph H is *$\mathcal{F}$-sharp* if there exists an almost extremal sequence $\{G_n\}_{n=1}^{\infty}$ for $\mathcal{F}$ such that $p(H; G_n) \neq o(1)$. If an $\mathcal{F}$-free 3-graph H is not $\mathcal{F}$-sharp we say it is *$\mathcal{F}$-negligible*. We denote the family of $\mathcal{F}$-sharp 3-graphs of order k by $\mathcal{H}_k^{\#}$.

A flag algebra proof of the Turán density of $\mathcal{F}$ also provides us with some information as to which 3-graphs are $\mathcal{F}$-sharp. Let $\{G_n\}_{n=1}^{\infty}$ be an almost extremal sequence for $\mathcal{F}$ and let k be fixed. Averaging over k-sets gives

$$d(G_n) = \sum_{H \in \mathcal{H}_k} p(H; G_n) d(H). \tag{1}$$

If we can find constants c_H for each $H \in \mathcal{H}_k$ such that

$$\sum_{H \in \mathcal{H}_k} c_H p(H; G_n) \geq o(1) \tag{2}$$

then summing (1) and (2), and using the fact that $\sum_{H \in \mathcal{H}_k} p(H; G_n) = 1$ we obtain

$$d(G_n) \leq \max_{H \in \mathcal{H}_k} (d(H) + c_H) + o(1).$$

Thus $\pi(\mathcal{F}) \leq \max_{H \in \mathcal{H}_k}(d(H) + c_H)$. Razborov's flag algebra semidefinite method provides a systematic way to find a choice of $\{c_H : H \in \mathcal{H}_k\}$ that minimises this upper bound for $\pi(\mathcal{F})$. In many cases it yields an upper bound that matches the best lower bound and so determines $\pi(\mathcal{F})$ exactly. In such cases it is easy to see that if $H \in \mathcal{H}_k$ and $d(H) + c_H < \pi(\mathcal{F})$ then H must be $\mathcal{F}$-negligible.

For a 3-graph G let $\mathcal{I}_k(G) = \{G[A] : A \subseteq V(G), |A| = k\}$. The strong hypergraph removal lemma of Rödl and Schacht [5] tells us that given any almost extremal sequence $\{G_n\}_{n=1}^{\infty}$ for $\mathcal{F}$ we can produce a new sequence of 3-graphs $\{G'_n\}_{n=1}^{\infty}$ by changing at most $o(n^3)$ edges in G_n, so that $\mathcal{I}_k(G'_n) \subseteq \mathcal{H}_k^{\#}$, i.e. all induced subgraphs of G'_n of order k are $\mathcal{F}$-sharp.

In many cases this observation, together with the information about $\mathcal{F}$-sharp graphs from a flag algebra proof of the Turán density, is enough to give a stability result. The reason for this is that many extremal constructions are determined by their small induced subgraphs (see Lemma 3).

We say that a 3-graph property $\mathcal{P}$ is *k-induced* if $\mathcal{I}_k(G) \subset \mathcal{P} \implies G \in \mathcal{P}$.

A 3-graph G is *complete tripartite* if there is a partition $V(G) = V_0 \cup V_1 \cup V_2$ such that the edges of G consist of all triples meeting each class once. A 3-graph G is *complete (2,1)-colourable* if there is a partition $V(G) = V_0 \cup V_1$ such that the edges of G consist of all triples meeting V_0 twice and V_1 once. A 3-graph G is *complete bipartite* if there is a partition $V(G) = V_0 \cup V_1$ such that the edges of G consist of all triples meeting V_0 and V_1.

Lemma 3 ([1])**.** *The following 3-graph properties are all 6-induced*

$$\mathcal{P}_S = \{G : G \text{ is complete tripartite}\},$$
$$\mathcal{P}_J = \{G : G \text{ is complete } (2,1)\text{-colourable}\},$$
$$\mathcal{P}_B = \{G : G \text{ is complete bipartite}\}.$$

The following is a sample of the results from [1].

Theorem 4 (Baber and Talbot [1])**.** *If*

$$F_1 = \{123, 124, 345, 156\}, F_2 = \{123, 124, 134, 235, 245, 156\} \text{ and}$$
$$F_3 = \{123, 124, 134, 125, 135, 235, 345, 126, 236, 146, 156, 456\}$$

then $\pi(F_1) = 2/9$*,* $\pi(F_2) = 4/9$*,* $\pi(F_3) = 3/4$ *and* $\pi(\{K_4^{(3)}, F_3\}) = 5/9$*. Moreover in each case we have stability, e.g if* $\{G_n\}_{n=1}^{\infty}$ *is almost extremal for* F_1 *then by changing* $o(n^3)$ *edges we can make* G_n *complete tripartite.*

In each case the proof proceeds by obtaining a flag algebra proof of the Turán density and then noting that this also tells us that the only potential $\mathcal{F}$-sharp 3-graphs are those that are found in the corresponding extremal construction. Lemma 3 then allows us to deduce that we must have stability.

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Turán numbers of expanded hypergraphs

Tao Jiang

(joint work with Axel Brandt, Robert Seiver)

Given an r-graph F on t vertices $v_1, \dots v_t$, let H_{p+1}^F, where $p+1 \geq t$, denote the r-graph obtained from F as follows. First, we add new vertices $v_{t+1}, \dots v_{p+1}$ and denote the set $\{v_1, \dots, v_{p+1}\}$ by S. Then for each pair $\{v_i, v_j\} \in \binom{S}{2}$ that is not covered by an edge of F, we add an r-edge $D_{i,j}$ such that $D_{i,j} \cap S = \{v_i, v_j\}$ and such that the $D_{i,j}$'s are pairwise disjoint outside S. For a rather wide family of r-graphs F, we show that the Turán number $ex(n, H_{p+1}^F)$ equals $|T^r(n,p)|$, where $T^r(n,p)$ denotes the p-partite Turán r-graph on n vertices. This gives an infinite family of r-graphs whose Turán number is precisely determined and generalizes or strengthens results in [1, 2, 3, 4, 5, 6]. Our exact results are established based on the corresponding stability properties.

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Problem Session

1. Complete partite subgraphs in dense hypergraphs (M. Schacht and **V. Rödl**)

It is known that any graph G with at least $\Omega(n^\ell)$ copies of K_ℓ already contains a t-blowup of K_ℓ with $t = \Omega(\log n)$.

For the hypergraphs this question is however wide open:

Conjecture 1. *For all ℓ, r and c there exists a γ such that the following holds. If H is an r-uniform hypergraph containing at least cn^ℓ copies of $K_\ell^{(r)}$, the complete r-uniform hypergraph with ℓ vertices, then H contains a complete ℓ-partite graph $K_{t,t,\ldots,t}$ (t-blowup of K_ℓ) with $t \geq \gamma(\log n)^{1/(r-1)}$.*

The essentially best known bound is $t = \Omega((\log n)^{1/\ell})$, which follows from an old result of Erdős.

2. Triangles in the union of two boolean lattices (P. Erdős, D. Gerbner, N. Lemons, **D. Mubayi**, C. Palmer, and B. Patkos)

Let G_n be the disjointness graph of the boolean lattice, namely, $V := V(G_n) = 2^{[n]}$ and $A, B \in V$ form an edge of G_n iff $A \cap B = \emptyset$. Let $a, b \geq 1$ and consider disjoint copies of G_a and G_b. Let $f(a,b)$ be the maximum number of edges that can be added between G_a and G_b such that in the resulting supergraph of $G_a \cup G_b$, there are no triangles that intersect both $V(G_a)$ and $V(G_b)$. Improve the bounds

$$\frac{1}{3}2^{a+b} \leq f(a,b) \leq \frac{3}{8}2^{a+b}.$$

After the talk, Dan Král' gave a construction showing that as long as one of $a, b \geq 3$, we have

$$f(a,b) \geq \frac{22}{64}2^{a+b}.$$

3. Decidability of the density function (**O. Pikhurko**)

Question 2. *Is $\pi(\mathcal{F}) \leq \alpha$ decidable where the input is a finite family $\mathcal{F}$ of r-uniform hypergraphs and $\alpha \in \mathbb{Q}$?*

4. Small Ramsey numbers (**V. Rödl**)

We write $G \longrightarrow (F)_r$ if no matter how one colors the edges of G with r colors there is a monochromatic copy of F in it. It is well-known that $K_6 \longrightarrow (K_3)_2$, and that there are K_4-free graphs G such that $G \longrightarrow (K_3)_2$ and moreover there is such a K_4-free G with at most 1000 vertices (i.e. and $G \longrightarrow (K_3)_2$).

Question 3. *How about* 3 *colors? Give good bounds on the order of* G *such that* $G \longrightarrow (K_3)_3$.

Daisies

Imre Leader
(joint work with Béla Bollobás, Claudia Malvenuto)

A *daisy*, or r*-daisy*, is a certain r-uniform hypergraph consisting of six sets: given an $(r-2)$-set P and a 4-set Q disjoint from P, the daisy on (P,Q) consists of the r-sets A with $P \subset A \subset P \cup Q$. We write this as $\mathcal{D}$, or $\mathcal{D}_r$. Our fundamental question is: how large can a family $\mathcal{A}$ of r-sets from an n-set be if $\mathcal{A}$ does not contain a daisy?

As usual, if $\mathcal{F}$ is a family of r-sets, we write $\mathrm{ex}(n,\mathcal{F})$ for the maximum size of a family of r-sets from an n-set that does not contain a copy of $\mathcal{F}$, and $\pi(\mathcal{F})$ or $\pi_r(\mathcal{F})$ for the limiting density, namely the limit of $\mathrm{ex}(n,\mathcal{F})/\binom{n}{r}$ as n tends to infinity – a standard averaging argument shows that this limit exists, and indeed that $\mathrm{ex}(n,\mathcal{F})/\binom{n}{r}$ is a decreasing function of n.

Conjecture 1. $\pi(\mathcal{D}_r) \to 0$ *as* $r \to \infty$.

What is unusual here is that we are not so concerned with the actual values of $\pi_r(\mathcal{D}_r)$ for particular r: our main interest is in the *limit* of these values.

This problem turns out to be related to the vertex-Turán problem in the hypercube, where almost nothing is known.

Since the hypergraph $\mathcal{D}_r$ is not r-partite, it follows that $\pi(\mathcal{D}_r) \geq r!/r^r$, as the complete r-partite r-graph does not contain a daisy. For $r = 2$, a daisy is precisely a K_4, and so Turán's theorem tells us that $\pi(\mathcal{D}_2) = 2/3$. Although even for $r = 3$ we do not know what the limiting density is, we believe we know what it should be.

Conjecture 2. $\pi(\mathcal{D}_3) = 1/2$.

To see where this conjecture comes from, note that the 3-graph on 7 vertices given by the complement of the Fano plane does not contain a daisy. Here, as usual, the Fano plane is the projective plane over the field of order 2; equivalently, it consists of the triples $\{a, a+1, a+3\}$, where the ground set is the integers mod 7. This gives $\mathrm{ex}(7,\mathcal{D}_3) \geq 28 = \frac{4}{5}\binom{7}{3}$. If we take a blow-up of this, thus dividing $[n]$ into 7 classes $C_0, \ldots, C_6$ each of size $\lfloor n/7 \rfloor$ or $\lceil n/7 \rceil$ and taking the 7-partite 3-graph consisting of all 3-sets whose 3 classes are not $\{C_a, C_{a+1}, C_{a+3}\}$

(with subscripts taken mod 7), we obtain $\mathrm{ex}(n, \mathcal{D}_3) \geq (1+o(1))\frac{24}{49}\binom{n}{3}$. But now we may iterate, taking a similar construction inside each class, and so on. This gives a limiting density of 24/49 times $1+1/49+1/49^2+\ldots$, which is exactly 1/2. This last conjecture (and the construction that motivates it) was made independently by Goldwasser.

The size of a hypergraph and its matching number

Po-Shen Loh
(joint work with Hao Huang, Benny Sudakov)

A k-uniform hypergraph is a pair $H = (V, E)$, where $V = V(H)$ is a finite set of vertices, and $E = E(H) \subseteq \binom{V}{k}$ is a family of k-element subsets of V called edges. A matching in H is a set of disjoint edges in $E(H)$. We denote by $\nu(H)$ the size of the largest matching, i.e., the maximum number of disjoint edges in H. The problem of finding the maximum matching in a hypergraph has many applications in various different areas of mathematics, computer science, and even computational chemistry. Yet although the graph matching problem is fairly well-understood, and solvable in polynomial time, most of the problems related to hypergraph matching tend to be very difficult and remain unsolved. Indeed, the hypergraph matching problem is known to be NP-hard even for 3-uniform hypergraphs, without any good approximation algorithm.

One of the most basic open questions in this area was raised in 1965 by Erdős [3], who asked to determine the maximum possible number of edges that can appear in any k-uniform hypergraph with matching number $\nu(H) < t \leq \frac{n}{k}$ (equivalently, without any t pairwise disjoint edges). He conjectured that this problem has only two extremal constructions. The first one is a clique consisting of all the k-subsets on $kt-1$ vertices, which obviously has matching number $t-1$. The second example is a k-uniform hypergraph on n vertices containing all the edges intersecting a fixed set of $t-1$ vertices, which also forces the matching number to be at most $t-1$. Neither construction is uniformly better than the other across the entire parameter space, so the conjectured bound is the maximum of these two possibilities. Note that in the second case, the complement of this hypergraph is a clique on $n-t+1$ vertices together with $t-1$ isolated vertices, and thus the original hypergraph has $\binom{n}{k} - \binom{n-t+1}{k}$ edges.

Conjecture *(Erdős.) Every k-uniform hypergraph H on n vertices with matching number $\nu(H) < t \leq \frac{n}{k}$ satisfies*

$$e(H) \leq \max\left\{\binom{kt-1}{k}, \binom{n}{k} - \binom{n-t+1}{k}\right\}.$$

In addition to being important in its own right, this Erdős conjecture has several interesting applications, which we discuss in the concluding remarks. Yet although it is more than forty years old, only partial results have been discovered so far. In the case $t = 2$, the condition simplifies to the requirement that every pair of edges intersects, so this conjecture is thus equivalent to a classical theorem of Erdős, Ko, and Rado [5]: that any intersecting family of k-subsets on $n \geq 2k$ elements has size at most $\binom{n-1}{k-1}$. The graph case ($k = 2$) was separately verified in [4] by Erdős and Gallai. For general fixed t and k, Erdős [3] proved his conjecture for sufficiently large n. Frankl [6] showed that the conjecture was asymptotically true for all n by proving the weaker bound $e(H) \leq (t-1)\binom{n-1}{k-1}$.

A short calculation shows that when $t \leq \frac{n}{k+1}$, we always have $\binom{n}{k} - \binom{n-t+1}{k} > \binom{kt-1}{k}$, so the potential extremal example in this case has all edges intersecting a fixed set of $t-1$ vertices. One natural question is then to determine the range of t (with respect to n and $k \geq 3$) for which the maximum is indeed equal to $\binom{n}{k} - \binom{n-t+1}{k}$, i.e., where the second case is optimal. Frankl, Rödl, and Ruciński [7] studied 3-uniform hypergraphs ($k = 3$), and proved that for $t \leq n/4$, the maximum was indeed $\binom{n}{3} - \binom{n-t+1}{3}$, establishing the conjecture in that range. Recently, Łuczak and Mieczkowska [8] resolved the 3-uniform case for all n larger than a certain absolute constant. For general $k \geq 4$, Bollobás, Daykin, and Erdős [2] explicitly computed the bounds achieved by the proof in [3], showing that the conjecture holds for $t < \frac{n}{2k^3}$. Frankl and Füredi [6] established the result in a different range $t < \left(\frac{n}{100k}\right)^{1/2}$, which improves the original bound when k is large relative to n. In this paper, we extend the range in which the Erdős conjecture holds to all $t < \frac{n}{3k^2}$.

Theorem *For any integers n, k, t satisfying $t < \frac{n}{3k^2}$, every k-uniform hypergraph on n vertices without t disjoint edges contains at most $\binom{n}{k} - \binom{n-t+1}{k}$ edges.*

To describe the idea of our proof, we first outline Erdős's original approach for the case $t < \frac{n}{2k^3}$. Let v be a vertex of maximum degree. By induction on t we find $t-1$ disjoint edges $F_1, \ldots, F_{t-1}$, none of which contain v. If $\deg(v)$ exceeds $k(t-1)\binom{n-2}{k-2}$, which is the maximum possible number of edges containing v which also meet a vertex in $\bigcup_{i=1}^{t-1} F_i$, then we can find t disjoint edges. Otherwise, the number of edges meeting any of F_i is at most $|\bigcup_{i=1}^{t-1} F_i| \cdot k(t-1)\binom{n-2}{k-2} = k(t-1) \cdot k(t-1)\binom{n-2}{k-2}$, which turns out to be less than the total number of edges when $n \geq 2k^3 t$. Any other edge will serve as the t-th edge in the matching.

To improve Erdős's bound, we show that in the first part of the argument, we are already done if the t-th largest degree exceeds $2t\binom{n-2}{k-2}$. This puts a tighter constraint on the sum of the degrees of the $k(t-1)$ vertices in $\bigcup_{i=1}^{t-1} F_i$, allowing the second stage to proceed under the relaxed assumption $n \geq 3k^2 t$. The fact that t vertices of degree at least $2t\binom{n-2}{k-2}$ are enough to find t disjoint edges leads naturally to the following multicolored version of the Erdős conjecture, which was also considered independently by Aharoni and Howard in [1].

Conjecture *Let $\mathcal{F}_1, \ldots, \mathcal{F}_t$ be families of subsets in $\binom{[n]}{k}$. If we have $|\mathcal{F}_i| > \max\left\{\binom{n}{k} - \binom{n-t+1}{k}, \binom{kt-1}{k}\right\}$ for all $1 \leq i \leq t$, then there is a "rainbow" matching of size t: one that contains exactly one edge from each family.*

The $k = 2$ case of this conjecture was established by Meshulam (see [1]). To obtain our theorem, we prove an asymptotic version of the above conjecture, by showing that a rainbow matching exists whenever $|\mathcal{F}_i| > (t-1)\binom{n-1}{k-1}$ for every $1 \le i \le t$.

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Powers of Hamiltonian cycles in pseudorandom graphs

Yury Person

(joint work with Peter Allen, Julia Böttcher, Hiệp Hàn and Yoshiharu Kohayakawa)

The study of quasi-random (a.k.a. pseudorandom) graphs was initiated by Thomason [5], whose motivation was to understand which properties make a deterministic graph to behave as a random one in many respects. Soon thereafter, a slightly general and less restrictive notion of quasi-randomness was studied by Chung, Graham and Wilson, who collected and proved in their seminal paper [2] a theorem that established equivalences of many properties shared by random graphs, and thus called further on quasi-random properties. Since then many beautiful results have been proven about quasi-random structures and even more such structures were found.

In the recent years it became more popular to study random graphs and their properties by defining (rather) artificial conditions, also called quasi-random, because they are satisfied by the random graph $G(n,p)$ for the probability p in question. Thus, showing that such a quasi-random (hyper-)graph has certain properties already implies the same statement for the *random graph* $G(n,p)$. While the probabilities p may be a good way away from optimal thresholds, it can be seen as a first step in the right direction.

The other theme in the study of quasi-random graphs is that there are known constructions of quasi-random graphs, most notably so-called (n,d,λ)-graphs or expander graphs, for an excellent survey by Krivelevich and Sudakov see [3], where Ramanujan graphs are most prominent among them.

An (n, d, λ)-graph is a d-regular graph with n vertices, whose second largest eigenvalue of the adjacency matrix (in the absolute value) is at most λ. The smaller λ is the "more random" an (n, d, λ)-graph appears. Krivelevich, Sudakov and Szabó [4] studied under what dependency of λ on d and n, such a graph possesses a triangle factor. There are however (n, d, λ)-graphs which look like quasi-random graphs but do not contain a single triangle, as shown by Alon [1].

A Hamiltonian cycle in a graph on n vertices is the set of n edges traversing all n vertices and a kth power of a graph G is obtained by adding to G all edges between any two vertices at distance at most k.

In joint work with Allen, Böttcher, Hàn and Kohayakawa, I showed first non-trivial conditions on (n, d, λ)-graphs to contain a k-th power of a Hamiltonian cycle. This in particularly implies the conditions for containment of K_{k+1}-factors ($\lfloor n/(k+1) \rfloor$ vertex-disjoint copies of K_{k+1}s) and improves upon the results of Krivelevich, Sudakov and Szabó [4]. To obtain our result we introduce a special notion of quasi-randomness and develop a technique that adapts known tools from the dense to sparse setting.

In the case of triangle factor our result reads as follows. There is a constant $c > 0$ and n_0 such that any (n, d, λ)-graph G with $n \geq n_0$ vertices and

$$\lambda \leq c \frac{d^{5/2}}{n^{3/2}}$$

contains a square of a Hamiltonian cycle, and thus a triangle factor if $3|n$. This improves the result of Krivelevich, Sudakov and Szabó [4] who showed a sufficient upper bound on λ to be $O(d^3/(n^2 \log n))$. On the other side, Alon [1] constructed (n, d, λ) graphs with $\lambda = \Theta(n^{1/3})$ and $d = \Theta(n^{2/3})$ (essentially the best quasi-random graphs) which are triangle-free. Our result holds for best known (n, d, λ)-graphs (where $\lambda = \Theta(\sqrt{d})$) when $d \gg n^{3/4}$.

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Hypergraph Ramsey Numbers: Triangles versus Cliques

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(joint work with Dhruv Mubayi, Jacques Verstraete)

By a *k-cycle* C_k we mean a hypergraph *loose k-cycle*, namely the hypergraph with edges $e_1, \dots, e_k$ such that for $i \neq j$, $|e_i \cap e_j| = 1$ if $|i - j| = 1 \mod k$ and $e_i \cap e_j = \emptyset$ otherwise. In particular, a *triangle* is a hypergraph consisting of three edges e, f, g such that $|e \cap f| = |f \cap g| = |g \cap e| = 1$ and $e \cap f \cap g = \emptyset$.

An *r-graph* is an r-uniform hypergraph. An *independent set* in a hypergraph is a set of vertices containing no edges of the hypergraph. Let K_t^r denote the t-vertex *complete r-graph*, i.e., the t-vertex r-graph whose edges are all r-element subsets of $[t]$. We consider the cycle versus complete hypergraph Ramsey numbers $R(C_k, K_t^r)$ – this is the minimum n such that every n-vertex r-graph contains either a cycle C_k or an independent set of t vertices. Our main effort will be on the triangle - complete hypergraph Ramsey number $R(C_3, K_t^r)$. A celebrated result of Kim [3] together with earlier bounds by Ajtai, Komlós and Szemerédi [1] shows

$$R(C_3, K_t) = \Theta\Big(\frac{t^2}{\log t}\Big).$$

This establishes the order of magnitude of these Ramsey numbers for graphs. Motivated by the triangle-complete graph Ramsey numbers, we determine the order of magnitude of the triangle-complete Ramsey numbers for triple systems up to logarithmic factors:

Theorem 1. *There exist constants $c_1, c_2 > 0$ such that for all $t \geq 1$,*

$$\frac{c_1 t^{3/2}}{(\log t)^{3/4}} \leq R(C_3, K_t^3) \leq c_2 t^{3/2}.$$

The lower bound in Theorem 1 comes from a *random block construction* that combines randomness and linear algebra. This construction extends in a straightforward manner to give lower bounds for $R(C_3, K_t^r)$ for all $r \geq 3$:

Theorem 2. *Let $r \geq 3$. Then for some constant $c > 0$ and all $t \geq 1$,*

$$c\Big(\frac{t}{\log t}\Big)^{3/2} \leq R(C_3, K_t^r) \leq (t+1)^2.$$

In light of Theorem 1, we make the following conjecture:

Conjecture 1.

$$R(C_3, K_t^3) = o(t^{3/2}).$$

A motivation for studying triangle-complete hypergraph Ramsey numbers is the notorious extremal problem for three-term arithmetic progressions. Let $r_3(N)$ denote the largest size of a subset of integers in $\{1, 2, \dots, N\}$ containing no three-term arithmetic progressions. This problem has attracted much attention, starting with the original theorems of van de Waerden and Roth. The best known bounds are as follows: for some constant $c > 0$,

$$\frac{N}{e^{c\sqrt{\log N}}} \leq r_3(N) \leq \frac{N}{(\log N)^{1-o(1)}}.$$

The lower bound, which comes from a construction of Behrend [2], is essentially unchanged for more than sixty years. The upper bound, due to Sanders improves many earlier results which gave smaller powers of $\log N$ in the denominator. Let $RL(C_3, K_t^3)$ denote the minimum n such that every n-vertex *linear* triangle-free r-graph has an independent set of size t. Then $RL(C_3, K_t^3) \leq R(C_3, K_t^3) \leq c_2 t^{3/2}$ by Theorem 1. We prove the following:

Theorem 3. *For some constants $c_1, c_2 > 0$,*

$$\frac{t^{3/2}}{e^{c_1\sqrt{\log t}}} \leq RL(C_3, K_t^3) \leq c_2 \frac{t^{3/2}}{(\log t)^{1/2}}.$$

This theorem is perhaps some support for believing $R(C_3, K_t^3) = o(t^{3/2})$. This relates to upper bounds on $r_3(N)$ as follows: if one is able to show

$$RL(C_3, K_t^3) = O\left(\frac{t^{3/2}}{(\log t)^{c+3/4}}\right)$$

for some $c > 0$, then we shall see that

$$r_3(N) = O\left(\frac{N}{(\log N)^{3c}}\right).$$

The random block construction for Theorem 1 extends more generally to give lower bounds on all *cycle-complete hypergraph Ramsey numbers*. For all $k, r \geq 3$ we give a construction of C_k-free r-graphs with low independence number based on known results on C_k-free bipartite Ramanujan graphs of Lubotzky, Phillips and Sarnak [4]. Specifically, we prove the following theorem by a suitable modification of the random block construction. We write $f = O^*(g)$ to denote that for some constant c, $f(t) = O((\log t)^c g(t))$, and $f = \Omega^*(g)$ denotes that $g = O^*(f)$.

Theorem 4. *For $r, k \geq 3$,*

$$R(C_k, K_t^r) \quad = \quad \Omega^*\left(t^{1+\frac{1}{3k-1}}\right).$$

The key point of this theorem is that the exponent $1+1/(3k-1)$ of t is bounded away from 1 by a constant independent of r, and strictly improves for all $r, k \geq 5$ the lower bounds given by considering appropriate random hypergraphs, namely

$$R(C_k, K_t^r) = \Omega^*\left(t^{1+\frac{1}{kr-k-r}}\right).$$

In fact, using more complicated results about the distribution of prime numbers, we can improve the exponent $1+1/(3k-1)$ slightly. Theorem 4 can be strengthened for pentagons:

Theorem 5. *For $r \geq 3$,*

$$R(C_5, K_t^r) \quad = \quad \Omega^*(t^{5/4}).$$

We suspect the exponent 5/4 above may be tight, and perhaps even more generally, $r(C_k, K_t^r) = \Theta^*(t^{k/(k-1)})$ for all $r, k \geq 3$.

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On Possible Turán Densities

Oleg Pikhurko

Let $\mathcal{F}$ be a (possibly infinite) family of *k-graphs* (that is, k-uniform set systems). We call elements of $\mathcal{F}$ *forbidden.* A k-graph G is *$\mathcal{F}$-free* if no member $F \in \mathcal{F}$ is a subgraph of G, that is, we cannot obtain F by deleting some vertices and edges from G. The *Turán function* $\mathrm{ex}(n, F)$ is the maximum number of edges that an $\mathcal{F}$-free k-graph on n vertices can have. This is one of the central questions of extremal combinatorics that goes back to the fundamental paper of Turán [17]. We refer the reader to the surveys of the Turán function by Füredi [9], Sidorenko [16], and Keevash [12].

As it was observed by Katona, Nemetz, and Simonovits [11] the ratio $\mathrm{ex}(n, \mathcal{F})/\binom{n}{k}$ is non-increasing in n. In particular, the limit

$$\pi(\mathcal{F}) := \lim_{n\to\infty} \frac{\mathrm{ex}(n, \mathcal{F})}{\binom{n}{k}}$$

exists. It is called the *Turán density* of $\mathcal{F}$. Let Π_∞^k consist of all possible Turán densities of k-graphs and let Π_{fin}^k be the set of all possible Turán densities when *finitely many* k-graphs are forbidden. Clearly, $\Pi_{\mathrm{fin}}^k \subseteq \Pi_\infty^k$.

For $k = 2$, the celebrated Erdős-Stone-Simonovits Theorem [7, 6] determines the Turán density for every family $\mathcal{F}$. In particular, we have

$$\Pi_{\mathrm{fin}}^2 = \Pi_\infty^2 = \left\{ \frac{m-1}{m} : m = 1, 2, 3, \ldots, \infty \right\}. \tag{1}$$

(It is convenient to allow empty families, so $1 \in \Pi_{\mathrm{fin}}^k$ for every k.)

Little is known about possible Turán densities for $k \geq 3$. Brown and Simonovits [3, Theorem 1] noted that for every $\mathcal{F}$ and $\epsilon > 0$ there is a finite $\mathcal{F}' \subseteq \mathcal{F}$ with $\pi(\mathcal{F}') \leq \pi(\mathcal{F}) + \epsilon$. It follows that Π_∞^k lies in the closure of Π_{fin}^k. We show that the set $\Pi_\infty^k \subseteq [0, 1]$ is closed (thus, it is, the closure of Π_{fin}^k.).

Also, we show that, for every $k \geq 3$, the set Π_∞^k has cardinality of the continuum.

Since the number of finite families of k-graphs (up to isomorphism) is countable, the last result implies that $\Pi^k_{\text{fin}} \neq \Pi^k_\infty$ for $k \geq 3$, answering one part of a question of Baber and Talbot [1, Question 6].

Very few explicit numbers were proved to belong to Π^k_{fin}. For example, before 2006 the only known members of Π^3_{fin} were 0, 2/9, 4/9, 3/4, and 1 (see [2, 5, 10]). Then Mubayi [13] showed that $(m-1)(m-2)/m^2 \in \Pi^3_{\text{fin}}$ for every $m \geq 4$. Very recently, Baber and Talbot [1] and Falgas-Ravry and Vaughan [8] determined a few further elements of Π^3_{fin}; their proofs are computer-generated, being based on the flag algebra approach of Razborov [14]. In all the cases when an explicit element of Π^k_{fin} is known, this limit density is achieved, informally speaking, by taking a finite pattern and blowing it up optimally. Here we generalize these results (as far as Π^k_{fin} is concerned) by showing that *every* finite pattern where, moreover, we are allowed to iterate the whole construction recursively inside a specified set of parts produces a density in Π^k_{fin}.

Chung and Graham [4, page 95] conjectured that Π^k_{fin} consists of rational numbers only. We disprove this conjecture for every $k \geq 3$. (Note that the conjecture is true for $k = 2$ by (1).) Independently, Chung and Graham's conjecture was disproved by Baber and Talbot [1] who discovered a family of only three forbidden 3-graphs whose Turán density is irrational. We should mention that our proof relies on the Strong Removal Lemma of Rödl and Schacht [15] so it produces families $\mathcal{F}$ of huge size even for some small concrete pattern P.

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Analogies and differences: colouring elements of $\mathbb{Z}_p$ and edges of K_n

Julia Wolf

It is a relatively well-known fact that given any 2-colouring of the cyclic group $\mathbb{Z}_p$ for p a prime, the number of monochromatic 3-term arithmetic progressions depends only on the densities of the colour classes R and B. Using discrete Fourier analysis, specifically the fact that $\widehat{1_R}(t) = -\widehat{1_B}(t)$ for $t \neq 0$, one easily obtains the result that the number of monochromatic 3-term progressions in such a colouring equals $(1 - 3\alpha + 3\alpha^2)p^2$, where one of the colour classes, R say, has size αp (see also [2]). Note that this is precisely the number of 3-term progressions we would expect if we were to choose the elements of the red colour class independently at random from $\mathbb{Z}_p$ with probability α.

While a similar formula holds for solutions to other equations in three variables, for example Schur triples of the form $x + y = z$, the same is not true for longer arithmetic progressions. It is not difficult to see that the number of monochromatic 4-term progressions in a given 2-colouring does not just depend on the density ratio of the colour classes. Instead, we may ask for the minimum number of monochromatic 4-term progressions in any 2-colouring of $\mathbb{Z}_p$, in the limit as $p \to \infty$. (We will normalize this quantity by the total number of arithmetic progressions in the discussion that follows.)

An easy bound can be derived from van der Waerden's theorem. This primitive estimate was significantly improved by Cameron, Cilleruelo and Serra [1] to a fraction of $\frac{2}{33}$. In [13] I improved this further to $\frac{1}{16}$. In the other direction, I exhibited a 2-colouring of $\mathbb{Z}_p$ with fewer than a proportion of $\frac{1}{8}(1 - \frac{1}{259200})$ of monochromatic 4-term progressions, which is slightly less than the proportion expected in a random colouring. This colouring was based on an example of Gowers [5], who constructed a set which is uniform in the sense of the Fourier transform but contains fewer than the expected number of 4-term progressions and is somewhat "quadratic" in nature. By purely computational efforts, Lu and Peng [8] recently improved the lower and upper bounds on this problem to $\frac{7}{96}$ and $\frac{17}{150}$, respectively. A gap remains.

This situation is very similar to the corresponding problem in graphs. A result of Goodman [4] states that any 2-colouring of the edges of the complete graph K_n contains at least the expected number of monochromatic triangles. In a next step, one wants to determine the minimum number of monochromatic K_4s in any 2-colouring of K_n, a problem which has resisted complete resolution for quite some time. Giraud [3] proved a lower bound of $\frac{1}{46}$ by combinatorial arguments, recently improved by Sperfeld [10] to $\frac{1}{35}$ using flag algebras. On the other hand, Thomason had disproved a conjecture of Erdős by constructing 2-colourings of graphs with fewer than the random proportion, namely $\frac{1}{33}$, of monochromatic K_4s. Interestingly, the first such construction also used a quadratic form, while later improvements were of a computational nature.

We observe that in $\mathbb{Z}_p$, a deep theorem from additive combinatorics (the so-called *inverse theorem for the U^3 norm*, due to Green and Tao [6], which implies Szemerédi's theorem for progressions of length 4) tells us that any colouring of $\mathbb{Z}_p$ that beats random for 4-term progressions must exhibit some (albeit relatively weak) quadratic structure. Does a similar statement hold in the case of graphs?

The analogous question in the interval $\{1, 2, \ldots, n\}$ is also of interest. Here the number of monochromatic 3-term progressions in a 2-colouring does not simply depend on the density of the colour classes, so even for the shortest progressions the question of determining the asymptotically minimal number is non-trivial and has not been resolved (for the best known upper and lower bounds see [9]).

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On a conjecture of Erdős and Simonovits

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(joint work with Benny Sudakov, Jacques Verstraëte)

Given a family $\mathcal{F}$ of graphs, a graph is *$\mathcal{F}$-free* if it contains no copy of a graph in $\mathcal{F}$ as a subgraph. The *Turán number* $\mathrm{ex}(n, \mathcal{F})$ is the maximum number of edges in an $\mathcal{F}$-free graph on n vertices. The *Zarankiewicz number* $\mathrm{z}(n, \mathcal{F})$ is the maximum number of edges in an $\mathcal{F}$-free *bipartite* graph on n vertices.

Let C_k denote a cycle of length k, and let $\mathcal{C}_k$ denote the set of cycles C_ℓ, where $3 \leq \ell \leq k$ and ℓ and k have the same parity. Erdős and Simonovits conjectured that for any family $\mathcal{F}$ consisting of bipartite graphs there exists an odd integer k such that $\mathrm{ex}(n, \mathcal{F} \cup \mathcal{C}_k) \sim \mathrm{z}(n, \mathcal{F})$ – here we write $f(n) \sim g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 1$. They proved this when $\mathcal{F} = \{C_4\}$ by showing that $\mathrm{ex}(n, \{C_4, C_5\}) \sim \mathrm{z}(n, C_4)$.

We extend this result by showing that if $\ell \in \{2, 3, 5\}$ and $k > 2\ell$ is odd, then $\mathrm{ex}(n, \mathcal{C}_{2\ell} \cup \{C_k\}) \sim \mathrm{z}(n, \mathcal{C}_{2\ell})$. Furthermore, if $k > 2\ell + 2$ is odd, then for infinitely many n we show that the extremal $\mathcal{C}_{2\ell} \cup \{C_k\}$-free graphs are bipartite incidence graphs of generalized polygons. We observe that this exact result does not hold for any odd $k < 2\ell$, and furthermore the asymptotic result does not hold when (ℓ, k) is $(3, 3)$, $(5, 3)$ or $(5, 5)$. Our proofs make use of pseudorandomness properties of nearly extremal graphs that are of independent interest.

Quasirandom permutations

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(joint work with Oleg Pikhurko)

Generally speaking, a combinatorial object is called quasirandom if it has properties that a random object has (asymptotically) almost surely. This notion has particularly been studied and developed for graphs. Extending earlier results of Rödl [19] and Thomason [21], Chung, Graham and Wilson [4] gave seven properties that a sufficiently large graph has one of them if and only if it has all of them and such that a random graph posseses them asymptotically almost surely (here, we consider the model where each edge is included to a graph with probability 1/2 independently of the other edges). These properties include relative densities of subgraphs, values of eigenvalues of the adjacency matrix or average size of a common neighborhood of two vertices.

The results of Chung et al. [4] imply that if the number of 2-vertex and 4-vertex subgraphs of a graph is asymptotically the same as in a random graph, then the number of all subgraphs is also asymptotically as in a random graph. Specifically, they show that if the number of edges of an n-vertex graph is $(1/4 + o(1))n^2$ and the number of induced cycles of length four is $(3/64 + o(1))n^4$, then the number of induced copies of any k-vertex graph H is $(k!2^{-\binom{k}{2}}/\pi + o(1))n^k$ where π is the number of automorphisms of H. Graham (see [5]) asked whether the same phenomenon also appears in the case of permutations.

We now give a precise description of the problem we study. Permutations are treated as two linear orders on the same set of points and the order of a permutation is the number of points in the underlying set. Let π_j be a sequence of permutations with orders n_j tending to infinity. The sequence is *asymptotically k-symmetric* if for every k-point permutation τ, $|\mathrm{Prob}[\pi_j|_k = \tau] - 1/k!| = o(1)$ where $\mathrm{Prob}[\pi_j|_k = \tau]$ is the probability that π_j restricted to a randomly chosen k points is isomorphic to τ. Graham asked whether there exists k_0 such that for every $k \geq k_0$, every asymptotically k-symmetric sequence is also asymptotically $(k+1)$-symmetric. Cooper [5] defined a stronger notion of asymptotic symmetry and he resolved the question under this stronger notion.

We answer the original question. We show for $k \geq 4$ that every asymptotically k-symmetric sequence of permutations is also asymptotically $(k+1)$-symmetric. We also give an example that the same statement is not true for $k = 3$ (the case $k = 1$ is trivial and the counterexample for $k = 2$ has been known).

In the proof of our main result, we employ the recent framework of flag algebras developed by Razborov [15]. This framework has been applied to many extremal problems related to graphs [6, 7, 8, 9, 12, 14, 16, 18, 20], hypergraphs [2, 17] or hypercubes [1, 3]. Our main result asserts that the "limit" of every asymptotically 4-symmetric sequence of permutations is the unique "uniform permutation limit" (this directly implies the statement). The definition of a permutation limit follows the lines of work on graph limits by Lovász and Szegedy [13] and it resembles the concept developed in [10, 11].

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Independent sets in hypergraphs

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(joint work with Robert Morris, Wojciech Samotij)

1. Introduction.

A great many questions researched in combinatorics fall into the following general framework: Given a finite set V and a collection $\mathcal{H} \subset \mathcal{P}(V)$ of *forbidden structures*, what can be said about sets $I \subseteq V$ that do not contain any member of $\mathcal{H}$? For example, the celebrated theorem of Szemerédi [12] states that if $V = \{1, \ldots, n\}$ and $\mathcal{H}$ is a collection of k-term arithmetic progressions in $[n] = \{1, \ldots, n\}$, then every set I that contains no member of $\mathcal{H}$ satisfies $|I| = o(n)$. The archetypal problem studied in extremal graph theory, dating back to Turán's theorem [13], is the problem of characterizing sets I as above when V is the edge set of the complete graph on n vertices and $\mathcal{H}$ is the collection of copies of some fixed graph H in K_n. In this setting, much more can be said. We now know not only the maximum size of I that contains no member of $\mathcal{H}$, but also what the largest such sets look like, how many of them there are, and what the structure of a typical such set is. Therefore, one might say that a large part of extremal combinatorics is concerned with studying independent sets in various hypergraphs.

The recently proved general transference theorems of Conlon-Gowers [2] and Schacht [10], which they used to prove, among other things, sparse random analogues of the classical theorems of Szemerédi [12], Erdős-Stone [5], and Turán [13], were stated in the language of hypergraphs. Roughly speaking, these transference theorems say that the independence number of a hypergraph whose edges are sufficiently uniformly distributed is 'well-behaved' with respect to taking subhypergraphs induced by random subsets of the vertex set with sufficiently large density. More precisely, given $p \in [0, 1]$ and a finite set V, we define the *p-random subset* of V, denoted V_p, to be the random subset of V, where each element of V is included with probability p, independently of all other elements. The results of Conlon-Gowers [2] and Schacht [10] imply, in particular, that if the distribution of the edges of some uniform hypergraph $\mathcal{H}$ is sufficiently balanced, then with probability tending to 1 as $v(\mathcal{H}) \to \infty$,

$$\alpha\big(\mathcal{H}[V(\mathcal{H})_p]\big) \leq p\alpha(\mathcal{H}) + o\big(pv(\mathcal{H})\big),$$

provided that p is sufficiently large.

In this work, we give a rough structural characterization of the family of all independent sets in uniform hypergraphs whose edge distribution satisfies a certain natural boundedness condition. We prove that the independent sets of each such hypergraph $\mathcal{H}$ exhibit a some type of clustering phenomenon. Our main result proves that the family $\mathcal{I}(\mathcal{H})$ of independent sets in $\mathcal{H}$ admits a partition into relatively few classes such that all members of each class are essentially contained in a single subset of $V(\mathcal{H})$ that is almost independent, that is, it induces only a tiny proportion of all the edges of $\mathcal{H}$. This somewhat abstract statement has surprisingly many deep and interesting consequences.

2. The main theorem.

Definition 1. *Let $\mathcal{H}$ be a uniform hypergraph with vertex set V, let $\mathcal{F}$ be an increasing family of subsets of V and let $\varepsilon \in (0,1]$. We say that $\mathcal{H}$ is $(\mathcal{F},\varepsilon)$*-dense *if*

$$e(\mathcal{H}[A]) \geq \varepsilon e(\mathcal{H})$$

for every $A \in \mathcal{F}$. Denote by $\mathcal{I}(\mathcal{H})$ the family of all independent sets in $\mathcal{H}$.

Example 2. *Consider the k-uniform hypergraph $\mathcal{H}_1$ on the vertex set $[n]$ whose edges are all k-term arithmetic progressions in $[n]$ and let $\mathcal{F}_1$ be the collection of all subsets of $[n]$ with at least δn elements. Clearly, $\mathcal{F}_1$ is an upset and it follows from the theorem of Szemerédi* [12] *that $\mathcal{H}_1$ is $(\mathcal{F}_1,\varepsilon)$-dense for some positive ε depending only on δ and k.*

Theorem 3. *For every $k \in \mathbb{N}$ and all positive c, c' and ε, there exists a positive constant C such that the following holds. Let $\mathcal{H}$ be a k-uniform hypergraph and let $\mathcal{F} \subseteq \mathcal{P}(V(\mathcal{H}))$ be an increasing family of sets such that $|A| \geq \varepsilon v(\mathcal{H})$ for all $A \in \mathcal{F}$. Suppose that $\mathcal{H}$ is $(\mathcal{F},\varepsilon)$-dense and $p \in (0,1)$ is such that $p^{k-1}e(\mathcal{H}) \geq c'v(\mathcal{H})$ and for every $\ell \in [k-1]$,*

$$\Delta_\ell(\mathcal{H}) \leq c \cdot \min\left\{p^{\ell-k}, p^{\ell-1}\frac{e(\mathcal{H})}{v(\mathcal{H})}\right\}.$$

Then there exists a family $\mathcal{S} \subseteq \binom{V(\mathcal{H})}{\leq Cp\cdot v(\mathcal{H})}$ and functions $f\colon \mathcal{S} \to \overline{\mathcal{F}}$ and $g\colon \mathcal{I}(\mathcal{H}) \to \mathcal{S}$ such that for every $I \in \mathcal{I}(\mathcal{H})$,

$$g(I) \subseteq I \qquad \text{and} \qquad I \setminus g(I) \subseteq f(g(I)).$$

For proof see [1].

2.1. **The number of sets with no k-term arithmetic progression.** The celebrated theorem of Szemerédi says that for every k, the largest subset of $[n]$ that contains no k-term arithmetic progression (AP) has $o(n)$ elements. Our first result can be viewed as a sparse analogue of this statement.

Theorem 4. *For every positive β and every positive integer k, there exist constants C and n_0 such that the following holds: For every n integer, if $n \geq n_0$ and $m \geq Cn^{1-1/(k-1)}$, then there are at most $\binom{\beta n}{m}$ m-subsets of $[n]$ that contain no k-term AP.*

Let us remark here that the sparse random analogue of Szemerédi's theorem, proved by Conlon and Gowers and independently by Schacht, follows as an easy corollary of Theorem 4. A set $A \subseteq \mathbb{N}$ is (δ, k)-*Szemerédi* if every subset $B \subseteq A$ with at least $\delta|A|$ elements contains a k-term AP. Recall that $[n]_p$ denotes the p-random subset of $[n]$.

Theorem 5. *For every $\delta \in (0,1)$ and every $k \in \mathbb{N}$, there exists a constant C such that if $p \geq Cn^{-1/(k-1)}$, then*

$$\lim_{n\to\infty} \Pr\big([n]_p \text{ is } (\delta,k)\text{-Szemerédi}\big) = 1.$$

We remark that both theorems are best possible up to the value of the constant C.

2.2. **Turán's theorem for random graphs.** The following result was proved by Conlon and Gowers (under the assumption that H is *strictly* 2-*balanced*[1]) and, independently, by Schacht.

Theorem 6. *For every graph H with $\Delta(H) \geq 2$ and every positive δ, there exists a positive constant C such that if $p_n \geq Cn^{-1/m_2(H)}$, then a.a.s.*

$$\mathrm{ex}\big(G(n,p_n),H\big) = \left(1 - \frac{1}{\chi(H)-1} \pm \delta\right)\binom{n}{2}p_n.$$

Our methods give yet another proof of this theorem in the case when H is 2-balanced. In fact, we deduce from our main result, a version of the general transference theorem of Schacht, which quite easily implies the above result.

Finally, our methods also yield the sparse random analogue of the famous stability theorem of Erdős and Simonovits [11].

2.3. **The typical structure of H-free graphs.** Let H be an arbitrary nonempty graph. We say that a graph G is H-*free* if G does not contain H as a subgraph. For an integer n, denote by $f_n(H)$ the number of labeled H-free graphs on the vertex set $[n]$. Since every subgraph of an H-free graph is also H-free, it follows that $f_n(H) \geq 2^{\mathrm{ex}(n,H)}$. Erdős, Frankl, and Rödl [3] proved that this crude lower bound is in a sense tight, namely that

$$f_n(H) = 2^{\mathrm{ex}(n,H)+o(n^2)}. \tag{1}$$

[1] A graph H is strictly 2-balanced if $m_2(H) > m_2(H')$ for every proper $H' \subset H$, where $m_2(H) = \max_{F\subset H} \frac{e(F)-2}{v(F)-1}$.

Our next result can be viewed as a 'sparse version' of (1). For integers n and m with $0 \le m \le \binom{n}{2}$, let $f_{n,m}(H)$ be the number of labeled H-free graphs on the vertex set $[n]$ that have exactly m edges. We remark that the statement below easily implies (1).

Theorem 7. *Let H be a 2-balanced graph and let δ be a positive constant. There exists a constant C such that for every n, if $m \ge Cn^{2-1/m_2(H)}$, then*

$$\binom{\mathrm{ex}(n,H)}{m} \le f_{n,m}(H) \le \binom{\mathrm{ex}(n,H)+\delta n^2}{m}.$$

Erdős, Kleitman, and Rothschild [4] proved that *almost all* triangle-free graphs are bipartite. Our next result is an approximate sparse version of this statement for an arbitrary 2-balanced graph H. Given a positive real δ and an integer k, let us call a graph G (δ,k)-*partite* if G can be made k-partite by removing from it at most $\delta e(G)$ edges.

Theorem 8. *Let H be a 2-balanced graph with $\chi(H) \ge 3$ and let δ be a positive constant. There exists a constant C such that for every n, if $m \ge Cn^{2-1/m_2(H)}$, then almost all n-vertex H-free graphs with m edges are $\big(\delta, \chi(H)-1\big)$-partite.*

3. The KŁR conjecture.

For sparse graphs, that is, n-vertex graphs with $o(n^2)$ edges, the original version of the regularity lemma is vacuous as all induced bipartite subgraphs in every partition of the vertex set of a sparse graph into a bounded number of parts are ε-regular, provided that n is sufficiently large. It was independently observed by Kohayakawa [6] and Rödl [9] that the notion of ε-regularity may be extended in a meaningful way to graphs with density tending to zero. Moreover, with this more general notion of regularity, they proved an associated regularity lemma which applies to a large class of sparse graphs.

Given a $p \in [0,1]$ and a positive ε, we say that a bipartite graph between sets V_1 and V_2 is (ε,p)-*regular* if for every $W_1 \subseteq V_1$ and $W_2 \subseteq V_2$ with $|W_1| \ge \varepsilon |V_1|$ and $|W_2| \ge \varepsilon|V_2|$, the density $d(W_1,W_2)$ of edges between W_1 and W_2 satisfies

$$\big|d(W_1,W_2) - d(V_1,V_2)\big| \le \varepsilon p.$$

A partition of the vertex set of a graph into r parts $V_1,\dots,V_r$ is said to be (ε,p)-*regular* if $\big||V_i|-|V_j|\big| \le 1$ for all i and j and for all but at most εr^2 pairs (V_i,V_j), the graph induced between V_i and V_j is (ε,p)-regular. The class of graphs to which the Kohayakawa-Rödl regularity lemma applies are the so-called upper-uniform graphs. Given positive η and K, we say that an n-vertex graph G is (η,p,K)-*upper-uniform* if for all $W \subseteq V(G)$ with $|W| \ge \eta n$, the density of edges within W satisfies $d(W) \le Kp$. This condition is satisfied in many natural classes of graphs, including all subgraphs of random graphs of given density p.

Let H be a graph on the vertex set $\{1,\dots,v(H)\}$, let ε and p be as above, and let n and m be integers satisfying $0 \le m \le n^2$. Let us denote by $\mathcal{G}(H,n,m,p,\varepsilon)$ the collection of all graphs G constructed in the following way. The vertex set of G is a disjoint union $V_1 \cup \ldots \cup V_{v(H)}$ of sets of size n, one for each vertex of H. For each edge $\{i, j\}$ of H, we add to G an (ε, p)-regular graph with m edges between the sets V_i and V_j. These are the only edges of G.

Intuitively, one would expect that all graphs in $\mathcal{G}(H, n, m, p, \varepsilon)$ contain a copy of H provided that $p \gg n^{-1/m_2(H)}$. It was observed by Łuczak [8] that this is not the case and for every $p = o(1)$, there are H-free graphs in $\mathcal{G}()$. Still, Kohayakawa, Łuczak, and Rödl [7] conjectured (hence the name KŁR conjecture) that there are very few such graphs. It has been verified only in some special cases: for all graphs H which do not contain a cycle, as well as when $H = K_3$, K_4, K_5 or C_ℓ. Our final result is a proof of the KŁR conjecture for all 2-balanced graphs.

Theorem 9. *Let H be a fixed 2-balanced graph and let*

$$\mathcal{G}^*(H, n, m, p, \varepsilon) = \{G \in \mathcal{G}(H, n, m, p, \varepsilon) \colon G \supset H\}.$$

Then for any positive β, there exist positive C, n_0, and ε such that for all n and m with $n \geq n_0$ and $m \geq Cn^{2-1/m_2(H)}$

$$\left|\mathcal{G}^*(H, n, m, m/n^2, \varepsilon)\right| \leq \beta^m \binom{n^2}{m}^{e(H)}.$$

Note that Theorem 9 easily implies Turán's theorem in random graphs, and its stability version.

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Linear trees in uniform hypergraphs

Zoltán Füredi

1. The main result, finding expanded forests in k-graphs

Given a graph H, the k-*blowup* (or k-expansion), denoted by $[H]^{(k)}$ (or $H^{(k)}$ for short), is the k-uniform hypergraph obtained from H by replacing each edge xy in H with a k-set E_{xy} that consists of x, y and $k-2$ new vertices such that for distinct edges $xy, x'y'$, $(E_{xy} - \{x, y\}) \cap (E_{x'y'} - \{x', y'\}) = \emptyset$. If H has p vertices and q edges, then $H^{(k)}$ has $p + q(k-2)$ vertices and q hyperedges. The resulting $H^{(k)}$ is a k-uniform hypergraph whose vertex set contains the vertex set of H.

Given a forest T, define the following

$$\sigma(T) := \min\{|X| + e(T \setminus X) : X \subset V(T) \text{ is independent in } T\}.$$

Here $T \setminus X$ is the forest left from T after deleting the vertices of X and the edges incident to them, $e(G)$ stands for the number of edges of the graph G.

Theorem 1. *Given a forest T with at least one edge and an integer $k \geq 4$. Then we have as $n \to \infty$, that*

$$\mathbf{ex}(n, T^{(k)}) = (\sigma(T) - 1 + o(1))\binom{n}{k-1}. \tag{1}$$

Our result, naturally, gives the same asymptotic as Theorem 5.3 in [4] whenever both can be applied to $T^{(k)}$. We **conjecture** that (1) holds for $k = 3$, too.

Let us note that Mubayi [11] and Pikhurko [12] determined precisely (for large n) the Turán number of the k-expansion of some other graphs, namely for the complete graph K_ℓ for $\ell > k \geq 3$. For smaller values of ℓ we know that $\mathbf{ex}_k(n, K_3^{(k)}) = \binom{n-1}{k-1}$ for $n > n(k)$, $k \geq 5$, a conjecture of Chvátal and Erdős established in [4].

2. Linear paths

Concerning linear paths of two edges for triple systems ($k = 3$) Erdős and Sós [1] determined $\mathbf{ex}_3(n, \mathcal{P}_2^{(3)})$ (it is $n-2$, or $n-1$, or n). For $k \geq 4$ they conjectured

$$\mathbf{ex}_k(n, \mathcal{P}_2^{(k)}) = \binom{n-2}{k-2}$$

for sufficiently large n with respect to k. This was proved by Frankl [2]. The case $k = 4$ was finished for all n by Keevash, Mubayi, and Wilson [9].

The case $\ell < k$ was asymptotically determined in [4].

Since the paper by G. Y. Katona and Kierstead [8] (1999) there is a renewed interest concerning paths and (Hamilton) cycles in uniform hypergraphs. Most of these are Dirac type results (large minimum degree implies the existence of the desired substructure) like in Kühn and Osthus [10], Rödl, Ruciński, and Szemerédi [13].

The present author, Tao Jiang, and Robert Seiver [7] determined $\mathbf{ex}_k(n, \mathcal{P}_\ell^{(k)})$ exactly, for **all** fixed k, ℓ, where $k \geq 4$, and sufficiently large n proving

$$(2) \qquad \mathbf{ex}_k(n, \mathcal{P}_{2t+1}^{(k)}) = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \ldots + \binom{n-t}{k-1},$$

where the only extremal family consists of all the k-sets in $[n]$ that meet some fixed set S of t elements, and

$$(3) \qquad \mathbf{ex}(n, \mathcal{P}_{2t+2}^{(k)}) = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \ldots + \binom{n-t}{k-1} + \binom{n-t-2}{k-2},$$

where the only extremal family consists of all the k-sets in $[n]$ that meet some fixed set S of t elements plus all the k-sets in $[n] \setminus S$ that contain some two fixed elements. 'Sufficiently large' n means that (2) and (3) hold when $kt = O(\log\log n)$. It is **conjectured** that they hold for all (or at least almost all) n's. The method in [7] does not quite work for the $k = 3$ case (cf. the remark after Lemma 3 below) but it is **conjectured** that still similar results hold for $k = 3$.

3. The product construction

Given two set systems (or hypergraphs) $\mathcal{A}$ and $\mathcal{B}$ their *join* is the family $\{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}$. We denote this new hypergraph by $\mathcal{A} \bowtie \mathcal{B}$.

Call a set Y 1-*cross-cut* of a family $\mathcal{C}$ if $|Y \cap E| = 1$ holds for all $E \in \mathcal{C}$. Define $\tau_1(\mathcal{C})$ as the minimum size of a 1-*cross-cut* of $\mathcal{C}$ (if such cross-cut exists, otherwise $\tau_1 := \infty$). One can see that every forest T and $k \geq 3$ the following holds.

$$\sigma(T) = \tau_1(T^{(k)}).$$

Thus $\sigma(T)$ is the minimum size of a set Y such that $T^{(k)}$ can be embedded into $\binom{Y}{1} \bowtie \binom{Z}{k-1}$ where Y and Z are disjoint sets, $|Z| \geq kq$. This means that in case of $Y := [\sigma - 1]$, $Z := [n] \setminus Y$ the hypergraph $\binom{Y}{1} \bowtie \binom{Z}{k-1}$ does not contain any copy of $T^{(k)}$. We obtain the lower bound

$$\mathbf{ex}(n, T^{(k)}) \geq |\{E : E \in \binom{[n]}{k}, |E \cap [\sigma - 1]| = 1\}| = (\sigma - 1)\binom{n - \sigma + 1}{k-1}.$$

4. The graph of 2-kernels, starting the proof with the delta-system method

Given a family $\mathcal{F} \subseteq \binom{[n]}{k}$, the *kernel-graph* with *threshold* s is a graph $G := G_{2,s}(\mathcal{F})$ on $[n]$ such that $\forall x, y \in [n]$, $xy \in E(G)$ if and only if $\deg^*_{\mathcal{F}}(\{x, y\}) \geq s$. Here $\deg^*_{\mathcal{F}}(W)$ stands for the number of pairwise disjoint edges in $\{F \setminus W : W \subset F \in \mathcal{F}\}$. The following (easy) lemma shows the importance of this definition.

Lemma 2 (see [7]). *Let H be a graph with q edges, $s = kq$, and let $\mathcal{F} \subseteq \binom{[n]}{k}$. Let G_2 be the kernel graph of $\mathcal{F}$ with threshold s. If $H \subseteq G_2$, then $\mathcal{F}$ contains a copy of $H^{(k)}$.* □

The *delta-system method* is a powerful tool for solving set system problems. Using a structural lemma from [5] and the method developed in [3, 4] the following theorem was obtained in [7] (see Theorem 3.8 and the proof of Lemma 4.3 there).

Lemma 3 (see [7])**.** *Let $\mathcal{F} \subseteq \binom{[n]}{k}$, T a forest of v vertices, $s = kv$, $G_2 := G_2(\mathcal{F})$, and suppose that $\mathcal{F}$ does not contain $T^{(k)}$. Then there is a constant $c := c(k, v)$ and a partition $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ with the following properties.*
— $|\mathcal{F}_1| \leq c\binom{n-2}{k-2}$.
— *Every edge $F \in \mathcal{F}_2$ has a center (not necessarily unique) $x(F) \in F$ such that $G_2|F$ contains a star of size $k-1$ with center $x(F)$. In other words, $\{x(F), y\} \in E(G_2)$ for all $y \in F \setminus \{x(F)\}$.* □

Actually, the delta-system method describes the intersection structure of $\mathcal{F}$ in a more detailed way, but for our purpose this lemma is sufficient. Lemma 3 and in fact the main result of this talk, Theorem 1 preceded (2)–(3), see [6].

Note that this is the only point where $k \geq 4$ is used. Lemma 3 is not true for $k = 3$. The 3-graph $\mathcal{F}^3$ obtained by joining a matching of size t and t one-element sets has $n = 3t$ vertices, $t^2 = n^2/9 = \Omega(n^{k-1})$ edges, it does not contain any linear tree except stars but $G_{2,s}(\mathcal{F}^3)$ forms a matching for every $s \geq 2$.

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Codegree densities of 3-graphs

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(joint work with Victor Falgas-Ravry, Oleg Pikhurko)

Given a 3-graph G, the codegree of a pair of distinct vertices $\{x, y\} \in V(G)$ is the number of 3-edges that contain $\{x, y\}$. We define $\delta_2(G)$ to be the minimum of the codegree over all pairs. For a forbidden 3-graph F, we define

$$\mathrm{coex}(n, F) = \max\{\delta_2(G) : G \text{ is an } F\text{-free 3-graph on } n \text{ vertices}\}.$$

The codegree density is defined to be

$$\gamma(F) = \lim_{n\to\infty} \frac{\mathrm{coex}(n, F)}{n}.$$

Mubayi and Zhao [3] showed that this limit always exists. Mubayi [2] showed that $\gamma(\text{Fano}) = 1/2$ and Keevash and Zhao [1] showed that for each $i \geq 1$ there is a 3-graph F for which $\gamma(F) = 1 - 1/i$.

Nagle [4] conjectured that $\gamma(K_4^-) = 1/4$ and $\gamma(K_4) = 1/2$, where K_4^- and K_4 are the 3-graphs on 4 vertices with 3 and 4 edges respectively.

Let T be a tournament on n vertices. Define $C_3(T)$ to be the 3-graph on n vertices that has an edge xyz iff $\{x, y, z\}$ induces an oriented 3-cycle in T. $C_3(T)$ has bipartite link graphs, and so does not contain K_4^- as a subgraph. If the arcs of T are chosen independently at random, with high probability

$$\delta_2(C_3(T)) = (1/4 + o(1))n.$$

Thus $\gamma(K_4^-) \geq 1/4$. Using a computer and flag algebras, we are able to show the following:

Proposition 1 (Density)**.** $\gamma(K_4^-) = 1/4$.

We can also show:

Proposition 2 (Stability)**.** *If G is K_4^--free and $\delta_2(G) \geq (1/4 + o(1))n$ then there is a tournament T such that G is at edit distance $o(n^3)$ from $C_3(T)$.*

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Loebl-Komlós-Sós Conjecture and structure of possibly sparse graphs

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(joint work with János Komlós, Diana Piguet, Miklós Simonovits, Maya J. Stein, Endre Szemerédi)

In the talk I outlined a technique to decompose graphs for the purpose of embedding trees. The motivation comes from the following two conjectures:

Conjecture 1 (Erdős-Sós Conjecture (1963, [2])). *Let G be a graph of order n with more than $(k-1)n/2$ edges. Then G contains each tree of order $k+1$ as a subgraph.*

Conjecture 2 (Loebl-Komlós-Sós Conjecture (1995, [3])). *Let G be a graph of order n with half of its vertices of degrees at least k. Then G contains each tree of order $k+1$ as a subgraph.*

The so-called "dense cases" of these conjectures seem to be approachable by the Szemerédi Regularity Lemma [6], that is when k is linear in n. Indeed, such a program was carried out in the context of the Loebl-Komlós-Sós Conjecture, see [5, 4, 1].

Ajtai, Komlós, Simonovits, and Szemerédi announced a solution of the Erdős-Sós Conjecture for all $k > k_0$ in the early 1990's. The key to this breakthrough is a decomposition of possibly sparse graphs which can be utilized for the purpose of embedding trees. In a relatively recent, and yet unpublished joint work with Komlós, Piguet, Simonovits, Stein, and Szemerédi we proved using a similar decomposition the following approximate version of the Loebl-Komlós-Sós Conjecture.

Theorem 3. *For each $\epsilon > 0$ there exists a k_0 such that for each $k > k_0$ and each graph G which has at least half of its vertices of degrees at least $(1+\epsilon)k$ contains all trees of order $k+1$ as subgraphs.*

In the talk I gave a high-level overview of the decomposition of n-vertex graphs with $\Theta(kn)$ edges, a key structural step in attacking the Erdős-Sós Conjecture, or the Loebl-Komlós-Sós Conjecture. This decomposition — like the Szemerédi Regularity Lemma — is not perfect, i.e., we allow $o(kn)$ edges to be lost. There are three main components of the decomposition: vertices of huge degree ($\gg k$), dense spots (which are dense graphs of order $\Theta(k)$ contained in the host graph), and an expander-like graph. The dense spots are further regularized using a variant of the Szemerédi Regularity Lemma. I tried to indicate why is each of these three components useful for embedding trees.

While this decomposition result is a very general one, it seems to be limited only to problems involving embeddings of trees. This is not surprising as there seems not to exist a "canonical" or "ultimate" regularity lemma for general sparse graphs. It would be of interest to find other applications of this technique, for example for hypergraphs.

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