Mathematisches Forschungsinstitut Oberwolfach

Report No. 23/2012

DOI: 10.4171/OWR/2012/23

Singularity Theory and Integrable Systems

Organised by
Boris Dubrovin, Trieste
Claus Hertling, Mannheim
Ian A. B. Strachan, Glasgow
Katrin Wendland, Freiburg

April 22nd – April 28th, 2012

Abstract. The workshop brought together three very different areas of mathematics, namely singularity theory, integrable systems and quantum cohomology. They are linked by their applications in topological quantum field theory and by constructions of (often isomorphic) Frobenius manifolds. The first and second are related by a version of mirror symmetry, the link of the second and third has attracted much attention. The connection of the first and third is the least developed and was at the focus of the workshop.

Mathematics Subject Classification (2000): 14B, 32S, 37K.

Introduction by the Organisers

Three very different areas lead to the same rich geometry of which Frobenius manifolds are the foundation. These are singularity theory, integrable systems and quantum cohomology. The connection between singularity theory and quantum cohomology via this geometry is one version of mirror symmetry, the connection between integrable systems and quantum cohomology has also been at the focus of a lot of activity. However, the direct connection between singularity theory and integrable systems has been somewhat neglected until recently. The aim of the workshop was to take up this recent work and to stimulate from different sides the further development of this connection.

This rich geometry is a twofold extension of the concept of a Frobenius manifold. In quantum cohomology Frobenius manifolds encode geometrically genus zero Gromov-Witten invariants without gravitational descendents. This twofold extension is given by including both Gromov-Witten invariants of arbitrary genus and gravitational descendents. Such extensions are naturally present in the theory of integrable systems: higher genus data and the dependence on infinitely many variables are naturally built into the general theory of such systems.

In contrast to this, in singularity theory only the geometry and meaning of the Frobenius manifolds is clear. The meaning of higher data is largely mysterious. Only the G-function F_1, which corresponds to genus one data without gravitational descendents, has been intensively studied and been used for predictions on the distribution of the spectral numbers of singularities.

This is unsatisfying as there is a well defined twofold extension for an arbitrary semisimple Frobenius manifold M. This was proposed in terms of a formula for a generating function $\mathcal{D}^M$ in infinitely many variables called *total descendent potential* by Givental and Dubrovin&Zhang in 2001. The generating function is obtained by applying a big operator to a product of $n(=\dim$ of the Frobenius manifold) copies of the generating function $\mathcal{D}^{A_1}$ of the quantum cohomology of a point. The big operator arises by a quantization (within Givental's symplectic loop space formalism) of a formula arising from the flat structure connection on $\mathbb{P}^1 \times M$ which is associated to the Frobenius manifold M and which has a logarithmic pole along $\{\infty\} \times M$ and an irregular pole along $\{0\} \times M$.

The workshop was a half size workshop with many young participants. There were 15 talks. All talks dealt with parts of the structures mentioned above and almost all with the connection between two of the three areas mentioned above.

Three talks presented mirror symmetry results related to singularities, one for the hyperbolic unimodal singularities, two for certain Laurent polynomials. These were in terms of isomorphism of Frobenius manifolds. The two for Laurent polynomials built on a good control of the corresponding global Gauss-Manin systems. Other talks went to the higher genus data for singularities, one starting with oscillating integrals, one starting with data of semisimple Frobenius manifolds. This one presented an impressive rather explicit (but still in an experimental status) formula for the genus 2 potential with gravitational descendents. Another talk also dealt with higher genus data for singularities, but by a construction which has some similarity with that of quantum cohomology. It leads to a new rich picture of connections between different sides, called Landau-Ginzburg mirror symmetry.

Several talks took a very general perspective. One resumed the Givental theory and many different techniques to control $\mathcal{D}^M$. Another aimed at a categorical construction of the central data for singularities, their Brieskorn lattices or, in other words, their non-commutative Hodge structure. Another related in a general way integrable systems and open symplectic manifolds.

Two talks discussed structures related to the differential geometry of Frobenius manifolds, one gave a survey of a many different variants of the underlying geometry including tt^* geometry, another reviewed the geometry on certain varieties which arise naturally in algebraic geometry. Another talk presented explicit global (without singularities) solutions of the tt^* equations and identified some of them via their Stokes structure with examples from singularity theory and quantum cohomology.

One talk concentrated on Stokes structures by themselves and interpreted the base spaces of the simple and the simple elliptic singularities as atlasses of Stokes data. Another used quantum cohomology, algebraic geometry and numerical methods to find several 3-dimensional Calabi-Yau manifolds with surprising similarities and differences. Finally, a lecture of double length was given by Varchenko on his work on critical points of master functions. He has relations to integrable hierarchies and to quantum cohomology (for flag varieties) which might lead to possible further applications of the main themes of the workshop.

The workshop did not solve the mystery of the meaning of the higher genus data within singularity theory. But it made the participants aware of other aspects of the landscape. It showed that mirror symmetry for singularities has developed a lot in very recent years and that several techniques are maturing right now. It makes us optimistic about a lot of progress in the near future.

Workshop: Singularity Theory and Integrable Systems

Table of Contents

Abstracts

Gromov-Witten theory of orbifold projective lines and integrable hierarchies

Todor Milanov
(joint work with Hsian-Hua Tseng, Yefeng Shen)

In my joint work with H.-H. Tseng [6] we constructed an integrable hierarchy that governs the Gromov–Witten invariants of the orbifold projective line equipped with two orbifold points. Part of our work was generalized by P. Rossi [7], who computed the quantum cohomology of $\mathbb{P}^1_{r_1,r_2,r_3}$. The latter is the projective line equipped with 3 orbifold points with isotropy group $\mathbb{Z}/r_i\mathbb{Z}$, where the weights r_i are positive integers satisfying:

$$\chi_{\rm orb}(\mathbb{P}^1_{r_1,r_2,r_3}) := \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-1>0.$$

The goal of our project is to construct an integrable hierarchy that governs the Gromov–Witten invariants of $\mathbb{P}^1_{r_1,r_2,r_3}$. To simplify the exposition, I presented our ideas for the case of $r_3=1$, which corresponds to our previous work with H.-H. Tseng. Even in that case our new construction looks quite different. Our starting point is the Landau–Ginzburg potential of $\mathbb{P}^1_{r_1,r_2,1}$

$$W(x)=x^{r_1}+(Q/x)^{r_2},$$

where $Q\neq 0$ is a complex parameter, and its miniversal deformation

$$f(t,x)=x^{r_1}+\Big(\frac{q}{x}\Big)^{r_2}+t_{01}+\sum_{i=1}^{r_1-1}t_{1i}x^i+\sum_{j=1}^{r_2-1}t_{2j}\Big(\frac{q}{x}\Big)^j,$$

where $q:=Qe^{t_{02}}$ and the remaining deformation parameters are viewed as coordinates on $B:=\mathbb{C}^*\times\mathbb{C}^N$, $N=r_1+r_2-1$. It was pointed out by B. Dubrovin (see [1]) that the space B has a Frobenius structure, i.e., the *period integrals*

$$I^{(-n)}_\beta(t,\lambda):=-d^B\int_{\beta_{t,\lambda}}\frac{(\lambda-f(t,x))^n}{n!}\omega,\quad\in\quad T^*_{t,\lambda}B,\quad \omega:=\frac{dx}{x},$$

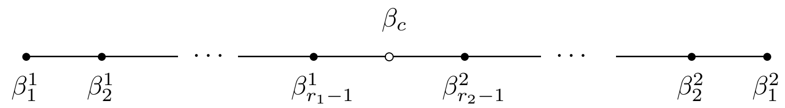
satisfy a system of differential equations in $(t,\lambda)\in B\times\mathbb{C}$. The notation in the above formula is as follows. Put $f_t(x):=f(t,x)$ and let $X_{t,\lambda}:=f_t^{-1}(\lambda)$ be the *Milnor fibers*; then d^B is the de Rham differential on B, the integration cycles, when t and λ very, form a flat family of relative homology cycles $\beta_{t,\lambda}\in H_1(\mathbb{C},X_{t,\lambda};\mathbb{C})$. Let us look at the periods corresponding to $n=0$ and $n=1$. For a fixed (t,λ) we have an isomorphism

$$I^{(-1)}(t,\lambda):\mathfrak{h}:=H_1(\mathbb{C},X_{t,\lambda};\mathbb{C})\longrightarrow T_t^*B,$$

while $I^{(0)}(t,\lambda)$ gives an embedding of $\overline{\mathfrak{h}}:=\widetilde{H}_0(X_{t,\lambda};\mathbb{C})$ in T_t^*B. The relation between $\mathfrak{h}$ and $\overline{\mathfrak{h}}$ is given by the standar long exact homology sequence of a pair:

$$0\longrightarrow H_1(\mathbb{C}^*,\mathbb{Z})\longrightarrow H_1(\mathbb{C},X_{t,\lambda};\mathbb{Z})\longrightarrow \widetilde{H}_0(X_{t,\lambda};\mathbb{Z})\longrightarrow 0$$

The vanishing cycles in $X_{t,\lambda}$ form a root system of type A_N. Moreover, if we let $\lambda \to \infty$ we see that there are 3 types of vanishing cycles: 1) supported near $x = \infty$; 2) supported near $x = 0$; and 3) supported at a point near $x = \infty$ and a point near $x = 0$. If we choose simple roots we get a Dynkin diagram as the one on the figure below, where β_i^1, β_j^2, and β_c are vanishing cycles of the types 1), 2), and 3) respectively.



The homology group $\overline{\mathfrak{h}}$ is also equipped with a monodromy action. Namely if we have a loop in $B \times \mathbb{C}$ based at (t, λ) and avoiding the discriminant of the Milnor fibration (where the periods acquire poles), then the parallel transport with respect to the Gauss-Manin connection of the vanishing homology induces a monodromy representation. In particular, if we take a big loop around the discriminant, i.e., a loop in $\{t\} \times \mathbb{C}$ near $\lambda = \infty$ we get a monodromy transformation σ, which can be expressed in terms of the simple reflections as follows:

$$\sigma = \Big(r_{\beta_1^1} r_{\beta_2^1} \cdots r_{\beta_{r_1-1}^1} \Big) \Big(r_{\beta_1^2} r_{\beta_2^2} \cdots r_{\beta_{r_2-1}^2} \Big).$$

Our main observation is that if we apply the vertex operator construction of [3]; then we obtain a Fock space realization of the basic representation of the affine Lie algebra of type $A_N^{(1)}$. This realization fits into the construction of V. Kac and D. Peterson (see [4]): they have constructed a realization of the basic representation corresponding to a choice of a conjugacy class in the Weyl group. In our settings the conjugacy class is the one determined by the monodromy transformation σ. Moreover, the Fock space realization leads naturally to an integrable system of Hirota bilinear equations known as *Kac-Wakimoto* hierarchy (see [5]). We can prove that the total descendant potential of $\mathbb{P}^1_{r_1,r_2,1}$ is a tau-function, i.e., solution to this hierarchy. Our project however is still unfinished, because the integrable hierarchy that we have constructed does not characterize the Gromov–Witten invariants, since our integrable hierarchy does not have dynamical variables corresponding to the descendant insertions of $1 \in H^*_{\mathrm{orb}}(\mathbb{P}^1_{r_1,r_2,1}; \mathbb{C})$. In other words, our conjecture is that the integrable hierarchy that governs the full Gromov–Witten theory of $\mathbb{P}^1$ is an extension of the Kac Wakimoto hierarchy corresponding to σ.

For the D and E cases a similar construction can be done by using the tri-polynomials of P. Rossi [7]. The outcome is the following: β_c is the branching node of the Dynkin diagram while σ will be the product of the remaining reflections. Again our conjecture is that the Gromov–Witten theory of the corresponding orbifold is governed by an integrable hierarchy which is an extension of the Kac-Wakimoto hierarchy corresponding to σ.

References

[1] B. Dubrovin, *Geometry of 2D topological field theories.* In: Integrable Systems and Quantum Groups. Lecture Notes in Mathematics, Vol. **1620**, Springer-Verlag, New York (1996), 120–348.

[2] B. Dubrovin and Y. Zhang, *Extended Affine Weyl groups and Frobenius manifolds.* Compos. Math. **111**(1998), 167–219.

[3] E. Frenkel, A. Givental, and T. Milanov, *Soliton equations, vertex operators, and simple singularities.* Functional analysis and other mathematics **3**(2010), 47–63.

[4] V. Kac and D. Peterson: 112 *constructions of the basic representation of the loop group of* E_8. Symposium on anomalies, geometry, topology (Chicago, Ill., 1985), 276–298, World Sci. Publishing, Singapore, 1985.

[5] V. Kac and M. Wakimoto: *Exceptional hierarchies of Soliton Equations.* Proc. Symp. Pure Math. **49**, pp. 138–177, AMS, 1989.

[6] T. Milanov and H.-H. Tseng: *The Spaces of Laurent Polynomials,* $\mathbb{P}^1$*-orbifolds, and Integrable Hierarchies.* Journal fuer die Reine und Angewandte Mathematik **622**(2008): 189–235.

[7] P. Rossi: *Gromov–Witten theory of orbi-curves, the space of tri-polynomials, and symplectic field theory of Seifert fibrations.* Math. Ann. 348.2(2010), 265–287.

Mirror symmetry between orbifold projective lines and cusp singularities

Atsushi Takahashi

Let A be a triple of positive integers (a_1, a_2, a_3). For the later use, we set

$$\mu_A := 2 + \sum_{i=1}^{3} (a_i - 1), \quad \chi_A := 2 + \sum_{i=1}^{3} \left(\frac{1}{a_i} - 1\right).$$

We can naturally associate three mathematical objects from completely different origins;

- an orbifold projective lines $\mathbb{P}^1_A := \mathbb{P}^1_{a_1,a_2,a_3}$, an orbifold $\mathbb{P}^1$ with at most three isotropic points of orders a_1, a_2, a_3.
- (Q_A, I) is the quiver with relations given in Figure 1.
- A polynomial $f_A(x_1, x_2, x_3) := x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - q^{-1}x_1x_2x_3$, $q \in \mathbb{C}\backslash\{0\}$.

At a glance, $\mathbb{P}^1_A$, (Q_A, I) and f_A are irrelevant. However, it turns out by mirror symmetry that they can be considered as three different realizations of more intrinsic object behind them. We here discuss their classical mirror symmetry, an isomorphism of Frobenius manifolds.

Conjecture.
There should exist isomorphisms of Frobenius manifolds among

- $M_{\mathbb{P}^1_A}$, the one constructed from the theory of Gromov–Witten invariants for $\mathbb{P}^1_A$,
- $M_{\widehat{W}_A}$, the one (should be) constructed from the invariant theory of the extended Weyl group $\widehat{W}_A$ associated to the quiver with relations (Q_A, I),
- M_{f_A,ζ_A}, the one constructed from the universal unfolding of f_A by the primitive form ζ_A.

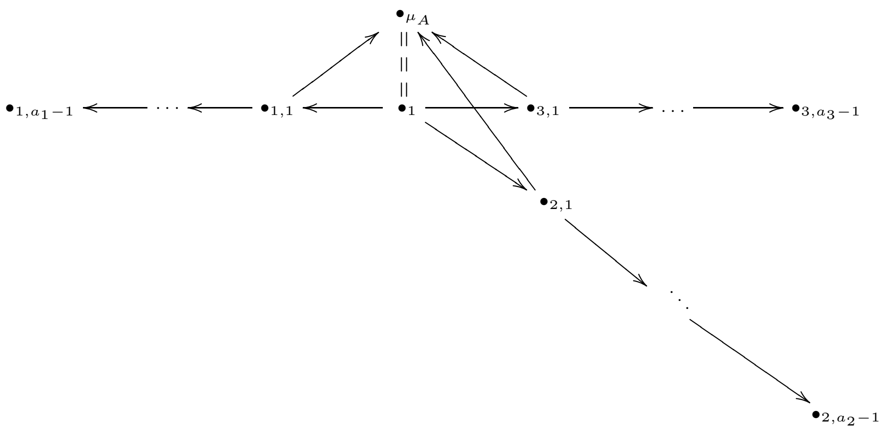


FIGURE 1. The quiver with relations (Q_A, I). The double dotted line denotes two generic linear combinations of three paths from $\bullet_1$ to $\bullet_{\mu_A}$ generating the ideal I.

Conjecture is known to hold if $a_i = 1$ for some $i = 1, 2, 3$ by Milanov–Tseng [3], if $\chi_A > 0$ by Rossi [4] and if $\chi_A = 0$ by Satake–Takahashi [6]. The following is our main result.

Theorem1 (Ishibashi–Shiraishi–Takahashi [2]). There exists a unique Frobenius manifold M with flat coordinates

$$t_1,\, t_{1,1}, \dots, t_{1,a_1-1},\, t_{2,1}, \dots, t_{2,a_2-1},\, t_{3,1}, \dots, t_{3,a_3-1},\, t_{\mu_A}$$

satisfying the following conditions:

(1) There exists a (local) isomorphism of complex manifolds:

$$M \simeq \mathbb{C} \times \mathbb{C}^{(a_1-1)+(a_2-1)+(a_3-1)} \times \mathbb{C}^*$$
$$s \mapsto \left(t_1(s), t_{1,1}(s), \dots, t_{3,a_3-1}(s), e^{t_{\mu_A}(s)} \right).$$

(2) The unit vector field e and the Euler vector field E are given by

$$e = \frac{\partial}{\partial t_1},\; E = t_1 \frac{\partial}{\partial t_1} + \sum_{i=1}^{3} \sum_{j=1}^{a_i-1} \frac{a_i - j}{a_i} t_{i,j} \frac{\partial}{\partial t_{i,j}} + \chi_A \frac{\partial}{\partial t_{\mu_A}}.$$

(3) The non-degenerate symmetric bilinear form η is normalized as

$$\eta\left(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_{\mu_A}}\right) = \eta\left(\frac{\partial}{\partial t_{\mu_A}}, \frac{\partial}{\partial t_1}\right) = 1,$$
$$\eta\left(\frac{\partial}{\partial t_{i_1,j_1}}, \frac{\partial}{\partial t_{i_2,j_2}}\right) = \begin{cases} \frac{1}{a_{i_1}}, & \text{if } i_1 = i_2 \text{ and } j_2 = a_{i_1} - j_1. \\ 0, & \text{otherwise.} \end{cases}$$

(4) The restriction of the Frobenius potential $\mathcal{F}$ to $\{t_1 = 0\}$ is homogeneous of degree 2 with respect to E. Namely, we have $E\mathcal{F}|_{t_1=0} = 2\mathcal{F}|_{t_1=0}$. $\mathcal{F}$ also satisfies

$$\mathcal{F}|_{t_1=0} \in \mathbb{C}\left[[t_{1,1},\dots,t_{1,a_1-1},t_{2,1},\dots,t_{2,a_2-1},t_{3,1},\dots,t_{3,a_3-1},e^{t_{\mu_A}}]\right]$$

(5) Assume the previous condition (iv). Then, the product $\circ$ can be extended to the limit $t_1 = t_{1,1} = \cdots = t_{3,a_3-1} = 0$, $e^{t_{\mu_A}} \to 0$ in the frame $\frac{\partial}{\partial t_1}$, $\frac{\partial}{\partial t_{1,1}},\dots,\frac{\partial}{\partial t_{3,a_3-1}}$, $\frac{\partial}{\partial t_{\mu_A}}$ of TM. The $\mathbb{C}$-algebra obtained in this limit is isomorphic to the quotient ring

$$\mathbb{C}[x_1,x_2,x_3]\,/(x_1x_2,x_2x_3,x_3x_1,a_1x_1^{a_1}-a_2x_2^{a_2},a_2x_2^{a_2}-a_3x_3^{a_3})\ ,$$

where $\partial/\partial t_{1,1},\partial/\partial t_{2,1},\partial/\partial t_{3,1}$ are mapped to x_1,x_2,x_3, respectively.

(6) The term $t_{1,1}t_{2,1}t_{3,1}e^{t_{\mu_A}}$ occurs with the coeffcient 1 in $\mathcal{F}$.

This Theorem enables us to simplify the proofs given by Milanov–Tseng [3] and Rossi [4] and to generalize them to the case $\chi_A \le 0$.

Corollary 1 ([2]). We have an isomorphism of Frobenius manifolds

$$M_{\mathbb{P}^1_A} \simeq M_{f_A,\zeta_A}. \tag{1}$$

We also have the following uniqueness theorem, which may be known for experts. We refer to Theorem 2.1 of [1] and Proposition 5.2 of [5] for the relevant statements.

Theorem 2 ([2]). A Frobenius manifold of rank μ and of dimension one with the following e and E is uniquely determined by the intersection form I :

$$e = \frac{\partial}{\partial t_1},\ E = t_1\frac{\partial}{\partial t_1} + \sum_{i=1}^{3}\sum_{j=1}^{a_i-1}\frac{a_i-j}{a_i}t_{i,j}\frac{\partial}{\partial t_{i,j}} + \chi_A\frac{\partial}{\partial t_\mu}. \tag{2}$$

By Theorem 2, it is now possible to give an isomorphism of Frobenius manifolds between $M_{\widehat{W}_A}$ and M_{f_A,ζ_A} :

Corollary 2 ([2]). Assume that $\chi_A > 0$. We have an isomorphism of Frobenius manifolds

$$M_{\widehat{W}_A} \simeq M_{f_A,\zeta_A}, \tag{3}$$

where $M_{\widehat{W}_A}$ is the one constructed in [1].

References

[1] B. Dubrovin and Y. Zhang, *Extended affine Weyl groups and Frobenius manifolds*, Compositio Mathematica, (1998) 167–219.

[2] Y. Ishibashi, Y. Shiraishi and A. Takahashi, in preparation.

[3] T. E. Milanov, H.-H. Tseng, *The space of Laurent polynomials, $\mathbb{P}^1$-orbifolds, and integrable hierarchies*, arXiv: 0607012v3.

[4] P. Rossi, *Gromov-Witten theory of orbicurves, the space of tri-polynomials and Symplectic Field Theory of Seifert fibrations*, arXiv:0808.2626.

[5] I. Satake, *Frobenius manifolds for elliptic root systems*, Osaka J. Math. **47** (2010) 301–330.

[6] I. Satake and A. Takahashi, *Gromov–Witten invariants for mirror orbifolds of simple elliptic singularities*, arXiv:1103.0951.

On the Hodge theory of differential graded categories

Dmytro Shklyarov

Given a smooth complex algebraic variety X there are isomorphisms [3, 9]

$$(1) \qquad HP_{2k}(\mathsf{Perf}(X)) \simeq \oplus_p H^{2p}_{DR}(X), \quad HP_{2k+1}(\mathsf{Perf}(X)) \simeq \oplus_p H^{2p+1}_{DR}(X)$$

where $\mathsf{Perf}(X)$ is the differential $\mathbb{Z}$-graded category of perfect complexes on X and $HP_\bullet$ are the periodic cyclic homology groups. The de Rham cohomology of an algebraic variety carries a (mixed) Hodge structure, and a natural question to ask is whether $\mathsf{Perf}(X)$ "remembers" any part of that structure or, more generally, whether the periodic cyclic homology of *any* differential graded category carries any functorial Hodge-like structure. The study of these questions has been initiated by M. Kontsevich and his collaborators (see [2] and references therein) in connection with the homological mirror symmetry program. Our aim here is to discuss one piece of the emerging picture, namely, an analog of the Hodge filtration in the categorical setting. We will recall the idea of the construction and present one application, obtained in [8], to isolated singularities.

The *negative* cyclic homology $HN_\bullet(\mathsf{D})$ of a differential $\mathbb{Z}$-graded category[1] D is the cohomology of the complex $(C_\bullet(\mathsf{D})[[u]]^{\mathrm{gr}}, b+uB)$ where $(C_\bullet(\mathsf{D}), b)$ is the Hochschild chain complex of D, B is the Connes differential, u is a formal variable (of degree 2), and $C_\bullet(\mathsf{D})[[u]]^{\mathrm{gr}}$ is the subspace in $C_\bullet(\mathsf{D})[[u]]$ (:= formal series with coefficients in $C_\bullet(\mathsf{D})$) spanned by homogeneous series. The *periodic* cyclic homology $HP_\bullet(\mathsf{D})$ is defined similarly but with $C_\bullet(\mathsf{D})[[u]]$ replaced by formal Laurent series $C_\bullet(\mathsf{D})((u))$ (since u is invertible now, all the groups $HP_{2k}(\mathsf{D})$ resp. $HP_{2k+1}(\mathsf{D})$ are isomorphic to each other).

With some obvious changes the definitions of the negative and periodic cyclic homology extend to the case of differential $\mathbb{Z}/2$-graded categories. Instead of the individual groups $HN_\bullet$ and $HP_\bullet$ one gets $\mathbb{C}[[u]]$-modules $HN_\pm$ and $\mathbb{C}((u))$-linear spaces $HP_\pm$. A standard example of a $\mathbb{Z}/2$-graded category is the category $\mathsf{MF}(w)$ of matrix factorizations of a series $w \in R = \mathbb{C}[[y_0, \dots, y_m]]$ with an isolated singularity at the origin (see [1] and references therein). The negative and periodic cyclic homology of $\mathsf{MF}(w)$ also have a geometric meaning: by [7] there is a quasi-isomorphism from the cyclic complex of $\mathsf{MF}(w)$ to the formal twisted de Rham complex associated with w

$$(2) \qquad (C_\bullet(\mathsf{MF}(w))((u)), b+uB) \simeq (\Omega^\bullet_{Spec\, R}((u)), -dw+ud)$$

which maps $C_\bullet(\mathsf{MF}(w))[[u]]$ to $\Omega^\bullet_{Spec\, R}[[u]]$.

For simplicity, we will assume from now on that the periodic cyclic homology groups of our categories are finite-dimensional[2].

In the case of differential $\mathbb{Z}$-graded categories the analog of the Hodge filtration is a filtration on the periodic cyclic homology groups defined via an analog of the Hodge-to-de Rham spectral sequence, in complete agreement with the classical picture. Namely, the complex computing $HP_\bullet(\mathsf{D})$ comes equipped with a canonical

[1] All our categories will be $\mathbb{C}$-linear.

[2] over $\mathbb{C}$ in the $\mathbb{Z}$-graded case and over $\mathbb{C}((u))$ in the $\mathbb{Z}/2$-graded case

decreasing filtration by the sub-complexes $u^n C_\bullet(\mathsf{D})[[u]]^{\mathrm{gr}}$, and the induced filtration on $HP_\bullet(\mathsf{D})$ is what we need. To see the connection to the classical case, note that the above filtration on the cyclic complex gives rise to a spectral sequence with $E_1 = HH_\bullet(\mathsf{D})((u))^{\mathrm{gr}}$ where $HH_\bullet$ is the Hochschild homology. Given that $HH_k(\mathsf{Perf}(X)) \simeq \prod_{q-p=k} H^q(X, \Omega_X^p)$(cf. [3, 9]), this spectral sequence is a close relative of the classical Hodge-to-de Rham spectral sequence.

In the example $\mathsf{D} = \mathsf{Perf}(X)$, where X is a smooth projective variety, the above filtration on, say, $HP_0(\mathsf{D})$ transforms under the isomorphism (1) into

$$F^n \left(\oplus_p H^{2p}_{DR}(X) \right) = \oplus_{p-q \geq 2n} H^{p,q}(X)$$

Note that on each $H^{2p}_{DR}(X)$ it differs from the usual Hodge filtration by a shift.

Let us explain now how all the filtered vector spaces $HP_\bullet(\mathsf{D})$ can be encoded in one simple additional structure on the cyclic complex. Let D, as before, be a differential $\mathbb{Z}$-graded category. Let us treat it as a $\mathbb{Z}/2$-graded category with the following additional property: its Hochschild complex carries an operator γ such that $[\gamma, b] = b$ and $[\gamma, B] = -B$ (γ is just the grading operator). Then the differential operator $\partial_u := \frac{d}{du} + \frac{\gamma}{2u}$ on $C_\pm(\mathsf{D})((u))$ satisfies the property

$$(3) \qquad [\partial_u, b + uB] = \frac{1}{2u}(b + uB)$$

and therefore descends to $HP_\pm(\mathsf{D})$. Recall (see, for instance, [5]) that given an integer N there is a functor from the category of triples $(\mathcal{H}, \nabla, \mathcal{H}^0)$, where $\mathcal{H}$ is a finite-dimensional vector space over $\mathbb{C}((u))$ equipped with a regular singular *connection* ∇ and a *lattice* $\mathcal{H}^0$, to the category of finite-dimensional filtered vector spaces (over $\mathbb{C}$) defined as follows:

$$(4) \qquad H_N := \oplus_{\frac{N}{2}-1 < a \leq \frac{N}{2}} \mathcal{H}^a, \; F^n H_N := \oplus_{\frac{N}{2}-1 < a \leq \frac{N}{2}} u^n \mathcal{H}^0 \cap \mathcal{V}^a / u^n \mathcal{H}^0 \cap \mathcal{V}^{>a}$$

where[3] $\mathcal{H}^a := \cup_n \mathrm{Ker}(u\nabla_{\frac{d}{du}} - a)^n$, $\mathcal{V}^a := \oplus_{a \leq b < a+1} \mathcal{H}^b[[u]]$, and $\mathcal{V}^{>a} := \cup_{b>a} \mathcal{V}^b$. A simple computation shows that for $N = 2k$ (resp. $2k+1$), $\mathcal{H} = HP_+(\mathsf{D})$ (resp. $HP_-(\mathsf{D})$), $\nabla_{\frac{d}{du}} = \partial_u$, and $\mathcal{H}^0 = \mathrm{Im}(HN_+(\mathsf{D}) \to HP_+(\mathsf{D}))$ (resp. $\mathrm{Im}(HN_-(\mathsf{D}) \to HP_-(\mathsf{D}))$) the functor reproduces $HP_{2k}(\mathsf{D})$ (resp. $HP_{2k+1}(\mathsf{D})$) with its filtration.

Theorem 1. *Let* D *be any differential* $\mathbb{Z}/2$*-graded category. There is an operator* $\partial_u = \frac{d}{du} + A(u)$ *on* $C_\pm(\mathsf{D})((u))$ *such that, firstly, the property (3) holds (in particular,* ∂_u *descends to* $HP_\pm(\mathsf{D})$*); and secondly, the induced operator on* $HP_\pm(\mathsf{D})$ *coincides with the one considered previously when* D *is* $\mathbb{Z}$*-graded.*

An explicit formula for ∂_u can be found in [8]. In general, the induced connection on the periodic cyclic homology has a pole of order 2 with respect to the lattice given by the image of the negative cyclic homology. For instance, in the case $D = \mathsf{MF}(w)$ one has:

[3] Let us assume for simplicity that the a's for which $\mathcal{H}^a$ is non-trivial are all real.

Theorem 2. ([8]) *Under the quasi-isomorphism (2) the operator from the previous theorem transforms into the operator*

$$\partial_u = \frac{d}{du} + \frac{w}{u^2} - \frac{\tilde{\gamma}}{2u} \tag{5}$$

on $\Omega^\bullet_{Spec\,R}((u))$ *where* w *is the operator of multiplication with* w *and* $\tilde{\gamma}$ *is the grading operator on* $\Omega^\bullet_{Spec\,R}$.

This result has a curious Hodge theoretic meaning. Let w be a germ of a holomorphic function. It is a classical fact (cf. [4, Sec.6]) that the connection on the formal twisted de Rham cohomology induced by (5) is, up to a shift, a Fourier-Laplace-like transform of the Gauss-Manin system of w (in particular, it has a regular singularity). Then Theorem 2, together with [6, Sec.5], implies that the construction (4), when applied to $HP_\pm(\mathsf{MF}(w))$, reproduces (again, up to a shift) the Steenbrink Hodge filtration on the vanishing cohomology of w. Moreover, the connection (5) encodes other invariants of the singularity (e.g. its spectrum or, rather, a shifted version of it) and, thus, Theorem 2 provides a natural way to extract those invariants from the category $\mathsf{MF}(w)$.

On a final note, the correct point of view seems to be that the connection (together with the lattice) from Theorem 1 should itself be viewed as a kind of "non-commutative Hodge filtration". For a review of the origin and development of this and related ideas see [2, 5].

References

[1] T. Dyckerhoff, *Compact generators in categories of matrix factorizations*, Duke Math. J. **159** (2011), no. 2, 223–274.

[2] L. Katzarkov, M. Kontsevich, T. Pantev, *Hodge theoretic aspects of mirror symmetry*, From Hodge theory to integrability and TQFT tt*-geometry, 87–174, Proc. Sympos. Pure Math., 78, Amer. Math. Soc., Providence, RI, 2008.

[3] B. Keller, *On the cyclic homology of ringed spaces and schemes*, Doc. Math. **3** (1998), 231–259 (electronic).

[4] B. Malgrange, *Intégrales asymptotiques et monodromie*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 405–430.

[5] C. Sabbah, *Non-commutative Hodge structures*, arXiv:1107.5890

[6] J. Scherk, J. H. M. Steenbrink, *On the mixed Hodge structure on the cohomology of the Milnor fibre*, Math. Ann. **271** (1985), no. 4, 641–665.

[7] E. Segal, *The closed state space of affine Landau-Ginzburg B-models*, arXiv:0904.1339.

[8] D. Shklyarov, *Non-commutative Hodge structures: Towards matching categorical and geometric examples*, arXiv:1107.3156

[9] C. Weibel, *The Hodge filtration and cyclic homology*, K-Theory **12** (1997), no. 2, 145–164.

Different interpretations of the Givental group action

Sergey Shadrin

Givental group action is nowadays one of the most important and useful tools in the theory of Frobenius manifolds, and it deserves a study on its own. Indeed, many of its interpretations come from completely different points of view on Frobenius manifolds and their applications. The interaction of different ways to define the

Givental group action reflect the interaction of different areas of mathematics, most notably geometry of the moduli space of curves, resolutions of BV algebras, and integrable hierarchies.

Here we briefly discuss four different ways to think about the Givental group action. To be more precise, we consider only a half of it, the action of the so-called upper-triangular group. In fact, it is the most interesting part since it is exactly the one that changes the underlying Frobenius structure. On subtlety related to differences in exposition below: in the first two interpretations we have decided to ignore the scalar product, for simplicity of the exposition, therefore we have a bigger group than in the last two interpretations.

Trivialization of the BV operator

The most essential part of the structure of formal Frobenius manifolds is the so-called hypercommutative algebra. That is, for $n \geq 2$ we have n-ary (graded) symmetric operation $m_n\colon V^{\otimes n} \to V$, and these operations satisfy the WDVV relations. The operations m_n should be thought of either as the representation of the fundamental class of the moduli space of curves $\overline{\mathcal{M}}_{0,1+n}$, or as the operations whose structure constants are given by expansion of the prepotential of the Frobenius manifold in flat coordinates.

The BV/Δ-structure on a vector space W is the structure of a differential $\mathbb{Z}_2$-graded commutative associative algebra with the differential $Q\colon W \to W$ and the product $\mu\colon W^{\otimes 2} \to W$. There is an extra odd operator, $\Delta\colon W \to W$, satisfying the conditions $\Delta^2 = 0$, $[Q, \Delta] = 0$, Δ is an operator of the second order with respect to μ. This would be a BV-algebra structure. We add an extra piece of the structure in order to homotopically resolve the BV operator Δ. That is, we have operators $\phi_i\colon W \to W$, $i = 1, 2, \dots$, such that $d + z\Delta = \exp(-\sum_{i=1}^{\infty} \phi_i) d \exp(\sum_{i=1}^{\infty} \phi_i)$.

In a recent preprint [5] with A. Khoroshkin and N. Markarian, we constructed an explicit quasi-isomorphism of the two operads behind these structures. This means that for any hypercommutative algebra V one can find a BV/Δ algebra W such that $V = H^*(W, Q)$ and the hypercommutative algebra structure on V comes from a higher BV/Δ structure on W. Moreover, there are explicit formulas that relate these two structures.

Now we consider an arbitrary formal power series $R(z) \in End(V) \otimes \mathbb{C}[[z]]$, $R(z) = Id + R_1 z + \dots$. We can lift it to a series $\bar{R}(z) \in End(W) \otimes \mathbb{C}[[z]]$ in such a way that $d = \bar{R}(z)^{-1} d \bar{R}(z)$. Then we can replace the series $\sum_{i=1}^{\infty} \phi_i z^i$ with $\sum_{i=1}^{\infty} \tilde{\phi}_i z^i = \log\left(\bar{R}(z) \exp(\sum_{i=1}^{\infty} \phi_i)\right)$.

This way we obtain a new BV/Δ-structure on W. It induces a new hypercommutative algebra structure on V, and it is the first possible interpretation of the Givental group action on the space of hypercommutative algebra structures on V.

More details are available in [5].

Multi-KP formalism

The structure of a formal Frobenius manifold can also be reduced to a pencil of flat connections, that is, the $s \times s$ matrix of the germs of 1-forms $A(u_1, \dots, u_s)$

such that $d + \zeta^{-1}A$ is a pencil of flat connections (here ζ is a formal variable). The equations are then $dA = 0$ and $A \wedge A = 0$.

We associate a particular matrix A to a formal power series of $s \times s$ matrices $R(z^{-1}) = \exp\left(\sum_{\ell=1}^{\infty} r_\ell z^{-\ell}\right)$ using the formalism of multi-KP hierarchies.

Let $V = \langle e_1, \dots, e_s\rangle$ be an n-dimensional vector space over $\mathbb{C}$. Let z be a formal variable. We denote by $\mathcal{V}$ the vector space $\Lambda^{\infty/2}\left(V \otimes \mathbb{C}[z^{-1}, z]\right)$ spanned by the semi-infinite wegde products

$$\omega = (e_{i_1} z^{d_1}) \wedge (e_{i_2} z^{d_2}) \wedge (e_{i_3} z^{d_3}) \wedge \dots$$

such that the tail of ω coinsides with the tail of vacuum vector

$$|0\rangle := (e_1 z^0) \wedge \cdots \wedge (e_s z^0) \wedge (e_1 z^1) \wedge \cdots \wedge (e_s z^1) \wedge \dots .$$

By tail of ω we call another basis vector in $\mathcal{V}$ that is obtained from ω by removing the first few factors in the wedge product.

We consider the matrices $r_\ell z^{-\ell}$ as the local Lie algebra elements acting on vectors in $\mathcal{V}$ by the Leibnitz rule. We denote by α_i the local Lie algebra element $E_{ii}z$. All basic objects that we are going to consider are some matrix elements of the operator $\mathcal{R} := \exp(\sum_{i=1}^{n} \alpha_i u_i) R(z^{-1})$, where $u_1, \dots, u_n$ are formal variables.

We denote by $(\Psi_0)_{ij} = (\Psi_0)_{ij}(R)$, $i, j = 1, \dots, s$, the following matrix elements of $\mathcal{R}$:

$$(\Psi_0)_{ij} := \frac{\left\langle (e_j z^{-1}) \wedge |0\rangle \,\middle|\mathcal{R}\,\middle|(e_i z^{-1}) \wedge |0\rangle \right\rangle}{\langle |0\rangle\, |\mathcal{R}\, ||0\rangle\rangle}.$$

Then the formula for a matrix of the germs of 1-forms A satisfying the property that $d + \zeta^{-1}A$ is a pencil of flat connections is simply $\Psi_0^{-1}\left(\sum_{i=1}^{s} E_{ii} du_i\right)\Psi_0$.

More details are available in [6, 7, 1, 9].

Deformation of classes of a CohFT

Formal solutions of the WDVV equation that has no terms of degree less than 3, satisfying the additional property of having a unit, are in one-to-one correspondence with cohomological field theories in genus 0. The formal definition is the following. We fix a vector space $V = \langle e_1, \dots, e_s\rangle$ (e_1 will play a special role) with a non-degenerate scalar product η. A cohomological field theory is a system of cohomology classes $C_n \in H^*(\overline{\mathcal{M}}_{0,n}) \otimes V^{\otimes n}$ satisfying the properties:

(1) C_n is equivariant with respect to the action of S_n on the labels of marked point and component of $V^{\otimes n}$.
(2) For any map $\rho\colon \overline{\mathcal{M}}_{0,n_1+1} \times \overline{\mathcal{M}}_{0,n_2+1} \to \overline{\mathcal{M}}_{0,n}$ that realizes a divisor in $\overline{\mathcal{M}}_{0,n}$ sewing the surfaces at the last marked points into a nodal surface, we have: $\rho^* C_n = \left(C_{n_1+1} \cdot C_{n_2+1}, \eta^{-1}\right)$ (we contract with the scalar product the two components of V corresponding to the two points in the preimage of the node under normalization).
(3) For the map $\pi\colon \overline{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{0,n}$ that forgets the last marked point, we have: $\pi^* C_n = (C_{n+1}, e_1)$ (we contract the component of V corresponding to the last marked point with e_1). Moreover, $(C_3, e_1 \otimes e_i \otimes e_j) = \eta_{ij}$.

Consider a series $\sum_{\ell=1}^{\infty} r_\ell z^\ell$, $r_\ell \colon V \to V$, r_ℓ is symmetric (resp., skewsymmetric) with respect to η for odd (respectively, even) ℓ. Following [4, 10], we associate to this series an infinitesimal deformation of CohFT. Denote by $(r_\ell z^\ell)\hat{}\, C_n$ the following class on $\overline{\mathcal{M}}_{0,n}$:

$$
\begin{aligned}
&-\pi_*\left(C_{n+1}\psi_{n+1}^{l+1}, r_\ell(e_1)\right) + \sum_{k=1}^{n} \psi_k^l r_\ell^{(k)} C_n \\
&+\frac{1}{2}\sum_{div}\sum_{i=0}^{\ell-1}(-1)^{i+1}\rho_*\left(C_{n_1+1}\psi_{n_1+1}^{i}\cdot C_{n_2+1}\psi_{n_2+1}^{\ell-1-i}, \eta^{-1}r_\ell\right).
\end{aligned}
$$

The last sum here ($\sum_{div}$) is taken over all irreducible boundary divisors, whose generic points are represented by two-component curves. By $r_\ell^{(k)}$ we denote the action of r_ℓ on the kth factor of V in $V^{\otimes n}$.

The classes $C'_n := \exp\left(\sum_{\ell=1}^{\infty}(r_\ell z^\ell)\hat{}\,\right) C_n$ form a CohFT.

More details are available in [4, 10, 8].

Lagrangian cones in the Fock space

One more way to think about a Frobenius structure is to say that we consider a formal solution of the string equation and the topological recursion relation. That is, we have a vector space $V = \langle e_1, \ldots, e_s\rangle$, a scalar product η, and we consider a formal power series $F_0(\{t_{i,d}\})$, $i = 1, \ldots, s$, $d = 0, 1, 2, \ldots$ satisfying the equations

$$
\begin{aligned}
\frac{\partial F_0}{\partial t_{1,0}} &= \sum_{i,d} t_{i,d+1}\frac{\partial F_0}{\partial t_{i,d}} + \frac{1}{2}\sum_{i,j=1}^{s} t_{i,0}t_{j,0}\eta_{ij}; \\
\frac{\partial^3 F_0}{\partial t_{i,a+1}\partial t_{j,b}\partial t_{k,c}} &= \sum \frac{\partial^2 F_0}{\partial t_{i,a}\partial t_{\alpha,0}}\eta_{\alpha\beta}\frac{\partial^3 F_0}{\partial t_{\beta,0}\partial t_{j,b}\partial t_{k,c}}.
\end{aligned}
$$

We can assume that the coefficient of monomial $\prod_{a=1}^{n} t_{i_a,d_a}$ is equal to zero if $\sum_{a=1}^{n} d_a > n-3$, it is a kind of tameness assumption.

In the space $V\otimes\mathbb{C}[z]$ we have a natural set of coordinates $q_{i,d}$ corresponding to the basis vectors $e_i z^d$. We consider F_0 as a function on $V\otimes\mathbb{C}[z]$ identifying two systems of coordinates by $q_{i,d} = t_{i,d} - \delta_{i1}\delta_{d1}$. The string equation and the topological recursion relation are equivalent to some geometric properties of the graph of dF_0, which is a germ of a Lagrangian cone in $T^*\left(V\otimes\mathbb{C}[z]\right) \cong V\otimes\mathbb{C}[[z^{-1}, z]$. These properties can be expressed in terms of symplectic linear algebra and the operator of multiplication by z. Thus the groups of symplectic transformations of $V\otimes\mathbb{C}[[z^{-1}, z]$ commuting with multiplication by z acts on such Lagrangian cones, and, therefore, on formal Frobenius structures.

In particular we have the action of a series $R(z)\in Hom(V,V)\otimes\mathbb{C}[[z]]$, satisfying $R(z) = Id + O(z)$ and $R^*(-z)R(z) = Id$ (here we should use tameness to define it properly). This is the original definition of Givental of the group action on Frobenius manifolds.

More details are available in [2, 3].

References

[1] E. Feigin, J. van de Leur, S. Shadrin Givental symmetries of Frobenius manifolds and multi-component KP tau-functions, arXiv:0905.0795.

[2] A. Givental, Gromov-Witten invariants and quantization of quadratic hamiltonians, arXiv: math/0108100.

[3] A. Givental, Symplectic geometry of Frobenius structures, arXiv:math/0305409.

[4] M. Kazarian, Deformations of cohomological field theories, preprint 2007.

[5] A. Khoroshkin, N. Markarian, S. Shadrin, Hypercommutative operad as a homotopy quotient of BV, in preparation.

[6] J. van de Leur, Twisted GL_n Loop Group Orbit and Solutions of the WDVV Equations, arXiv:nlin/0004021.

[7] A. Losev, I. Polyubin, On compatibility of tensor products on solutions to commutativity and WDVV equations, JETP Lett. 73 (2001), no. 2, 53–58.

[8] S. Shadrin, BCOV theory via Givental group action on cohomological field theories, arXiv:0810.0725.

[9] S. Shadrin, D. Zvonkine, A group action on Losev-Manin cohomological field theories, arXiv:0909.0800.

[10] C. Teleman, The structure of 2D semi-simple field theories, arXiv:0712.0160.

Landau-Ginzburg A-models, Mirror Symmetry, and the Calabi-Yau/Landau-Ginzburg Correspondence

Tyler Jarvis

Following some ideas of E. Witten, my collaborators H. Fan and Y. Ruan and I developed a cohomological field theory for each quasi-homogeneous, non-degenerate hypersurface singularity W and each admissible subgroup G of automorphisms of W [FJR1, FJR2]. This theory, sometimes known as *FJRW theory*, plays the role of an "orbifold" Landau-Ginzburg A-model in physics. Recently, algebraic constructions of this theory were given by Polischuk-Vaintrob [PV] using matrix factorization and by Jun Li using cosection localization.

This construction is interesting for at least three reasons: (1) because it fits into a mirror symmetry picture for Landau-Ginzburg models; (2) because it provides a key piece in the so-called *Landau-Ginzburg/Calabi-Yau Correspondence*, [CR, CIR, KS, MR]; and (3) because it provides the foundation for the study of a new system of nonlinear elliptic PDEs called the *Witten Equation* [FJR2, FJR3]. In this abstract we focus on the first two of these.

Landau-Ginzburg mirror symmetry. Given a quasi-homogeneous polynomial $W \in \mathbb{C}[x_1, \dots, x_n]$, with an isolated singularity at the origin, the (un-orbifolded) Landau-Ginzburg B-model produces the local algebra (or Milnor ring) of W. Making a suitable choice of primitive form, Saito constructed a Frobenius manifold which may be thought of as the genus-zero Landau-Ginzburg B-model corresponding to W. Since the Saito construction is semi-simple, Givental's theory of "higher genus Frobenius manifolds" determines a potential function for the theory. This potential should agree with the potential that arises from a cohomological field theory which, in genus zero, agrees with Saito's Frobenius manifold.

Givental-Milanov [GM] showed that in the case of simple singularities (A, D, or E), this potential function satisfies the corresponding (A, D, or E) Drinfeld-Sokolov/Kac-Wakimoto integrable hierarchy.

Based on the work in [IV, Ka, Tu] it is natural to conjecture the existence of a an *orbifolded Landau-Ginzburg B-model* for pairs W and G, where $W \in \mathbb{C}[x_1, \ldots, x_n]$ is a non-degenerate, quasi-homogeneous polynomial, and $G \leq SL(n)$ is an appropriate group of automorphisms of W. The combined work of [IV, Ka, Kr, Tu] has produced a construction of the B-model orbifolded Frobenius algebra. It seems reasonable to expect that a full higher-genus construction for the orbifolded case will also exist.

The *Landau-Ginzburg (LG) Mirror Symmetry Conjecture* predicts that for a large class of polynomials W (called invertible) with a group G of admissible symmetries of W, there is a dual polynomial W^T and dual group G^T of symmetries of W^T with the following mirror propert: the (FJRW) Landau-Ginzburg A-model for the pair (W, G) is isomorphic to the orbifolded B-model construction for the pair (W^T, G^T).

The dual polynomial W^T was described by Berglund and Hübsch [BH], and Mark Krawitz [Kr] constructed the corresponding dual group G^T. When G is the maximal group of admissible symmetries of W, the dual group G^T is the trivial group. In this case the full Landau-Ginzburg mirror symmetry conjecture has been verified for the simple (ADE) singularities [JKV, Lee, FSZ, FJR1] and the unimodal singularities P_8, X_9, and J_{10} in all genera [KS, MR].

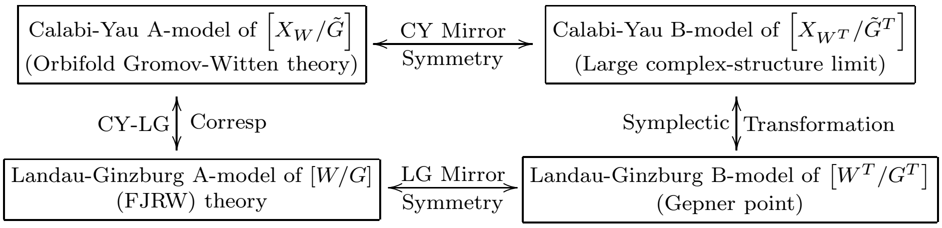
However, when G is not maximal, then G^T is not trivial and it is much harder to verify mirror symmetry. Nevertheless, Landau-Ginzburg Mirror Symmetry has been proven for a large class of orbifold models at the level of Frobenius algebras [FJJS].

The Calabi-Yau/Landau-Ginzburg correspondence. A quasi-homogeneous polynomial W with an isolated singularity at the origin also defines a smooth hypersurface $X_W = \{W = 0\}$ in weighted projective space. If G is an admissible group of automorphisms of W, let $J = G \cap \mathbb{C}^*$, where $\mathbb{C}^*$ is the 1-parameter subgroup of $(\mathbb{C}^*)^n$ defining the weighted projective space. There is an induced action of the group $\tilde{G} = G/J$ on X_W.

In the case that the quotient orbifold $[X_W/\tilde{G}]$ is Calabi-Yau, the *Calabi-Yau/ Landau-Ginzburg (CY/LG) Correspondence* predicts that the analytic continuation of the FJRW (Landau-Ginzburg A-model) potential of the pair (W, G), after a suitable Givental symplectic transformation, will match precisely with the orbifold Gromov-Witten potential (Calabi-Yau A-model) of the orbifold $[X_W/\tilde{G}]$ in all genera.

The correspondence has been verified at the level of genus-zero potentials for un-orbifolded (i.e., $\tilde{G} = 1$), Gorenstein, Calabi-Yau hypersurfaces [CR, CIR]. The correspondence has also been verified in all genera for the ellipic curves defined by the polynomials P_8, X_9, and J_{10} when G is maximal [KS].

The Big Picture. Combining the mirror symmetry conjectures for both Calabi-Yau and Landau-Ginzburg theories with the CY-LG correspondence, we have the following conjectural picture:



Additional progress has been made by many people on many aspects of this conjectural diagram, beyond the work described here, but much work remains to to be done.

References

[BH] P. Berglund and T. Hübsch, *A Generalized Construction of Mirror Manifolds*, Nucl. Phys. B **393** (1993) 377–391.

[CR] A. Chiodo and Y. Ruan. *Landau-Ginzburg/Calabi-Yau correspondence of quintic three-fold via symplectic transformations*, Invent. Math. **182** (2010), no. 1, 117–165.

[CIR] A. Chiodo, H. Iritani, and Y. Ruan, *Landau-Ginzburg/Calabi-Yau correspondence, global mirror symmetry and Orlov equivalence*, `arXiv:1201.0813v2 [math.AG]`

[DZ] B. Dubrovin, and Y. Zhang, *Normal forms of heirarchies of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants*, `arxiv: math. DG/0108160`.

[FJR1] H. Fan, T. Jarvis, and Y. Ruan, *The Witten equation, mirror symmetry and quantum singularity theory.* Preprint. arXiv:0712.4021v3 [math.AG]

[FJR2] ———, *The Witten Equation and Its Virtual Fundamental Cycle*, arXiv:0712.4025v3 [math.AG]

[FJR3] H. Fan, T. Jarvis and Y. Ruan, *Geometry and analysis of spin equations*, Comm. Pure Applied Math 61(2008), 715-788.

[FJR4] ———, *Quantum Singularity Theory for A_{r-1} and r-Spin Theory*, To appear in Annales de l'Institut Fourier, Special Volume on Geometry and Physics of the Landau-Ginzburg model (2011). arXiv:1012.0066v1 [math.AG]

[FJMR] H. J. Fan, T. J. Jarvis, E. Merrell, & Y. Ruan, (2010) *D_4 Integrable Hierarchies.* arXiv:1008.0927v1 [math.AG].

[FGM] E. Frenkel, A. Givental and T. Milanov, *Soliton equations, vertex operators, and simple singularities.* Funct. Anal. Other Math. **3** (2010), no. 1, 47–63.

[FJJS] A. Francis, T. Jarvis, D. Johnson, and R. Suggs, *Landau-Ginzburg Mirror Symmetry for Orbifolded Frobenius Algebras.* arXiv:1111.2508v1 [math.AG]

[FSZ] C. Faber, S. Shadrin and D. Zvonkine, *Tautological relations and the r-spin Witten conjecture*, Annales Scientifiques de l'École Normal Supérieure. Quatrième Série. **43** (2010), no. 4, 621–658.

[GM] A. Givental and T. Milanov, *Simple singularities and integrable hierarchies*, The breadth of symplectic and Poisson geometry, Progr. Math., 232, Birkhauser Boston, Boston, MA, (2005) 173–201.

[IV] K. Intriligator and C. Vafa, *Landau-Ginzburg orbifolds*, Nuclear Phys. B **339** (1990), no. 1, 95–120.

[JKV] T. Jarvis, T. Kimura and A. Vaintrob, *Moduli spaces of higher spin curves and integrable hierarchies*, Compositio Math. **126** (2001), 157–212.

[Ka] R. Kaufmann, *Singularities with symmetries, orbifold Frobenius algebras and mirror symmetry*, Cont. Math. 403, 67–116.

[Kr] M. Krawitz, *FJRW rings and Landau-Ginzburg Mirror Symmetry.* PhD Dissertation, University of Michigan, 2010.

[KS] Marc Krawitz, Yefeng Shen. *Landau-Ginzburg/Calabi-Yau Correspondence of all Genera for Elliptic Orbifold* $\mathbb{P}^1$. arXiv:1106.6270v1 [math.AG].

[KW] V. Kac and M. Wakimoto, *Exceptional hierarchies of soliton equations*, Proc. Sym. in Pure Math. Vol 49(1989), Part 1, 191-237.

[Lee] Lee, Y.-P. *Witten's conjecture and the Virasoro conjecture for genus up to two*, Gromov-Witten theory of spin curves and orbifolds, Contemp. Math., 403, Amer. Math. Soc., Providence, RI, (2006) 31–42.

[MR] T. Milanov and Y. Ruan. *Gromov-Witten theory of elliptic orbifold* P^1 *and quasi-modular forms.* Preprint `arXiv:1106.2321v1 [math.AG]`

[PAKWR] N. Priddis, P. Acosta, M. Krawitz, N. Wilde, and H. Rathnakamura, *The FJRW-ring and Mirror Symmetry for Unimodal and Bimodal Singularities.* Comm. Math. Phys. **296** (2010), no. 1, 145–174.

[PV] A. Polishchuk and A. Vaintrob, *Matrix factorizations and Cohomological Field Theories.* Preprint. arXiv:1105.2903v1 [math.AG]

[Sa1] K. Saito, *Primitive forms for a universal unfolding of a function with an isolated critical point*, J. Fac. Sci. Univ. Tokyo Sect. IA 28 (1982)

[Sa2] ———, *Frobenius manifolds: isomonodromic deformations and infinitesimal period mappings*, Expo. Math. **16** (1998), 1-58.

[Se] P. Seidel, *Fukaya categories and Picard-Lefschetz theory*, Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[ST] K. Saito, and Atsushi Takahashi, *From primitive forms to Frobenius manifolds*, From Hodge theory to integrability and TQFT tt*-geometry, 31–48, Proc. Sympos. Pure Math., **78**, Amer. Math. Soc., Providence, RI, 2008.

[Te1] C. Teleman, *The structure of 2D semi-simple field theories*, Invent. Math. **188**, Number 3 (2012), 525–588,

[Tu] V. Turaev. *Homotopy field theory in dimension 2 and group-algebras.* Preprint. `arXiv:math/9910010v1 [math.QA]`

[Wi] ———, *Algebraic geometry associated with matrix models of two-dimensional gravity*, Topological methods in modern mathematics (Stony Brook, NY, 1991), Publish or Perish, Houston, TX, 1993, 235–269.

On the genus two free energies for simple singularities

Youjin Zhang
(joint work with Boris Dubrovin, Si-Qi Liu)

My talk, based on joint works with Boris Dubrovin and Si-Qi Liu [1], is on properties of the genus two free energies for the Frobenius manifolds defined on the spaces of miniversal deformations of simple singularities.

For a semisimple Frobenius manifold, the genus zero free energy is the logarithm of a particular tau function of the integrable hierarchy of hydrodynamic type associated to the Frobneius manifold [2], this integrable hierarchy possesses an infinite number of Virasoro symmetries which acts nonlinearly on the tau function, the procedure to linearize these Virasoro symmetries enables one to define the higher genus free energies of the semisimple Frobenius manifold by solving the so called loop equation [3]. In particular, an explicit formula for the genus two free energy was given in the preprint [3]. However, this formula is given in terms of canonical coordinates of the Frobenius manifold, it is complicated and is not convenient for applications.

In my talk I present, for the Frobenius manifolds defined on the spaces of miniversal deformations of simple singularities, a conjectural formula for the genus two free energy given in terms of sixteen dual graphs of genus two stable curves with marked points, each dual graph is associated in a simple way to a function represented in terms of the flat coordinates and the potential of the Frobenius manifold. I show evidences to support the validity of the conjecture.

References

[1] B. Dubrovin, S.-Q. Liu, Y. Zhang, *On the Genus Two Free Energies for Semisimple Frobenius Manifolds*, in preparation.

[2] B. Dubrovin, *Geometry of 2D topological field theories, Integrable systems and quantum groups*, Montecatini Terme, 1993, 120–348, Lecture Notes in Math., 1620, Springer, Berlin, 1996.

[3] B. Dubrovin, Y. Zhang, *Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants*, arXiv: math.DG/0108160.

Global solutions of the tt^*-equations and their integral Stokes data

Martin Guest

Unfoldings of singularities and the quantum cohomology algebras of Fano manifolds produce examples of "Frobenius manifolds with real structure". These, in turn, give examples of solutions of the tt^*-equations (topological-antitopological fusion equations) of Cecotti and Vafa.

"Physical" solutions are predicted to be globally defined on the appropriate domain (a certain moduli space of quantum field theories). Furthermore they correspond to "initial data" at certain distinguished points of the domain, and such initial data is a priori integral (because of its physical interpretation). Cecotti and Vafa used this in [1] as a strategy to classify deformations of $N = 2$ supersymmetric quantum field theories, and carried out this strategy for some simple examples. However, it is a nontrivial task to justify this mathematically.

Dubrovin reformulated the tt^*-equations as the equations for pluriharmonic maps into $GL_N(\mathbb{R})/O_N$. We have characterized and investigated a special case of these equations which we call the tt^*-Toda equations ([2], [3]). With $N = n + 1$, these are

$$(w_i)_{z\bar{z}} = -e^{(w_{i+1}-w_i)} + e^{(w_i-w_{i-1})}, \quad 0 \le i \le n$$

for $w_i : \mathbb{C}^* \to \mathbb{R}$, $w_i = w_i(z, \bar{z}) = w_i(|z|)$, with the additional conditions

(i) $w_i + w_{n-i} = 0, \quad 0 \le i \le n$

or

(ii) $w_i + w_{k-i-1} = 0, \quad 0 \le i \le k - 1$ and $w_i + w_{n+k-i} = 0, \quad k \le i \le n$

(case (i) was already considered by Cecotti and Vafa).

For these equations we can describe the global solutions (in the above sense), the Stokes data of the associated isomonodromic system of o.d.e., and the geometrical/physical solutions corresponding to integral Stokes data.

For example, in case (i) when $n+1=4$, the global solutions are parametrized by $(\gamma_0,\gamma_1,\gamma_2,\gamma_3)\in\mathbb{R}$ such that $\gamma_i-\gamma_{i-1}\geq -2$. We have $\gamma_3=-\gamma_0$, $\gamma_0=-\gamma_2$, so this is a convex set. The monodromy data reduces to a single Stokes matrix, in fact to two real numbers s_1,s_2. This gives another parametrization of the global solutions. Explicitly, we have

$$s_1 = 2\cos\tfrac{\pi}{4}(\gamma_0+1) - 2\cos\tfrac{\pi}{4}(\gamma_1-1)$$
$$s_2 = -2 + 4\cos\tfrac{\pi}{4}(\gamma_0+1)\,\cos\tfrac{\pi}{4}(\gamma_1-1).$$

Precisely 19 pairs (s_1,s_2) belong to $\mathbb{Z}\times\mathbb{Z}$, and these are easily computed. To understand the geometric significance of these 19 solutions we consider the "holomorphic data" in the sense of harmonic map theory, which is of the form

$$\frac{1}{\lambda}\begin{pmatrix} & & & z^{k_0} \\ z^{k_1} & & & \\ & z^{k_2} & & \\ & & z^{k_3} & \end{pmatrix}\frac{dz}{z}.$$

Explicitly, we have

$$\gamma_0+1=\frac{k_0+1}{4+\sum k_i},\ \gamma_1-1=\frac{k_2+1}{4+\sum k_i}$$

with $k_i\geq -1$. We may normalize so that $4+\sum k_i=1$. Amongst the 19 examples we recognise the quantum cohomology rings of $\mathbb{P}^3$, $\mathbb{P}(1,3)$, $\mathbb{X}^2(1,1,4)$ (hypersurface of degree 2 in $\mathbb{P}(1,1,4)$), and $\mathbb{X}^{2,3}(1,1,1,6)$ (intersection of hypersurfaces of degree 2,3 in $\mathbb{P}(1,1,1,6)$), as well as the Milnor ring of A_4. Full details will appear in [3]; the basic p.d.e. existence result can be found in [2].

References

[1] S. Cecotti and C. Vafa, *On classification of $N=2$ supersymmetric theories*, Commun. Math. Phys. **158** (1993), 569–644.

[2] M. A. Guest and C.-S. Lin *Nonlinear PDE aspects of the tt* equations of Cecotti and Vafa*, preprint (arXiv:1010.1889)

[3] M. A. Guest and C.-S. Lin *tt* structures and their integral Stokes data*, in preparation

Integrable systems as homological invariants for open symplectic manifolds

Oliver Fabert

It was pointed out by Eliashberg in his ICM 2006 plenary talk ([2]) that the integrable systems of rational Gromov-Witten theory very naturally appear in the rich algebraic formalism of symplectic field theory (SFT) ([3]). Carefully generalizing the definition of gravitational descendants from Gromov-Witten theory to SFT ([5]), one can assign to every closed stable Hamiltonian manifold a Hamiltonian system with symmetries on SFT homology and the question of its integrability arises ([7],[8]).

Examples for closed manifolds with a stable Hamiltonian structure are contact manifolds, circle bundles over closed symplectic manifolds and symplectic mapping tori. In my current research I focus on the last example. They are determined by a symplectomorphism on a closed symplectic manifold and hence provide new invariants for symplectomorphisms, generalizing the well-known invariants from Floer theory ([4]).

Floer theory plays a very prominent role in symplectic topology and can be viewed as a generalization of Gromov-Witten theory. Guided by Eliashberg-Givental-Hofer's symplectic field theory, I explain how the relation between Gromov-Witten theory, Frobenius manifolds and integrable systems is generalized to the Floer theory of symplectomorphisms.

To be more precise, the pair-of-pants product in Floer theory of symplectomorphisms ([11]) can be viewed as a generalization of the small quantum product in Gromov-Witten theory. While the latter just counts holomorphic spheres with three marked points, there also exists a big version of the quantum cup product. It involves the full rational Gromov-Witten potential and leads to the geometrical notion of Frobenius manifolds ([10]). In my talk I define the corresponding big version of the pair-of-pants product in Floer theory and show that it leads to cohomology F-manifolds, a generalization of Frobenius manifolds introduced by Merkulov ([9]). This is the content of the main theorem of my talk, see also [6].

Theorem. The tuple $(\mathbf{Q}, X, \star, E)$, consisting of the coordinate super space $\mathbf{Q}$ and the differential vector field X of contact homology, the big pair-of-pants product $\star$ and the Euler vector field E, defines a *cohomology F-manifold* in the sense that

- $X \in \mathcal{T}^{1,0}\,\mathbf{Q}$ is an odd homological vector field, $[X, X] = 2X^2 = 0$, so that $(\mathbf{Q}, X)$ defines a differential graded manifold $\mathbf{Q}_X$,
- $\star \in \mathcal{T}^{1,2}\,\mathbf{Q}$ satisfies $\mathcal{L}_X \star = 0$, and the induced map $\star : \mathcal{T}^{1,0}\,\mathbf{Q}_X \otimes \mathcal{T}^{1,0}\,\mathbf{Q}_X \to \mathcal{T}^{1,0}\,\mathbf{Q}_X$ is a graded commutative and associative product,
- $E \in \mathcal{T}^{1,0}\,\mathbf{Q}$ is an Euler vector field in the sense that $Ef = |f|f$ for all homogeneous functions $f \in \mathcal{T}^{0,0}\,\mathbf{Q}$, with $\mathcal{L}_E X = [E, X] = -X$ and $\mathcal{L}_E \star = 0$. In particular, it induces a grading operator $\mathcal{L}_E : \mathcal{T}^{r,s}\,\mathbf{Q}_X \to \mathcal{T}^{r,s}\,\mathbf{Q}_X$, $r, s \geq 1$ on all tensor fields.

References

[1] S. Dostoglou, D. Salamon: *Instanton homology and symplectic fixed points*, in Symplectic Geometry, edited by D. Salamon, Proceedings of a Conference, LMS Lecture Notes Series 192, Cambridge University Press, pp. 57-94, 1993.

[2] Y. Eliashberg, *Symplectic field theory and its applications*, Proceedings of the ICM 2006.

[3] Y. Eliashberg, A. Givental, H. Hofer, *Introduction to symplectic field theory*, GAFA 2000 Visions in Mathematics special volume, part II, 2000.
[4] O. Fabert, *Contact Homology of Hamiltonian Mapping Tori*, Comm. Math. Helv. **85**, 2010.
[5] O. Fabert, *Gravitational descendants in symplectic field theory*, Comm. Math. Phys. **302**, 2011.
[6] O. Fabert, *Floer theory and cohomology F-manifolds*, in preparation.
[7] O. Fabert, P. Rossi, *String, dilaton and divisor equation in symplectic field theory*, Int. Math. Res. Not. **2011**, 2011.
[8] O. Fabert, P. Rossi, *Topological recursion relations in non-equivariant cylindrical contact homology*, to appear in J. Symp. Geom.
[9] S. Merkulov, *Operads, deformation theory and F-manifolds*, in Frobenius manifolds, quantum cohomology, and singularities, eds. C. Hertling and M. Marcolli, Vieweg, 2004.
[10] D. McDuff, D. Salamon, *J-holomorphic curves and symplectic topology*, AMS Colloquium Publications 52, 2004.
[11] M. Schwarz, M.: *Cohomology operations from S^1-cobordisms in Floer homology*, Ph.D. thesis, Swiss Federal Inst. of Techn. Zurich, Diss. ETH No. 11182, 1995.

Laurent Polynomials, Hypergeometric Systems and Mirror symmetry

Thomas Reichelt

(joint work with Christian Sevenheck)

Let A be a $n \times m$ integer matrix so that its columns, seen as elements of $\mathbb{Z}^n$, generate $\mathbb{Z}^n$ over $\mathbb{Z}$. Consider the following Laurent polynomial built from the entries (a_{ki}) of A:

$$\varphi : T \times \mathbb{C}^m \longrightarrow \mathbb{C}_{\lambda_0} \times \mathbb{C}^m$$
$$(y_1, \ldots, y_n, \lambda_1, \ldots, \lambda_m) \mapsto (-\sum_{i=1}^{m} \lambda_i \prod_{k=1}^{n} y_k^{a_{ki}}, \lambda_1, \ldots, \lambda_m)\,,$$

where we have denoted the torus $(\mathbb{C}^*)^n$ by T. Define the following matrix

$$\widetilde{A} := \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix}$$

and assume that the semigroup $\mathbb{N}\widetilde{A}$ is saturated. Then the following relation between the (compact) Gauß-Manin system of φ and the GKZ-system holds.

Theorem 1. [8] *For all $\beta \in \mathbb{N}A$ and $\beta' \in \mathbb{N}A \cap (\mathbb{R}_+ A)^\circ$, there exist the following exact sequences, which are related by holonomic duality:*

$$0 \longrightarrow \mathbb{V}^{n-1} \longrightarrow \mathcal{H}^0(\varphi_+ \mathcal{O}_{T\times\mathbb{C}^n}) \longrightarrow \mathcal{M}^{\beta}_{\widetilde{A}} \longrightarrow \mathbb{V}^n \longrightarrow 0\,,$$
$$0 \longleftarrow \mathbb{V}^{n+1}_c \longleftarrow \mathcal{H}^0(\varphi_\dagger \mathcal{O}_{T\times\mathbb{C}^n}) \longleftarrow \mathcal{M}^{-\beta'}_{\widetilde{A}} \longleftarrow \mathbb{V}^n_c \longleftarrow 0\,,$$

where we denoted $H^i(T, \mathbb{C}) \otimes \mathcal{O}_{T\times\mathbb{C}^n}$ resp. $H^i_c(T, \mathbb{C}) \otimes \mathcal{O}_{T\times\mathbb{C}^n}$ by $\mathbb{V}^i$ resp. $\mathbb{V}^i_c$.

We are especially interested in the Fourier-Laplace transformed Gauß-Manin system of φ. The upper sequence above gives us the following isomorphism:

$$\mathrm{FL}_{\lambda_0}^{z^{-1}}(\mathcal{H}^0(\varphi_+\mathcal{O}_{T\times\mathbb{C}^n}[\partial_{\lambda_0}^{-1}]) \simeq \mathrm{FL}_{\lambda_0}^{z^{-1}}\mathcal{M}^{\beta}_{\widetilde{A}}[\partial_{\lambda_0}^{-1}] \simeq \mathrm{FL}_{\lambda_0}^{z^{-1}}\mathcal{M}^{\beta}_{\widetilde{A}}\,.$$

where the last isomorphism follows from a theorem of [6].

This isomorphism turns out to be extremely useful in the computation of the mirror Landau-Ginzburg model of an n-dimensional, smooth, toric, Fano variety X. Let Σ be the corresponding fan and let m be the number of one-dimensional cones. Denote by $a_1,\ldots,a_m$ the primitive, integral generators of the one-dimensional cones and let A be the matrix with $a_1,\ldots,a_m$ as columns. Then there exists the following exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & Div_T(X) & \xrightarrow{\pi} & Pic(X) & \longrightarrow & 0 \\ & & \wr\| & & \wr\| & & \wr\| & & \\ & & \mathbb{Z}^n & \xrightarrow{A^t} & \mathbb{Z}^m & \underset{G}{\rightleftarrows} & \mathbb{Z}^r & & \end{array}$$

where the matrix $G=(g_{ia})$ defines a section of π. The Landau-Ginzburg mirror model of X is then the following family of Laurent polynomials

$$W_X : T\times\mathcal{K} \longrightarrow \mathbb{C}_{\lambda_0}\times\mathcal{K}$$
$$(y_1,\ldots,y_n,q_1,\ldots,q_r) \mapsto (-\sum_{i=1}^{m}\prod_{a=1}^{r} q_a^{g_{ia}}\prod_{k=1}^{n} y_k^{a_{ki}}, q_1,\ldots,q_r)$$

where $\mathcal{K}\simeq H^2(X,\mathbb{C})/2\pi i H^2(X,\mathbb{C})\simeq(\mathbb{C}^*)^r$ is the so-called complexified Kähler moduli space. The matrix G defines an embedding of the complexified Kähler moduli space

$$i:\mathcal{K}\longrightarrow\mathbb{C}^m$$
$$(q_1,\ldots,q_r)\mapsto(\prod_{a=1}^{r} q_a^{g_{1a}},\ldots,\prod_{a=1}^{r} q_a^{g_{ma}})\,,$$

so that we can use Theorem 1 to explicitly calculate the Fourier-Laplace transformed Gauß-Manin system of W_X, which we denoted by $\mathcal{QM}_{\widetilde{A}}$, by base change with respect to i.

The ultimate goal of the paper [9] with C. Sevenheck was to establish mirror symmetry as an isomorphism of logarithmic Frobenius manifolds. A logarithmic Frobenius manifold is a Frobenius manifold outside a normal crossing divisor D, where the multiplication, the metric, the unit and Euler vector field is defined on the locally free sheaf of logarithmic vector fields with respect to D. On the so-called A-side the construction of a Frobenius manifold from the big quantum cohomology of X has been carried out by Dubrovin in [2] and the construction of the logarithmic Frobenius manifold is given in [7]. On the B-side there exists a construction of a Frobenius manifold starting from a single Laurent polynomial by Douai and Sabbah [1]. However it is was only possible for some special cases, e.g.

the complex projective spaces, to prove an isomorphism between these Frobenius manifolds.

To establish an isomorphism for a general smooth, toric, nef variety we pursued a different approach. If one restricts the Frobenius manifold coming from the big quantum cohomology to the complexified Kähler moduli space $\mathcal{K}$, on which the small quantum cohomology is defined, one gets a so-called Frobenius structure which consists of a trivial vector bundle on $\mathbb{P}^1 \times \mathcal{K}$ together with a flat connection, which acquires a pole of order 2 along $z = 0$ and a logarithmic pole along $z = \infty$, and which is called Givental connection in the context of the A-model. This connection degenerates with a logarithmic pole along a normal crossing divisor D which is the boundary of a partial compactification $\overline{\mathcal{K}}$ of $\mathcal{K}$ containig the so-called large radius limit point. The underlying $\mathcal{D}$-module of the Givental connection is called the quantum $\mathcal{D}$-module. It is now possible, first, to reconstruct the Givental connection from the quantum $\mathcal{D}$-module by a method of Guest [4] and second to reconstruct the Frobenius manifold from the Givental connection by a theorem of Hertling and Manin [5]. Our philosophy was to find the object corresponding to the quantum $\mathcal{D}$-module on the B-side and then to work our way up using the various reconstruction theorems.

In the paper [3] Givental identified the solutions of the quantum $\mathcal{D}$-module, which are given by the so-called J-function, with hypergeometric functions encoded by the so-called I-function. Now having an explicit description of the Landau-Ginzburg mirror model we use Givental's theorem to establish an isomorphism between the quantum $\mathcal{D}$-module and $\mathcal{QM}_{\widetilde{A}}$. To construct the connection on the B-side, which corresponds to the Givental connection on the A-side, we computed the so-called Brieskorn lattice in $\mathcal{QM}_{\widetilde{A}}$ and adapted the method of Guest which basically consists of finding a solution of the Birkhoff-problem for this lattice. Here the logarithmic degeneration property turns out to be crucial. We extend the Brieskorn lattice to the boundary divisor D and use the fact that the residue connection has a regular singularity at the large radius limit. This gives a canonical logarithmic extension at $z = \infty$, which we are able to extend to the whole Kähler moduli space. Applying the reconstruction theorem of Hertling and Manin, we get a (logarithmic) Frobenius manifold.

Theorem 2. [9]

Logarithmic Frobenius manifold from big quantum cohomology of X_Σ	$\simeq$	*Logarithmic Frobenius manifold from Landau-Ginzburg model of* W_{X_Σ}

References

[1] A. Douai, C. Sabbah, *Gauss-Manin systems, Brieskorn lattices and Frobenius structures I*, Ann. Inst. Fourier (Grenoble) 53 (2003), no. 4, 1055-1116.

[2] B. Dubrovin, *Integrable systems in topological field theory*, Nuclear Phys. B 379 (1992), no. 3, 627-689.

[3] A. Givental, *A mirror theorem for toric complete intersections*, Topological field theory, primitive forms and related topics (Kyoto, 1996), Progr. Math., vol. 160, Birkhäuser Boston, Boston, MA, 1998, pp. 141-175.

[4] M. Guest, *Quantum cohomology via D-modules*, Topology 44 (2005), no. 2, 263-281.
[5] C. Hertling and Y. Manin, *Unfoldings of meromorphic connections and a construction of Frobenius manifolds*, Frobenius manifolds, Aspects Math., E36, Vieweg, Wiesbaden, 2004, pp. 113-144.
[6] M. Schulze, U. Walther, *Hypergeometric D-modules and twisted Gauß-Manin systems*, J. Algebra 322 (2009), no. 9, 3392–3409.
[7] T. Reichelt, *A construction of Frobenius manifolds with logarithmic poles and applications*, Comm. Math. Phys. 287 (2009), no. 3, 1145–1187.
[8] T. Reichelt, *Laurent polynomials, Hypergeometric Systems and Mixed Hodge modules*, to appear.
[9] T. Reichelt, C. Sevenheck, *Logarithmic Frobenius manifolds, hypergeometric systems and quantum D-modules*, arXiv math.AG/1010.2118 .

Landau-Ginzburg models for complete intersections

Christian Sevenheck
(joint work with Thomas Reichelt)

We report on ongoing work concerning the B-model for nef complete intersections in a smooth toric variety X_Σ. Such an intersection is given as the zero locus of a generic section of a sum $\mathcal{E} = \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_s$ of s say, ample, line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_s$. We suppose also that $-K_{X_\Sigma} - \mathcal{L}_1 - \ldots - \mathcal{L}_s$ is numerically effective. One can consider the so-called twisted Gromov-Witten invariants for these data, that is, correlators

$$\langle \alpha_1, \ldots, \widetilde{\alpha_k}, \ldots, \alpha_n \rangle_{0,n,\beta} := \int_{[\overline{\mathcal{M}}_{0,n,\beta}(X)]^{virt}} \bigcup_{i=1}^{n} \mathrm{ev}_i(\alpha_i) \cup e(\widetilde{\mathcal{E}}_k),$$

where $\widetilde{\mathcal{E}}_k$ is the bundle on $\overline{\mathcal{M}}_{0,n,\beta}(X)$ with fibre at $[C,(x_1,\ldots,x_n),f]$ being the subspace of $H^0(C, f^*\mathcal{E})$ of sections which vanishes at x_k. This gives rise to the twisted Gromov-Witten quantum product. Notice that the bilinear pairing used in the definition of this twisted quantum product defined by $(\alpha,\beta) := \int_X \alpha \cup \beta \cup e(\mathcal{E})$, hence, it is in general degenerate. One may consider the quotient $H^*(X_\Sigma, \mathbb{C})/(ker(e(\mathcal{E})\cdot -))$ on which this pairing is non-degenerate. Then the twisted quantum product descends to the so-called reduced one. It is known (see [CG07]) that the reduced one is the quantum product on the *ambient cohomology* of the subvariety defined by a generic section of $\mathcal{E}$.

Classical constructions of Dubrovin and Givental associate a family of holomorphic vector bundles on $\mathbb{P}^1$ to the quantum cohomology of a smooth projective variety, this is the so-called quantum $\mathcal{D}$-module. A similar construction exists for the twisted and reduced invariants, and a concrete description of these $\mathcal{D}$-modules in the toric case has recently been given in [MM11]. The aim of our work is to reconstruct these differential systems from an appropriate Landau-Ginzburg model. For this, we rely on our earlier paper [RS10] in which we describe the Gauß-Manin system of a generic family of Laurent polynomials by hypergeometric differential equations. For a complete intersection, the corresponding Laurent polynomials are constructed from an extended fan Σ', which is the fan of the total bundle of $\oplus_{j=1}^s \mathcal{L}_j^{-1}$.

For the family of Laurent polynomials $\varphi_{\Sigma'} : (\mathbb{C}^*)^{n+s} \times \mathbb{C}^{m+s} \to \mathbb{C} \times \mathbb{C}^{m+s}$ obtained we consider its so-called twisted de Rham cohomology, namely, the $n+s$-th cohomology $H^{n+s}(\varphi)$ of the complex $(\Omega^\bullet_{pr}[z], zd - d\varphi\wedge)$, where $\Omega^\bullet_{pr}$ is the relative de Rham complex $\Omega^\bullet_{(\mathbb{C}^*)^{n+s}\times\mathbb{C}^{m+s}/\mathbb{C}^{m+s}}$. We can relate this $\mathcal{D}$-module to the twisted quantum-$\mathcal{D}$-module from [MM11], and the reduced quantum-$\mathcal{D}$-module from loc.cit. can also be described as a certain intersection complex. We can conclude, using [Sab08], that it underlies a variation of non-commutative pure polarized Hodge structures.

References

[CG07] Tom Coates and Alexander Givental, *Quantum Riemann-Roch, Lefschetz and Serre*, Ann. of Math. (2) **165** (2007), no. 1, 15–53.

[MM11] Etienne Mann and Thierry Mignon, *Quantum $\mathcal{D}$-modules for toric nef complete intersections*, Preprint arXiv:1112.1552, 2011.

[RS10] Thomas Reichelt and Christian Sevenheck, *Logarithmic frobenius manifolds, hypergeometric systems and quantum $\mathcal{D}$-modules*, Preprint math.AG/1010.2118, 2010.

[Sab08] Claude Sabbah, *Fourier-Laplace transform of a variation of polarized complex Hodge structure.*, J. Reine Angew. Math. **621** (2008), 123–158.

F-manifolds and eventual identities

Liana David

(joint work with Ian A. B. Strachan)

In [5] Dubrovin associated to a Frobenius manifold a new object, called an almost Frobenius manifold, which shares most of the properties of a Frobenius manifold. In particular, from a solution of the WDVV-equations one obtains a new dual solution. More specifically, given a Frobenius manifold $(M, \circ, e, \tilde{g}, E)$ with multiplication $\circ$, unit field e, flat invariant metric $\tilde{g}$ and Euler field E, one may define a new multiplication and metric by

$$X * Y := X \circ Y \circ E^{-1}, \quad g(X,Y) := \tilde{g}^{-1}(X \circ E^{-1}, Y).$$

It is clear that $*$ is commutative, associative, with unit field E and, as it turns out, the metric g is flat. However, the flatness of the unit field is not preserved: while e is flat on $(M, \tilde{g})$, E is not flat on (M, g), and, in general, various other relations do not hold in the almost dual setting. The purpose of this talk is to extend Dubrovin's almost duality for Frobenius manifolds to the larger setting of F-manifolds with eventual identities. An F-manifold [7] is a manifold M with a multiplication $\circ$ on TM, which is commutative, associative, with unit field e, such that the integrability condition

$$L_{X\circ Y}(\circ) = X \circ L_Y(\circ) + Y \circ L_X(\circ)$$

holds, for any vector fields $X, Y \in \mathcal{X}(M)$. Any Frobenius manifold without metric and Euler field is an F-manifold. A vector field $\mathcal{E}$ on an F-manifold $(M, \circ, e)$ is called an eventual identity [9] if it is invertible (i.e. there is a vector field $\mathcal{E}^{-1}$ such that $\mathcal{E} \circ \mathcal{E}^{-1} = e$) and the multiplication

$$X * Y = X \circ Y \circ \mathcal{E}^{-1} \tag{1}$$

defines a new F-manifold structure on M. Our main result is the following.

Theorem 1. *i) Let $(M, \circ, e)$ be an F-manifold and $\mathcal{E}$ an invertible vector field. Then $\mathcal{E}$ is an eventual identity if and only if*

$$L_{\mathcal{E}}(\circ)(X, Y) = [e, \mathcal{E}] \circ X \circ Y, \quad \forall X, Y \in \mathcal{X}(M). \tag{2}$$

ii) Suppose that (2) holds. Then e is an eventual identity on the F-manifold $(M, *, \mathcal{E})$, where $*$ is related to $\circ$ by (1), and the map

$$(M, \circ, e, \mathcal{E}) \to (M, *, e, \mathcal{E})$$

is an involution on the set of F-manifolds with eventual identities.

In the followings we develop applications of Theorem 1.

1) Connections. A connection ∇ on an F-manifold $(M, \circ, e)$ is called compatible, if

$$\nabla_X(\circ)(Y, Z) = \nabla_Y(\circ)(X, Z), \quad \forall X, Y, Z \in \mathcal{X}(M).$$

The Levi-Civita connection of a Frobenius manifold is a compatible connection for the underlying F-manifold. It turns out that the dual of an F-manifold with a compatible, torsion-free, connection inherits, in a canonical way, a family of compatible, torsion-free connections [3]. If we restrict to connections which preserve the unit fields, we can state [3]:

Proposition 2. *The map*

$$(M, \circ, e, \mathcal{E}, \tilde{\nabla}) \to (M, *, \mathcal{E}, e, \nabla)$$

where $*$ is related to $\circ$ by (1) and ∇ is related to $\tilde{\nabla}$ by

$$\nabla_X(Y) = \mathcal{E} \circ \tilde{\nabla}_X(\mathcal{E}^{-1} \circ Y) - \tilde{\nabla}_{\mathcal{E}^{-1} \circ Y}(\mathcal{E}) \circ X + \frac{1}{2}[\mathcal{E}^{-1}, \mathcal{E}] \circ X \circ Y$$

is an involution on the set of F-manifolds with eventual identities and compatible, torsion-free connections preserving the unit fields.

2) Metrics. Riemannian F-manifolds were used in [8] to interpret arguments from the theory of integrable systems in a coordinate free way. A Riemannian F-manifold is an F-manifold $(M, \circ, e)$ together with an invariant metric $\tilde{g}$, such that the following two conditions are satisfied: the coidentity $\tilde{g}(e) \in \Omega^1(M)$, which is the 1-form $\tilde{g}$-dual to unit field e, is closed; the curvature of $\tilde{g}$ satisfies the relation:

$$X \circ R^{\tilde{g}}_{Y,Z}(V) + Z \circ R^{\tilde{g}}_{X,Y}(V) + Y \circ R^{\tilde{g}}_{Z,X}(V) = 0, \quad \forall X, Y, Z, V \in \mathcal{X}(M). \tag{3}$$

For example, (3) holds when $\tilde{g}$ has constant sectional curvature.

Proposition 3. *Let $(M, \circ, e, \mathcal{E}, \tilde{g})$ be an F-manifold with an eventual identity $\mathcal{E}$ and invariant metric $\tilde{g}$. Define*

$$g(X, Y) := \tilde{g}(X, \mathcal{E}^{-1} \circ Y), \quad \forall X, Y \in \mathcal{X}(M). \tag{4}$$

Then $(M, \circ, e, \tilde{g})$ *is a Riemannian* F*-manifold if and only if* $(M, *, \mathcal{E}, g)$ *is a Riemannian* F*-manifold.*

Using Proposition 3 we may construct non-flat invariant metrics on F-manifolds, which satisfy condition (3). Indeed, if in Proposition 3 one takes $\tilde{g}$ - flat, then g is not flat in general and relation (3) holds with multiplication $*$ and metric g.

3) Pairs of metrics. Compatible pairs of metrics were defined in [6] and used to construct bi-hamiltonian structures of non-local type. The geometry of compatible pairs of metrics was studied in [1], where it was shown that the main features of the Dubrovin's correspondendence [4] between Frobenius manifolds and flat pencils of metrics (with some additional conditions), are encoded in the compatibility, rather than the flatness property of the metrics. Let $(g, \tilde{g})$ be two metrics on a manifold M and assume that, for any λ, $g^*_\lambda := g^* + \lambda \tilde{g}^*$ is non-degenerate. (For a metric g on M, g^* denotes the induced metric on T^*M). We denote by $\nabla, \tilde{\nabla}, \nabla^\lambda$ and $R^g, R^{\tilde{g}}, R^\lambda$ the Levi-Civita connections and the curvatures of $g, \tilde{g}, g_\lambda$. The metrics $(g, \tilde{g})$ are called almost compatible [6] if

$$g^*_\lambda(\nabla^\lambda_X \alpha, \beta) = g^*(\nabla_X \alpha, \beta) + \lambda \tilde{g}^*(\tilde{\nabla}_X \alpha, \beta), \quad \forall \alpha, \beta \in \Omega^1(M), \quad \forall X \in \mathcal{X}(M), \quad \forall \lambda.$$

If, moreover,

$$g^*_\lambda(R^\lambda_{X,Y} \alpha, \beta) = g^*(R^g_{X,Y} \alpha, \beta) + \lambda \tilde{g}^*(R^{\tilde{g}}_{X,Y} \alpha, \beta), \quad \forall \alpha, \beta \in \Omega^1(M), \quad \forall X, Y \in \mathcal{X}(M),$$

for any constant λ, then $(g, \tilde{g})$ are called compatible. The following proposition shows how eventual identities may be used to construct compatible pairs of metrics.

Proposition 4. *Let* $(M, \circ, e, \mathcal{E}, \tilde{g})$ *be an* F*-manifold with an eventual identity* $\mathcal{E}$ *and invariant metric* $\tilde{g}$*. Define*

$$g(X, Y) := \tilde{g}(X \circ \mathcal{E}^{-1}, Y), \quad \forall X, Y \in \mathcal{X}(M).$$

Then $(g, \tilde{g})$ *are almost compatible. If, moreover, the coidentity* $\tilde{g}(e)$ *is closed, then* $(g, \tilde{g})$ *are compatible.*

4) tt^***-geometry.** We assume that the initial F-manifold $(M, \circ, e)$ comes with an invariant metric $\tilde{g}$ and real structure $\tilde{k}$, such that $\tilde{h}(X, Y) := \tilde{g}(X, \tilde{k}Y)$ is a pseudo-Hermitian metric. On $(M, \circ, e, \tilde{g}, \tilde{k})$ we impose various conditions from tt^*-geometry (e.g. $(\tilde{h}, \tilde{g})$ are compatible; or $(TM, \tilde{C}, \tilde{h})$ is a harmonic Higgs bundle, where $\tilde{C}_X(Y) := X \circ Y$ is the Higgs field; or $(\tilde{C}, \tilde{g}, \tilde{k})$ extends to a CV structure). In [2] we found obstructions for these conditions to be preserved under the duality. In the restricted setting - semisimple initial multiplication and diagonal real structure, all obstructions vanish.

Acknowledgements. This work was partially supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project no. PN-II-ID-PCE-2011-3-0362 and an EPSRC research grant EP/H019553/1.

References

[1] L. David, I. A. B. Strachan: *Compatible pairs of metrics and non-local bi-Hamiltonian structures*, Internat. Math. Res. Notices. no 66 (2004), p. 3533-3557.
[2] L. David, I. A. B. Strachan: *Dubrovin's duality for F-manifolds with eventual identities*, Adv. Math. (5), vol. 266, p. 4031-4060.
[3] L. David, I. A. B. Strachan: *Symmetries of F-manifolds with eventual identities and special families of connections*, arxiv:1103.2045.
[4] B. Dubrovin: *Flat pencils of metrics and Frobenius manifolds*, in Integrable systems and algebraic geometry (Kobe/Kyoto 1997), 47-72, World Sci. Publishing, River Edge, NJ, 1998.
[5] B. Dubrovin: *On almost duality for Frobenuis manifolds*, Geometry, Topology and Mathematical Physics, 75-132, Am. Math. Soc. Transl. Ser. 2, 212.
[6] E. V. Ferapontov: *Nonlocal Hamiltonian operators of hydrodynamic type: differential geometry and applications*, Am. Math. Soc. Trans. 170 (1995), p. 33-58.
[7] C. Hertling, Y. I. Manin: *Weak Frobenius manifolds*, Internat. Math. Res. Notices, no. 6 (1999), p. 277-286.
[8] P. Lorenzoni, M. Pedroni, A. Raimondo: *F-manifolds and integrable systems of hydrodinamic type*, arxiv:0905.4052.v2.
[9] Y. I. Manin: *F-manifolds with flat structure and Dubrovin's duality*, Adv. Math., vol. 198 (1) (2005), p. 5-26.

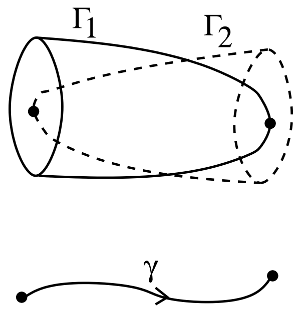
Conifold Period Expansions

Duco van Straten
(joint work with Slawek Cynk)

In the talk two different 1-parameter families of Calabi-Yau 3-folds with $h^{12} = 1, h^{11} = 54$ were described. Both arise as small resolutions of fibre products of rational elliptic surfaces of the type described originally by C. Schoen. Their Picard-Fuchs differential operators are determinded by calculating a conifold power series expansion: the integral of the holomorphic 3-form over a 3-sphere that vanishes at the point of expansion.

The operator of the first family has a point of maximal unipotent monodromy (MUM-point) as is usually encountered in mirror symmetry, whereas the operator of the second family has no such point. W. Zudilin remarked that such operators should be considered as "orphans", as "every Calabi-Yau equation needs a MUM" in order to be listed in the *Table of Calabi-Yau operators of* [AESZ]. Examples of such families without MUM-points were described earlier by J. Rohde [R] and studied further by A. Garbagnati and B. van Geemen [GG]. But in our example the Picard-Fuchs operator has the full Sp_4 as differential Galois group. As such it seems to be the first example and thereby answers a question posed by J. Rohde.

The two families in the talk are constructed as follows. A rational elliptic surface with singular fibres I_1, I_1, I_1, I_1, I_8 is the pull-back of a surfaces with fibres I_1, I_1, II^* under a quadratic map of $\mathbf{P}^1$, which ramifies at the II^* fibre and a variable point, which is the modulus of the surface. Two I_1 fibres are said to be 'in involution' if they are pull-pack of the same I_1-fibre. As a second rational elliptic surface we take the Beauville surface with singular fibres I_1, I_2, I_3, I_6. We use two using different "matchings" of the singular fibres of these surfaces. For both we match the I_8 and I_6, but for the first family we match the I_2 and I_3 fibre with two I_1-fibres 'in involution', for the second with two I_1 fibres that are *not* 'in involution'. A conifold degeneration occurs, when for a special choice of the modulus the I_1-fibre of the Beauville-surface collides with an I_1 fibre of the other surface.



In general, we obtain a vanishing sphere S^3 as 'fibre product' of Lefschetz thimbles Γ_1 and Γ_2 in the two factors of fibre product, taken over a path γ connecting nearby critical points of the two fibrations. The conifold expansion then takes the form

$$\Phi(x) = \int_{S^3} \omega = \int_\gamma F(t,x)G(t,x)dt$$

where F and G are the period integrals of the two factors. After a tedious computation the Picard-Fuchs operator for the second family is found to be the following

$$\begin{aligned}
&\theta^2(\theta-1)(\theta+1) - \tfrac{1}{45}x\theta(\theta+1)(1687\theta^2 - 1121\theta + 10)\\
&+\tfrac{4}{75}x^2(\theta+1)(10645\theta^3 - 4079\theta^2 + 304\theta + 570)\\
&-x^3\left(\tfrac{544}{45} + \tfrac{11856}{25}\theta - \tfrac{748864}{1125}\theta^2 + \tfrac{269152}{75}\theta^3 + \tfrac{1654928}{375}\theta^4\right)\\
&-x^4\left(\tfrac{429952}{225} + \tfrac{1118272}{1125}\theta + \tfrac{395008}{125}\theta^2 - \tfrac{17839232}{1125}\theta^3 - \tfrac{20686912}{1125}\theta^4\right)\\
&+x^5\left(\tfrac{14239232}{1125} + \tfrac{2029312}{75}\theta + \tfrac{11358464}{375}\theta^2 - \tfrac{8363008}{375}\theta^3 - \tfrac{4775424}{125}\theta^4\right)\\
&-x^6\left(\tfrac{221184}{5} + \tfrac{9739264}{75}\theta + \tfrac{161364992}{1125}\theta^2 + \tfrac{15122432}{375}\theta^3 - \tfrac{9969664}{375}\theta^4\right)\\
&+x^7\left(\tfrac{14106624}{125} + \tfrac{127442944}{375}\theta + \tfrac{426057728}{1125}\theta^2 + \tfrac{8126464}{45}\theta^3 + \tfrac{29229056}{1125}\theta^4\right)\\
&-\tfrac{65536}{375}x^8(\theta+1)(260\theta^3 + 1092\theta^2 + 1627\theta + 834) + \tfrac{524288}{125}x^9(\theta+2)(\theta+1)(2\theta+3)^2
\end{aligned}$$

We thank M. Bogner for checking that the differential Galois group of above operator is the full symplectic group Sp_4. It has the following **Riemann scheme**

$$\left\{\begin{array}{ccccccc} -1 & 0 & \frac{1}{8} & \frac{5}{12} & \frac{9}{8} & \frac{5}{4} & \infty \\ \hline 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 \\ 1 & 0 & \frac{1}{2} & 1 & 1 & 1 & \frac{3}{2} \\ 1 & -1 & \frac{3}{2} & 3 & 1 & 1 & \frac{3}{2} \\ 2 & 1 & \frac{3}{2} & 4 & 2 & 2 & 2 \end{array}\right\}$$

One can see that the operator has 4 singular points of 'conifold type' $(-1, 0, 9/8, 5/4)$ where the local monodromy is a symplectic reflection, one apparent singularity $(5/12)$ and two further singular points $(1/8, \infty)$ with Jordan blocks of size two, so there are no MUM-points.

We thank J. Hofmann for computing the following tuple of **monodromy matrices**, which in an appropriate basis of H^3 with integral coefficients take the form:

$$\left(\tfrac{5}{4}, \tfrac{9}{8}, \tfrac{5}{12}, \tfrac{1}{8}, 0, -1\right) \sim \left(\begin{pmatrix} 14 & 0 & -143 & 26 \\ -37 & 1 & 407 & -74 \\ 1 & 0 & -10 & 2 \\ -1 & 0 & 11 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 17 & -1 \\ -1 & 0 & -17 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 20 & 9 & 51 & 9 \\ -42 & -19 & -96 & -18 \\ 0 & 0 & -1 & 0 \\ -7 & -3 & -23 & -4 \end{pmatrix}, \begin{pmatrix} 9 & 4 & 22 & 2 \\ -16 & -7 & -44 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 218 & 93 & 372 & 124 \\ -469 & -200 & -804 & -268 \\ 7 & 3 & 13 & 4 \\ -49 & -21 & -84 & -27 \end{pmatrix} \right)$$

We remark that our varieties are non-projective small resolutions of nodal projective varieties. If we replace the I_8, I_1, I_1, I_1, I_1 surface by the isogenous I_2, I_4, I_4, I_1, I_1 surface (where the two I_4-fibres are in involution), the families are replaced by projective varieties with Hodge numbers $h^{1,2} = 1, h^{1,1} = 33$ for the first, but non-projective varieties with $h^{1,2} = 4, h^{1,1} = 30$ for the second family. There are other operators of this type arising from various fibre products of rational elliptic surfaces and their systematic determination is part of work in progress with S. Cynk.

References

[AESZ] G. Almkvist, C. van Enckevort, D. van Straten, W. Zudilin, *Tables of Calabi–Yau equations*: arXiv:math/0507430.

[CS] S. Cynk, D. van Straten, *Calabi-Yau operators from Conifold Expansions*, in preparation.

[GG] A. Garbagnati, B. van Geemen, *Examples of Calabi-Yau threefolds parametrised by Shimura varieties*. Rend. Semin. Mat. Univ. Politec. Torino 68 (2010), no. 3, 271–287.

[R] J. Rohde, *Maximal automorphisms of Calabi-Yau manifolds versus maximally unipotent monodromy*. Manuscripta Math. 131 (2010), no. 3-4, 459–474.

[S] C. Schoen, *On fiber products of rational elliptic surfaces with section*. Math. Z. 197 (1988), no. 2, 177–199.

Critical Points of Master Functions and Integrable Hierarchies

Alexander Varchenko
(joint work with Daniel Wright)

Generation of critical points. For functions $f(x), g(x)$, denote $W(f,g) = f'(x)g(x) - f(x)g'(x)$ the Wronskian determinant. An N-tuple of polynomials $y = (y_1(x), \dots, y_N(x))$ is called generic if each polynomial has no multiple roots and for any i the polynomials y_i and y_{i+1} have no common roots. Here $y_{N+1} = y_1$.

Consider $\mathbb{C}^k$, $k = k_1 + \dots + k_N$, with coordinates u collected into N groups, the j-th group consists of k_j variables, $u = (u^{(1)}, \dots, u^{(N)})$, $u^{(j)} = (u_1^{(j)}, \dots, u_{k_j}^{(j)})$. The master function on $\mathbb{C}^k$ is

$$\begin{aligned}\Phi(u,z) = \quad & 2\sum_{j=1}^{N} \sum_{0<i<j<k_j+1} \log(u_i^{(j)} - u_{i'}^{(j)}) - \\ & - \sum_{j=1}^{N-1} \sum_{i,i'} \log(u_i^{(j)} - u_{i'}^{(j+1)}) - \sum_{i,i'} \log(u_i^{(N)} - u_{i'}^{(1)}).\end{aligned}$$

Given u, introduce an N-tuple of polynomials in x, $y = (y_1(x), \dots, y_N(x))$, $y^j(x) = \prod_{i=1}^{k_j}(x - u_i^{(j)})$.

Theorem 1 ([3]) *A generic N-tuple y represents a critical point if and only if for any $j = 1, \dots, N$, there exists a polynomial $\tilde{y}_j(x)$ satisfying*

$$W(\tilde{y}_j, y_j) = y_{j-1}y_{j+1}.$$

For $j = 1$, this is $W(\tilde{y}_1, y_1) = y_N y_2$ and for $j = N$, this is $W(\tilde{y}_N, y_N) = y_{N-1}y_1$.

This equation is a first order inhomogeneous differential equation with respect to $\tilde{y}_j$. The solution is $\tilde{y}_j = y_j \int \frac{y_{j-1}y_{j+1}}{y_j^2} dx + c y_j$, where c is an arbitrary number.

Theorem 2 ([3]) *If a generic N-tuple y represents a critical point, then for almost all numbers c, the N-tuple $(y_1, \dots, \tilde{y}_j, \dots, y_N)$ represents a critical point, the exceptional numbers c form a finite set. For exceptional c the N-tuple is not generic.*

We get a one-parameter family of critical points parameterized by $c \in \mathbb{P}^1$. For exactly one value $c \in \mathbb{P}^1$, the the degree of the polynomial $\tilde{y}_j$ drops. For that c some roots of $\tilde{y}_j$ disappear at infinity. All critical points of this family but one are critical points for the same master function, and the one with the smaller degree of $\tilde{y}_j$ is a critical point of the master function with a smaller number of variables.

This theorem gives us a way to generate new critical points. Starting with a critical point we can generate N one-parameter families of critical points. Then we may apply the same procedure to each of the obtained critical points and so on.

Let us choose a starting critical point to be the critical point represented by the tuple $y^\emptyset = (1, \dots, 1)$. Each of the polynomials of the tuple does not have roots. The tuple $y^\emptyset$ represents the critical point of the master function with no variables.

Choose a sequence of integers $J = (j_1, \dots, j_m)$, $j_a \in \{1, \dots, N\}$, and apply to the tuple $y^\emptyset$ the sequence of generation procedures at the indices of J. As a result we obtain an m-parameter family of N-tuples of polynomials

$$y^J(x, c) \;=\; (y_1(x, c), \dots, y_N(x, c)),$$

depending on m integration constants $c = (c_1, \dots, c_m)$.

Example. Let $N = 3$, $J = (1, 2, 3, 1, 2, 3, ...)$. Then $y^\emptyset = (1, 1, 1)$ is transformed to $(x+c_1, 1, 1)$, then to $(x+c_1, \frac{(x+c_1)^2}{2}+c_2, 1)$, then to $(x+c_1, \frac{(x+c_1)^2}{2}+c_2, \frac{(x+c_1)^4}{8}+(x + c_1)c_2 + c_3)$, and so on. The triple $(1, 1, 1)$ represents the critical point of the master function with no variables. The triple $(x+c_1, 1, 1)$ represents critical points of the master function of one variable, namely, the constant function $\Phi : t_1^{(1)} \mapsto 1$. All points of the line are critical points. The triple $(x+c_1, \frac{(x+c_1)^2}{2}+c_2, 1)$ represents the critical points of the master function $\Phi = \log\left(\frac{(t_1^{(2)}-t_2^{(2)})^2}{(t_1^{(2)}-t_1^{(1)})(t_2^{(2)}-t_1^{(1)})}\right)$.

Miura opers and mKdV hierarchy. An $N \times N$ Miura oper is a differential operator of the form $L = \frac{d}{dx} + \Lambda + V$, where $\Lambda = \begin{pmatrix} 0 & 0 & \dots & 0 & \lambda \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$ and $V = \operatorname{diag}(f_1(x), \dots, f_N(x))$ is a diagonal matrix, λ is a parameter of the differential operator.

Drinfeld and Sokolov [2] considered the space of Miura opers and defined on it an infinite sequence of commuting flows $\partial_{t_i}, i = 1, 2, \dots$ called the mKdV hierarchy of $\mathfrak{sl}_N$-type.

Lemma ([2]). *Let $L = \frac{d}{dx} + \Lambda + V$ be a Miura oper. Then there exists $T = 1 + \sum_{i=1}^{\infty} T_i\Lambda^i$, where T_i are diagonal matrices depending on x, such that $T^{-1}LT$ has the form $\frac{d}{dx} + \Lambda + \sum_{i<1} b_i\Lambda^i$, with b_i being scalar functions of x.*

Let L be a Miura oper and $r \in \mathbb{N}$. The differential equation

$$\frac{\partial L}{\partial t_r} = [L, \left(T\Lambda^r T^{-1}\right)^+]$$

is called the *r-th mKdV equation.* Notation $\left(T\Lambda^r T^{-1}\right)^+$ has the following meaning. Given $M = \sum_{i\in\mathbb{Z}} d_i\Lambda^i$ with diagonal matrices d_i, we set $M^+ = \sum_{i>-1} d_i\Lambda^i$.

This equation defines vector fields ∂_{t_r}, $r = 1, 2, \dots$ on the space of Miura opers, the vector fields commute [2].

Main result. To every N-tuple y of polynomials assign the Miura opera L with

$$V = \operatorname{diag}\left(\log'\left(\frac{y_1}{y_N}\right), \log'\left(\frac{y_2}{y_1}\right), \dots, \log'\left(\frac{y_N}{y_{N-1}}\right)\right).$$

For any sequence $J = (j_1, \ldots, j_m)$, consider the corresponding m-parameter family of critical points of the master functions, represented by the family of N-tuples of polynomials $y^J(x,c)$. Let $L^J(c)$ be the corresponding family of Miura opers.

Theorem ([4]) *This family $L^J(c)$ of Miura opers is invariant under every flow of the mKdV hierary and is point-wise fixed by every flow ∂_{t_i} with $i > 2m$.*

Our starting point was the famous paper by Adler-Moser [1] (1978) where this theorem was proved for $N = 2$.

Identities for Schur functions. We know two proofs of the theorem. The first is straightforward: for any flow of the hierarchy we can deform the constants of integration of the generation procedure to move the oper in the direction of the flow. The second proof uses tau-functions and, in particular, Schur functions. Due to lack of space I will formulate only examples of identities for Schur functions that we have obtained. Schur functions $S_\lambda(t_1, t)$ are labeled by partitions $\lambda = (\lambda_0, \lambda_1, \ldots)$ and are functions of t_1 and $t = (t_2, t_3, \ldots)$. Define functions $h_i(t_1, t)$, $i = 0, 1, \ldots$, by
$\exp\left(-\sum_{j=1}^{\infty} t_j z^j\right) = \sum_{i=0}^{\infty} h_i z^i$, and set $S_\lambda = \det\left(h_{\lambda_i - i + j}\right)$.

A. Certain 4-tuples of Schur functions satisfy the Wronskian identity

$$W_{t_1}(S_{\lambda_1}, S_{\lambda_2}) = S_{\lambda_3} S_{\lambda_4}.$$

For example, $W_{t_1}(S_{(2,1)}, S_{(0)}) = S_{(1)}S_{(1)}$ or $W_{t_1}(S_{(4,2,1)}, S_{(2,2,1)}) = S_{(3,2,2,1)}S_{(2,1)}$.

B. For any N, certain N-tuples of Schur functions

$$y(t_1, t) = (S_{\lambda_1}(t_1, t), \ldots, S_{\lambda_N}(t_1, t))$$

are such that for almost all t, the tuple $y(x,t)$ represents a critical point of the master function.

For example for $N = 3$ the triple $(S_{(1,1)}, S_{(2,1,1)}, S_{(1)})$ is such a triple.

Conclusion. We have observed that the critical points of simplest master functions (which are objects of the CFT and Gaudin model theory) are related to the simplest integrable hierarchies. The question is how far can this relation be extended.

References

[1] M. Adler, J. Moser, *On a class of polynomials connected with the Korteweg-de Vries equation*, Comm. Math. Phys. **61** (1978), no. 1, 1–30.

[2] V. Drinfeld, V. Sokolov, *Lie algebras and equations of Korteweg-de Vries type*, (Russian) Current problems in mathematics, Vol. **24** (1984), 81–180.

[3] E. Mukhin, A. Varchenko, *Critical points of master functions and flag varieties*, Commun. Contemp. Math. **6** (2004), no. 1, 111–163.

[4] A. Varchenko, D. Wright, *Critical Points of Master Functions and Integrable Hierarchies*, in preparation.

Distinguished bases and marked singularities

Claus Hertling

The set $\mathcal{B}$ of all distinguished bases (of the Milnor lattice $H_n(f^{-1}(r), \mathbb{Z}) \cong \mathbb{Z}^\mu$ with $r > 0$) of an isolated hypersurface singularity $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is obtained by choosing one fixed generic deformation $F_t : X \to \Delta$ of f (so it has μ A_1-singularities with μ different critical values) and choosing all possible distinguished systems of paths (each path from a critical value to the regular value $r > 0$, $r \in \partial\Delta$). Pushing vanishing cycles along these paths to $H_n(F_t^{-1}(r), \mathbb{Z}) \cong H_n(f^{-1}(r), \mathbb{Z})$ gives one distinguished basis. It is unique up to the signs of the vanishing cycles. See [AGV88] or [Eb01] for all these notions and facts.

Looijenga [Lo73] and Deligne [De74] considered in 1973/74 the ADE-singularities and changed there the point of view. They fixed one tuple $\underline{u} = (u_1, ..., u_\mu)$ of μ different critical values and *one* distinguished system of paths, but they considered all parameters

$$t \in M \cong \mathbb{C}^\mu \cong \text{(base space of the universal unfolding of an ADE-singularity)}$$

with

$$\text{(critical values of } F_t) = \underline{u}.$$

Using the Lyashko-Looijenga map

$$LL : M \to \mathbb{C}^\mu / S_\mu, \quad t \mapsto (\text{critical values of } F_t \mod S_\mu),$$

these are the parameters in the set $LL^{-1}(\underline{u})$. They showed that the map

$$LL^{-1}(\underline{u}) \to \mathcal{B}/\text{(signs)}$$

is a bijection. My slogan of this result is that *the base space of the universal unfolding is an atlas of distinguished bases (up to signs)* for the ADE-singularities.

Here the surjectivity follows from the fact that here LL is a branched covering. Their proof of the injectivity reduces then to showing that

$$|LL^{-1}(\underline{u})| = |\mathcal{B}/\text{(signs)}|.$$

The number $|LL^{-1}(\underline{u})|$ is the degree of the Lyashko-Looijenga map and is determined easily in the case of the ADE-singularities. The number $|\mathcal{B}/\text{(signs)}|$ is determined by a careful analysis of the set $\mathcal{B}$, which is an orbit of the braid group Br_μ.

I have a more conceptual proof using that

$$\begin{aligned} &\mathrm{Aut}(M \text{ as an } F\text{-manifold with Euler field}) \\ (1) \qquad &\cong \mathrm{Aut}(\text{Milnor lattice with Seifert form})/\{\pm \mathrm{id}\} \end{aligned}$$

which follows from considerations of mine on symmetries of singularities in [He02].

C. Roucairol and I showed in 2007 the same result (= the slogan) for the simple elliptic singularities. But here the base space M is the universal cover of the global base space $M^{Jaw} \cong \mathbb{C}^{\mu-1} \times (\mathbb{C} - \{0; 1\})$ of a global unfolding of Jaworski [Ja86] for all simple elliptic singularities in one μ-constant family. Again (1) holds. We also use and improve the results of Jaworski [Ja86] on the Lyashko-Looijenga map for this global unfolding. Especially we determine the (generic) degree of LL on M^{Jaw} by constructing a partial compactification of M^{Jaw} to an orbibundle with generic fibers $\mathbb{C}^{\mu-1}$ and base $\mathbb{P}^1$ such that LL is well defined and finite outside of the zero section. This work is not yet published.

I have 2 steps towards the same goal for all hypersurface singularities.

(1) [He12] A construction of a moduli space of all *marked singularities* in one μ-homotopy class. It is locally isomorphic to a μ-constant stratum.
(2) (Work in progress) A thickening of this moduli space to a μ-dimensional F-manifold which is locally isomorphic to the base space of a universal unfolding of a singularity and such that (1) holds. In the case of the simple elliptic singularities this is the universal covering of M^{Jaw}. I hope that it is the right space in general.

Though the properties of LL and the question whether $LL^{-1}(\underline{u}) \to \mathcal{B}/(\text{signs})$ is bijective are wide open.

References

[AGV88] V.I. Arnold, S.M. Gusein-Zade, A.N. Varchenko: Singularities of differentiable maps, volume II. Birkhäuser, Boston 1988.

[Br83] E. Brieskorn: Milnor lattices and Dynkin diagrams. In: Proceedings of Symposia in Pure Mathematics, **40.1** (1983), 153–165.

[De74] P. Deligne: Letter to Looijenga on March 9, 1974, Reprinted in the diploma thesis of P. Kluitmann, pages 101–111.

[Eb01] W. Ebeling: Funktionentheorie, Differentialtopologie und Singularitäten. Vieweg, Wiesbaden Braunschweig, 2001.

[He02] Hertling, C.: Frobenius manifolds and moduli spaces for singularities. Cambridge Tracts in Mathematics **251**, Cambridge University Press, 2002.

[He12] Hertling, C.: mu-constant monodromy groups and marked singularities. arXiv:1108.0546, 38 pages. To appear in Ann. Inst. Fourier, Grenoble.

[Ja86] P. Jaworski: Distribution of critical values of miniversal deformations of parabolic singularities. Invent. Math. **86** (1986), 19–33.

[Lo73] E. Looijenga: The complement of the bifurcation variety of a simple singularity. Invent. Math. **23** (1974), 105–116.

Geometric structures obtained from almost Frobenius structures

Eduard Looijenga

(joint work with Gert Heckman, Wim Couwenberg, Dali Shen)

Let R be a root system (assumed to be irreducible, reduced). We let $\mathcal{G}$ be a commutative algebraic group scheme of relative dimension one with base field $\mathbb{C}$. More specifically, $\mathcal{G}$ will be the additive group $\mathbb{C}$, the multiplicative group $\mathbb{C}^\times$ or a non-isotrivial family of elliptic curves $\mathcal{E}\ell/\mathsf{S}$ over a connected smooth complex curve S. Note that $\mathrm{Aut}(\mathcal{G})$ is $\mathbb{C}^\times$ for $\mathcal{G} = \mathbb{C}$, but equal to $\{\pm 1\}$ in the other two cases.

Let $\mathsf{Q}(\mathsf{R})$ stand for the the root lattice of R and denote by $\mathrm{Hom}(\mathsf{Q}(\mathsf{R}), \mathcal{G})^\circ$ the group of homomorphisms which have no root in their kernel (in the case of $\mathcal{E}\ell/\mathsf{S}$

we let this be the geometric realization of $\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{O}_S \otimes Q(R), \mathcal{E}\ell/S)^\circ$ so that we get a family over S). For a subgroup $A \subset \mathrm{Aut}(R)$, we put

$$S_A(R, \mathcal{G}) := \mathrm{Aut}(\mathcal{G})\backslash \mathrm{Hom}(Q(R), \mathcal{G})^\circ / A,$$

but omit A from the notation in case $A = \mathrm{Aut}(R)$. For example, for $R = A_r$ and $A \subset \mathfrak{S}_{r+1} \subset \mathrm{Aut}(A_r)$, $S_A(A_r, \mathbb{C})$ can be identified with the quotient of the moduli space $\mathcal{M}_{0,r+2}$ under A by letting A act as permuting the first $r+1$ points.

It has turned out that for many choices of R, A and $\mathcal{G}$, $S_A(R, \mathcal{G})$ has a modular interpretation via a period map and that $S_A(R, \mathcal{G})$ thus acquires a Kähler metric with constant holomorphic curvuture (making it locally isometric to complex projective space with the Fubini-Study metric, a flat space, or to a complex ball with its complex-hyperbolic metric). For instance, the Deligne-Mostow ball quotients are all obtained for some $R = A_r$ and A as above. Another class of examples is furnished by the moduli space consisting of pairs (C, l), where C is a smooth quartic curve and l a line meeting C with prescribed multiplicities is of this type. For example, for the case when l is a double tangent of C resp. a flex line, we get $S(E_7, \mathbb{C})$ resp. $S(E_6, \mathbb{C}^\times)$ and either orbifold gets via this interpretation a complex-hyperbolic metric. Our goal is to produce such metrics in a universal manner.

The general scheme by which this is accomplished is as follows. Let $U \to M$ be a principal $\mathbb{C}^\times$-bundle over a connected complex manifold of dimension n. An *affine structure* on U amounts to giving a flat torsion free connection ∇ on its holomorphic (co)tangent bundle (the subsheaf of $\mathcal{O}_U$ of affine-linear functions is then the set of $f \in \mathcal{O}_U$ with $\nabla(df) = 0$). We assume this affine structure to be homogeneous of a fixed degree $\lambda \in \mathbb{C}$ in the sense that if E is the infinitesimal generator of the $\mathbb{C}^\times$-action (its Euler vector field), and $f \in \mathcal{O}_U$ is affine-linear, then $E(f) - \lambda f$ is constant. In case $\lambda \neq 0$, the f for which this constant is zero, then even define a linear structure on U, but when $\lambda = 0$, then the $\mathbb{C}^\times$-action is in local coordinates like a translation (the affine structure on an $\mathbb{C}^\times$-orbit is then that of $\mathbb{C}/\mathbb{Z}$). This induces on M a projective resp. affine structure. Suppose that on the tangent bundle of U we are further given a flat Hermitian form H such that H is positive definite on the normal bundle, in the sense that we are in one of the following three cases:

- **elliptic:** H is positive definite , $\lambda \in \mathbb{R} - \{0\}$ and M inherits a Fubini-Study metric,
- **parabolic:** H is positive semidefinite with kernel spanned by E, $\lambda = 0$ and M inherits a flat metric,
- **hyperbolic:** H has hyperbolic signature, $H(E, E) < 0$, $\lambda \in \mathbb{R} - \{0\}$ and M inherits a complex hyperbolic metric.

So M is then modelled after the hermitian symmetric space $(D, G) = (\mathbb{P}^n, \mathrm{PU}(n))$, $(\mathbb{C}^n, U(n) \ltimes \mathbb{C}^n)$, $(\mathbb{B}^n, \mathrm{PU}(n, 1))$ respectively and we have a *developing map* $\tilde{M} \to D$, where $\tilde{M} \to M$ is the holonomy cover of M. If Γ denotes the covering group, then this defines a homomorphism $\Gamma \to G$ such that the developing map is Γ-equivariant. We say that the *Schwarz condition* is fulfilled if Γ is discrete in G and that it is *strictly* so if the developing map is an embedding so that M may be identified with an open subset of $\Gamma\backslash D$.

We encounter this situation in the setting of an almost Frobenius structure: then $\nabla^0 = \nabla$ moves in a family of affine structures on U: $\nabla^\kappa = \nabla^0 + \kappa A$, where A is a End(Ω_U)-valued differential of the above type (and with $\lambda = \lambda(\kappa)$ depending affine-linearly on κ) and we are given $H = H^0$ such that we are in the nonhyperbolic case. In the cases of interest we can show that H^0 then deforms with ∇^κ to a family $(\nabla^\kappa, H^\kappa)$ over an interval $[0, \kappa_1)$ such that for some $0 \le \kappa_0 < \kappa_1$, H^κ is elliptic, parabolic resp. elliptic according to whether $\kappa < \kappa_0$, $\kappa = \kappa_0$ or $\kappa > \kappa_0$. We also find values for κ for which the (strict) Schwarz condition is fulfilled.

We carried this program out in detail for $U = \mathrm{Hom}(Q(R), \mathbb{C})^\circ$ (with the $\mathbb{C}^\times$-action given by scalar multiplication) in [2], although at the time we were not aware of the almost Frobenius setting (which was later observed by Feigin and Veselov [4]). The case of the trivial bundle $U = \mathbb{C}^\times \times \mathrm{Hom}(Q(R), \mathbb{C}^\times)^\circ \to \mathrm{Hom}(Q(R), \mathbb{C}^\times)^\circ$ is outlined in [3] and [5] and as Bryan and Gholampour have shown, appears also in Gromov-Witten theory. The case where U is $\mathbb{C}^\times$-bundle over $\mathrm{Hom}(Q(R), \mathcal{E}\ell/S)^\circ$, but that case has yet to treated.

References

[1] J. Bryan, A. Gholampour, *Root systems and the quantum cohomology of ADE resolutions*, Algebra Number Theory 2 (2008), no. 4, 369–390.

[2] W. Couwenberg, G. Heckman, E. Looijenga, *Geometric structures on the complement of a projective arrangement*, Publ. Math. Inst. Hautes Études Sci. No. 101 (2005), 69–161.

[3] W. Couwenberg, G. Heckman, E. Looijenga, *On the geometry of the Calogero-Moser system*, Indag. Math. 16 (2005), no. 3-4, 443–459.

[4] M.V. Feigin, A.P. Veselov, *Logarithmic Frobenius structures and Coxeter discriminants*, Adv. Math. 212 (2007), no. 1, 143–162.

[5] G. Heckman, E. Looijenga, *Hyperbolic structures and root systems*, in: Casimir force, Casimir operators and the Riemann hypothesis, 211–228, Walter de Gruyter, Berlin, 2010.

Reporter: Thomas Reichelt

Participants

Prof. Dr. Sergey Agafonov
rua Capitao Severino Cesarino da Nobrega
215, apto. 201 / Bancarios
Joao Pessoa 58051-220 - PB
BRAZIL

Alexey Basalaev
Institut für Mathematik
Universität Hannover
Welfengarten 1
30167 Hannover

Dr. Liana David
Institute of Mathematics
"Simion Stoilow" of the
Romanian Academy
P.O.Box 1-764
014 700 Bucharest
ROMANIA

Prof. Dr. Boris A. Dubrovin
Lab. of Geometrical Meth. & Math. Physic
Dept. of Mechanics & Mathematics
Moscow State University
Leninskie Gory, 1
Moscow 119 991
RUSSIA

Dr. Oliver Fabert
Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg

Prof. Dr. Martin A. Guest
Department of Mathematics
Tokyo Metropolitan University
Minami-Ohsawa 1-1
Hachioji-shi
Tokyo 192-0397
JAPAN

Prof. Dr. Claus Hertling
Lehrstuhl für Mathematik VI
Fak. für Wirtschaftsinformatik & W.-Math
Universität Mannheim
Seminargebäude A 5
68159 Mannheim

Prof. Dr. Tyler Jarvis
Department of Mathematics
Brigham Young University
Provo , UT 84602-6539
USA

Prof. Dr. Si-Qi Liu
Department of Mathematical Sciences
Tsinghua University
Beijing 100 084
P.R. CHINA

Prof. Dr. Eduard J.N. Looijenga
Mathematisch Instituut
Rijksuniversiteit Utrecht
P.O.Box 80.010
6 Budapestlaan
NL-3584 CD Utrecht

Prof. Dr. Todor Milanov
Institute for the Physics and
Mathematics of the Universe (IPMU)
The University of Tokyo
5-1-5 Kashiwanoha, Kashiwa
Chiba 277-8568
JAPAN

Dr. Thomas Reichelt
Fakultät für Mathematik und
Informatik
Universität Mannheim
68131 Mannheim

Dr. Christian Sevenheck
Lehrstuhl für Mathematik VI
Fak. für Mathematik und Informatik
Universität Mannheim
Seminargebäude A 5
68159 Mannheim

Dr. Sergey Shadrin
Korteweg-de Vries Instituut
Universiteit van Amsterdam
Postbus 94248
NL-1090 GE Amsterdam

Yuuki Shiraishi
Lehrstuhl VI für Mathematik
Universität Mannheim
Seminargebäude A5
68131 Mannheim

Dr. Dmytro Shklyarov
Institut für Mathematik
Universität Augsburg
86135 Augsburg

Prof. Dr. Ian A.B. Strachan
Department of Mathematics
University of Glasgow
University Gardens
GB-Glasgow G12 8QW

Prof. Dr. Duco van Straten
Institut für Mathematik
Johannes-Gutenberg Universität Mainz
Staudingerweg 9
55128 Mainz

Prof. Dr. Atsushi Takahashi
Department of Mathematics
Graduate School of Science
Osaka University
Machikaneyama 1-1, Toyonaka
Osaka 560-0043
JAPAN

Prof. Dr. Alexander Varchenko
Department of Mathematics
University of North Carolina
Phillips Hall
Chapel Hill NC 27599-3250
USA

Prof. Dr. Katrin Wendland
Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg

Prof. Dr. Youjin Zhang
Department of Mathematical Sciences
Tsinghua University
Beijing 100 084
P.R. CHINA